ON SERRIN'S OVERDETERMINED PROBLEM IN SPACE FORMS

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Abstract. We consider Serrin's overdetermined problem for the equation $\Delta v + nKv = -1$ in space forms, where K is the curvature of the space, and we prove a symmetry result by using a P-function approach. Our approach generalizes the one introduced by Weinberger to space forms and, as in the Euclidean case, it provides a short proof of the symmetry result which does not make use of the method of moving planes.

1. Introduction

In the seminal paper [26] Serrin proved that if there exists a solution to

$$\Delta v + f(v) = 0 \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ such that

$$v = 0 \quad \text{and} \quad v_\nu = const \quad \text{on } \partial\Omega\,,$$

then Ω must be a ball and v radially symmetric. The proof in [26] makes use of the method of moving planes and actually applies to more generally uniformly elliptic operators (see [26]).

In [28] Weinberger considerably simplified the proof of Serrin's result in the case $\Delta v = -1$ by considering what is nowadays called P-function and using some integral identities. The approach of Weinberger, as well as the use of a P-function, inspired several works in the context of elliptic partial differential equations (see e.g. [1, 10, 11, 12, 13, 20, 25, 27] and references therein).

In the present paper we investigate such symmetry results from a broader perspective of the ambient space. We consider overdetermined problems in space forms by assuming the ambient space to be a complete simply-connected Riemannian manifold with constant sectional curvature K. Up to homoteties we may assume $K = 0, -1, 1$; the case $K = 0$ corresponds to the case of the Euclidean space, $K = -1$ is the Hyperbolic space and $K = 1$ is the unitary sphere with the round metric.

In space forms, Serrin's symmetry result was proved in [14] and [18] by adapting the proof of Serrin [26], i.e. by using the method of moving planes. The aim of the present paper is to prove Serrin's result in space forms by using an approach analogous to the one of Weinberger by using a suitable P-function associated to the equation $\Delta v + nKv = -1$. As in the Euclidean case, our approach is suitable only for the equation that we are considering, and does not fit with more general equations of the form (1.1).

Let (M, g) be a Riemannian manifold isometric to one of the following three models: the Eucliden space $\mathbb{R}^n$, the Hyperbolic space $\mathbb{H}^n$, the hemisphere $\mathbb{S}^n_+$. The three models can be described as the warped product space $M = I \times \mathbb{S}^{n-1}$ equipped with the rotationally symmetric metric

$$g = dr^2 + h^2\, g_{\mathbb{S}^{n-1}}\,,$$

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where $g_{\mathbb{S}^{n-1}}$ is the round metric on the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ and

- $I = [0, \infty)$ and $h(r) = r$ in the Euclidean case $(K = 0)$;
- $I = [0, \infty)$ and $h(r) = \sinh(r)$ in the hyperbolic case $(K = -1)$;
- $I = [0, \pi/2)$ and $h(r) = \sin(r)$ in the spherical case $(K = 1)$.

Our main result is the following.

Theorem 1.1. *Let $\Omega \subset M$ be a bounded connected domain with boundary $\partial\Omega$ of class C^1. Let v be the solution to*

$$\begin{cases} \Delta v + nKv = -1 & \text{in } \Omega\,, \\ v = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.2}$$

and assume that

$$|\nabla v| = c \quad \text{on } \partial\Omega\,, \tag{1.3}$$

for some positive constant c. Then Ω is a geodesic ball B_R and v depends only on the distance from the center of B_R.

In theorem 1.1 we may assume, up to isometries, that B_R is centered at the origin. In this case v is given by

$$v(r) = \frac{H(R) - H(r)}{n\dot{h}(R)}\,,$$

with $H = \int_0^r h(s)ds$. Indeed, since the Laplacian of a radial radial function $u = u(r)$ is given by $\Delta u = \ddot{u} + (n-1)\dot{h}h^{-1}\dot{u}$, a straightforward computation yields that v solves (1.2). Furthermore, by computing the first derivative of v, we deduce that c and R are related by

$$c = \frac{h(R)}{n\dot{h}(R)}\,.$$

We notice that for $K = 0$, (1.2) reduces to the classical model problem $\Delta v = -1$ in the Euclidean space. The extra term nKv is the one needed to obtain that the Hessian of the solution in the radial case is proportional to the metric. Moreover, this allows us to consider the P-function

$$P(v) = |\nabla v|^2 + \frac{2}{n}v + Kv^2\,, \tag{1.4}$$

which is subharmonic when v solves (1.2). An analogous approach was exploited in [21] for $K = 1$.

Equation (1.2) arises from the study of constant mean curvature hypersurfaces in space forms. Indeed, it is known from Reilly's paper [23] that a possible approach to prove Alexandrov *soap bubble theorem* in the Euclidean space is by considering the torsion potential, i.e. the solution to $\Delta v = -1$, and apply Reilly's identity. In space forms this approach was generalized by Qui and Xia in [21] by replacing equation $\Delta v = -1$ with $\Delta v + nKv = -1$.

We also mention that Alexandrov's soap bubble theorem in the Euclidean space can be proved via Serrin's overdetermined problem for the equation $\Delta v = -1$ (see [23][remark at p. 468]). Hence, theorem 1.1 can be used to give an alternative proof to Alexandrov theorem in space forms by using the generalization of Reilly's identity in [21]. Further connections between Alexandrov soap bubble theorem and Serrin's overdetermined problem can be found in [4, 5, 7, 8, 9, 15, 16, 17].

In the next section we write ∇^2 to denote the Hessian of a function and, for X, Y vector fields, we write $X \cdot Y$ instead of $g(X, Y)$.

2. Proof of the result

We first prove that the P-function (1.4) is subharmonic.

Lemma 2.1. *Let v be a solution to*

$$\Delta v + nKv = -1$$

and let P be given by (1.4). Then

$$\Delta P(v) \geq 0 \quad \text{in } \Omega\,.$$

Moreover, $\Delta P(v) = 0$ if and only if

$$\nabla^2 v = -\left(\frac{1}{n} + Kv\right) g\,. \tag{2.1}$$

Proof. From the Bochner-Weitzenböck formula in space forms

$$\frac{1}{2}\Delta|\nabla v|^2 = |\nabla^2 v|^2 + \nabla(\Delta v)\cdot\nabla v + (n-1)K\nabla v\cdot\nabla v$$

and from Cauchy-Schwartz inequality we obtain that

$$\frac{1}{2}\Delta|\nabla v|^2 \geq \frac{1}{n}(\Delta v)^2 + D(\Delta v)\cdot\nabla v + (n-1)K\nabla v\cdot\nabla v\,.$$

From (1.2) we find that

$$\begin{aligned}\frac{1}{2}\Delta|\nabla v|^2 &\geq \frac{1}{n}(\Delta v)(-1-nKv) + D(-1-nKv)\cdot\nabla v + (n-1)K\nabla v\cdot\nabla v\\ &= -\frac{1}{n}\Delta v - Kv\Delta v - K\nabla v\cdot\nabla v\\ &= -\frac{1}{n}\Delta v - \frac{K}{2}\Delta v^2\,,\end{aligned}$$

where in the last inequality we have used that $\Delta(v^2/2) = v\Delta v + |\nabla v|^2$. Hence $\Delta P(v) \geq 0$.

From the argument above, we readily see that $\Delta P(v) = 0$ if and only if

$$n|\nabla^2 v|^2 = (\Delta v)^2\,,$$

which implies that $\nabla^2 v$ is a multiple of the metric g. Since v satisfies (1.2) then we obtain (2.1). □

Lemma 2.1 will be used in the following form.

Corollary 2.2. *Let v be the solution to (1.2) and assume that (1.3) holds. Then either*

$$P(v) = c^2 \quad \text{in } \bar{\Omega} \tag{2.2}$$

or

$$c^2\int_\Omega \dot{h} > \left(1+\frac{2}{n}\right)\left(\int_\Omega \dot{h}v - K\int_\Omega hvv_r\right)\,. \tag{2.3}$$

Proof. From lemma 2.1 we have that $\Delta P(v) \geq 0$. Since $P(v) = c^2$ on $\partial\Omega$, by the strong maximum principle either $P(v) = c^2$ in $\bar{\Omega}$ or $P(v) < c^2$ in Ω. If we assume that $P(v) < c^2$ in Ω then, being $\dot{h} > 0$,

$$c^2\int_\Omega \dot{h} > \int_\Omega \dot{h}|\nabla v|^2 + \frac{2}{n}\int_\Omega \dot{h}v + K\int_\Omega \dot{h}v^2$$

and since

$$\operatorname{div}(\dot{h}v\nabla v) = \dot{h}|\nabla v|^2 + \dot{h}v\Delta v + \ddot{h}vv_r \tag{2.4}$$

and $v=0$ on $\partial\Omega$, being $\ddot h=-Kh$, we obtain that

$$\begin{aligned} c^2\int_\Omega \dot h &> -\int_\Omega \dot h v\Delta v - \int_\Omega \ddot h v v_r + \frac{2}{n}\int_\Omega \dot h v + K\int_\Omega \dot h v^2 \\ &= (n+1)K\int_\Omega \dot h v^2 + \left(1+\frac{2}{n}\right)\int_\Omega \dot h v + K\int_\Omega h v v_r\,. \end{aligned}$$

Since $\mathrm{div}\,(h\partial_r) = n\dot h$, we have

$$\mathrm{div}\,(v^2 h\partial_r) = n\dot h v^2 + 2hvv_r\,, \tag{2.5}$$

and from $v=0$ on $\partial\Omega$ we obtain (2.3). □

As we will see, (2.3) will give a contradiction which follows from Pohožaev identity.

Lemma 2.3. *Let v be the solution to (1.2) and assume that v satisfies (1.3). Then*

$$c^2\int_\Omega \dot h = \left(1+\frac{2}{n}\right)\left(\int_\Omega \dot h v - K\int_\Omega h v v_r\right)\,. \tag{2.6}$$

Proof. We first consider a generic sufficiently smooth function v (not necessarily a solution to (1.2)). We consider the Pohožaev identity in space forms (see e.g. [6])

$$\mathrm{div}\left(\frac{|\nabla v|^2}{2}X - hv_r\nabla v\right) = \frac{n-2}{2}\dot h|\nabla v|^2 - hv_r\Delta v\,, \tag{2.7}$$

where X is the radial vector field

$$X = h\partial_r.$$

Since

$$hv_r = X\cdot\nabla v = \mathrm{div}\,(vX) - n\dot h v\,,$$

we have that

$$\begin{aligned} \frac{1}{n}\mathrm{div}\,\big(|\nabla v|^2X - 2(X\cdot\nabla v)\nabla v\big) - \frac{n-2}{n}\Big(\mathrm{div}\,(\dot h v\nabla v) - \dot h v\Delta v + KvX\cdot\nabla v\Big)& \\ + \frac{2}{n}(\mathrm{div}\,(vX) - n\dot h v)\Delta v = 0\,,& \end{aligned}$$

which we write as

$$\begin{aligned} &\frac{1}{n}\mathrm{div}\,\big(|\nabla v|^2X - 2(X\cdot\nabla v)\nabla v\big) \\ &\qquad - \frac{n-2}{n}\Big(\mathrm{div}\,(\dot h v\nabla v) - \dot h v(\Delta v + nKv) + nK\dot h v^2 + KvX\cdot\nabla v\Big) \\ &\qquad\qquad + \frac{2}{n}(\mathrm{div}\,(vX) - n\dot h v)(\Delta v + nKv) - 2(\mathrm{div}\,(vX) - n\dot h v)Kv = 0\,, \end{aligned}$$

i.e.

$$\begin{aligned} &\frac{1}{n}\mathrm{div}\,\big(|\nabla v|^2X - 2(X\cdot\nabla v)\nabla v\big) \\ &\qquad - \frac{n-2}{n}\mathrm{div}\,(\dot h v\nabla v) - \frac{n+2}{n}\dot h v(\Delta v + nKv) + \frac{2}{n}(\mathrm{div}\,(vX))(\Delta v + nKv) \\ &\qquad\qquad + (n+2)K\dot h v^2 - \frac{n-2}{n}KvX\cdot\nabla v - 2Kv\mathrm{div}\,(vX) = 0\,. \end{aligned} \tag{2.8}$$

Now we assume that v is a solution to (1.2) satisfying (1.3), and we integrate (2.8)

$$-\frac{c^2}{n}\int_{\partial\Omega} X\cdot\nu + \frac{n+2}{n}\int_\Omega \dot h v + (n+2)K\int_\Omega \dot h v^2 - \frac{n-2}{n}K\int_\Omega vX\cdot\nabla v - 2K\int_\Omega v\mathrm{div}\,(vX) = 0\,, \tag{2.9}$$

i.e.

$$-\frac{c^2}{n}\int_{\partial\Omega} X\cdot\nu + \frac{n+2}{n}\int_\Omega \dot h v - (n-2)K\int_\Omega \dot h v^2 + \left(\frac{2}{n}-3\right)K\int_\Omega vX\cdot\nabla v = 0\,,$$

and since $\mathrm{div}\,X = n\dot h$ we obtain

$$c^2\int_\Omega \dot h = \frac{n+2}{n}\int_\Omega \dot h v - (n-2)K\int_\Omega \dot h v^2 + \left(\frac{2}{n}-3\right)K\int_\Omega vX\cdot\nabla v\,.$$

From (2.5) and $v=0$ on $\partial\Omega$ we obtain (2.6). □

Although the following result can be deduced from [3][Section 1] (see also [2][Theorem 1.1] and [24][Lemma 3]), we provide a proof for reader's convenience.

Lemma 2.4. *Let Ω be a bounded connected domain in M and assume that there exists a function $v\colon\bar\Omega\to\mathbb{R}$, with $v\in C^1(\overline{\Omega})\cap C^2(\Omega)$, such that*

$$\begin{cases}\nabla^2 v = (-\frac{1}{n} - Kv)g & \text{in } \Omega\,,\\ v=0 & \text{on } \partial\Omega\,.\end{cases} \tag{2.10}$$

Then Ω is a geodesic ball B_R and v depends only on the center of B_R.

Proof. We first notice that $v>0$ in Ω. This follows from the standard maximum principles when $K=0,-1$ (see e.g. [27][theorem 2.6]). Now we consider the case $K=1$. We recall that the first eigenvalue of the Dirichlet Laplacian on the hemisphere is n and the corresponding eigenfunction ϕ is strictly positive. By writing $v=w\phi$ we see that w satisfies

$$\begin{cases}\Delta w + 2\frac{\nabla\phi}{\phi}\cdot\nabla w < 0 & \text{in } \Omega\\ w=0 & \text{on } \partial\Omega\,,\end{cases}$$

which implies that $w>0$ in Ω again by [27][Theorem 2.6]. Hence $v>0$ in Ω.

Since $v>0$ it achieves the maximum at a point $p\in\Omega$, with $v(p)=a>0$. Let $\gamma\colon I\to M$ be a unit speed maximal geodesic satisfying $\gamma(0)=p$ and let $f(s)=v(\gamma(s))$. From (2.10) it follows

$$\ddot f(s) = -\frac{1}{n} - Kf(s)\,,\quad \dot f(0)=0\,,\quad f(0)=a\,,$$

and therefore

$$f(s) = \left(a-\frac{1}{n}\right)H(s) - \frac{1}{n}\,.$$

This implies that v has the same expression along any geodesic starting from p, and hence v depends only on the distance from p, which completes the proof. □

Proof of Theorem 1.1. Corollary 2.2 and Lemma 2.3 imply that $P(v)=c^2$ and from Lemma 2.1 we find that v satisfies (2.1). Lemma 2.4 gives that Ω is a geodesic ball B_R and v depends only on the distance from the center of B_R. □

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