Mechanism Design for Optimal Consensus Problems

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***Abstract*— We consider stationary consensus protocols for networks of dynamic agents with fixed and switching topologies. At each time instant, each agent knows only its and its neighbors' state, but must reach consensus on a group decision value that is function of all the agents' initial state.We show that our protocol design is the solution of individual optimizations performed by the agents. This notion suggests a game theoretic interpretation of consensus problems as mechanism design problems. Under this perspective a supervisor entails the agents to reach a consensus by imposing individual objectives. We prove that such objectives can be chosen so that rational agents have a unique optimal protocol, and asymptotically reach consensus on a desired group decision value.**

I. INTRODUCTION

Coordination of agents/vehicles is an important task in several applications including autonomous formation flight [3], [4], cooperative search of unmanned air-vehicles (UAVs) [5], swarms of autonomous vehicles or robots [6], [7], [8], multi-retailer inventory control [9], [10], [11] and congestion/flow control in communication networks [12]. Distributed *consensus protocols* are distributed control policies based on local information that allow the coordination of multi-agent systems. Agents implement a consensus protocol to reach *consensus*, that is to (make their states) converge to a same value, called *consensus-value*, or *group decision value* [1], [2].

Particularly interesting is the progress in the design and analysis of consensus protocols obtained merging notions and tools from the Graph Theory and Control Theory [13]. Actually, a central point in consensus problems is the connection between the graph topology, possibly switching, and delays or distortions in communication links [14]. Switching topology and directional communications are studied in [1], [15], [16], [17], [18], [19], while cooperation based on the notion of *coordination variable* and *coordination function* in [20], [21]. There, coordination variable is referred to as the *minimal amount of information needed to effect a specific coordination objective*, whereas a coordination function *parameterizes the effect of the coordination variable on the myopic objectives of each agent*.

In this paper, n dynamic agents reach consensus on a group decision value by implementing *optimal*, *distributed* and *stationary control policies* based on neighbors' state feedback. In [22], neighborhood relations are defined by the existence of communication links between nearby agents. we assume that the set of communication links are bidirectional and define a time-invariant connected communication network. In this context, we argue that agents asymptotically reach consensus on the desired group decision value by studying equilibrium properties and stability of the group decision value via Lyapunov theory. Then, we consider networks with switching topology and directional communications. To generalize our previous results here we assume *dwell time* [15], [23] between switchings and exploit an analysis tool for switched systems known as *common Lyapunov function* [1], [24], [25], [26]. Similarly to [1], [3], [13], our agents follow a first-order dynamics.

Our contribution to the study on consensus problems is two-fold. In [22], we show that consensus can be reached if the agents' state trajectories satisfy a certain time invariancy property. In doing this we consider both linear and non-linear protocols. On the basis of such a result, we prove that the group decision values considered are sufficiently general to include any mean of order p of the agents' initial states, and not only the arithmetic/min/max means usually dealt with in the literature (see, e.g., [13], [27]). The referred result are extended here to systems with switching topology.

Moreover, we show that the distributed/individual optimality of the considered protocols allows interpreting our consensus problem as a *non cooperative differential game* [28], [29] where a supervisor entails the agents to reach a consensus by imposing individual objectives. This perspective reminds the *mechanism design*, or inverse game theory [30]. Indeed, the main topic of the mechanism design is the definition of game rules or incentive schemes that induce self-interested players to cooperate and reach Pareto optimal solutions [28]. To justify the use of mechanism design consider the general case when it is not employed. The distributed protocols are generally planned at high level by a supervisor. The supervisor communicates the planned control policies to the agents that are in charge to implement them at a low level. The agents are only said "what to do" by the supervisor and do not exhibit any decision capability. This aspect is actually a drawback in systems with large dimensions and complex structures. Indeed, if agents are not motivated to behave in the desired way, then a costly monitoring system may be necessary. In addition, computing the optimal protocol can be too onerous for the supervisor. The situation is even worse if the system operates in a hostile environment. In this case, loosing the supervisor dramatically deprives the entire system of its central intelligence, with foreseeable negative consequences. The advantage of the mechanism design is that intelligence as well as implementation capabilities are distributed. Reviewing the asymptotically consensus reaching as a *team objective*, the supervisor decomposes this team objective in n *individual objectives*. By doing this, the agents are said "what to aim at" instead of "what to do", and are free to find the best solution to their subtasks. Obviously, the advantage of distributing intelligence has on one hand a non indifferent cost due to the necessity of equipping each agent with computational and processing units, but, on the other hand, reduces the monitoring costs. We show that, if the supervisor imposes convex penalty functions, rational agents have a unique optimal protocol. We prove this result through the Pontryagin Minimum Principle (see, e.g., [31]). From a slightly different point of view, the solution of a mechanism design problem allows to determine whether a set of agents with given individual objective functions will implement the considered protocol.

The present paper states consensus protocol definition and mechanism design as two separate problems for the sake of clarity. However, it must be noted that these two problems may be seen as two faces of the same coin. The consensus protocol definition problem answers to the question of determining the policies that the agents must implement to reach a given consensus. The mechanism design problem answers to the question of which policy is implemented, and hence which consensus value is reached, by selfish agents with given individual objectives.

In solving the mechanism design problem, we need to make

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some assumptions on the information available to each agent and its computational capabilities. In particular, we assume that, at each time instant, the generic agent can observe the state of its neighbor agents but cannot predict their future behavior and hence uses a *naive expectation* of the state of the neighbor agents in evaluating its individual objective function. In accordance with [32], naive expectation means that the agents choose their optimal control policy assuming the neighbors' state variables as constants. Needless to say that the naive expectation hypothesis is a major one, but it is reasonable when agents: cannot keep record of the past behavior of their neighbors, are so numerous that they can infer little or none on the group decision value from the punctual observation of their neighbors' state and, possibly, are organized in a network with a switching topology whose switching times and pattern are unpredictable. Obviously, under different assumptions on the information available and the computational capabilities of each agent, the agents may choose different policies and then converge to a different group decision values. However, note that at least when the considered group decision value is the arithmetic mean of the agents' initial states, the optimal protocol that we obtain imposing the naive expectation hypothesis is the same that we would obtain if we have assumed that the agents could also know the first derivative of the state of its neighbors. In this case, the optimal policy can be determined by observing that the problem is an *affine quadratic game*. Then, it belongs to one of the few classes of differential games that can be solved exactly [28].

The paper is organized as follows. In Section II, we formulate the consensus problem (Problem 1) and the mechanism design problem (Problem 2). In Section III, we generalize our results to systems with switching topology. Section IV addresses the mechanism design problem, whose solution is derived starting from the results on the consensus problem. More specifically, we exploit the Pontryagin Minimum Principle to derive necessary and sufficient optimality conditions. Then, we merge the results on time invariancy, stability, and optimality to design a mechanism for the distributed optimal consensus. Finally, in Section V, we draw some conclusions.

II. Consensus and Mechanism Design Problems

We consider a system of n dynamic agents $\Gamma = \{1,\dots,n\}$ and model the interaction topology among agents through a time-invariant connected network (graph) $G = (\Gamma, E)$, where each edge (i,j) in the edgeset E represents a bidirectional communication link. The network is undirected since we assume that if agent i can receive information from agent j then also agent j can receive information from agent i. The network is connected since we assume that for any agent $i \in \Gamma$ there exists a path, i.e., a sequence of edges in E, $(i,k_1)(k_1,k_2)\dots(k_r,j)$, that connects it with any other agent $j \in \Gamma$. Finally, the network G is not complete since each agent i exchanges information only with a subset of other agents $N_i = \{j : (i,j) \in E\}$ called *neighborhood of* i.

Each agent i has a (simplified) *first-order dynamics* controlled by a *distributed* and *stationary* control policy

$$\dot{x}_i = u_i(x_i, x^{(i)}) \quad \forall i \in \Gamma, \tag{1}$$

where x_i is the state of agent i and $x^{(i)}$ is the state vector of the agents in N_i with generic component j defined as follows, $x_j^{(i)} = \begin{cases} x_j & \text{if } j \in N_i, \\ 0 & \text{otherwise,} \end{cases}$

and such that (1) has unique solutions. The policy is distributed since, for each agent i, it depends only on the local information available to it, which is x_i and $x^{(i)}$. No other information on the current or past system state is available to agent i. (We discuss the limitation of this assumption at the beginning of Section 3). The policy is stationary since it does not depend explicitly on time t. In other words, the policy is a time-invariant and memoryless function of the state. Define the system state vector $x(t) = \{x_i(t), i \in \Gamma\}$, then the system initial state $x(0)$ is the collection of the the agents' initial states. Define $u(x) = \{u_i(x_i, x^{(i)}) : i \in \Gamma\}$ as a *distributed stationary protocol* or simply *a protocol*. Let $\hat{\chi} : \mathbb{R}^n \to \mathbb{R}$ be a generic continuous function of n variables $x_1,\dots,x_n$ which is permutation invariant, i.e., $\hat{\chi}(x_1, x_2, \dots, x_n) = \hat{\chi}(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$ for any one to one (permutation) mapping $\sigma(.)$ from the set Γ to the set Γ. Henceforth $\hat{\chi}$ is also called *agreement function*. Putting together slightly different definitions in [1], [2], [27], we say that a protocol $u(.)$ makes the agents asymptotically reach consensus on a *group decision value* $\hat{\chi}(x(0))$ function of their initial states if $\|x_i - \hat{\chi}(x(0))\| \longrightarrow 0$ as $t \longrightarrow \infty$. When this happens we also say that the system converges to $\hat{\chi}(x(0))\mathbf{1}$. Here and in the following, $\mathbf{1}$ stands for the vector $(1,1,\dots,1)^T$.

Notwithstanding each agent i has only a local information $(x_i, x^{(i)})$ about the system state x, we are interested in making the agents reach consensus on group decision values that are functions of the whole system initial state $x(0)$. In particular, we are interested in agreement functions verifying

$$\min_{i\in\Gamma}\{y_i\} \le \hat{\chi}(y) \le \max_{i\in\Gamma}\{y_i\}, \quad \text{for all } y \in \mathbb{R}^n. \tag{2}$$

The above condition means that the group decision value must be confined between the minimum and the maximum values of the agents' initial states.

Finally, we define an *individual objective* for an agent i, i.e.,

$$J_i(x_i, x^{(i)}, u_i) = \lim_{T\longrightarrow\infty} \int_0^T \left(F(x_i, x^{(i)}) + \rho u_i^2 \right) dt \tag{3}$$

where $\rho > 0$ and $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a nonnegative *penalty function* that measures the deviation of x_i from neighbors' states. We say that a protocol is *optimal* if each u_i optimizes the corresponding individual objective.

In the above context, we face the following problem.

Problem 1: (Consensus Problem) Consider a network $G = (\Gamma, E)$ of dynamic agents with first-order dynamics. For any agreement function $\hat{\chi}$ satisfying condition (2, determine a (distributed stationary) protocol, whose components have the feedback form (1), that makes the agents asymptotically reach consensus on $\hat{\chi}(x(0))$ for any initial state $x(0)$.

A protocol that solves the consensus problem is also referred to as a *consensus protocol*. In [22] we generalized our consensus problem to any agreement function $\hat{\chi}(x)$.

Assumption 1: (Structure of $\hat{\chi}(.)$) Assume that the generic agreement function $\hat{\chi}(.)$ satisfies condition (2) and is such that $\hat{\chi}(x) = f(\sum_{i\in\Gamma} g(x_i))$, for some $f, g : \mathbb{R} \to \mathbb{R}$ with $\frac{dg(x_i)}{dx_i} \neq 0$ for all x_i.

A point of interest is that the above family of agreement function is more general than the arithmetic/min/max means already reported in the literature (see, e.g., Tab. I). In this sense, observe that the structure of the agreement function is general to the extent that any value in the range between the minimum and the maximum values of the agents' initial states can be chosen as a group decision value. To see this, it is sufficient to consider mean of order p with p varying between $-\infty$ and ∞. In [22] it is proved that the agents asymptotically reach consensus on $\hat{\chi}(x(0))\mathbf{1}$ when function $g(.)$ is strictly increasing, i.e., $\frac{dg(y)}{dy} > 0$ for all $y \in \mathbb{R}$.

We recall hereafter the main results proved in [22]:

Lemma 1: Consider a network $G = (\Gamma, E)$ of dynamic agents with first-order dynamics and implement a distributed and stationary protocol $u(.)$ whose components have the feedback form

$$u_i(x_i, x^{(i)}) = \alpha \frac{1}{\frac{dg}{dx_i}} \sum_{j\in N_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)), \quad \text{for all } i \in \Gamma. \tag{4}$$

Then, for any initial state $x(0)$, the system may not converge (asymptotically) to any equilibrium point different from $\hat{\chi}(x(0))\mathbf{1}$.

mean	$\hat{\chi}(x)$	$f(y)$	$g(z)$
arithmetic	$\sum_{i\in\Gamma}\frac{1}{n}x_i$	$\frac{1}{n}y$	z
geometric	$\sqrt[n]{\prod_{i\in\Gamma}x_i}$	$e^{\frac{1}{n}y}$	$\log z$
harmonic	$\frac{1}{\sum_{i\in\Gamma}\frac{n}{x_i}}$	$\frac{n}{y}$	$\frac{1}{z}$
mean of order p	$\sqrt[p]{\sum_{i\in\Gamma}\frac{1}{n}x_i^p}$	$\sqrt[q]{\frac{1}{n}y}$	z^p

TABLE I

MEANS UNDER CONSIDERATION AND THEIR REPRESENTATIONS IN TERMS OF f AND g

Theorem 1: Consider a network $G=(\Gamma,E)$ of dynamic agents with first-order dynamics and implement a distributed and stationary protocol whose components have the feedback form (4). If function $g(.)$ is strictly increasing, the agents asymptotically reach consensus on $\hat{\chi}(x(0))\mathbf{1}$ for any initial state $x(0)$.

In addition a consensus protocol is said *optimal*, if its components are the optimal controls $u_i(.)$ corresponding to minimizing (3).

Problem 2: (Mechanism design problem) Consider a network $G=(\Gamma,E)$ of dynamic agents with first-order dynamics. For any agreement function $\hat{\chi}(.)$ determine a penalty function $F(.)$ such that there exists an optimal consensus protocol $u(.)$ with respect to $\hat{\chi}(x(0))$ for any initial state $x(0)$.

Notice that a pair $(F(.),u(.))$ solving Problem 2 must be such that all individual objectives (3) converge to a finite value. Then, it is necessary that the integrand in (3) be null if computed in $\chi\mathbf{1}$. We will check later on that this necessary condition is verified by our candidate penalty function $F(.)$.

III. SYSTEMS WITH SWITCHING TOPOLOGY

In [22] we prove that protocols (4) make the agents asymptotically reach consensus on the group decision value where the system topology is fixed. Here we discuss whether we can generalize our results to systems with a *switching topology* [1], [15], [24], [25], [26], i.e., systems where existing communication links between agents may fail and new communication links may be created over the time. In this situation, the network G has a time variant edgeset E. More formally, let $\mathcal{E}$ be the finite set of all the possible edgesets connecting the agents in Γ. The elements $E\in\mathcal{E}$ define a family of dynamical systems (1)

$$\dot{x}_i=u_{iE}(x_i,x^{(i)})\quad\forall i\in\Gamma, \tag{5}$$

where the control policy is still stationary but depends on the edgsets of the system network as described by index E.

To define a switched system from the above family of dynamical system we must define a *switching function*. Let $\mathcal{I}$ be the index set associated with the elements of $\mathcal{E}$. Then $E=E_k$, with $E_k\in\mathcal{E}$ and $k=s(t)$, where $s:\mathbb{R}^+\to\mathcal{I}$ is the switching function. Also, let us call t_r a *switching time* if $s(t_r^-)\neq s(t_r^+)$. In the rest of this section, to avoid pathological behaviors arising when the switching times have a finite accumulation point (see, e.g., the *zeno behavior* in [24]) and in accordance with [15], [25], we assume that the switching function s has a finite number of switching times on every bounded time interval, and takes constant value on every interval between two consecutive switching times (see, e.g., the notion of *dwell time* in [15], [23]).

Following the same line of reasoning as in [1], we observe that the proofs of all the theorems in the previous sections assume only that network G is connected, that the communication links are bidirectional, and that $\hat{\chi}(.)$ is time-invariant. All these properties still hold even if the network G has a switching topology but is always connected. Unfortunately, once one allows switching, the right hand side of (5) becomes discontinuous. It is not (locally) Lipschitz any more, and one cannot guarantee uniqueness of solutions. One way for the existence and uniqueness of solution to hold is to demand that f be *piecewise continuous* in t [24]. In this case one needs to work with a weaker concept of solution, namely, a function $x(.)$ that is piecewise differentiable and satisfies the differential equation almost everywhere. Such functions are known as *absolutely continuous* and provide solutions *in the sense of Carathéodory*. With this in mind, one is induced to think that Lyapunov second method cannot be applied in his standard form. This is true only in part. Indeed, let us start from the observation that the Lyapounov direct (second) method leads to the same conclusions even *if V is merely continuous and not necessarily continuously differentiable* (see, e.g., comments on the Lyapunov direct method, Appendix A, [24]). Then, for the asymptotical stability it suffices that V is strictly decreasing along nonzero solutions. Even the latter condition, i.e., V being strictly decreasing along nonzero solutions is not necessary to the global stability of a switched system [23]. Convergence to zero is, indeed, guaranteed even under the weaker condition that at every switching time the sequence $v(x(t_k))$ for $k=0,\ldots,\infty$ converges to zero.

With the above considerations in mind, one way of studying stability of switched systems is based on the notion of *common Lyapunov function* [24], [25], [26].

Then, to prove that Theorem 1 holds even if the network G has a switching topology we must prove that $V(\eta)=\frac{1}{2}\sum_{i\in\Gamma}\eta_i^2$ is a common Lyapunov function for the family of systems (5). To do this we will make use of the notion of *minimum spanning tree*, defined on graph G with time varying edge costs

$$c_{ij}=\alpha(g(x_j)-g(x_i))\hat{\phi}(\vartheta(x_j)-\vartheta(x_i)),\quad\forall(i,j)\in E.$$

In particular, let us define the subset $\mathcal{Q}\subseteq\mathcal{E}$ of the edgsets defining graphs (Γ,Q) that are (connected) trees. For each $Q\in\mathcal{Q}$ we introduce the function

$$\begin{aligned} W_Q(\eta) &= \alpha\{\sum_{(i,j)\in Q}(\eta_j-\eta_i)\hat{\phi}(\vartheta(g^{-1}(\eta_j+g(\hat{\chi}(x(0))) & \text{(6a)}\\ &\quad -\vartheta(g^{-1}(\eta_i+g(\hat{\chi}(x(0))))= & \text{(6b)}\\ &= \alpha\{\sum_{(i,j)\in Q}(g(x_j)-g(x_i))\hat{\phi}(\vartheta(x_j)-\vartheta(x_i))\} & \text{(6c)}\\ &= \sum_{(i,j)\in Q}c_{ij}. & \text{(6d)}\end{aligned}$$

Note that $g^{-1}(.)$ is a well defined continuous function as $g(.)$ is a continuous strictly increasing function. It is immediate to observe that, for each $Q\in\mathcal{Q}$, function $W_Q(\eta)$ is continuous, as it is composition of continuous functions, and *positive definite*, that is, $W_Q(\eta)\geq 0$ and $W_Q(\eta)=0$ if and only if $\eta=0$. Now, the minimum spanning tree solves $W(\eta)=\min_{Q\in\mathcal{Q}}\{W_Q(\eta)\}$, where function $W(\eta)$ is still continuous, and positive definite.

Finally, we observe that the Lyapunov function $V(\eta)=\frac{1}{2}\sum_{i\in\Gamma}\eta_i^2$ defined in the proof of Theorem 1 is a positive definite continuously differentiable function such that

$$\begin{aligned} \dot{V}(\eta) &= \sum_{i\in\Gamma}\eta_i\frac{dg(x_i)}{dx_i}u_{iE} & \text{(7a)}\\ &= -\alpha\sum_{(i,j)\in E}(g(x_j)-g(x_i))\hat{\phi}(\vartheta(x_j)-\vartheta(x_i)) & \text{(7b)}\\ &\leq -W(\eta),\quad\forall\eta\in\mathbb{R}^n,\,E\in\mathcal{E} & \text{(7c)}\end{aligned}$$

In fact, each edgeset $E\in\mathcal{E}$ defines a connected graph, thus there surely exist $Q\in\mathcal{Q}$ such that $Q\subseteq E$. Since for each $(i,j)\in E$ we have $(g(x_j)-g(x_i))\hat{\phi}(\vartheta(x_j)-\vartheta(x_i))\geq 0$, we obtain that the inequality between $\dot{V}(\eta)$ and $W(\eta)$ must hold in (7) for each $E \in \mathcal{E}$.

Since, $W(\eta)$ is continuous and positive definite, in accordance with [25], the switched system is globally uniformly asymptotically stable in the quotient space $\mathbb{R}^n/span\{\mathbf{1}\}$.

IV. Penalty Functions and Optimal Protocols

Hereafter we discuss whether we can make the agents asymptotically reach consensus on the group decision value by assigning to each agent an individual objective function to optimize.

In solving our mechanism design problem, we need to make some assumptions on the information available to each agent and its computational capabilities. In particular, we assume that,

Assumption 2: (Naive Expectation) At each time $t \geq 0$, each agent $i \in \Gamma$ can observe only the state $x^{(i)}$ of its neighbor agents $j \in N_i$, but cannot predict their future behavior nor, in case of switching topology, it can predict the possible evolution of the topology. For these reasons agents optimize its individual objective function (3) using a receding horizon approach where, at each time $t \geq 0$, the states of the neighbors and the topology of the network are considered constant from time t on.

The naive assumption is nothing but an alternative way of dealing with uncertainties similar to what done in the min-max approach (when the uncertainty is unknown but bounded, one assumes that the uncertain variable takes on the worst value at each time) or the min-exp approach (when some stochastic property of the uncertainty is known, one assumes that the uncertain variable takes on the expected value at each time). In the introduction we have already pointed out that the above naive expectation assumption [32] "a la Cournot" [33], although strong, is justified by the information set up of this work. An opposite assumption, in the authors' opinion as much strong as the naive expectation assumption, is that the agents keep track of the recent evolution of neighbors' state up to current time t and use it to refine their expectations on the neighbors' states [34], [35] (for instance by introducing a derivative term in the prediction or receiving an assumed control trajectory from each neighbor). However, such a refinement would affect the stationarity of the protocol and increase the computation and communication requirements on the part of the agents.

Now, let us introduce the receding approach formally [34], [35].

Assume update time interval (usually the sample interval) $\delta \in (0, T)$. The receding horizon update times are $t_k = t_0 + \delta k$, where $k = 0, 1, \ldots$. The cost of agent i depends on his state and others' state trajectories as well. Let us denote with $\hat{x}_i(\tau, t_k)$ and $\hat{x}^{(i)}(\tau, t_k)$, $\tau \geq t_k$ respectively the predicted state of agent i and of his neighbors.

Problem 3: (Receding Horizon) For all agents $i \in \Gamma$ and times $t_k, k = 0, 1, \ldots$, given the initial state $x_i(t_k)$, and $x^{(i)}(t_k)$ and assuming that no switching occurs between t_k and t_{k+1}, find

$$\hat{u}_i^\star(\tau, t_k) = \arg\min \mathcal{J}_i(x_i(t_k), x^{(i)}(t_k), \hat{u}_i(\tau, t_k)),$$

where

$$\begin{aligned}&\mathcal{J}_i(x_i(t_k), x^{(i)}(t_k), \hat{u}_i(\tau, t_k)) = \\ &\lim_{T\longrightarrow\infty} \int_{t_k}^{T} \left(\mathcal{F}(\hat{x}_i(\tau, t_k), \hat{x}^{(i)}(\tau, t_k)) + \rho \hat{u}_i^2(\tau, t_k)\right) d\tau\end{aligned} \tag{8}$$

subject to

$$\begin{aligned}
\dot{\hat{x}}_i(\tau, t_k) &= \hat{u}_i(\tau, t_k) && \text{(9a)}\\
\dot{\hat{x}}_j(\tau, t_k) &= \hat{u}_j(\tau, t_k) := 0, \quad \forall j \in N_i, && \text{(9b)}\\
\hat{x}_i(t_k, t_k) &= x_i(t_k) && \text{(9c)}\\
\hat{x}_j(t_k, t_k) &= x_j(t_k), \qquad \forall j \in N_i. && \text{(9d)}
\end{aligned}$$

In the above problem, equations (9a) and (9b) predict respectively the evolution of the state of agent i and of his neighbors, and conditions (9c) and (9d) represent the initial state at time t_k. The hypothesis on the invariance of the network topology as well as equation (9b) are based on the naive expectation assumption. Indeed, whereas agent i may predict with a certain approximation the evolution of its state as described by (9a), nothing can he know about the evolution of the states of his neighbors (9b). In other words, agent i at time $t_k, k = 0, 1, \ldots$ assumes that the states of his neighbors take on the current value at each future time, that is $\hat{x}^{(i)}(\tau, t_k) = x^{(i)}(t_k)$, $\forall \tau > t_k$ as described in (15b). If we review the naive assumption in a receding horizon context, at time t_k, given the initial neighbors' state $x^{(i)}(t_k)$, the naive assumption represents a way to predict the evolution of the state of his neighbors $\hat{x}^{(i)}(\tau, t_k), \tau > t_k$ until a next initial state update $x^{(i)}(t_{k+1})$ is available. Thus, though the naive assumption may appear a strong assumption, one must note that it can be mitigated by choosing a smaller update time interval δ.

According to the standard receding horizon scheme, the agents update the receding horizon control policy when a new initial state update $x^{(i)}(t_{k+1})$ is available. As a result, for all $i \in \Gamma$, we have the following *closed-loop* system

$$\dot{x}_i = u_{i_{RH}}(\tau), \quad \tau \geq t_0,$$

where the applied receding horizon control law $u_{i_{RH}}(\tau)$ satisfies

$$u_{i_{RH}}(\tau) = \hat{u}_i^\star(\tau, t_k), \quad \tau \in [t_k, t_{k+1}).$$

Once we assume (9b) we have $\hat{x}^{(i)}$ constant in (8) and the receding horizon problem (Problem 3) reduces to one dimension (in fact, n one-dimensional problems). This is evident, if we explicit dependence of $\mathcal{F}(.)$ only on the state $\hat{x}_i(\tau, t_k)$, and simplify the expression of the individual objective function (8) as

$$J_i = \lim_{T\longrightarrow\infty} \int_{t_k}^{T} \left(\mathcal{F}(\hat{x}_i(\tau, t_k)) + \rho \hat{u}_i^2(\tau, t_k)\right) d\tau. \tag{10}$$

Hence, the problem reduces to determine the control $\hat{u}_i(\tau, t_k)$ that optimizes (10).

Now, to verify that a given control $\hat{u}_i(\tau, t_k)$ is optimal, we use the Pontryagin Minimum Principle. To do this, first, we must construct the Hamiltonian function (for sake of simplicity dependence on τ and t_k is dropped)

$$H(\hat{x}_i, \hat{u}_i, p_i) = (\mathcal{F}(\hat{x}_i) + \rho \hat{u}_i^2) + p_i \hat{u}_i. \tag{11}$$

Second, we must impose the Pontryagin necessary conditions.

Optimality condition:
$$\frac{\partial H(\hat{x}_i, \hat{u}_i, p_i)}{\partial \hat{u}_i} = 0 \tag{12}$$
$$\Rightarrow \quad p_i = -2\rho \hat{u}_i. \tag{13}$$

Multiplier condition:
$$\dot{p}_i = -\frac{\partial H(\hat{x}_i, \hat{u}_i, p_i)}{\partial \hat{x}_i}. \tag{14}$$

State equation:
$$\dot{\hat{x}}_i = \frac{\partial H(\hat{x}_i, \hat{u}_i, p_i)}{\partial p_i} \tag{15}$$
$$\Rightarrow \quad \dot{\hat{x}}_i = \hat{u}_i. \tag{16}$$

Minimality condition:
$$\left.\frac{\partial^2 H(\hat{x}_i, \hat{u}_i, p_i)}{\partial \hat{u}_i^2}\right|_{\hat{x}_i=\hat{x}_i^*, \hat{u}_i=\hat{u}_i^*, p_i=p_i^*} \tag{17}$$
$$\geq 0 \Rightarrow \rho \geq 0. \tag{18}$$

Boundary condition:
$$H(\hat{x}_i^*, \hat{u}_i^*, p_i^*) = 0. \tag{19}$$

This last condition requires that the Hamiltonian must be null along any optimal path $\{\hat{x}_i^*(t), \forall t \geq 0\}$ (see, e.g., [31], Section 3.4.3).

We recall that the Pontryagin Minimum Principle provides *necessary* but not sufficient optimality conditions [31]. The above conditions become also sufficient to identify a unique optimal solution if also the following condition holds [31].

$$\textbf{Uniqueness condition:} \quad \mathcal{F}(x_i) \text{ is convex.} \tag{20}$$

The following theorem state sufficient conditions on the structure of $\mathcal{F}(x_i)$ that allow us to determine analytically a unique optimal control policy $\hat{u}_i(.)$.

Theorem 2: Consider an agent i with first-order dynamics, at times $t_k = 0, 1, \ldots$, assign it an objective function as (8) whose penalty function is

$$\mathcal{F}(\hat{x}_i(\tau, t_k)) = \rho \left(\frac{1}{\frac{dg}{dx_i}} \sum_{j \in N_i} (\vartheta(x_j(t_k)) - \vartheta(\hat{x}_i(\tau, t_k))) \right)^2 \tag{21}$$

where $g(.)$ is increasing, $\vartheta(.)$ is concave, and $\frac{1}{\frac{dg(y)}{dy}}$ is convex. Then the following control policy is the unique optimal solution to Problem 3

$$\hat{u}_i^\star(\tau, t_k) = u_i(x_i(\tau)) = \alpha \frac{1}{\frac{dg}{dx_i(\tau)}} \sum_{j \in N_i} (\vartheta(x_j(t_k)) - \vartheta(x_i(\tau))), \quad \alpha = 1. \tag{22}$$

The above theorem holds trivially for $\alpha = -1$ if $\frac{dg}{dx_i} < 0$ for all $x_i(0)$.

Note that the optimal control policy (22) is a feedback policy with respect to the only state $\hat{x}_i$ whereas is an open-loop one with respect to the neighbors' states.

Then, an immediate consequence of Theorem 2 and of the above assumption is the following corollary.

Corollary 1: Consider a network $G = (\Gamma, E)$ of dynamic agents with first-order dynamics, at times $t_k = 0, 1, \ldots$, assign it an objective function as (8) whose penalty function is

$$\mathcal{F}(\hat{x}_i(\tau, t_k)) = \rho \left(\frac{1}{\frac{dg}{dx_i}} \sum_{j \in N_i} (\vartheta(x_j(t_k)) - \vartheta(\hat{x}_i(\tau, t_k))) \right)^2 \tag{23}$$

where $g(.)$ is increasing, $\vartheta(.)$ is concave, and $\frac{1}{\frac{dg(y)}{dy}}$ is convex. If the update time interval $\delta \longrightarrow 0$ then the following conditions hold

i) the penalty function

$$\mathcal{F}(x_i(\tau, t_k)) \longrightarrow F(x_i, x^{(i)}) = \rho \left(\frac{1}{\frac{dg}{dx_i}} \sum_{j \in N_i} (\vartheta(x_j) - \vartheta(x_i)) \right)^2 \tag{24}$$

ii) the applied receding horizon control law

$$u_{i_{RH}}^\star(\tau) \longrightarrow u_i(x_i, x^{(i)}) = \frac{1}{\frac{dg}{dx_i}} \sum_{j \in N_i} (\vartheta(x_j) - \vartheta(x_i)). \tag{25}$$

From the above corollary it is straightforward to derive a solution to the mechanism design problem (Problem 2). Indeed, a supervisor can make the agents asymptotically reach consensus on the group decision value $\hat{\chi}(x) = f(\sum_{i \in \Gamma} g(x_i))$ by assigning them an individual objective function (3) with penalty function (24) to optimize, provided that $g(.)$ is increasing, $\frac{1}{\frac{dg(y)}{dy}}$ is convex and the update time interval δ is "sufficiently" small in comparison with the speed of variation of state $x(.)$.

It is worth to observe that, in general, control (25) cannot be proved to be optimal for (3) if the naive expectation assumption is dropped. However, to the best of authors' knowledge, even in presence of more information about the evolution of $x^{(i)}(t)$, it would be difficult to determine the optimal control for the generic agent i as conditions (12)-(20) are in general difficult to solve. A notable exception is when $\frac{dg}{dx_i} = 1$ and $\vartheta(x_i) = x_i$ for all $i \in \Gamma$. In this case, agents asymptotically reach consensus on the arithmetic mean of the values of their initial states and it is immediate to verify that control (25) is optimal even when the naive expectation assumption is dropped (see, e.g., *linear quadratic differential games* in [28], ch. 6).

V. Conclusions

In this paper we have considered a set of agents with a simple first-order dynamics and we have faced the problem of making the agents' states reach consensus on a group decision value of interest. For group decision values with a quite general structure as established in Assumption 1, we have shown that the agents can reach consensus using a distributed and stationary linear or non-linear protocol, provided that the networks defined by the communication links between agents is connected though possibly time variant. Also, we have proposed a game theoretic approach to solve consensus problems. Under this perspective, consensus is the result of a mechanism design. A supervisor imposes individual objectives. Then, the agents reach asymptotically consensus as a side effect of the optimization of their own individual objectives on a local basis.

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APPENDIX

Proof of Theorem 2 We first show that the problem of minimizing (10) is well posed by proving that, given a penalty function as in (21), there exists at least a control policy for which objective (10) converges. Second we certificate the optimality of control policy (22) via Pontryagin Minimum Principle and, finally, we prove the uniqueness of the optimal solution by showing that $\mathcal{F}(.)$ is convex.

To prove that the problem is well posed, let us start by showing that there exists at least one reachable state x_i^* under a stationary control policy function only of local information $(x_i, x^{(i)})$ in which both the penalty (21) and the control itself are null. Now, the penalty (21) is null in a state $\hat{x}_i^*$ such that $\sum_{j\in N_i}(\vartheta(x_j(t_k)) - \vartheta(x_i^*)) = 0$. From the latter we also have $x_i^* = \vartheta^{-1}\left(\frac{\sum_{j\in N_i}\vartheta(x_j(t_k))}{|N_i|}\right)$ which means that x_i^* can be determined on the basis of the local information $(\hat{x}_i(\tau, t_k), x^{(i)}(t_k))$ available to agent i. Therefore there trivially exists a control policy that is null in x_i^* and makes the objective function (10) converge.

Now, to certificate the optimality of control policy (22) with $\alpha = 1$, let us show that it satisfies conditions (12)-(20) imposed by the Pontryagin Minimum Principle. Our hypothesis on the agent dynamics and on the structure of the agent objective function trivially satisfy (15) and (17). By computing $\dot{p}_i$ from (12) and substituting the obtained value in (14) we have

$$2\rho\dot{\hat{u}}_i = \frac{\partial H(\hat{x}_i, \hat{u}_i, p_i)}{\partial \hat{x}_i}. \tag{26}$$

In (15), we can write $\dot{\hat{u}}_i = \frac{\partial \hat{u}_i}{\partial \hat{x}_i}\dot{\hat{x}}_i = \frac{\partial \hat{u}_i}{\partial \hat{x}_i}\hat{u}_i$. Hence condition (26) becomes $2\rho\frac{\partial \hat{u}_i}{\partial \hat{x}_i}\hat{u}_i = \frac{\partial H(\hat{x}_i, \hat{u}_i, p_i)}{\partial \hat{x}_i}$. Integrating and imposing condition (19) we obtain that a possible solution of (26) must satisfy

$$\rho\hat{u}_i^2 = \mathcal{F}(\hat{x}_i). \tag{27}$$

It is immediate to verify that $\hat{u}_i(\tau, t_k) = \frac{1}{\frac{dg}{d\hat{x}_i}}\sum_{j\in N_i}(\vartheta(x_j(t_k)) - \vartheta(\hat{x}_i(\tau, t_k)))$ satisfies the above condition.

Finally, to prove that control policy (22) is the unique optimal solution, let us show that $\mathcal{F}(\hat{x}_i)$ is convex. To do this, let us write $\mathcal{F} = \mathcal{F}_3(F_1(\hat{x}_i), \mathcal{F}_2(\hat{x}_i))$ where function $\mathcal{F}_1(\hat{x}_i) = \left(\frac{\partial g}{\partial x_i}\right)^{-1}$, function $\mathcal{F}_2(\hat{x}_i) = \sum_{j\in N_i}(\vartheta(x_j(t_k)) - \vartheta(\hat{x}_i))$ and $\mathcal{F}_3 = (\mathcal{F}_1(\hat{x}_i)\cdot \mathcal{F}_2(\hat{x}_i))^2$. With $\mathcal{F}_3(.)$ being non decreasing in each argument, function $\mathcal{F}_3(.)$ is convex if both functions $\mathcal{F}_1(.)$ and $\mathcal{F}_2(.)$ are also convex [36]. Function $\mathcal{F}_1(.)$ is convex as $\left(\frac{dg}{d\hat{x}_i}\right)^{-1}$ is convex by hypothesis. Analogously, $\mathcal{F}_2(.)$ is convex as $\vartheta(.)$ is concave. $\square$