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Positive solutions of Dirichlet and homoclinic type for a class of singular equations

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ABSTRACT

We study a nonlinear singular boundary value problem and prove that, depending on a relationship between exponents of power terms, the problem has either solutions of Dirichlet type or homoclinic solutions. We make use of shooting techniques and lower and upper solutions.

1. Introduction and main results

In recent decades many authors have studied the solvability of singular differential equations under different boundary conditions. A wealth of general results for singular ordinary differential equations can be found in monographs such as [1] or [14]. It is worth mentioning also that in [18] the reader may get acquainted with a rich collection of singular problems, arising in the applied sciences, whose solutions illustrate a wide variety of mathematical techniques.

In the present note, we deal with a one dimensional singular problem of p-Laplacian type which, specifically, can be put in the form

$$(|u'|^{p-2}u')' = \frac{|u'|^k}{u^s} - f(t, u, u'), \tag{1}$$

where $p > 1$, k, $s > 0$, $I \subset \mathbb{R}$ is an interval and $f : I \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is positive and continuous.

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Motivation for the study of this kind of equations may be traced back at least to [11] or [3] and is given also in some more recent articles where this or similar equations are studied; we refer the reader to [5,7,19,21,22,20,16,23] and their references. Let us just mention that the problem appears in physics in connection with (possibly degenerate) parabolic equations from fluid flow theory, in particular involving non-Newtonian models.

In the articles we have mentioned, all concerning ordinary differential equations, the research is focused on the two-point boundary value problem for (1) in a finite interval $I = [0, T]$, namely the problem of finding a positive function $u(t)$, solving (1) in $(0, T)$ and satisfying the boundary conditions

$$u(0^+) = 0, \ \ u(T^-) = 0.$$

For brevity, we shall refer to such solutions as *Dirichlet type solutions*.

Some authors have also pointed out the existence of the so-called *T-periodic solutions*, that is, solutions of Dirichlet type whose derivative also vanishes at the endpoints of its domain

$$u'(0^+) = 0, \ \ u'(T^-) = 0$$

(a feature already accounted in [19,20,23]).

With the present paper we add a contribution to understanding the nature of the solutions of (1) in a number of aspects. First, we wish to extend in some way the range of powers k, s that have been considered in the literature. Second, we include new information about the appearance of the T-periodic solutions.

Finally, we intend to highlight the fact that the order relation between s and k determines the type of solutions that one can expect: roughly speaking, if $s < k$, (1) has "Dirichlet solutions", while if $I = \mathbb{R}$ and $s \geq k$, positive homoclinic solutions appear (a definition is recalled before the statement of Theorem 5).

The Dirichlet solutions will be presented, for simplicity, in case $p = 2$ only. It will be apparent that, differently from other results in the literature, we show that s and k may take any values as long as $s < k$; in particular the size of k is not restricted by p.

We wish also to point out that, taking advantage from the particular structure of our one-dimensional problem, we can adopt suitable techniques based on well known results of classical nonlinear analysis. Nevertheless let us remark that in the recent literature one can find results about either the n-dimensional Dirichlet problem, or homoclinics for n-dimensional systems, that are clearly related to those that concern us here (see [2,4,7,10,12,13,24]).

The paper is organized in such a way that Dirichlet solutions are studied through Sections 2–6 and the approach to homoclinics is done in the final section.

In order to state our results for Dirichlet solutions let us write the equation, for notational convenience, as

$$u'' = \frac{|u'|^{2\beta}}{u^{\mu}} - f(t, u, u'). \tag{2}$$

In the statements of Theorems 1–4 we assume

(H) $f : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous and

$$M_f = \sup_{[0,T]\times\mathbb{R}^+\times\mathbb{R}} f < +\infty, \quad m_f := \inf_{[0,T]\times\mathbb{R}^+\times\mathbb{R}} f > 0.$$

Theorem 1. *Let $2\beta > \mu$. Then equation (2) has a solution of Dirichlet type on the interval $(0, T)$.*

Recall that a T-periodic positive solution is a function $u \in C^2(0, T) \cap C^1([0, T])$ such that $u(t) > 0$ for $t \in (0, T)$, satisfying (2) in $(0, T)$ and $u(0+) = 0 = u(T-)$, $u'(0+) = 0 = u'(T-)$.

Theorem 2. *If $\mu = \beta$, then equation (2) has a T-periodic solution under one of the following conditions:* (i) $\beta \geq 1$, *or* (ii) $\beta \in (0,1)$ *and*

$$M_f \leq (2\beta)^{\frac{\beta}{1-\beta}}(1-\beta). \tag{3}$$

Theorem 3. *If $\beta > \mu \geq 1$, then equation (2) has a T-periodic solution.*

Theorem 4. *If $2\beta > \mu > \beta$, then equation (2) has a T-periodic solution under one of the following conditions:* (i) $\mu \geq 1$, *or* (ii) $\mu \in (0,1)$ *and either*

$$M_f \leq \min\left\{\frac{8}{T^2}, (2\beta)^{\frac{\beta}{1-\beta}}(1-\beta)\right\} \tag{4}$$

or

$$T \leq 2\sqrt{2}M_f^{-\frac{1+\mu-2\beta}{2(\mu-\beta)}}\left(\frac{2\beta(1-\beta)}{1-\mu}\right)^{\frac{\beta}{2(\mu-\beta)}}(1-\beta)^{\frac{1-\beta}{2(\mu-\beta)}}. \tag{5}$$

Remark 1. Theorems 2 and 4 include the case when $\mu \in (0,1)$, as far as we know this is the first work where an equation put in the form (2) is considered with a weak singularity. We refer the reader to [6,15,17] to review works dealing with this type of singularity. In Theorem 3 an analogous result cannot be proven (see Remark 3).

Remark 2. The existence of T-periodic solutions to the equation (2) when $\mu \in (0,1)$ and $\beta \leq \mu/(1-\mu)$ seems to be a more difficult problem, even in the autonomous case (see Section 5). The analysis of this case can be a nice open problem.

Finally we state a result concerning homoclinic solutions of (1). By a (positive) homoclinic solution of (1) we mean a solution $u(t)$ defined in $\mathbb{R}$ such that $u(t) > 0\ \forall t \in \mathbb{R}$ and

$$\lim_{t\to\pm\infty} u(t) = 0, \qquad \lim_{t\to\pm\infty} u'(t) = 0.$$

Theorem 5. *Let $1 < p < \infty$, $k > 1$ and $s \geq k$, and assume in addition*

(i) *there exist positive constants α, γ, r and a positive, continuous function $\beta : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, with $0 < \alpha \leq \beta$, such that*

$$0 < r < k \tag{6}$$

and

$$\alpha \leq f(t,x,y) \leq \beta(t,x) + \gamma|y|^r, \tag{7}$$

for every $(t,x,u) \in \mathbb{R} \times [0,+\infty) \times \mathbb{R}$ and moreover

$$\sup_{t\in\mathbb{R}} \max_{x\in[0,M]} \beta(t,x) < \infty \tag{8}$$

for every $M > 0$.

Then for every $M > 0$ equation (1) admits at least one positive homoclinic solution u such that $\max_{t\in\mathbb{R}} u = M$.

2. The method of lower and upper solutions

We start by recalling the meaning of lower and upper solutions to a general equation

$$u'' = h(t, u, u'), \tag{9}$$

where $h \in C([a, b] \times D \times \mathbb{R}; \mathbb{R})$, with $D \subseteq \mathbb{R}$ an open interval. The following definition is a particular case of the definitions of lower and upper functions introduced in [8] (see also [9]).

Definition 1. The continuous function $\sigma : (a, b) \to D$ is said to be a lower (upper) solution to equation (9) if, for some $a < t_1 < \ldots < t_n < b$, $\sigma \in C^2((a, b) \setminus \{t_1, \ldots, t_n\}; D)$, there exist finite limits $\sigma(a+), \sigma(b-), \sigma'(t_i+)$, and $\sigma'(t_i-)$, $i = 1, \ldots, n$, such that

$$\sigma'(t_i-) \le \sigma'(t_i+) \quad (\text{resp. } \sigma'(t_i-) \ge \sigma'(t_i+)), \qquad i = 1, \ldots, n,$$

and

$$\sigma'' \ge h(t, \sigma, \sigma') \quad (\text{resp. } \sigma'' \le h(t, \sigma, \sigma')) \qquad \text{for } t \in (a, b) \setminus \{t_1, \ldots, t_n\}.$$

The following lemma deals with the existence of a solution to equation (9) satisfying the boundary conditions

$$u(a+) = c_1, \qquad u(b-) = c_2. \tag{10}$$

The result is a simple modification of the Scorza-Dragoni Theorem and its proof can be found in [9].

Lemma 1. *Assume $D = \mathbb{R}$. Let σ_1 and σ_2, respectively, lower and upper solutions to equation (9) such that*

$$\sigma_1(t) \le \sigma_2(t) \qquad \text{for } t \in (a, b), \tag{11}$$

and

$$|h(t, x, y)| \le K \qquad \text{for } t \in (a, b), \quad \sigma_1(t) \le x \le \sigma_2(t), \quad y \in \mathbb{R},$$

where $K > 0$. Let $c_1 \in [\sigma_1(a+), \sigma_2(a+)]$ and $c_2 \in [\sigma_1(b-), \sigma_2(b-)]$. Then, problem (9)–(10) has a solution $u \in C^2((a, b); \mathbb{R})$ satisfying

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \qquad \text{for } t \in (a, b). \tag{12}$$

We are now in a position to prove the following result for equation (2).

Proposition 1. *Let $\sigma_1 > 0$ and σ_2 be lower and upper solutions to equation (2) such that (11) holds with $a = 0$, $b = T$, and*

$$\sigma_1(0+) = 0 = \sigma_1(T-), \qquad \sigma_2(0+), \sigma_2(T-) \ge 0$$

hold. Then, equation (2) has a solution $u \in C^2((0, T); \mathbb{R}^+)$ satisfying (12) (with $a = 0$, $b = T$) and $u(0+) = 0 = u(T-)$.

Proof. Consider the equation

$$u'' = \chi(u')\left[\frac{|u'|^{2\beta}}{u^{\mu}} - f(t,u,u')\right], \tag{13}$$

where

$$\chi(y) = \begin{cases} 1, & \text{for } |y| \le \rho_1, \\ 2 - \frac{|y|}{\rho_1}, & \text{for } \rho_1 < |y| < 2\rho_1, \\ 0 & \text{for } 2\rho_1 \le |y|, \end{cases}$$

and $\rho_1 := TM_f + \|\sigma_1\|_{C^1} + \|\sigma_2\|_{C^1}$ (we denote the C^1-norm by $\|\cdot\|_{C^1} := \|\cdot\|_\infty + \|\cdot'\|_\infty$). Define $\widetilde{\sigma}_{1n} \equiv \sigma_1|_{[t_{1n},t_{2n}]}$. Here, $(t_{1n})_{n\in\mathbb{N}} \subseteq (0,T/2)$, $(t_{2n})_{n\in\mathbb{N}} \subseteq (T/2,T)$ are sequences of points satisfying

$$t_{1n} \searrow 0, \qquad t_{2n} \nearrow T,$$

and verifying

$$\sigma_1'(t_{1n}) \ge 0 \ge \sigma_1'(t_{2n}). \tag{14}$$

Observe that $\widetilde{\sigma}_{1n}$ is a lower solution of (13) in the interval $[t_{1n}, t_{2n}]$ for any $n \in \mathbb{N}$. Since σ_2 is also an upper solution of (13), by Lemma 1, for any $n \in \mathbb{N}$ equation (13) has a solution u_n defined on $[t_{1n}, t_{2n}]$ such that

$$u_n(t_{1n}) = \sigma_1(t_{1n}), \qquad u_n(t_{2n}) = \sigma_1(t_{2n}), \tag{15}$$

$$\sigma_1(t) \le u_n(t) \le \sigma_2(t) \qquad \text{for } t \in [t_{1n}, t_{2n}]. \tag{16}$$

Furthermore, in view of (14), (15) and (16) one observes that

$$u_n'(t_{1n}) \ge 0 \ge u_n'(t_{2n}). \tag{17}$$

The proof will be completed by checking three claims.

Claim 1. *u_n is a solution of (2) on $[t_{1n}, t_{2n}]$. We notice that*

$$u_n'' \ge -f(t, u_n, u_n') \qquad \textit{for } t \in [t_{1n}, t_{2n}].$$

Taking into account (17) one easily proves that

$$\max_{t\in[t_{1n},t_{2n}]} |u_n'(t)| < \rho_1 \tag{18}$$

so that, indeed, u_n is solution of (2) in the interval $[t_{1n}, t_{2n}]$.

Claim 2. *$u_n \to u$ uniformly on every compact interval of $[0,T]$. Moreover, u is solution of (2) in the interval $(0,T)$. Let $[\alpha, \delta] \subseteq (0,T)$ be a compact interval. We take $n_0 \in \mathbb{N}$ sufficiently large such that $t_{1n} < \alpha$, $t_{2n} > \delta$ for any $n \ge n_0$. It is obvious that*

$$\max_{t\in[\alpha,\delta]} u_n(t) < \rho_1, \qquad \max_{t\in[\alpha,\delta]} |u_n'(t)| < \rho_1.$$

Moreover,

$$|u_n''| \leq \frac{\rho_1^{2\beta}}{\sigma_{1*}^{\mu}} + \|f\|_\infty \qquad \textit{for } t \in [\alpha, \delta],$$

where $\sigma_{1*} := \min_{t\in[\alpha,\delta]} \sigma_1(t)$. *According to the Arzelà–Ascoli Theorem we can assume without loss of generality that*

$$u_n \to u \qquad \textit{uniformly on } [\alpha, \delta],$$
$$u_n' \to v \qquad \textit{uniformly on } [\alpha, \delta].$$

Now, it is standard to verify that $u \in C^2([\alpha, \delta]; \mathbb{R})$ *with* $u' = v$ *and it is a solution to* (2). *This concludes the proof of the Claim.*

Claim 3. $u(0+) = 0 = u(T-)$. *With respect to* (15) *and* (18), *for every* $n \in \mathbb{N}$ *we have*

$$|u_n(t) - \sigma_1(t_{1n})| = \left|\int_{t_{1n}}^{t} u_n'(s)ds\right| \leq \rho_1|t - t_{1n}|,$$

$$|u_n(t) - \sigma_1(t_{2n})| = \left|\int_{t_{2n}}^{t} u_n'(s)ds\right| \leq \rho_1|t - t_{2n}|,$$

for $t \in [t_{1n}, t_{2n}]$. *Hence, taking limits as* $n \to +\infty$ *in the last inequalities one obtains that*

$$|u(t)| \leq \rho_1 \min\{t, T - t\} \qquad \textit{for } t \in (0, T).$$

Therefore, $u(0+) = 0 = u(T-)$.

The proof follows immediately from Claims 2 and 3. □

3. Periodic solutions

Throughout this section we shall assume that the equation (2) has a solution u defined on $(0, T)$ satisfying $u(0+) = 0 = u(T-)$. The goal of this section consists in studying under what conditions we can ensure that u is T-periodic.

The lemma below shows that if the singularity of equation (2) is "strong" (i.e., $\mu \geq 1$), then the solutions of Dirichlet type defined on the interval $(0, T)$ satisfy $u'(0+) = 0 = u'(T-)$.

Proposition 2. *If* $\mu \geq 1$, *then* $u'(0+) = 0 = u'(T-)$.

Proof. First we point out that u has derivatives at the endpoints of the interval $[0, T]$. Since there exist $t_{1n} \to 0^+$ and $t_{2n} \to T^-$ such that $u'(t_{1n}) \geq 0$ and $u'(t_{2n}) \leq 0$ we see that $\frac{|u'(s)|^{2\beta}}{u^\mu(s)}$ is integrable and

$$\int_0^T \frac{|u'(s)|^{2\beta}}{u^\mu(s)} ds \leq TM_f < +\infty. \tag{19}$$

Hence $u'(0)$ and $u'(T)$ exist. Assume now, for instance, without loss of generality, that $u'(0) > 0$. Thus there exists $t_u > 0$ such that

$$2u'(0) > u'(t) > \frac{u'(0)}{2} \qquad \text{for } t \in [0, t_u]. \tag{20}$$

Hence, the function u has inverse on $[0, t_u]$ and moreover

$$\frac{u'(0)}{2} < u'(u^{-1}(y)) < 2u'(0) \qquad \text{for } y \in [0, u(t_u)] \tag{21}$$

holds. Therefore, according to (20) and (21), the following computations can be easily verified:

$$\begin{aligned}\int_0^T \frac{|u'(s)|^{2\beta}}{u^\mu(s)} ds &\geq \int_0^{t_u} \frac{|u'(s)|^{2\beta}}{u^\mu(s)} ds \\ &\geq \left(\frac{u'(0)}{2}\right)^{2\beta} \int_0^{t_u} \frac{ds}{u^\mu(s)} \\ &= \left(\frac{u'(0)}{2}\right)^{2\beta} \int_0^{t_u} \frac{u'(s)}{u'(s)u^\mu(s)} ds \\ &= \left(\frac{u'(0)}{2}\right)^{2\beta} \int_0^{u(t_u)} \frac{dy}{y^\mu u'(u^{-1}(y))} \\ &> \left(\frac{u'(0)}{2}\right)^{2\beta} \frac{1}{2u'(0)} \int_0^{u(t_u)} \frac{dy}{y^\mu}.\end{aligned}$$

Thus, because $\displaystyle\int_0^{u(t_u)} \frac{dy}{y^\mu} = +\infty$, the previous inequalities group contradicts (19).

The proof of relation $u'(T-) = 0$ is identical and it will be omitted. □

Solving the problem posed in this section when $\mu \in (0, 1)$ becomes a difficult task and requires a deeper treatment. There are examples proving that under this assumption solutions of Dirichlet type may exist which are not periodic.

4. Construction of lower and upper solutions

We now describe the construction of lower and upper solutions to the equation (2). Throughout this subsection the equation

$$u'' = \frac{|u'|^{2\beta}}{u^\mu} - \alpha, \tag{22}$$

where $\alpha > 0$, will be considered. We take advantage of the fact that the equation is autonomous and perform a change of variables that leads to a first order differential equation. The information obtained in this way will be used to construct lower (or upper) solutions to equation (2).

Given $M > 0$, the classical theory of the Cauchy problem ensures that the solution of

$$u'' = \frac{|u'|^{2\beta}}{u^\mu} - \alpha, \qquad u(0) = M, \qquad u'(0) = 0 \tag{23}$$

is even and it is defined in some interval $(-\tau_\alpha(M), \tau_\alpha(M))$ and $u(-\tau_\alpha(M)^+) = 0 = u(\tau_\alpha(M)^-)$. To better justify the feature of $\tau_\alpha(M)$, we take advantage of the autonomous character of the problem (23) to reduce it to a first order Cauchy problem. In fact, observing that the solution $t \mapsto u(t)$ is strictly increasing, and therefore invertible, in $(-\tau_\alpha(M), 0]$, we infer that u'^2 may be written as a function of u, say

$$u'(t)^2 = \psi(u(t)) \qquad \text{for } t \in (-\tau_\alpha(M), 0], \tag{24}$$

where $\psi = \psi(u)$ solves the following first order problem in $(0, M]$:

$$\psi' = 2\left[\frac{\psi^\beta(u)}{u^\mu} - \alpha\right], \qquad \psi(M) = 0, \tag{25}$$

the letter u now denoting the independent variable. Conversely, the solution of (25) yields the restriction of the solution $u(t)$ of (23) to $(-\tau_\alpha(M), 0]$ by solving $u' = \sqrt{\psi(u)}$ with the initial condition $u(0) = M$. (We remark that, since the square root is an increasing function, this equation is uniquely solvable backwards.)

The following classical proposition holds. Its proof is omitted here.

Proposition 3. *Let $f : (a, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $y_1, y_2 \in C^1((a, b])$ such that:*

(i) $y_1(b) = y_2(b)$,
(ii) $y_2'(u) = f(t, y_2(u))$ *for* $u \in (a, b]$,
(iii) $y_1'(u) > f(t, y_1(u))$ *for* $u \in (a, b]$.

Then, $y_1(u) < y_2(u)$ *for* $u \in (a, b)$.

Lemma 2. *The solution of (25) is defined on the interval $(0, M]$. In particular, if $\mu < 2\beta$ equation (22) has a solution of Dirichlet type in some interval $(-\tau_\alpha(M), \tau_\alpha(M))$.*

Proof. The existence of a unique maximal positive solution $\psi_{\alpha,M}$ of (25) defined in the interval $(0, M]$ is established by applying the classical theory of the Cauchy problem. A further reasoning based on the following direct observation

$$\psi'_{\alpha,M}(u) \begin{cases} < 0, & \text{if } \psi_{\alpha,M}(u) < \alpha^{\frac{1}{\beta}} u^{\frac{\mu}{\beta}}, \\ = 0, & \text{if } \psi(u)_{\alpha,M} = \alpha^{\frac{1}{\beta}} u^{\frac{\mu}{\beta}}, \\ > 0 & \text{if } \psi(u)_{\alpha,M} > \alpha^{\frac{1}{\beta}} u^{\frac{\mu}{\beta}}, \end{cases} \tag{26}$$

shows that there exists (a unique) $u_M \in (0, M)$ such that

$$(\psi_{\alpha,M}(u) - \alpha^{1/\beta} u^{\mu/\beta})(u_M - u) > 0 \qquad \text{for } u \in (0, M) \setminus \{u_M\} \tag{27}$$

Indeed, (25) shows that $\psi_{\alpha,M}(u)$ cannot vanish in $(0, M)$ and (26) implies that $\psi_{\alpha,M}(u) < \alpha^{1/\beta} u^{\mu/\beta}$ $\forall u \in (0, M]$ cannot hold. Hence, from (26) again we infer that the equation $\psi_{\alpha,M}(u) = \alpha^{1/\beta} u^{\mu/\beta}$ has exactly one solution in $(0, M)$.

In particular, we stress that from (27) it follows that

$$\psi_{\alpha,M}(u) \text{ is concave in } (u_M, M). \tag{28}$$

Indeed, a direct computation shows that

$$\psi''_{\alpha,M}(u) = 2\frac{u^{\mu-1}\psi^{\beta-1}_{\alpha,M}(u)}{u^{2\mu}} \left(\beta u \psi'_{\alpha,M}(u) - \mu \psi_{\alpha,M}(u)\right) < 0 \qquad \text{for } u \in (u_M, M).$$

Finally, put

$$\tau_\alpha(M) := \int_0^M \frac{du}{\sqrt{\psi_{\alpha,M}(u)}}. \tag{29}$$

First of all, observe that (28) leads to

$$\psi_{\alpha,M}(u) \geq \frac{\psi_{\alpha,M}(u_M)}{M-u_M}(M-u) \qquad \text{for } u \in [u_M, M]. \tag{30}$$

Hence, from (27) and (30), having in mind that $\mu < 2\beta$, one has

$$\begin{aligned}\tau_\alpha(M) &\leq \frac{1}{\sqrt{\alpha^{1/\beta}}}\int_0^{u_M}\frac{du}{u^{\mu/2\beta}} + \sqrt{\frac{M-u_M}{\psi_{\alpha,M}(u_M)}}\int_{u_M}^M \frac{du}{\sqrt{M-u}} \\ &= \frac{1}{\sqrt{\alpha^{1/\beta}}}\frac{2\beta}{2\beta-\mu}u_M^{(2\beta-\mu)/2\beta} + 2\frac{M-u_M}{\sqrt{\psi_{\alpha,M}(u_M)}} < +\infty.\end{aligned}$$

The above estimate of $\tau_\alpha(M)$ concludes the proof. □

Now let us estimate the (time-map) function $\tau_\alpha(\cdot)$ associated to the problem (25). We set the function $\tau_\alpha : (0,+\infty) \to (0,+\infty)$, $\tau_\alpha(M)$ being as in (29) for all $M > 0$.

Lemma 3. $\lim_{M\to+\infty} \tau_\alpha(M) = +\infty$; $\lim_{M\to 0^+} \tau_\alpha(M) = 0$.

Proof. We integrate (25) over the interval $[u, M] \subseteq (0, M]$ in order to obtain

$$-\psi_{\alpha,M}(u) = 2\int_u^M \left[\frac{\psi^\beta_{\alpha,M}(s)}{s^\mu} - \alpha\right] ds.$$

Consequently,

$$\psi_{\alpha,M}(u) < 2\alpha(M-u) \qquad \text{for } u \in (0, M].$$

Hence

$$\tau_\alpha(M) = \int_0^M \frac{du}{\sqrt{\psi_{\alpha,M}(u)}} > \frac{1}{\sqrt{2\alpha}}\int_0^M \frac{du}{\sqrt{M-u}}, \tag{31}$$

whence $\tau_\alpha(M) \to +\infty$ as $M \to +\infty$.

We continue now with the second part of the proof. This task will be divided into two cases:

Case 1. $0 < \mu \leq \beta$. Let us fix $\gamma \in (0, \beta)$ and we define the auxiliary function

$$\eta_M(u) := \frac{u(M-u)}{M^{\gamma/\beta}} \qquad \text{for } u \in [0, M].$$

By a direct calculation we have

$$\int_0^M \frac{du}{\sqrt{u(M-u)}} = \pi. \tag{32}$$

Now we shall prove that there exists $\bar{M} < 1$ such that for every $M \in (0, \bar{M})$,

$$\eta_M(u) < \psi_{\alpha,M}(u) \qquad \text{for } u \in (0, M). \tag{33}$$

Indeed, observe that there exists $\bar{M} \in (0,1)$ such that

$$2M^{2\beta-\mu-\gamma} + M^{1-\gamma/\beta} < 2\alpha \qquad \text{for } M < \bar{M}.$$

At this point, given $M < \bar{M}$, a direct computation shows that, for every $u \in (0, M)$ one has

$$\begin{aligned}
\eta_M'(u) &= M^{1-\gamma/\beta} - 2\frac{1}{M^{\gamma/\beta}}u > -M^{1-\gamma/\beta} > 2\left[M^{2\beta-\mu-\gamma} - \alpha\right] \\
&= 2\left[\frac{1}{M^\gamma}M^{\beta-\mu}M^\beta - \alpha\right] > 2\left[\frac{1}{M^\gamma}\frac{u^\beta(M-u)^\beta}{u^\mu} - \alpha\right] \\
&= 2\left[\frac{\eta_M(u)^\beta}{u^\mu} - \alpha\right].
\end{aligned}$$

The inequality (33) follows directly from Proposition 3. Finally, for $M \in (0, \bar{M})$, according to (32), (33) it follows that

$$\tau_\alpha(M) < M^{\frac{\gamma}{2\beta}} \int_0^M \frac{du}{\sqrt{u(M-u)}} = \pi M^{\frac{\gamma}{2\beta}}.$$

Thus $\tau_\alpha(M) \to 0$ as $M \to 0$. The proof is complete in this case.

Case 2. $\beta < \mu < 2\beta$. Let us now fix $r > 0$ such that $1 < \mu/\beta < r < 2$ and define the auxiliary function

$$\zeta_M(u) := \frac{u^r(M-u)}{M} \qquad \text{for } u \in [0, M].$$

Then

$$\int_0^M \frac{du}{\sqrt{u^r(M-u)}} = kM^{\frac{1-r}{2}}, \tag{34}$$

where $k > 0$ is independent of M. This follows easily by the substitution $u = Mv$, and

$$k := \int_0^1 \frac{dv}{\sqrt{v^r(1-v)}} < +\infty.$$

Following the same arguments of Case 1, now we shall prove that there exists $\bar{M} \in (0,1)$ such that for every $M \in (0, \bar{M})$,

$$\zeta_M(u) < \psi_{\alpha,M}(u) \qquad \text{for } u \in (0, M). \tag{35}$$

In fact, there exists $\bar{M} \in (0,1)$ such that

$$2M^{r\beta-\mu} + (r+1)M^{r-1} < 2\alpha \qquad \text{for } M < \bar{M}.$$

For a fixed $M < \bar{M}$ one has

$$\begin{aligned}
\zeta_M'(u) &= ru^{r-1} - (r+1)\frac{u^r}{M} > -(r+1)M^{r-1} > 2\left[M^{r\beta-\mu} - \alpha\right] \\
&= 2\left[\frac{1}{M^\beta}M^{r\beta-\mu}M^\beta - \alpha\right] \\
&> 2\left[\frac{1}{M^\beta}u^{r\beta-\mu}(M-u)^\beta - \alpha\right] \\
&= 2\left[\frac{1}{M^\beta}\frac{u^{r\beta}(M-u)^\beta}{u^\mu} - \alpha\right] \\
&= 2\left[\frac{\zeta_M(u)^\beta}{u^\mu} - \alpha\right].
\end{aligned}$$

The inequality (35) follows directly from Proposition 3. Finally, for $M \in (0, \bar{M})$, according to (34), (35) it follows that

$$\tau_\alpha(M) < M^{1/2}\int_0^M \frac{du}{\sqrt{u^r(M-u)}} = kM^{1-\frac{r}{2}}.$$

Thus $\tau_\alpha(M) \to 0$ as $M \to 0$. The proof is complete. □

Using Lemma 3 and the continuity of the map $\tau_\alpha(M)$ we can state the following assertion.

Corollary 1. *Im* $\tau_\alpha = (0, +\infty)$.

The next step is devoted to compare the time maps associated to the following problems

$$\psi' = 2\left[\frac{\psi^\beta}{u^\mu} - \alpha_i\right], \qquad \psi(M_i) = 0, \qquad \text{for } i = 1, 2;$$

assuming that $\alpha_2 > \alpha_1$.

Lemma 4. $\tau_{\alpha_1}(M) > \tau_{\alpha_2}(M)$ *for* $M := M_1 = M_2$.

Proof. Observing that $\psi'_{\alpha_2,M}(M) < \psi'_{\alpha_1,M}(M)$, from Proposition 3 follows that $\psi_{\alpha_2,M}(\cdot) > \psi_{\alpha_1,M}(\cdot)$ on the interval $(0, M)$. Hence,

$$\tau_{\alpha_2}(M) = \int_0^M \frac{du}{\sqrt{\psi_{\alpha_2,M}(u)}} < \int_0^M \frac{du}{\sqrt{\psi_{\alpha_1,M}(u)}} = \tau_{\alpha_1}(M). \quad \square$$

Lemma 5. *For every* $M_1 > 0$ *there exists* $M_2 > M_1$ *such that* $\tau_{\alpha_1}(M_1) = \tau_{\alpha_2}(M_2)$. *Moreover, the inequality*

$$\psi_{\alpha_2,M_2}(u) > \psi_{\alpha_1,M_1}(u) \qquad \text{for } u \in (0, M_1) \tag{36}$$

holds.

Proof. Given $M_1 > 0$, Lemma 4 assures that $\tau_{\alpha_2}(M_1) < \tau_{\alpha_1}(M_1)$. In view of Lemma 3, $\lim_{M\to+\infty} \tau_{\alpha_2}(M) = +\infty$. Hence, by the continuity of the function $\tau_{\alpha_2}(\cdot)$ one has that the set $\tau_{\alpha_2}([M_1, +\infty))$ is an unbounded interval containing $\tau_{\alpha_1}(M_1)$ in its interior. This leads to the existence of $M_2 > M_1$ such that $\tau_{\alpha_2}(M_2) = \tau_{\alpha_1}(M_1)$.

Now let us prove that (36) holds. Following the same arguments as in the proof of Lemma 2, if we put $\Lambda = \{\omega \in (0, M_1) : \psi_{\alpha_2,M_2}(u) \geq \psi_{\alpha_1,M_1}(u)$ for $u \in (\omega, M_1]\}$, as a consequence of Proposition 3 one can verify that

$$\Lambda = (0, M_1). \tag{37}$$

Moreover, if by contradiction there exists $\bar{u} \in (0, M)$ such that $\psi_{\alpha_2,M_2}(\bar{u}) = \psi_{\alpha_1,M_1}(\bar{u})$, then from $\alpha_1 < \alpha_2$ one deduces that $\psi'_{\alpha_2,M_2}(\bar{u}) < \psi'_{\alpha_1,M_1}(\bar{u})$. Hence, there exists $\omega \in (\bar{u}, M)$ such that $\psi_{\alpha_2,M_2}(u) < \psi_{\alpha_1,M_1}(u)$ for every $u \in (\bar{u}, \omega)$, in contradiction with (37). □

Combining Lemma 2 and Corollary 1 and introducing a translation of time, the following assertion is obtained.

Proposition 4. *Assume that $\mu < 2\beta$. For any $T > 0$, equation (22) has a solution of Dirichlet type in the interval $(0, T)$. That solution is symmetric about the midpoint of the interval.*

We close this section with some comments on the construction of lower (also upper) solutions to equation (2) based on Proposition 4.

Corollary 2. *There exists a lower solution σ_1 (resp. an upper solution σ_2) to the equation (2) such that $\sigma_1(0+) = 0 = \sigma_1(T-)$ (resp. $\sigma_2(0+) = 0 = \sigma_2(T-)$).*

Proof. Observe that σ_1 can be defined as the solution of Dirichlet type defined in the interval $(0, T)$ for the equation (22) with $\alpha = m_f$ (see Proposition 4). Analogously we construct σ_2, the only difference being to consider $\alpha = M_f$ instead of $\alpha = m_f$. □

Next we investigate the well-ordering of the lower and upper solutions constructed in Corollary 2 i.e., letting σ_1 and σ_2, respectively, be the lower and upper solution of (2) connected to the equation (25) with $\alpha_1 = m_f$, resp. with $\alpha_2 = M_f$ (see Corollary 2), we want to know whether $\sigma_1 \leq \sigma_2$ on the interval $(0, T)$. For this purpose we denote by $\tau_{\alpha_1}(\cdot)$ and $\tau_{\alpha_2}(\cdot)$, respectively, their associated time maps. The lemma below provides the desired order.

Proposition 5. *With the above notation the inequality*

$$\sigma_1(t) \leq \sigma_2(t) \qquad \textit{for } t \in (0, T) \tag{38}$$

holds.

Proof. Notice that there is no loss of generality in assuming that $\alpha_1 < \alpha_2$ (otherwise the proof is trivial). By Corollary 1 there exists $M_1 > 0$ such that $\tau_{\alpha_1}(M_1) = T/2$. From Lemma 5 there exists $M_2 > M_1$ such that $\tau_{\alpha_2}(M_2) = \tau_{\alpha_1}(M_1)$ and (36) holds. Hence, according to the discussion done in this section with respect to the relation between σ_i and the solutions of the problem (25) (remember that $\alpha_1 = m_f$ and $= \alpha_2 = M_f$), we have the estimate $\sigma_2(T/2) = M_2 > M_1 = \sigma_1(T/2)$. If (38) does not hold, we can assume without loss of generality that there exists $t^* \in (0, T/2)$ such that $0 < \sigma_2(t^*) = \sigma_1(t^*) < M_1$. From (36) remember that $\psi_{\alpha_2,M_2}(u) > \psi_{\alpha_1,M_1}(u)$ for all $u \in (0, \sigma_1(t^*))$. Hence, we achieve the contradiction

$$t^* = \int_0^{\sigma_1(t_*)} \frac{ds}{\sqrt{\psi_{\alpha_1,M_1}(s)}} > \int_0^{\sigma_2(t_*)} \frac{ds}{\sqrt{\psi_{\alpha_2,M_2}(s)}} = t^*.$$

The proof is complete. □

5. Periodic solutions in the autonomous case

In this section we shall analyze equation (22) with respect to the existence of T-periodic solutions. This will be done by exploring the fact that solutions of this type are related to solutions of (25) such that $\psi(0+) = 0$ (see (24) in Section 4).

5.1. *Case $\mu = \beta$*

This is the easiest case, since our problem (25) then concerns a first order homogeneous equation:

$$\psi' = 2\left[\left(\frac{\psi}{u}\right)^{\beta} - \alpha\right], \qquad \psi(M) = 0.$$

According to the elementary technique applicable to such equations, we introduce a new dependent variable z by the transformation $\psi_{\alpha,M}(u) = uz(u)$ and we obtain

$$z' = \frac{2z^\beta - z - 2\alpha}{u}, \qquad z(M) = 0,$$

and we find explicitly

$$u = M \exp\left[\int_0^{z(u)} \frac{ds}{2s^\beta - s - 2\alpha}\right]. \tag{39}$$

Lemma 6. *Assume that $\mu = \beta$. Under one of the following conditions*

1. $\beta \geq 1$,
2. $(2\beta)^{\frac{\beta}{1-\beta}}(1-\beta) \geq \alpha$,

it follows that $\psi_{\alpha,M}(0+) = 0$. In other words, the equation (22) possesses a T-periodic solution.

Proof. Consider the function

$$\xi : [0, +\infty) \to \mathbb{R}, \qquad \xi(x) := 2x^\beta - x - 2\alpha.$$

Observe that under the hypotheses *1.* or *2.* there exists $r > 0$ such that $\xi(r) = 0$ and $\xi(x) < 0$ for all $x \in (0, r)$. In view of (39) we infer that $z(0+) = r$, whence it follows that $\psi_{\alpha,M}(0+) = 0$. □

5.2. *Case $\beta > \mu$*

In this case examples may be given which show that T-periodic solutions are not always available. The following remark better explains this circumstance.

Remark 3. Assume that $\beta > \mu/(1-\mu)$, $\mu \in (0,1)$. Then the solution of (25) is such that $\psi_{\alpha,M}(0+) > 0$. In other words, the equation (22) does not possess T-periodic solutions. Indeed, if $\psi_{\alpha,M}(0+) = 0$ then there exists an interval $(0, u^*)$ where $\psi_{\alpha,M}(u) < 1$. Hence, in such an interval one has

$$\begin{aligned}\psi_{\alpha,M}(u) = \psi_{\alpha,M}(u) - \psi_{\alpha,M}(0^+) &= 2\int_0^u \frac{\psi^\beta_{\alpha,M}(s)}{s^\mu}ds - 2\alpha u \\ &< 2\int_0^u \frac{ds}{s^\mu} \\ &= \frac{2}{1-\mu}u^{1-\mu}.\end{aligned}$$

On the other hand, in view of (27), one has

$$\alpha^{1/\beta}u^{\mu/\beta} < \psi_{\alpha,M}(u) < \frac{2}{1-\mu}u^{1-\mu}$$

for all $u \in (0, u^*)$ small enough, a contradiction since $\mu/\beta < 1-\mu$.

Nevertheless, after combining Propositions 2 and 4, the result below follows easily.

Lemma 7. *Assume that $1 \le \mu < \beta$. Then (22) has a T-periodic solution.*

5.3. Case $\beta < \mu < 2\beta$

In this case (25) will be compared with a homogeneous problem to be solved for $w = w(u)$:

$$w' = 2\left[\left(\frac{w}{u}\right)^\beta - \alpha\right], \qquad w(M) = 0,$$

using the results from Subsection 5.1.

Lemma 8. *Assume that $\beta < \mu < 2\beta$. Under one of the following conditions*

1. $\mu \ge 1$,
2. $\alpha \le \min\left\{8/T^2, (2\beta)^{\frac{\beta}{1-\beta}}(1-\beta)\right\}$,

it follows that $\psi_{\alpha,M}(0+) = 0$. In other words, the equation (22) possesses a T-periodic solution.

Proof. Taking into account that $\alpha \le 8/T^2$ and $\tau_\alpha(M) = T/2$, by (31) one observes that $0 < M < 1$. Since $u^\mu < u^\beta$ if $u \in (0,1)$ (observe that $\mu > \beta$ and w is defined above), we obtain $\psi_{\alpha,M}(u) \le w(u)$ for $u \in (0,1)$. According to Lemma 6 (or Proposition 2) we have $w(0+) = 0$, and we deduce that $\psi_{\alpha,M}(0+) = 0$. □

Now we show that for a suitable small $T > 0$ there exists a T-periodic solution of (22) assuming that $\mu \in (0,1)$. More precisely,

Lemma 9. *Assume that $\beta < \mu < 2\beta$ and $\mu \in (0,1)$. Then, the equation (22) has a T-periodic solution provided that*

$$T \leq 2\sqrt{2}\left(\frac{1}{\alpha}\right)^{\frac{1+\mu-2\beta}{2(\mu-\beta)}}\left[\frac{2\beta(1-\beta)}{1-\mu}\right]^{\frac{\beta}{2(\mu-\beta)}}(1-\beta)^{\frac{1-\beta}{2(\mu-\beta)}}. \tag{40}$$

Proof. Consider the problem (25). By the transformation $z = \psi_{\alpha,M}(u^\gamma)$ we obtain the equivalent problem

$$z' = 2\gamma\left(\left(\frac{z}{u}\right)^\beta - \alpha u^{\gamma-1}\right), \qquad z(M^{1/\gamma}) = 0,$$

here $\gamma := (1-\beta)/(1-\mu) > 1$. Compare this problem with

$$w' = 2\gamma\left(\left(\frac{w}{u}\right)^\beta - \alpha M^{\frac{\gamma-1}{\gamma}}\right), \qquad w(M^{1/\gamma}) = 0. \tag{41}$$

Since $\gamma > 1$ one easily checks that $z \leq w$ in some neighborhood of 0. Again, we introduce a new dependent variable v by the transformation $w(u) = uv(u)$ in order to reduce the problem (41) to

$$v' = \frac{2\gamma v^\beta - v - 2\gamma\alpha M^{\frac{\gamma-1}{\gamma}}}{u}, \qquad v(M^{1/\gamma}) = 0,$$

and we obtain

$$u = M^{1/\gamma}\exp\left[-\int_0^{v(u)}\frac{ds}{s - 2\gamma s^\beta + 2\gamma\alpha M^{\frac{\gamma-1}{\gamma}}}\right]. \tag{42}$$

Consider the function

$$\xi : [0, +\infty) \to \mathbb{R}, \qquad \xi(x) := x - 2\gamma x^\beta + 2\gamma\alpha M^{\frac{\gamma-1}{\gamma}}.$$

Observe that there exists $r > 0$ such that $\xi(r) = 0$ and $\xi(x) > 0$ for all $x \in (0, r)$ if we assume that

$$M \leq \left(\frac{1}{\alpha}\right)^{\frac{1-\beta}{\mu-\beta}}\left(\frac{2\beta(1-\beta)}{1-\mu}\right)^{\frac{\beta}{\mu-\beta}}(1-\beta)^{\frac{1-\beta}{\mu-\beta}}. \tag{43}$$

Hence, according to (42) we deduce that $v(0+) = r$, whence it follows that $z(0+) = 0$; therefore $\psi_{\alpha,M}(0+) = 0$. The remaining of the proof is devoted to check that (40) implies (43). Indeed, in view of the discussions done above it turns out that $\tau_\alpha(M) = T/2$, and by (31) we obtain that $M \leq \alpha T^2/8$. The latter inequality combined with (40) yields (43). The proof is complete. □

6. Proof of Theorems 1–4

We now assume the framework of Section 1.

Proof of Theorem 1. Considering the equation (22) with $\alpha = m_f$, by Proposition 4 we obtain σ_1 a solution of Dirichlet type in the interval $(0, T)$, which is a lower solution of (2). Now we apply Proposition 4 with $\alpha = M_f$ in order to find σ_2 an upper solution to equation (2). In view of Proposition 5, the inequality $\sigma_2 \geq \sigma_1$ holds on the interval $(0, T)$. Finally the result follows applying Proposition 1. □

Proof of Theorem 2. The first part follows immediately from Theorem 1 and Proposition 2. The second part is a direct consequence of Proposition 1 after combining (3), Lemma 6, Corollary 2 and Proposition 5 in order to obtain σ_1 and σ_2, respectively, T-periodic lower and upper solutions to the equation (2) verifying (38). □

Proof of Theorem 3. This follows immediately from Theorem 1 and Proposition 2. □

Proof of Theorem 4. The first part follows by using Theorem 1 and Proposition 2. Now, by combining (4) (resp. (5)), Lemma 8 (resp. Lemma 9), Corollary 2 and Proposition 5 we find σ_1 and σ_2, respectively, T-periodic lower and upper solutions to the equation (2) satisfying (38). Consequently, the second part of the statement follows on the basis of Proposition 1. □

7. Proof of Theorem 5

Let us start with some preliminary results.

Given $M > 0$ we will consider the following Cauchy problem

$$\begin{cases} (|u'|^{p-2}u')' - \dfrac{|u'|^k}{u^s} + f(t,u,u') = 0, \\ u(0) = M, \\ u'(0) = 0. \end{cases} \tag{44}$$

Let us recall that a solution (around zero) of problem (44) is any function $w \in C^1(a,b)$, with $0 \in (a,b) \subseteq \mathbb{R}$, such that $|w'|^{p-2}w' \in C^1(a,b)$ and

$$\begin{cases} (|w'(t)|^{p-2}w'(t))' - \dfrac{|w'(t)|^k}{w^s(t)} + f(t,w(t),w'(t)) = 0, \quad \forall t \in (a,b) \\ w(0) = M, \\ w'(0) = 0. \end{cases}$$

The next two propositions stress some nice properties of the positive solutions of (44) that will be crucial in the proof of Theorem 5.

Proposition 6. *Let $w : (a,b) \to \mathbb{R}$ be a positive solution of (44) and put*

$$K(w) = \{t \in (a,b) : \ w'(t) = 0\}, \quad \mathcal{M}(w) = \{t \in (a,b) : \ t \text{ is a local maximizer of } w\}.$$

Then $K(w) \subset \mathcal{M}(w)$.

Proof. First observe that $K(w) \neq \emptyset$ since $0 \in K(w)$. Pick $t_0 \in K(w)$, then for every $t \in (t_0,b)$ one has

$$|w'(t)|^{p-2}w'(t) = \int_{t_0}^{t} \left(\frac{|w'(\tau)|^k}{w^s(\tau)} - f(\tau,w(\tau),w'(\tau)) \right) d\tau.$$

Hence, if t is in a suitable right neighborhood U^+ of t_0, in view of the fact that $w' \in C^1$, recalling that $t_0 \in K(w)$ and exploiting the positivity of w and of f, it is clear that

$$w'(t) < 0 \quad \forall t \in U^+.$$

Reasoning in a similar way, it is possible to find a left neighborhood U^- of t_0 such

$$w'(t) > 0 \quad \forall t \in U^-.$$

Finally, it is easy to conclude that $t_0 \in \mathcal{M}(w)$. □

Proposition 7. *Let $w : (a, b) \to \mathbb{R}$ be a positive solution of (44). Then w can not be locally constant.*

Proof. Assume that there exist $t_0 \in (a, b)$, $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subset (a, b)$ and $w(t) = w(t_0)$ for all $t \in (t_0 - \delta, t_0 + \delta)$. Then,

$$f(t, w(t), w'(t)) = -(|w'(t)|^{p-2}w'(t))' + \frac{|w'(t)|^k}{w^s(t)} = 0$$

for every $t \in (t_0 - \delta, t_0 + \delta)$, in contradiction with (i). □

Proposition 8. *Let $w : (a, b) \to \mathbb{R}$ be a positive solution of (44). Then,*

$$K(w) = \{0\}. \tag{45}$$

Proof. It has been already observed that $0 \in K(w)$. Assume that there exists $t_0 \in K(w) \setminus \{0\}$ and suppose $t_0 < 0$ (the other case is analogous). From Proposition 6 one has that 0 and t_0 are both local maximizers of w. Thus, there exists $t_1 \in (t_0, 0)$ that is a global minimum of $w|_{[t_0,0]}$. Hence, if we put $\mathcal{N}(w) = \{t \in (a, b) : \ t \text{ is a local minimum of } w\}$, having in mind Proposition 6, we obtain

$$t_1 \in \mathcal{M}(w) \cap \mathcal{N}(w),$$

namely w is constant in a neighborhood of t_1, in contradiction with Proposition 7. □

Proposition 9. *Let $w : (a, b) \to \mathbb{R}$ be a positive solution of (44). Then, w is increasing in $(a, 0)$ and decreasing in $(0, b)$ and, in particular, $w(0) = \max_{t\in(a,b)} w(t)$.*

Proof. From Propositions 6 and 8 it follows that 0 is the only critical point that, in particular, is a local maximum of w.

We claim that

$$w'(t) > 0 \quad \forall t \in (a, 0), \tag{46}$$

that leads to the first part of the conclusion. If (46) does not hold, because of Proposition 8, there exist $t_0 \in (a, 0)$ such that $w'(t_0) < 0$. But, again from Proposition 8, this means that $w'(t) < 0$ for every $t \in (a, 0)$, namely w is decreasing in $(a, 0)$ which implies that w is constant in left neighborhood of 0, in contradiction with Proposition 7. Hence claim (46) is true.

Reasoning in a similar way one can verify that w is decreasing in $(0, b)$. Finally, it is obvious that 0 is the unique global maximum of w. □

We are now in a position to prove Theorem 5.

Proof of Theorem 5. Fix $M > 0$ and divide the proof in three steps.

Step 1: existence of the local solution.
Observe that, since f is continuous, after writing (44) as a system of equations, the classical theory of the Cauchy problem ensures that there exists $T > 0$ and a positive function $v \in C^1(-T, T)$ that is a solution of problem (44) in $(-T, T)$.

Step 2: existence of the maximal solution.
For every interval $(a, b) \subseteq \mathbb{R}$, with $-\infty \le a \le -T < T \le b \le +\infty$, put $I_a^b = (a, b)$ and

$$X_{(a,b)} = \{w \in C^1(I_a^b) \mid w \text{ is positive and solves (44) in } I_a^b\},$$

$$\mathcal{E} = \left\{ (I_a^b, w) \,\middle|\, w \in X_{(a,b)} \text{ and } w|_{I_{-T}^T} = v \right\}.$$

Of course $(I_{-T}^T, v) \in \mathcal{E}$. Moreover, let us point out that for every $(I_a^b, w) \in \mathcal{E}$, with $-\infty < a$ ($b < +\infty$), one has

$$w(a^+) = \lim_{t \to a^+} w(t) > 0, \qquad \left(w(b^-) = \lim_{t \to b^-} w(t) > 0 \right). \tag{47}$$

In fact, consider the case $-\infty < a$ (the case $b < +\infty$ is the same) from Proposition 9 we already know that $w(a^+)$ exists. Moreover, because w is positive, it is clear that

$$w(a^+) \geq 0.$$

By contradiction assume that (47) does not hold, namely w can be extended up to $t_0 = a$ with continuity by putting

$$w(a) = w(a^+) = 0. \tag{48}$$

Letting $0 < \widetilde{b} < b$, for every $\varepsilon > 0$ small enough, because w solves (44) in $I_a^{\tilde{b}}$, integrating one has

$$\begin{aligned}
\int_{a+\varepsilon}^{\widetilde{b}-\varepsilon} \frac{|w'(t)|^k}{w(t)^s}\, dt &= \int_{a+\varepsilon}^{\widetilde{b}-\varepsilon} \left((|w'(t)|^{p-2} w'(t))' + f(t, w(t), w'(t)) \right) dt \\
&\leq |w'(\widetilde{b}-\varepsilon)|^{p-2} w'(\widetilde{b}-\varepsilon) - |w'(a+\varepsilon)|^{p-2} w'(a+\varepsilon) \\
&\qquad + \int_{a+\varepsilon}^{\widetilde{b}-\varepsilon} \left(\beta(t, w(t)) + \gamma |w'(t)|^r \right) dt \\
&\leq (\widetilde{b} - a) \max_{[a,\widetilde{b}] \times [0,M]} \beta(t,x) + \gamma \int_{a+\varepsilon}^{\widetilde{b}-\varepsilon} \frac{|w'(t)|^r}{w^{\frac{sr}{k}}(t)} w^{\frac{sr}{k}}(t)\, dt \\
&\leq (\widetilde{b} - a) \max_{[a,\widetilde{b}] \times [0,M]} \beta(t,x) + c_1 \gamma \left(\int_{a+\varepsilon}^{\widetilde{b}-\varepsilon} \frac{|w'(t)|^k}{w^s(t)}\, dt \right)^{r/k}
\end{aligned} \tag{49}$$

with c_1 a positive constant independent from ε and where the fundamental theorem of calculus, condition (i) of the statement of Theorem 5, as well as Proposition 9 and the Hölder inequality have been exploited. Passing to the limit as $\varepsilon \to 0^+$ in (49) and recalling condition (6), one has

$$\frac{|w'|^k}{w^s} \in L^1(a, \widetilde{b}).$$

Hence, if k' is the conjugate exponent of k and t is such that $w(\tau) < 1$ for every $\tau \in [a, t]$, we have

$$\begin{aligned}
\int_a^t |w'(\tau)|\, d\tau &= \int_a^t \frac{|w'(\tau)|}{|w^{s/k}(\tau)|} |w^{s/k}(\tau)|\, d\tau \\
&\leq \left(\int_a^t \frac{|w'(\tau)|^k}{w^s(\tau)}\, d\tau \right)^{1/k} \left(\int_a^t w^{sk'/k}(\tau)\, d\tau \right)^{1/k'}
\end{aligned}$$

$$\leq \left(\int_a^{\tilde{b}} \frac{|w'(\tau)|^k}{w^s(\tau)} \, d\tau \right)^{1/k} \left(\int_a^t w^{k'}(\tau) \, d\tau \right)^{1/k'} .$$

This implies that, for every $\tau \in (a, t)$, $w(t) \leq \int_a^t |w'(\tau)| \, d\tau \leq c_2 \left(\int_a^t w^{k'}(\tau) \, d\tau \right)^{1/k'}$, so that

$$w^{k'}(t) \leq c_2^{k'} \int_a^t w^{k'}(\tau) \, d\tau,$$

where c_2 is a constant independent from t. Thus, Gronwall's lemma leads to $w(\tau) = 0$ for every $\tau \in [a, t]$ which is absurd, and (47) holds.

Let us introduce the following order in $\mathcal{E}$

$$(I_{a_1}^{b_1}, w_1) \leq (I_{a_2}^{b_2}, w_2) \Leftrightarrow (a_1, b_1) \subseteq (a_2, b_2) \text{ and } w_2|_{(a_1,b_1)} = w_1,$$

namely $(I_{a_2}^{b_2}, w_2)$ is greater than $(I_{a_1}^{b_1}, w_1)$ if w_2 is a solution of (44) that extends w_1 to the interval (a_2, b_2).

Fix a chain (that is, a totally ordered subset) $\mathcal{C}$ in $\mathcal{E}$ and put

$$I_{\mathcal{C}} = \cup \left\{ I_a^b : \ (I_a^b, w) \in \mathcal{C} \text{ for some } w \in X_{(a,b)} \right\}.$$

That is $I_{\mathcal{C}}$ is an open interval containing $(-T, T)$, let us say

$$I_{\mathcal{C}} = I_{\bar{a}}^{\bar{b}},$$

with $-\infty \leq \bar{a} \leq -T < T \leq \bar{b} \leq +\infty$. Define $\bar{z} : I_{\bar{a}}^{\bar{b}} \to \mathbb{R}$ by putting

$$\bar{z}(t) = w(t)$$

for every $t \in I_{\bar{a}}^{\bar{b}}$, where $w \in X_{(a,b)}$ with $t \in I_a^b$ for some I_a^b and $(I_a^b, w) \in \mathcal{C}$. The function $\bar{z}$ is well defined, because if $I_{a_1}^{b_1}, I_{a_2}^{b_2}$ are such that $t \in I_{a_1}^{b_1} \cap I_{a_2}^{b_2}$ and $(I_{a_1}^{b_1}, w_1)$, $(I_{a_2}^{b_2}, w_2) \in \mathcal{C}$, then, assuming that $I_{a_1}^{b_1} \leq I_{a_2}^{b_2}$ (the opposite case is analogous) one has $w_1(t) = w_2(t)$. It is easy to check that

$$(I_{\bar{a}}^{\bar{b}}, \bar{z}) \in \mathcal{E} \quad \text{and} \quad (I_a^b, w) \leq (I_{\bar{a}}^{\bar{b}}, \bar{z}) \ \forall (I_a^b, w) \in \mathcal{C}.$$

Hence, Zorn's lemma ensures the existence of a maximal element $(I_{a^*}^{b^*}, u^*) \in \mathcal{E}$.

We claim that

$$I_{a^*}^{b^*} = \mathbb{R}. \tag{50}$$

By contradiction, assume that (50) is false. Namely, suppose $-\infty < a^*$ (the case $b < +\infty$ is analogous). From (47) u^* can be extended with continuity up to a^* by putting

$$u^*(a^*) = \lim_{t \to a^{*+}} u^*(t),$$

and $u^*(a^*) > 0$. Let us distinguish the cases

$({u^*}')_1$ $\liminf_{t \to a^{*+}} {u^*}'(t) \leq \limsup_{t \to a^{*+}} {u^*}'(t) < +\infty$,

$({u^*}')_2$ $\liminf_{t\to a^{*+}} {u^*}'(t) < \limsup_{t\to a^{*+}} {u^*}'(t) = +\infty$,

$({u^*}')_3$ $\lim_{t\to a^{*+}} {u^*}'(t) = +\infty$.

Assume $({u^*}')_1$. Then, from Proposition 9,

$$0 \le \liminf_{t\to a^{*+}} {u^*}'(t) \le \limsup_{t\to a^{*+}} {u^*}'(t) < +\infty,$$

that is, ${u^*}'$ is bounded on $(a^*, 0)$. Hence, since for every $t \in (a^*, 0)$

$$(|{u^*}'(t)|^{p-2}{u^*}'(t))' = \frac{|{u^*}'(t)|^k}{(u^*)^s(t)} - f(t, u^*(t), {u^*}'(t)),$$

putting

$$v^* = |{u^*}'|^{p-2}{u^*}', \tag{51}$$

there exists $K > 0$ such that

$$\sup_{t\in(a^*,0)} |(v^*)'(t)| \le K.$$

Thus, for every $\{t_n\}$ in $I_{a^*}^{b^*}$ with $t_n \to a^*$ one has

$$|v^*(t_m) - v^*(t_n)| = \left|\int_{t_n}^{t_m} (v^*)'(t)\, dt\right| \le K|t_m - t_n|,$$

namely $\{v^*(t_n)\}$ is a Cauchy sequence and

$$0 \le \lim_{t\to a^{*+}} v^*(t) < +\infty.$$

Put ${u^*}'(a^*) = (\lim_{t\to a^{*+}} v^*(t))^{1/(p-1)}$. It is clear that $u^* \in C^1[a^*, 0]$ and, in particular,

$${u^*}'(a^*) > 0.$$

In fact, if ${u^*}'(a^*) = 0$, then reasoning as in Proposition 6, a^* is a local maximum. Recalling that u^* is increasing in $(a^*, 0)$ we obtain that u^* is locally constant near a^* in contradiction with Proposition 7.

Let us now consider the Cauchy problem

$$\begin{cases} (|v'|^{p-2}v')' - \dfrac{|v'|^k}{v^s} + f(t, v, v') = 0, \\ v(a^*) = u^*(a^*), \\ v'(a^*) = {u^*}'(a^*), \end{cases}$$

that yields a solution $z \in C^1(a^* - \varepsilon, a^* + \varepsilon)$, with $\varepsilon > 0$ such that

$$z(t) > 0, \quad z'(t) > 0$$

for every $t \in (a^* - \varepsilon, a^* + \varepsilon)$. At this point, the function $u_\varepsilon : (a^* - \varepsilon, b^*) \to (0, M]$ defined by

$$u_\varepsilon(t) = \begin{cases} z(t) & \text{if } t \in (a^* - \varepsilon, a^*] \\ u^*(t) & \text{if } t \in [a^*, b^*) \end{cases}$$

is such that $(I_{a^*-\varepsilon}^{b^*}, u_\varepsilon) \in \mathcal{E}$. In particular, in order to verify that $|u'_\varepsilon|^{p-2}u'_\varepsilon \in C^1(I_{a^*-\varepsilon}^{b^*})$, it is enough to check the regularity at a^*. Hence,

$$\begin{aligned} \lim_{t \to a^*-} (|u'_\varepsilon(t)|^{p-2}u'_\varepsilon(t))' &= \lim_{t \to a^*-} \left(\frac{|z'(t)|^k}{z^s(t)} - f(t, z(t), z'(t)) \right) \\ &= \frac{|{u^*}'(a^*)|^k}{(u^*(a^*))^s} - f(a^*, u^*(a^*), {u^*}'(a^*)) \\ &= \lim_{t \to a^*+} \frac{|{u^*}'(t)|^k}{(u^*(t))^s} - f(t, u^*(t), {u^*}'(t)) \\ &= \lim_{t \to a^*+} (|u'_\varepsilon(t)|^{p-2}u'_\varepsilon(t))'. \end{aligned}$$

Moreover, $(I_{a^*}^{b^*}, u^*) \le (I_{a^*-\varepsilon}^{b^*}, u_\varepsilon)$, against the maximality of $(I_{a^*}^{b^*}, u^*)$. Namely $({u^*}')_1$ can not occur.

Assume $({u^*}')_2$. In this case we have that v^* defined in (51) is a C^1 oscillatory function and one can find a sequence $\{t_n\}$ in $(a^*, 0)$ with $t_n \to a^*$ and such that

$$\lim_{n \to +\infty} {u^*}'(t_n) = +\infty,$$
$$(v^*)'(t_n) = 0 \quad \forall n.$$

Thus, for every n,

$$0 = \frac{(v^*)'(t_n)}{|{u^*}'(t_n)|^k} = \frac{1}{(u^*(t_n))^s} - \frac{f(t_n, u^*(t_n), {u^*}'(t_n))}{|{u^*}'(t_n)|^k}.$$

Exploiting (i) and passing to the limit in the previous condition we achieve again a contradiction, namely $({u^*}')_2$ can not occur.

Assume $({u^*}')_3$. Then, because u^* solves (44) and taking into account (i), it is clear that v^* as defined in (51) is such that

$$\lim_{t \to a^*+} (v^*)'(t) = +\infty.$$

Let $\bar{t} \in (a^*, 0)$ such that $(v^*)'(t) > 1$ for every $t \in (a^*, \bar{t})$. Then, one has

$$v^*(\bar{t}) - v^*(t) = \int_t^{\bar{t}} (v^*)'(\tau)\, d\tau > \bar{t} - t$$

for each $t \in (a^*, \bar{t})$. Hence, v^* is bounded in $(a^*, \bar{t})$, as well as ${u^*}'$, in contradiction with $({u^*}')_3$.

Finally, all these contradictions assure that (50) is true.

Step 3: the maximal solution is a homoclinic.

Summarizing, with Steps 1 and 2 we have proved that the original positive local solution $v \in C^1(-T, T)$ of (44) can be extended by u^* up to the real line still preserving the positivity. Moreover, in view of Proposition 9, we are sure that u^* is increasing in $(-\infty, 0)$ and decreasing in $(0, +\infty)$, and moreover $M = u^*(0) = \max_{\mathbb{R}} u^*$. Hence, the proof will be concluded choosing $u = u^*$ and verifying that

$$L = \lim_{|t| \to +\infty} u(t) = 0 \tag{52}$$

and

$$L' = \lim_{|t| \to +\infty} u'(t) = 0. \tag{53}$$

It is clear that, by the monotonicity and the sign properties of u, L exists and $L \geq 0$. Moreover, because for every $t \in \mathbb{R}$ there exists $\tau_t \in (t-1, t)$ such that

$$u(t) - u(t-1) = u'(\tau_t),$$

passing to the limit for $t \to -\infty$ and having in mind that $u'(t) > 0$ in $(-\infty, 0)$, we find

$$\liminf_{t \to -\infty} u'(t) = 0. \tag{54}$$

By contradiction, if $L > 0$ let us distinguish the cases

$(u')_1$ $\limsup_{t \to -\infty} u'(t) > 0$,
$(u')_2$ $\limsup_{t \to -\infty} u'(t) = 0$.

Assume $(u')_1$. Then $|u'|^{p-2}u'$ is a C^1 function having an oscillatory behavior at $-\infty$, that is, from (54) one can find a sequence $\{t_n\}$ with $t_n \to -\infty$ such that

$$u'(t_n) \to 0,$$
$$(|u'(t_n)|^{p-2}u'(t_n))' = 0 \quad \forall n.$$

Hence, for every n

$$0 = (|u'(t_n)|^{p-2}u'(t_n))' = \frac{|u'(t_n)|^k}{(u(t_n))^s} - f(t_n, u(t_n), u'(t_n)).$$

Passing to the limit inferior in the previous condition and taking into account assumption (i), one obtains

$$0 < \alpha \leq \liminf_{n \to \infty} f(t_n, u(t_n), u'(t_n)) = 0$$

which is a contradiction. Namely, $(u')_1$ does not hold.

Assume $(u')_2$. In this case $\lim_{t \to -\infty} u'(t) = 0$, which implies, arguing as in the proof of (54), but with $|u'|^{p-2}u'$ in place of u, that

$$\liminf_{t \to -\infty} |(|u'(t)|^{p-2}u'(t))'| = 0.$$

Pick a sequence $\{t_n\}$ such that $t_n \to -\infty$ and $(|u'(t_n)|^{p-2}u'(t_n))' \to 0$. Then, since for every n one has

$$(|u'(t_n)|^{p-2}u'(t_n))' = \frac{|u'(t_n)|^k}{(u(t_n))^s} - f(t_n, u(t_n), u'(t_n)),$$

passing to the limit inferior we achieve again the contradiction

$$0 < \alpha \le \liminf_{n\to+\infty} f(t_n, u(t_n), u'(t_n)) = \frac{0}{L^2} = 0.$$

Thus also $(u')_2$ does not hold and we conclude that (52) is true.

Finally, we already observed that (54) holds. If we assume that $(u')_1$ is true, let us pick a sequence $\{t_n\}$ with $t_n \to -\infty$ such that

$$u'(t_n) \to \limsup_{t\to-\infty} u'(t) = \tilde{L} > 0,$$
$$(|u'(t_n)|^{p-2}u'(t_n))' = 0 \quad \forall n.$$

Then, for every n

$$0 = (|u'(t_n)|^{p-2}u'(t_n))' = \frac{|u'(t_n)|^k}{(u(t_n))^s} - f(t_n, u(t_n), u'(t_n)). \tag{55}$$

Distinguish the cases:

$(\tilde{L})_1$ $0 < \tilde{L} < +\infty$
$(\tilde{L})_2$ $\tilde{L} = +\infty$.

If $(\tilde{L})_1$ holds, passing to the limit superior as $n \to +\infty$ in (55), it follows that

$$\limsup_{n\to+\infty} f(t_n, u(t_n), u'(t_n)) = +\infty,$$

in contradiction with (i).

Otherwise, if $(\tilde{L})_2$ holds, from (55) we derive that, for every n

$$0 = \frac{1}{(u(t_n))^s} - \frac{f(t_n, u(t_n), u'(t_n))}{|u'(t_n)|^k}$$

and passing to the limit on n, having in mind i), one obtains a contradiction.

Thus the case $(u')_1$ is excluded, so $(u')_2$ holds together with (54). That is, (53) holds and the proof is complete. □

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