Indag. Mathem., N.S., 17 (1), 115–127 March 27, 2006

Irreducibility of Hurwitz spaces of coverings with one special fiber

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Communicated by Prof. J.J. Duistermaat at the meeting of February 28, 2005

ABSTRACT

Let Y be a smooth, projective complex curve of genus $g \geqslant 1$. Let d be an integer $\geqslant 3$, let $\underline{e} = \{e_1, e_2, \dots, e_r\}$ be a partition of d and let $|\underline{e}| = \sum_{i=1}^{r}(e_i - 1)$. In this paper we study the Hurwitz spaces which parametrize coverings of degree d of Y branched in n points of which $n-1$ are points of simple ramification and one is a special point whose local monodromy has cyclic type $\underline{e}$ and furthermore the coverings have full monodromy group S_d. We prove the irreducibility of these Hurwitz spaces when $n - 1 + |\underline{e}| \geqslant 2d$, thus generalizing a result of Graber, Harris and Starr [A note on Hurwitz schemes of covers of a positive genus curve, Preprint, math. AG/0205056].

1. INTRODUCTION

Let Y be a smooth, connected, projective complex curve of genus $g \geqslant 1$ and let $b_0 \in Y$. Let $d \geqslant 3$ be an integer and let $\underline{e} = \{e_1, e_2, \dots, e_r\}$ be a partition of d, $e_1 + e_2 + \cdots + e_r = d$, where $e_1 \geqslant e_2 \geqslant \cdots \geqslant e_r \geqslant 1$. Let $|\underline{e}| = \sum_{i=1}^{r}(e_i - 1)$.

Let us denote by $H_{d,n-1,\underline{e}}(Y, b_0)$ the Hurwitz space that parametrizes equivalence classes of pairs $[\pi, \phi]$ of a covering $\pi : X \to Y$ and a bijection $\phi : \pi^{-1}(b_0) \to \{1, \dots, d\}$ satisfying the following: π is a covering of degree d of Y, the cover X is smooth and connected, π is unramified at b_0 and is branched in $n > 0$ points, $n-1$ of which are points of simple branching and one is a special point whose local monodromy has cyclic type $\underline{e}$. Denote by D the branch locus of π and denote by $m : \pi_1(Y - D, b_0) \to S_d$ the associated monodromy homomorphism. Because X is irreducible, the image of m is a transitive subgroup

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of S_d. Associated to $[\pi, \phi]$ is an ordered $(n+2g)$-tuple of elements of S_d, $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$, satisfying the following: for some j the permutation t_j has cyclic type $\underline{e}$, t_i are transpositions for each $i \neq j$ and $t_1 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g]$. We call $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ a *Hurwitz system* and the group generated by t_i, λ_k, μ_k the *monodromy group* of the Hurwitz system. In this paper we are interested in $H^o_{d,n-1,\underline{e}}(Y, b_0)$, the subset of $H_{d,n-1,\underline{e}}(Y, b_0)$ parameterizing pairs $[\pi, \phi]$ whose monodromy group is all S_d. In a similar manner one defines the Hurwitz space $H^o_{d,n-1,\underline{e}}(Y)$ which parametrizes coverings of the considered type without fixing a bijection ϕ. We prove the following theorem:

Theorem 1. *Let Y be a smooth, connected, projective curve of genus $g \geqslant 1$ and let $b_0 \in Y$. If $n - 1 + |\underline{e}| \geqslant 2d$ then the Hurwitz spaces $H^o_{d,n-1,\underline{e}}(Y, b_0)$ and $H^o_{d,n-1,\underline{e}}(Y)$ are irreducible.*

Coverings of curves of positive genus were studied by Graber, Harris, Starr in [4] and by Kanev in [6]. Graber, Harris and Starr considered Hurwitz spaces parameterizing irreducible degree d covers of a genus $g \geqslant 1$ curve with n simple branch points. When $n \geqslant 2d$, they proved the Hurwitz spaces is irreducible. Kanev sharpened this result and proved the irreducibility of these spaces in the case $n \geqslant \max\{2, 2d-4\}$ if $g \geqslant 1$ and $n \geqslant \max\{2, 2d-6\}$ if $g = 1$. Kanev also proved the irreducibility of $H^o_{d,n-1,\underline{e}}(Y)$ when $n - 1 \geqslant 2d - 2$.

The result of this paper is a generalization of that of Graber, Harris and Starr. Namely, we prove the irreducibility of the Hurwitz spaces for the same values of the genera of X and Y as they do, but furthermore we allow one special fiber. The irreducibility of $H^o_{d,n-1,\underline{e}}(Y)$ follows immediately from the irreducibility of $H^o_{d,n-1,\underline{e}}(Y, b_0)$. We prove the irreducibility of $H^o_{d,n-1,\underline{e}}(Y, b_0)$ by proving the transitivity of the action of the braid group $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ on the set of Hurwitz systems $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ with monodromy group S_d. We follow the key idea of [4], i.e., we prove that applying a finite number of braid moves it is possible to replace every $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ by a new system of type $(\bar{t}_1, \dots, \bar{t}_n; 1, 1, \dots, 1, 1)$. Then using only elementary transformations of the Artin's braid group, we reduce $(\bar{t}_1, \dots, \bar{t}_n)$ to a normal form.

It seem likely the inequality in the hypothesis of Theorem 1 may be replace by the weaker one $n - 1 + |\underline{e}| \geqslant 2d - 2$. This inequality is necessary for coverings whose Hurwitz systems are braid equivalent to ones with $\lambda'_1 = \mu'_1 = \cdots = \lambda'_g = \mu'_g = 1$. Unfortunately our method of proof does not allow to cover also the limiting case $n - 1 + |\underline{e}| = 2d - 2$.

2. PRELIMINARIES AND BRAID MOVES

Let Y and X be smooth, connected, projective complex curves of genus $\geqslant 0$. Let $\pi : X \to Y$ be a covering of Y, i.e., π is a finite holomorphic mapping.

A *branch point* is a point $b \in Y$ such that some point of $\pi^{-1}(b)$ is a ramification point of π. A branch point $b \in Y$ is called a *point of simple branching* for π if π is ramified at only one point $x \in \pi^{-1}(b)$ and the ramification index $e(x)$ of π at x is 2.

A branch point $b \in Y$ is called a *special point* if it is not simple. The set of branch points is called the *branch locus* of π.

Let d be a positive integer. Two d-sheeted branched coverings $\pi_1 : X_1 \to Y$ and $\pi_2 : X_2 \to Y$ are called *equivalent* if there exists a biholomorphic map $f : X_1 \to X_2$ such that $\pi_2 \circ f = \pi_1$. The equivalence class containing π_1 is denoted by $[\pi_1]$.

Let $\underline{e} = (e_1, e_2, \dots, e_r)$ be a partition of d where $e_1 \geqslant e_2 \geqslant \cdots \geqslant e_r \geqslant 1$. Associate to $\underline{e}$ the following element in S_d having cycle type $\underline{e}$,

$$(1) \qquad \varepsilon := (12\dots e_1)(e_1+1\dots e_1+e_2)\cdots\big((e_1+\cdots+e_{r-1})+1\dots d\big).$$

Let b_0 be a point of Y, let us denote by $H_{d,n-1,\underline{e}}(Y, b_0)$ the Hurwitz space that parametrizes equivalence classes of pairs $[\pi, \phi]$ of a covering $\pi : X \to Y$ and a bijection $\phi : \pi^{-1}(b_0) \to \{1, \dots, d\}$ satisfying the following: π is a covering of degree d of Y, π is unramified at b_0 and it is branched in $n > 0$ points, $n - 1$ of which are points of simple branching and one is a special point whose local monodromy belongs to the conjugacy class of ε.

Let $Y^{(n)}$ be the n-fold symmetric product of Y and let Δ be the codimension 1 locus of $Y^{(n)}$ consisting of nonsimple divisors. Let $\Psi : H_{d,n-1,\underline{e}}(Y, b_0) \to (Y - b_0)^{(n)} - \Delta$ be the map which assigns to each $[\pi, \phi]$ the reduced branch locus of π.

Convention. The natural action of S_d on $\{1, \dots, d\}$ here is on the *right* and multiplication of permutations is by $\sigma \cdot \tau = \tau \circ \sigma$, e.g., $(12)(13) = (123)$.

Let $[\pi, \phi] \in H_{d,n-1,\underline{e}}(Y, b_0)$, let D be the reduced branch divisor, let $[\gamma] \in \pi_1(Y - D, b_0)$, and for every $i = 1, \dots, d$, denote $x_i = \phi^{-1}(i)$ in $\pi^{-1}(b_0)$. For every $i = 1, \dots, d$, i^t equals $\phi(y)$, where y is the terminal point of the unique lift of γ whose initial point is x_i.

For the rest of the paper we suppose $n \geqslant 2$. Let $D = \{b_1, \dots, b_n\}$ and let $\gamma_1, \gamma_2, \dots, \gamma_n, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ be the closed arcs oriented counterclockwise represented in Fig. 1.

The corresponding homotopy classes of these arcs yield a system of generators for $\pi_1(Y - D, b_0)$ which satisfy the only relation

$$\gamma_1\gamma_2\cdots\gamma_n \simeq [\alpha_1, \beta_1]\cdots[\alpha_g, \beta_g].$$

Definition 1. An ordered sequence $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ of permutations of S_d such that $t_i \neq 1$ for each $i = 1, \dots, n$ and $t_1 t_2 \cdots t_n = [\lambda_1, \mu_1]\cdots[\lambda_g, \mu_g]$ is called a *Hurwitz system*. The subgroup $G \subseteq S_d$ generated by t_i, λ_k, μ_k with $i = 1, \dots, n$ and $k = 1, \dots, g$ is called the *monodromy group* of the Hurwitz system. An *$\underline{e}$-Hurwitz system* is a Hurwitz system such that 1 of $t_1, \dots, t_n$ has cycle type $\underline{e}$, and the other $n - 1$ elements in $t_1, \dots, t_n$ are transpositions.

The images via the monodromy homomorphisms m of $\gamma_1, \dots, \gamma_n, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ determine $\underline{e}$-Hurwitz systems

$$\big(m(\gamma_1), \dots, m(\gamma_n), m(\alpha_1), m(\beta_1), \dots, m(\alpha_g), m(\beta_g)\big),$$

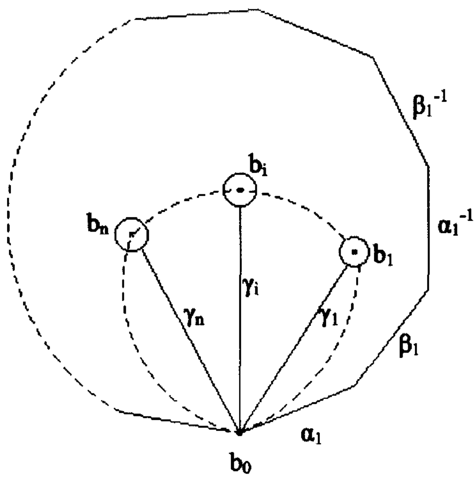

Figure 1.

with transitive monodromy group.

Let us denote by $A_{d,n+2g}$ the set of all $\underline{e}$-Hurwitz systems $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ with transitive monodromy group. The Riemann existence theorem determines a bijection from the fiber of Ψ over D to $A_{d,n+2g}$.

Definition 2. Let $G \subset S_d$ be a transitive subgroup. A *decomposition* for G is a partition $(\Sigma_1, \dots, \Sigma_k)$ of $\{1, \dots, d\}$ into sets of equal size $v \neq 1, d$ such that $(\Sigma_i)^g \in \{\Sigma_1, \dots, \Sigma_k\}$ for every $g \in G$ and $i = 1, \dots, k$. If there exists a decomposition for G, G is *imprimitive*, otherwise G is *primitive*.

Let $A^o_{d,n+2g}$ be the set of all $\underline{e}$-Hurwitz systems $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ in $A_{d,n+2g}$ with primitive monodromy group. We denote by $H^o_{d,n-1,\underline{e}}(Y, b_0)$ the set of all the pairs $[\pi, \phi]$ in $H_{d,n-1,\underline{e}}(Y, b_0)$ such that if D is the reduced branch locus of π and $\gamma_1, \gamma_2, \dots, \gamma_n, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ is a system of closed arcs as in the figure, then the monodromy group of $(m(\gamma_1), \dots, m(\gamma_n), m(\alpha_1), m(\beta_1), \dots, m(\alpha_g), m(\beta_g))$ is a primitive group. Therefore by Riemann's existence theorem we can identify the fiber of $H^o_{d,n-1,\underline{e}}(Y, b_0) \to (Y - b_0)^{(n)} - \Delta$ over D with $A^o_{d,n+2g}$.

There is a unique topology on $H^o_{d,n-1,\underline{e}}(Y, b_0)$ such that $H^o_{d,n-1,\underline{e}}(Y, b_0) \to (Y - b_0)^{(n)} - \Delta$ is a topological covering map, cf. [3]. Therefore the braid group $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ acts on $A^o_{d,n+2g}$. If this action is transitive then $H^o_{d,n-1,\underline{e}}(Y, b_0)$ is connected.

Shortly we recall some notion on braid groups.

The braid groups of orientable 2-manifolds of genus $g \geqslant 1$ were studied by J.S. Birman, E. Fadell and G.P. Scott (see [1,2,9]). Let Y be a smooth, connected, projective complex curve of genus $g \geqslant 1$. The generators of $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ are the elementary braids σ_i with $i = 1, \dots, n-1$ and the braids ρ_{ak}, τ_{bk} with $1 \leqslant a, b \leqslant n$ and $1 \leqslant k \leqslant g$. The calculation of the action of the elementary braids σ_i on Hurwitz systems is due to Hurwitz [5].

The elementary moves σ_i', relative to the elementary braids σ_i, bring

$$(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$$

to

$$(t_1, \ldots, t_{i-1}, t_i t_{i+1} t_i^{-1}, t_i, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g).$$

Therefore their inverses bring $(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ to

$$(t_1, \ldots, t_{i-1}, t_{i+1}, t_{i+1}^{-1} t_i t_{i+1}, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g).$$

The braid moves that correspond to the generators ρ_{ik}, τ_{ik} were studied by Graber, Harris, Starr in [4] and by Kanev in [6]. We make use of some results proved in [6]. In this paper to each generator ρ_{ik} or τ_{ik} is associated a pair of braid moves ρ_{ik}', $\rho_{ik}'' = (\rho_{ik}')^{-1}$ and τ_{ik}', $\tau_{ik}'' = (\tau_{ik}')^{-1}$, respectively.

Let $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ be a Hurwitz system. The braid move ρ_{ik}' leaves unchanged λ_l for each l, t_j for each $j \neq i$ and μ_l for each $l \neq k$, while changing t_i and μ_k. Analogously the braid move τ_{ik}'' changes t_i and λ_k, leaving unchanged μ_l for each l, λ_l for each $l \neq k$ and t_j for each $j \neq i$.

We use the following result.

Proposition 1 [6, Corollary 1.9]. *Let $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ be a Hurwitz system. Let $u_k = [\lambda_1, \mu_1] \cdots [\lambda_k, \mu_k]$ for $k = 1, \ldots, g$ and let $u_0 = 1$. The following formulae hold*:

(i) *For ρ_{1k}'*:

$$\rho_{1k}' : \mu_k \to \mu_k' = (b_1^{-1} t_1^{-1} b_1) \mu_k,$$

where $b_1 = u_{k-1} \lambda_k$.

(ii) *For τ_{1k}''*:

$$\tau_{1k}'' : \lambda_k \to \lambda_k'' = (u_{k-1}^{-1} t_1^{-1} u_{k-1}) \lambda_k.$$

In particular

$$\tau_{11}'' : \lambda_1 \to t_1^{-1} \lambda_1.$$

3. IRREDUCIBILITY OF $H^o_{d,n-1,\underline{e}}(Y, b_0)$

In this section we will prove the irreducibility of $H^o_{d,n-1,\underline{e}}(Y, b_0)$ for $n - 1 + |\underline{e}| \geqslant 2d$. Since $H^o_{d,n-1,\underline{e}}(Y, b_0)$ is smooth in order to prove its irreducibility it suffices to prove it is connected. In Section 1 we observed that if $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ acts transitively on $A^o_{d,n+2g}$ then $H^o_{d,n-1,\underline{e}}(Y, b_0)$ is connected. In order to prove the transitivity of the action of $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ on $A^o_{d,n+2g}$ it is sufficient to prove that, acting by braid moves, it is possible to bring every $\underline{e}$-Hurwitz system in $A^o_{d,n+2g}$ to a given normal form. So first we prove that every $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ in $A^o_{d,n+2g}$ can be transformed into $(t'_1, \ldots, t'_{n-1}, t'_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ where t'_{n-1}, t'_n are equal transpositions and $\langle t'_1, \ldots, t'_{n-2}\rangle = \langle t'_1, \ldots, t'_{n-2}, t'_{n-1}, t'_n\rangle$. Then we apply the Main Lemma of [6] which states that $(t'_1, \ldots, t'_{n-1}, t'_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ can be replaced by $(t'_1, \ldots, (t'_{n-1})^h, (t'_n)^h; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ where $h \in \langle t'_1, \ldots, t'_{n-2}, \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g\rangle$.

We remember that the monodromy group G of a $\underline{e}$-Hurwitz system in $A^o_{d,n+2g}$ is a primitive group which contains a transposition. In [6] it is proved that a primitive group $G \subseteq S_d$ which contains a transposition is all S_d. Therefore the monodromy group of every $\underline{e}$-Hurwitz system in $A^o_{d,n+2g}$ is S_d.

Using these results and braid moves we are ready to normalize $(\lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$. The proof follows by applying a sequence of braid moves and inverse braid moves and then using Mochizuchi's proposition [8, pp. 369–370].

Definition 3. We call two Hurwitz systems *braid equivalent* if one is obtained from the other by a finite sequence of braid moves $\sigma'_i, \rho'_{jk}, \tau'_{jk}, (\sigma'_i)^{-1}, \rho''_{jk}, \tau''_{jk}$ where $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant g$. We denote the braid equivalence by $\sim$.

Definition 4. Two ordered n-tuples (or sequences) of permutations $(t_1, \ldots, t_n)$ and $(t'_1, \ldots, t'_n)$ are called *braid equivalent* if $(t'_1, \ldots, t'_n)$ is obtained from $(t_1, \ldots, t_n)$ by a finite sequence of braid moves of type σ'_i, $(\sigma'_i)^{-1}$. Note that if $t_1 \cdots t_n = s$ then $t'_1 \cdots t'_n = s$.

Lemma 1 [6, Main Lemma 2.1]. *Let $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ be a Hurwitz system of permutations of S_d. Suppose that $t_i t_{i+1} = 1$. Let H be the subgroup of S_d generated by $\{t_1, \ldots, t_{i-1}, t_{i+2}, \ldots, t_n, \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g\}$. Then for every $h \in H$ the given Hurwitz system is braid equivalent to*

$$(t_1, \ldots, t_{i-1}, t_i^h, t_{i+1}^h, t_{i+2}, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g).$$

For the rest of the paper we suppose $d \geqslant 3$. We now want to prove that every $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ in $A^o_{d,n+2g}$ can be transformed, by a finite number of braid moves σ'_i and of their inverses, into $(t'_1, t'_2, \ldots, t'_{n-2}, t'_{n-1}, t'_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ where t'_1 has cyclic type $\underline{e}$, $t'_2, \ldots, t'_n$ are transpositions, $t'_{n-1} = t'_n$ and

$$\langle t'_1, t'_2, \ldots, t'_{n-2}\rangle = \langle t'_1, \ldots, t'_{n-2}, t'_{n-1}, t'_n\rangle.$$

Lemma 2. *Let (t_1, t_2) be an ordered 2-tuple such that t_1 is a d-cycle and t_2 a transposition. Let a'_k be a fixed element of the set $\{1, \ldots, d\}$. Then (t_1, t_2) is braid equivalent to (t'_1, t'_2) where t'_1 is a d-cycle and t'_2 a transposition that moves a'_k.*

Proof. It is not restrictive to assume that a'_k is the element that occupies the first place in t_1. Let $(t_1, t_2) = ((a_1 \ldots a_i \ldots a_j \ldots a_d), (a_i a_j))$ where $a_1 = a'_k$. Acting twice with the elementary move σ'_1 we obtain

$$\big((a_1 \ldots a_i \ldots a_j \ldots a_d), (a_i\ a_j)\big) \sim \big((b_1 \ldots b_{i-1} \ldots b_{j-1} \ldots b_d), (b_{i-1} b_{j-1})\big)$$

where $(b_1, \ldots, \hat{b}_{i-1}, \ldots, \hat{b}_{j-1}, \ldots, b_d) = (a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_d)$ and $(b_{i-1}, b_{j-1}) = (a_j, a_i)$.

Acting with $(\sigma_1')^2$ another $i-2$ times we obtain the required result, i.e., (t_1, t_2) is braid equivalent to (t_1', t_2') where t_1' is a d-cycle and t_2' a transposition that moves a_1. □

Lemma 3. *Let (t_1, τ, τ) be a sequence such that t_1 is an arbitrary permutation of S_d and τ a transposition. Then (t_1, τ, τ) is braid equivalent to (τ, τ, t_1).*

Proof. Applying the elementary moves $(\sigma_1')^{-1}$, $(\sigma_2')^{-1}$ we obtain

$$(t_1, \tau, \tau) \sim (\tau, \tau^{-1} t_1 \tau, \tau) \sim (\tau, \tau, t_1). \qquad \square$$

Lemma 4. *Let (t_1, τ, τ) be a sequence such that t_1 is the d-cycle $(a_1 \ldots a_i \ldots a_j \ldots a_d)$ and τ the transposition $(a_i a_j)$. Then (t_1, τ, τ) is braid equivalent to (t_1, τ', τ') where $\tau' = (a_i a_{j'})$ and $j' \equiv (2i - j) \pmod{d}$. If $j - i \neq d/2$ then $\tau' \neq \tau$.*

Proof. Applying successively the elementary moves σ_1', σ_2' and using Lemma 3 we obtain

$$\begin{aligned}\big(t_1, (a_i a_j), (a_i a_j)\big) &\sim \big((a_{i-1} a_{j-1}), (a_{i-1} a_{j-1}), t_1\big) \\ &\sim \big(t_1, (a_{i-1} a_{j-1}), (a_{i-1} a_{j-1})\big).\end{aligned}$$

Applying the sequence of elementary moves σ_1', σ_2' and using Lemma 3 another $(j - i) - 1$ times we obtain the lemma. □

Lemma 5. *Let $(t_1, \ldots, t_i, t_{i+1}, \ldots, t_n)$ be a sequence of permutations in S_d such that t_i, t_{i+1} are two equal transpositions of S_d. Then we can move to the right (respectively, to the left) the pair (t_i, t_{i+1}) leaving unchanged other permutations of the sequence.*

Proof. The proof follows by Lemma 3. □

Notice that applying braid moves σ_i' or their inverses we can move one arbitrary transposition of the sequence $(t_1, \ldots, t_i, t_{i+1}, \ldots, t_n)$ where we want. In this way, however, we change also other permutations of the sequence.

Lemma 6. *Let t_1 be a d-cycle, let $t_2, \ldots, t_l$ be transpositions and let a be an element moved by at least one of the transpositions. Then $(t_1, t_2, \ldots, t_l)$ is braid equivalent to a sequence $(\tilde{t}_1, \alpha_1, \alpha_1, \ldots, \alpha_z, \alpha_z, \beta_1, \beta_2, \ldots, \beta_m)$ such that*

(i) *$\tilde{t}_1$ is a d-cycle,*
(ii) *for every $i = 1, \ldots, z$, respectively, $j = 1, \ldots, m$, the element α_i, respectively, β_j, is a transposition moving a,*
(iii) *the elements $\beta_1, \ldots, \beta_m$ are distinct, and*

(iv) *if* $l-1 \geqslant d$ *then* z *is at least* 1.

Proof. We prove the lemma by induction on l. If $l=2$, t_2 is a transposition moving a. So (t_1, t_2) is a sequence as we want in which $z=0$ and $m=1$. Let $l>2$. By way of induction, we suppose the lemma is proved for all smaller values of l. Acting with braid moves σ_j', we bring the transpositions that move the element a to the end of the sequence, obtaining

$$\big(t_1, t_2', \dots, t_{v-1}', t_v' = (a\ *), \dots, (a\ *)\big).$$

Applying Lemma 2 we replace (t_1, t_2') by (t_1', t_2'') where t_1' is a d-cycle and t_2'' is a transposition that moves a. By braid moves σ_j' we move t_2'' to the left of t_v'. Proceeding in this way successively for every transposition of the sequence that does not move a we obtain

(2) $\qquad \big(t_1'', (a\ *), \dots, (a\ *)\big).$

If the transpositions in (2) are all distinct, the (2) is a sequence as we want in which $z=0$. If instead in (2) there are two equal transpositions, using inverses of elementary moves σ_j', we move them to the front obtaining

$$(\alpha_1, \alpha_1, \bar{t}_1, \bar{t}_2, \dots, \bar{t}_{l-2}).$$

We can then apply the induction hypothesis to the sequence $(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{l-2})$. The proof follows by applying Lemma 5. Observe that if $l-1 \geqslant d$, because there are only $d-1$ distinct transpositions that move a, some transposition occurs twice in (2). So if $l-1 \geqslant d$, z is at least 1. □

Proposition 2. *Let* $(t_1, t_2, \dots, t_l)$ *be a sequence such that* t_1 *is a d-cycle and* $t_2, \dots, t_l$ *are transpositions. If d is odd and* $l-1 \geqslant d$ *or if d is even and* $l-1>d$ *then* $(t_1, t_2, \dots, t_l)$ *is braid equivalent to* $(\bar{t}_1, \tau, \tau, \tau, \bar{t}_5, \dots, \bar{t}_l)$ *where* $\bar{t}_1$ *is a d-cycle and* $\tau, \bar{t}_5, \dots, \bar{t}_l$ *are transpositions.*

Proof. We suppose by way of contradiction there is not a sequence braid equivalent to $(t_1, t_2, \dots, t_l)$ with three equal transpositions. By Lemmas 6 and 5 we have that $(t_1, t_2, \dots, t_l)$ is braid equivalent to a sequence $(\alpha_1, \alpha_1, \dots, \alpha_z, \alpha_z, \tilde{t}_1, \beta_1, \beta_2, \dots, \beta_m)$ where the elements α_i, β_j are transpositions moving a, $\beta_j \neq \beta_h$ for $h \neq j$ and z is at least 1.

Let $\alpha_1', \dots, \alpha_z'$ be the transpositions that one obtains applying Lemma 4 to the sequence $(\tilde{t}_1, \alpha_i, \alpha_i)$ for $i=1, \dots, z$. Let $\alpha_i = (ac_i)$, $\alpha_i' = (ac_i')$ and $\beta_j = (ad_j)$. Let $C=\{c_1, \dots, c_z\}$, $C'=\{c_1', \dots, c_z'\}$ and $D=\{d_1, \dots, d_m\}$. The following inequality holds

(3) $\qquad \sharp(C \cup C') + \sharp D > d-1$

whatever d is, odd or even.

In fact, let d be odd. In this case $C \cap C' = \emptyset$, because otherwise there would be a braid equivalent sequence with 4 equal transpositions, so $\sharp(C \cup C') = \sharp C + \sharp C' = 2z$. $\sharp D = (l-1) - 2z$, thus

$$\sharp(C \cup C') + \sharp D = 2z + (l-1) - 2z = l - 1.$$

By hypothesis $l - 1 \geqslant d$ therefore (3) holds.

Let d be even. In this case $0 \leqslant \sharp(C \cap C') \leqslant 1$, so $2z - 1 \leqslant \sharp(C \cup C') \leqslant 2z$. Then $l - 2 \leqslant \sharp(C \cup C') + \sharp D \leqslant l - 1$. Because, by hypothesis, $l - 1 > d$ the inequality (3) holds in this case as well.

Since the element a is not in $C \cup C' \cup D$, the inequality (3) assures that $\sharp((C \cup C') \cap D) \geqslant 1$. Then there exist $j, j \in \{1, \dots, m\}$, such that either $\beta_j = \alpha_i$ or $\beta_j = \alpha_i'$, for some $1 \leqslant i \leqslant z$. If $\beta_j = \alpha_i$ we obtain a braid equivalent sequence with three equal transpositions which is a contradiction. If $\beta_j = \alpha_i'$ we arrive at a contradiction in the same way applying Lemma 4 to $(\tilde{t}_1, \alpha_i, \alpha_i)$. This proves the proposition. □

Definition 5. Let σ be a permutation of S_d and let $\sigma = \sigma_1 \cdots \sigma_r$ be a factorization of σ into a product of independent cycles. Define the *norm* of σ as follows

$$|\sigma| := \sum_{i=1}^{r} (\sharp\sigma_i - 1).$$

Lemma 7 [10, Corollary 4.1]. *Let $(t_1, t_2, \dots, t_n)$ be a sequence such that $t_2, \dots, t_n$ are transpositions, t_1 is an arbitrary permutation of S_d and $G = \langle t_1, \dots, t_n \rangle$ is transitive. Then $(t_1, t_2, \dots, t_n)$ is braid equivalent to $(t_1, \tilde{t}_2, \dots, \tilde{t}_k, \dots, \tilde{t}_n)$ where $\tilde{t}_2, \dots, \tilde{t}_n$ are transpositions and*

$$|t_1| < |t_1\tilde{t}_2| < \cdots < |t_1\tilde{t}_2 \cdots \tilde{t}_k| = d - 1.$$

Proof. Let $\Gamma_1, \dots, \Gamma_m$ be the domains of transitivity of the permutation t_1. We prove the lemma by induction on m. If $m = 1$ then t_1 is a d-cycle, so $|t_1| = d - 1$. Let $m > 1$. By way of induction, we suppose the lemma is proved for $m > 1$ and we prove it for $m + 1$. Because by hypothesis G is transitive at least one transposition t_i, $2 \leqslant i \leqslant n$, is such that $t_i = (ab)$ with $a \in \Gamma_s$, $b \in \Gamma_l$ and $s \neq l$. Acting by inverses of elementary moves σ_i' we bring (ab) to the right of t_1, obtaining a new sequence

$$(t_1, \tilde{t}_2, t_3', \dots, t_n')$$

where $\tilde{t}_2 = (ab)$, $|t_1| < |t_1\tilde{t}_2|$ and $t_1\tilde{t}_2$ has m domains of transitivity. We can then apply the induction hypothesis and so we obtain the lemma. □

Proposition 3. *Let $(t_1, t_2, \dots, t_n)$ be a sequence of permutations in S_d such that t_1 is a permutation that belongs to the conjugacy class of ε (cf. Eq. (1)) and $t_2, \dots, t_n$ are transpositions.*

If $n - 1 + |\underline{e}| \geqslant 2d$ then $(t_1, t_2, \dots, t_n)$ is braid equivalent to

$$(t_1', t_2', \dots, t_{n-2}', t_{n-1}', t_n')$$

where t_1' belongs to the conjugacy class of ε, $t_2', \ldots, t_n'$ are transpositions, $t_{n-1}' = t_n'$ and

$$\langle t_1', t_2', \ldots, t_{n-2}' \rangle = \langle t_1', \ldots, t_{n-2}', t_{n-1}', t_n' \rangle.$$

Proof. Let $G = \langle t_1, \ldots, t_n \rangle$ and let $\Sigma_1, \ldots, \Sigma_v$ be the domains of transitivity of G. Acting by braid moves σ_i' and their inverses we place the transpositions $t_2, \ldots, t_n$ so that $(t_1, t_2, \ldots, t_n)$ is braid equivalent to

$$(t_1, t_{11}, \ldots, t_{1n_1}, t_{21}, \ldots, t_{2n_2}, \ldots, t_{v1}, \ldots, t_{vn_v})$$

where $t_{i1}, \ldots, t_{in_i}$ are the only transpositions that move the elements of Σ_i.

For every $i = 1, \ldots, v$, denote by $\bar{t}_i$ the permutation preserving the partition $(\Sigma_1, \ldots, \Sigma_v)$ set-by-set whose restriction to Σ_i equals $t_{1|\Sigma_i}$, and whose restriction to Σ_j is the identity for $j \neq i$. For every $i = 1, \ldots, v$, denote $d_i = \sharp\Sigma_i$; observe $d_1 + d_2 + \cdots + d_v = d$.

Notice that $\sum_{i=1}^{v} |\bar{t}_i| = |\underline{e}|$ and $n - 1 = \sum_{i=1}^{v} n_i$.

By Lemma 7 we have that $(\bar{t}_i, t_{i1}, \ldots, t_{in_i})$ is braid equivalent to a sequence $(\bar{t}_i, \tilde{t}_{i1}, \ldots, \tilde{t}_{ik_i}, \tilde{t}_{ik_i+1}, \ldots, \tilde{t}_{in_i})$ where $\bar{t}_i\tilde{t}_{i1} \cdots \tilde{t}_{ik_i}$ is a d_i-cycle and moreover only braid moves among $t_{i1}, \ldots, t_{in_i}$ were used.

Because $\sum_{i=1}^{v}(n_i - k_i) = n - 1 + |\underline{e}| + v - d$, which is greater than $n - 1 + |\underline{e}| - d$, and because $\sum_{i=1}^{v} d_i = d$, the hypothesis $n - 1 + |\underline{e}| \geqslant 2d$ implies $\sum_{i=1}^{v}(n_i - k_i) > \sum_{i=1}^{v} d_i$, in particular there exists j such that $n_j - k_j > d_j$. Moving the transpositions $t_{j1}, \ldots, t_{jn_j}$ to the front we obtain that $(t_1, t_2, \ldots, t_n)$ is braid equivalent to $(t_1, t_{j1}, \ldots, t_{jn_j}, \ldots)$.

By Lemma 7, the sequence $(t_1, t_{j1}, \ldots, t_{jn_j})$ is braid equivalent to a sequence,

$$(t_1, \tilde{t}_{j1}, \ldots, \tilde{t}_{jk_j}, \ldots, \tilde{t}_{jn_j}),$$

such that $\bar{t}_1\tilde{t}_{j1} \cdots \tilde{t}_{jk_j}$ is a d_j-cycle. Since $n_j - k_j > d_j$, by Proposition 2 one has that

$$(t_1, t_2, \ldots, t_n) \sim (t_1', t_{j1}', \ldots, t_{jk_j}', \tau, \tau, \tau, \ldots, t_{jn_j}', \ldots).$$

In this way we obtain a new sequence where there are three equal transpositions. Therefore cancelling two of these three transpositions the group generated by the remaining ones remains unchanged. The proof follows by moving two of these three transpositions to the end of the sequence. □

Remark 1. If d is odd or if d is even and $v \geqslant 2$, Proposition 3 is true for $n - 1 + |\underline{e}| \geqslant 2d - 1$. If d is odd and $v \geqslant 2$ or if d is even and $v \geqslant 3$, Proposition 3 is true for $n - 1 + |\underline{e}| \geqslant 2d - 2$.

Let $(t_1, \ldots, t_{n-1})$ be a sequence of transpositions such that $t_1 \cdots t_{n-1} = s$ and $\langle t_1, \ldots, t_{n-1} \rangle$ is transitive. Let $s = s_1 \cdots s_q$ be a factorization of s into a product of independent cycles and let $\Gamma_1, \ldots, \Gamma_q$ be the domains of transitivity of s. If

$\sharp\Gamma_i = e_i$ for each $1 \leqslant i \leqslant q$ and 1_i is the minimal number in Γ_i, then we write $s_i = (1_i 2_i \dots (e_i)_i)$. Let us order the Γ_i so that $1_1 < 1_2 < \cdots < 1_q$ and denote by Z_i the sequence $((1_i 2_i), (1_i 3_i), \dots, (1_i (e_i)_i))$. Let Z be the concatenation $Z_1 Z_2 \dots Z_q$. We use the following result.

Proposition 4 ([7] or [8, pp. 369–370]). *Let $(t_1, \dots, t_{n-1})$ be a sequence of transpositions such that $t_1 \cdots t_{n-1} = s$ and $\langle t_1, \dots, t_{n-1}\rangle$ is transitive. Then $(t_1, \dots, t_{n-1})$ is braid equivalent to*

$$(Z, \bar{t}_{N+1}, \dots, \bar{t}_{n-1})$$

where $(n-1) - N \equiv 0 \pmod 2$ and

(i) *if $q = 1$ then $\bar{t}_i = (1_1 2_1)$ for each $i \geqslant N+1$,*
(ii) *if $q > 1$ then*

$$(\bar{t}_{N+1}, \dots, \bar{t}_{n-1}) = \big((1_1\ 1_2), (1_1 1_2), (1_1 1_3), (1_1 1_3), \dots, (1_1 1_q), (1_1\ 1_q)\big)$$

where each $(1_1 1_i)$ appears twice if $2 \leqslant i \leqslant q-1$ and $(1_1 1_q)$ appears an even number of times.

Proof of Theorem 1. The forgetful map $H^o_{d,n-1,\underline{e}}(Y, b_0) \to H^o_{d,n-1,\underline{e}}(Y)$ given by $[X \to Y, \phi] \to [X \to Y]$ has a dense image, so it suffices to prove the irreducibility of $H^o_{d,n-1,\underline{e}}(Y, b_0)$. Since $H^o_{d,n-1,\underline{e}}(Y, b_0)$ is smooth in order to prove its irreducibility it suffices to prove it is connected. If we show that every $\underline{e}$-Hurwitz system in $A^o_{d,n+2g}$ is braid equivalent to the normal form $(Z, \bar{t}_{N+1}, \dots, \bar{t}_{n-1}, \varepsilon^{-1}; 1, \dots, 1)$, where $(Z, \bar{t}_{N+1}, \dots, \bar{t}_{n-1})$ is the sequence in Proposition 4, then $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ acts transitively on $A^o_{d,n+2g}$ and so $H^o_{d,n-1,\underline{e}}(Y, b_0)$ is connected.

Step 1. Let τ be an arbitrary transposition of S_d and let $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g) \in A^o_{d,n+2g}$. Because $n - 1 + |\underline{e}| \geqslant 2d$, by Proposition 3 we can replace $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ by $(t'_1, t'_2, \dots, t'_{n-2}, t'_{n-1}, t'_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ where t'_1 belongs to the conjugacy class of ε, $t'_2, \dots, t'_n$ are transpositions, $t'_{n-1} = t'_n$ and

$$\langle t'_1, t'_2, \dots, t'_{n-2}\rangle = \langle t'_1, t'_2, \dots, t'_{n-2}, t'_{n-1}, t'_n\rangle.$$

Therefore we can apply Lemma 1 and because the monodromy group of a Hurwitz system in $A^o_{d,n+2g}$ is S_d we obtain that $(t'_1, t'_2, \dots, t'_{n-2}, t'_{n-1}, t'_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ is braid equivalent to $(t'_1, t'_2, \dots, t'_{n-2}, \tau, \tau; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$.

Step 2. We claim that every $\underline{e}$-Hurwitz system in $A^o_{d,n+2g}$ is braid equivalent to $(t'''_1, t'''_2, \dots, t'''_n; 1, \dots, 1)$ where for some j the permutation t'''_j has cycle type $\underline{e}$ and t'''_i is a transposition for every $i \neq j$.

We prove this using induction on $\sum_{h=1}^{g}(|\lambda_h| + |\mu_h|)$. If $\sum_{h=1}^{g}(|\lambda_h| + |\mu_h|) = 0$ then $\lambda_h = 1$ and $\mu_h = 1$ for each $h = 1, \dots, g$.

Let $\sum_{h=1}^{g}(|\lambda_h| + |\mu_h|) > 0$. At least one λ_h or one μ_h is different by 1. If $\lambda_1 \neq 1$, let $\lambda_1 = r_1 \cdots r_s$ be a factorization of λ_1 as a product of nontrivial independent cycles. Let us choose a transposition σ such that $|\sigma r_1| = |r_1| - 1$. According to step 1 one has

$$(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g) \sim (t'_1, t'_2, \dots, t'_{n-2}, \sigma, \sigma; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g).$$

Moving σ to the front we obtain $(\sigma, t''_2, \dots, t''_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$. Applying the braid move τ''_{11} (see Proposition 1) we transform, without changing others, λ_1 into $\lambda'_1 = \sigma\lambda_1$ where $|\lambda'_1| < |\lambda_1|$, so the proof follows by applying the induction.

If $\lambda_1 = 1$ and $\mu_1 \neq 1$, we choose a transposition σ such that $|\sigma\mu_1| < |\mu_1|$. Again by step 1 and acting with inverses of elementary moves we obtain $(\sigma, t''_2, \dots, t''_n, \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$. Applying the braid move ρ'_{11} we transform μ_1 into $\mu'_1 = \sigma\mu_1$. The proof follows by applying the induction hypothesis.

If $\lambda_k \neq 1$ and $\lambda_1 = \dots = \lambda_{k-1} = 1$, $\mu_1 = \dots = \mu_{k-1} = 1$, one has $u_{k-1} = 1$. Proceeding in the same way and applying the braid move τ''_{1k} one transforms λ_k into $\sigma\lambda_k$. If $\mu_k \neq 1$ and $\lambda_1 = \dots = \lambda_k = 1$, $\mu_1 = \dots = \mu_{k-1} = 1$ one applies ρ'_{1k} which transforms μ_k into $\sigma\mu_k$. In both cases by the induction hypothesis we obtain the claim of step 2.

Step 3. Starting by $(t_1, t_2, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ and applying step 2 we obtain $(t'''_1, t'''_2, \dots, t'''_n; 1, 1, \dots, 1)$. Acting with braid moves σ'_i and their inverses, we may replace this Hurwitz system with a new system $(\tilde{t}_1, \dots, \tilde{t}_{n-1}, \tilde{t}_n; 1, \dots, 1)$ such that the cycle type of $\tilde{t}_n$ is $\underline{e}$. The permutation ε^{-1} has the same cyclic type of $\tilde{t}_n$, so $\varepsilon^{-1} = a^{-1}\tilde{t}_n a$ with $a \in S_d$. Let $a = \gamma_1 \cdots \gamma_r$ with γ_i transpositions.

Because $\langle \tilde{t}_1, \dots, \tilde{t}_{n-1}, \tilde{t}_n \rangle = S_d$ and $\tilde{t}_1 \cdots \tilde{t}_{n-1} = \tilde{t}_n^{-1}$ we have that $\langle \tilde{t}_1, \dots, \tilde{t}_{n-1} \rangle = S_d$. Then applying Mochizuchi's lemma [8, Lemma 2.4] and braid moves σ'_i, we obtain that

$$(\tilde{t}_1, \dots, \tilde{t}_{n-1}) \sim (\dots, \gamma_1), \quad \text{so} \quad (\tilde{t}_1, \dots, \tilde{t}_{n-1}, \tilde{t}_n) \sim (\dots, \gamma_1, \tilde{t}_n).$$

Acting twice by σ'_{n-1} we have

$$(\gamma_1, \tilde{t}_n) \sim (\gamma_1 \tilde{t}_n \gamma_1, \gamma_1) \sim (\gamma'_1, \gamma_1 \tilde{t}_n \gamma_1)$$

and therefore one has that $(\tilde{t}_1, \dots, \tilde{t}_{n-1}, \tilde{t}_n) \sim (\dots, \gamma_1 \tilde{t}_n \gamma_1)$.

Proceeding in this way also for $\gamma_2, \dots, \gamma_r$, we obtain that $(\tilde{t}_1, \dots, \tilde{t}_{n-1}, \tilde{t}_n; 1, 1, \dots, 1)$ is braid equivalent to $(\bar{t}_1, \dots, \bar{t}_{n-1}, \varepsilon^{-1}; 1, 1, \dots, 1)$ where $\langle \bar{t}_1, \dots, \bar{t}_{n-1} \rangle = S_d$ and $\bar{t}_1 \cdots \bar{t}_{n-1} = \varepsilon$. To conclude it is sufficient to apply Proposition 4 to the sequence $(\bar{t}_1, \dots, \bar{t}_{n-1})$.

ACKNOWLEDGEMENT

The author wishes to thank the referee for very useful suggestions and remarks.

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(Received July 2004)