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RIEMANN-TYPE DEFINITION OF THE IMPROPER INTEGRALS

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Abstract. Riemann-type definitions of the Riemann improper integral and of the Lebesgue improper integral are obtained from McShane's definition of the Lebesgue integral by imposing a Kurzweil-Henstock's condition on McShane's partitions.

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1. Introduction

Let $F\colon [a,b] \to \mathbb{R}$ be a differentiable function and let f be its derivative. The problem of recovering F from f is called *the problem of primitives*.

In 1912, the problem of primitives was solved by A. Denjoy with an integration process (called *totalization*) that includes the Lebesgue integral and the Lebesgue improper integral. Equivalent solutions are due to O. Perron, J. Kurzweil and R. Henstock (see for example [4], [6], [7], and [8]).

In 1986, A. M. Bruckner, R. J. Fleissner and J. Foran [3] remarked that the solution provided by Denjoy, Perron, Kurzweil and Henstock possesses a generality which is not needed for the problem of primitives. In fact the function $F(x) = x\sin(1/x^2)$ for $x \in (0,1]$ and $F(0) = 0$ is ACG^* (i.e. a primitive for the Denjoy-Perron-Kurzweil-Henstock integral) but, for any absolutely continuous function G, the function $F - G$ is not differentiable at $x = 0$.

Note that the function $f(x) = F'(x)$ for $x \in (0,1]$ and $f(0) = 0$ is a Riemann improper integrable function. So the minimal integral which includes Lebesgue integrable functions and derivatives (defined descriptively in [3]) does not contain the

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Riemann improper integral. Now this minimal integral can be obtained from McShane's definition of the Lebesgue integral (see for example [4], [5], and [7]) by imposing a mild regularity condition on McShane's partitions (see [1] and [2]).

In this note we prove that the Riemann improper integral and the Lebesgue improper integral can be also obtained from McShane's definition of the Lebesgue integral by imposing a Kurzweil-Henstock's condition on McShane's partitions.

2. Preliminaries

The set of all real numbers is denoted by $\mathbb{R}$. If $E \subset \mathbb{R}$ then $|E|$ denotes the Lebesgue measure of E, and $\overline{E}$ its closure. In this paper $[a,b]$ is a fixed interval of $\mathbb{R}$. By *McShane's partition* of $[a,b]$ we mean any finite collection $\{(A_h, x_h)\}_{h=1}^{p}$ of pairwise disjoint intervals A_h and points $x_h \in [a,b]$ such that $[a,b] = \bigcup_h A_h$. We say that a McShane's partition $\{(A_h, x_h)\}_{h=1}^{p}$ satisfies the Kurzweil-Henstock condition (br. K-H condition) on a set $E \subset [a,b]$ whenever $x_h \in A_h$ if $x_h \in E$.

A *McShane's partition* satisfying the K-H condition on $[a,b]$ is called a *Kurzweil-Henstock's partition* of $[a,b]$.

Let δ be a gauge (i.e. a positive function) on $[a,b]$. A *McShane's partition* of $[a,b]$, say $\{(A_h, x_h)\}_{h=1}^{p}$, is said to be δ-fine whenever $A_h \subset (x_h - \delta(x_h), x_h + \delta(x_h))$, for each h.

We recall that a function $f\colon [a,b] \to \mathbb{R}$ is said to be McShane (resp. Kurzweil-Henstock) integrable on $[a,b]$ if there is a real number I satisfying the following condition: for each $\varepsilon > 0$ there exists a gauge δ such that

$$\left|\sum_{h=1}^{p} f(x_h)|A_h| - I\right| < \varepsilon, \tag{1}$$

for each δ-fine McShane's (resp. Kurzweil-Henstock's) partition P of $[a,b]$ (see [4], [5], [6] and [7]).

I is called the McShane (resp. Kurzweil-Henstock) integral of f on $[a,b]$. The McShane integral is equivalent to the Lebesgue integral, and the Kurzweil-Henstock integral is equivalent to the Denjoy-Perron integral (see [4], [7]).

In this paper the Kurzweil-Henstock integral of f on $[a,b]$ is denoted by $\int_a^b f$. Since the Riemann integral, the Riemann improper integral, the Lebesgue integral and the Lebesgue improper integral are contained in the Kurzweil-Henstock integral, then by $\int_a^b f$ we also denote each of the mentioned integrals on $[a,b]$.

For simplicity, in the sequel we set $\sum_P f = \sum_{h=1}^{p} f(x_h)|A_h|$.

We also recall that a function $f\colon [a,b]\to\mathbb{R}$ is said to be Lebesgue (resp. Riemann) improper integrable on $[a,b]$ if there exist $a_i\in[a,b]$, $i=0,1,\dots,k$ with $a=a_0<a_1<\dots<a_k=b$ such that f is Lebesgue (resp. Riemann) integrable on each compact subinterval $[\alpha,\beta]$ of (a_i,a_{i+1}), and the limits

$$\lim_{\substack{\alpha\searrow a_i\\ \beta\nearrow a_{i+1}}}\int_\alpha^\beta f \tag{2}$$

exist finite, for $i=0,1,\dots,k-1$.

3. The Lebesgue improper integral

In this section we prove

Theorem 1. *A function $f\colon [a,b]\to\mathbb{R}$ is Lebesgue improper integrable on $[a,b]$ if and only if there exists a finite set $E\subset[a,b]$ and a real number I such that:*

(LI) *for each $\varepsilon>0$ there exists a gauge δ so that $\left|\sum_P f-I\right|<\varepsilon$ for each δ-fine McShane's partition P of $[a,b]$ satisfying the* K-H *condition on E.*

Proof. Assume that f is Lebesgue improper integrable on $[a,b]$. Then there exist $a=a_0<a_1<\dots<a_k=b$, such that f is Lebesgue integrable on each compact subinterval $[\alpha,\beta]$ of (a_i,a_{i+1}), and the limits (2) exist finite for each i.

Given $i\in\{0,1,\dots,k-1\}$, let $\{\alpha_{i,j}\}_{j\in\mathbb{Z}}\subset(a_i,a_{i+1})$ be an increasing sequence such that $\lim\limits_{j\to-\infty}\alpha_{i,j}=a_i$ and $\lim\limits_{j\to\infty}\alpha_{i,j}=a_{i+1}$.

Fixed $\varepsilon>0$, $i\in\{0,1,\dots,k-1\}$ and $j\in\mathbb{Z}$, by Henstock's lemma there exists a gauge $\delta_{i,j}$ on $(\alpha_{i,j},\alpha_{i,j+1})$ such that

$$\sum_{(A,x)\in P_{i,j}}\left|f(x)|A|-\int_A f\right|<\frac{\varepsilon}{k2^{|j|+3}} \tag{3}$$

for each $\delta_{i,j}$-fine McShane's partition $P_{i,j}$ of $(\alpha_{i,j},\alpha_{i,j+1})$.

For $x\neq a_i$, $0\leqslant i\leqslant k$ we set

$$\delta(x)=\begin{cases}\min\{\delta_{i,j}(x),x-\alpha_{i,j},\alpha_{i,j+1}-x\}, & \text{if } \alpha_{i,j}<x<\alpha_{i,j+1};\\ \min\{\delta_{i,j-1}(x),\delta_{i,j}(x)\}, & \text{if } x=\alpha_{i,j}.\end{cases}$$

Moreover, for $x=a_i$, $0\leqslant i\leqslant k$, we define $\delta(a_i)>0$ such that

$$\delta(a_i)|f(a_i)|<\frac{\varepsilon}{8(k+1)}, \tag{4}$$

and

$$\left|\int_{a_i}^{\alpha} f\right| < \frac{\varepsilon}{8(k+1)} \tag{5}$$

for each $\alpha \in [a,b]$ with $|\alpha - a_i| < \delta(a_i)$.

Let $\{(A_h, x_h)\}_{h=1}^{p}$ be a δ-fine McShane's partition of $[a,b]$ satisfying the K-H condition on $\{a_0,\dots,a_k\}$. Then, by (4) and (5) we have

$$\begin{aligned}\sum_{x_h\in\{a_0,\dots,a_k\}}\left|f(x_h)|A_h| - \int_{A_h} f\right| &\leqslant \sum_{x_h\in\{a_0,\dots,a_k\}}\left(|f(x_h)|\,|A_h| + \left|\int_{A_h} f\right|\right)\\ &\leqslant \sum_{x_h\in\{a_0,\dots,a_k\}}\left(2\cdot\frac{\varepsilon}{8(k+1)} + 2\cdot\frac{\varepsilon}{8(k+1)}\right)\\ &\leqslant (k+1)\cdot\left(\frac{\varepsilon}{4(k+1)} + \frac{\varepsilon}{4(k+1)}\right) = \frac{\varepsilon}{2}.\end{aligned}$$

So, by (3) we get

$$\begin{aligned}\left|\sum_{h=1}^{p} f(x_h)|A_h| - \int_a^b f\right| &\leqslant \sum_{x_h\in\{a_0,\dots,a_k\}}\left|f(x_h)|A_h| - \int_{A_h} f\right|\\ &\quad + \sum_{i=0}^{k-1}\sum_{j=-\infty}^{+\infty}\sum_{x_h\in(\alpha_{i,j},\alpha_{i,j+1})}\left|f(x_h)|A_h| - \int_{A_h} f\right|\\ &\leqslant \frac{\varepsilon}{2} + \sum_{i=0}^{k-1}\sum_{j=-\infty}^{+\infty}\frac{\varepsilon}{k2^{|j|+3}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Therefore condition (LI) is satisfied.

Vice versa, assume that there exist $a = a_0 < a_1 \dots < a_k = b$, and a real number I satisfying condition (LI). Then f is Kurzweil-Henstock integrable on $[a,b]$ with $I = \int_a^b f$. Moreover, by an easy adaptation of Henstock's lemma (see [4, Lemma 9.11]) we have

$$\sum_{h=1}^{p}\left|f(x_h)|A_h| - \int_{A_h} f\right| < 2\varepsilon,$$

for each δ-fine McShane's partition $\{(A_h, x_h)\}_{h=1}^{p}$ of $[a,b]$ satisfying the K-H condition on E. This implies that f is McShane integrable (hence Lebesgue integrable) on each compact subinterval of (a_i, a_{i+1}), $i = 0, 1, \dots, k-1$ and, by the continuity of the Kurzweil-Henstock integral, the limits

$$\lim_{\substack{\alpha\searrow a_i\\ \beta\nearrow a_{i+1}}}\int_\alpha^\beta f$$

exist finite, for $i=0,1,\ldots,k-1$.

In conclusion f is Lebesgue improper integrable on $[a,b]$. □

4. The Riemann improper integral

In this section we prove

Theorem 2. *A function $f\colon [a,b]\to\mathbb{R}$ is Riemann improper integrable on $[a,b]$ if and only if there exists a finite set $E\subset[a,b]$ and a real number I such that*

(RI) *for each $\varepsilon>0$ there exists a gauge δ so that*
  (1) *δ is continuous on $[a,b]\setminus E$;*
  (2) $\Bigl|\sum\limits_P f-I\Bigr|<\varepsilon$, *for each δ-fine McShane's partition P of $[a,b]$ satisfying the* K-H *condition on E.*

The proof is based on the following lemma:

Lemma 1. *If $f\colon [a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$, then for each $\varepsilon>0$ there exists a constant $\eta>0$ such that*

$$\sum_{h=1}^{p}\Bigl|f(x_h)|A_h|-\int_{A_h}f\Bigr|<\varepsilon,$$

for each η-fine McShane's partition $\{(A_h,x_h)\}_{h=1}^p$ of $[a,b]$.

Proof. By the definition of Riemann integral, to each $\varepsilon>0$ there exists a constant $\eta_1>0$ such that

$$\Bigl|\sum_{h=1}^{p}f(y_h)|A_h|-\int_a^b f\Bigr|<\frac{\varepsilon}{8},$$

for each η_1-fine Kurzweil-Henstock's partition $\{(A_h,y_h)\}_{h=1}^p$ of $[a,b]$. Then, by an easy adaptation of Henstock's lemma, we have

$$\sum_{h=1}^{p}\Bigl|f(y_h)|A_h|-\int_{A_h}f\Bigr|<\frac{\varepsilon}{4}, \tag{6}$$

for each η_1-fine Kurzweil-Henstock's partition $\{(A_h, y_h)\}_{h=1}^p$ of $[a, b]$. Now, for each natural n, let

$$E_n = \left\{x \in [a, b]\colon \omega\Big(f, \Big(x - \frac{1}{n}, x + \frac{1}{n}\Big)\Big) \geqslant \frac{\varepsilon}{4(b-a)}\right\},$$

where $\omega(f, A)$ stands for the oscillation of f on the set A.

It is easy to see that $\overline{E}_n \subset E_{n+1}$. Moreover, since the set of points of discontinuity of a Riemann integrable function is a Lebesgue null set, we have $\left|\bigcap\limits_{n=1}^{\infty} E_n\right| = 0$. Hence $|\overline{E}_n| \to 0$. Let $M = \sup\{|f(x)|\colon x \in [a, b]\}$, and let n_0 be such that $|\overline{E}_{n_0}| < \varepsilon/(4M)$.

Now, let $\{(\alpha_i, \beta_i)\}_{i=1}^{\infty}$ be a covering of $\overline{E}_{n_0}$ with

$$\sum_{i=1}^{\infty} (\beta_i - \alpha_i) < \frac{\varepsilon}{4M}. \tag{7}$$

Then, there is i_0 such that $\overline{E}_{n_0} \subset \bigcup\limits_{i=1}^{i_0} (\alpha_i, \beta_i)$. Set

$$\eta_2 = \inf\left\{|x - y|\colon x \in \overline{E}_{n_0} \text{ and } y \notin \bigcup_{i=1}^{i_0} (\alpha_i, \beta_i)\right\},$$

and

$$\eta = \min\{\eta_1, \eta_2, 1/n_0\}.$$

Let $\{(A_h, x_h)\}_{h=1}^p$ be an η-fine McShane's partition of $[a, b]$, and let $y_h \in A_h$ for $h = 1, \ldots, p$. Then (6) holds. Moreover, by definitions of E_{n_0} and η, for $h = 1, \ldots, p$ we have

$$|f(x_h) - f(y_h)| < \frac{\varepsilon}{4(b-a)}, \text{ if } x_h \notin E_{n_0},$$

and

$$A_n \subset \bigcup_{i=1}^{i_0} (\alpha_i, \beta_i), \text{ if } x_h \in E_{n_0}.$$

Thus, by (7) we get

$$\sum_{x_h \in E_{n_0}} |A_h| < \frac{\varepsilon}{4M}.$$

In conclusion

$$\begin{aligned}
\sum_{h=1}^{p}\Big|f(x_h)|A_h| - \int_{A_h} f\Big| \\
&\leqslant \sum_{h=1}^{p} |f(x_h)-f(y_h)|\,|A_h| + \sum_{h=1}^{p}\Big|f(y_h)|A_h| - \int_{A_h} f\Big| \\
&\leqslant \sum_{h=1}^{p} |f(x_h)-f(y_h)|\,|A_h| + \frac{\varepsilon}{4} \\
&= \sum_{x_h\in E_{n_0}} |f(x_h)-f(y_h)|\,|A_h| + \sum_{x_h\notin E_{n_0}} |f(x_h)-f(y_h)|\,|A_h| + \frac{\varepsilon}{4} \\
&< 2M \sum_{x_h\in E_{n_0}} |A_h| + \frac{\varepsilon}{4(b-a)} \sum_{x_h\notin E_{n_0}} |A_h| + \frac{\varepsilon}{4} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

□

Corollary 1. *A function $f\colon [a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$, if and only if there exists a real number I satisfying the following condition: for each $\varepsilon>0$ there exists a positive constant η such that $\Big|\sum_P f - I\Big| < \varepsilon$, for each η-fine McShane's partition $P=\{(A_h,x_h)\}_{h=1}^{p}$ of $[a,b]$.*

Proof of Theorem 2. Assume that f is Riemann improper integrable on $[a,b]$. Then there exist $a=a_0<a_1<\ldots<a_k=b$ such that f is Riemann integrable on each interval $[\alpha,\beta]\subset(a_i,a_{i+1})$, and the limits (2) exist finite, for $i=0,1,\ldots,k-1$.

Given $i\in\{0,1,\ldots,k-1\}$, let $\{\alpha_{i,j}\}_{j\in\mathbb{Z}}\subset(a_i,b_i)$ be an increasing sequence such that $\lim\limits_{j\to-\infty}\alpha_{i,j}=a_i$ and $\lim\limits_{j\to\infty}\alpha_{i,j}=a_{i+1}$.

For $i\in\{0,1,\ldots,k-1\}$, $j\in\mathbb{Z}$, and $\varepsilon>0$, by Lemma 1 there exist $0<\tau_{i,j}<\min\{\alpha_{i,j+1}-\alpha_{i,j},\alpha_{i,j+2}-\alpha_{i,j+1},\alpha_{i,j+3}-\alpha_{i,j+2}\}$ such that condition (3) is satisfied for each $\tau_{i,j}$-fine McShane's partition $P_{i,j}$ of $[\alpha_{i,j},\alpha_{i,j+3}]$. Define $\delta_{i,j}\leqslant\min\{\tau_{i,j-2},\tau_{i,j-1},\tau_{i,j}\}$ such that $\delta_{i,j}\leqslant\delta_{i,j+1}$ for $j<0$, $\delta_{i,-1}=\delta_{i,0}$ and $\delta_{i,j}\geqslant\delta_{i,j+1}$, for $j\geqslant0$. Moreover, for $i=0,1,\ldots,k$ take $\delta(a_i)>0$ such that conditions (4) and (5) are satisfied. Finally, for $i\in\{0,1,\ldots,k-1\}$ and $x\in(a_i,a_{i+1})$ define

$$\delta(x)=\begin{cases}\delta_{i,j-1}+\dfrac{x-\alpha_{i,j}}{\alpha_{i,j+1}-\alpha_{i,j}}(\delta_{i,j}-\delta_{i,j-1}), & \text{if } x\in[\alpha_{i,j},\alpha_{i,j+1}) \text{ and } j<0;\\[2ex] \delta_{i,j}+\dfrac{x-\alpha_{i,j}}{\alpha_{i,j+1}-\alpha_{i,j}}(\delta_{i,j+1}-\delta_{i,j}), & \text{if } x\in[\alpha_{i,j},\alpha_{i,j+1}) \text{ and } j\geqslant0.\end{cases}$$

Note that δ is continuous on $[a,b] \setminus \bigcup_{i=1}^{p} \{a_i\}$ and $\delta(x) \leqslant \delta_{i,j}$ for $x \in [\alpha_{i,j}, \alpha_{i,j+1})$, $i = 0, 1, \dots, k-1$, $j \in \mathbb{Z}$.

To verify condition $(\mathrm{RI})_2$, let $\{(A_h, x_h)\}_{h=1}^{p}$ be a δ-fine McShane's partition of $[a,b]$ satisfying the K-H condition on $\{a_0, \dots, a_k\}$. Then, by (4) and (5) we have

$$\sum_{x_h \in \{a_0,\dots,a_k\}} \left| f(x_h)|A_h| - \int_{A_h} f \right| \leqslant \frac{\varepsilon}{2}. \tag{8}$$

Now, whenever $x_h \in [\alpha_{i,j}, \alpha_{i,j+1}]$ with $j \geqslant 0$, then

$$\begin{aligned} A_h &\subset (x_h - \delta(x_h), x_h + \delta(x_h)) \subset (\alpha_{i,j} - \delta_{i,j}, \alpha_{i,j+1} + \delta_{i,j}) \\ &\subset (\alpha_{i,j} - \tau_{i,j-1}, \alpha_{i,j+1} + \tau_{i,j-1}) \subset [\alpha_{i,j-1}, \alpha_{i,j+2}], \end{aligned}$$

and whenever $x_h \in [\alpha_{i,j}, \alpha_{i,j+1}]$ with $j < 0$, then

$$\begin{aligned} A_h &\subset (x_h - \delta(x_h), x_h + \delta(x_h)) \subset (\alpha_{i,j} - \delta_{i,j+1}, \alpha_{i,j+1} + \delta_{i,j+1}) \\ &\subset (\alpha_{i,j} - \tau_{i,j-1}, \alpha_{i,j+1} + \tau_{i,j-1}) \subset [\alpha_{i,j-1}, \alpha_{i,j+2}]. \end{aligned}$$

Therefore the family $\{(A_h, x_h)\colon\ x_h \in [\alpha_{i,j}, \alpha_{i,j+1}]\}$ is a $\tau_{i,j-1}$-fine McShane's partial partition in $[\alpha_{i,j-1}, \alpha_{i,j+2}]$. Thus, by (3) and (8) we have

$$\begin{aligned} \left| \sum_{h=1}^{p} f(x_h)|A_h| - \int_a^b f \right| &\leqslant \sum_{x_h \in \{a_0,\dots,a_k\}} \left| f(x_h)|A_h| - \int_{A_h} f \right| \\ &\quad + \sum_{i=0}^{k-1} \sum_{j=-\infty}^{+\infty} \sum_{x_h \in (\alpha_{i,j}, \alpha_{i,j+1})} \left| f(x_h)|A_h| - \int_{A_h} f \right| \\ &\leqslant \frac{\varepsilon}{2} + \sum_{i=0}^{k-1} \sum_{j=-\infty}^{+\infty} \frac{\varepsilon}{k 2^{|j|+3}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This completes the proof of condition $(\mathrm{RI})_2$.

Vice versa, assume that there exist $a = a_0 < a_1 \dots < a_k = b$, and a real number I satisfying condition (RI). Since condition (RI) implies condition (LI), then f is Lebesgue improper integrable on $[a,b]$, by Theorem 1. Thus we have only to prove that f is Riemann integrable on each compact subinterval of (a_i, a_{i+1}), for $i = 0, 1, \dots, k-1$. Let $[\alpha, \beta] \subset (a_i, a_{i+1})$, $i = 0, 1, \dots, k-1$. By $(\mathrm{RI})_2$ and by an easy adaptation of Henstock's lemma, we have $\left| \sum_P f - I \right| < 2\varepsilon$, for each δ-fine McShane's partition P of $[\alpha, \beta]$. Now, by continuity of δ on $[\alpha, \beta]$, we have $\eta = \min\{\delta(x)\colon\ x \in [\alpha, \beta]\} > 0$. Therefore $\left| \sum_P f - I \right| < 2\varepsilon$, for each η-fine McShane's (hence Kurzweil-Henstock's) partition P of $[\alpha, \beta]$. Thus f is Riemann integrable on $[\alpha, \beta]$, and the proof is complete. □

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