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Cite this article: Bologna E, Zingales M. 2018 Stability analysis of Beck's column over a fractional-order hereditary foundation. *Proc. R. Soc. A* **474**: 20180315. http://dx.doi.org/10.1098/rspa.2018.0315

Received: 16 May 2018
Accepted: 27 September 2018

Subject Areas:
structural engineering, differential equations

Keywords:
follower forces, fractional calculus, Routh–Hurwitz criterion, state space approach

Author for correspondence:
M. Zingales
e-mail: massimiliano.zingales@unipa.it

Stability analysis of Beck's column over a fractional-order hereditary foundation

E. Bologna[1,2] and M. Zingales[1,2]

[1]Dipartimento di Ingegneria Civile, Ambientale, Aerospaziale, dei Materiali, Viale delle Scienze, ed. 8, 90100 Palermo, Italy
[2]Bio/NanoMechanics for Medical Sciences Laboratory (BNM[2]-Lab) Advanced Technology Network (ATeN)-Center, CHAB Pole, Viale delle Scienze, Ed. 18, 90128 Palermo, Italy

EB, 0000-0002-6209-1599

This paper considers the case of Beck's column resting on a hereditary bed of independent springpots. The springpot possesses an intermediate rheological behaviour among linear spring and linear dashpot. It is defined by means of couple (C_β, β) that characterize the material of the element and is ruled by a Caputo's fractional derivative. In this paper, we investigate the critical load of the column under the action of a follower load by means of a novel complex transform that allows to use the Routh–Hurwitz theorem in the complex half-plane for the stability analysis.

1. Introduction

Stability analysis of a cantilevered column subjected to constant intensity follower force applied to its free end is known as Beck's problem [1]. Since the first introduction [2], the problem has been investigated by several authors [3,4] for different engineering application [5–16]. Models of follower agencies involves rocket thrust [17,18], MEMS [19], spine biomechanical problems [20,21], civil engineering application. In several physical applications, however, the cantilever is externally restrained to model the external medium surrounding the column. External restraints are often represented as a bed of elastic independent springs [22] or of linear viscous elements [23]. In the former case, the presence of elastic restraints yields the well-known Hermann–Smith paradox [22]. In the latter case, the viscous external restraints provide, instead, the Ziegler paradox [3,24]. Models of external restraints in

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terms of linear springs and linear dashpots represents however limiting cases of real material behaviour known as Hookean or Newtonian description [25,26].

In order to capture experimental data on complex materials, combinations of spring and dashpots with specific arrangements have been proposed in scientific literature [27]. However, these models are often not suitable to described material behaviour in both displacement control (relaxation) or force control (creep) tests.

Recently, hereditary material description has been provided by means of the so-called springpot. This element is represent by a linear hereditary mathematical operators defined by two parameters $C_\beta > 0$ and $\beta \in [0, 1]$ [27,28]. Such mathematical operator is defined as fractional derivative [29–32] and it is basically a convolution integral with hypersingular kernel. Stability of dynamical systems ruled by fractional-order differential equations has been dealt in recent papers [33–37] by resorting to a Laplace domain approach. More recently the analysis of Beck's column resting different time of fractional-order external restraint has also been investigated to assess the influence of different kinds of external supports on the stability boundaries [38–40].

The existing approaches, however, do not allow for a parametric analysis since no exact solution of the eigenvalues problem may be obtained and the well-celebrated Routh–Hurwitz criterion is not applicable. Indeed, the Routh–Hurwitz approach yields the number of eigenvalues χ_j with positive real parts. Such a condition is not sufficient to enforce stability of dynamical systems with fractional derivative of order β since the stability is assessed in a sector of a complex Cauchy plane that satisfies the condition $\arg(\chi_j) \le |\pi\beta|/2 \, \forall\, j = 1, 2, \ldots n$, with n the number of state variables of the dynamical system [25,41].

In this paper, the authors deal with the study of the cantilever Beck's column resting over a fractional-order hereditary foundation with a different approach extending the Routh–Hurwitz criterion to the stability of fractional differential dynamical systems. The influence of the differentiation order β and the anomalous dissipation coefficient C_β are investigated showing some an unexpected effect of β on the critical load. The paper is organized as follows: §2 deals with the derivation of the column dynamic equilibrium equations over a hereditary foundation. In §3, the stability analysis of the column in terms of the state-space variables is provided §3b. The generalization of the Routh–Hurwitz criterion for the stability of fractional-order dynamical equilibrium is reported in §3c and parametric analysis is shown in §4. Some conclusions have been reported in §5.

2. Dynamic equilibrium of an elastic cantilever over a fractional-order foundation

Let us consider a Bernoulli–Euler elastic column with cross section A, length L and rotary inertial properties across the centroidal axis denoted as J_1. Young modulus of elasticity has been dubbed as E and the material density as ρ, both assumed homogeneous along the column axis, dubbed x_3 in the following (figure 1). The column is clamped at $x_3 = 0$ and it rests on a bed of vertical and independent springpots defined by the couples (C_β, β) assumed homogeneous along the x_3-axis. The column is subjected to a follower axial load, dubbed P at the free end (figure 1). In the following, we restrict our analysis to the onset of bifurcations from the undeformed (straight) configuration (figure 1) in the $(x_2 - x_3)$ plane defined by the vertical displacement field $w(x_3)$ of the column axis.

The governing equation of column may be obtained in classical fashion with the equilibrium of a column element (figure 2) where we denoted $[\bullet]^{\cdot\cdot} = \partial^2/\partial t^2$ and F_f the reaction of the fractional-order hereditary support of the column foundation.

The equilibrium equation of the column reads (omitting time argument):

$$\begin{aligned} &N(x_3 + \Delta x_3)\cos(\phi(x_3) + \Delta\phi(x_3)) - N(x_3)\cos(\phi(x_3)) \\ &\quad + T(x_3)\sin(\phi(x_3)) - T(x_3 + \Delta x_3)\sin(\phi(x_3) + \Delta\phi(x_3)) = 0 \end{aligned} \tag{2.1a}$$

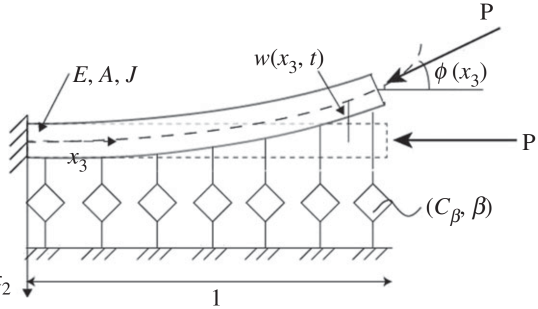

Figure 1. Onset of the column bifurcations and its generic section.

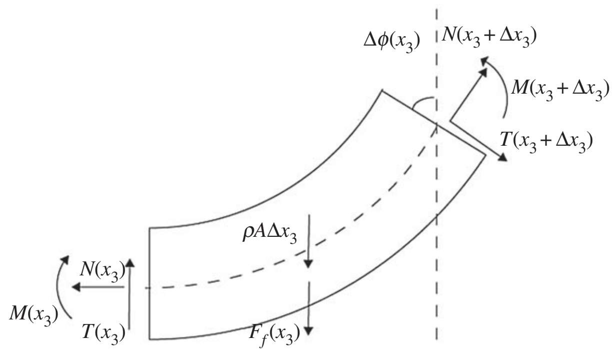

Figure 2. Onset of the column bifurcation.

$$\begin{aligned}T(x_3)\sin(\phi(x_3)) + N(x_3)\sin(\phi(x_3)) - T(x_3+\Delta x_3)\cos(\phi(x_3)+\Delta\phi(x_3))\\ - N(x_3+\Delta x_3)\sin(\phi(x_3)+\Delta\phi(x_3)) + \rho A\frac{\partial^2 w}{\partial t^2}\Delta x_3 + F_f(x_3) = 0\end{aligned} \tag{2.1b}$$

and

$$\begin{aligned}T(x_3+\Delta x_3)\cos(\phi(x_3)+\Delta\phi(x_3))\Delta x_3 + T(x_3+\Delta x_3)\sin(\phi(x_3)\\ + \Delta\phi(x_3))\Delta w(x_3) + M(x_3) - M(x_3+\Delta x_3) = 0,\end{aligned} \tag{2.1c}$$

where in the latter equation, we neglected the contribution of the axial stress moment that is a higher-order infinitesimal. Since onset of bifurcation is analysed the following approximations hold time $\sin(x) \cong x$; $\cos(x) \cong 1$ yielding the equilibrium equations in linearized formulation:

$$N(x_3+\Delta x_3) - N(x_3) + T(x_3)\phi(x_3) - T(x_3+\Delta x_3)(\phi(x_3)+\Delta\phi(x_3)) = 0 \tag{2.2a}$$

$$\begin{aligned}T(x_3)\phi(x_3) + N(x_3)\phi(x_3) - T(x_3+\Delta x_3) - N(x_3+\Delta x_3)(\phi(x_3)+\Delta\phi(x_3))\\ + \rho A\frac{\partial^2 w}{\partial t^2}\Delta x_3 + F_f(x_3) = 0\end{aligned} \tag{2.2b}$$

and

$$T(x_3+\Delta x_3)\Delta x_3 + T(x_3+\Delta x_3)(\phi(x_3)+\Delta\phi(x_3))\Delta w(x_3) + M(x_3) - M(x_3+\Delta x_3) = 0. \tag{2.2c}$$

After some straightforward manipulations, and neglecting higher-order contributions, the governing equations of the onset of bifurcation read:

$$\frac{\partial N(x_3,t)}{\partial x_3} = 0 \tag{2.3a}$$

$$\rho A\frac{\partial^2 w(x_3,t)}{\partial t^2} + F_f(x_3,t) - \frac{\partial T(x_3,t)}{\partial x_3} - \frac{\partial N(x_3,t)}{\partial x_3} = 0 \tag{2.3b}$$

and

$$\frac{\partial M(x_3,t)}{\partial x_3} = T(x_3,t), \tag{2.3c}$$

yielding, after substitutions $\phi(x_3,t) = -\partial w(x_3,t)/\partial x_3$; $N(x_3,t) = -P$; $M(x_3,t) = -EJ(\partial^2 w(x_3,t)/\partial {x_3}^2)$ the governing equation of the transverse displacement of the column over a fractional-order foundation as:

$$\rho A \frac{\partial^2 w(x_3,t)}{\partial t^2} + EJ \frac{\partial^4 w(x_3,t)}{\partial {x_3}^4} + P \frac{\partial^2 w(x_3,t)}{\partial {x_3}^2} + C_\beta (D^\beta_{0+} w)(x_3,t) = 0. \tag{2.4}$$

In equation (2.4), the fractional-derivative terms generalizes the case of Newtonian viscous foundation and it contains the anomalous viscosity coefficient $[C_\beta] = F/LT^\beta$ that depends on the order of differentiation and that reduces to the Newtonian viscosity as $\beta = 1$. The boundary value problem associated with equation (2.4) is defined with the aid of the initial and boundary conditions:

$$w(0,t) = \left.\frac{\mathrm{d}w}{\mathrm{d}x_3}\right|_0 = 0; \quad EJ \left.\frac{\partial^2 w}{\partial {x_3}^2}\right|_L = EJ \left.\frac{\partial^3 w}{\partial {x_3}^3}\right|_L = 0 \tag{2.5}$$

and

$$w(x_3,0) = 0; \quad \dot{w}(x_3,0) = 0. \tag{2.6}$$

It may be observed that (2.1*a*)–(2.1*c*) rules the mechanics of the column resting on a fractional-order foundation that is formally analogous to a well-known column on an elastic [2] and viscous [1] foundation.

The onset of stability of the Beck's column resting on the fractional-order foundation is analysed hereinafter in non-dimensional form obtained by casting equation (2.4) in the form:

$$\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + \frac{\partial^4 \bar{w}}{\partial \zeta^4} + \lambda^2 \frac{\partial^2 \bar{w}}{\partial \zeta^2} + \tau_0^{1-\beta} \tau_\beta (D^\beta_{0+} \bar{w})(\zeta,\bar{t}) = 0, \tag{2.7}$$

where $\zeta = x_3/L$; $\bar{t} = t/\tau_0$; $\tau_0 = PAL^4/EJ$; $\bar{w} = w/L$; $\lambda^2 = PL^2/EJ$; and $\tau_\beta = C_\beta(L^4/EJ)$, with the associate boundary conditions:

$$\bar{w}(0,t) = \left.\frac{\partial \bar{w}}{\partial \zeta}(\zeta,t)\right|_0 = \left.\frac{\partial^2 \bar{w}}{\partial \zeta^2}(\zeta,t)\right|_1 = \left.\frac{\partial^3 \bar{w}}{\partial \zeta^3}(\zeta,t)\right|_1 = 0. \tag{2.8}$$

Despite these formal analogies, stability analysis of the column in the presence of fractional order time-derivative needs a specific core as will be discussed in the next section.

3. Stability analysis of Beck's column over fractional-order hereditary foundation

In this section, the evaluation of the critical instability load of the Beck's column is outlined. Stability analysis is conducted, as usual, introducing the variable separation $\bar{w}(\zeta,\bar{t}) = v(\zeta)y(\bar{t})$ yielding, after some straightforward manipulations:

$$\frac{\mathrm{d}^4 v(\zeta)}{\mathrm{d}\zeta^4} + \lambda^2 \frac{\mathrm{d}^2 v(\zeta)}{\mathrm{d}\zeta^2} - \omega^2 v(\zeta) = 0 \tag{3.1a}$$

and

$$\frac{\mathrm{d}^2 y(\bar{t})}{\mathrm{d}\bar{t}^2} + \tau_0^{1-\beta} \tau_\beta (D^\beta y)(\bar{t}) + \omega^2 y(\bar{t}) = 0, \tag{3.1b}$$

with function ω^2 an unknown separation constant and the boundary conditions for the shape function $v(\zeta)$ as:

$$v(0) = \left.\frac{\mathrm{d}v}{\mathrm{d}\zeta}\right|_0 = \left.\frac{\mathrm{d}^2 v}{\mathrm{d}\zeta^2}\right|_1 = \left.\frac{\mathrm{d}^3 v}{\mathrm{d}\zeta^3}\right|_1 = 0 \tag{3.2}$$

in passing we observe that the governing equation for the shape function involves the parameter ω^2 that has the role of unknown dynamical frequency shown in equation (3.1*b*). The separation constant $\omega^2 \in \mathbb{C}$, with $\mathbb{C}$ the complex field, so that $\omega^2 = \omega_r + \mathrm{i}\omega_i$ with $\omega_r, \omega_i \in \mathbb{R}$ and $\mathrm{i} = \sqrt{-1}$ the imaginary unit.

(a) The characteristic polynomial

Exact solution of equation (3.1*a*) involves linear combinations of transcendental and trigonometric functions that as we introduce the boundary condition yields the eigenvalue equation among the axial load λ and the dynamic frequency ω. Such a relation, yielding a condition $\omega = \omega(\lambda)$ corresponds for each choice of λ, to a different time evolution of the column displacement. The explicit relation $\omega = \omega(\lambda)$ is very cumbersome and it is beyond the scope of the paper. In the following, we resort to an approximate representation of the shape function $v(\zeta)$ assuming, as two-terms linear combination as:

$$v(\zeta) = A_1\phi_1(\zeta) + A_2\phi_2(\zeta) = \phi_j(\zeta)A_j, \quad j = 1, 2 \tag{3.3}$$

where in the latter equation (3.3) we use the sum on the repeated index convention. Shape functions $\phi_j(\zeta)$ in equation (3.3) belong to a set of functions that satisfy the boundary conditions $\phi_j(\zeta) = \mathrm{d}\phi_j(\zeta)/\mathrm{d}\zeta|_{\zeta=0} = 0$ and A_j are the unknown coefficients. The use of the two-terms approximation of the space-dependent solution $v(\zeta)$ is a well-established procedure in the context of elastic stability of non-conservative system. The reduction method introduced in equation (3.3) yields, in the case of purely viscous foundation obtained for $\beta = 1$ in the proposed analysis an overestimation of the critical load of almost 4% with respect to the exact solution of equation (3.1*a*). A similar consideration holds true also for the case of elastic foundation where the overestimation of the critical load obtained with the two-terms expansion with respect to the exact solution is almost 1% [1]. Introducing equation (3.3) into equation (3.1*a*) yields:

$$\varepsilon(\zeta) = A_j\left[\frac{\partial^4\phi_j(\zeta)}{\partial\zeta^4} + \lambda^2\frac{\partial^2\phi_j(\zeta)}{\partial\zeta^2} - \omega^2 A_j\phi_j(\zeta)\right] \neq 0, \tag{3.4}$$

with $j = 1, 2$ and the sum convention has been used. The unbalance function, $\varepsilon(\zeta)$, projected onto manifold defined by two functions $\phi_j(\zeta)$ yields two algebraic equations in terms of constants A_j that read:

$$\int_0^1 \varepsilon(\zeta)\phi_1(\zeta)\,\mathrm{d}\zeta = \varepsilon_{11}(\omega^2,\lambda^2)A_1 + \varepsilon_{21}(\omega^2,\lambda^2)A_2 = 0 \tag{3.5a}$$

and

$$\int_0^1 \varepsilon(\zeta)\phi_2(\zeta)\,\mathrm{d}\zeta = \varepsilon_{12}(\omega^2,\lambda^2)A_1 + \varepsilon_{22}(\omega^2,\lambda^2)A_2 = 0, \tag{3.5b}$$

where the coefficients ε_{jk} are reported in appendix A. The system of algebraic equations in equations (3.5*a*), (3.5*b*) may be solved by an inhomogeneous set of constants A_j only if the determinant:

$$r(\omega^2,\lambda^2) = \varepsilon_{11}(\omega^2,\lambda^2)\varepsilon_{22}(\omega^2,\lambda^2) - \varepsilon_{21}(\omega^2,\lambda^2)\varepsilon_{12}(\omega^2,\lambda^2) = 0 \tag{3.6}$$

that corresponds to a specific condition among the separation constant ω^2 and the square of the non-dimensional load λ. For any specified value of the axial load, a value of dynamic frequency $\omega^2(\lambda^2)$ is obtained by (3.6) and the solution of equation (3.1*b*) is obtained as a series of two-parameters Mittag–Leffler functions as:

$$y(\bar{t}) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\omega^2}{\tau_0^{1-\beta}\tau_\beta}\right)^k t^{2(k+1)-1}E^{(k)}_{2-\beta,2+\beta k}\left(-\frac{\omega^2}{\tau_0^{1-\beta}\tau_\beta}t^{2-\beta}\right), \tag{3.7}$$

where $E^{(k)}_{\lambda,\mu}(\cdot)$ is two-parameters Mittag–Leffler function, with $k = (0, 1, 2, \ldots)$ [29]. Stability of the solution in equation (3.7) can not be inferred by direct inspection and, in such a case, several strategies to assess the stability of the dynamical system ruled by fractional differential equations (FDE) have been proposed. In more details, Laplace transform of equation (3.1*b*) yields conditions on the stability by checking poles of the Laplace transform $\mathcal{L}[y(\bar{t})] = \hat{y}(s)$ [5,38]. In the following, a different approach to stability of the FDE is proposed under the assumption of rational values of the derivation order.

(b) State-space representation of the dynamic equilibrium equation

The transverse displacement $w(\zeta, t)$ of a cantilever column introduced in the previous section is ruled by the two differential equations (3.1*a*), (3.1*b*). Separated solutions involve the Mittag–Leffler series that requires the knowledge of the function $\omega^2 = \omega^2(\lambda)$ by equation (3.1*b*) but no considerations on the stability of the solution may be inferred. In the section, we aim to provide a solution to the stability problem of the fractional-order Beck's column with an original approach. Let us assume, in the following, that the differentiation order $\beta = p/q$, with $p, q \in \mathbb{N}$ and $p \leq q$ since $0 \leq \beta \leq 1$, may be expanded. Under these circumstances, the time variations of the transverse displacement $y(\bar{t})$ is ruled by FDE [42]:

$$\frac{d^2}{d\bar{t}^2} y(\bar{t}) + (D_{0^+}^{p/q} y)(\bar{t}) + \omega^2(\lambda) y(\bar{t}) = 0. \tag{3.8}$$

The composition rule of the fractional-order operators namely $D_{0^+}^{\beta_1}(D_{0^+}^{\beta_2} f) = D_{0^+}^{\beta_1+\beta_2}$ allows to recast equation (3.8) in the form:

$$\sum_{l=1}^{2q} C_l (D_{0^+}^{l\bar{\beta}} y)(\bar{t}) + \omega^2(\lambda) y(\bar{t}) = 0, \tag{3.9}$$

with the order $\bar{\beta} = 1/q$ and the coefficients $C_l = 0 \ \forall\, l \neq p, 2p$ and $C_p = C_\beta$, $C_{2p} = 1$. Under these circumstances equation (3.9) may be expanded in terms of a state variable vector $\mathbf{y}(\bar{t})$ with elements:

$$\mathbf{y}(\bar{t}) = \begin{bmatrix} y_1(\bar{t}) \\ y_2(\bar{t}) \\ y_3(\bar{t}) \\ \vdots \\ y_{p+1}(\bar{t}) \\ \vdots \\ y_m(\bar{t}) \end{bmatrix} = \begin{bmatrix} y(\bar{t}) \\ (D_{0^+}^{\bar{\beta}} y) \\ (D_{0^+}^{2\bar{\beta}} y) \\ \vdots \\ (D_{0^+}^{p\bar{\beta}} y) \\ \vdots \\ (D_{0^+}^{(m-1)\bar{\beta}} y) \end{bmatrix} \tag{3.10}$$

with the supplementary $2q - 1$ equations:

$$(D_{0^+}^{\bar{\beta}} y_l)(\bar{t}) - y_l(\bar{t}) = 0 \quad l = 1, 2, \ldots, m-1, \tag{3.11}$$

with $m = 2q$ The introduction of state variables vector and the state identities in equation (3.10), (3.11) allows to express equations (3.9) as a system of $\bar{\beta}$-order FDE as:

$$\left.\begin{aligned} &(D_{0^+}^{\bar{\beta}} y_2)(\bar{t}) + C_{22} y_3 = 0, \\ &(D_{0^+}^{\bar{\beta}} y_3)(\bar{t}) + C_{34} y_4 = 0, \\ &\qquad\vdots \\ &(D_{0^+}^{\bar{\beta}} y_{m-1})(\bar{t}) = y_m(\bar{t}) \\ \text{and}\quad &(D_{0^+}^{\bar{\beta}} y_m)(\bar{t}) - C_{mp+1} y_{p+1}(\bar{t}) + C_m y_m(\bar{t}) = 0, \end{aligned}\right\} \tag{3.12}$$

with $C_{mp+1} = C_\beta$, $C_{m1} = \omega^2$, $C_{12} = C_{23} = C_{34} = \cdots = 1$. System of equation (3.12) may be written in matrix form as:

$$(D_{0^+}^{\bar{\beta}} \mathbf{y})(\bar{t}) - \mathbf{C}(\omega^2) \mathbf{y}(\bar{t}) = 0, \tag{3.13}$$

and in the matrix form:

$$\mathbf{C}(\omega^2)=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots & 0\\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0\\ 0 & 0 & 0 & 1 & \cdots & \cdots & 0\\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1\\ -\omega^2 & 0 & 0 & 0 & \cdots & -C_\beta & 0 \end{bmatrix}. \tag{3.14}$$

System of FDE in equation (3.13) may be obtained:

$$\mathbf{y}(\bar{t})=\boldsymbol{\Phi} E_\beta(-\bar{t}^{-\bar{\beta}}\chi), \tag{3.15}$$

with $\boldsymbol{\Phi}\in\mathbb{C}^m$ and $\chi\in\mathbb{C}$ are the eigenvectors and eigenvalues of the matrix $\mathbf{C}(\omega^2)$. Substitution of equation (3.15) into equation (3.13) yields the algebraic system:

$$(\mathbf{I}\chi-\mathbf{C}(\omega^2))\boldsymbol{\Phi}=\mathbf{0}, \tag{3.16}$$

where $\mathbf{I}$ is the identity matrix to the eigenvalue equation:

$$\det[\mathbf{C}(\omega^2)-\mathbf{I}\chi]=0. \tag{3.17}$$

Equation (3.16) is an algebraic equation for the problem eigenvalues $\chi(\omega^2)$ and it can always be expressed as:

$$\chi^{m+1}+\chi^{m-1}C_\beta+A_m(\omega^2)=0, \tag{3.18}$$

where A_m is a β-dependent coefficient that will be specified as the order of fractional derivative is prescribed. The explicit values of the roots $\chi_1(\omega),\ldots,\chi_m(\omega)$ of the algebraic secular equation in equation (3.18) may not be obtained for any values of the order β, and some explicit values have been obtained in previous papers by Bologna *et al.* [43]. In the following, we introduce a different approach to the stability of fractional-order Beck's column based on the extension of the Routh–Hurwitz criterion.

As far as the assumption of rational order derivative is removed, then the real- (or irrational) order of derivation may be decomposed, as an example, in terms of the series of partial fractions and an unbounded number of state functions will be involved in the analysis.

(c) Stability analysis of fractional-order Beck's column via extend Routh–Hurwitz criterion

Direct solution of the secular equation in equation (3.18) provides values $\chi_j(\omega^2)=\chi_j(\omega^2(\lambda))$ $j=1,2\ldots,m$ of the system eigenvalues that may be, $\chi_j\in\mathbb{R}$ or $\chi_j,\chi_j^*\in\mathbb{C}$ where $*$ denotes complex conjugate. Dynamic stability of the fractional-order set of differential equations in equation (3.12) is provided as [33,44,45]:

$$\arg(\chi_j)\le\left|\frac{\pi\bar{\beta}}{2}\right|\quad\forall j=1,2,\ldots,m, \tag{3.19}$$

where we denoted $\arg(\chi_j)$ the argument of the complex number χ_j evaluated in the principal Riemannian manifold that is $\arg(\chi_j)=tg^{-1}(\chi_{j,I}/\chi_{j,R})$ with $\chi_{j,I}$ and $\chi_{j,R}$ the imaginary and real component of complex number χ_j, respectively.

Stability analysis conducted by direct evaluation of the roots of the characteristic equation (3.18) is a formidable task since no closed form expression of the roots $\chi(\omega^2)$ is available under general differential order β. The seek for eigenvalues $\chi_j(\omega^2)$ that fall into the instability region figure 3 may be observed under cases $\beta=1$, resorting to the well-established Routh–Hurwitz criterion $\arg(\chi_j)\le|\pi/2|$ $\forall j$ that corresponds to $\chi_{j.R}\le 0$ $\forall j$ [44]. Similar considerations holds to also for case $\beta=0$ yielding dynamic stability as $\chi_j\in\mathbb{R}^-$ $\forall j$. In passing, we observe that under such condition the Hermann–Smith paradox is involved yielding that stability load λ is not affected by the external elastic restraints ($\beta=0$) [2].

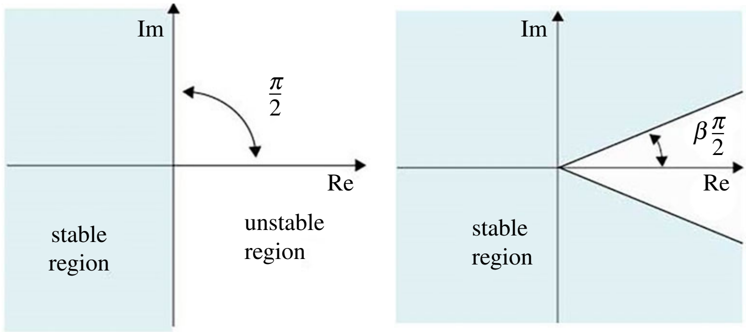

Figure 3. Stability region on the complex plane. (Online version in colour.)

Indeed, it is well known that the evaluation of the number of roots $\chi_j\, j = 1, 2 \ldots m$ with positive real parts is provided by the evaluation of the number of change of signs of the determinant sequence [44] $\Delta_1, \Delta_2, \ldots \Delta_n$, namely $V(1, \Delta_1, \Delta_2, \ldots, \Delta_n)$ with V the number of sign changed. Routh–Hurwitz theorem in [44] may be obtained by the Sturm sequence applied the characteristic equation followed by an appropriate rotation of the complex plane counterclockwise by phase $\theta = \mathrm{i}\pi/2$.

A different case is involved as fractional-order hereditary restraints are considered with $0 \le \bar{\beta} \le 1$ as in equation (3.9). Indeed, under such circumstances, a vector of the complex phase with origin in $x = 0$ corresponds to dynamic instability as $-\bar{\beta}\pi/2 \le \arg(\chi_j) \le \bar{\beta}\pi/2$ as shown in figure 3

In order to extend the Routh–Hurwitz stability theorem to this latter case we observe that, considering complex of conjugated χ_j roots on the borders of the sector with arguments:

$$\arg(\chi_j) = \frac{\bar{\beta}\pi}{2} \quad \arg(\chi_j^*) = -\frac{\bar{\beta}\pi}{2}. \tag{3.20}$$

The introduction of phase shifts

$$\varphi_j = (1-\beta)\frac{\pi}{2} \quad \varphi_j^* = (1+\beta)\frac{\pi}{2}, \tag{3.21}$$

as:

$$\bar{\chi}_j = \chi_j \mathrm{e}^{\mathrm{i}\varphi_j}; \quad \bar{\chi}_j^* = \chi_j^* \mathrm{e}^{\mathrm{i}\varphi_j{}^*}, \tag{3.22}$$

yields arguments of the couples $\bar{\chi}_j$ as:

$$\arg(\bar{\chi}_j) = \frac{\pi}{2}; \quad \arg(\bar{\chi}_j^*) = -\frac{\pi}{2}, \tag{3.23}$$

that corresponds to complex roots on the imaginary axis. Complex conjugated roots in the instability region with $\arg(\chi_j) \le |\bar{\beta}\pi/2|$, yields after the phase shifts, $\mathrm{Re}(\bar{\chi}_j) > 0$; $\mathrm{Re}(\bar{\chi}_j^*) > 0$. Complex conjugated roots within the stable region with $\arg(\chi_j) \ge \beta\pi/2$ and $\arg(\chi_j^*) \ge \beta\pi/2$ yields, after phase shifts: $\mathrm{Re}(\bar{\chi}_j) \le 0$; $\mathrm{Re}(\bar{\chi}_j^*) \le 0$. Under these circumstances, as we introduced phase shifts in equation (3.21) as:

$$\chi_j = \bar{\chi}_j\, \mathrm{e}^{-\mathrm{i}\varphi_j}, \tag{3.24}$$

and we replace in to equation (3.18) we get a complex coefficients eigenvalue equation in terms of roots $\bar{\chi}_j$ as:

$$s(\omega^2) = N_{m+1}\bar{\chi}^{m+1} + N_{m-1}\bar{\chi}^{m-1}C_\beta + A_m(\omega^2) = 0, \tag{3.25}$$

where $N_{m+1} = \mathrm{e}^{-\mathrm{i}\varphi_j(m+1)}$ and $N_{m-1} = \mathrm{e}^{-\mathrm{i}\varphi_j(m-1)}$.

In passing, we observe that introducing the complementary rotation $\chi_j^* = \bar{\chi}_j^* \mathrm{e}^{\mathrm{i}\varphi_j^*}$ a similar equation with complex conjugated coefficients, $s^*(\omega^2)$ is obtained for the conjugated roots $\bar{\chi}_j^*$ (figure 4).

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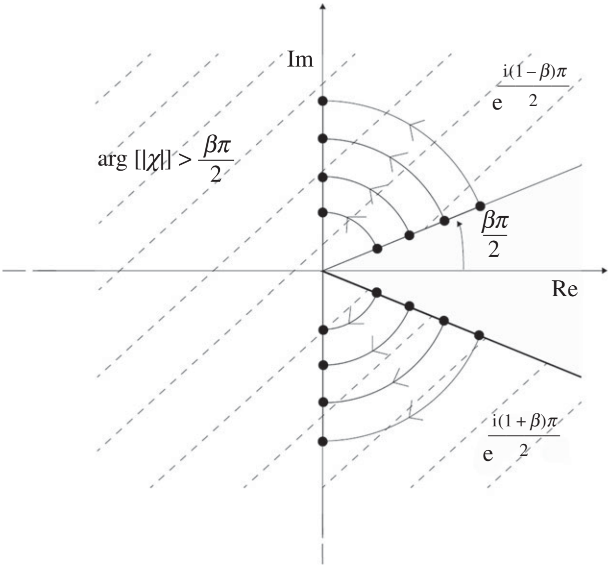

Figure 4. Stability region and complex map.

Indeed, straightforward algebraic manipulation yields a characteristic equation coalescing equation (3.25). The sign of the real parts $\bar{\chi}_{jR}$ of the roots of equation (3.25) corresponds, henceforth, to roots χ_j of the characteristic equation in equation (3.18) that may be: (i) $\text{Re}(\bar{\chi}_j) < 0$ $\forall j \Rightarrow \arg(\chi_j) \geq |\bar{\beta}\pi|/2\, \forall j$; (ii) $\text{Re}(\chi_j) = 0 \Rightarrow \arg(\chi_j) = |\bar{\beta}\pi|/2\, \forall j$. Previous considerations yields that phase shift of the stability sector in the complex plane represents a conformal mapping of the $\mathbb{C}$ plane with one-to-one correspondence. The aforementioned considerations suggest that stability of a fractional-order dynamical system may be studied via the Routh–Hurwitz criterion applied to the complex characteristic equation in equation (3.25). Routh–Hurwitz criterion will be applied over the real coefficient characteristic equation $S^2(\omega^2(\lambda)) = 0$ of order 2 m obtained by the product $S^2(\omega^2(\lambda)) = s(\omega^2)s^*(\omega^2)$. Signs of the real part of the roots of the equation $S^2(\omega^2(\lambda)) = 0$ will be obtained by the sign of the sequence of determinants.

$$p = V(1, \Delta_1, \ldots, \Delta_n) \tag{3.26a}$$

and

$$q = V(1, -\Delta_1, \ldots, (-1)^n \Delta_n), \tag{3.26b}$$

where Δ_k [44] has the structure of:

$$\Delta_k = \begin{vmatrix} m_1 & m_3 & \cdots & m_{2k-1} & -n_2 & -n_4 & \cdots & -n_{2k-2} \\ 1 & m_2 & \cdots & m_{2k-2} & -n_1 & -n_3 & \cdots & -n_{2k-3} \\ 0 & m_1 & \cdots & m_{2k-3} & 0 & -n_2 & \cdots & -n_{2k-4} \\ 0 & 1 & \cdots & \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & m_{2k} & 0 & 0 & \cdots & -n_{k-1} \\ 0 & n_2 & \cdots & n_{2k-2} & m_1 & m_3 & \cdots & -m_{k-3} \\ \vdots & n_1 & \cdots & n_{2k-3} & 1 & m_2 & \cdots & \vdots \\ 0 & 0 & \cdots & n_k & 0 & 0 & \cdots & -m_{k-1} \end{vmatrix} \tag{3.27}$$

for $k = 1, \cdot, n$, with m_j and n_j, with $j = 1, \ldots, n$ are the coefficients of the characteristic polynomial, m_j are the coefficients of the real part and n_j are the coefficients of the imaginary part.

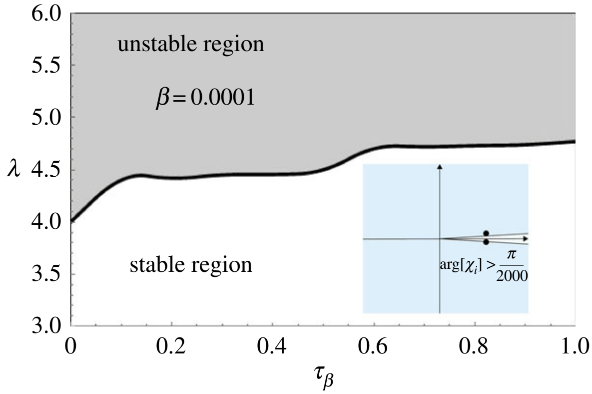

Figure 5. Plot of critical load for $\beta = 0.0001$. (Online version in colour.)

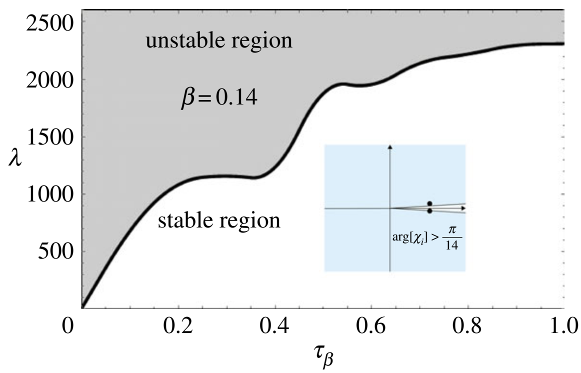

Figure 6. Plot of critical load for $\beta = 0.14$. (Online version in colour.)

4. Parametric analysis

The effect of the derivation order β as well as of the anomalous damping coefficient C_β on the critical load λ is investigated in this section by means of a specific computational code. The analysis is conducted introducing functions $\phi_j(s)$ in terms of Duncan Polynomials [12,46] as:

$$\phi_1 = 12\varsigma^2 - 8\varsigma^3 + 2\varsigma^4 \tag{4.1a}$$

and

$$\phi_2 = 20\varsigma^3 - 20\varsigma^4 + 6\varsigma^5 \tag{4.1b}$$

the use of the function in equations (3.5*a*), (3.5*b*), yields the equation of the column, namely $r(\omega^2, \lambda^2) = 0$ reads, after substitution:

$$\frac{41728\omega^4}{40425} + \frac{316672\lambda^2\omega^2}{24255} - \frac{41984\omega^2}{77} + \frac{60\lambda^4}{7} + \frac{13824\lambda^2}{49} + \frac{46080}{7} = 0 \tag{4.2}$$

equation (4.2) is obtained by replacing the coefficients ε_{jk} reported in appendix A in equations (A 1*a*)–(A 1*d*) allowing to express the dynamic frequency $\omega^2(\lambda^2)$:

$$\omega^2(\lambda^2) = \frac{38254225\lambda^4}{132933023442} + \frac{2937825\lambda^2}{5499968} + \frac{1039500}{85937} + \frac{30925\sqrt{24482704\lambda^4 + 90888432129\lambda^2 + 2058198024960\lambda^2}}{531732093768}. \tag{4.3}$$

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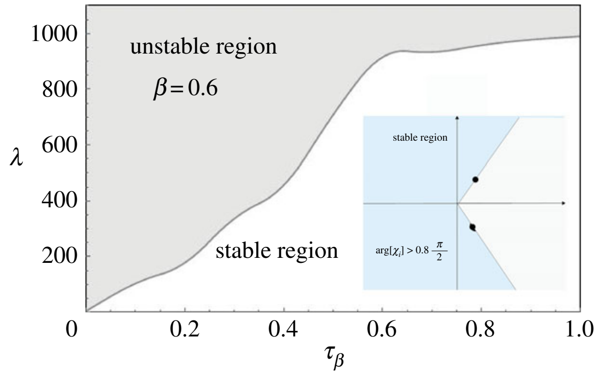

Figure 7. Plot of critical load for $\beta = 0.6$. (Online version in colour.)

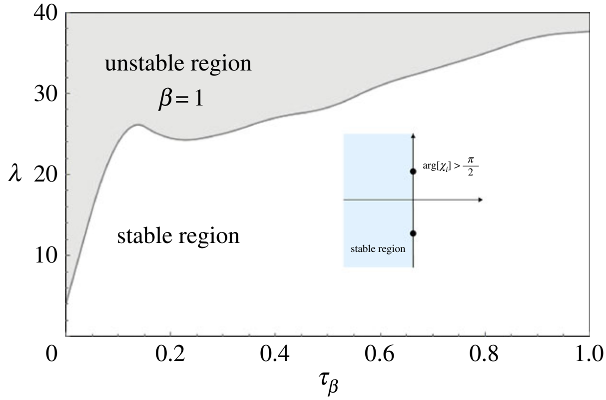

Figure 8. Plot of critical load for $\beta = 1$. (Online version in colour.)

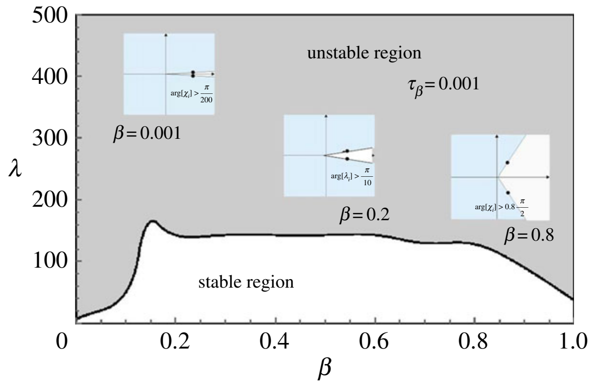

Figure 9. Plot of critical load with different stability when β varying $\tau_\beta = 0.001$. (Online version in colour.)

In passing, note that the well-known paradox of Hermann–Smith and Ziegler are particular cases ($\beta = 0$ and $\tau_\beta \to 0$) of the parametric analysis discussed in this section.

In more detail, the effect of the dissipation coefficient for different orders of differentiation $\beta = 10^{-4}$, $\beta = 0.14$, $\beta = 0.6$ and $\beta = 1$ on the stability load λ have been presented in figures 5–8.

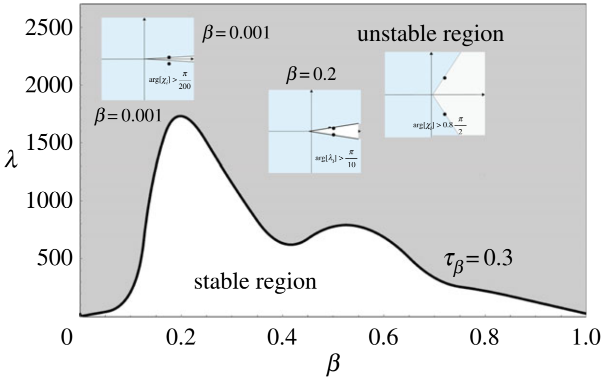

Figure 10. Plot of critical load with different stability when β varying $\tau_\beta = 0.3$. (Online version in colour.)

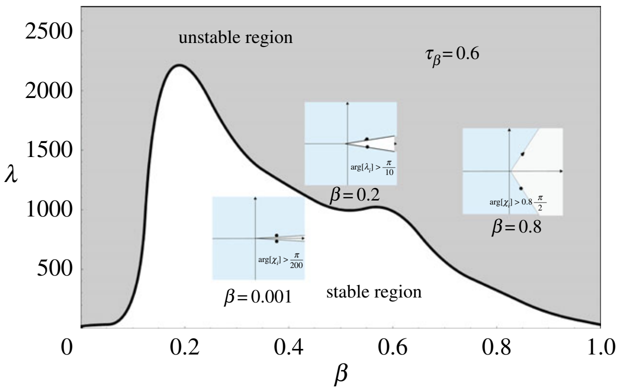

Figure 11. Plot of critical load with different stability when β varying $\tau_\beta = 0.6$. (Online version in colour.)

It may be observed from figure 5 that as $\beta = 10^{-4}$ ($\beta \to 0$ external elastic restraints), the load λ is almost insensitive to the presence of the external restraints as predicted by the Hermann–Smith paradox. At the same time, however, the value of critical load λ does not match those predicted in the presence of elastic restraints ($\lambda = 20.05$ [1]) for $\tau_\beta \to 0$. Indeed, the common value $\lambda = 4.13$ is obtained as $\tau_\beta \to 0$ for any value of the differentiation order β as predicted by Ziegler paradox [3]. The observation of the critical loads for increments of the differentiation order β shows a non-uniform peak. Such a behaviour is shown contrasting figures 6 and 8 observing that, non-monotonic increments of stability load λ is experienced increasing the value of τ_β. Additionally, a shaper increment of the critical load is achieved for $\beta = 0.14$ that is almost 80 times higher that the case of classical damping for $\beta = 1$. The effect is less evident for other values of the differentiation order β and the maximum values are achieved in correspondence of other values of the dissipation coefficient τ_β.

The mechanical behaviour of the column is also influenced by order of differentiation β as shown by figures 9–12 predicting the trend of the critical load λ for prescribed values of the coefficient τ_β. It may be observed that the critical load λ is maximum for order $\beta = 0.18$ and it increases with the differentiation τ_β. The Ziegler paradox is shown by the observation of figure 8 for smaller values of τ_β that correspond, for $\beta = 1$, to a value of critical load $\lambda = 4.13$ that is observed in figure 5. Hermann–Smith paradox is observable in the stability regions (figures 9–12) for $\beta = 0$ since, $\lambda = 20.05$ as in the presence of elastic external restraints. The coupled effect of τ_β and β is described by the stability surface in figure 13. In figure 14, the contour plot associated with the stability surface in figure 13 has also reported to provide an immediate perception of the range of parameter that maximize the stability load.

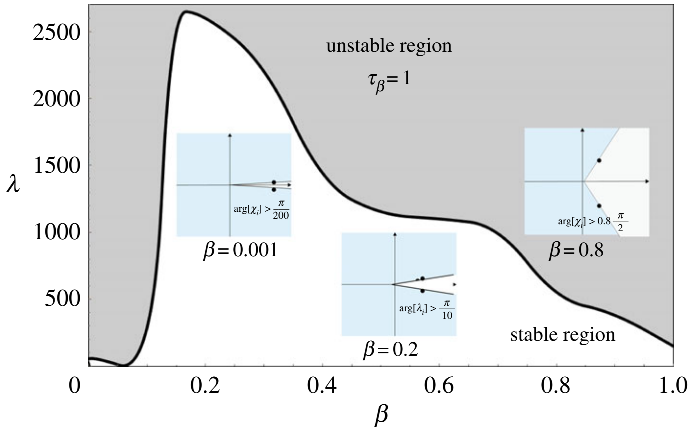

Figure 12. Plot of critical load with different stability when β varying $\tau_\beta = 1$. (Online version in colour.)

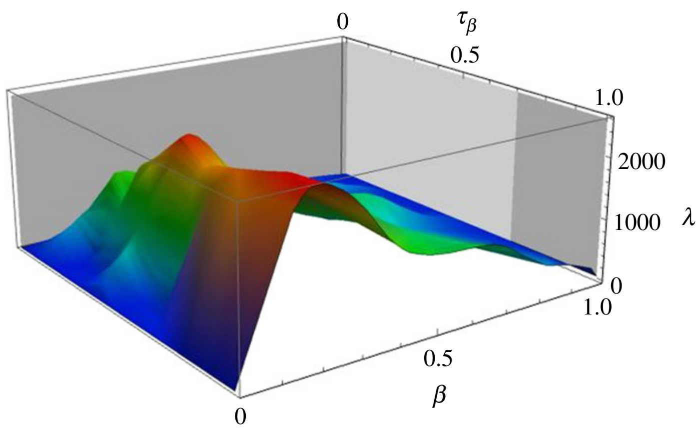

Figure 13. Stability surface of critical load $\lambda_{cr}(\beta, \tau_\beta)$. (Online version in colour.)

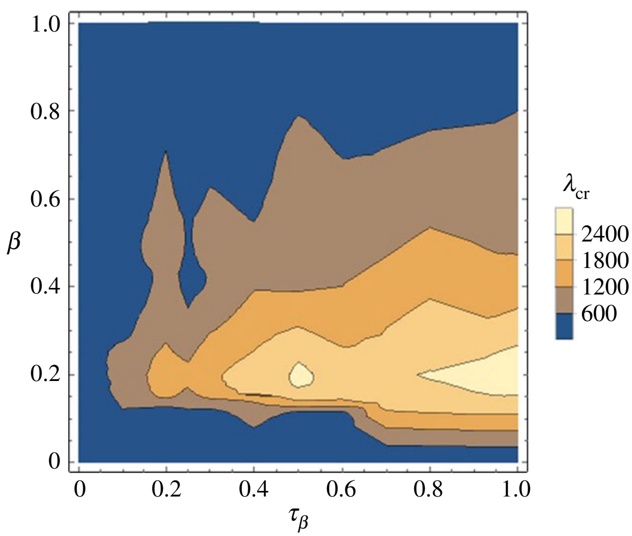

Figure 14. Contour plot $\lambda_{cr}(\beta, \tau_\beta)$.

5. Conclusion

In this paper, the authors investigate the stability of the Beck's column resting on a bed of independent linear springpots. The analysis has been conducted resorting to a state-space representation of the fractional-order differential equation ruling the time evaluation of the displacement field of a column. As the fractional differential equation system is established, the stability is assessed by means of a novel approach based on a conformal mapping of the complex plane followed by the application of the Routh–Hurwitz method. The results obtained by the present approach showed that the critical load is strongly influenced by the dissipation coefficient and by the order of differentiation ruling the external springpots. Such effect is evidenced in the enhancement of the critical stability load for values of differentiation order between 0 and 1. The results obtained by the analysis show that the well-known paradoxes observed by Hermann–Smith (for vanishing differentiation order) and Ziegler (for small dissipation and Newtonian damping) are not removed introducing the fractional-order damping. These results have been confirmed both in the present analysis, involving a single external springpot as well as in the previous paper involving the fractional Zener model studied in the context of Laplace transform methods. The influence of generalized fractional-order foundations, as the fractional Zener foundation model, has been also studied in the context of the proposed state-variable approach to stability of Beck's column and it will be reported in a forthcoming paper.

Ethics. This work did not involve any active collection of human data, but only computer simulations of human behaviour.

Data accessibility. This article does not contain any additional data.

Authors' contributions. M.Z. and E.B. conceived the mathematical models, interpreted the computational results and wrote the paper. All the authors gave their final approval for publication.

Competing interests. We declare we have no competing interests.

Funding. This work was supported by PON FSE-FESR ricerca e innovazione 2014–2020 DOT1320558 and MIUR grant PRIN 2015 'Advanced mechanical models of new materials and structures for Horizon 2020 challenges'.

Acknowledgements. The authors are very grateful to the MIUR grant PON FSE-FESR ricerca e innovazione 2014–2020 DOT1320558, with coordinator Prof. Massimiliano Zingales, and MIUR grant no. PRIN 2015, 'Advanced mechanical models of new materials and structures for Horizon 2020 challenges', with national coordinator Prof. Mario Di Paola; this financial support is gratefully acknowledged.

Appendix A

In this appendix, we reported the coefficients of the expression equation (3.6) that read, in general:

$$\varepsilon_{11}(\omega^2,\lambda^2)=\int_0^1 \phi_1{}^{\mathrm{IV}}(\zeta)\phi_1(\zeta)\,\mathrm{d}\zeta+\lambda^2\int_0^1 \phi_1{}^{\mathrm{II}}(\zeta)\phi_1(\zeta)\,\mathrm{d}\zeta-\omega^2\int_0^1 \phi_1{}^2(\zeta)\,\mathrm{d}\zeta, \tag{A 1a}$$

$$\varepsilon_{21}(\omega^2,\lambda^2)=\int_0^1 \phi_1{}^{\mathrm{IV}}(\zeta)\phi_2(\zeta)\,\mathrm{d}\zeta+\lambda^2\int_0^1 \phi_1{}^{\mathrm{II}}(\zeta)\phi_2(\zeta)\,\mathrm{d}\zeta+ \\ -\omega^2\int_0^1 \phi_1(\zeta)\phi_2(\zeta)\,\mathrm{d}\zeta, \tag{A 1b}$$

$$\varepsilon_{12}(\omega^2,\lambda^2)=\int_0^1 \phi_2{}^{\mathrm{IV}}(\zeta)\phi_1(\zeta)\,\mathrm{d}\zeta+\lambda^2\int_0^1 \phi_2{}^{\mathrm{II}}(\zeta)\phi_1(\zeta)\,\mathrm{d}\zeta+ \\ -\omega^2\int_0^1 \phi_1(\zeta)\phi_2(\zeta)\,\mathrm{d}\zeta \tag{A 1c}$$

and

$$\varepsilon_{22}(\omega^2,\lambda^2)=\int_0^1 \phi_2{}^{\mathrm{IV}}(\zeta)\phi_2(\zeta)\,\mathrm{d}\zeta+\lambda^2\int_0^1 \phi_2{}^{\mathrm{II}}(\zeta)\phi_2(\zeta)\,\mathrm{d}\zeta-\omega^2\int_0^1 \phi_2{}^2(\zeta)\,\mathrm{d}\zeta \tag{A 1d}$$

as we assume for functions ϕ_1 and ϕ_2 the Duncan polynomials as reported in equations (4.1*a*), (4.1*b*) the coefficients in equations (A 1*a*)–(A 1*d*) read as straightforward conclusion:

$$\varepsilon_{11} = \frac{48\lambda^2}{7} - \frac{416\omega^2}{45} + \frac{576}{5}, \tag{A 2a}$$

$$\varepsilon_{12} = \frac{114\lambda^2}{7} - \frac{2608\omega^2}{315} + 96, \tag{A 2b}$$

$$\varepsilon_{21} = \frac{30\lambda^2}{7} - \frac{2608\omega^2}{315} + 96 \tag{A 2c}$$

and

$$\varepsilon_{22} = \frac{80\lambda^2}{7} - \frac{5216\omega^2}{693} + \frac{960}{7}. \tag{A 2d}$$

The coefficients in the previous equations have been used in §4 to yield equation (4.2) and their specific expression depends on the set of function used in the analysis.

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