On the numerical implementation of a 3D fractional viscoelastic constitutive model

G. Alotta[1†], O. Barrera[2], A.C.F. Cocks[2], M. Di Paola[3]

[1]Dipartimento di Ingegneria Civile, Ambientale, Aerospaziale, dei Materiali (DICAM), Università degli Studi di Palermo, Viale delle Scienze Ed. 8, 90128 Palermo, Italy, currently visiting student at the Department of Engineering Science, University of Oxford

[2]Department of Engineering Science, University of Oxford, Parks Road, OX1 3PJ, Oxford, United Kingdom

[3]Dipartimento di Ingegneria Civile, Ambientale, Aerospaziale, dei Materiali (DICAM), Università degli Studi di Palermo, Viale delle Scienze Ed. 8, 90128 Palermo, Italy

†Email: gioacchino.alotta@unipa.it, gioacchino.alotta@eng.ox.ac.uk

Abstract: *The aim of this paper is the implementation of a 3D fraction al viscoelastic constitutive law in a user material subroutine (UMAT) in the finite element software Abaqus. Essential to the implementation of the model is access to the strain history at each Gauss point of each element in a constructive manner. Details of the UMAT and comparison with some analytical results are presented in order to show that the fractional viscoelastic constitutive law has been successfully implemented.*

Keywords: *Fractional viscoelasticity, Constitutive model, Abaqus/Standard, Creep, Relaxation*

1. Introduction

Real viscoelastic materials like rubbers, polymers, biological tissues, asphalt mixtures, soils etc exhibit power law creep and relaxation behaviour (Nutting, 1921; Di Paola, 2014; Deseri, 2013; Di Mino, 2014; Bagley, 1984). Relaxation and creep of this type of material has been modelled in the scientific literature, mainly, by means of single and/or linear combinations of exponential functions, in an attempt to capture the contributions of both solid and fluid phases. This approach does not allow for a correct fit of experimental results. Power law creep and relaxation leads to fractional viscoelastic constitutive models that are characterized by the presence of so-called fractional derivatives and integrals, namely derivatives and integrals of non-integer order; when the order of derivation (or integration) is integer, the fractional operators restore the classical differential operators. The most interesting aspect of fractional operators is that they have a long “fading” memory. In this context the term “hereditariness” is usually used in the sense that the actual response in terms of stress/displacement depends on the previous stress/strain history. If a relaxation or creep test is well fitted by a power law decay then the fractional constitutive law can be directly derived. Such a constitutive law is defined by a small number of parameters to avoid the conventional use of combinations of simple models which can require a much larger number of parameters to capture both the creep and relaxation behaviour. The aim of this paper is to describe the implementation of a 3D fractional viscoelastic constitutive model in a user material routine (UMAT). The use of other subroutines is necessary to access the strain history at each gauss point to evaluate the stress response.

2. Preliminary concepts

In this section we introduce some preliminary concepts on fractional viscoelasticity and fractional differentiation and integration.

It is well known that a viscoelastic material can be characterized, for one dimensional problems, by its Relaxation and Creep functions, $R(t)$ and $C(t)$ respectively. These functions describe the behaviour of the material when a constant strain and a constant stress are applied, respectively.

Classical models are characterized by exponential type relaxation and creep functions. This happens when viscoelastic materials are modelled by different combinations of elastic elements (springs) and viscous elements (dashpots); the simplest models of this kind are Maxwell and Kelvin-Voigt models in which a spring and a dashpot are in series and in parallel, respectively. Although these models are able to describe the time-dependent behaviour of viscoelastic materials, they fail to capture both the relaxation and the creep behaviour; for this reason more complicated models, with combinations of springs and dashpots are used, but this leads to more complex creep and relaxation functions and governing equations; furthermore these classical models are not able to describe the long-time memory of real viscoelastic materials.

Creep and relaxation tests on real viscoelastic materials, such as polymers, rubbers, asphalt mixtures, biological tissues, have shown that creep and relation tests are well fitted by power laws of real order (Nutting, 1921; Deseri, 2013; Di Mino, 2014; Di Paola, 2014; Bagley, 1984) rather than exponential functions. These functions can be written as follows:

$$R(t) = \frac{C_\beta\ t^{-\beta}}{\Gamma\left(1-\beta\right)}; \qquad C(t) = \frac{t^{\beta}}{C_\beta \Gamma\left(1+\beta\right)} \tag{1a,b}$$

where $\Gamma\left(\bullet\right)$ is the Euler gamma function. For viscoelastic materials $0 \le \beta \le 1$.

It is well known that, within the framework of linear viscoelasticity, the Boltzmann superposition principle is valid; this principle allows us to obtain the response of a material when the imposed stress or strain history is not constant and can be expressed in two forms:

$$\sigma(t) = \int_0^t R(t-\tau)\dot{\varepsilon}(\tau)d\tau; \qquad \varepsilon(t) = \int_0^t C(t-\tau)\dot{\sigma}(\tau)d\tau \tag{2a,b}$$

These integrals are often labelled as "hereditary" integrals, because the actual value of $\sigma(t)$ (or $\varepsilon(t)$) depends on the previous history of $\varepsilon(t)$ (or $\sigma(t)$). By taking the Laplace transform of Equations 2, an interesting relationship between the relaxation and creep functions is obtained in Laplace domain:

$$\hat{R}(s)\hat{C}(s) = \frac{1}{s^2} \tag{3}$$

where the 'hat' means Laplace transform and $s \in \mathbb{C}$ is the variable in the Laplace domain. This implies that it is sufficient to perform a single creep or relaxation test to determine all the relevant parameters of the viscoelastic model.

Substitution of Equations 1 into 2 leads to constitutive laws that involve fractional operators, namely derivatives and integrals of real order (Podlubny, 1999). This is straightforward for the case in which we apply a strain history (Equation 2a) and we want to evaluate the corresponding stress history:

$$\sigma(t) = \frac{C_\beta}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} \dot{\varepsilon}(\tau) d\tau = C_\beta \left({}_0^C D_t^\beta \varepsilon \right)(t) \tag{4}$$

In Equation 4 $\left({}_0^C D_t^\beta \bullet \right)$ is the so called Caputo fractional derivative (Podlubny, 1999), which is a convolution integral with a power law kernel. If we consider the case in which we apply a stress history (Equation 2b), performing an integration by parts and after some manipulations we obtain the Riemann-Liouville (RL) fractional integral $\left({}_0 I_t^\beta \bullet \right)$ (Podlubny, 1999):

$$\varepsilon(t) = \frac{1}{C_\beta \Gamma(1+\beta)} \int_0^t (t-\tau)^{\beta} \dot{\sigma}(\tau) d\tau = \frac{1}{C_\beta \Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \sigma(\tau) d\tau = \frac{1}{C_\beta} \left({}_0 I_t^\beta \sigma \right)(t) \tag{5}$$

These constitutive laws do not correspond to a springs or dashpot or a simple combination of springs and dashpots and is generally referred to as a *springpot.* in the scientific literature (Scott Blair, 1949).

Caputo's fractional derivative and the Riemann-Liouville fractional integral are considered integro-differential operators because all rules of integer order derivatives and integrals are still valid (Podlubny, 1999). Moreover, when the value of β reaches the limit values of 0 and 1, derivatives of order 0 and 1 are obtained. This illustrates a very important feature of these equations: when $\beta \to 0$ the fractional viscoelastic constitutive law of Equations 4 and 5 reduces to the purely elastic (one-dimensional) Hooke's law, while for $\beta \to 1$ the fractional constitutive law becomes the constitutive law of a dashpot. For this reason, as the fractional operators are generalization of integro-differential operators of integer order, the constitutive law of the springpot can be seen as a generalization of the constitutive laws of springs and dashpots. This concept is summarized in Figure 1.

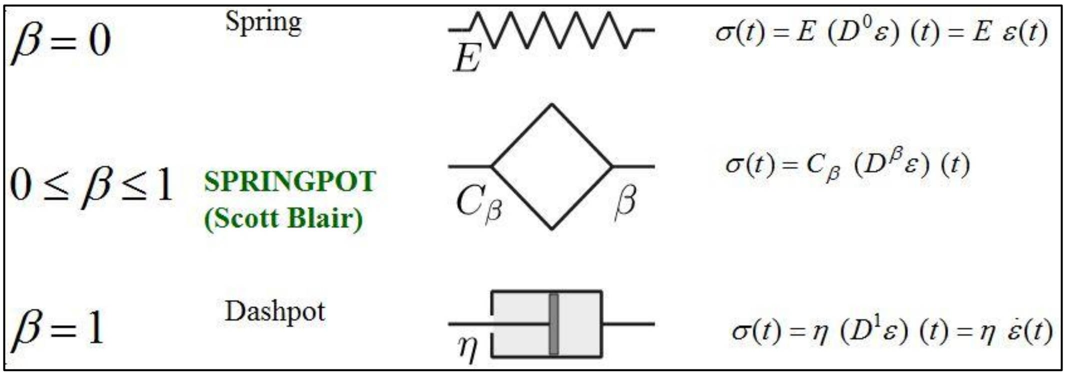

Figure 1. Spring, springpot and dashpot and related constitutive law

For numerical purposes there is a discrete version of the fractional derivative, namely the Grünwald-Letnikov (GL) fractional derivative; after some manipulation the GL fractional derivative can be written in this form:

$$\left({}_{0}^{GL}D_t^{\beta} f \right)(t) = \lim_{\Delta t \to 0} \Delta t^{-\beta} \sum_{k=0}^{N} \lambda_k f(t - k\Delta t) \qquad (6)$$

where λ_k are coefficients evaluated in a recursive way in the form:

$$\lambda_k = \frac{k-1-\beta}{k} \lambda_{k-1}; \qquad \lambda_0 = 1 \qquad (7)$$

The fractional derivative of Equation 6 becomes an integral if $\beta < 0$. The summation of the GL fractional derivative is a discrete convolution and for sufficiently small Δt Equation 6 gives the same results of the Caputo fractional derivative.

An important feature of fractional operators is that the fractional derivative (or integral) depends on the past history of the function, hence they are able to describe the “fading” memory of real viscoelastic materials.

In some cases it is necessary to introduce a more complete model to better represent the behaviour of real viscoelastic materials. This model is the so-called fractional Kelvin-Voigt model and it is constituted by a springpot in parallel with a spring with elastic modulus E (see Figure 2).

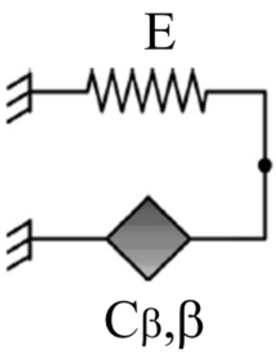

Figure 2. Fractional Kelvin-Voigt model

This ensures that the relaxation and creep responses asymptote towards constant stress and strain, respectively, and reflect a more realistic material behaviour, especially when the stress tensor has a hydrostatic component. For this model, the relaxation and creep laws are:

$$R(t) = E + \frac{C_\beta \, t^{-\beta}}{\Gamma(1-\beta)}; \qquad C(t) = \frac{1}{E}\left(1 - E_\beta\left(-\frac{E}{C_\beta}t^\beta\right)\right) \tag{8}$$

where $E_\beta(\bullet)$ is the one parameter Mittag-Leffler function defined as

$$E_\beta(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\beta j + 1)} \tag{9}$$

In the next section the extension to a 3D fractional constitutive law is presented.

3. 3D fractional constitutive law

The constitutive model is obtained by means of a generalization of the elastic constitutive law (Hooke Law); in that case only two parameters are required to define the whole stiffness (or compliance) matrix for an isotropic material and these two parameters can be chosen as Young's modulus and the shear modulus, Young's modulus and Poisson's ratio, the Lamé constants, or shear modulus and Bulk modulus. In this case we choose to write the stiffness matrix in terms of shear modulus and Bulk modulus, for their clear physical meaning (deviatoric and volumetric part of the stiffness). The terms of the stiffness matrix $\mathbf{D}$ can be written as follows:

$$D_{ijkh} = \left(K - \frac{2}{3}G\right)\delta_{ij}\delta_{kh} + G\left(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}\right) \tag{10}$$

where $K = \frac{E}{3(1-2\nu)}$ is the Bulk modulus, G is the shear modulus, δ_{ij} is the Kronecker delta, E is Young's modulus and ν is Poisson's ratio.

To generalize these elastic laws and obtain a fractional viscoelastic constitutive model, it is sufficient to substitute the shear modulus and Bulk modulus with appropriate relaxation functions; we choose to consider both the behaviour of the deviatoric part (shear relaxation function) and the behaviour of the volumetric part (Bulk relaxation function) as given by the fractional Kelvin-Voigt model:

$$G(t) = G_\infty + \frac{G_\alpha \, t^{-\alpha}}{\Gamma(1-\alpha)}$$
$$K(t) = K_\infty + \frac{K_\beta \, t^{-\beta}}{\Gamma(1-\beta)} \qquad (11a,b)$$

where G_∞ and K_∞ are the elastic parts of the deviatoric and volumetric relaxation functions, respectively, while G_α, G_β, α and β are parameters of the time varying parts of the deviatoric and volumetric relaxation functions, respectively.

By assuming the relaxation functions in Equation 11, a six parameter mechanical model is obtained; the model can be particularized in many ways, simply by changing values of these parameters. The strain-stress relationship can be obtained simply by applying the Boltzmann superposition principle:

$$\boldsymbol{\sigma}(t) = \int_0^t \mathbf{R}(t-\tau)\dot{\boldsymbol{\varepsilon}}(\tau)\,\mathrm{d}\tau \qquad (12)$$

where $\mathbf{R}(t)$ is the relaxation matrix and $\boldsymbol{\sigma}(t)$ and $\boldsymbol{\varepsilon}(t)$ are the stress and strain tensor, respectively. The relaxation matrix $\mathbf{R}(t)$ can be written in the same way as the stiffness matrix of Equation 10, in which G is substituted with $G(t)$ of Equation 11a and K is substituted with $K(t)$ of Equation 11b.

Since relaxation functions contain power laws, Equation 12 contains relationships that involve fractional derivatives of order α and β. It is obvious that this model is also able to reproduce Hooke's law (for $\alpha \to 0$ and $\beta \to 0$), a generalized 3D Kelvin-Voigt model (for $\alpha \to 1$ and $\beta \to 1$) or a generalized 3D viscous law (for $\alpha \to 1, \beta \to 1,\ G_\infty \to 0$ and $K_\infty \to 0$).

In order to obtain the inverse relationship of Equation 12 we need to obtain the creep matrix $\mathbf{C}(t)$ by using Equation 3. $\mathbf{C}(t)$ is evaluated by performing a Laplace transformation of the relaxation matrix (that is Laplace transforms element by element) and evaluating its inverse:

$$\hat{\mathbf{C}}(s) = \frac{\hat{\mathbf{R}}^{-1}(s)}{s^2} \qquad (13)$$

4. Numerical implementation of the 3D fractional constitutive law

The constitutive model in equation 12 has been implemented in a user material subroutine UMAT in Abaqus/Standard. The subroutine calculates the increment of stress at the end of each increment and the Jacobian.

When the UMAT is called, the following information is available as 'Input': the stress at the beginning of the increment, the strain at the beginning of the increment and the increment of strain. We also need to have access to the history of strain i.e. the values of strain in all the increments of the analysis. We will give some details of how this is achieved in the next paragraph.

We start evaluating the direct component of stress σ_{11} at the end of increment k, which is the first component of the stress vector in Equation 12:

$$\begin{aligned}\sigma_{11,k+1} = K_\infty \varepsilon_{V,k+1} + K_\beta \Delta t^{-\beta}\left(\sum_{j=1}^{k+1}\lambda_j^{(\beta)}\varepsilon_{V,k-j+2}\right) + \frac{4}{3}G_\infty\left(\varepsilon_{11,k+1} - \frac{\varepsilon_{22,k+1}+\varepsilon_{33,k+1}}{2}\right) + \\ + \frac{4}{3}G_\alpha \Delta t^{-\alpha}\sum_{j=1}^{k+1}\lambda_j^{(\alpha)}\left(\varepsilon_{11,k-j+2} - \frac{\varepsilon_{22,k-j+2}+\varepsilon_{33,k-j+2}}{2}\right)\end{aligned} \tag{14}$$

where $\varepsilon_V = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ is the volumetric strain. The stress at the beginning of the increment (which is known when the UMAT is called) can be written as follow:

$$\begin{aligned}\sigma_{11,k} = K_\infty \varepsilon_{V,k} + K_\beta \Delta t^{-\beta}\left(\sum_{j=1}^{k}\lambda_j^{(\beta)}\varepsilon_{V,k-j+1}\right) + \frac{4}{3}G_\infty\left(\varepsilon_{11,k} - \frac{\varepsilon_{22,k}+\varepsilon_{33,k}}{2}\right) + \\ + \frac{4}{3}G_\alpha \Delta t^{-\alpha}\sum_{j=1}^{k}\lambda_j^{(\alpha)}\left(\varepsilon_{11,k-j+1} - \frac{\varepsilon_{22,k-j+1}+\varepsilon_{33,k-j+1}}{2}\right)\end{aligned} \tag{15}$$

We obtain the increment of stress by evaluating the difference between Equations 14 and 15 that will be useful to calculate the Jacobian:

$$\begin{aligned}\Delta\sigma_{11,k+1} = K_\infty \Delta\varepsilon_{V,k+1} + K_\beta \Delta t^{-\beta}\left(\sum_{j=1}^{k}\lambda_j^{(\beta)}\Delta\varepsilon_{V,k-j+2} + \lambda_{k+1}^{(\beta)}\varepsilon_{V,1}\right) + \frac{4}{3}G_\infty\left(\Delta\varepsilon_{11,k+1} - \frac{\Delta\varepsilon_{22,k+1}+\Delta\varepsilon_{33,k+1}}{2}\right) + \\ + \frac{4}{3}G_\alpha \Delta t^{-\alpha}\left[\sum_{j=1}^{k}\lambda_j^{(\alpha)}\left(\Delta\varepsilon_{11,k-j+2} - \frac{\Delta\varepsilon_{22,k-j+2}+\Delta\varepsilon_{33,k-j+2}}{2}\right) + \lambda_{k+1}^{(\alpha)}\left(\varepsilon_{11,1} - \frac{\varepsilon_{22,1}+\varepsilon_{33,1}}{2}\right)\right]\end{aligned} \tag{16}$$

where $\Delta\varepsilon_{i,k-j+2} = \varepsilon_{i,k-j+2} - \varepsilon_{i,k-j+1}$, with $i = 11, 22, 33$, and $\Delta\varepsilon_{V,k+1} = \varepsilon_{V,k+1} - \varepsilon_{V,k}$. In the UMAT, we code Equation 14; the other direct components of stress can be obtained simply by rotating indices of the strain components.

To evaluate all three direct components of the increment of stress, we define a scalar quantity related to the volumetric deformation (*TERMV*) and a three components vector(*TERMK*) each related to one direct component of strain:

$$TERMV = K_\infty \Delta\varepsilon_{V,k+1} + K_\beta \Delta t^{-\beta}\left(\sum_{j=1}^{k}\lambda_j^{(\beta)}\Delta\varepsilon_{V,k-j+2} + \lambda_{k+1}^{(\beta)}\varepsilon_{V,1}\right) \tag{17}$$

$$TERMS_j = \frac{4}{3}G_\infty \Delta\varepsilon_{jj,k+1} + \frac{4}{3}G_\alpha \Delta t^{-\alpha}\left(\sum_{j=1}^{k}\lambda_j^{(\alpha)}\Delta\varepsilon_{jj,k-j+2} + \lambda_{k+1}^{(\alpha)}\varepsilon_{jj,1}\right) \tag{18}$$

With these two quantities we are able to evaluate the direct component of stress as follows:

$$\Delta\sigma_{ll} = TERMV + TERMS_l - \frac{1}{2}\left[\sum_j TERMS_j - TERMS_l\right] \tag{19}$$

In an analogous way it is possible to compute of shear components of stress and their increments:

$$\sigma_{lm,k+1} = G_\infty \gamma_{lm,k+1} + G_\alpha \Delta t^{-\alpha} \sum_{j=1}^{k+1} \lambda_j^{(\alpha)} \gamma_{lm,k-j+2} \tag{20a}$$

$$\Delta\sigma_{lm,k+1} = G_\infty \Delta\gamma_{lm,k+1} + G_\alpha \Delta t^{-\alpha} \left(\sum_{j=1}^{k} \lambda_j^{(\alpha)} \Delta\gamma_{lm,k-j+2} + \lambda_{k+1}^{(\alpha)} \gamma_{lm,1} \right) \tag{20b}$$

Where $\Delta\gamma_{lm,k-j+2} = \gamma_{lm,k-j+2} - \gamma_{lm,k-j+1}$ and $\Delta\gamma_{lm,k+1} = \gamma_{lm,k+1} - \gamma_{lm,k}$, with $lm = 12, 13, 23$. These terms can be computed directly one by one.

At this point, from Equation 16 and 20b, we can evaluate the components of the Jacobian as:

$$\frac{\partial\Delta\sigma_{jj,k+1}}{\partial\Delta\varepsilon_{jj,k+1}} = K_\infty + K_\beta \Delta t^{-\beta} \lambda_1^{(\beta)} + \frac{4}{3} G_\infty + \frac{4}{3} G_\alpha \Delta t^{-\alpha} \lambda_1^{(\alpha)}; \qquad \lambda_1^{(\beta)} = \lambda_1^{(\alpha)} = 1 \tag{21}$$

$$\frac{\partial\Delta\sigma_{jj,k+1}}{\partial\Delta\varepsilon_{ll,k+1}} = K_\infty + K_\beta \Delta t^{-\beta} \lambda_1^{(\beta)} - \frac{2}{3} G_\infty - \frac{2}{3} G_\alpha \Delta t^{-\alpha} \lambda_1^{(\alpha)}; \qquad \lambda_1^{(\beta)} = \lambda_1^{(\alpha)} = 1 \tag{22}$$

$$\frac{\partial\Delta\sigma_{jl,k+1}}{\partial\Delta\gamma_{jl,k+1}} = G_\infty + G_\alpha \Delta t^{-\alpha} \lambda_1^{(\alpha)} \tag{23}$$

To code this terms we need two quantities; one related to the volumetric relaxation function (*TERM1*) and the other related to the deviatoric function (*TERM2*)

$$TERM1 = K_\infty + K_\beta \Delta t^{-\beta} \tag{24}$$

$$TERM2 = G_\infty + G_\alpha \Delta t^{-\alpha} \tag{25}$$

and then

$$\frac{\partial\Delta\sigma_{jj,k+1}}{\partial\Delta\varepsilon_{jj,k+1}} = TERM1 + \frac{4}{3} TERM2 \tag{26}$$

$$\frac{\partial\Delta\sigma_{jj,k+1}}{\partial\Delta\varepsilon_{ll,k+1}} = TERM1 - \frac{2}{3} TERM2 \tag{27}$$

$$\frac{\partial\Delta\sigma_{jl,k+1}}{\partial\Delta\varepsilon_{jl,k+1}} = TERM2 \tag{28}$$

The main issue in the implementation of the fractional viscoelasticity law is that we need to have access to the history of strains in order to obtain the increment of stress. To overcome this problem we store the values of the components of strain at each increment in a COMMONBLOCK. that keeps track of these values when the UMAT is called.

5. Testing of the UMAT

In order to validate the UMAT a number of simple tests have been carried out for which we are able to evaluate the analytical solutions.

Here we show a viscoelastic cube (Figure 3) subjected to creep and relaxation tests. The mechanical properties of the cube are: $K_\beta = 5x10^8$ Pa sec$^\beta$, $K_\infty = 10^9$ Pa, $G_\alpha = 3.75x10^8$ Pa sec$^\alpha$, $G_\infty = 7.5x10^8$ Pa, $\alpha = \beta = 0.3$.

In the creep test of Figure 3a the cube has one of its faces normal to the direction x constrained to prevent motion in the x direction. On the opposite face a uniform and constant tensile stress $\sigma_{xx} = \bar{\sigma} = 10$ MPa is applied in the x direction.

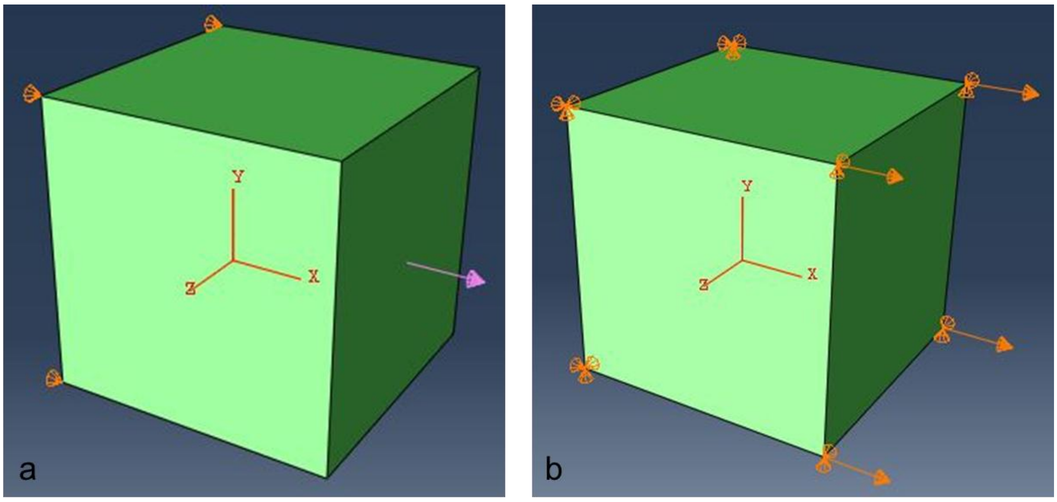

Figure 3. Viscoelastic cube for the creep test (a) and relaxation test (b).

The analytical solution is obtained by using Equation 13 and the convolution of Equation 5 as follows:

$$\varepsilon_{xx}(t) = \frac{\bar{\sigma}}{3G_\infty}\left[1 - E_\beta\left(-\frac{G_\infty}{G_\beta}t^\beta\right)\right] + \frac{\bar{\sigma}}{9K_\infty}\left[1 - E_\beta\left(-\frac{K_\infty}{K_\beta}t^\beta\right)\right] \quad (29)$$

$$\varepsilon_{yy}(t) = \varepsilon_{zz}(t) = -\frac{\bar{\sigma}}{6G_\infty}\left[1 - E_\beta\left(-\frac{G_\infty}{G_\beta}t^\beta\right)\right] + \frac{\bar{\sigma}}{9K_\infty}\left[1 - E_\beta\left(-\frac{K_\infty}{K_\beta}t^\beta\right)\right] \quad (30)$$

Figure 4 shows a comparison between the Abaqus/Standard result and the analytical solution of Equations (29) and (30); red dashed lines are responses evaluated with Abaqus/Standard with a constant time step of $0.1\,\text{sec}$, while black continuous lines represent the solutions of Equations (29) and (30). From this figure it is possible to appreciate that the numerical procedure reproduces the analytical results.

In the relaxation test in Figure 3b all of the faces of the cube but one are fixed only in the normal direction. We then apply a displacement of 1 mm to the free face (normal to the x direction), which corresponds to a strain $\varepsilon_{xx} = \bar{\varepsilon} = 0.01 = 1\%$ (see Figure 5); the displacement was applied with a linear ramp of 1 sec and then held for another 9 secs, as shown in Figure 5. With the

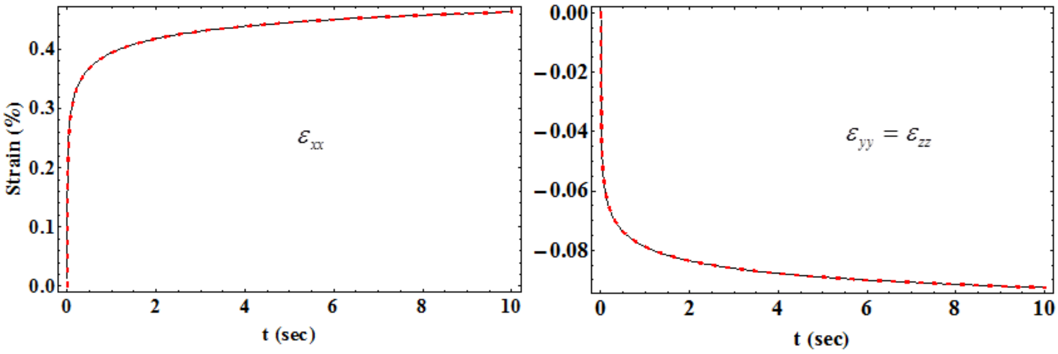

Figure 4. Comparison between analytical and Abaqus/Standard responses for creep test of the cube in Figure 3.

boundary conditions described above, $\varepsilon_{yy}(t) = \varepsilon_{zz}(t) = 0$ and all the direct components of stress are different from zero. The history of the superimposed strain can be written as follows:

$$\varepsilon_{xx}(t) = \bar{\varepsilon}\left[t\left(U(t) - U(t-1)\right) + U(t-1)\right] \tag{31}$$

where $U(\bullet)$ is the Unit-step function; then by inserting Equation (31) in Equation (12) we obtain:

$$\begin{aligned}\sigma_{xx} = \frac{\bar{\varepsilon}}{3\Gamma(2-\beta)}\Big\{ & t\left[\left(4G_\beta + 3K_\beta\right)t^{-\beta} + \left(4G_\infty + 3K_\infty\right)\Gamma(2-\beta)\right] + \\ & -(t-1)\left[\left(4G_\beta + 3K_\beta\right)(t-1)^{-\beta} + \left(4G_\infty + 3K_\infty\right)\Gamma(2-\beta)\right]U(t-1)\Big\}\end{aligned} \tag{32}$$

$$\begin{aligned}\sigma_{yy} = \sigma_{zz} = \frac{\bar{\varepsilon}}{3\Gamma(2-\beta)}\Big\{ & t\left[\left(2G_\beta - 3K_\beta\right)t^{-\beta} + \left(2G_\infty - 3K_\infty\right)\Gamma(2-\beta)\right] + \\ & +(t-1)\left[\left(-2G_\beta + 3K_\beta\right)(t-1)^{-\beta} + \left(-2G_\infty + 3K_\infty\right)\Gamma(2-\beta)\right]U(t-1)\Big\}\end{aligned} \tag{33}$$

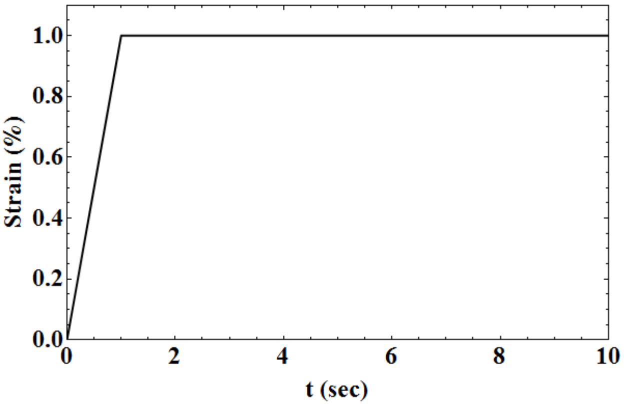

Figure 5. Strain history during the relaxation test

The analytical solution given in Equations (32) and (33) have been compared with the solution obtained by running the UMAT and it can be seen in Figure 6 that the analytical results are reproduced by the numerical calculations .

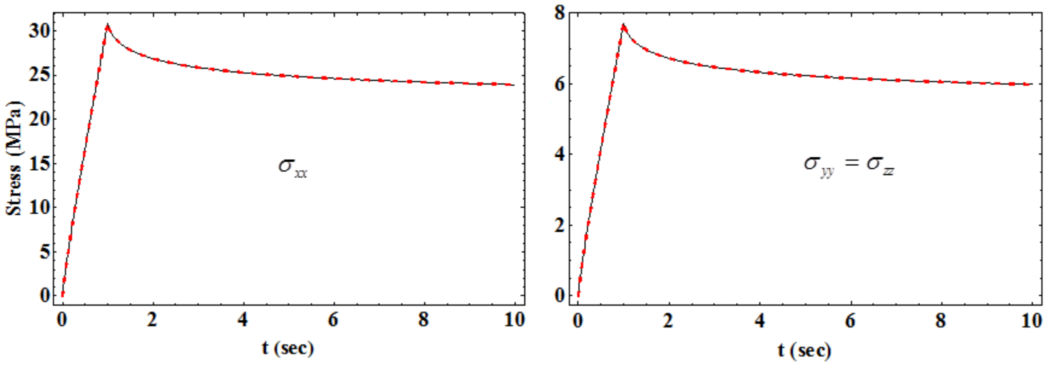

Figure 6. Comparison between analytical and Abaqus/Standard responses for relaxation test of the cube in Figure 5.

A 2D plain strain model of a viscoelastic Euler-Bernoulli beam (Figure 7) under a uniformly distributed load and constant over the time has been analysed. The beam is 5 m long, has a rectangular cross section with base 10 cm and height of 20 cm and the same material properties used of the previous examples; the beam is simply supported. Figure 7 shows the Abaqus model of the beam modelled with 100 4-noded plain strain element (CPE4) with dimension 10x10 cm.

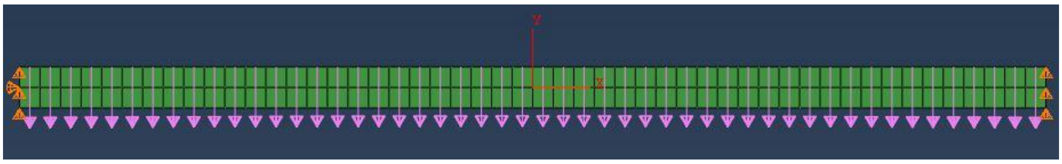

Figure 7. 2D plain strain model of the viscoelastic beam

Displacements of two points of the beam were monitored, located at 1 m (A) and 2 m (B) from the left end of the beam, respectively; Abaqus results have been compared with analytical results evaluated with the same approach of (Di Paola, 2013) as shown in Figure 8.

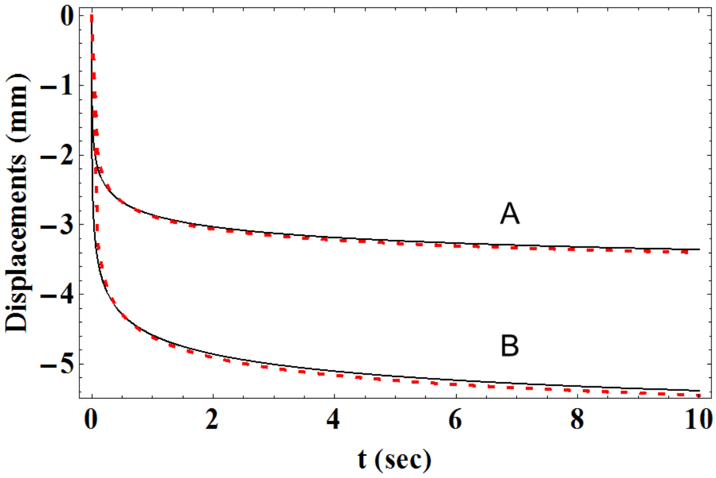

Figure 8. Comparison between analytical and Abaqus results for displacements of two points of the beam

The analytical and Abaqus results are in very good agreement; a contour plot of σ_{11} (stress in x direction) is also reported in the deformed configuration in Figure 9.

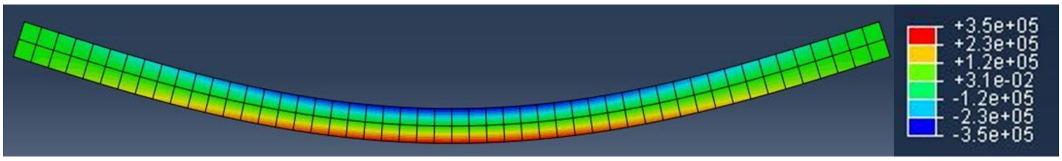

Figure 9. Contour plot of longitudinal normal stress in the deformed configuration of the beam. Values of stress in Pa

A number of other tests, such as creep tests with initial ramps and relaxation tests with a range of different boundary conditions have been carried out and they all give the same responses as the analytical solutions. These results are discussed elsewhere.

6. Conclusions

In this paper the implementation of a fractional viscoelastic constitutive model in Abaqus/Standard has been presented. This constitutive model is derived by fitting creep or relaxation data of real materials like rubbers, polymers, asphalt mixtures, biological tissues, and many others and the application of the Boltzmann superposition principle (linear viscoelasticity). Moreover it has the feature that the actual strain (or stress) at a point of the solid depends on the previous history of stress (or strain) and not only on the current stress (or strain). In order to access the history of strain, the components of strain at each increment are stored in a

COMMONBLOCK. Comparison between analytical results and results obtained with the UMAT shows the accuracy of the implementation of the fractional viscoelastic model presented here.

We believe that this novel fractional viscoelastic model simulates the viscoelastic behaviour of real material in a more realistic way. Examples of the use of the UMAT to solve a range of practical engineering problems are presented elsewhere.

7. References

1. Bagley, R.L., Torvik, P. J., "On the appearance of the fractional derivative in the behaviour of real materials", Journal of Applied Mechanics, no. 51, pp. 294-298, 1984.

2. Deseri, L., Di Paola, M., Pollaci, P., Zingales, M., "Power law hereditariness of hierarchical fractal bones", International Journal for Numerical Method in Biomedical Engineering, no 29 (12), pp. 1338-1360, 2013.

3. Di Mino, G., Airey, G., Di Paola, M., Pinnola, F. P., D'Angelo, G., Lo Presti, D., "Linear and non linear fractional hereditary constitutive laws of asphalt mixture", Journal of Civile Engineering and Managements, doi:10.3846/13923730.2014.914104, 2014.

4. Di Paola, M., Fiore, V., Pinnola, F. P., Valenza, A., "On the influence of the initial ramp for a correct definition of the parameters of fractional viscoelastic materials", Mechanics of Materials, no. 69 (1), pp. 63-70, 2014.

5. Di Paola, M., Heuer, R., Pirrotta, A., "Fractional visco-elastic Euler-Bernoulli beam", International Journal of Solids and Structures", no. 50, pp. 3505-3510, 2013.

6. Nutting, P. G., "A new general law of deformation", Journal of the Franklin Institute, no. 191, pp.679-685, 1921.

7. Podlubny, I., Fractional Differential Equations, Academic Press, 1999.

8. Scott Blair, G. W., Caffyn, J. E., "An application of the theory of quasi-properties to the treatment of anomalous stain-stress relations", The Philosophical Magazine, no. 40 (300), pp. 679-685,1949.