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On the qualitative analysis of the solutions of a mathematical model of social dynamics

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Abstract

This work deals with a family of dynamical systems which were introduced in [M.L. Bertotti, M. Delitala, From discrete kinetic and stochastic game theory to modelling complex systems in applied sciences, Math. Models Methods Appl. Sci. 7 (2004) 1061–1084], modelling the evolution of a population of interacting individuals, distinguished by their social state. The existence of certain uniform distribution equilibria is proved and the asymptotic trend is investigated.

Keywords: Kinetic theory; Discretization; Boltzmann models; Population models; Nonlinearity; Asymptotic stability

1. Introduction and the mathematical model

Several recent papers show an interest of applied mathematicians in developing methods of non-equilibrium statistical mechanics and mathematical kinetic theory for active particles [1] in life sciences: among others [2–9], with reference to social sciences, politics, psychology and biology.

Along this line of research, a mathematical model describing social dynamics of interacting individuals with different social positions, e.g. different levels of wealth, was proposed in [10], corresponding to a society where interactions express competition and/or cooperation. The model refers to the generalized kinetic theory for active particles whose microscopic state includes the mechanical variables, typically position and velocity, but also an additional variable, called "activity", corresponding to a non-mechanical function of the particles. Mathematical frameworks for models with a continuous microscopic state are proposed e.g. in [11] with special emphasis on modelling biological systems. In contrast, the mathematical structures considered in [10] concern the case of discrete sociobiological states.

Specifically, in [10] the following structure has been used toward modelling of social systems:

$$\frac{\mathrm{d} f_i}{\mathrm{d}t} = J_i[f] = \sum_{h=1}^{n} \sum_{k=1}^{n} \eta_{hk} A_{hk}^{i} f_h f_k - f_i \sum_{k=1}^{n} \eta_{ik} f_k, \quad i = 1, \ldots, n, \tag{1.1}$$

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where f_i denotes the fraction (with respect to the overall number of individuals) with social state u_i belonging to the set

$$I_u = \{u_1, \ldots, u_h, \ldots, u_n\},$$

and the **interaction rate** is given by

$$\eta_{hk} = \eta(u_h, u_k) : \quad I_u \times I_u \to \mathbb{R}_+,$$

while the **transition probability density** is given by

$$A_{hk}^i = A(u_h, u_k; u_i) : \quad I_u \times I_u \times I_u \to \mathbb{R}_+, \quad \text{with } \sum_{i=1}^n A_{hk}^i = 1, \quad \forall h, k = 1, \ldots, n.$$

The above terms have been modelled, inspired by the following phenomenological observation (and with the consciousness that this is nothing more than a conceivable example among several):

- when two individuals have close social states, then a competition occurs: the individual placed in the higher social position improves its situation, while the one in a lower position faces a further decrease (**competitive behavior**);
- when the social state of the individuals is sufficiently far away, the opposite behavior (**altruistic behavior**) occurs.

To translate these concepts into formulas, besides the number n of social classes, a parameter m has been introduced, possibly attaining any integer value between 0 and $n - 1$, which represents the distance between classes themselves and which distinguishes the competitive and the altruistic behavior. This allows one to assign, in correspondence to any natural number n and to any integer m between 0 and $n - 1$, the value of A_{hk}^i for every $i, h, k = 1, \ldots, n$, constructing in this way a so-called **table of games**. The non-null elements are

$$h = k : \quad A_{hk}^{i=h} = 1$$

$$h \neq k : \begin{cases} |h-k| \leq m : \begin{cases} h = 1 \text{ or } h = n : \quad A_{hk}^{i=h} = 1 \\ h \neq 1, h \neq n : \begin{cases} h < k : \quad A_{hk}^{i=h-1} = 1 \\ h > k : \quad A_{hk}^{i=h+1} = 1 \end{cases} \end{cases} \\ |h-k| > m : \begin{cases} h < k : \quad A_{hk}^{i=h+1} = 1 \\ h > k : \quad A_{hk}^{i=h-1} = 1 \end{cases} \end{cases}$$

In this note we prove for this family of dynamical systems, for suitable values of the parameters n and m, the existence of a uniform distribution equilibrium, discussing as well some qualitative properties of the flow.

2. Existence and stability of equilibrium configurations

The model summarized in Section 1 is characterized by the two parameters n and m. Its application to the analysis of real world systems contemplates performing a qualitative analysis as well as computational simulations for the initial value problem

$$\begin{cases} \dfrac{\mathrm{d} f_i}{\mathrm{d}t} = J_i[f], \\ f_{i0} = f_i(t = 0), \end{cases} \tag{2.1}$$

where $J_i[f]$ is defined in (1.1) and the set $\{f_{i0}\}$ is a discrete probability density,

$$\sum_{i=1}^n f_{i0} = 1. \tag{2.2}$$

Recall, with reference to [10], that the global existence of solutions is proved by the following theorem:

Theorem 2.1. *Assume $\eta_{hk} \le M$ for some positive constant $M < +\infty$, for any $h, k = 1, \dots, n$. Then, for any given set $\{f_{i0}\}$ such that $f_{i0} \ge 0$ for $i = 1, \dots, n$ and the set $\{f_{i0}\}$ is a discrete probability density as indicated in (2.2), the solution $f(t) = (f_1(t), \dots, f_n(t))$ of system (2.1) exists and is unique for all $t \in [0, +\infty)$. In particular, one has*

$$\forall t \ge 0: \quad f_i(t) \ge 0 \quad \text{for any } i = 1, \dots, n \quad \text{and} \quad \sum_{i=0}^{n} f_i(t) = 1. \tag{2.3}$$

A corresponding theorem was also proved in [3] for models with continuous distribution over the microscopic state. In particular, the positive invariance of the standard $(n-1)$-simplex is proved for the flow of system (2.1). Moreover, one easily sees that the solution claimed in Theorem 2.1 is of class C^∞ and the continuous dependence on the initial conditions holds. If, moreover, the encounter rate is taken to be constant:

$$\eta_{hk} = c \quad \forall h, k = 1, \dots, n \tag{2.4}$$

for some positive constant c, then the existence of at least one equilibrium solution f of system (1.1), with $f_i \ge 0$ for all $i = 1, \dots, n$, is guaranteed ([10]). We consider this case and, for simplicity, we assume the constant c in (2.4) to be equal to one. The system (1.1) becomes then

$$\frac{\mathrm{d} f_i}{\mathrm{d}t} = \sum_{h=1}^{n} \sum_{k=1}^{n} A_{hk}^{i} f_h f_k - f_i \sum_{k=1}^{n} f_k, \quad i = 1, \dots, n. \tag{2.5}$$

An analytical study was carried out in [10] for the cases $n = 3$ (with $m = 0, 1, 2$) and $n = 4$ (with $m = 0, 1, 2, 3$). At the level of computational simulations, a great number of cases were examined, corresponding to several values of n and m. Analytically, both for $n = 3$ and for $n = 4$, degenerate (non-isolated) equilibria were found to exist in the two extreme cases when $m = 0$ and $m = n - 1$, all sharing the property of having some component equal to zero. On the other hand, when $m \ne 0$ and $m \ne n - 1$, namely for $n = 3$ and $m = 1$, for $n = 4$ and $m = 1, 2$, only isolated equilibria were proved to exist. In particular, in each one of these three cases only one equilibrium exists, having all components different from zero. This "positive" equilibrium was proved to be "globally asymptotically stable", namely stable and attractive with respect to all solutions $f(t)$ with initial data $\{f_0\}$ satisfying $f_{i0} \ge 0$ for $i = 1, \dots, n$ and (2.2), and different from any of the equilibria coinciding with the vertices of the unitary simplex.

Remark 2.1. The study of the model under consideration seems to be more significant when the number n of social classes is odd. It is indeed in such a case that a middle class exists.

Focusing attention on the general case of odd n, we prove now the following fact.

Theorem 2.2. *If n is odd and the parameter m takes the value $m = (n-1)/2$, an equilibrium configuration corresponding to the constant distribution exists; in other words, the point $P = (f_1, \dots, f_n)$ with $f_i = 1/n$ for all $i = 1, \dots, n$ is an equilibrium.*

Proof of Theorem 2.2. If $n = 3$ (and $m = 1$), this fact is proved in [10]. So, it suffices here to assume $n \ge 5$. Our goal is to show that, in the case under consideration,

$$\sum_{h=1}^{n} \sum_{k=1}^{n} A_{hk}^{i} = n$$

for any $i = 1, \dots, n$. From this fact, the vanishing of the right hand side of Eq. (2.5) follows when the value $f_i = 1/n$ for all $i = 1, \dots, n$ is substituted into the equation itself, proving the claim. We will distinguish three cases, respectively $i = 1$, $i = n$ and $2 \le i \le n - 1$:

– if $i = 1$, the only nonzero elements A_{hk}^{i} turn out to be:
  $A_{11}^{1} = 1$,
  $A_{1k}^{1} = 1$ for any $k : 2 \le k \le (n+1)/2$,
  $A_{2k}^{1} = 1$ for any $k : 3 \le k \le \min\{(3+n)/2, n\}$.

Hence,

$$\sum_{h=1}^{n}\sum_{k=1}^{n} A^i_{hk} = 1 + (n+1)/2 - 1 + (3+n)/2 - 2 = n.$$

– If $i = n$, the only nonzero elements A^i_{hk} turn out to be:
$A^n_{nn} = 1$,
$A^n_{nk} = 1$ for any $k : (n+1)/2 \le k \le n-1$,
$A^n_{n-1k} = 1$ for any $k : \max\{(n-1)/2, 1\} \le k \le n-2$.
Hence,

$$\sum_{h=1}^{n}\sum_{k=1}^{n} A^i_{hk} = 1 + n - 1 + n - 2 - (n-1)/2 + 1 = n.$$

– If $2 \le i \le n-1$, the only nonzero elements A^i_{hk} are:
$A^i_{ii} = 1$,
$A^i_{i-1k} = 1$ for any $k : i + (n-1)/2 \le k \le n$,
$A^i_{i-1k} = 1$ for any $k : \max\{i - (n+1)/2, 1\} \le k \le i-2$,
$A^i_{i+1k} = 1$ for any $k : 1 \le k \le i - (n-1)/2$,
$A^i_{i+1k} = 1$ for any $k : i + 2 \le k \le \min\{i + (n+1)/2, n\}$.

Skipping some steps in the calculations, one gets

$$\sum_{h=1}^{n}\sum_{k=1}^{n} A^i_{hk} = 1 - \max\{i - (n+1)/2, 1\} + \min\{i + (n+1)/2, n\} = n. \tag{2.6}$$

The last equality in (2.6) requires some care, in view of the fact that the “max” and the “min” involved in it actually depend on the value of i. To see where it comes from, just notice that

$$\max\{i - (n+1)/2, 1\} = \begin{cases} i - (n+1)/2 & \text{if } i > (n+3)/2 \\ 1 & \text{if } i = (n+3)/2 \\ 1 & \text{if } i < (n+3)/2 \end{cases}$$

$$\min\{i + (n+1)/2, n\} = \begin{cases} n & \text{if } i > (n-1)/2 \\ n & \text{if } i = (n-1)/2 \\ i + (n+1)/2 & \text{if } i < (n-1)/2. \end{cases}$$

The five subcases $i < (n-1)/2$, $i = (n-1)/2$, $i = (n+1)/2$, $i = (n+3)/2$, $i > (n+3)/2$ can then be separately handled to get the conclusion. □

3. Simulations, comments and perspectives

All the computational simulations performed confirm a scenario similar to the one analytically proved for the cases $n = 3$ and $n = 4$. Indeed, they constantly indicate that, if $m \ne 0$ and $m \ne n-1$, for any initial condition a unique asymptotic state appears. In contrast, if $m = 0$ (totally cooperative systems) or $m = n-1$ (totally competitive systems), several asymptotic states appear, depending on the initial conditions. In particular, we want to stress the following interesting output: when m is relatively small, i.e. when the system behaves with altruistic behavior, the asymptotic configurations appear to be characterized by large concentrations on central values; conversely, when m is relatively large, i.e. when the interactions are predominantly competitive, the asymptotic configurations show large concentrations on the extreme values $u = 0$ and $u = 1$. This appears clearly in Fig. 3.1, which shows the asymptotic configuration displayed by simulations in the cases $n = 7, 9, 11$ for different values of the parameter m. The same “structure” for the solutions can be recognized.

Also, we point out, on the basis of several simulations, that in the cases with n odd and $m = (n-1)/2$, the equilibrium configuration $P = (f_1, \ldots, f_n)$ with $f_i = 1/n$ for all $i = 1, \ldots, n$ established in Theorem 2.1 seems to be the asymptotic state for all the initial conditions $\{f_0\}$ on the standard $(n-1)$-simplex but the simplex vertices. In the case $n = 3$, this fact is proved in [10]. For general odd n, the asymptotic stability of P remains a conjecture. In

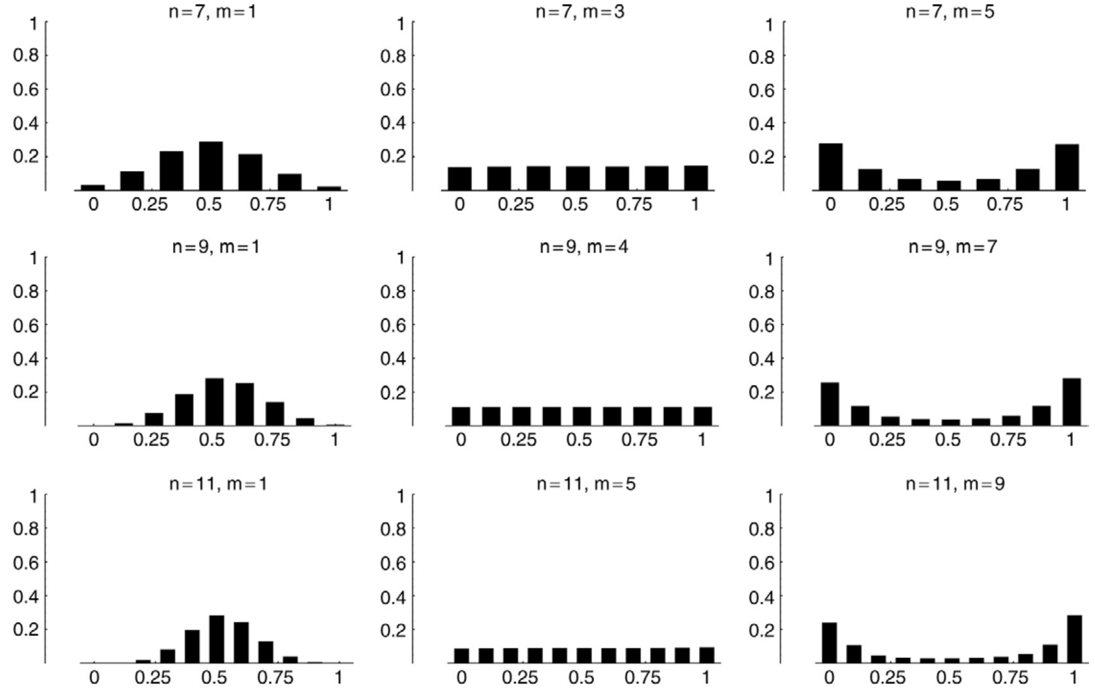

Fig. 3.1. Asymptotic configuration in the cases $n = 7, 9, 11$, for different values of m.

spite of what one could hope, an analytical proof for general odd n does not seem to be so easy to find, the difficulty being related to the attempt at generality. Indeed, one should deal simultaneously with a countable family of systems of increasing dimension and involving coefficients A^i_{hk} depending on n. Looking instead at a specific system, i.e. fixing n (odd, with $m = (n-1)/2$), at least the asymptotic stability of the uniform distribution equilibrium is expected to be an accessible result. It is convenient in this connection to study the system of $n-1$ differential equations, obtained by substituting $f_{\hat{n}} = 1 - \sum_{i \neq \hat{n}} f_i$, where $\hat{n} = (n+1)/2$, in the $n-1$ equations of the system (2.5) corresponding to the indices $i \neq \hat{n}$. The eigenvalues of the linearized vector field at equilibrium can then be easily evaluated by means of a symbolic calculation program. As a matter of fact, they turn out to have negative real part for all values of n which have been tested, say certainly for $n = 5, 7, 9, \ldots$. From the analytic point of view, a deeper understanding of the common qualitative properties of these flows is a challenging problem. But, maybe, from the point of view of applications, getting the information for fixed values of n can be sufficient.

Finally, let us remark that the model analyzed in this work does not preserve the overall wealth given by the first-order momentum. Indeed, the model corresponds to an open system where the outer environment acts on the wealth of the lower and higher classes. An interesting perspective is considering a modified model which delivers a description corresponding to a closed system where the above-mentioned overall wealth is preserved.

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