Quasilinear degenerate parabolic equations in unbounded domains

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ABSTRACT: We prove the existence of bounded solutions of Cauchy-Dirichlet problem associated to a degenerate parabolic equation of second order in divergence form in unbounded domain.

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1 Introduction

In this paper we study the weak solvability of the Cauchy-Dirichlet problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\sum_{i=1}^{m} \frac{\partial}{\partial x_i} a_i(x,t,u,\nabla u) - c_0 u - f(x,t,u,\nabla u) & \text{in } \Omega \times (0,+\infty) \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = 0 & \text{in } \Omega \end{cases} \tag{1}$$

where Ω is an unbounded open set of $\mathbb{R}^m$, c_0 is a positive constant, ∇u is the gradient of unknown function u and f is nonlinear function which has quadratic growth with respect to gradient ∇u. We shall suppose that the following degenerate ellipticity condition is satisfied:

$$\lambda(|u|) \sum_{i=1}^{m} a_i(x,t,u,p) p_i \geq \nu(x)\psi(t)|p|^2, \tag{2}$$

where $p = (p_1, p_2, ..., p_m)$, $|p|$ denotes the modul of p and $\nu : \Omega \to \mathbb{R}$, $\psi :]0,+\infty[\to \mathbb{R}$, $\lambda : [0,+\infty[\to [1,+\infty[$ are functions with properties precised later on.

For bounded domains, results of this type, in non degenerate case, are established e.g. in [10] and, using the method of sub and supersolutions, in [3], while the degenerate case have been studied widely by Guglielmino-Nicolosi in [7]; just of this paper our note may be regarded as a continuation and completion. In linear degenerate case, let us mention also the papers [2] and [13].

The present paper is organized as follows. In section 2 we formulate the hypotheses, we state our problem and the main existence theorem. Section 3 consists of preliminary assertions which are sufficient in the proof of our main result. Finally, in section 4 we prove the existence theorem.

2 Hypotheses and formulation of the main results

Let $\mathbb{R}^m$ the Euclidean m-space $(m > 2)$ with generic point $x = (x_1, x_2, ..., x_m)$, Ω an open nonempty subset of $\mathbb{R}^m$. If $0 < T \leq +\infty$, let us denote by $Q(0,T)$ the cylinder $\Omega \times]0,T[$. For any $n \in \mathbb{N}$ we denote

$$\Omega_n = \{x \in \Omega : |x| < n\} \ , \ Q_n = \Omega_n \times]0,n[.$$

Hypothesis (1) *Let $\nu(x)$ be a positive and measurable function defined in Ω such that:*

$$\nu(x) \in L^1_{loc}(\Omega) \ , \ \nu^{-1}(x) \in L^1_{loc}(\Omega).$$

The symbol $H^1(\nu,\Omega)$ stands for the set of all real valued functions $u \in L^2(\Omega)$ such that their derivatives (in the sense of distributions) are functions $\frac{\partial u}{\partial x_i}$ $(i = 1, 2, ..., m)$ which have the following property

$$\sqrt{\nu}\frac{\partial u}{\partial x_i} \in L^2(\Omega)\ ,\ i = 1, 2, ..., m.$$

Then $H^1(\nu,\Omega)$ is a Hilbert space with respect to the norm

$$\|u\|_1 = \left(\int_\Omega \left(|u|^2 + \sum_{i=1}^m \nu \left|\frac{\partial u}{\partial x_i}\right|^2\right) dx\right)^{\frac{1}{2}};$$

$H^1_0(\nu,\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in $H^1(\nu,\Omega)$. For details concerning the above assertion see e.g. [12].

Remark (1) *If Ω is bounded, there exists a positive number k_0 such that for any $u \in H^1_0(\nu,\Omega)$ it is also $\min_{x\in\Omega}(u(x), k) \in H^1_0(\nu,\Omega)$ for any $k \geq k_0$ (see [11]).*

Remark (2) *If Ω is bounded, for any $u \in H^1_0(\nu,\Omega)\cap L^\infty(\Omega)$ and for any $\gamma > 0$ it is $u(x)|u(x)|^\gamma \in H^1_0(\nu,\Omega) \cap L^\infty(\Omega)$ (see [5]).*

Hypothesis (2) *For any $n \in I\!N$ there exists a real number $g_n > \frac{m}{2}$ such that $\nu^{-1}(x) \in L^{g_n}(\Omega_n)$.*

Let us observe that hypothesis (2) implies the existence of two real numbers α_n, β_n such that $\alpha_n \in]2, +\infty[$, $\beta_n \in]0, +\infty[$ and

$$\left(\int_{\Omega_n} |u|^{\alpha_n} dx\right)^{\frac{1}{\alpha_n}} \leq \beta_n \left(\int_{\Omega_n} |u|^2 + \nu\,|\nabla u|^2\, dx\right)^{\frac{1}{2}}$$

for any $u \in H^1_0(\nu,\Omega_n)$.

Hypothesis (3) *Let $\psi(t)$ be a positive measurable monotone nondecreasing function defined in $]0, +\infty[$. There exists a real positive number $\tilde{g}_n$ such that $\dfrac{1}{\psi} \in L^{\tilde{g}_n}(0, n)$, for any $n \in I\!N$.*

The symbol $H^{1,0}(\nu\psi, Q)$ stands for the set of all real valued functions $u \in L^2(Q)$ such that their derivatives (in the sense of distributions) are functions $\frac{\partial u}{\partial x_i}$ $(i = 1, 2, ..., m)$ which have the following property

$$\sqrt{\nu\psi}\frac{\partial u}{\partial x_i} \in L^2(Q)\ ,\ i = 1, 2, ..., m.$$

Then $H^{1,0}(\nu\psi, Q)$ is a Hilbert space with respect to the norm

$$\|u\|_{1,0} = \left(\int_Q \left(|u|^2 + \sum_{i=1}^m \nu\psi \left|\frac{\partial u}{\partial x_i}\right|^2\right) dxdt\right)^{\frac{1}{2}}.$$

$H^{1,1}(\nu\psi, Q)$ is the subset of $H^{1,0}(\nu\psi, Q)$ of all functions u such that $\dfrac{\partial u}{\partial t}$ (in the sense of distributions) belongs to $L^2(Q)$. We can suppose that any functions of $H^{1,1}(\nu\psi, Q)$ is continuous in $[0, T]$ if $T < +\infty$, in $[0, +\infty[$ if $T = +\infty$.

$V^{1,0}(\nu\psi, Q)$ is the space of all functions $u \in H^{1,0}(\nu\psi, Q)$ such that, a.e. $t \in]0, T[$, $u(x, t)$ belongs to $H^1_0(\nu,\Omega)$. $V^{1,1}(\nu\psi, Q) = H^{1,1}(\nu\psi, Q) \cap V^{1,0}(\nu\psi, Q)$.

If $T < +\infty$, $V^{1,1}_T(\nu\psi, Q)$, denotes the following subset of $V^{1,1}(\nu\psi, Q)$:

$$V^{1,1}_T(\nu\psi, Q) = \left\{u \in V^{1,1}(\nu\psi, Q) : u(x, T) = 0 \text{ a.e. in } \Omega\right\}.$$

Finally, if Ω is unbounded and $T = +\infty$, $\tilde{V}^{1,1}(\nu\psi, Q)$ stands for the set of all function $w \in V^{1,1}(\nu\psi, Q)$ such that

$$\text{support } w \subseteq H_w \times [0, +\infty[$$

where H_w (depending on w) is a close subset of Ω.

Hypothesis (4) *The functions* $f(x,t,u,p)$, $a_i(x,t,u,p)$ $(i = 1,2,...,m)$ *are Caratheodory's functions in* $Q \times I\!R \times I\!R^n$, *i.e. measurable with respect to* (x,t) *for any* $(u,p) \in I\!R \times I\!R^n$, *continuous with respect to* (u,p) *for a.e.* (x,t) *in* Q.

Hypothesis (5) *There exists a function* $f^\star(x,t) \in L^1(Q)$ *such that*

$$|f(x,t,u,p)| \leq \lambda(|u|)\left[f^\star(x,t) + \nu(x)\psi(t)|p|^2\right] \tag{3}$$

holds for almost every $(x,t) \in Q$ *and for all real numbers* $u, p_1, p_2, ..., p_m$.

Hypothesis (6) *There exist nonnegative real number* $c_1 < c_0$ *and a function* $f_0(x,t) \in L^1(Q) \cap L^\infty(Q)$ *such that for almost every* $(x,t) \in Q$ *and for all real numbers* $u, p_1, p_2, ..., p_m$ *the inequality*

$$uf(x,t,u,p) + c_1^2 + \lambda(|u|)\nu(x)\psi(t)|p|^2 + f_0(x,t) \geq 0 \tag{4}$$

holds.

Hypothesis (7) *There exists a function* $a^\star(x,t) \in L^2(Q)$ *such that, for almost every* $(x,t) \in Q$, *we have*

$$\frac{|a_i(x,t,u,p)|}{\sqrt{\nu\psi}} \leq \lambda(|u|)\left[a^\star(x,t) + \sqrt{\nu\psi}|p|\right] \tag{5}$$

for any real numbers $u, p_1, p_2, ..., p_m$.

Hypothesis (8) *The condition (2) is satisfied for almost every* $(x,t) \in Q$ *and for all real numbers* $u, p_1, p_2, ..., p_m$*; the function* $\lambda : [0,+\infty[\to [1,+\infty[$ *is monotone nondecreasing.*

Hypothesis (9) *For almost every* $(x,t) \in Q$ *we have*

$$\sum_{i=1}^{m} \left[a_i(x,t,u,p) - a_i(x,t,u,q)\right](p_i - q_i) \geq 0 \tag{6}$$

for any real numbers $u, p_1, p_2, ..., p_m, q_1, q_2, ..., q_m$*; the inequality holds if and only if* $p \neq q$.

In this paper we shall study the following

Problem *Find a function* $u \in V^{1,0}(\nu\psi, Q) \cap L^\infty(Q)$ *such that the relation*

$$\int_0^{+\infty}\int_\Omega \left\{\sum_{i=1}^{m} a_i(x,t,u,\nabla u)\frac{\partial w}{\partial x_i} + c_0 uw + f(x,t,u,\nabla u)w - u\frac{\partial w}{\partial t}\right\} dxdt = 0$$

holds for any $w \in \tilde{V}^{1,1}(\nu\psi, Q) \cap L^\infty(Q)$.

In section 4 we will prove the following
Theorem *Let* Ω *be unbounded and Hypotheses (1) - (9) be satisfied. Then the* **Problem** *has at least one solution.*

3 Preliminaries

The following lemmas will be usefull in the proof of **Theorem**.

Lemma 1 *Let Hypotheses (1) - (9) be satisfied. Then there exists a function*

$$u_n(x,t) \in V^{1,0}(\nu\psi, Q_n) \cap L^\infty(Q_n)$$

such that

$$\int_{Q_n} \left\{ \sum_{i=1}^m a_i(x,t,u_n,\nabla u_n) \frac{\partial w}{\partial x_i} + c_0 u_n w + f(x,t,u_n,\nabla u_n) w - u_n \frac{\partial w}{\partial t} \right\} dxdt = 0 \tag{7}$$

for any $w \in V_n^{1,1}(\nu\psi, Q_n) \cap L^\infty(Q_n)$.

Moreover

$$ess \sup_{Q_n} |u| \leq \left(\frac{||f_0||_\infty}{c_0 - c_1} \right)^{\frac{1}{2}} = K, \tag{8}$$

$$\left(\int_{Q_n} \left(|u_n|^2 + \sum_{i=1}^m \nu\psi \left| \frac{\partial u_n}{\partial x_i} \right|^2 \right) dxdt \right)^{\frac{1}{2}} \leq \left\{ \frac{K\lambda(K) \left[||f^\star||_1 + ||f_0||_1^{\frac{1}{\chi}} \right]}{\min\left(\frac{1}{\lambda(K)}, c_0 \right)} \right\}^{\frac{\chi}{2}} +1 \tag{9}$$

where $\chi \geq 1$ *is such that* $\dfrac{\chi+1}{\lambda(K)} - \lambda(K) > 1$ *(* $\| \cdot \|_\beta$ $(1 \leq \beta \leq +\infty)$ *denotes the norm in* $L^\beta(Q)$*).*

Proof

See theorem (5.1) of [7]; let us observe that the estimates (8) and (9), are obtained by a sligth modification of the proof of Lemma (2.1) and Lemma (2.2) of [7], taking into account that the hypothesis (6) holds instead of the hypothesis (1.8) of cited paper.

Now, let Ω be unbounded. Let Ω_0 be an open bounded subset of Ω and b be a real positive number; $Q_0 = \Omega_0 \times]0, b[$.

Lemma 2 *Let Hypotheses (1), (7), (8) be satisfied. Let* $u(x,t) \in H^{1,0}(\nu\psi, Q_0)$ *and* $\{u_n\}$ *be a sequence in* $H^{1,0}(\nu\psi, Q_0)$ *such that there exists a constant* $\mu > 0$ *for which* $\int_{Q_0} |u_n|^2 + \nu\psi|\nabla u_n|^2 dxdt \leq \mu$ *and* $\lambda(|u_n(x,t)|) \leq \mu$ *for almost* $(x,t) \in Q_0$ *and for any* $n = 1, 2, \ldots$. *Moreover, let us suppose*

$$\lim_{n\to+\infty} \int_{Q_0} |u_n(x,t) - u(x,t)|^2 dxdt = 0,$$

$$\lim_{n\to+\infty} \int_{Q_0} \sum_{i=1}^m \left[a_i(x,t,u_n(x,t),\nabla u_n(x,t)) - a_i(x,t,u_n(x,t),\nabla u(x,t)) \right] \frac{\partial(u_n - u)}{\partial x_i} dxdt = 0.$$

Then

$$\lim_{n\to+\infty} \int_{Q_0} \nu\psi \sum_{i=1}^m \left| \frac{\partial(u_n - u)}{\partial x_i} \right|^2 dxdt = 0.$$

Proof

See lemma (3.1) of [7].

Lemma 3 *Let Hypotheses (1), (3) - (8) be satisfied,* $\bar{\Omega}_0 \subset \Omega$ *and let* $u(x,t) \in V^{1,0}(\nu\psi, Q) \cap L^\infty(Q)$ *such that*

$$\int_0^{+\infty} \int_\Omega \left\{ \sum_{i=1}^m a_i(x,t,u,\nabla u) \frac{\partial w}{\partial x_i} + c_0 u w + f(x,t,u,\nabla u) w - u \frac{\partial w}{\partial t} \right\} dxdt = 0$$

for any $w \in \tilde{V}^{1,1}(\nu\psi, Q) \cap L^\infty(Q)$ *with support* $w \subseteq \bar{\Omega}_0 \times [0, b]$.

Then, u *has derivative with respect to* t *belonging to* $L^1_{loc}(0, b; W^{-1,1}(\Omega_0))$ *and, for any* τ *:* $0 < \tau < b$ *it results*

$$\int_0^\tau \left\| \frac{\partial u}{\partial t} \right\|_{W^{-1,1}(\Omega_0)} dt \leq \left[c_0 \left(meas \Omega_0 \right)^{\frac{1}{2}} \sqrt{\tau} + \lambda(\|u\|_\infty) + m\lambda\left(\|u\|_\infty\right) \sqrt{\psi(\tau)} \left(\int_{\Omega_0} \nu(x) dx \right)^{\frac{1}{2}} \right] \cdot$$

$$\cdot \|u\|_{1,0} + \lambda\left(\|u\|_\infty\right) \left[\|f^\star\|_1 + m\sqrt{\psi(\tau)} \left(\int_{\Omega_0} \nu(x) dx \right)^{\frac{1}{2}} \|a^\star\|_2 \right]. \tag{10}$$

The proof is an easy modification of the proof of lemma (2.3) of [7].

4 Proof of the Theorem

The proof is performed in *three steps*.

For any positive integer n, let u_n be the function defined in Lemma (1).

We shall extend this function outside of Q_n defining $u_n(x,t) = 0$ for $(x,t) \in \{\Omega \setminus \Omega_n\} \times \{]-\infty, +\infty[\setminus]0,n[\}$. From estimates (8) and (9) we have

$$\|u_n\|_\infty + \|u_n\|_{1,0} \leq L \tag{11}$$

(note that L does not depend on n), hence it is possible to find a subsequence (denoted again by $\{u_n\}$) which converges weakly in $V^{1,0}(\nu\psi, Q)$ and weakly* in $L^\infty(Q)$ to an element $u \in V^{1,0}(\nu\psi, Q) \cap L^\infty(Q)$ for which $||u||_\infty \leq L$ and $||u||_{1,0} \leq L$.

First step:

Let us take arbitrary but fixed $w \in V^{1,1}(\nu\psi, Q) \cap L^\infty(Q)$ with

$$\text{support } w \subseteq H_w \times [0, l],$$

being H_w a close bounded subset of Ω and l is a positive real number.

We proceed to show that

$$\int_0^{+\infty}\!\!\int_\Omega \left\{\sum_{i=1}^m a_i(x,t,u,\nabla u)\frac{\partial w}{\partial x_i} + c_0 uw + f(x,t,u,\nabla u)w - u\frac{\partial w}{\partial t}\right\} dxdt = 0. \tag{12}$$

Let us introduce open bounded sets A and B and the real positive number b such that $H_w \subseteq B \subseteq \bar{B} \subseteq A \subseteq \bar{A} \subset \Omega$, $b > l$.

For enough large n (such that $A\times]0,b[\subset Q_n$) we obtain

$$\int_0^{+\infty}\int_\Omega \{\sum_{i=1}^m a_i(x,t,u_n,\nabla u_n)\frac{\partial w}{\partial x_i} + c_0 u_n w f(x,t,u_n,\nabla u_n)w - u_n\frac{\partial w}{\partial t}\} dxdt = 0$$

for any $w \in V^{1,1}(\nu\psi, Q) \cap L^\infty(Q)$ with support $w \subseteq \bar{A} \times [0,b]$, hence, from Lemma (3), for any $0 < \tau < b$ and for any n, we obtain

$$\left\|\frac{\partial u_n}{\partial t}\right\|_{L^1(0,\tau;W^{-1,1}(A))} \leq C(A,\tau,L)$$

where C is independent on n [1] .

On the other hand, for any $n \in \mathbb{N}$ we have

$$\int_\tau^b \int_A \left\{|u_n| + \sum_{i=1}^m \left|\frac{\partial u_n}{\partial x_i}\right|\right\} dxdt \leq \Big[\sqrt{b \cdot measA} + \\ + \sqrt{\frac{mb}{\psi(\tau)}}\left(\int_A \frac{1}{\nu} dx\right)^{\frac{1}{2}}\Big] ||u_n||_{1,0} = C_1(L),$$

then (see Corollary 6 of [14] and pg. 112 of [7]), there exists a subsequence of $\{u_n\}$ (denoted again by $\{u_n\}$) such that u_n converges to u almost everywhere in $A\times]0,b[$ and, from (11), in $L^2(A\times]0,b[)$.

Consequently

$$\lim_{n\to+\infty} \int_0^l \int_B |u_n - u|^2 dxdt = 0; \tag{13}$$

We shall prove that

$$\lim_{n\to+\infty} \int_0^l \int_B \nu\psi\, |\nabla u_n - \nabla u|^2\, dxdt = 0 \tag{14}$$

[1] for definition of $W^{-1,1}(A)$ see e.g. [4]

and the relation (12) will be the consequence of (13), (14) and support $w \subseteq B \times [0, l]$.

In order to prove (14) we apply Lemma (2). We have (13) and it is also

$$\int_0^l \int_B |u_n|^2 + \nu\psi|\nabla u_n|^2 dxdt \leq L,$$

$$ess \sup_{B\times]0,b[} |u_n(x,t)| \leq L.$$

It remains to verify the assumption

$$\lim_{n\to+\infty} \int_0^l \int_B \sum_{i=1}^m [a_i(x,t,u_n(x,t),\nabla u_n(x,t)) - a_i(x,t,u_n(x,t),\nabla u(x,t))] \frac{\partial(u_n-u)}{\partial x_i} dxdt = 0.$$

It is known that

$$\int_0^b \int_A \Big\{ \sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_r,\nabla u_r)] \frac{\partial w}{\partial x_i} + c_0(u_n-u_r)w +$$

$$+ [f(x,t,u_n,\nabla u_n) - f(x,t,u_r,\nabla u_r)]\, w - (u_n-u_r)\frac{\partial w}{\partial t} \Big\} dxdt = 0 \tag{15}$$

for any $w \in V^{1,1}(\nu\psi, Q) \cap L^\infty(Q)$ with support $w \subseteq \bar{A} \times [0, b]$.

It follows from partition of the unity that there exists a function $\phi \in C_0^\infty(\mathbb{R}^m)$ such that $0 \leq \phi(x) \leq 1$ for any $x \in \mathbb{R}^m$, support $\phi \subseteq A$ and $\phi(x) = 1$ for any $x \in B$.

For any $p,s \in \mathbb{N}$ we define $\chi_p(t)$ and $\omega^s_{n,r}(x,t)$ by

$$\chi_p(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ pt & \text{if } 0 < t \leq \frac{1}{p} \\ 1 & \text{if } \frac{1}{p} < t \leq l \\ \frac{b+l-2t}{b-l} & \text{if } l < t \leq \frac{b+l}{2} \\ 0 & \text{if } t > \frac{b+l}{2} \end{cases},$$

$$\omega^s_{n,r}(x,t) = s \int_t^{t+\frac{1}{s}} U_{n,r}\, |U_{n,r}|^\gamma\, \chi_p^{\gamma+1}(\lambda) d\lambda,$$

where $U_{n,r}(x,t) = u_n(x,t) - u_r(x,t)$ and $\gamma > 0$ which will be fixed later.

Taking in (15) as test function $w = \phi(x)\chi_p(t)\omega^s_{n,r}(x,t)$, according to

$$\int_0^b \int_A (u_n - u_r)\phi(x)\chi_p(t) \frac{\partial \omega^s_{n,r}(x,t)}{\partial t} dxdt \leq 0$$

we get

$$\int_0^b \int_A \Big\{ \sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_r,\nabla u_r)]\phi(x)\chi_p(t)\frac{\partial \omega^s_{n,r}(x,t)}{\partial x_i} +$$
$$+ c_0(u_n-u_r)\phi(x)\chi_p(t)\omega^s_{n,r}(x,t) + [f(x,t,u_n,\nabla u_n) -$$
$$- f(x,t,u_r,\nabla u_r)]\phi(x)\chi_p(t)\omega^s_{n,r}(x,t) \Big\} dxdt \leq$$
$$\leq \int_0^b \int_A (u_n-u_r)\phi(x)\chi'_p(t)\omega^s_{n,r}(x,t)dxdt -$$
$$- \int_0^b \int_A \sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_r,\nabla u_r)] \frac{\partial \phi}{\partial x_i}\chi_p(t)\omega^s_{n,r}(x,t)dxdt.$$

From this, letting $s \to +\infty$, it follows

$$\int_0^b \int_A (\gamma+1) \sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_r,\nabla u_r)]\phi(x)\chi_p^{\gamma+2}(t)\,|U_{n,r}|^\gamma \cdot$$
$$\cdot \frac{\partial U_{n,r}}{\partial x_i} dxdt \le \int_0^b \int_A |U_{n,r}|^{\gamma+1}\,\phi(x)\chi_p^{\gamma+2}(t)[\,|f(x,t,u_n,\nabla u_n)| +$$
$$+ |f(x,t,u_r,\nabla u_r)|\,]dxdt + \int_0^b \int_A |U_{n,r}|^{\gamma+2}\,\phi(x)\chi_p'(t)\chi_p^{\gamma+1}(t)dxdt - \int_0^b \int_A U_{n,r}\cdot$$
$$\cdot |U_{n,r}|^\gamma \sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_r,\nabla u_r)] \frac{\partial \phi(x)}{\partial x_i}\chi_p^{\gamma+2}(t)dxdt. \tag{16}$$

Taking into account that

$$0 \le \int_0^l \int_A |U_{n,r}|^{\gamma+2}\,\phi(x)\chi_p'(t)\chi_p^{\gamma+1}(t)dxdt \le p\int_0^{\frac{1}{p}} \int_A |U_{n,r}|^{\gamma+2}\,\phi(x)dxdt$$

from (16), passing to the limit as $p \to +\infty$, we get

$$\int_0^b \int_A (\gamma+1) \sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_r,\nabla u_r)]\phi(x)\chi^{\gamma+2}(t)\,|U_{n,r}|^\gamma \cdot$$
$$\cdot \frac{\partial U_{n,r}}{\partial x_i} dxdt \le \int_0^b \int_A |U_{n,r}|^{\gamma+1}\,\phi(x)\chi^{\gamma+2}(t)[\,|f(x,t,u_n,\nabla u_n)| +$$
$$+ |f(x,t,u_r,\nabla u_r)|\,]dxdt - \int_0^b \int_A U_{n,r}\,|U_{n,r}|^\gamma \cdot$$
$$\cdot \sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_r,\nabla u_r)] \frac{\partial \phi(x)}{\partial x_i}\chi^{\gamma+2}(t)dxdt \tag{17}$$

where

$$\chi(t) = \begin{cases} 0 & \text{if } t \le 0 \\ 1 & \text{if } 0 < t \le l \\ \frac{b+l-2t}{b-l} & \text{if } l < t \le \frac{b+l}{2} \\ 0 & \text{if } t > \frac{b+l}{2} \end{cases}.$$

It follows from Hypothesis (8) and from (11)

$$(\gamma+1)\sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_r,\nabla u_r)]\phi(x)\chi^{\gamma+2}(t)\,|U_{n,r}|^\gamma \frac{\partial U_{n,r}}{\partial x_i} \ge$$
$$\ge \frac{(\gamma+1)}{\lambda(L)}\phi(x)\chi^{\gamma+2}(t)\nu(x)\psi(t)(\,|\nabla u_n|^2 + |\nabla u_r|^2\,)\,|U_{n,r}|^\gamma -$$
$$-(\gamma+1)\sum_{i=1}^m a_i(x,t,u_n,\nabla u_n)\frac{\partial u_r}{\partial x_i}\phi(x)\chi^{\gamma+2}(t)\,|U_{n,r}|^\gamma -$$
$$-(\gamma+1)\sum_{i=1}^m a_i(x,t,u_r,\nabla u_r)\frac{\partial u_n}{\partial x_i}\phi(x)\chi^{\gamma+2}(t)\,|U_{n,r}|^\gamma \tag{18}$$

By (11) and by Hypothesis (5) we get

$$|U_{n,r}|^{\gamma+1}\,\phi(x)\chi^{\gamma+2}(t)[\,|f(x,t,u_n,\nabla u_n)| + |f(x,t,u_r,\nabla u_r)|\,] \le$$
$$\le \lambda(L)|U_{n,r}|^{\gamma+1}\,\phi(x)\chi^{\gamma+2}(t)\left[2|f^\star(x,t)| + \nu(x)\psi(t)\left(|\nabla u_n|^2 + |\nabla u_r|^2\right)\right]. \tag{19}$$

Then it follows from (17)- (19) the following inequality

$$\frac{(\gamma+1)}{\lambda(L)}\int_0^b\int_A \phi(x)\chi^{\gamma+2}(t)\nu(x)\psi(t)(\,|\nabla u_n|^2+|\nabla u_r|^2\,)\,|U_{n,r}|^\gamma\,dxdt \le$$
$$\le \int_0^b\int_A(\gamma+1)\sum_{i=1}^m a_i(x,t,u_n,\nabla u_n)\frac{\partial u_r}{\partial x_i}\phi(x)\chi^{\gamma+2}(t)\,|U_{n,r}|^\gamma\,dxdt+$$
$$+\int_0^b\int_A(\gamma+1)\sum_{i=1}^m a_i(x,t,u_r,\nabla u_r)\frac{\partial u_n}{\partial x_i}\phi(x)\chi^{\gamma+2}(t)\,|U_{n,r}|^\gamma\,dxdt+$$
$$+\lambda(L)\int_0^b\int_A|U_{n,r}|^{\gamma+1}\,\phi(x)\chi^{\gamma+2}(t)[2|f^\star(x,t)|+\nu(x)\psi(t)\left(|\nabla u_n|^2+\right.$$
$$\left.+|\nabla u_r|^2\right)]dxdt-\int_0^b\int_A U_{n,r}\,|U_{n,r}|^\gamma\cdot$$
$$\cdot\sum_{i=1}^m[a_i(x,t,u_n,\nabla u_n)-a_i(x,t,u_r,\nabla u_r)]\frac{\partial\phi(x)}{\partial x_i}\chi^{\gamma+2}(t)dxdt.$$

Now, we choose

$$\gamma=2L\,[\lambda(L)]^2+\lambda(L)+1.$$

The above choose implies

$$\int_0^b\int_A \phi(x)\chi^{\gamma+2}(t)\nu(x)\psi(t)(\,|\nabla u_n|^2+|\nabla u_r|^2\,)\,|U_{n,r}|^\gamma\,dxdt \le$$
$$\le \int_0^b\int_A(\gamma+1)\sum_{i=1}^m a_i(x,t,u_n,\nabla u_n)\frac{\partial u_r}{\partial x_i}\phi(x)\chi^{\gamma+2}(t)\,|U_{n,r}|^\gamma\,dxdt+$$
$$+\int_0^b\int_A(\gamma+1)\sum_{i=1}^m a_i(x,t,u_r,\nabla u_r)\frac{\partial u_n}{\partial x_i}\phi(x)\chi^{\gamma+2}(t)\,|U_{n,r}|^\gamma\,dxdt+$$
$$+4L\lambda(L)\int_0^b\int_A|U_{n,r}|^\gamma\,\phi(x)\chi^{\gamma+2}(t)|f^\star(x,t)|dxdt-\int_0^b\int_A U_{n,r}\,|U_{n,r}|^\gamma\cdot$$
$$\cdot\sum_{i=1}^m[a_i(x,t,u_n,\nabla u_n)-a_i(x,t,u_r,\nabla u_r)]\frac{\partial\phi(x)}{\partial x_i}\chi^{\gamma+2}(t)dxdt. \tag{20}$$

We shall prove that $\lim\limits_{n\to+\infty}\int_0^b\int_A\sum_{i=1}^m[a_i(x,t,u_n(x,t),\nabla u_n(x,t))-$

$$-a_i(x,t,u_n(x,t),\nabla u(x,t))]\frac{\partial(u_n-u)}{\partial x_i}\phi(x)\chi^2(t)dxdt=0, \tag{21}$$

in this way, by definition of $\phi(x)$ and $\chi(t)$, we will obtain (14), too.

Let us denote by $\bar\omega$ the maximum limit of the sequence in the left side, we can find a subsequence (denoted again by $\{u_n\}$) such that

$$\lim_{n\to+\infty}\int_0^b\int_A\sum_{i=1}^m[a_i(x,t,u_n(x,t),\nabla u_n(x,t))-$$
$$-a_i(x,t,u_n(x,t),\nabla u(x,t))]\;\frac{\partial(u_n-u)}{\partial x_i}\phi(x)\chi^2(t)dxdt=\bar\omega,$$

and, moreover, such that the sequences

$$\left\{\frac{a_i\left(x,t,u_n(x,t),\nabla u_n(x,t)\right)}{\sqrt{\nu(x)\psi(t)}}\right\}\quad(i=1,2,...,m)$$

converge weakly in $L^2(A\times]0,b[)$ to some functions $\Lambda_i(x,t)$ $(i=1,2,...,m)$. From Hypothesis (9) we get $\bar{\omega}\geq 0$, therefore it will be sufficient to prove that $\bar{\omega}\leq 0$.

Denoted by $B_{n,r}$ the secon term of (20), it results:

$$\lim_{r\to+\infty} B_{n,r}=B_n$$

where

$$B_n=(\gamma+1)\int_0^b\int_A\sum_{i=1}^m a_i(x,t,u_n,\nabla u_n)\frac{\partial u}{\partial x_i}\phi(x)\chi^{\gamma+2}(t)\left|u_n-u\right|^\gamma dxdt+$$
$$+(\gamma+1)\int_0^b\int_A\sum_{i=1}^m \Lambda_i(x,t)\sqrt{\nu(x)\psi(t)}\frac{\partial u_n}{\partial x_i}\phi(x)\chi^{\gamma+2}(t)\left|u_n-u\right|^\gamma dxdt+$$
$$+4L\lambda(L)\int_0^b\int_A\left|u_n-u\right|^\gamma\phi(x)\chi^{\gamma+2}(t)|f^\star(x,t)|dxdt-\int_0^b\int_A(u_n-u)\left|u_n-u\right|^\gamma\cdot$$
$$\cdot\sum_{i=1}^m\left[a_i(x,t,u_n,\nabla u_n)-\Lambda_i(x,t)\sqrt{\nu(x)\psi(t)}\right]\frac{\partial\phi(x)}{\partial x_i}\chi^{\gamma+2}(t)dxdt.$$

To pass to the limit, it is convenient to write

$$\left|U_{n,r}\right|^\gamma=\left(\left|u_n(x,t)-u(x,t)\right|^\gamma\right)+\left(\left|U_{n,r}\right|^\gamma-\left|u_n(x,t)-u(x,t)\right|^\gamma\right).$$

For instance, we prove

$$\lim_{r\to+\infty}\int_0^b\int_A\left|U_{n,r}\right|^\gamma U_{n,r}\sum_{i=1}^m a_i(x,t,u_r(x,t),\nabla u_r(x,t))\frac{\partial\phi(x)}{\partial x_i}\chi^{\gamma+2}(t)dxdt=$$
$$=\int_0^b\int_A|u_n-u|^\gamma(u_n-u)\sum_{i=1}^m\Lambda_i(x,t)\sqrt{\nu(x)\psi(t)}\frac{\partial\phi(x)}{\partial x_i}\chi^{\gamma+2}(t)dxdt. \qquad (22)$$

We can write the first term of (22) as

$$\int_0^b\int_A|u_n-u|^\gamma(u_n-u)\sum_{i=1}^m a_i(x,t,u_r(x,t),\nabla u_r(x,t))\frac{\partial\phi(x)}{\partial x_i}\chi^{\gamma+2}(t)dxdt+$$
$$+\int_0^b\int_A\left(|u_n-u_r|^\gamma(u_n-u_r)-|u_n-u|^\gamma(u_n-u)\right)\cdot$$
$$\cdot\sum_{i=1}^m a_i(x,t,u_r(x,t),\nabla u_r(x,t))\frac{\partial\phi(x)}{\partial x_i}\chi^{\gamma+2}(t)dxdt. \qquad (23)$$

Now, taking into account that by

$$w\longrightarrow\int_0^b\int_A\sum_{i=1}^m\frac{a_i\left(x,t,u_n(x,t),\nabla u_n(x,t)\right)}{\sqrt{\nu(x)\psi(t)}}wdxdt,$$

we define a continuous linear functional on $L^2(A\times]0,b[)$, it follows

$$\lim_{r\to+\infty}\int_0^b\int_A|u_n-u|^\gamma(u_n-u)\sum_{i=1}^m a_i(x,t,u_r(x,t),\nabla u_r(x,t))\frac{\partial\phi(x)}{\partial x_i}\chi^{\gamma+2}(t)dxdt=$$
$$\int_0^b\int_A|u_n-u|^\gamma(u_n-u)\sum_{i=1}^m\Lambda_i(x,t)\sqrt{\nu(x)\psi(t)}\frac{\partial\phi(x)}{\partial x_i}\chi^{\gamma+2}(t)dxdt.$$

Next, from the Lebesgue theorem and (11), we have that the second term of (22) goes to zero.

Hence (22) is established. Moreover, a trivial verification shows that

$$\lim_{n\to+\infty} B_n = 0.$$

From (17) with $\gamma = 0$ and hypothesis (5) we get

$$\int_0^b \int_A \sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_r,\nabla u_r)]\phi(x)\chi^2(t)\frac{\partial U_{n,r}}{\partial x_i} dxdt \leq$$
$$\leq \lambda(L) \int_0^b \int_A |U_{n,r}|\, \phi(x)\chi^2(t)[2|f^\star(x,t)| + \nu(x)\psi(t)\left(|\nabla u_n|^2 + |\nabla u_r|^2\right)]dxdt -$$
$$-\int_0^b \int_A U_{n,r} \sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_r,\nabla u_r)]\frac{\partial \phi(x)}{\partial x_i}\chi^2(t)dxdt.$$

On the other hand

$$\int_0^b \int_A |U_{n,r}|\, \phi(x)\chi^2(t)\nu(x)\psi(t)\left(|\nabla u_n|^2 + |\nabla u_r|^2\right) dxdt \leq$$
$$\leq \left(\int_0^b \int_A |U_{n,r}|^\gamma\, \phi(x)\chi^{\gamma+2}(t)\nu(x)\psi(t)\left(|\nabla u_n|^2 + |\nabla u_r|^2\right) dxdt\right)^{\frac{1}{\gamma}} \cdot$$
$$\cdot \left(\int_0^b \int_A \phi(x)\chi^{\frac{\gamma-2}{\gamma-1}}(t)\nu(x)\psi(t)\left(|\nabla u_n|^2 + |\nabla u_r|^2\right) dxdt\right)^{\frac{\gamma-1}{\gamma}} \leq$$
$$\leq (B_{n,r})^{\frac{1}{\gamma}}\left(2L^2\right)^{\frac{\gamma-1}{\gamma}}$$

and then

$$\int_0^b \int_A \sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_r,\nabla u_r)]\phi(x)\chi^2(t)\frac{\partial U_{n,r}}{\partial x_i} dxdt \leq$$
$$\leq 2\lambda(L) \int_0^b \int_A |U_{n,r}|\,|f^\star(x,t)|dxdt + \lambda(L)\,(B_{n,r})^{\frac{1}{\gamma}}\left(2L^2\right)^{\frac{\gamma-1}{\gamma}} + 2m\lambda(L)\cdot$$
$$\cdot \int_0^b\int_A |U_{n,r}|\sqrt{\nu(x)\psi(t)}|a^\star(x,t)|dxdt + 2m\lambda(L)L\left(\int_0^b\int_A |U_{n,r}|^2\,\nu\psi dxdt\right)^{\frac{1}{2}}. \tag{24}$$

Since

$$[a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_r,\nabla u_r)]\frac{\partial U_{n,r}}{\partial x_i} = a_i(x,t,u_n,\nabla u_n)\frac{\partial u_n}{\partial x_i} -$$
$$-a_i(x,t,u_n,\nabla u_n)\frac{\partial u_r}{\partial x_i} - a_i(x,t,u_r,\nabla u_r)\frac{\partial (u_n - u)}{\partial x_i} + a_i(x,t,u_r,\nabla u_r)\frac{\partial (u_r - u)}{\partial x_i}$$

from (24) letting $r \to +\infty$ we get

$$\int_0^b \int_A \sum_{i=1}^m [a_i(x,t,u_n,\nabla u_n) - \Lambda_i(x,t)\sqrt{\nu(x)\psi(t)}]\frac{\partial (u_n - u)}{\partial x_i}\phi(x)\chi^2(t)dxdt + \bar{\omega} \leq$$
$$\leq 2\lambda(L) \int_0^b \int_A |u_n - u||f^\star(x,t)|dxdt + \lambda(L)\,(B_n)^{\frac{1}{\gamma}}\left(2L^2\right)^{\frac{\gamma-1}{\gamma}} + 2m\lambda(L)\cdot$$
$$\cdot \int_0^b \int_A |u_n - u|\sqrt{\nu\psi}|a^\star(x,t)|dxdt + 2m\lambda(L)L\left(\int_0^b \int_A |u_n - u|^2\nu(x)\psi(t)dxdt\right)^{\frac{1}{2}}.$$

From this, for $n \to +\infty$, it follows $2\bar{\omega} \leq 0$.

Second step:

Now, let us fix a function $w_1 \in V^{1,1}(\nu\psi, Q) \cap L^\infty(Q)$ with

$$\text{support } w_1 \subseteq H_{w_1} \times [0, +\infty[,$$

being H_{w_1} a close bounded subset of Ω.

For any $n \in \mathbb{N}$ we introduce

$$z_n(t) = \begin{cases} 1 & \text{if } t \leq n-1 \\ n-t & \text{if } n-1 < t \leq n \\ 0 & \text{if } t > n \end{cases} .$$

It follows immediately that support $w_1 z_n \subseteq H_{w_1} \times [0, n]$, hence from (12) we get

$$\int_0^{+\infty} \int_\Omega \Big\{ \sum_{i=1}^m a_i(x,t,u,\nabla u) \frac{\partial w_1}{\partial x_i} z_n(t) + c_0 u w_1 z_n(t) + f(x,t,u,\nabla u) w_1 z_n(t) + \\ -u \frac{\partial w_1}{\partial t} z_n(t) \Big\} dx dt + \int_{n-1}^n \int_\Omega u w_1 dx dt = 0,$$

for any $n \in \mathbb{N}$.

Letting $n \to +\infty$ we obtain that $u(x,t)$ satisfies (12) also for a such w_1.

Third step:

Finally, let w_2 be such that $w_2 \in V^{1,1}(\nu\psi, Q) \cap L^\infty(Q)$ and

$$\text{support } w_2 \subseteq H_{w_2} \times [0, +\infty[,$$

where H_{w_2} is a close (not necessarily bounded) subset of Ω.

Let us define in $\mathbb{R}^m$ a function $\Theta(\tau) \in C_0^\infty(\mathbb{R}^m)$ such that

$$\Theta(\tau) = \Theta(|\tau|) = \begin{cases} 1 & \text{if } |\tau| \leq \frac{1}{2} \\ \\ 0 & \text{if } |\tau| \geq 1 \end{cases} , \quad 0 \leq \Theta(\tau) \leq 1 \quad \text{if } \frac{1}{2} < |\tau| < 1.$$

For any integer n we put

$$\Theta_n(x) = \Theta_n(|x|) = \begin{cases} 1 & \text{if } |x| \leq n-1 \\ \\ \Theta(|x| - n + 1) & \text{if } |x| > n-1 \end{cases} .$$

The function $w_2 \Theta_n$ belongs to $V^{1,1}(\nu\psi, Q) \cap L^\infty(Q)$, moreover

$$\text{support } w_2\Theta_n \subseteq \{x \in H_{w_2} : |x| \leq n\} \times [0, +\infty[,$$

then, according to conclusion of *second step*, we have

$$\int_0^{+\infty} \int_\Omega \Big\{ \sum_{i=1}^m a_i(x,t,u,\nabla u) \frac{\partial w_2}{\partial x_i} \Theta_n(x) + c_0 u w_2 \Theta_n(x) + f(x,t,u,\nabla u) w_2 \Theta_n(x) + \\ -u \frac{\partial w_2}{\partial t} \Theta_n(x) \Big\} dx dt + \int_0^{+\infty} \int_\Omega \sum_{i=1}^m a_i(x,t,u,\nabla u) \frac{\partial \Theta_n(x)}{\partial x_i} w_2 dx dt.$$

Therefore, we can conclude the proof of theorem passing to the limit for $n \to +\infty$ in the above relation.

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