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Quantum Non-Markovian Piecewise Dynamics from Collision Models

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Abstract. Recently, a large class of quantum non-Markovian piecewise dynamics for an open quantum system obeying closed evolution equations has been introduced [1]. These dynamics have been defined in terms of a waiting-time distribution between quantum jumps, along with quantum maps describing the effect of jumps and the system evolution between them. Here, we present a quantum collision model with memory, whose reduced dynamics in the continuous-time limit reproduces the above class of non-Markovian piecewise dynamics, thus providing an explicit microscopic realization.

Keywords: Open quantum system; collision model; Lindblad equation; non-Markovian dynamics.

1. Introduction

Prompted by the growing impact of quantum technologies, the study of non-Markovian (NM) quantum dynamics is currently a topical field [2 – 5]. Be-

sides the goal of defining, witnessing and even quantifying on a rigorous basis the degree of quantum "non-Markovianity" of an open system dynamics, efforts are under way to advance the longstanding quest for the non-Markovian counterpart of the celebrated Gorini-Kossakowski-Lindblad-Sudarshan master equation (ME) [6, 7]. As a pivotal requisite, which may be easily violated [8, 9], a well-defined NM ME must entail a completely positive and trace-preserving (CPT) dynamics for an arbitrary initial state and for suitably large classes of operators and parameters appearing in its expression. While the set of known NM dynamics described by well-behaved MEs (in the above sense) is still relatively small, remarkable progress was made in the last few years.

A relatively new approach to quantum NM dynamics is based on quantum collision models (CMs) [10 – 18]. A CM is a simple microscopic framework for describing the open dynamics of a system S in contact with a bath, where the latter is assumed to consist of a large number of elementary subsystems, the "ancillas". The open dynamics of S results from its successive pairwise *collisions* with the bath ancillas, each collision being typically described by a bipartite unitary on S and the involved ancilla.

In the continuous-time limit, a CM leads to a Lindblad ME with no need to resort to the Born-Markov approximation [19]. Such appealing property prompted NM generalizations of the simplest memoryless CM, whose continuous-time-limit dynamics is ensured by construction to be CPT. A significant instance is the CM in [13, 20], recently extended in [16], which produced a new NM memory-kernel ME. The peculiar structure of this memory-kernel ME and the corresponding dynamical map inspired further investigations [21–23,1,24] from different *viewpoints*, which allowed to further enlarge the class of known NM dynamics governed by well-defined MEs.

One of these viewpoints builds on the well-known quantum-jumps picture of the Lindblad ME [25 – 28] to devise a far larger, NM class of piecewise dynamics characterized by a waiting time distribution, a CPT map describing the effect of jumps and a collection of CPT maps accounting for the evolution between jumps [21, 22, 1]. This class of piecewise dynamics obeys a memory-kernel ME [1]. Given that this general ME encompasses the reduced ME of the CM in [16] only as a special case, it is natural to wonder whether a generalized CM can be constructed giving rise to the piecewise-dynamics ME with no restrictions. In this work, we prove that such a CM indeed exists and show that it can be defined as a non-trivial generalization of [16] where collisions occur in the form of probabilistic SWAP operations. Among its major distinctive features are the *doubling* of each ancilla into a pair of subancillas, which allows to introduce the jump map that was fully absent in [16], and the introduction of *time-step-dependent* swap probabilities, which allows to reproduce waiting time distributions of arbitrary shape unlike [16] that was restricted to exponential ones. This extension is of particular importance

to comply with possible experimental implementations as well as encompass all the different features of the interaction dynamics that might give rise to non-Markovianity.

This paper is organized as follows. In Sect. 2, we review the class of NM quantum dynamics introduced in [21, 22, 1]. As anticipated, the main purpose of this work is demonstrating the existence of a quantum CM with memory, which in the continuous-time limit reproduces the above class of NM dynamics. Since this CM is an extension of the one in [16], the latter is reviewed in Sect. 3 and a brief introduction to quantum CMs is provided. These introductory sections, in particular, allow us to introduce most of the notation and formalism that we use later in Sect. 4, where the main results of this work are presented. Owing to its central importance, Sect. 4 is structured in a number of subsections so as to better highlight the different essential aspects of the proposed CM: the initial state, the way system-ancilla collisions are modelled, the discrete dynamics, its continuous-time limit and, at last, the reduced dynamics of the open system. Our conclusions along with some comments and outlook are given in Sect. 5. Some technical proofs are presented in Appendix A.

2. Review of Non-Markovian Piecewise Quantum Dynamics

The prototypical Markovian dynamics of an open quantum system S is described by the Gorini-Kossakowski-Lindblad-Sudarshan ME [6, 7], which reads

$$\dot{\rho} \;=\; -i[\hat{H},\rho] + \sum_k \gamma_k \Big(\hat{L}_k \rho \hat{L}_k^\dagger - \frac{1}{2}\{\hat{L}_k^\dagger \hat{L}_k, \rho\} \Big)\,, \tag{2.1}$$

where $\rho(t)$ is the S density operator, $\{\,\cdot\,,\,\cdot\,\}$ stands for the anticommutator, $\hat{H}$ is a Hermitian operator, $\{\gamma_k\}$ are positive rates, and where $\{\hat{L}_k\}$ are jump operators. By introducing the maps

$$\mathcal{R}_t[\rho] \;=\; e^{\hat{R}t}\rho\, e^{\hat{R}^\dagger t}\,, \qquad \mathcal{J}[\rho] \;=\; \sum_k \gamma_k\, \hat{L}_k \rho \hat{L}_k^\dagger\,, \tag{2.2}$$

where we defined the non-Hermitian operator $\hat{R} = -i\hat{H} - \frac{1}{2}\sum_k \gamma_k \hat{L}_k^\dagger \hat{L}_k$, the solution of the Lindblad ME (2.1) can be written as the Dyson series [26]

$$\rho_t \;=\; \mathcal{R}_t[\rho_0] + \sum_{j=1}^{\infty} \int_0^t \mathrm{d}t_j \ldots \int_0^{t_2} \mathrm{d}t_1 \;\ldots \mathcal{R}_{t-t_j}\mathcal{J} \ldots \mathcal{J}\mathcal{R}_{t_2-t_1}\mathcal{J}\mathcal{R}_{t_1}[\rho_0] \tag{2.3}$$

with $0 \le t_1 \le t_2 \le \ldots \le t$. Equation (2.3) shows that the time evolution of S can be viewed as an underlying dynamics described by the evolution map $\mathcal{R}_t$ interrupted by jumps each transforming the system state according

to the jump map $\mathcal{J}$. Index j in (2.3) indeed represents the number of jumps occurred up to time t at instants $\{t_1, t_2, \ldots, t_j\}$ such that $0 \leq t_1 \leq t_2 \leq \ldots \leq t_j \leq t$. Note that the maps (2.2) are *not* trace-preserving.

Both the Lindblad ME (2.1) and the representation (2.3) for its exact solution have been taken as a starting point for possible generalizations leading to well-defined dynamics to be described by means of memory kernel MEs, which can describe memory effects in the time evolution. Starting from the seminal work in [29], different approaches have been devised along this line [30 – 34]. One of us recently extended these results investigating a NM generalization of (2.3) [21, 22, 1], which in its most general form can be expressed as [1]

$$\begin{aligned} \rho_t \;&=\; g(t)\bar{\mathcal{E}}_t[\rho_0] + \sum_{j=1}^{\infty} \int_0^t \mathrm{d}t_j \ldots \int_0^{t_2} \mathrm{d}t_1 \\ &\quad \times f(t-t_j)\ldots f(t_2-t_1) g(t_1) \mathcal{E}_{t-t_j} \mathcal{Z} \ldots \mathcal{Z}\mathcal{E}_{t_2-t_1} \mathcal{Z}\bar{\mathcal{E}}_{t_1}[\rho_0]\,. \end{aligned} \tag{2.4}$$

Compared to (2.3), the jump map $\mathcal{J}$ (see (2.2)) is turned into the CPT map $\mathcal{Z}$, while $\mathcal{R}_t$ is replaced by the CPT evolution map $\bar{\mathcal{E}}_t$ *before* any jump has occurred and by the CPT evolution map $\mathcal{E}_t$ *after* the first jump (if any) has taken place. Maps $\mathcal{Z}$, $\bar{\mathcal{E}}_t$ and $\mathcal{E}_t$ are fully unspecified, but for the requirement of being CPT. Importantly, while in (2.3) the statistical weight of each possible trajectory is determined by the non-trace-preserving maps $\mathcal{R}_t$, $\mathcal{J}$ and the initial state [22], in (2.4) these statistical weights are assigned independently of the maps $\bar{\mathcal{E}}_t$, $\mathcal{E}_t$, $\mathcal{Z}$ and the initial state. Indeed, the functions $f(t)$ and $g(t)$, appearing in (2.4), stand for an arbitrarily chosen *waiting time distribution*, namely the probability density for the distribution in time of the jumps, and its associated *survival probability* $g(t)=1-\int_0^t \mathrm{d}t' f(t')$, that is the probability that no jump has taken place up to time t. The waiting time distribution and the associated survival probability can always be expressed in the form

$$g(t) \;=\; \exp\Big[-\int_0^t \mathrm{d}s\,\phi(s)\Big]\,, \qquad f(t) \;=\; \phi(t)\,\exp\Big[-\int_0^t \mathrm{d}s\,\phi(s)\Big]\,, \tag{2.5}$$

where the positive function

$$\phi(t) \;=\; \frac{f(t)}{g(t)} \tag{2.6}$$

is known as *hazard rate function* or simply *hazard function* [35]. The meaning of this is that $\phi(t)\mathrm{d}t$ provides the probability for a jump to take place in the time interval $(t, t+\mathrm{d}t]$, given that no jump has taken place up to time t. Accordingly, the time-dependent coefficient $f(t-t_j)\ldots f(t_2-t_1)g(t_1)$ in (2.4)

gives the probability density that j jumps take place at times $\{t_1, t_2, \ldots, t_j\}$, while the pre-factor of the first term on the rhs is the probability that no jumps occurred up to time t (this pre-factor indeed multiplies the jump-free evolution map $\bar{\mathcal{E}}_t$). The jumps are thus distributed in time according to a renewal process, which in particular entails that after each jump the process starts anew. Note that, in fact by construction, the dynamical map defined by (2.4) is ensured to be CPT. Importantly, it can be shown [1] that it obeys the memory-kernel ME

$$\dot{\rho} \;=\; \int_0^t \mathrm{d}t'\, \mathcal{W}(t-t')[\rho(t')] + \mathcal{I}(t)[\rho_0]\,, \tag{2.7}$$

where

$$\mathcal{W}(t) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left[f(t)\mathcal{E}_t\right]\mathcal{Z} + \delta(t) f(0)\mathcal{E}_0\mathcal{Z}\,, \qquad \mathcal{I}(t) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left[g(t)\bar{\mathcal{E}}_t\right]. \tag{2.8}$$

The corresponding open dynamics, at variance with the Lindbladian case, see (2.1) and (2.3), is in general NM [21].

3. Collision Models with Memory

A quantum CM [36, 37, 38] is a simple microscopic model for describing the open dynamics of a system S in contact with a bath B. In its prototypical version, a CM assumes that B comprises a huge number of elementary, identical and non-interacting ancillas all initialized in the same state η. The S-B interaction process occurs via successive pairwise "collisions" between S and the ancillas, each of these collisions being described by a bipartite unitary operation $\hat{U}_n$. By hypothesis, S can collide with each ancilla only once. After n collisions, the state of S is given by $\rho_n{=}\Phi^n[\rho_0]$, where the CPT map Φ is defined as $\Phi[\rho] = \mathrm{Tr}_n\{\hat{U}_n(\rho\otimes\eta_n)\hat{U}_n^\dagger\}$. Note that, despite the apparent dependance on n (see e.g. the partial trace over the nth ancilla), the map Φ does not depend on n since the bath initial state and system-ancilla interaction Hamiltonian are fully homogeneous. It can be shown [19] that in the continuous-time limit the dynamics of such a simple CM is described by a Lindblad ME of the form (2.1), a result which can be expected based on the discrete semigroup property enjoyed by the collision map, $\Phi^{n+m} = \Phi^n\Phi^m$. The open dynamics of S corresponding to such a paradigmatic CM is thereby fully Markovian.

There are several ways to endow the basic CM just described with memory so as to give rise to a NM dynamics [10,13–15,17,18]. The one of concern to us, given the goals of this paper, is the CM with memory of [16], which can be regarded as a generalization of a model first put forward in [13, 20]. The

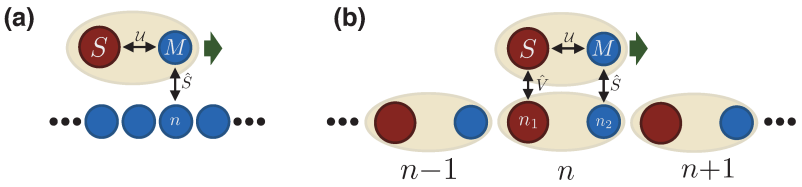

Fig. 1: (a) Sketch of the considered CMs: the system undergoing collisions with the bath ancillas is bipartite, comprising the very open system S and the memory M, with the latter having the same dimension as each bath ancilla. Only M is directly involved in collisions with ancillas. Each collision swaps the states of M and the nth ancilla in a probabilistic way by means of the transformation $\hat{S}$ acting on M and n. (b) Sketch of the generalized CM with memory of Sect. 4: ancillas are now bipartite, the nth of which comprising a subancilla n_1 (n_2) having the same dimension as S (M). Now both S and M are directly involved in collisions with ancillas. At each collision with some probability the states of M and n_2 are swapped, and at the same time the bipartite unitary $\hat{V}$ is applied on S and n_1.

general structure of the CM in [16] is in many respects analogous to the basic memoryless CM described in the previous paragraph except that the system undergoing collisions with the bath ancillas is now bipartite, comprising the very open system under study S *plus* an auxiliary system M, the "memory", whose Hilbert space dimension is the same as each ancilla's one. A sketch of the CM is given in Fig. 1(a). Systems S and M interact all the time according to the pairwise unitary evolution map

$$\mathcal{U}_\tau[\sigma] \;=\; e^{-i\hat{H}_{SM}\tau}\sigma\, e^{i\hat{H}_{SM}\tau}\,, \tag{3.1}$$

where $\hat{H}_{SM}$ is the joint S-M Hamiltonian. Here and throughout this paper, σ stands for a joint state of the S-M system *and* all the bath ancillas.

By hypothesis, only M is in direct contact with the bath (see Fig. 1(a)). This interaction takes place through successive collisions, each being described by the pairwise *non-unitary* quantum map

$$\mathcal{S}_n[\sigma] \;=\; p\,\sigma+(1-p)\hat{S}_{Mn}\sigma\,\hat{S}_{Mn}\,, \tag{3.2}$$

where $\hat{S}_{Mn}$ is the swap unitary operator exchanging the states of M and the nth ancilla. The CPT map (3.2), which depends parametrically on the probability p, can be interpreted as a probabilistic partial SWAP gate: the memory and ancilla states are either swapped or left unchanged with probability p.

The initial state of the overall system (S, M and the bath ancillas) is assumed to be

$$\sigma_0 \;=\; (\rho_0\otimes\bar{\eta}_M)\otimes\eta_1\otimes\eta_2\otimes\cdots \tag{3.3}$$

where in particular ρ_0 ($\bar{\eta}_M$) is the initial state of S (M). Throughout this paper, tensor product symbols will be omitted whenever possible to avoid using too cumbersome notation.

By calling σ_n the overall state at step n, the dynamics proceeds according to

$$\sigma_n \;=\; \mathcal{S}_n\,\mathcal{U}_\tau\,\mathcal{S}_{n-1}\,\mathcal{U}_\tau\ldots\mathcal{S}_2\,\mathcal{U}_\tau\,\mathcal{S}_1\,\mathcal{U}_\tau[\sigma_0]\,, \tag{3.4}$$

namely an S-M unitary dynamics goes on all the time, being interrupted at each *fixed* time step τ by a collision between M and a "fresh" ancilla (i.e. one still in the initial state η) described by the non-unitary map (3.2). Equivalently, one can view each $\mathcal{U}_\tau$ itself as embodying the effect of a unitary collision that is however internal to the joint S-M system in such a way that the overall CM dynamics results from subsequent M-ancilla collisions interspersed with internal ones that involve S and M only [16].

Like for any CM, the dynamics just defined is discrete. One can, however, define a *continuous-time limit* by assuming that the duration of each time step τ becomes very small while the step number n gets very large in such a way that $n\tau \to t$, where t is a continuous time variable. The assumption of a very large number of steps demands an additional prescription for the continuous-time limit of the probability p entering (3.2) since (3.4) clearly features p's powers $\{p^k\}$ for all positive integers $k \leq n$. This task is carried out by first defining a rate Γ that allows to express p as

$$p \;=\; e^{-\Gamma\tau} \tag{3.5}$$

(which is always possible) and assuming next that $\Gamma\tau \ll 1$ in such a way that $p \simeq 1$. This ensures that p^k, for any k smaller than n and yet large enough so that $k\tau \to t' < t$ is finite, be not washed out in the continuous-time limit. Indeed, this yields

$$p^k \;=\; (p^{\frac{1}{\tau}})^{k\tau} \;\longrightarrow\; e^{-\Gamma t'}. \tag{3.6}$$

By finally noting that, consistently with the hypothesis $\Gamma\tau \ll 1$, $1-p = 1-e^{-\Gamma\tau} \simeq \Gamma\tau$ (cf. (3.2)) and that since the CM is well defined for any choice of η, $\bar{\eta}$ and $\mathcal{U}_t$ it is possible to describe the reduced evolution of S by the following CPT map

$$\rho_t \;=\; e^{-\Gamma t}\bar{\mathcal{E}}_t[\rho_0] + \sum_{j=1}^{\infty}\Gamma^j e^{-\Gamma t}\int_0^t \mathrm{d}t_j \ldots \int_0^{t_2}\mathrm{d}t_1\mathcal{E}_{t-t_j}\ldots\mathcal{E}_{t_2-t_1}\bar{\mathcal{E}}_{t_1}[\rho_0]\,, \tag{3.7}$$

which is a special case of (2.4) for

$$\begin{aligned}\bar{\mathcal{E}}_t\rho \;&=\; \mathrm{Tr}_M\left\{\mathcal{U}_t[\rho\,\bar{\eta}_M]\right\}, \quad \mathcal{E}_t\rho \;=\; \mathrm{Tr}_M\left\{\mathcal{U}_t[\rho\,\eta_M]\right\}, \quad \mathcal{Z} \;=\; \mathbb{I},\\ f(t) \;&=\; \Gamma e^{-\Gamma t}, \quad g(t) \;=\; e^{-\Gamma t}, \quad \phi(t) \;=\; \Gamma\,.\end{aligned} \tag{3.8}$$

In fact it can be shown [16] that the map (3.7) obeys a memory-kernel ME of the form (2.7).

Yet, the CM in fact lacks the jump map $\mathcal{Z}$ and is, in addition, apparently constrained to a purely exponential waiting time distribution $f(t) = \Gamma e^{-\Gamma t}$ (the corresponding hazard function $\phi(t)$ being thus constant).

In the next section, we show how to construct a CM with memory whose continuous-time limit yields ME (2.7) in the most general case, including in particular an arbitrary jump map $\mathcal{Z}$ and an arbitrary waiting time distribution $f(t)$.

4. A Generalized Collision Model with Memory

The CM to be defined here is a non-trivial generalization of the CM of [16] reviewed in the last section. Just like in [16], the system undergoing collisions with the bath ancillas comprises S and a memory M that are subject to a coherent mutual coupling giving rise to the unitary evolution map (3.1). At variance with [16], however, now each bath ancilla is *bipartite* as well, consisting of a pair of "subancillas": one subancilla has the same Hilbert space dimension as S, while the other subancilla has the same dimension as M. A sketch of this generalized CM with memory is displayed in Fig. 1(b).

4.1. Initial state

The initial joint state reads

$$\sigma_0 \;=\; (\rho_0 \otimes \bar{\eta}_M) \otimes (\xi_1 \otimes \eta_1) \otimes (\xi_2 \otimes \eta_2) \otimes \dots \,, \tag{4.1}$$

where ξ (η) is the initial state of the subancilla having the same dimension as S (M). In full analogy with (3.3), ρ_0 ($\bar{\eta}_M$) is the initial state of S (M).

4.2. System-ancilla collisions

A further distinctive feature of the generalized CM with memory is that the collisions with the ancillas now involve S as well. By definition, the collision between S-M and the nth bipartite ancilla is described by the non-unitary *four*-partite CPT map

$$\mathcal{S}_n[\sigma] \;=\; p_n\sigma + (1-p_n)\,\hat{V}_{Sn_1}\hat{S}_{Mn_2}\sigma\hat{S}^{\dagger}_{Mn_2}\hat{V}^{\dagger}_{Sn_1}\,, \tag{4.2}$$

where n_1 and n_2 are the two n's subancillas that are isodimensional to S and M, respectively, while $\hat{V}_{Sn_1}$ is a unitary operator acting on S and subancilla n_1. Map (4.2) therefore swaps the states of M and n_2 and, at the same time, applies the unitary $\hat{V}_{Sn_1}$ on S and n_1, or leaves unchanged the state

of S, M, n_1 and n_2 with probability p_n. Note that, unlike the CM of the previous Section (cf. (3.2)), now we allow the probability p_n to be in general step-dependent. The reason for this will become clear later on.

Based on (4.2) and the ancilla's initial state (cf. (4.1)), it is convenient to define a bipartite CPT map on S and M as

$$\begin{aligned}\widetilde{\mathcal{Z}}\,[\rho_{SM}] &= \mathrm{Tr}_{n_1 n_2}\left\{\hat{V}_{Sn_1}\hat{S}_{Mn_2}\,(\rho_{SM}\otimes\xi_{n_1}\otimes\eta_{n_2})\;\hat{S}^{\dagger}_{Mn_2}\hat{V}^{\dagger}_{Sn_1}\right\}\\ &= \mathcal{Z}[\mathrm{Tr}_M\{\rho_{SM}\}]\otimes\eta_M\,,\end{aligned}\tag{4.3}$$

where $\mathcal{Z}$ is the CPT map on S defined by

$$\mathcal{Z}[\rho] \;=\; \mathrm{Tr}_{n_1}\left\{\hat{V}_{Sn_1}\rho\otimes\xi_{n_1}\hat{V}^{\dagger}_{Sn_1}\right\}.\tag{4.4}$$

The proof of the last step in (4.3) is given in Appendix A.

Equations (4.3) and (4.4) entail that the collision with the nth ancilla (see (4.2)) changes the *reduced* state of S and M, ρ_{SM}, according to

$$\begin{aligned}\mathrm{Tr}_{n_1 n_2}\{\mathcal{S}_n(\rho_{SM}\,\xi_{n_1}\,\eta_{n_2})\} &= p_n\,\rho_{SM}+(1-p_n)\,\widetilde{\mathcal{Z}}[\rho_{SM}]\\ &= p_n\,\rho_{SM}+(1-p_n)\,\mathcal{Z}[\mathrm{Tr}_M\{\rho_{SM}\}]\otimes\eta_M\,.\end{aligned}\tag{4.5}$$

The essential effect of the collision, thereby, is to either leave with probability p_n the S-M state unchanged or, with probability $1-p_n$, to apply the CPT map $\mathcal{Z}$ on S by simultaneously resetting the M's state to η.

4.3. Discrete dynamics

Similarly to the CM in [16] (see previous section), the initial state (4.1) evolves through an underlying S-M unitary dynamics that is interrupted at each *fixed* time step τ by a collision described by (4.2) involving a fresh bipartite ancilla that is still in state $\xi\otimes\eta$. Accordingly, the overall state at the nth step is given by $\sigma_n=\mathcal{S}_n\,U_\tau\,\mathcal{S}_{n-1}\,\mathcal{U}_\tau\,\ldots\mathcal{S}_2\,\mathcal{U}_\tau\,\mathcal{S}_1\,\mathcal{U}_\tau[\sigma_0]$.

Starting from $\rho^{(0)}_{SM}=\rho_0\otimes\bar{\eta}_M$ (see (4.1)), at the end of the first step the reduced S-M state is turned into

$$\rho^{(1)}_{SM} \;=\; \mathcal{U}_\tau[\rho^{(0)}_{SM}]\,.\tag{4.6}$$

Next, the collision with ancilla 1 described by map $\mathcal{S}_1$ (see (4.2)) takes place followed by another application of the S-M unitary. At the end of the second step, the S-M state thus reads

$$\begin{aligned}\rho^{(2)}_{SM} &= \mathrm{Tr}_{1_1 1_2}\left\{\mathcal{U}_\tau\mathcal{S}_1\left[\rho^{(1)}_{SM}\,\xi_{1_1}\eta_{1_2}\right]\right\}=\mathrm{Tr}_{1_1 1_2}\left\{\left(p_1\mathcal{U}_\tau+q_1\mathcal{U}_\tau\tilde{\mathcal{Z}}\right)\left[\rho^{(1)}_{SM}\,\xi_{1_1}\eta_{1_2}\right]\right\}\\ &= p_1\,\mathcal{U}_\tau\left[\rho^{(1)}_{SM}\right]+q_1\,\mathcal{U}_\tau\tilde{\mathcal{Z}}\left[\rho^{(1)}_{SM}\right]\,,\end{aligned}\tag{4.7}$$

where the trace is taken over the nth ancilla for $n = 1$ and to simplify the notation we set $q_n = 1 - p_n$. By replacing in the last identity the state at the end of the first step (4.6), (4.7) can be expressed as a function of the initial S-M state only as

$$\rho_{SM}^{(2)} = \underbrace{p_1 \mathcal{U}_\tau^2}_{0\,\text{jumps}}[\rho_{SM}^{(0)}] + \underbrace{q_1 \mathcal{U}_\tau \widetilde{\mathcal{Z}} \mathcal{U}_\tau}_{1\,\text{jump}}[\rho_{SM}^{(0)}]. \tag{4.8}$$

Since the elapsed time of the process is an integer multiple of the time step τ and given that a jump (if any) occurs at the end of each time step τ, at the second step either 0 or 1 jumps have taken place. The former and latter cases correspond to the terms featuring zero or one $\widetilde{\mathcal{Z}}$ in (4.8) as highlighted by the captions. At the end of the 3rd step, after the application of maps $\mathcal{S}_2$ and $\mathcal{U}_\tau$, an analogous calculation leads to

$$\begin{aligned} \rho_{SM}^{(3)} &= p_2 \mathcal{U}_\tau[\rho_{SM}^{(2)}] + q_2 \mathcal{U}_\tau \widetilde{\mathcal{Z}}[\rho_{SM}^{(2)}] \\ &= \underbrace{p_2 p_1 \mathcal{U}_\tau^3}_{0\,\text{jumps}}[\rho_{SM}^{(0)}] + \underbrace{(p_2 q_1 \mathcal{U}_\tau^2 \widetilde{\mathcal{Z}} \mathcal{U}_\tau + q_2 p_1 \mathcal{U}_\tau \widetilde{\mathcal{Z}} \mathcal{U}_\tau^2)}_{1\,\text{jump}}[\rho_{SM}^{(0)}] \\ &\quad + \underbrace{q_2 q_1 \mathcal{U}_\tau \widetilde{\mathcal{Z}} \mathcal{U}_\tau \widetilde{\mathcal{Z}} \mathcal{U}_\tau}_{2\,\text{jumps}}[\rho_{SM}^{(0)}]\,, \end{aligned} \tag{4.9}$$

showing that, as expected, 0, 1 or 2 jumps are possible in this case corresponding to as many applications of the map $\widetilde{\mathcal{Z}}$. In a similar fashion, at the 4th step we get

$$\begin{aligned} \rho_{SM}^{(4)} &= p_3 \mathcal{U}_\tau[\rho_{SM}^{(3)}] + q_3 \mathcal{U}_\tau \widetilde{\mathcal{Z}}[\rho_{SM}^{(3)}] \\ &= \underbrace{p_3 p_2 p_1 \mathcal{U}_\tau^4}_{0\,\text{jumps}}[\rho_{SM}^{(0)}] \\ &\quad + \underbrace{(p_3 p_2 q_1 \mathcal{U}_\tau^3 \widetilde{\mathcal{Z}} \mathcal{U}_\tau + p_3 q_2 p_1 \mathcal{U}_\tau^2 \widetilde{\mathcal{Z}} \mathcal{U}_\tau^2 + q_3 q_2 p_1 \mathcal{U}_\tau \widetilde{\mathcal{Z}} \mathcal{U}_\tau^3)}_{1\,\text{jump}}[\rho_{SM}^{(0)}] \\ &\quad + \underbrace{(p_3 q_2 q_1 \mathcal{U}_\tau^2 \widetilde{\mathcal{Z}} \mathcal{U}_\tau \widetilde{\mathcal{Z}} \mathcal{U}_\tau + q_3 p_2 q_1 \mathcal{U}_\tau \widetilde{\mathcal{Z}} \mathcal{U}_\tau^2 \widetilde{\mathcal{Z}} \mathcal{U}_\tau + q_3 q_2 p_1 \mathcal{U}_\tau \widetilde{\mathcal{Z}} \mathcal{U}_\tau \widetilde{\mathcal{Z}} \mathcal{U}_\tau^2)}_{2\,\text{jumps}}[\rho_{SM}^{(0)}] \\ &\quad + \underbrace{q_3 q_2 q_1 \mathcal{U}_\tau \widetilde{\mathcal{Z}} \mathcal{U}_\tau \widetilde{\mathcal{Z}} \mathcal{U}_\tau \widetilde{\mathcal{Z}} \mathcal{U}_\tau}_{3\,\text{jumps}}[\rho_{SM}^{(0)}]\,. \end{aligned} \tag{4.10}$$

In order to write down the n-step state in a compact form, having in mind the structure of (2.4), we first note that based on (3.1) any k-th power of the map $\mathcal{U}_\tau$ is given by $\mathcal{U}_\tau^k = \mathcal{U}_{k\tau}$ (in the following we will further set $\mathcal{U}_k \equiv \mathcal{U}_{k\tau}$ to simplify the notation).

By induction, the n-step state for arbitrary $n \geq 2$ is given by

$$\rho_{SM}^{(n)} = \Big(\prod_{\ell=1}^{n-1} p_\ell\Big)\,\mathcal{U}_n[\rho_{SM}^{(0)}] + \sum_{j=1}^{n-1}\sum_{k_j=1}^{n-1}\sum_{k_{j-1}=1}^{k_j-1}\cdots\sum_{k_1=1}^{k_2-1}\pi(k_j,\ldots,k_1)\,\mathcal{U}_{n-k_j}\,\tilde{\mathcal{Z}}\,\mathcal{U}_{k_j-k_{j-1}}\tilde{\mathcal{Z}}\ldots\tilde{\mathcal{Z}}\,\mathcal{U}_{k_2-k_1}\,\tilde{\mathcal{Z}}\,\mathcal{U}_{k_1}[\rho_{SM}^{(0)}]\,, \tag{4.11}$$

where $\pi(k_j,\ldots,k_1)$ stands for the probability to perform exactly j jumps at specific steps $\{k_j,\ldots,k_1\}$ and reads

$$\pi(k_j,\ldots,k_1) = \Big(\prod_{\ell=k_j+1}^{n-1} p_\ell\Big)q_{k_j}\Big(\prod_{\ell=k_{j-1}+1}^{k_j-1} p_\ell\Big)q_{k_{j-1}}\cdots q_{k_2}\Big(\prod_{\ell=k_1+1}^{k_2-1} p_\ell\Big)q_{k_1}\Big(\prod_{\ell=1}^{k_1-1} p_\ell\Big)\,. \tag{4.12}$$

4.4. Continuous-time limit

In order to perform the continuous-time limit, in analogy with (3.5) we introduce the quantities

$$p(t_k - t_{k-1}) = e^{-\int_0^{t_k-t_{k-1}}\mathrm{d}s\phi(s)}, \quad q(t_k - t_{k-1}) = 1 - e^{-\int_0^{t_k-t_{k-1}}\mathrm{d}s\phi(s)},$$

corresponding respectively to the probability of no jump or one jump to take place in each small time interval $t_k - t_{k-1}$, which in the case of constant hazard function $\phi(s)$ reduces to a Poisson distribution for the jumps. According to the definition of a renewal process, the jump probabilities thereby depend only on the elapsed time. In this representation, the various contributions appearing in (4.12), in the limit of a large number of short steps such that the time intervals between steps become increasingly small, can be written as

$$\begin{aligned}\Big(\prod_{\ell=k+1}^{j} p_\ell\Big)q_k &= \prod_{\ell=k+1}^{j} e^{-\int_0^{t_l-t_{l-1}}\mathrm{d}s\phi(s)}\Big(1 - e^{-\int_0^{t_k-t_{k-1}}\mathrm{d}s\phi(s)}\Big)\\ &\approx e^{-\int_0^{t_j-t_k}\mathrm{d}s\phi(s)} - e^{-\int_0^{t_j-t_{k-1}}\mathrm{d}s\phi(s)}\\ &\approx \phi(t_j - t_{k-1})\,e^{-\int_0^{t_j-t_{k-1}}\mathrm{d}s\phi(s)}(t_k - t_{k-1})\,.\end{aligned}$$

This shows that the function $\phi(t)$ has indeed the role of hazard rate function (cf. (2.5)), which determines the renewal process describing the time distribution of jumps. Hence, we can thus finally identify

$$\Big(\prod_{\ell=k+1}^{j} p_\ell\Big)q_k \approx f(t_j - t_{k-1})\mathrm{d}t_{k-1}\,. \tag{4.13}$$

The first term in (4.11) accordingly becomes

$$\prod_{\ell=1}^{k} p_l \;\longrightarrow\; e^{-\int_0^{t_k}\phi(s)ds} \;=\; g(t_k)\,. \tag{4.14}$$

Therefore, (4.11) in the continuous-time limit reads

$$\begin{aligned}\rho_{SM}(t) \;&=\; g(t)\,\mathcal{U}_t[\rho_{SM}(0)] \\ &\quad+\sum_{j=1}^{\infty}\int_0^t \mathrm{d}t_j \ldots \int_0^{t_2}\mathrm{d}t_1 f(t-t_j)\ldots f(t_2-t_1)g(t_1) \\ &\qquad\qquad\times\; \mathcal{U}_{t-t_j}\,\widetilde{\mathcal{Z}}\,\ldots\,\widetilde{\mathcal{Z}}\,\mathcal{U}_{t_2-t_1}\,\widetilde{\mathcal{Z}}\,\mathcal{U}_{t_1}[\rho_{SM}(0)]\,.\end{aligned} \tag{4.15}$$

4.5. Reduced dynamics

So far, we have focused on the bipartite system S-M, working out its evolution. We now consider the resulting reduced dynamics for the system S, which embodies the degrees of freedom of the open quantum system of interest. We first recall that $\rho_{SM}(0) = \rho_0 \otimes \bar{\eta}_M$ (see (4.1)), which ensures the existence of the reduced dynamical map of S. When this expression is replaced in (4.15) upon taking the trace over M we get

$$\begin{aligned}\rho(t) \;&=\; g(t)\mathrm{Tr}_M\{\mathcal{U}_t[\rho_0\otimes\bar{\eta}_M]\} \\ &\quad+\sum_{j=1}^{\infty}\int_0^t \mathrm{d}t_j \ldots \int_0^{t_2}\mathrm{d}t_1 f(t-t_j)\ldots f(t_2-t_1)g(t_1) \\ &\qquad\qquad\times\; \mathrm{Tr}_M\left\{\mathcal{U}_{t-t_j}\,\widetilde{\mathcal{Z}}\,\ldots\,\widetilde{\mathcal{Z}}\,\mathcal{U}_{t_2-t_1}\,\widetilde{\mathcal{Z}}\,\mathcal{U}_{t_1}[\rho_0\otimes\bar{\eta}_M]\right\}.\end{aligned} \tag{4.16}$$

By next introducing, according to (3.8), the CPT maps $\mathcal{E}_t$ and $\bar{\mathcal{E}}_t$, whose definition is thus identical to the model in [16], and recalling (4.3) and (4.4), we get

$$\begin{aligned}\rho(t) \;&=\; g(t)\bar{\mathcal{E}}_t[\rho_0] + \sum_{j=1}^{\infty}\int_0^t \mathrm{d}t_j \ldots \int_0^{t_2}\mathrm{d}t_1 (t-t_j)\ldots f(t_2-t_1)g(t_1) \\ &\qquad\times\; \mathrm{Tr}_M\left\{\mathcal{U}_{t-t_j}\,\widetilde{\mathcal{Z}}\,\ldots\,\widetilde{\mathcal{Z}}\,\mathcal{U}_{t_2-t_1}[\mathcal{Z}[\bar{\mathcal{E}}_t[\rho_0]]\otimes\eta_M]\right\}.\end{aligned} \tag{4.17}$$

The argument of the partial trace can be expressed by iteration according to

$$\begin{aligned}&\mathcal{U}_{t-t_j}\,\widetilde{\mathcal{Z}}\,\ldots\,\widetilde{\mathcal{Z}}\,\mathcal{U}_{t_2-t_1}\left[\mathcal{Z}\left[\bar{\mathcal{E}}_t\left[\rho_0\right]\right]\otimes\eta_M\right] \\ &=\mathcal{U}_{t-t_j}\,\widetilde{\mathcal{Z}}\,\ldots\left[\mathcal{Z}\left[\,\mathcal{E}_{t_2-t_1}\left[\mathcal{Z}\,\left[\bar{\mathcal{E}}_t\left[\rho_0\right]\right]\right]\right]\right]\otimes\eta_M\right],\end{aligned}$$

which finally leads to the expression

$$\mathrm{Tr}_M \left\{ \mathcal{U}_{t-t_j} \widetilde{\mathcal{Z}} \dots \widetilde{\mathcal{Z}} \mathcal{U}_{t_2-t_1} \left[\mathcal{Z} \left[\bar{\mathcal{E}}_t[\rho_0] \right] \otimes \eta_M \right] \right\} = \mathcal{E}_{t-t_j} \mathcal{Z} \dots \mathcal{Z} \mathcal{E}_{t_2-t_1} \mathcal{Z} \bar{\mathcal{E}}_t[\rho_0] \,,$$

where for the sake of simplicity we have removed the nested square brackets in the last expression. When this result is replaced in (4.17), we end up with (2.4). Accordingly, the reduced dynamics of S in the continuous-time limit necessarily obeys ME (2.7) with no restrictions.

We can therefore conclude that the generalized collision model with memory constructed here is indeed able to reproduce altogether the piecewise NM dynamics with jumps considered in [1].

5. Conclusions and Outlook

The Gorini-Kossakowski-Lindblad-Sudarshan ME has been for over 40 years the workhorse of open quantum systems theory. It embodies the basic reference for open dynamics that lack memory effects. Clearly, though, in the case of strong coupling and/or structured reservoirs a memoryless Markovian description fails to faithfully capture the relevant features of the dynamics. Many non-trivial challenges follow, in particular the need for more general *evolution equations* that ensure a well-defined (i.e., CPT) dynamics and, at the same time, effectively describe memory effects. On top of this, it is highly desirable that these theoretical descriptions be associated with corresponding environmental models thus providing an underlying *microscopic* interpretation and, possibly, a controlled implementation of such non-Markovian dynamics.

Both the above aspects were the focus of this paper. Starting from a recently proposed family of memory-kernel MEs corresponding to a large class of generally non-Markovian time evolutions, we showed that *any* such ME admits a microscopic CM from which it can be obtained as the equation governing its continuous-time-limit reduced dynamics.

Specifically, the considered time evolutions consist of piecewise dynamics in which a continuous, generally non-Markovian, time evolution is interrupted at random times, distributed according to a general waiting time distribution, by a quantum jump described by a general CPT transformation. These dynamics obey a closed memory-kernel ME. In this work, we showed that one such ME can be obtained as the continuous-time limit of a CM where memory effects are due to auxiliary degrees of freedom (which we indeed called memory) mediating the action of the environment on the system. As a distinctive feature of the CM, each bath ancilla is *bipartite* comprising a pair of subancillas. Each collision occurs in the form of a map that, with some probability, swaps the state of the memory and one subancilla, while a unitary is at the same time applied on the system under study and the other

subancilla. As a further hallmark of the considered CM, the probability for such swap-and-unitary operation can *depend* at will on the time step.

As remarked in the main text, the ancillas' doubling along with the step-dependance of the aforementioned probability are the crucial features marking the difference between the CM in [16] and the one addressed here (which can thus be viewed as a non-trivial generalization of the former). They allow to introduce a jump map as well as a waiting time distribution of arbitrary shape.

It is interesting to note that the term "collisional model" was at times used in the literature (see e.g. [29]) to denote a quantum dynamics that is interrupted at random times by "collisions" — that is jumps in fact — just like in the framework addressed in [1]. In this respect, our work provides a connection between this definition of CM, based on *random*-time collisions, and the one used throughout the paper, where instead collisions occur at *fixed* times.

We finally point out that ME (2.7) was obtained in [1] within a general framework based on the quantization of a family of classical stochastic dynamics. Since this quantization involves *non-commuting* operators, ME (2.7) arises only as one of *two* possible cases corresponding to different operator orderings. The question whether or not a class of underlying CMs can be devised even for the ME arising in the other case [1] — which is qualitatively different from ME (2.7) — is under ongoing investigations.

Appendix A

We here provide the proof of the last identity in (4.3). Let us first recall the starting point, namely the definition of the map $\widetilde{\mathcal{Z}}$ given in the first line of (4.3), omitting the tensor product symbol to simplify the notation

$$\widetilde{\mathcal{Z}}[\rho_{SM}] \;=\; \mathrm{Tr}_{n_1 n_2}\left\{\hat{V}_{Sn_1}\hat{S}_{Mn_2}\rho_{SM}\xi_{n_1}\eta_{n_2}\,\hat{V}^{\dagger}_{Sn_1}\hat{S}^{\dagger}_{Mn_2}\right\}, \qquad \text{(A.1)}$$

and consider two orthonormal bases $\{|\mu\rangle_M\}$ and $\{|\nu\rangle_{n_2}\}$ in the Hilbert spaces of M and n_2, respectively. In terms of these vectors, the swap operator $\hat{S}_{Mn_2}$ is expressed as

$$\hat{S}_{Mn_2} \;=\; \sum_{\mu,\nu} |\mu\rangle\langle\nu|_M \otimes |\nu\rangle\langle\mu|_{n_2}\,.$$

Using this expression in (A.1) the rhs explicitly reads

$$\sum_{\substack{\mu,\nu\\ \mu',\nu'}} \mathrm{Tr}_{n_1 n_2}\Big\{\hat{V}_{Sn_1}|\mu\rangle\langle\nu|_M \otimes |\nu\rangle\langle\mu|_{n_2}\rho_{SM}\xi_{n_1}\eta_{n_2}|\nu'\rangle\langle\mu'|_M \otimes |\mu'\rangle\langle\nu'|_{n_2}\hat{V}^{\dagger}_{Sn_1}\Big\}$$

$$= \sum_{\substack{\mu,\nu\\ \mu',\nu'}} \mathrm{Tr}_{n_1 n_2}\Big\{\hat{V}_{Sn_1}|\mu\rangle_M \otimes |\nu\rangle_{n_2}\langle\nu|\rho_{SM}|\nu'\rangle_M\xi_{n_1}\langle\mu|\eta_{n_2}|\mu'\rangle_{n_2}\,{}_M\langle\mu'|\,{}_{n_2}\langle\nu'|\hat{V}^{\dagger}_{Sn_1}\Big\},$$

so that taking the partial trace over n_2 we end up with

$$\widetilde{\mathcal{Z}}[\rho_{SM}] \;=\; \sum_{\mu,\mu'} \mathrm{Tr}_{n_1}\Big\{\hat{V}_{Sn_1}|\mu\rangle_M \,\mathrm{Tr}_M\{\rho_{SM}\}\xi_{n_1}\langle\mu|\eta_{n_2}|\mu'\rangle_{n_2\,M}\langle\mu'|\,\hat{V}^\dagger_{Sn_1}\Big\}\,.$$

By recalling that the state ρ of the reduced system is just the marginal of ρ_{SM} and by noting that the expression in square brackets swaps η_{n_2} and η_M, we finally get

$$\begin{aligned}\widetilde{\mathcal{Z}}[\rho_{SM}] &= \mathrm{Tr}_{n_1}\Big\{\hat{V}_{Sn_1}\rho\,\xi_{n_1}\eta_M\hat{V}^\dagger_{Sn_1}\Big\}\\ &= \mathrm{Tr}_{n_1}\Big\{\hat{V}_{Sn_1}\rho\,\xi_{n_1}\hat{V}^\dagger_{Sn_1}\Big\}\otimes\eta_M \;=\; \mathcal{Z}[\rho]\otimes\eta_M\,,\end{aligned}$$

which according to the definition (4.4) of the map $\mathcal{Z}$ concludes the proof.

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Bibliography

[1] B. Vacchini, Phys. Rev. Lett. **117**, 230401 (2016).
[2] H.-P. Breuer, J. Phys. B **45**, 154001 (2012).
[3] A. Rivas, S. F. Huelga, and M. B. Plenio, Rep. Prog. Phys. **77**, 094001 (2014).
[4] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Rev. Mod. Phys. **88**, 021002 (2016).
[5] I. de Vega and D. Alonso, Rev. Mod. Phys. **89**, 015001 (2017).
[6] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).
[7] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
[8] S. M. Barnett and S. Stenholm, Phys. Rev. A **64**, 033808 (2001).
[9] S. Maniscalco and F. Petruccione, Phys. Rev. A **73**, 012111 (2006).
[10] T. Rybar, S. N. Filippov, M. Ziman, and V. Bužek, J. Phys. B **45**, 154006 (2012).
[11] V. Giovannetti and G. M. Palma, Phys. Rev. Lett. **108**, 040401 (2012).
[12] V. Giovannetti and G. M. Palma, J. Phys. B **45**, 154003 (2012).
[13] F. Ciccarello, G. M. Palma, and V. Giovannetti, Phys. Rev. A **87**, 040103 (2013).
[14] N. K. Bernardes, A. R. R. Carvalho, C. H. Monken, and M. F. Santos, Phys. Rev. A **90**, 032111 (2014).
[15] A. L. Grimsmo, Phys. Rev. Lett. **115**, 060402 (2015).
[16] S. Lorenzo, F. Ciccarello, and G. M. Palma, Phys. Rev. A **93**, 052111 (2016).
[17] S. Kretschmer, K. Luoma, and W. T. Strunz, Phys. Rev. A **94**, 012106 (2016).
[18] S. Lorenzo, F. Ciccarello, and G. M. Palma, arXiv:1705.03215 (2017).

[19] M. Ziman, P. Stelmachovic, and V. Bužek, Open Sys. Information Dyn. **12**, 81 (2005).

[20] F. Ciccarello and V. Giovannetti, Phys. Scr. **T153**, 014010 (2013).

[21] B. Vacchini, Phys. Rev. A **87**, 030101 (2013).

[22] B. Vacchini, Internat. J. Quant. Information **12**, 1461011 (2014).

[23] D. Chruściński and A. Kossakowski, Phys. Rev. A **94**, 020103(R) (2016).

[24] D. Chruściński and A. Kossakowski, Phys. Rev. A **95**, 042131 (2017).

[25] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, Oxford Univ. Press, 2002.

[26] A. S. Holevo, *Statistical Structure of Quantum Theory*, Lect. Notes Phys. **67**, Springer, 2001.

[27] A. Barchielli and M. Gregoratti, *Quantum Trajectories and Measurements in Continuous Time*, Lect. Notes Phys. **782**, Springer, 2009.

[28] A. Barchielli, *Some stochastic differential equations in quantum optics and measurement theory: the case of counting processes*, in *Stochastic Evolution of Quantum States in Open Systems and in Measurement Processes*, L. Diósi, B. Lukàcs, eds., World Scientific, 1994.

[29] A. A. Budini, Phys. Rev. A **69**, 042107 (2004).

[30] A. A. Budini, Phys. Rev. E **72**, 056106 (2005).

[31] H.-P. Breuer and B. Vacchini, Phys. Rev. Lett. **101**, 140402 (2008).

[32] A. Kossakowski and R. Rebolledo, Open Sys. Information Dyn. **16**, 259 (2009).

[33] A. A. Budini, Phys. Rev. A **88**, 032115 (2013).

[34] A. A. Budini, Phys. Rev. A **88**, 012124 (2013).

[35] S. M. Ross, *Introduction to Probability Models*, Academic Press, 2007.

[36] J. Rau, Phys. Rev. **129**, 1880 (1963).

[37] V. Scarani, M. Ziman, P. Štelmachovič, N. Gisin, and V. Bužek, Phys. Rev. Lett. **88**, 097905 (2002).

[38] M. Ziman, P. Štelmachovič, V. Bužek, M. Hillery, V. Scarani, and N. Gisin, Phys. Rev. A **65**, 042105 (2002).