AN UNCOUNTABLE FAMILY OF ALMOST NILPOTENT VARIETIES OF POLYNOMIAL GROWTH

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Abstract. A non-nilpotent variety of algebras is almost nilpotent if any proper subvariety is nilpotent. Let the base field be of characteristic zero. It has been shown that for associative or Lie algebras only one such variety exists. Here we present infinite families of such varieties. More precisely we shall prove the existence of
1) a countable family of almost nilpotent varieties of at most linear growth and
2) an uncountable family of almost nilpotent varieties of at most quadratic growth.

1. Introduction

Let F be a field of characteristic zero and $F\{X\}$ the free non associative algebra on a countable set X over F. If $\mathcal{V}$ is a variety of not necessarily associative algebras and $Id(\mathcal{V})$ is the T-ideal of polynomial identities of $\mathcal{V}$, then $F\{X\}/Id(\mathcal{V})$ is the relatively free algebra of countable rank of the variety $\mathcal{V}$. It is well known that in characteristic zero every identity is equivalent to a system of multilinear ones, and an important invariant is provided by the sequence of dimensions $c_n(\mathcal{V})$ of the n-multilinear part of $F\{X\}/Id(\mathcal{V})$, $n = 1, 2, \ldots$. More precisely, for every $n \geq 1$ let P_n be the space of multilinear polynomials in the variables $x_1, \ldots, x_n$. Since char $F = 0$, $F\{X\}/Id(\mathcal{V})$ is determined by the sequence of subspaces $\{P_n/(P_n \cap Id(\mathcal{V}))\}_{n\geq 1}$ and the integer $c_n(\mathcal{V}) = \dim P_n/(P_n \cap Id(\mathcal{V}))$ is called the n-th codimension of $\mathcal{V}$. The growth function determined by the sequence of integers $\{c_n(\mathcal{V})\}_{n\geq 1}$ is the growth of the variety $\mathcal{V}$.

In general a variety $\mathcal{V}$ has overexponential growth, i.e., the sequence of codimensions cannot be bounded by any exponential function. Recall that $\mathcal{V}$ has exponential growth if $c_n(\mathcal{V}) \leq a^n$, for all $n \geq 1$, for some constant a. For instance any variety generated by a finite dimensional algebra has exponential growth. For such varieties the limit $\lim_{n\to\infty} \sqrt[n]{c_n(\mathcal{V})} = exp(\mathcal{V})$, is called the PI-exponent of the variety $\mathcal{V}$, provided it exists.

We say that a variety $\mathcal{V}$ has polynomial growth if there exist constants $\alpha, t \geq 0$ such that asymptotically $c_n(\mathcal{V}) \simeq \alpha n^t$. When $t = 1$ we speak of linear growth and when $t = 2$, of quadratic growth.

Moreover $\mathcal{V}$ has intermediate growth if for any $k > 0$, $a > 1$ there exist constants C_1, C_2, such that for any n the inequalities

$$C_1 n^k < c_n(\mathcal{V}) < C_2 a^n$$

hold. Finally we say that a variety $\mathcal{V}$ has subexponential growth if for any constant B there exists n_0 such that for all $n > n_0$, $c_n(\mathcal{V}) < B^n$. Clearly varieties with polynomial growth or intermediate growth have subexponential growth and it can be shown that varieties realizing each growth can be constructed. For instance a class of varieties of intermediate growth was constructed in [5].

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The purpose of this note is the study of the almost nilpotent varieties. Recall that a variety $\mathcal{V}$ is almost nilpotent if it is not nilpotent but all proper subvarieties are nilpotent.

About previous results, if we consider varieties of associative algebras, it is easily seen that the only almost nilpotent variety is the variety $\mathcal{V}$ of commutative algebras, (the sequence of codimensions is $c_n(\mathcal{V}) = 1, n \geq 1$). In the case of varieties of Lie algebras it has been shown that there is also only one almost nilpotent variety: the variety $\mathcal{A}^2$ of metabelian Lie algebras and in this case $c_n(\mathcal{A}^2) = n - 1$. In [3] it was proved that there exist only two almost nilpotent varieties of Leibniz algebras and both varieties have at most linear growth. For general non associative algebras, in [11] an almost nilpotent variety of exponent two was constructed. Later in [10] it was proved that for any integer m an almost nilpotent variety with exponent m exists. Recently in [8] it was proved the existence of almost nilpotent varieties with fractional exponent.

An algebra satisfying the identity $x(yz) \equiv 0$ will be called left nilpotent of index two. In [12] two almost nilpotent varieties with linear growth were constructed and it was proved that they represent a full list of almost nilpotent varieties with subexponential growth in the class of left nilpotent algebras of index two. For commutative (anticommutative) metabelian algebras similar result were obtained in [1], [9].

The purpose of this note is to prove the existence of two families of almost nilpotent varieties. The first one is a countable family of at most linear growth and the second one is an uncountable family of at most quadratic growth.

2. The general setting

Throughout A will be a non necessarily associative algebra over a field F of characteristic zero and $F\{X\}$ the free non associative algebra on a countable set $X = \{x_1, x_2, \ldots\}$. The polynomial identities satisfied by A form a T-ideal $Id(A)$ of $F\{X\}$ and by the standard multilinearization process, we consider only the multilinear polynomials lying in $Id(A)$. To this end, for every $n \geq 1$, we set P_n to be the space of multilinear polynomials in $x_1, \ldots, x_n$, and we let the symmetric group S_n act on P_n be setting $\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, for $\sigma \in S_n, f \in P_n$.

The space $P_n(A) = P_n/(P_n \cap Id(A))$ has an induced structure of S_n-module and we let $\chi_n(A)$ be its character, called the n-th cocharacter of A. By complete reducibility we write

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$$

where χ_λ is the irreducible S_n-character corresponding to the partition $\lambda \vdash n$ and $m_\lambda \geq 0$ is the corresponding multiplicity (we refer the reader to [6] for an account of this approach).

We next recall some basic properties of the representation theory of the symmetric group that we shall use in the sequel. Let $\lambda \vdash n$ and let T_λ be a Young tableau of shape $\lambda \vdash n$. We denote by e_{T_λ} the corresponding essential idempotent, i.e., $e_{T_\lambda}^2 = \alpha e_{T_\lambda}$, $0 \neq \alpha \in F$, of the group algebra FS_n. Recall that $e_{T_\lambda} = R_{T_\lambda}^+ C_{T_\lambda}^-$ where $R_{T_\lambda}^+ = \sum_{\sigma \in R_{T_\lambda}} \sigma$, and $C_{T_\lambda}^- = \sum_{\tau \in C_{T_\lambda}} (\mathrm{sgn}\tau)\tau$ and R_{T_λ}, C_{T_λ} are the groups of row and column stabilizers of T_λ, respectively. Recall that if M_λ is an irreducible S_n-submodule of $P_n(A)$ corresponding to λ, there exists a polynomial $f(x_1, \ldots, x_n) \in P_n$ and a tableau T_λ such that $e_{T_\lambda} f(x_1, \ldots, x_n) \notin Id(A)$. Let $e'_{T_\lambda} = C_{T_\lambda}^- R_{T_\lambda}^+ C_{T_\lambda}^-$. Since $R_{T_\lambda}^+ C_{T_\lambda}^- R_{T_\lambda}^+ C_{T_\lambda}^- \neq 0$ then e'_{T_λ} is a nonzero essential idempotent that generates the same irreducible module and so also $e'_{T_\lambda} f(x_1, \ldots, x_n) \notin Id(A)$.

In what follows we shall also denote by $g(\lambda)$ the polynomial obtained from the essential idempotent corresponding to a tableau of shape λ by identifying the elements in each row. Recall that $g(\lambda)$ is an highest weight vector of the general linear group $GL_k(F)$ where k is the number of distinct part of λ (see [2])

Now, for a fixed arrangement of the parentheses T, let us denote by P_n^T the subspace of P_n spanned by the monomials whose arrangement of the parentheses is T. Let also $P_n^T(A) = P_n^T/(P_n^T \cap Id(A))$. Then clearly $P_n(A) = \sum_T P_n^T(A)$.

Since the S_n-module $P_n^T(A)$ is a homomorphic image of $P_n^T \equiv FS_n$, the regular S_n-representation, it follows that, if $\chi_n(A)^T$ is the S_n-character of $P_n^T(A)$, then

$$\chi_n(A)^T = \sum_{\lambda \vdash n} m_\lambda^T \chi_\lambda$$

and $m_\lambda^T \leq d_\lambda = \deg \chi_\lambda$. Clearly $m_\lambda \leq \sum_T m_\lambda^T$.

Throughout we shall also use the following convention: we shall write the same symbol (e.g. $\bar{}$, $\tilde{}$) over two or more variables of a polynomial to indicate that the polynomial is alternating on these variables.

For instance $x_3\bar{x}_1\bar{x}_2 = x_3x_1x_2 - x_3x_2x_1$.

We also need to recall some results from the theory of infinite words (see [7]). Recall that, given an infinite (associative) word w in the alphabet $\{0,1\}$ the complexity Comp_w of w is defined as the function $\mathrm{Comp}_w : \mathbb{N} \to \mathbb{N}$, where $\mathrm{Comp}_w(n)$ is the number of distinct subwords of w of length n.

Also, an infinite word $w = w_1w_2\cdots$ is *periodic* with period T if $w_i = w_{i+T}$ for $i = 1, 2, \ldots$. It is easy to see that for any such word $\mathrm{Comp}_w(n) \leq T$. Moreover, an infinite word w is called a *Sturmian* word if $\mathrm{Comp}_w(n) = n + 1$ for all $n \geq 1$.

For a finite word x, the height $h(x)$ of x is the number of occurrences of the symbol 1 appearing in x. Also, if $|x|$ denotes the length of the word x, the *slope* of x is defined as $\pi(x) = \frac{h(x)}{|x|}$. In some cases this definition can be extended to infinite words as follows. Let w be some infinite word and let $w(1,n)$ denote its prefix subword of length n. If the sequence $\frac{h(w(1,n))}{n}$ converges for $n \to \infty$ and the limit

$$\pi(w) = \lim_{n\to\infty} \frac{h(w(1,n))}{n}$$

exists then $\pi(w)$ it is called the slope of w. Examples of infinite words for which the slope is not defined can be given. Nevertheless for periodic and Sturmian words the slope is well defined. In the next proposition we reassume the main properties of these words that we shall use here.

Theorem 1. ([7, Section 2.2]) *Let w be a Sturmian or periodic word. Then there exists a constant C such that*

1) *$|h(x) - h(y)| \leq C$, for any finite subwords x, y of w with $|x| = |y|$;*
2) *the slope $\pi(w)$ of w exists;*
3) *$|\pi(u) - \pi(w)| \leq \frac{C}{|u|}$, for any non-empty subword u of w;*
4) *for any real number $\alpha \in (0,1)$ there exists a word w with $\pi(w) = \alpha$ and w is Sturmian or periodic according as α is irrational or rational, respectively.*

If w is Sturmian we can take $C = 1$, and if w is periodic of period t, then $\pi(w) = \frac{h(w(1,t))}{t}$.

3. Algebras constructed from periodic or Sturmian words

Our aim in this section is to prove the existence of two families of almost nilpotent varieties. The first is a countable family of varieties of at most linear growth and the second is an uncountable family of at most quadratic growth. To do this we will make use of an algebra constructed in [4].

Throughout A will be the algebra generated by one element a such that every word in A containing two or more subwords equal to a^2 must be zero.

Note that in particular the algebra A is metabelian, i.e., it satisfies the identity

$$(x_1x_2)(x_3x_3) \equiv 0.$$

A partial decomposition of the cocharacter of A was given in [4] and we recall it here.

Let L_a and R_a denote the linear transformations on A of left and right multiplication by a, respectively. We shall usually write $bL_a = L_a(b) = ab$ and $bR_a = R_a(b) = ba$.

We have the following

Remark 1.

1) $\chi_n(A) = m_{(n)}\chi_{(n)} + m_{(n-1,1)}\chi_{(n-1,1)}$
2) $c_n(A) \geq 2^{n-2}$.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$ be a partition of n such that $n - \lambda_1 \geq 2$. This says that either the first column of λ has at least three boxes or the first two columns of λ have at least two boxes each. Hence, if f_λ is an highest weight vector associated to λ, either f is alternating on three variables or f is alternating on two distinct pairs of variables. In both cases every monomial of f_λ evaluated in A contains at least two subwords equal to a^2. Hence $f_\lambda \in Id(A)$ and this implies that χ_λ appears with zero multiplicity in the decomposition of $\chi_n(A)$. It follows tha

$$\chi_n(A) = m_{(n)}\chi_{(n)} + m_{(n-1,1)}\chi_{(n-1,1)}$$

is the decomposition of $\chi_n(A)$ into irreducibles.

In order to prove 2) we compute the multiplicity $m_{(n)}$ in $\chi_n(A)$.

Let $w(L_a, R_a) \in End(A)$ be a word in L_a and R_a of length $n-2$. Clearly $a^2v(L_a, R_a) = v(L_a, R_a)(a^2)$ is the evaluation of an highest weight vector associated to the partition (n) which is not an identity of A. Since there are 2^{n-2} distinct such words, we get 2^{n-2} highest weight vectors which are linearly independent mod $Id(A)$. Thus since $\deg \chi_{(n)} = 1$, from $\chi_n(A) = m_{(n)}\chi_{(n)} + m_{(n-1,1)}\chi_{(n-1,1)}$, we have that $c_n(A) \geq 2^{n-2}$. □

Next we shall compute the decomposition of the cocharacter $\chi_n^T(A)$ for a fixed arrangement T of the parentheses of P_n.

We have the following

Proposition 1. *For any arrangement T of the parentheses in P_n we have*

$$\chi_n(A)^T = \chi_{(n)} + 2\chi_{(n-1,1)}. \tag{1}$$

Proof. If $P_n^T(A) \neq 0$ then any monomial of P_n^T is of the form

$$x_{\sigma(1)}x_{\sigma(2)}T_{1,x_{\sigma(3)}} \ldots T_{n-2,x_{\sigma(n)}} \pmod{Id(A)},$$

where $T_{j,x_i} = L_{x_i}$ or $T_{j,x_i} = R_{x_i}$, for any i, j.

It follows that, mod $Id(A)$, the highest weight vectors corresponding to standard tableaux of shape $(n-1,1)$ are

$$g_0(x_1, x_2) = (\overline{x}_1\overline{x}_2)T_{x_1} \ldots T_{x_1}$$

and

$$g_i(x_1, x_2) = (\overline{x}_1x_1)T_{1,x_1} \ldots T_{i-1,x_1}\overline{T}_{i,x_2}T_{i+1,x_1} \ldots T_{n-2,x_1}, \quad 1 \leq i \leq n-2.$$

Recall that the symbol $^-$ over two or more variables of a polynomial means that the polynomial is alternating on these variables.

We claim that for any $1 \leq i, j \leq n-2$ the elements $g_i(x_1, x_2)$ and $g_j(x_1, x_2)$ are linearly dependent mod $Id(A)$. In fact, since any word containing two subwords equal to a^2 is zero in A, in a non-zero evaluation φ we must set $\varphi(x_1) = a$ and $\varphi(x_2) = a^2v(L_a, R_a)$, for some monomial $v(L_a, R_a) \in End(A)$.

We get

$$\varphi(g_i(x_1,x_2)) = \varphi(g_j(x_1,x_2)) = -a^2 v(L_a,R_a)R_a T_{1,a}\dots T_{n-2,a},$$

and the claim is established.

Next our aim is to prove that the polynomials $g_0(x_1,x_2)$ and $g_1(x_1,x_2)$ are linearly independent mod $Id(A)$. In fact suppose that $\alpha g_0(x_1,x_2)+\beta g_1(x_1,x_2)$ is an identity of A, for some $\alpha,\beta\in F$. If we consider the evaluation $\varphi(x_1)=a$ and $\varphi(x_2)=a^2$, we get

$$\alpha g_0(a,a^2)+\beta g_1(a,a^2) = \alpha a^2 L_a T_{1,a}\dots T_{n-2,a} - (\alpha+\beta)a^2 R_a T_{1,a}\dots T_{n-2,a},$$

and the right hand side is zero only if $\alpha=\beta=0$.

We have proved that $\chi_{(n-1,1)}$ appears with multiplicity 2 in the decomposition of $\chi_n(A)^T$. Since $m^T_{(n)}=1$ we get that $\chi_n(A)^T=\chi_{(n)}+2\chi_{(n-1,1)}$ and the proposition is proved. □

Next for every real number between 0 and 1 we shall construct a quotient algebras of A. To this end we keep in mind the terminology of the previous section.

We are going to associate to every finite word in the alphabet $\{0,1\}$ a monomial in $End(A)$ in left and right multiplications: if $u(0,1)$ is such a word we associate to u the monomial $u(L_a,R_a)$ obtained by substituting 0 with L_a and 1 with R_a.

Let α be a real number, $0<\alpha<1$, and let w_α be a Sturmian or periodic infinite word in the alphabet $\{0,1\}$ whose slope is $\pi(w_\alpha)=\alpha$.

Let I_α be the ideal of the algebra A generated by the elements $a^2u(L_a,R_a)$ where $u(0,1)$ is not a subword of the word w_α.

Let $A_\alpha=A/I_\alpha$ denote the corresponding quotient algebra and let $\mathcal{V}_\alpha$ be the variety generated by the algebra A_α.

We have

Lemma 1. *For any real number α, $0<\alpha<1$, the variety $\mathcal{V}_\alpha$ has linear or quadratic growth according as w_α is a periodic or a Sturmian word.*

Proof. We are going to find an upper and a lower bound of the codimensions of the algebra A_α. To this end we start from the decomposition of the cocharacter of A given in (1).

Let $n\geq 3$ be any integer and let $u(0,1)$ be a subword of the word w_α of length $n-1$. We may clearly assume that 0 is the leftmost symbol of such word and, so, we write $u(0,1)=0v(0,1)$ for some subword $v(0,1)$ of w_α.

Since u and v are subwords of w_α, $a^2v(L_a,R_a), a^2L_av(L_a,R_a)\notin I_\alpha$. This implies that the polynomial

$$g_0(x_1,x_2) = (\overline{x}_1\overline{x}_2)v(L_{x_1},R_{x_1})$$

is not an identity of the algebra A_α. In fact, recall that the evaluation $\varphi(x_1)=a, \varphi(x_2)=a^2$ gives $\varphi(g_0(x_1,x_2)) = a^2L_av(L_a,R_a) - a^2R_av(L_a,R_a)\notin I_\alpha$.

Since the word $u(0,1)$ is an arbitrary subword of the word w_α of length $n-1$, this says that, for any corresponding arrangement T of the parentheses in

$$\chi_n(A_\alpha)^T = \chi_n + m^T_{(n-1,1)}\chi_{n-1,1} \tag{2}$$

we must have $m^T_{(n-1,1)}>0$. Moreover compare the last equality with (1) and recall that, since A_α is a quotient algebra of A, the multiplicities in $\chi^T_n(A_\alpha)$ are bounded by the multiplicities in $\chi^T_n(A)$. It follows that $0<m^T_{(n-1,1)}\leq 2$.

Now, the different arrangements of the parentheses in nonzero words of length n in A_α correspond to the subwords of w_α of length n. Recalling that $\mathrm{Comp}_{w_\alpha}(n)$ is either constant or equal to $n+1$ according as w_α is periodic or Sturmian respectively, it follows that their number is bounded by a constant in case α is rational (i.e., w_α is periodic) and by a linear function of n in case α is irrational (i.e., w_α is Sturmian).

Since $\deg \chi_{(n)} = 1$ and $\deg \chi_{(n-1,1)} = n-1$, from (2) and the above discussion we can find constants C_1, C_2 such that for any n we have

$$C_1 n \leq c_n(A_\alpha) \leq C_2 n,$$

if α is rational, and

$$C_1 n^2 \leq c_n(A_\alpha) \leq C_2 n^2$$

if α is irrational.

Recalling that the growth of $\mathcal{V}_\alpha$ is the growth of the sequence $c_n(A_\alpha)$ the proof of the lemma is complete. □

Proposition 2. *For $0 < \alpha < \beta < 1$, the variety $\mathcal{V}_\alpha \cap \mathcal{V}_\beta$ is nilpotent.*

Proof. Let $K_n(w_\gamma)$ denote the set of different subwords of length n of a word w_γ. Now, the slope of the words w_α and w_β is equal to α and β, respectively. Since $\alpha \neq \beta$, by Theorem 1 there exist m such that for any $n \geq m$ the intersection $K_n(w_\alpha) \cap K_n(w_\beta)$ is the empty set. In particular there exist m such that any word $u(0,1)$ of length m is not a subword either of the word w_α or of the word w_β.

Let for instance $u(0,1)$ be a word of length m which is not a subword of the word w_α, and consider the monomial $y_1 y_2 u(L_x, R_x)$. Construct the multilinear element $y_1 y_2 \overline{u}$ on $y_1, y_2, x_1, \ldots, x_m$ where $\overline{u}$ is obtained by substituting $x_1, \ldots, x_m$ instead of x inside $u(L_x, R_x)$. Hence $y_1 y_2 \overline{u} \equiv 0$ is an identity of the variety V_α. It follows that $y_1 y_2 \overline{u} \equiv 0$ is also an identity of $V_\alpha \cap V_\beta$, and so $c_{m+2}(V_\alpha \cap V_\beta) = 0$. From this it follows that $P^T_{m+2}(\mathcal{V}_\alpha \cap \mathcal{V}_\beta) = 0$ for any arrangement of the parentheses T and the variety $\mathcal{V}_\alpha \cap \mathcal{V}_\beta$ is nilpotent. □

We can now prove the main result of this note.

Theorem 2. *Over a field of characteristic zero there are countable many almost nilpotent metabelian varieties of at most linear growth and uncountable many almost nilpotent metabelian varieties of at most quadratic growth.*

Proof. Recall that by [11, Theorem 1] every non-nilpotent variety has an almost nilpotent subvariety. Hence for any real number α, $0 < \alpha < 1$, the variety $\mathcal{V}_\alpha$ contains an almost nilpotent subvariety. Let $\mathcal{U}_\alpha$ be such subvariety. Since $c_n(\mathcal{U}_\alpha) \leq c_n(\mathcal{V}_\alpha)$, then $c_n(\mathcal{U}_\alpha) \leq Cn$ or $c_n(\mathcal{U}_\alpha) \leq Cn^2$, according as α is rational or irrational, respectively. Hence $\mathcal{U}_\alpha$ has at most quadratic growth.

Now, by Proposition 2 for any $0 < \alpha < \beta < 1$ $\mathcal{U}_\alpha \neq \mathcal{U}_\beta$, and this says that there are countable many almost nilpotent metabelian varieties of at most linear growth and uncountable many almost nilpotent metabelian varieties of at most quadratic growth. □

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