

Research Article

On the Sets of Regularity of Solutions for a Class of Degenerate Nonlinear Elliptic Fourth-Order Equations with L^1 Data

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We establish Hölder continuity of generalized solutions of the Dirichlet problem, associated to a degenerate nonlinear fourth-order equation in an open bounded set $\Omega \subset \mathbb{R}^n$, with L^1 data, on the subsets of Ω where the behavior of weights and of the data is regular enough.

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1. Introduction

In this paper, we will deal with equations involving an operator $A : \overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega) \rightarrow (\overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega))^*$ of the form

$$Au = \sum_{|\alpha|=1,2} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \nabla_2 u), \quad (1.1)$$

where Ω is a bounded open set of \mathbb{R}^n , $n > 4$, $2 < p < n/2$, $\max(2p, \sqrt{n}) < q < n$, ν and μ are positive functions in Ω with properties precised later, $\overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$ is the Banach space of all functions $u : \Omega \rightarrow \mathbb{R}$ with the properties $|u|^q, \nu |D^\alpha u|^q, \mu |D^\beta u|^p \in L^1(\Omega)$, $|\alpha| = 1$, $|\beta| = 2$, and “zero” boundary values; $\nabla_2 u = \{D^\alpha u : |\alpha| \leq 2\}$.

The functions A_α satisfy growth and monotonicity conditions, and in particular, the following strengthened ellipticity condition (for a.e. $x \in \Omega$ and $\xi = \{\xi_\alpha : |\alpha| = 1, 2\}$):

$$\sum_{|\alpha|=1,2} A_\alpha(x, \xi) \xi_\alpha \geq c_2 \left\{ \sum_{|\alpha|=1} \nu(x) |\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x) |\xi_\alpha|^p \right\} - g_2(x), \quad (1.2)$$

where $c_2 > 0$, $g_2(x) \in L^1(\Omega)$.

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We will assume that the right-hand sides of our equations, depending on unknown function, belong to $L^1(\Omega)$.

A model representative of the given class of equations is the following:

$$\begin{aligned}
 - \sum_{|\alpha|=1} D^\alpha \left[\nu \left(\sum_{|\beta|=1} |D^\beta u|^2 \right)^{(q-2)/2} D^\alpha u \right] + \sum_{|\alpha|=2} D^\alpha \left[\mu \left(\sum_{|\beta|=2} |D^\beta u|^2 \right)^{(p-2)/2} D^\alpha u \right] \\
 = -|u|^{\sigma-1} u + f \quad \text{in } \Omega,
 \end{aligned} \tag{1.3}$$

where $\sigma > 1$ and $f \in L^1(\Omega)$.

The assumed conditions and known results of the theory of monotone operators allow us to prove existence of generalized solutions of the Dirichlet problem associated to our operator (see, e.g., [1]), bounded on the sets $G \subset \Omega$ where the behavior of weights and of the data of the problem is regular enough (see [2]).

In our paper, following the approach of [3], we establish on such sets a result on Hölder continuity of generalized solutions of the same Dirichlet problem.

We note that for one high-order equation with degenerate nonlinear operator satisfying a strengthened ellipticity condition, regularity of solutions was studied in [4, 5] (non-degenerate case) and in [6, 7] (degenerate case). However, it has been made for equations with right-hand sides in L^t with $t > 1$.

2. Hypotheses

Let $n \in \mathbb{N}$, $n > 4$, and let Ω be a bounded open set of \mathbb{R}^n . Let p, q be two real numbers such that $2 < p < n/2$, $\max(2p, \sqrt{n}) < q < n$.

Let $\nu : \Omega \rightarrow \mathbb{R}^+$ be a measurable function such that

$$\nu \in L^1_{\text{loc}}(\Omega), \quad \left(\frac{1}{\nu} \right)^{1/(q-1)} \in L^1_{\text{loc}}(\Omega). \tag{2.1}$$

$W^{1,q}(\nu, \Omega)$ is the space of all functions $u \in L^q(\Omega)$ such that their derivatives, in the sense of distribution, $D^\alpha u$, $|\alpha| = 1$, are functions for which the following properties hold: $\nu^{1/q} D^\alpha u \in L^q(\Omega)$ if $|\alpha| = 1$; $W^{1,q}(\nu, \Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,q,\nu} = \left(\int_{\Omega} |u|^q dx + \sum_{|\alpha|=1} \int_{\Omega} \nu |D^\alpha u|^q dx \right)^{1/q}. \tag{2.2}$$

$\overset{\circ}{W}{}^{1,q}(\nu, \Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\nu, \Omega)$.

Let $\mu(x) : \Omega \rightarrow \mathbb{R}^+$ be a measurable function such that

$$\mu \in L^1_{\text{loc}}(\Omega), \quad \left(\frac{1}{\mu} \right)^{1/(p-1)} \in L^1_{\text{loc}}(\Omega). \tag{2.3}$$

$W^{1,q}_{2,p}(\nu, \mu, \Omega)$ is the space of all functions $u \in W^{1,q}(\nu, \Omega)$, such that their derivatives, in the sense of distribution, $D^\alpha u$, $|\alpha| = 2$, are functions with the following properties:

$\mu^{1/p} D^\alpha u \in L^p(\Omega)$, $|\alpha| = 2$; $W_{2,p}^{1,q}(\nu, \mu, \Omega)$ is a Banach space with respect to the norm

$$\|u\| = \|u\|_{1,q,\nu} + \left(\sum_{|\alpha|=2} \int_{\Omega} \mu |D^\alpha u|^p dx \right)^{1/p}. \quad (2.4)$$

$\overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W_{2,p}^{1,q}(\nu, \mu, \Omega)$.

Hypothesis 2.1. Let $\nu(x)$ be a measurable positive function:

$$\begin{aligned} \frac{1}{\nu} &\in L^t(\Omega) \quad \text{with } t > \frac{nq}{q^2 - n}, \\ \nu &\in L^{\bar{t}}(\Omega) \quad \text{with } \bar{t} > \frac{nt}{qt - n}. \end{aligned} \quad (2.5)$$

We put $\tilde{q} = nqt/(n(1+t) - qt)$. We can easily prove that a constant $c_0 > 0$ exists such that if $u \in W^{1,q}(\nu, \Omega)$, the following inequality holds:

$$\int_{\Omega} |u|^{\tilde{q}} dx \leq c_0 \left\{ \int_{\text{supp } u} \left(\frac{1}{\nu} \right)^t dx \right\}^{\tilde{q}/qt} \left\{ \sum_{|\alpha|=1} \int_{\Omega} \nu |D^\alpha u|^q dx \right\}^{\tilde{q}/q}. \quad (2.6)$$

We set $\tilde{\nu} = \mu^{q/(q-2p)} (1/\nu)^{2p/(q-2p)}$.

Hypothesis 2.2. $\tilde{\nu} \in L^1(\Omega)$.

Hypothesis 2.3. There exists a real number $r > \tilde{q}(q-1)/(\tilde{q}(q-1)(p-1) - q)$ such that

$$\frac{1}{\mu} \in L^r(\Omega). \quad (2.7)$$

For more details about weight functions, see [8, 9].

Let Ω_1 be a nonempty open set of \mathbb{R}^n such that $\Omega_1 \subset \Omega$.

Definition 2.4. It is said that G closed set of \mathbb{R}^n is a “regular set” if $\overset{\circ}{G}$ is nonempty and $G \subset \Omega_1$.

Denote by $\mathbb{R}^{n,2}$ the space of all sets $\xi = \{\xi_\alpha \in \mathbb{R} : |\alpha| = 1, 2\}$ of real numbers; if a function $u \in L^1_{\text{loc}}(\Omega)$ has the weak derivatives $D^\alpha u$, $|\alpha| = 1, 2$ then $\nabla_2 u = \{D^\alpha u : |\alpha| = 1, 2\}$. Suppose that $A_\alpha : \Omega \times \mathbb{R}^{n,2} \rightarrow \mathbb{R}$ are Carathéodory functions.

Hypothesis 2.5. There exist $c_1, c_2 > 0$ and $g_1(x), g_2(x)$ nonnegative functions such that $g_1, g_2 \in L^1(\Omega)$ and, for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^{n,2}$, the following inequalities

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hold:

$$\begin{aligned} & \sum_{|\alpha|=1} [\nu(x)]^{-1/(q-1)} |A_\alpha(x, \xi)|^{q/(q-1)} + \sum_{|\alpha|=2} [\mu(x)]^{-1/(p-1)} |A_\alpha(x, \xi)|^{p/(p-1)} \\ & \leq c_1 \left\{ \sum_{|\alpha|=1} \nu(x) |\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x) |\xi_\alpha|^p \right\} + g_1(x), \end{aligned} \quad (2.8)$$

$$\sum_{|\alpha|=1,2} A_\alpha(x, \xi) \xi_\alpha \geq c_2 \left\{ \sum_{|\alpha|=1} \nu(x) |\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x) |\xi_\alpha|^p \right\} - g_2(x). \quad (2.9)$$

Moreover, we will assume that for almost every $x \in \Omega$ and every $\xi, \xi' \in \mathbb{R}^{m,2}$, $\xi \neq \xi'$,

$$\sum_{|\alpha|=1,2} [A_\alpha(x, \xi) - A_\alpha(x, \xi')] (\xi_\alpha - \xi'_\alpha) > 0. \quad (2.10)$$

Let $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that

- (a) for almost every $x \in \Omega$, the function $F(x, \cdot)$ is nonincreasing in \mathbb{R} ;
- (b) for every $x \in \Omega$, the function $F(\cdot, s)$ belongs to $L^1(\Omega)$.

Let $A : \overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega) \rightarrow (\overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega))^*$ be the operator such that for every $u, v \in \overset{\circ}{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$,

$$\langle Au, v \rangle = \int_{\Omega} \left\{ \sum_{|\alpha|=1,2} A_\alpha(x, \nabla_2 u) D^\alpha v \right\} dx. \quad (2.11)$$

We consider the following Dirichlet problem:

$$(P) = \begin{cases} Au = F(x, u) & \text{in } \Omega \\ D^\alpha u = 0, \quad |\alpha| = 0, 1, & \text{on } \partial\Omega. \end{cases} \quad (2.12)$$

Definition 2.6. A W -solution of problem (P) is a function $u \in \overset{\circ}{W}^{2,1}(\Omega)$ such that

- (i) $F(x, u) \in L^1(\Omega)$;
- (ii) $A_\alpha(x, \nabla_2 u) \in L^1(\Omega)$, for every $\alpha : |\alpha| = 1, 2$;
- (iii) $\langle Au, \phi \rangle = \langle F(x, u), \phi \rangle$ in distributional sense.

It is well known that Hypotheses 2.1–2.3, 2.5, and assumptions on $F(x, s)$ imply the existence of a W -solution of problem (P) (see [1]). Moreover, a boundedness local result for such solution has been established in [2] under more restrictive hypotheses on data and weight functions.

More precisely, the following holds (see [2, Theorem 5.1]).

THEOREM 2.7. *Suppose that Hypotheses 2.1–2.3 and 2.5 are satisfied. Let $q_1 \in (q, \tilde{q}(q-1)/q)$, $\tau > \tilde{q}/(\tilde{q} - q_1)$. Assume that restrictions of the functions $\nu^{q_1/(q_1-q)}$, $\tilde{\nu}$, g_1 , g_2 , and $|F(\cdot, 0)|^{q_1/(q_1-1)}$ on G belong to $L^\tau(G)$, for every “regular set” G .*

Then there exists \bar{u} W -solution of problem (P) such that for every G , $\text{ess}_G \sup |\bar{u}| \leq M_G < +\infty$, with M_G positive constant depending only on known values.

3. Main result

In the sequel of paper, G will be a “regular set.” In order to obtain our regularity result on G , we need the following further hypotheses.

Hypothesis 3.1. There exists a constant $c' > 0$ such that for all $y \in \overset{\circ}{G}$ and for all $\rho > 0$, with $\overline{B(y, \rho)} \subset G$, we have

$$\left\{ \rho^{-n} \int_{B(y, \rho)} \left(\frac{1}{\nu} \right)^t dx \right\}^{1/t} \left\{ \rho^{-n} \int_{B(y, \rho)} \nu^\tau dx \right\}^{1/\tau} \leq c'. \quad (3.1)$$

With regard to this assumption, see [3].

Hypothesis 3.2. There exist a real positive number σ and two real functions $h(x) (\geq 0)$, $f(x) (> 0)$ defined on G , such that

$$|F(x, s)| \leq h(x)|s|^\sigma + f(x), \quad \text{for almost every } x \in G \text{ and every } s \in \mathbb{R}. \quad (3.2)$$

Moreover, we assume that

$$h(x), f(x) \in L^\tau(G), \quad (3.3)$$

with τ defined as above.

Using considerations stated in [1], following the approach of [3], we establish the following result.

THEOREM 3.3. *Let all above-stated hypotheses hold and let conditions of Theorem 2.7 be satisfied. Then, the W -solution \bar{u} of Dirichlet problem (P), essentially bounded on G , is also locally Hölderian on G .*

More precisely, there exist positive constant C and λ ($0 < \lambda < 1$) such that for every open set $\Omega', \overline{\Omega'} \subset G$, and every $x, y \in \Omega'$

$$|\bar{u}(x) - \bar{u}(y)| \leq C[d(\Omega', \overset{\circ}{\partial G})]^{-\lambda} |x - y|^\lambda, \quad (3.4)$$

where C and λ depend only on $c_1, c_2, c_0, c', n, q, p, t, \tau, \sigma, M_G, \text{diam } G, \text{meas } G, \|f\|_{L^\tau(G)}, \|h\|_{L^\tau(G)}, \|g_1\|_{L^\tau(G)}, \|g_2\|_{L^\tau(G)}, \|\tilde{\nu}\|_{L^\tau(G)}$, and $\|1/\nu\|_{L^1(\Omega)}$.

Proof. For every $l \in \mathbb{N}$, we define the function $F_l : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$F_l(x, s) = \begin{cases} -l & \text{if } F(x, 0) - F(x, s) < -l, \\ F(x, 0) - F(x, s) & \text{if } |F(x, 0) - F(x, s)| \leq l, \\ l & \text{if } F(x, 0) - F(x, s) > l, \end{cases} \quad (3.5)$$

and the function $f_l : \Omega \rightarrow \mathbb{R}$ by

$$f_l(x) = \begin{cases} F(x, 0) & \text{if } |F(x, 0)| \leq l, \\ 0 & \text{if } |F(x, 0)| > l. \end{cases} \quad (3.6)$$

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By Lebesgue's theorem and property (b) of $F(x, s)$, we have that $f_l(x)$ goes to $F(x, 0)$ in $L^1(\Omega)$.

Next, inequalities (2.6), (2.8)–(2.10), property (a) of $F(x, s)$, and known results of the theory of monotone operators (see, e.g., [10]) imply that for any $l \in \mathbb{N}$, there exists $u_l \in W_{2,p}^{1,q}(\nu, \mu, \Omega)$ such that

$$\int_{\Omega} \left\{ \sum_{|\alpha|=1,2} A_{\alpha}(x, \nabla_2 u_l) D^{\alpha} v + F_l(x, u_l) v \right\} dx = \int_{\Omega} f_l v dx, \quad (3.7)$$

for every $v \in \mathring{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$.

From considerations stated in [1, Section 3], we deduce that there exists a W -solution \bar{u} of problem (P) such that

$$u_l \rightarrow \bar{u} \quad \text{a.e. in } \Omega. \quad (3.8)$$

Moreover, see proof of Theorem 2.7,

$$\text{ess sup}_G |u_l| \leq M_G, \quad \text{for every } l \in \mathbb{N}. \quad (3.9)$$

We set $\bar{n} = q^2 / (q - 2p)$, $a = (1/\bar{n})(q - n/t - n/\tau)$.

Let us fix $y \in G$, $\rho > 0$ and $\overline{B}(y, 2\rho) \subset G$. Let us put

$$\begin{aligned} \omega_{1,l} &= \text{ess inf}_{B(y,2\rho)} u_l, & \omega_{2,l} &= \text{ess sup}_{B(y,2\rho)} u_l, \\ \omega_l &= \omega_{2,l} - \omega_{1,l}. \end{aligned} \quad (3.10)$$

We will show that

$$\text{osc} \{u_l, B(y, \rho)\} \leq \tilde{c} \omega_l + \rho^a, \quad (3.11)$$

with $\tilde{c} \in]0, 1[$ independent of $l \in \mathbb{N}$.

To this aim, we fix $l \in \mathbb{N}$ and we set

$$\begin{aligned} \Phi_l &= \sum_{|\alpha|=1} \nu |D^{\alpha} u_l|^q + \sum_{|\alpha|=2} \mu |D^{\alpha} u_l|^p, \\ \psi(x) &= \rho^{-a\bar{n}} (1 + f(x) + h(x) + g_1(x) + g_2(x) + \tilde{\nu}(x)) + \rho^{-q} \nu. \end{aligned} \quad (3.12)$$

Obviously, we will assume that

$$\omega_l \geq \rho^a \quad (\text{otherwise, it is clear that (3.11) is true}). \quad (3.13)$$

We introduce now the following functions:

$$F_{1,l}(x) = \begin{cases} \frac{2e\omega_l}{u_l(x) - \omega_{1,l} + \rho^a} & \text{if } x \in B(y, 2\rho), \\ e & \text{if } x \in \Omega \setminus B(y, 2\rho); \end{cases} \quad (3.14)$$

$\varphi \in C_0^\infty(\Omega)$: $0 \leq \varphi \leq 1$ in Ω , $\varphi = 0$ in $\Omega \setminus B(y, 2\rho)$ and satisfying

$$|D^\alpha \varphi| \leq \bar{c} \rho^{-|\alpha|}, \quad |\alpha| = 1, 2, \quad (3.15)$$

where the positive constant \bar{c} depends only on n .

Let us fix $s > q$ and $r \geq 0$ and define

$$\begin{aligned} v_l &= (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^s, \\ z_l &= -\frac{1}{2e\omega_l} [r(\lg F_{1,l})^{r-1} + (q-1)(\lg F_{1,l})^r] F_{1,l}^q \varphi^s. \end{aligned} \quad (3.16)$$

From Hypothesis 2.2 and (3.15), we have that $v_l \in \mathring{W}_{2,p}^{1,q}(\nu, \mu, \Omega)$ and the next inequalities are true:

$$|D^\alpha v_l - z_l D^\alpha u_l| \leq \bar{c} s \varphi^{s-1} (\lg F_{1,l})^r F_{1,l}^{q-1} \rho^{-1} \quad \text{if } |\alpha| = 1 \text{ a.e. in } B(y, 2\rho), \quad (3.17)$$

$$\begin{aligned} |D^\alpha v_l - z_l D^\alpha u_l| &\leq 5q^2 s(r+1)^2 (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^s \left\{ \sum_{|\beta|=1} \frac{|D^\beta u_l|^2}{(u_l - \omega_{1,l} + \rho^a)^2} \right\} \\ &+ 2nq s^2 \bar{c}^2 \rho^{-2} (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^{s-2} \quad \text{if } |\alpha| = 2 \text{ a.e. in } B(y, 2\rho). \end{aligned} \quad (3.18)$$

Since $u_l(x)$ satisfies (3.7), for $v = v_l$, we obtain

$$\int_\Omega \left\{ \sum_{|\alpha|=1,2} A_\alpha(x, \nabla_2 u_l) D^\alpha v_l + F_l(x, u_l) v_l \right\} dx = \int_\Omega f_l v_l dx. \quad (3.19)$$

From this, taking into account (3.9) and Hypothesis 3.2, we have

$$\int_\Omega \sum_{|\alpha|=1,2} A_\alpha(x, \nabla_2 u_l) D^\alpha v_l dx \leq (3 + M_G^c) \int_\Omega \{1 + f(x) + h(x)\} v_l dx. \quad (3.20)$$

Hence

$$\int_\Omega \sum_{|\alpha|=1,2} \{A_\alpha(x, \nabla_2 u_l) D^\alpha u_l\} (-z_l) dx \leq (3 + M_G^c) \int_\Omega \{1 + f(x) + h(x)\} v_l dx + I_1 + I_2, \quad (3.21)$$

where

$$I_i = \int_\Omega \sum_{|\alpha|=i} |A_\alpha(x, \nabla_2 u_l)| |D^\alpha v_l - z_l D^\alpha u_l| dx, \quad i = 1, 2. \quad (3.22)$$

Using Hypothesis 2.5 and definition of z_l , we have

$$\begin{aligned} \frac{(q-1)c_2}{2e\omega_l} \int_\Omega \Phi_l (\lg F_{1,l})^r F_{1,l}^q \varphi^s dx &\leq (3 + M_G^c) \int_\Omega \{1 + f(x) + h(x)\} (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^s dx \\ &+ \int_\Omega g_2(x) (-z_l) dx + I_1 + I_2. \end{aligned} \quad (3.23)$$

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Note that

$$\begin{aligned} F_{1,l}^{q-1} &\leq (\text{diam } G)^a (2e\omega_l)^{q-1} \rho^{-aq}, \\ -z_l &\leq (q-1)(r+1)(2e\omega_l)^{q-1} \rho^{-aq} \varphi^s (\lg F_{1,l})^r \quad \text{a.e. in } B(y, 2\rho), \end{aligned} \quad (3.24)$$

consequently, from (3.23), we obtain

$$\begin{aligned} &\frac{c_2}{2e\omega_l} \int_{B(y, 2\rho)} \Phi_l (\lg F_{1,l})^r F_{1,l}^q \varphi^s dx \\ &\leq c_3 (r+1) (2e\omega_l)^{q-1} \int_{B(y, 2\rho)} \rho^{-aq} \{1 + f(x) + h(x) + g_2(x)\} (\lg F_{1,l})^r \varphi^s dx + I_1 + I_2, \end{aligned} \quad (3.25)$$

where $c_3 = (q-1)(3 + M_G^\sigma)(\text{diam } G + 1)$.

Let us fix $|\alpha| = 1$. Let $\epsilon > 0$, then, applying Young's inequality and using (2.8) and (3.17), we establish

$$\begin{aligned} I_1 &\leq \frac{c_1 \epsilon}{2e\omega_l} \int_{B(y, 2\rho)} \Phi_l F_{1,l}^q (\lg F_{1,l})^r \varphi^s dx \\ &\quad + c_1 \epsilon (2e\omega_l)^{q-1} \int_{B(y, 2\rho)} \rho^{-aq} g_1(x) (\lg F_{1,l})^r \varphi^s dx \\ &\quad + \epsilon^{1-q} (2e\omega_l)^{q-1} n(\bar{c}s)^q \int_{B(y, 2\rho)} \rho^{-q} \nu (\lg F_{1,l})^r \varphi^{s-q} dx. \end{aligned} \quad (3.26)$$

Let us fix $|\alpha| = 2$ and estimate I_2 . To this aim, it will be useful to observe that the following equalities are true:

$$\frac{p-1}{p} + \frac{2}{q} + \frac{q-2p}{qp} = 1, \quad q-1 = \frac{p-1}{p} q + \left(\frac{q}{p} - 1\right). \quad (3.27)$$

Moreover,

$$\rho^{-aq-2p} \mu \leq \rho^{-a\bar{n}} \tilde{\nu} + \rho^{-q} \nu \quad \text{in } \Omega. \quad (3.28)$$

Furthermore, due to (2.8), (3.18), and Young's inequality, we have

$$\begin{aligned} I_2 &\leq \frac{c_4 \epsilon}{2e\omega_l} \int_{B(y, 2\rho)} \Phi_l F_{1,l}^q (\lg F_{1,l})^r \varphi^s dx \\ &\quad + c_5 (2e\omega_l)^{q-1} \epsilon \left(1 + \frac{1}{\epsilon}\right)^{\bar{n}} s^{\bar{n}} (r+1)^{\bar{n}} \int_{B(y, 2\rho)} \{\rho^{-a\bar{n}} (g_1(x) + \tilde{\nu}(x)) + \rho^{-q} \nu\} (\lg F_{1,l})^r \varphi^{s-q} dx, \end{aligned} \quad (3.29)$$

where c_4 depends only on c_1, n, q ; and c_5 depends only on c_1, n, q, p, \bar{c} , and $\text{diam } G$.

From (3.25), (3.26), and (3.29), we get

$$\begin{aligned} & \frac{c_2}{2e\omega_l} \int_{B(y,2\rho)} \Phi_l(\lg F_{1,l})^r F_{1,l}^q \varphi^s dx \\ & \leq \frac{(c_1 + c_4)\epsilon}{2e\omega_l} \int_{B(y,2\rho)} \Phi_l F_{1,l}^q (\lg F_{1,l})^r \varphi^s dx \\ & \quad + (2e\omega_l)^{q-1} c_6 (r+1) \bar{n} s^{\bar{n}} \left(1 + \epsilon + \frac{1}{\epsilon}\right)^{\bar{n}+1} \int_{B(y,2\rho)} \psi (\lg F_{1,l})^r \varphi^{s-q} dx, \end{aligned} \quad (3.30)$$

where the constant c_6 depends only on $c_1, \bar{c}, n, q, p, M_G, \sigma$, and $\text{diam } G$.

Setting

$$\epsilon = \frac{c_2}{2(c_1 + c_4)}, \quad (3.31)$$

from the last inequality, we deduce

$$\int_{B(y,2\rho)} \Phi_l(\lg F_{1,l})^r F_{1,l}^q \varphi^s dx \leq c_7 (2e\omega_l)^q (r+1) \bar{n} s^{\bar{n}} \int_{B(y,2\rho)} \psi (\lg F_{1,l})^r \varphi^{s-q} dx, \quad (3.32)$$

where the constant c_7 depends only on $c_1, c_2, \bar{c}, n, q, p, M_G, \sigma$, and $\text{diam } G$.

Now, if we choose φ such that $\varphi = 1$ in $B(y, (4/3)\rho)$, from (3.32), with $r = 0$ and $s = q + 1$, we get

$$\int_{B(y, (4/3)\rho)} \left\{ \sum_{|\alpha|=1} \nu |D^\alpha u_l|^q \right\} F_{1,l}^q dx \leq c_7 (2e\omega_l)^q (q+1) \bar{n} \int_{B(y,2\rho)} \psi dx. \quad (3.33)$$

Moreover, if we take in (3.32) instead of φ the function $\varphi_1 \in C_0^\infty(\Omega)$ with the properties $0 \leq \varphi_1 \leq 1$ in Ω , $\varphi_1 = 0$ in $\Omega \setminus B(y, (4/3)\rho)$, $\varphi_1 = 1$ in $B(y, \rho)$, and $|D^\alpha \varphi| \leq \bar{c} \rho^{-|\alpha|}$ in Ω , $|\alpha| = 1, 2$, we obtain that for every $r > 0$ and $s > q$,

$$\int_{B(y,2\rho)} \left\{ \sum_{|\alpha|=1} \nu |D^\alpha u_l|^q \right\} (\lg F_{1,l})^r F_{1,l}^q dx \leq c_7 (2e\omega_l)^q s^{\bar{n}} (r+1) \bar{n} \int_{B(y,2\rho)} \psi (\lg F_{1,l})^r \varphi_1^{s-q} dx. \quad (3.34)$$

We fix arbitrary $r > 0$ and $s > \tilde{q}$, and let

$$z_l = (\lg F_{1,l})^{r/\tilde{q}} \varphi_1^{s/\tilde{q}}. \quad (3.35)$$

By means of Hypothesis 2.1, we establish that $z_l \in \dot{W}^{1,q}(\nu, \Omega)$ and for $|\alpha| = 1$,

$$\begin{aligned} \nu |D^\alpha z_l|^q & \leq 2^{q-1} \left(\frac{r}{\tilde{q}}\right)^q (\lg F_{1,l})^{(r/\tilde{q}-1)q} (F_{1,l})^q \frac{1}{(2e\omega_l)^q} |D^\alpha u_l|^q \nu \varphi_1^{s/\tilde{q}} \\ & \quad + 2^{q-1} \left(\frac{s}{\tilde{q}}\right)^q (\lg F_{1,l})^{r/\tilde{q}} \varphi_1^{(s/\tilde{q}-1)q} \bar{c}^q \rho^{-q} \nu. \end{aligned} \quad (3.36)$$

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Now, it is convenient to observe that $\tilde{q}/(\tilde{q} - q_1) > nt/(qt - n)$, then $\tau > nt/(qt - n)$; moreover, $\psi(x) \in L^\tau(G)$. From (3.34) and (3.36), we deduce

$$\begin{aligned} & \int_{\Omega} \nu |D^\alpha z_l|^q dx \\ & \leq c_8 s^{\bar{n}} (r+1)^{\bar{n}+q} \left(\int_{B(y,2\rho)} \psi^\tau dx \right)^{1/\tau} \left(\int_{B(y,2\rho)} (\lg F_{1,l})^{r(q/\tilde{q})(\tau/(\tau-1))} \varphi_1^{(s/\tilde{q}-1)q(\tau/(\tau-1))} dx \right)^{(\tau-1)/\tau}, \end{aligned} \quad (3.37)$$

where the constant c_8 depends only on $c_1, c_2, \bar{c}, n, q, p, M_G, \sigma$, and $\text{diam } G$.

We set

$$\theta = \frac{\tilde{q}(\tau-1)}{q\tau}, \quad m = \frac{q\tau}{\tau-1}, \quad (3.38)$$

and for every $r, s > 0$, we define

$$I(r, s) = \int_{B(y,2\rho)} (\lg F_{1,l})^r \varphi_1^s dx. \quad (3.39)$$

Consequently, last inequality can be rewritten in this manner:

$$\int_{\Omega} \nu |D^\alpha z_l|^q dx \leq c_8 s^{\bar{n}} (r+1)^{\bar{n}+q} \left(\int_{B(y,2\rho)} \psi^\tau dx \right)^{1/\tau} \left[I\left(\frac{r}{\theta}, \frac{s}{\theta} - m\right) \right]^{(\tau-1)/\tau}. \quad (3.40)$$

Due to Hypothesis 2.1,

$$I(r, s) = \int_{B(y,2\rho)} z_l^{\tilde{q}} dx \leq c_0 \left[\int_{B(y,2\rho)} \left(\frac{1}{\nu}\right)^t dx \right]^{\tilde{q}/qt} \left[\sum_{|\alpha|=1} \int_{\Omega} \nu |D^\alpha z_l|^q dx \right]^{\tilde{q}/q}. \quad (3.41)$$

Let us denote by \prod_G the norm of $(1 + f(x) + h(x) + g_1(x) + g_2(x) + \tilde{\nu}(x))$ in $L^\tau(G)$. By simple computation, we have

$$\left(\int_{B(y,2\rho)} \psi^\tau dx \right)^{1/\tau} \leq \rho^{-q} \left(\int_{B(y,2\rho)} \nu^\tau dx \right)^{1/\tau} + \prod_G \rho^{-a\bar{n}}. \quad (3.42)$$

Now, it is convenient to observe that $(q - n/t - n/\tau)(\tilde{q}/q) = n(\theta - 1)$.

Then, from (3.40)–(3.42), using Hypothesis 3.1, we get

$$I(r, s) \leq M(r+s)^{\bar{m}} \rho^{n(1-\theta)} \left[I\left(\frac{r}{\theta}, \frac{s}{\theta} - m\right) \right]^\theta, \quad \text{for every } r > 0, s > \tilde{q}, \quad (3.43)$$

where $\bar{m} = 2(q + \bar{n})\tilde{q}$ and the positive constant M depends only on $c_1, c_2, \bar{c}, c_0, c', n, q, p, t, \|1/\nu\|_{L^t(\Omega)}, M_G, \sigma, \text{meas } G, \text{diam } G$, and \prod_G .

We set for $i = 0, 1, 2, \dots$ that

$$r_i = \frac{tq}{t+1} \theta^i, \quad s_i = \frac{m\theta}{\theta-1} (\theta^{i+1} - 1). \quad (3.44)$$

Then by (3.43), it is trivial to establish the following iterative relation:

$$I(r_i, s_i) \leq Mc_9 \rho^{n(1-\theta)} \theta^{i\bar{m}} [I(r_{i-1}, s_{i-1})]^\theta \quad \text{for every } i \in \mathbb{N}, \quad (3.45)$$

where c_9 depends only on n, q, p, t , and τ .

Using this recurrent relation, we obtain that for every $i \in \mathbb{N}$,

$$I(r_i, s_i) \leq \left[(Mc_9 + 1)^{1/(1-\theta)} \theta^{S\bar{m}} (\text{diam } G + 1)^n \rho^{-n} I(r_0, s_0) \right]^{\theta^i}, \quad (3.46)$$

where S is a positive constant depending only on n, q, t , and τ .

Now, we assume that

$$\text{meas} \left\{ x \in B\left(y, \frac{4}{3}\rho\right) : u_l(x) \geq \frac{\omega_{1,l} + \omega_{2,l}}{2} \right\} \geq \frac{1}{2} \text{meas} B\left(y, \frac{4}{3}\rho\right). \quad (3.47)$$

We observe that if $x \in B(y, (4/3)\rho)$ satisfies $u_l(x) \geq (\omega_{1,l} + \omega_{2,l})/2$, then $F_{1,l}(x) \leq 4e$, so by [11, Lemma 4], we deduce

$$\int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx \leq c\rho^n + \frac{c\rho r_0}{2e\omega_l} \int_{B(y, (4/3)\rho)} \left\{ \sum_{|\alpha|=1} |D^\alpha u_l| (\lg F_{1,l})^{r_0-1} F_{1,l} \right\} dx, \quad (3.48)$$

where c depends only on n .

Then, using Young's inequality, we get

$$\int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx \leq cr_0 \rho^n + r_0 \left(\frac{cr_0 \rho}{2e\omega_l} \right)^{r_0} \int_{B(y, (4/3)\rho)} \left\{ \sum_{|\alpha|=1} |D^\alpha u_l| \right\}^{r_0} F_{1,l}^{r_0} dx. \quad (3.49)$$

Last inequality, using Hölder's inequality and (3.33), gives

$$\begin{aligned} \int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx &\leq cr_0 \rho^n + r_0 [cr_0]^{r_0} 2^{r_0-1} [c_7(q+1)\bar{n}]^{t/(t+1)} \rho^{r_0} \\ &\quad \times \left(\int_{B(y, 2\rho)} \psi dx \right)^{t/(t+1)} \left(\int_{B(y, 2\rho)} \left(\frac{1}{\nu} \right)^t dx \right)^{1/(t+1)}. \end{aligned} \quad (3.50)$$

Observe that due to (3.42) and Hypothesis 3.1,

$$\left(\int_{B(y, 2\rho)} \psi dx \right)^{t/(t+1)} \left(\int_{B(y, 2\rho)} \left(\frac{1}{\nu} \right)^t dx \right)^{1/(t+1)} \leq c_{10} (1+M) \rho^{n-r_0}, \quad (3.51)$$

where c_{10} depends only on measure of the unit ball in \mathbb{R}^n .

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Consequently, from (3.50), we obtain

$$\int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx \leq \left(c_{10}(1+M)r_0 [cr_0]^{r_0} 2^{r_0-1} [c_7(q+1)^{\bar{n}}]^{t/(t+1)} + cr_0 \right) \rho^n. \quad (3.52)$$

Taking into account that

$$I(r_0, s_0) \leq \int_{B(y, (4/3)\rho)} (\lg F_{1,l})^{r_0} dx, \quad (3.53)$$

from (3.46) we get

$$I(r_i, s_i) \leq [c_{11}]^{\theta^i}, \quad \text{for every } i \in \mathbb{N}. \quad (3.54)$$

Last inequality allow us to conclude that

$$\operatorname{ess\,sup}_{B(y,\rho)} F_{1,l}(x) \leq (1 + c_{11}), \quad (3.55)$$

and so

$$\operatorname{osc} \{u_l, B(y, \rho)\} \leq (1 - 2e^{-1-c_{11}}) \omega_l + \rho^a. \quad (3.56)$$

Recall that we proved (3.11) under assumption (3.47). If (3.47) is not true, we take instead of $F_{1,l}$ the function $F_{2,l} : \Omega \rightarrow \mathbb{R}^n$ such that $F_{2,l} = 2e\omega_l(\omega_{2,l} - u_l + \rho^a)^{-1}$ in $B(y, 2\rho)$, and arguing as above, we establish (3.11) again.

It is important to observe that the positive constant c_{11} depends only on $c_1, c_2, c, \bar{c}, c_0, c', n, q, p, t, \|1/\nu\|_{L^t(\Omega)}, M_G, \sigma, \operatorname{diam} G$, and $\prod G$, and is independent of $l \in \mathbb{N}$.

Now from (3.11), taking into account [12, Chapter 2, Lemma 4.8], we deduce that there exist positive constant C and $\lambda (< 1)$ depending on c_{11} and a but independent of $l \in \mathbb{N}$ such that

$$\operatorname{osc} \{u_l, B(y, \rho)\} \leq C[d(y, \partial \overset{\circ}{G})]^{-\lambda} \rho^\lambda, \quad \text{for every } \rho \in]0, d(y, \partial \overset{\circ}{G})]. \quad (3.57)$$

This and (3.8) imply that

$$\operatorname{osc} \{\bar{u}, B(y, \rho)\} \leq C[d(y, \partial \overset{\circ}{G})]^{-\lambda} \rho^\lambda, \quad \text{for every } \rho \in]0, d(y, \partial \overset{\circ}{G})]. \quad (3.58)$$

The proof is complete. \square

4. An example

Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, $0 < \gamma < \min(q - n/q, q/2)$, and let ν, μ be the restriction in $\Omega \setminus \{0\}$ of real functions

$$|x|^\gamma, \quad |x|^{2p\gamma/q}. \quad (4.1)$$

According to considerations stated in [3, Section 7], we have that functions ν, μ satisfy Hypotheses 2.1 and 2.3.

Now, we will verify that $\nu(x)$ satisfies Hypothesis 3.1, for all $t: nq/(q^2 - n) < t < n/\gamma$. To this aim, let $G \subset \Omega \setminus \{0\}$ be a “regular set,” and fix $y \in G, \rho > 0: \overline{B(y, \rho)} \subset G$.

If $|y| < 2\rho$, it follows that $B(y, \rho) \subset B(0, 3\rho)$. Hence, we have

$$\begin{aligned} \int_{B(y, \rho)} \frac{1}{|x|^{\gamma t}} dx &\leq \int_{B(0, 3\rho)} \frac{1}{|x|^{\gamma t}} dx = n\chi_n \int_0^{3\rho} r^{n-1-\gamma t} dr = n\chi_n \frac{3^{n-\gamma t}}{n-\gamma t} \rho^{n-\gamma t}, \\ \int_{B(y, \rho)} |x|^{\gamma \tau} dx &\leq \int_{B(0, 3\rho)} |x|^{\gamma \tau} dx = n\chi_n \frac{3^{n+\gamma \tau}}{n+\gamma \tau} \rho^{n+\gamma \tau}. \end{aligned} \quad (4.2)$$

From (4.2), taking into account that $\tau > nt/(qt - n)$, we get

$$\left(\rho^{-n} \int_{B(y, \rho)} \frac{1}{|x|^{\gamma t}} dx \right)^{1/t} \left(\rho^{-n} \int_{B(y, \rho)} |x|^{\gamma \tau} dx \right)^{1/\tau} \leq (n\chi_n + 1) 3^n \left(\frac{1}{n-\gamma t} + 1 \right) \quad \text{if } |y| < 2\rho. \quad (4.3)$$

Instead if $|y| \geq 2\rho$, we denote by Ξ that

$$\Xi = \left\{ k \in \mathbb{N} : \frac{|y|}{\rho} \geq k \right\}. \quad (4.4)$$

Note that $\Xi \neq \emptyset$ and is bounded from above. Consequently, if we denote $\bar{k} = \max \Xi$, we obtain

$$\bar{k}\rho \leq |y| < \rho(\bar{k} + 1). \quad (4.5)$$

Last inequality implies that for every $x \in B(y, \rho)$, it results that

$$(\bar{k} - 1)\rho \leq |x| \leq (\bar{k} + 2)\rho. \quad (4.6)$$

From (4.6), we obtain

$$\begin{aligned} \int_{B(y, \rho)} \frac{1}{|x|^{\gamma t}} dx &\leq \frac{\chi_n}{(\bar{k} - 1)^{\gamma t}} \rho^{n-\gamma t}, \\ \int_{B(y, \rho)} |x|^{\gamma \tau} dx &\leq \chi_n (\bar{k} + 2)^{\gamma \tau} \rho^{n+\gamma \tau}, \end{aligned} \quad (4.7)$$

where χ_n is the measure of the unit ball in \mathbb{R}^n .

Therefore, we get

$$\left(\rho^{-n} \int_{B(y,\rho)} \frac{1}{|x|^{\gamma t}}\right)^{1/t} \left(\rho^{-n} \int_{B(y,\rho)} |x|^{\gamma t} dx\right)^{1/\tau} \leq 4^n (\chi_n + 1) \quad \text{if } |y| \geq 2\rho. \quad (4.8)$$

We can conclude that (3.1) holds with $c' = 4^n(n\chi_n + 1)(1/(n - \gamma t) + 1)$.

Next, let $f : \Omega \rightarrow \mathbb{R}$ be the function such that for every $x \in \Omega \setminus \{0\}$,

$$f(x) = \frac{|x|^{-n}}{(1 - \lg|x|)^2} + \frac{1}{\sqrt{1 - |x|}}. \quad (4.9)$$

Observe that $f(x) \in L^1(\Omega)$ but $f(x)$ does not belong to $L^\gamma(\Omega)$, for every $\gamma > 1$.

Let $\sigma > 1$, we consider the following Dirichlet problem:

$$\begin{aligned} - \sum_{|\alpha|=1} D^\alpha \left[\nu \left(\sum_{|\beta|=1} |D^\beta u|^2 \right)^{(q-2)/2} D^\alpha u \right] + \sum_{|\alpha|=2} D^\alpha \left[\mu \left(\sum_{|\beta|=2} |D^\beta u|^2 \right)^{(p-2)/2} D^\alpha u \right] \\ = -|u|^{\sigma-1} u + f \quad \text{in } \Omega, \\ D^\alpha u = 0, \quad |\alpha| = 0, 1, \text{ on } \partial\Omega. \end{aligned} \quad (4.10)$$

By Theorem 2.7, we establish that there exists a W -solution \bar{u} of problem (4.10), bounded in every “regular set” $G \subset \Omega \setminus \{0\}$, and moreover, applying our result, Hölderian in every open set $A : \bar{A} \subset \Omega \setminus \{0\}$.

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