



Combinatorial aspects of L -convex polyominoes

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Abstract

We consider the class of L -convex polyominoes, i.e. those polyominoes in which any two cells can be connected with an “ L ” shaped path in one of its four cyclic orientations. The paper proves bijectively that the number f_n of L -convex polyominoes with perimeter $2(n + 2)$ satisfies the linear recurrence relation $f_{n+2} = 4f_{n+1} - 2f_n$, by first establishing a recurrence of the same form for the cardinality of the “2-compositions” of a natural number n , a simple generalization of the ordinary compositions of n . Then, such 2-compositions are studied and bijectively related to certain words of a regular language over four letters which is in turn bijectively related to L -convex polyominoes. In the last section we give a solution to the open problem of determining the generating function of the area of L -convex polyominoes.

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1. L -convex polyominoes: Basic definitions

A *polyomino* is a finite union of elementary cells of the lattice $\mathbb{Z} \times \mathbb{Z}$, whose interior is connected (see Fig. 1(a)). Polyominoes are defined up to translations. Invented by Golomb [14] who coined the term *polyomino*, these well-known combinatorial objects are related to many challenging problems, such as tilings [3,13], games [12], among many others.

A *column* (*row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells whose centers lie on a vertical (horizontal) line. In a polyomino the *semi-perimeter* is half the length of the border, while the *area* is the number of its cells.

Enumerating polyominoes is one of the most famous problems in combinatorics, and despite several efforts made recently by physicists and mathematicians, it remains unsolved. The number

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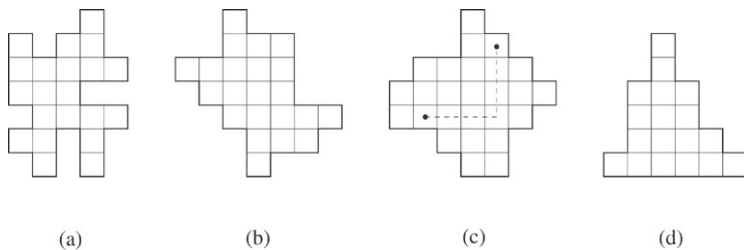


Fig. 1. (a) A polyomino; (b) a convex polyomino; (c) an *L*-convex polyomino; (d) a stack polyomino.

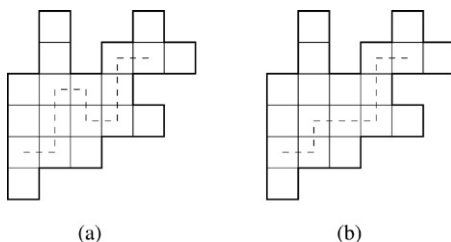


Fig. 2. (a) A path between two cells in a polyomino; (b) a monotone path made only of north and east steps, having four changes of direction.

a_n of polyominoes with n cells is known up to $n = 56$ [17] and the asymptotic behavior of the sequence $(a_n)_{n \geq 0}$ is partially known due to the relation $\lim_{n \rightarrow \infty} (a_n)^{1/n} = \mu$, where $3.96 < \mu < 4.64$, where the lower bound is a recent improvement of [2].

Nevertheless, several subclasses were enumerated by imposing *convexity* constraints [4,11]. A polyomino is *h-convex* (resp. *v-convex*) if each of its rows (resp. columns) is connected. A polyomino is *hv-convex*, or simply *convex*, if it is both *h-convex* and *v-convex* (see Fig. 1(b)).

Let us now introduce the basic concept of our paper, i.e. the notion of *k-convexity*. We define a *path* in a polyomino to be a self-avoiding sequence of unit steps of four types: *north* $N = (0, 1)$, *south* $S = (0, -1)$, *east* $E = (1, 0)$, and *west* $W = (-1, 0)$, entirely contained in the polyomino. A path connecting two distinct cells, A and B , starts from the center of A , and ends at the center of B (see Fig. 2(a)). We say that a path is *monotone* if it is constituted only of steps of two types (see Fig. 2(b)). Given a path $w = u_1 \dots u_k$, each pair of steps $u_i u_{i+1}$ such that $u_i \neq u_{i+1}$, $0 < i < k$, is called a *change of direction*.

In [8] the authors observed that convex polyominoes have the property that every pair of cells is connected by a monotone path, and proposed a classification of convex polyominoes based on the number of changes of direction in the paths connecting any two cells of the polyomino. More precisely, a polyomino is *k-convex* if every pair of its cells can be connected by a monotone path with at most k changes of direction. For $k = 1$ we have the *L-convex polyominoes*, such that each two cells can be connected by a path with at most one change of direction (see Fig. 1(c)).

In the sequel we present a different characterization of this class based on the notion of *maximal rectangles* [8].

A *rectangle*, that we denote by $[x, y]$, with $x, y \geq 1$, is a rectangular polyomino with x columns and y rows. For any polyomino P , we say that $[x, y]$ is *maximal* in P if

$$\forall [x', y'], \quad [x, y] \subseteq [x', y'] \subseteq P \Rightarrow [x, y] = [x', y'].$$

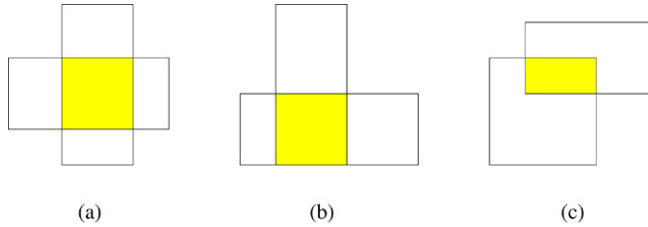


Fig. 3. (a), (b) Two rectangles having a crossing intersection; (c) two rectangles having a non-crossing intersection.

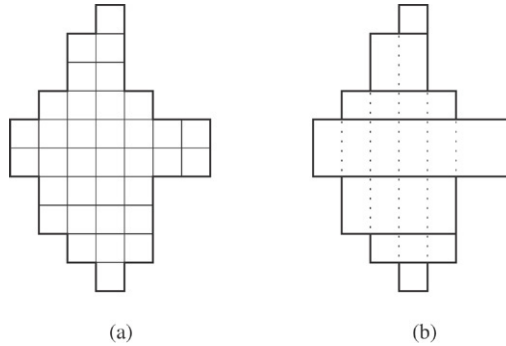


Fig. 4. The L -convex polyomino in (a) can be defined by the overlapping of five non-comparable rectangles, each pair having crossing intersections, as depicted in (b).

By abuse of notation, for any two polyominoes P and P' we will write $P \subseteq P'$ to mean that P is geometrically included in P' . Two rectangles $[x, y]$ and $[x', y']$ contained in P have a *crossing intersection* if their intersection is a non-trivial rectangle whose basis is the smallest of the two bases and whose height is the smallest of the two heights, i.e.

$$[x, y] \cap [x', y'] = [\min(x, x'), \min(y, y')];$$

see Fig. 3 for an example. The following theorem gives a characterization of L -convex polyominoes in terms of maximal rectangles [8].

Theorem 1. *A convex polyomino P is L -convex if and only if any two of its maximal rectangles have a non-void crossing intersection.*

Thus, an L -convex polyomino can be seen as the overlapping of maximal rectangles (see Fig. 4 for an example).

In the literature L -convex polyominoes have been considered from several points of view: in [9] it is shown that they are a well-ordering according to the sub-picture order; in [7] the authors have investigated some tomographical aspects, and have discovered that L -convex polyominoes are uniquely determined by their horizontal and vertical projections. Finally, in [6] it is proved that the number f_n of L -convex polyominoes having semi-perimeter equal to $n + 2$ satisfies the recurrence relation

$$f_{n+2} = 4f_{n+1} - 2f_n \quad (n \geq 1), \tag{1}$$

with $f_0 = 1, f_1 = 2, f_2 = 7$, giving the sequence 1, 2, 7, 24, 82, 280, 956, 3264, ... (sequence A003480 in [22]).

A special class of L -convex polyominoes is the well-known class of *stack polyominoes* [19] [23, p. 76] [24], i.e. convex polyominoes such that the lowest row touches both the left and the right sides of the minimal bounding rectangle (see Fig. 1(d)). Indeed such polyominoes are characterized, among L -convex polyominoes, by the property that any two cells are connected by a path $E^{h_1} N^{h_2}$ or $S^{h_1} E^{h_2}$, $h_1, h_2 \geq 0$. That is, stack polyominoes are defined by two particular orientations of the L .

In this paper we extend the study of the combinatorial properties of the class of L -convex polyominoes and of the sequence $(f_n)_{n \geq 0}$. In particular, the main results of the paper are the following:

1. in Section 2 we introduce and study the class of *2-compositions*, a natural extension of the ordinary compositions; in particular we prove that this class is enumerated by the sequence $(f_n)_{n \geq 0}$, and then we establish several properties of this sequence;
2. in Section 3 we determine a bijection between 2-compositions and L -convex polyominoes, thus giving a combinatorial explanation that L -convex polyominoes satisfy the simple recurrence in (1); such a bijection allows us also to count L -convex polyominoes according to other interesting statistics, such as the number of rows and columns and the number of maximal rectangles;
3. in Section 4, we determine the generating function for L -convex polyominoes according to the area; this problem was also considered in [6] using the ECO method, which is a constructive method for producing all the objects of a given class, according to the growth of a certain parameter (the *size*) of the objects; basically, the idea is to perform local expansions on each object of size n , thus constructing a set of objects of the successive size (see [1] for more details); the authors of [6] determined a system of algebraic equations involving the area generating function, without solving it by means of other than standard analytical techniques.

Finally we point out that some unpublished results on the class of L -convex polyominoes have been obtained independently by W. James in [16]. In particular W. James introduces a general method of representing polyominoes by their column structure (the so called *dot notation*), and applies this method to determine the generating function for the area of L -convex polyominoes and to discover some combinatorial properties of the class.

2. The class of 2-compositions

A *composition* of a natural number n is an ordered partition of n , that is a k -tuple (x_1, \dots, x_k) of positive integers such that $x_1 + \dots + x_k = n$ (see [10]).

We now extend the definition of composition to the 2-dimensional case. A *2-composition* of n is a $2 \times k$ matrix whose entries are non-negative integers that add up to n and such that each column has at least one non-zero entry. The number of columns, $k \geq 0$, is called the *length* of the 2-composition. Let U_n be the class of 2-compositions of n and let $u_n = |U_n|$. Notice that U_0 contains only the empty 2-composition, while

$$U_1 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \quad U_2 = \left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

Hence $u_0 = 1$, $u_1 = 2$ and $u_2 = 7$. More generally we have the following result.

Proposition 1. *The numbers u_n satisfy the recurrence*

$$u_{n+2} = 4u_{n+1} - 2u_n \tag{2}$$

for $n \geq 1$, with the initial values $u_0 = 1$, $u_1 = 2$, $u_2 = 7$.

Proof. Let $n \geq 1$. The 2-compositions in U_{n+2} can be all obtained by performing the following operations on each 2-composition $M \in U_{n+1}$:

1. add a column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ on the left of M ;
2. add a column $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ on the left of M ;
3. increase by one the first element on the first row of M ;
4. increase by one the first element on the second row of M .

By performing the four operations on the 2-compositions of U_{n+1} we obtain a set of $4u_{n+1}$ elements of U_{n+2} . However, some 2-compositions are obtained twice, and they are precisely those containing no null elements in the first column, that is:

1. those whose first column is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$;
2. those whose first column is $\begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$, with $x, y \geq 0$ and $(x, y) \neq (0, 0)$.

Since the number of elements in these two classes is clearly given by $2u_n$ it follows that $u_{n+2} = 4u_{n+1} - 2u_n$. \square

Thus we have the remarkable fact that the number of the L -convex polyominoes with semi-perimeter $n+2$ is equal to the number of the 2-compositions of n . In Section 3 we will determine a simple bijection between these two classes.

As proved in the following proposition, the numbers u_n also satisfy a recurrence which is not linear yet has positive constant coefficients.

Proposition 2. *The numbers u_n satisfy the recurrence*

$$u_{n+2} = 3u_{n+1} + u_n + u_{n-1} + \cdots + u_0 \quad (3)$$

with the initial values $u_0 = 1$ and $u_1 = 2$.

Proof. The 2-compositions of $n+2$ can be partitioned according to the form of the first column. Indeed the first column can be:

1. $\begin{bmatrix} 0 \\ y \end{bmatrix}$ with $y = 1, 2, \dots, n+2$; deleting from each the first column, we obtain U_{n+2-k} ;
2. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$; deleting from each the first column, we obtain U_{n+1} ;
3. $\begin{bmatrix} x+2 \\ 0 \end{bmatrix}$ with $x \geq 0$ or $\begin{bmatrix} y \\ z \end{bmatrix}$ with $y, z \geq 1$; subtracting 1 from the top entry of the first column, we obtain U_{n+1} .

This implies recurrence (3). \square

Moreover we have that the numbers u_n satisfy a recurrence relation with non-constant coefficients.

Proposition 3. *The numbers u_n satisfy the recurrence*

$$u_{n+1} = \sum_{k=0}^n (k+2)u_{n-k} \quad (4)$$

with the initial value $u_0 = 1$.

| n/k | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|---|---|----|-----|-----|-----|----|
| 0 | 1 | | | | | | |
| 1 | 0 | 2 | | | | | |
| 2 | 0 | 3 | 4 | | | | |
| 3 | 0 | 4 | 12 | 8 | | | |
| 4 | 0 | 5 | 25 | 36 | 16 | | |
| 5 | 0 | 6 | 44 | 102 | 96 | 32 | |
| 6 | 0 | 7 | 70 | 231 | 344 | 240 | 64 |

| n/k | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|----|-----|-----|-----|-----|-----|----|
| 0 | 1 | | | | | | |
| 1 | 1 | 1 | | | | | |
| 2 | 2 | 3 | 2 | | | | |
| 3 | 4 | 8 | 8 | 4 | | | |
| 4 | 8 | 20 | 26 | 20 | 8 | | |
| 5 | 16 | 48 | 76 | 76 | 48 | 16 | |
| 6 | 32 | 112 | 208 | 252 | 208 | 112 | 32 |

Fig. 5. On the left: table of the numbers $u_{n,k}$. On the right: table of the numbers $v_{n,k}$, $n, k \geq 6$.

Proof. Consider those 2-compositions of $n + 1$ for which the first column is $\left[\begin{smallmatrix} x \\ k-x+1 \end{smallmatrix} \right]$ with fixed values of k and x and $0 \leq k \leq n$, $0 \leq x \leq k + 1$. Deleting these first columns we obtain U_{n-k} . This happens $k + 2$ times (for each $x = 0, 1, \dots, k + 1$). Hence identity (4). \square

The numbers u_n satisfy the following property which resembles the well-known *Cassini identity* for Fibonacci numbers [18].

Proposition 4. *The numbers u_n have the property*

$$u_{n+1}^2 - u_{n+2}u_n = 2^{n-1} \quad (n \geq 1). \tag{5}$$

Proof. Let

$$d_n = u_{n+1}^2 - u_{n+2}u_n = \begin{vmatrix} u_{n+1} & u_n \\ u_{n+2} & u_{n+1} \end{vmatrix}.$$

Using the recurrence (2), we have

$$d_n = \begin{vmatrix} u_{n+1} & u_n \\ 4u_{n+1} - 2u_n & 4u_n - 2u_{n-1} \end{vmatrix} = \begin{vmatrix} u_{n+1} & u_n \\ -2u_n & -2u_{n-1} \end{vmatrix} = 2 \begin{vmatrix} u_n & u_{n-1} \\ u_{n+1} & u_n \end{vmatrix} = 2d_{n-1}.$$

Hence

$$d_n = 2^{n-1}d_1 = 2^{n-1} \begin{vmatrix} 7 & 2 \\ 24 & 7 \end{vmatrix} = 2^{n-1}. \quad \square$$

Let $u_{n,k}$ be the number of the elements of U_n having length k . The first values are presented in the table on the left, in Fig. 5.

Proposition 5. *The numbers $u_{n,k}$ satisfy the following recurrences:*

$$u_{n+2,k+1} = 2u_{n+1,k+1} + 2u_{n+1,k} - u_{n,k+1} - u_{n,k} \tag{6}$$

$$u_{n+1,k+1} = u_{n,k+1} + 2u_{n,k} + u_{n-1,k} + \dots + u_{0,k} \tag{7}$$

$$u_{n+1,k+1} = \sum_{i=0}^n (i + 2)u_{n-i,k}. \tag{8}$$

Proof. The recurrences (6)–(8) can be obtained with the same combinatorial arguments as are used in the proofs of Propositions 1–3, respectively. \square

Proposition 6. *The generating series for the numbers $u_{n,k}$ is*

$$u(x, y) = \sum_{n,k \geq 0} u_{n,k} x^n y^k = \frac{1}{1 - xh(x)y} \tag{9}$$

where

$$h(x) = \sum_{n \geq 0} (n + 2)x^n = \frac{2 - x}{(1 - x)^2}.$$

Proof. The series $u(x, y)$ could be obtained using one of the recurrences in Proposition 5. However, we prefer to obtain it in the following way. Since every 2-composition can be viewed as the concatenation of its columns, it follows that the set of all 2-composition is the language \mathcal{A}^* on the infinite alphabet $\mathcal{A} = \{a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, \dots\}$, where the letter a_{ij} corresponds to the column $\begin{bmatrix} i \\ j \end{bmatrix}$. Then (see [20]) the generating series of \mathcal{A}^* is

$$u(x_{10}, x_{01}, x_{20}, x_{11}, x_{02}, \dots) = \frac{1}{1 - x_{10} - x_{01} - x_{20} - x_{11} - x_{02} - \dots}.$$

Now $u(x, y)$ can be obtained setting $x_{ij} = x^{i+j}y$. \square

Expanding the series (9) we can obtain the following formula.

Proposition 7. *The numbers $u_{n,k}$ have the explicit expression (for $n, k \geq 1$):*

$$u_{n,k} = \sum_{j=0}^k \binom{k}{j} \binom{n+k-j-1}{2k-1} (-1)^j 2^{k-j}.$$

Proposition 8. *The infinite lower triangular matrix $\mathcal{U} = [u_{n,k}]_{n,k \geq 0}$ is a Riordan matrix with spectrum*

$$\left(1, \frac{2x - x^2}{(1 - x)^2} \right).$$

Proof. From (9) it immediately follows that the generating series for the columns of \mathcal{U} are

$$u_k(x) = \left(\frac{2x - x^2}{(1 - x)^2} \right)^k$$

for every non-negative integer k . Hence \mathcal{U} is a Riordan matrix [21]. \square

For $y = 1$ in (9), we reobtain the generating series $u(x)$ of the sequence $(u_n)_{n \geq 0}$. We also retrieve that $u(x)$ is the *quasi-inversion* of the series $xh(x)$ as pointed out in [5], in a completely different study.

Notice that from (9) there follows the identity $u(x, y) = 1 + xyh(x)u(x, y)$ which implies the recurrence (8). Similarly, for $y = 1$ we have $u(x) = 1 + xh(x)$ and hence (4).

Another interesting statistic can be obtained in the following way. For any $n \geq 1$, the *projection* (here the term is used in the sense of the discrete tomography [15]) of the 2-composition

$$M = \begin{bmatrix} x_1 & x_2 & \dots & \dots & x_k \\ y_1 & y_2 & \dots & \dots & y_k \end{bmatrix} \in U_n$$

is the 2-composition

$$\pi(M) = \begin{bmatrix} x_1 + x_2 + \dots + x_k \\ y_1 + y_2 + \dots + y_k \end{bmatrix}.$$

Clearly $\pi(M)$ is still an element of U_n . Moreover, for any $M \in U_n$ let us define

$$[M] = \{Q \in U_n : \pi(Q) = \pi(M)\}.$$

For instance,

$$\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

One easily observes that for any $n \geq 0$, there are $n + 1$ distinct classes $[M]$ in U_n . For $0 \leq k \leq n$, let $v_{n,k}$ be the number of elements of U_n whose projection is equal to $\begin{bmatrix} n-k \\ k \end{bmatrix}$. The first terms of $v_{n,k}$ are presented in the table in Fig. 5, on the right (sequence A059576 in [22]).

Proposition 9. *The numbers $v_{n,k}$ satisfy the recurrences*

$$v_{n+2,k+1} = 2v_{n+1,k+1} + 2v_{n+1,k} - 2v_{n,k} \tag{10}$$

$$v_{n+2,k+1} = 2v_{n+1,k+1} + v_{n+1,k} + v_{n,k-1} + \dots + v_{n-k,0}. \tag{11}$$

Proof. The recurrences (10) and (11) can be obtained with the same combinatorial arguments as are used in the proofs of Propositions 1 and 2, respectively. \square

The recurrence (10) implies the following

Proposition 10. *The generating series for the numbers $v_{n,k}$ is*

$$v(x, y) = \sum_{n,k \geq 0} v_{n,k} x^n y^k = \frac{1 - x - xy + x^2y}{1 - 2x - 2xy + 2x^2y}.$$

In particular

$$\sum_{n \geq 0} v_{n,0} x^n = \frac{1 - x}{1 - 2x}, \quad \sum_{n \geq k} v_{n,k} x^n = \frac{2^{k-1}(x - x^2)^k}{(1 - 2x)^{k+1}} \quad (k \geq 1).$$

Finally, expanding the series obtained in the previous proposition, one can obtain an explicit formula for the numbers $v_{n,k}$.

Proposition 11. *For $(n, k) \neq (0, 0)$, we have the explicit form*

$$v_{n,k} = \sum_{j=0}^{\min(k,n-k)} \binom{k}{j} \binom{n-j}{k} (-1)^j 2^{n-j-1}.$$

3. A bijection between L-convex polyominoes and 2-compositions

In this section we will present a bijection between the set L_n of L-convex polyominoes with semi-perimeter equal to $n + 2$ and the set of 2-compositions of n . In order to do this, we need first to represent L-convex polyominoes in terms of 2-colored stacks.

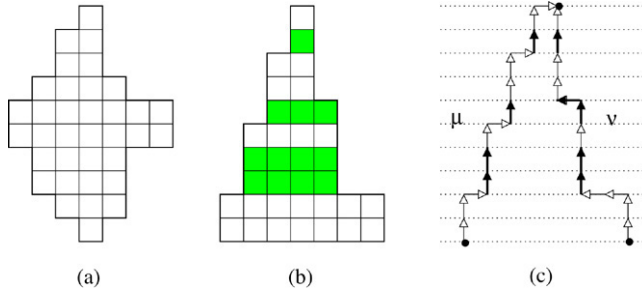


Fig. 6. (a) an L -convex polyomino; (b) the corresponding 2-colored stack polyomino; (c) the paths μ and ν ; for simplicity we represent the north, east and west steps by means of 2-colored arrows.

A 2-colored stack polyomino is a stack polyomino whose rows can have two colors, say black and white, and satisfy the following priority conditions:

1. if a row is white then all the other rows of the same length placed above it (if any) have the same color;
2. the rows having maximal length are colored white.

Starting from an L -convex polyomino, we give the black color to the rows placed below the horizontal basis, and then vertically translate them above the basis respecting condition 1 (see Fig. 6(b)). We observe that by the definition of L -convexity, the polyomino obtained is actually a 2-colored stack polyomino. In the same way, to each 2-colored stack polyomino there corresponds a unique L -convex polyomino.

The boundary of a 2-colored stack polyomino is uniquely determined by two non-intersecting (except at the end points) lattice paths μ and ν (see Fig. 6(c)):

1. μ runs from the origin to the rightmost point having maximal ordinate in the polyomino, and uses 2-colored north and east unit steps, that are black (resp. white) when it meets a black (resp. white) cell; for simplicity of representation, in Fig. 6(c) we have depicted black and white steps using black and white arrows, respectively.
2. ν runs from the rightmost point having minimal ordinate to the rightmost point having maximal ordinate in the polyomino, and uses 2-colored north and west unit steps, that are black (resp. white) when it meets a black (resp. white) cell.

Both μ and ν start with a white north step, μ ends with an east step, and ν with a north step.

Now we give a coding of 2-colored stacks in terms of words of regular language on four letters. The word representation of the polyomino is obtained by following the two paths, μ and ν , level by level from the bottom to the top of the polyomino. At each level one can meet:

1. a pair of north steps, one in μ , and the other in ν ; in this case we write a (resp. d) if the steps are white (resp. black);
2. a sequence of east steps in μ and, on the same horizontal line, a sequence of west steps in ν ; in this case we write a for each east step and a c for each west step; by convention, we assume that, at the same level, we read east steps before west steps; the coloring of the east and west steps is irrelevant.

Using such a coding we have that any L -convex polyomino P having semi-perimeter $n + 2$ can be represented as a word $w(P)$ (briefly, w) on the alphabet $\Sigma = \{a, b, c, d\}$, having the same length. The language of all such words will be referred to as \mathcal{L} . For example, the word

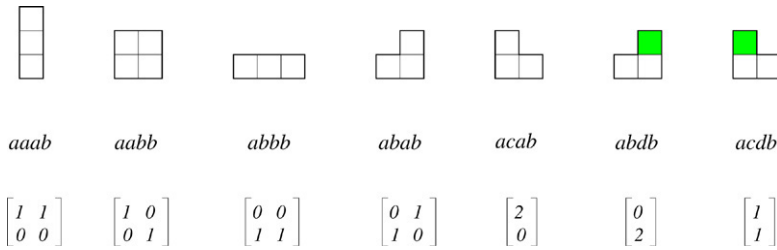


Fig. 7. The bijection between L_2 and U_2 .

corresponding to the polyomino in Fig. 6(a) is $aabccddabdcaabdab$. The number of rows (resp. columns) of the polyomino is given by the number of a plus the number of d (resp. the number of b plus the number of c) in the corresponding word of \mathcal{L} .

The words of \mathcal{L} are characterized by the property that they begin with an a and end with a b , and contain neither the factor ad nor the factor cb .

Remark 1. These simple observations lead us to state that \mathcal{L} is a regular language, described by the regular expression:

$$a(a + b + c^+a + bd^+ + c^+d^+)^* b. \tag{12}$$

For a proof of the statement, and for further properties of the language \mathcal{L} (not needed in this section) see Section 3.1. Notice that using the same coding we can represent stack polyominoes in terms of words on the alphabet $\{a, b, c\}$, beginning with an a and ending with a b , and not containing the factor cb .

To conclude defining our bijection, we now give a representation of the words of \mathcal{L} of length $n + 2$ in terms of 2-compositions in U_n .

Let P be a polyomino in L_n , w the word of \mathcal{L} corresponding to P , and \tilde{w} be the word obtained from w by removing the first and the last symbol. We observe that \tilde{w} can be uniquely factorized into the factors

$$c^h a, bd^j, c^r d^s, \quad h, j \geq 0, r, s \geq 1,$$

and then it can be transformed into a 2-composition using the following coding:

$$c^h a \rightarrow \begin{bmatrix} h + 1 \\ 0 \end{bmatrix} \quad (h \geq 0), \quad bd^k \rightarrow \begin{bmatrix} 0 \\ k + 1 \end{bmatrix} \quad (k \geq 0), \quad c^r d^s \rightarrow \begin{bmatrix} r \\ s \end{bmatrix} \quad (r, s \geq 1).$$

For instance, the word $w = aabccddabdcaabdab$ (corresponding to the polyomino in Fig. 6(a)) is translated into the 2-composition

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 & 0 & 0 & 2 & 0 \end{bmatrix}.$$

The reader can easily observe that using the above coding is also easy to pass from a 2-composition to a word of \mathcal{L} , and then to a L -convex polyomino, which completes the bijection. Fig. 7 shows the bijection between L_2 and U_2 .

3.1. Some statistics on L -convex polyominoes

In this section we extend the study of the regular language \mathcal{L} associated with L -convex polyominoes in order to derive several statistics. We start by giving a deterministic automaton

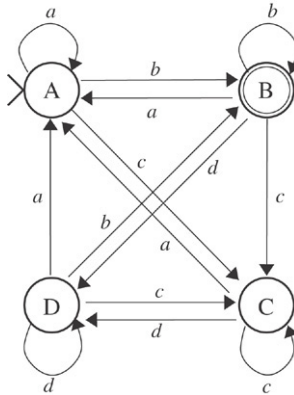


Fig. 8. The automaton recognizing the language \mathcal{L} ; A is the starting state, and B is the final one.

accepting the words of \mathcal{L} . The set of states is $S = \{A, B, C, D\}$, the alphabet is Σ , A is the start state and B is the final state. The transition function is defined according to the characterization of \mathcal{L} : in a word, the symbol $y \in \Sigma$ determines a transition from the state X to the state Y, with $X, Y \in S$, if and only if $xy \neq ad$, or $xy \neq cb$ (see Fig. 8). It is easy to prove that the defined automaton recognizes all words, and only those words of \mathcal{L} , thus \mathcal{L} is defined by the unambiguous regular expression in (12).

We use such a characterization in order to determine the generating function of L -convex polyominoes according to rows, columns, and maximal rectangles.

We have already noticed that passing from an L -convex polyomino P to the corresponding word $w(P)$, the number of rows (columns) is given by the sum of the numbers of a and d (resp. b and c). Moreover, referring to Fig. 6 we observe that, given an L -convex polyomino P , each of its maximal rectangles determines a set of rows having the same length in the 2-colored stack representing P . In the word $w(P) \in \mathcal{L}$ associated with P , each group of rows having the same length corresponds to an occurrence of a factor ab, ac, db or dc . For instance, the polyomino P in Fig. 6(a) has five maximal rectangles and in the corresponding word

$$w(P) = a|\mathbf{ab}|ccdd|\mathbf{ab}|d\mathbf{c}|a|\mathbf{ab}|d|\mathbf{ab}|$$

we have five occurrences of such factors (emphasized in bold).

Let $f_{i,j,k}$ be the number of L -convex polyominoes having i rows, j columns and made of exactly k maximal rectangles, and let $f(x, y, z) = \sum_{i \geq 1, j \geq 1, k \geq 1} f_{i,j,k} x^i y^j z^k$, the associated generating series.

From the automaton depicted in Fig. 8, using simple basics from the theory of formal power series [20] we are able to write the following system of equations:

$$\begin{aligned} A(x, y, z) &= x + xA(x, y, z) + xB(x, y, z) + xC(x, y, z) + xD(x, y, z) \\ B(x, y, z) &= yzA(x, y, z) + yB(x, y, z) + yzD(x, y, z) \\ C(x, y, z) &= yzA(x, y, z) + yB(x, y, z) + yC(x, y, z) + yzD(x, y, z) \\ D(x, y, z) &= xB(x, y, z) + xC(x, y, z) + xD(x, y, z). \end{aligned} \tag{13}$$

Observe that the variable z , counting the maximal rectangles, is increased only when a factor ab, ac, db or dc is generated. The desired solution $f(x, y, z)$ is given by the series associated

| i/j | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------|---|----|-----|------|-------|--------|--------|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 5 | 11 | 19 | 29 | 41 | 55 |
| 3 | 1 | 11 | 42 | 110 | 235 | 441 | 756 |
| 4 | 1 | 19 | 110 | 402 | 1135 | 2709 | 5740 |
| 5 | 1 | 29 | 235 | 1135 | 4070 | 11982 | 30618 |
| 6 | 1 | 41 | 441 | 2709 | 11982 | 42510 | 128534 |
| 7 | 1 | 55 | 756 | 5740 | 30618 | 128534 | 452900 |

| n/k | 0 | 1 | 2 | 3 |
|-------|---|-----|------|-----|
| 0 | 1 | | | |
| 1 | 2 | | | |
| 2 | 3 | 4 | | |
| 3 | 4 | 20 | | |
| 4 | 5 | 61 | 16 | |
| 5 | 6 | 146 | 128 | |
| 6 | 7 | 301 | 584 | 64 |
| 7 | 8 | 560 | 1992 | 704 |

Fig. 9. On the left: table of the numbers $l_{i,j}$. On the right: table of the numbers $H_{n,k}$.

with the final state of the automaton, i.e. $B(x, y, z)$:

$$f(x, y, z) = \frac{z(1-x)(1-y)xy}{1-2x-2y+x^2+4xy+y^2-2x^2y+x^2y^2-2xy^2-xyz(4-2x-2y+xy)} \tag{14}$$

which can be rewritten as

$$f(x, y, z) = \frac{z(1-x)(1-y)xy}{(1-x)^2(1-y)^2-xyz(2-x)(2-y)}. \tag{15}$$

3.1.1. Enumeration of L -convex polyominoes according to the number of maximal rectangles

Substituting $x = y$ into (15), we obtain the generating series for L -convex polyominoes according to the semi-perimeter and the number of maximal rectangles:

$$f(x, z) = \frac{zx^2(1-x)^2}{(1-x)^4-zx^2(2-x)^2}. \tag{16}$$

For simplicity, let $H_{n,k}$ be the number of all L -convex polyominoes with semi-perimeter $n + 2$ and with exactly $k + 1$ maximal rectangles. Then let $H_k(x) = \sum_{k \geq 0} H_{n,k}x^k$ and $H(x, z) = \sum_{k \geq 0} H_k(x)z^k$. Since $H(x, z) = f(x, z)/zx^2$, it follows that

$$H_k(x) = \frac{x^{2k}(2-x)^{2k}}{(1-x)^{2(2k+1)}} = \frac{1}{(1-x)^2} \left(\frac{2x-x^2}{(1-x)^2} \right)^{2k}$$

and consequently

$$H_{n,k} = \sum_{j=0}^{2k} \binom{2k}{j} \binom{n+2k-j+1}{4k+1} 2^{2k-j} (-1)^j$$

(see table in Fig. 9). The series $H_k(x)$ can also be written as

$$H_k(x) = \frac{x^{2k}}{(1-x)^{2k+2}} \left(1 + \frac{1}{1-x} \right)^{2k} = \sum_{j=0}^{2k} \binom{2k}{j} \frac{x^{2k}}{(1-x)^{2k+j+2}}$$

from which one can obtain the following formula:

$$H_{n,k} = \sum_{j=0}^{2k} \binom{2k}{j} \binom{n+j+1}{2k+j+1}.$$

From the two expressions obtained for the coefficients $H_{n,k}$ we have the binomial identity

$$\sum_{j=0}^{2k} \binom{2k}{j} \binom{n+2k-j+1}{4k+1} 2^{2k-j} (-1)^j = \sum_{j=0}^{2k} \binom{2k}{j} \binom{n+j+1}{2k+j+1}.$$

3.1.2. Enumeration of L -convex polyominoes according to rows and columns

From (14) we can also obtain the generating series of L -convex polyominoes according to the number of rows and columns. More precisely, if $l_{i,j}$ denotes the number of L -convex polyominoes with $i + 1$ rows and $j + 1$ columns, then

$$l(x, y) = \sum_{i,j \geq 0} l_{i,j} x^i y^j = \frac{f(x, y, 1)}{xy} = \frac{(1-x)(1-y)}{1-2x-2y+x^2+y^2}. \tag{17}$$

This agrees basically with the generating function obtained in [6] via generating trees.

From the form of this series we immediately have the recurrence

$$l_{i+2,j+2} = 2l_{i+1,j+2} + 2l_{i+2,j+1} - l_{i,j+2} - l_{i+2,j}.$$

Expanding the series (17) it is possible to obtain an explicit formula for the coefficients $l_{i,j}$. Indeed writing

$$l(x, y) = \frac{(1-x)(1-y)}{(1-x-y)^2 - 2xy} = \frac{(1-x)(1-y)}{(1-x-y)^2} \frac{1}{1 - \frac{2xy}{(1-x-y)^2}}$$

then it is possible to obtain the following formula for $m, n \geq 1$:

$$l_{m,n} = \sum_{k=0}^{\min(m,n)} \binom{m+n-2k}{m-k} \binom{m+n}{2k} \frac{k^2 + (m+n)k + mn + m + n}{(m+n)(2k+1)} 2^k.$$

See Fig. 9 for a table of these numbers. In particular, for $m = n$ we have the number $l_n = l_{n,n}$ of all L -convex polyominoes inscribed in a $(n + 1) \times (n + 1)$ box, for $n \geq 1$:

$$l_n = \sum_{k=0}^n \binom{2n-2k}{n-k} \binom{2n}{2k} \frac{(n+k)^2 + 2n}{2n(2k+1)} 2^k.$$

Finally, using standard techniques, it is possible to obtain the generating series $l(x)$ for these numbers as the diagonal of the bivariate series $l(x, y)$. Specifically $l(x)$ is an algebraic series, unlike $f(x)$, which is rational:

$$\begin{aligned} l(x) &= \frac{1}{2} \sqrt{\frac{2+5x-2x^2+(2-x)\sqrt{1-12x+4x^2}}{1-12x+4x^2}} \\ &= 1 + 5x + 42x^2 + 402x^3 + 4070x^4 + 42510x^5 + 452900x^6 + 4891988x^7 \\ &\quad + 53376966x^8 + \dots \end{aligned}$$

The sequence $(l_n)_{n \geq 0}$ does not appear in [22].

4. Enumeration according to area

In this section we determine the generating function of L -convex polyominoes according to the area. In order to do this, we first observe that every L -convex polyomino can be obtained with a sequence of the following operations starting from the polyomino formed by one cell:

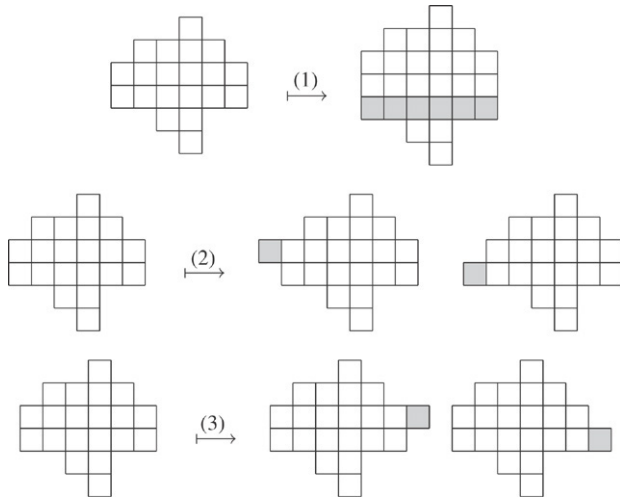


Fig. 10. Generation of *L*-convex polyominoes.

1. add a new row of maximal length;
2. add a new cell on the left of a row of maximal length;
3. add a new cell on the right of a row of maximal length.

(See Fig. 10 for an example.) In this way, however, every polyomino with exactly one row of maximal length with a cell protruding on the left and a cell protruding on the right can be obtained two times applying operations 2 and 3 first in this order and then in the inverse order.

Let $L_{n,i,j}$ be the set of all *L*-convex polyominoes with semi-perimeter $n + 2$ and $i + 1$ rows of maximal length $j + 1$ and let

$$a_{n,i,j}(q) = \sum_{P \in L_{n,i,j}} q^{a(P)}$$

where $a(P)$ is the area (i.e. the number of cells) of P . It follows that

$$a_{n+1,i+1,j}(q) = q^{j+1} a_{n,i,j}(q) \tag{18}$$

$$a_{n+2,0,j+2}(q) = \sum_{k=0}^{n+1} (k + 1)(2q a_{n+1,k,j+1}(q) - q^2 a_{n,k,j}(q)). \tag{19}$$

Consider now the generating series

$$a_{i,j}(q; x) = \sum_{n \geq 0} a_{n,i,j}(q) x^n, \quad a_j(q; x, y) = \sum_{n,i \geq 0} a_{n,i,j}(q) x^n y^i.$$

Since $a_{0,i+1,j}(q) = 0$ for every $i, j \geq 0$, Eq. (18) implies that

$$a_{i+1,j}(q; x) = q^{j+1} x a_{i,j}(q; x)$$

and consequently

$$a_j(q; x, y) = \frac{b_j(q; x)}{1 - q^{j+1} x y} \tag{20}$$

where $b_j(q; x) = a_{0,j}(q; x)$.

From Eq. (19) it follows that

$$\mathcal{R}_x^2 b_{j+2}(q; x) = 2q[\mathcal{R}_x(\theta_y + 1)a_{j+1}(q; x, y)]_{y=1} - q^2[(\theta_y + 1)a_j(q; x, y)]_{y=1} \quad (21)$$

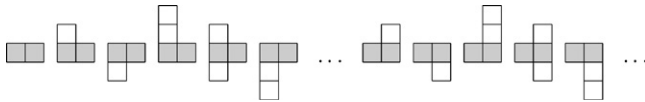
where \mathcal{R}_x is the operator defined by $\mathcal{R}_x f(x) = (f(x) - f(0))/x$ and $\theta_y = y \frac{d}{dy}$. From Eq. (20) it follows that

$$[\theta_y a_j(q; x, y)]_{y=1} = \frac{q^{j+1} x b_j(q; x)}{(1 - q^{j+1} x)^2}.$$

Since $b_{0,j+2}(q) = a_{0,0,j+2}(q) = 0$ and $b_{1,j+2}(q) = a_{1,0,j+2}(q) = 0$ for every j , Eq. (21) becomes

$$b_{j+2}(q; x) = \frac{2qx}{(1 - q^{j+2} x)^2} b_{j+1}(q; x) - \frac{q^2 x^2}{(1 - q^{j+1} x)^2} b_j(q; x). \quad (22)$$

Finally we need the initial values. For $j = 0$ there is only the unit square, and hence $b_0(q; x) = q$. For $j = 1$ we have all the following polyominoes:



and then

$$b_1(q; x) = 2 \sum_{h,k \geq 0} q^{h+k+2} x^{h+k+1} - q^2 x = \frac{q^2 x (1 + 2qx - q^2 x^2)}{(1 - qx)^2}.$$

Recurrence (22), with the given initial values, completely determines the sequence $b_j(q; x)$ and easily yields

$$b_j(q; x) = \frac{q^{j+1} x^j f_j(q; x)}{(1 - qx)^2 (1 - q^2 x)^2 \dots (1 - q^j x)^2} \quad (23)$$

for suitable polynomials $f_j(q; x)$. Substituting the expression of $b_j(q; x)$ given by (23) in (22), it follows that the polynomials $f_j(q; x)$ satisfy the recurrence

$$f_{j+2}(q; x) = 2f_{j+1}(q; x) - (1 - q^{j+2} x)^2 f_j(q; x) \quad (24)$$

with the initial conditions $f_0(q; x) = 1$ and $f_1(q; x) = 1 + 2qx - q^2 x^2$. This completely defines the polynomials $f_k(q; x)$.

Then we have that

$$a_j(q; x, y) = \frac{q^{j+1} x^j f_j(q; x)}{(1 - qx)^2 (1 - q^2 x)^2 \dots (1 - q^j x)^2 (1 - q^{j+1} xy)}$$

and finally

$$\begin{aligned} a(q; x, y, z) &= \sum_{n,i,j \geq 0} a_{n,i,j}(q) x^n y^i z^j \\ &= \sum_{k \geq 0} \frac{q^{k+1} x^k z^k f_k(q; x)}{(1 - qx)^2 (1 - q^2 x)^2 \dots (1 - q^k x)^2 (1 - q^{k+1} xy)}. \end{aligned} \quad (25)$$

From this series we can deduce several other generating series. First of all for $q = 1$ Eq. (24) becomes

$$f_{j+2}(1; x) = 2f_{j+1}(1; x) - (1 - x)^2 f_j(1; x) \tag{26}$$

and $f_0(1; x) = 1$ and $f_1(1; x) = 1 + 2x - x^2$. Then the generating series for these polynomials is

$$f(1; x, t) = \frac{1 - (1 - x)^2 t}{1 - 2t + (1 - x)^2 t^2}. \tag{27}$$

Hence for $q = 1$ the series (25) becomes

$$a(1; x, y, z) = \frac{1}{1 - xy} f\left(1; x, \frac{xz}{(1 - x)^2}\right) = \frac{(1 - x)^2(1 - xz)}{(1 - xy)(1 - 2x + x^2 - 2xz + x^2z^2)}.$$

In particular, for $y = z = 1$, we obtain again the generating series

$$a(1; x, 1, 1) = \frac{(1 - x)^2}{1 - 4x + 2x^2}$$

for the number of L -convex polyominoes according to the semi-perimeter.

Finally let a_n be the number of all L -convex polyominoes with area n . The first few terms of this sequence are

1, 1, 2, 6, 15, 35, 76, 156, 310, 590, 1098, 1984, 3515, 6094, 10398, 17434, 28837, 47038, 75820.

This sequence is not in the *Encyclopedia of Integer Sequences* [22]. From (25) it immediately follows that its generating series is

$$a(q) = \sum_{n \geq 0} a_n q^n = 1 + \sum_{k \geq 0} \frac{q^{k+1} f_k(q)}{(1 - q)^2(1 - q^2)^2 \dots (1 - q^k)^2(1 - q^{k+1})}$$

where $f_k(q) = f_k(q; 1)$. This series is very similar to the generating series [24]

$$s(q) = \sum_{n \geq 0} s_n q^n = 1 + \sum_{k \geq 0} \frac{q^{k+1}}{(1 - q)^2(1 - q^2)^2 \dots (1 - q^k)^2(1 - q^{k+1})},$$

for the numbers s_n of stacks with area n (sequence A001523 in [22]). They only differ by the presence of the polynomials $f_k(q)$.

It might be interesting to investigate more deeply the (combinatorial and formal) structure of such polynomials. By definition it immediately follows that the $f_k(q)$ are continuants of order $k + 1$ [25] of the form

$$f_k(q) = \begin{vmatrix} 1 & (1 - q)^2 & & & \\ 1 & 2 & (1 - q^2)^2 & & \\ & 1 & 2 & (1 - q^3)^2 & \\ & & \dots & \dots & \dots \\ & & 1 & 2 & (1 - q^k)^2 \\ & & & 1 & 2 \end{vmatrix}.$$

Moreover, from the recurrence (26) it follows that their generating series $f(q; x)$ is defined by the equation

$$f(q; x) = \frac{1 - (1 - q)^2 x}{(1 - x)^2} + \frac{2q^2 x^2}{(1 - x)^2} f(q; qx) - \frac{q^4 x^2}{(1 - x)^2} f(q; q^2 x)$$

which yields the following expression:

$$f(q; x) = \sum_{k \geq 0} \frac{1 - (1 - q)^2 q^k x}{(1 - q^k x)^2} g_k(q; x)$$

where the $g_k(q; x)$ are defined by the recurrence

$$g_{k+2}(q; x) = \frac{2q^{2k+4} x^2}{(1 - q^{k+1} x)^2} g_{k+1}(q; x) - \frac{q^{2k+6} x^2}{(1 - q^{k+1} x)^2} g_k(q; x)$$

with the initial conditions $g_0(q; x) = 1$ and $g_1(q; x) = \frac{2q^2 x^2}{(1-x)^2}$. Also the series $g_k(q; x)$ are also continuants.

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