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A STATE-SPACE APPROACH TO DYNAMIC STABILITY OF FRACTIONAL-ORDER SYSTEMS: THE EXTENDED ROUTH-HURWITZ THEOREM

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Abstract. *This paper considers the case of Beck's column, a linear elastic cantilever column subjected to a constant follower load at its free end. The column foundation is modeled as bed of hereditary elements that react with a vertical force distributed along the beam axis. The reacting supports are modeled with spring-pot element that is a two parameters mechanical elements* (C_β, β) *with an intermediate behavior between spring and dashpot. The constitutive equation of the spring-pot involves the so called fractional order derivatives and dynamic stability problem in presence of fractional-order operator must be faced for the Beck's column. In this study , the authors generalize Routh-Hurwitz theorem of stability on the fractional order differential equation (FODE), system that governs the dynamic stability. Some numerical examples has been reported in the paper for two-degree of freedom system.*

1 INTRODUCTION

Several physics and engineering problems that involves elastic systems under follower agencies may be found in in aerospace and/or in biomechanics setting [1, 2]. Under such non-conservative agencies the condition of equilibrium stability can not be faced in terms of well-known Euler-static condition and the dynamic equilibrium stability must be accounted for [3]. Dynamic stability of elastic columns under follower agencies traces back to the fundamental paper by Beck [4] that involves a linear elastic column resting on an elastic external bed of independent springs under constant follower force applied to the free end. Stability analysis show that the presence of the elastic foundation does not affect the critical load [5]. In this study the external restraints will be represented by independent spring-pot elements characterized by the couples of real parameters (C_β, β) with $C_\beta \geq 0$ and $0 \leq \beta \leq 1$. Indeed it is known that the material time-dependent behavior is well described by power-laws as t^β or $t^{-\beta}$ (for creep and relaxation) [6, 7] as observed in rubbers, foams and biological tissues. This mathematical representation of the material corresponds to the introduction of fractional order differential operator in the constitutive equations. As a consequence the governing equation of the system involves fractional-order derivatives yielding a FODE system that rules the dynamical equilibrium. Stability of FODE is nowadays a widely investigated issue [8] since in such a case the stability can not be observed with the classical Routh-Hurwitz criterion that gives informations on the sign of the real part of the system eigenvalues. A suitable extension of Routh criterion is reported in the paper for stability of FODE system. Such result has been achieved resorting to a state-space approach to the dynamical equilibrium stability equation of fractional-order by means of sequential fractional derivatives [9, 10]. Some numerical examples have been reported for a two degree of freedom column under follower force at the column tip.

2 FRACTIONAL ORDER CALCULUS

The time dependent behavior of viscoelastic material may be introduced starting from the so-called relaxation function $G(t)$ that is the stress history $\sigma(t)$ for an assigned strain $\gamma(t) = U(t)$ being $U(t)$ the unit step function. Alternatively viscoelastic material may be characterized by the creep function $J(t)$ namely the strain history for the assigned stress history $\sigma(t) = U(t)$. In virtue of the Boltzmann superposition principle the stressstrain relations is expressed as:

$$\sigma(t) = G(t)\gamma(0) + \int_0^t G(t-\tau)d\gamma(\tau) = G(t)\gamma(0) + \int_0^t G(t-\tau)\dot{\gamma}(\tau)\,d\tau \tag{1}$$

or in its inverse form

$$\gamma(\tau) = J(\tau)\sigma(0) + \int_0^t J(t-\tau)d\sigma = J(\tau)\sigma(0) + \int_0^t J(t-\tau)\dot{\sigma}(\tau)\,d\tau \tag{2}$$

where $\gamma(0)$ and $\sigma(0)$ are the strain and the stress in $t = 0$,respectively. Creep and relaxation function are related to each other by the following relationship in the Laplace domain:

$$\hat{G}(s)\,\hat{J}(s) = \frac{1}{s^2} \tag{3}$$

being $\hat{G}(s)$ and $\hat{J}(s)$ the Laplace transform of $G(t)$ and $J(t)$, respectively.

Let us now suppose that from experimental relaxation test G(t) is well fitted by

$$G(t) = \frac{C_\beta}{\Gamma(1-\beta)} t^{-\beta} \tag{4}$$

where the $\Gamma(\cdot)$ is the Euler-Gamma function, C_β and β are are characteristic coefficients depending on the material at hand that must satisfy the thermodynamic restrictions $\beta \in [0,1]$ and $C_\beta > 0$ yielding, after substitution in eq.(1):

$$\sigma(t) = \frac{C_\beta}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} \dot{\gamma}(\tau)\, d\tau = C_\beta \left({}^C D_{0^+}^{\beta} \gamma \right)(t) \tag{5}$$

where $\left({}^C D_{0^+}^{\beta} \gamma \right)(t)$ is Caputo's fractional derivative of order β [11]. Furthermore as we define the creep function in the power-law class as:

$$J(t) = \frac{1}{C_\beta \Gamma(1-\beta)} t^{\beta} \tag{6}$$

$$\gamma(t) = \frac{1}{C_\beta \Gamma(1-\beta)} \int_0^t (t-\tau)^{\beta} \dot{\sigma}(\tau) d\tau = \frac{1}{C_\beta} \left(I_{0^+}^{\beta} \sigma \right)(t) \tag{7}$$

where $\left(I_{0^+}^{\beta} \sigma \right)(t)$ is the Riemann-Liouville fractional integral. Values of coefficient β coinciding with the extrema of the closed interval $[0,1]$ correspond to asymptotic behavior of the model expressed by power-law assumption that is: elastic solid for $\beta = 0$ and viscous fluid for $\beta = 1$. Values of $\beta \in [0,1]$ correspond to an intermediate behavior between elastic solid and viscous fluid as for complex-structured materials and soft matter.

3 GOVERNING EQUATON AND STATE SPACE APPROACH

The problem about the stability of Bernoulli-Euler elastic column is studied in this paper as a two degree of freedom of the column, with cross-section A and length 2l . The governing equation obtained with the Newtonian mechanics approach of two-degree of freedom system at the onset of bifurcation in (fig.1) reads:

$$\begin{aligned} ml^2(\ddot{\varphi}_1 + \ddot{\varphi}_2) + l^2 C_\beta \left[{}^C D_{0^+}^{\beta} (\varphi_1 + \varphi_2) \right] + K_2(\varphi_1 + \varphi_2) = 0 \\ ml^2(2\ddot{\varphi}_1 + \ddot{\varphi}_2) + l^2 C_\beta \left[{}^C D_{0^+}^{\beta} (\varphi_1 + \varphi_2) \right] + C_\beta \left[{}^C D_{0^+}^{\beta} (\varphi_1) \right] + \\ (K_1 + K_2)\varphi_1 - K_2 \varphi_2 + P(\varphi_2 - \varphi_1) = 0 \end{aligned} \tag{8}$$

where m is the mass, K_1 and K_2 are the values of the stiffness, φ_1 and φ_2 are the slope of the two body, P is the axial load and l is the length of each element of the column. In the following we confine the analysis to rational value of $\beta = \frac{p}{q} \in \mathbb{Q}$ with $p, q \in \mathbb{N}$ and $q > p$, since $0 \leq \beta \leq 1$

Stability analysis of the FODE system in eq.(8), may be obtained by Laplace transform $\mathcal{L}[\varphi(t)] = \hat{\varphi}(s)$ [12, 13]. In this paper the authors study the stability of the FODE system by means of a state-space approach extending the Routh-Hurwitz criterion to the stability analysis of fractional-order dynamical systems.

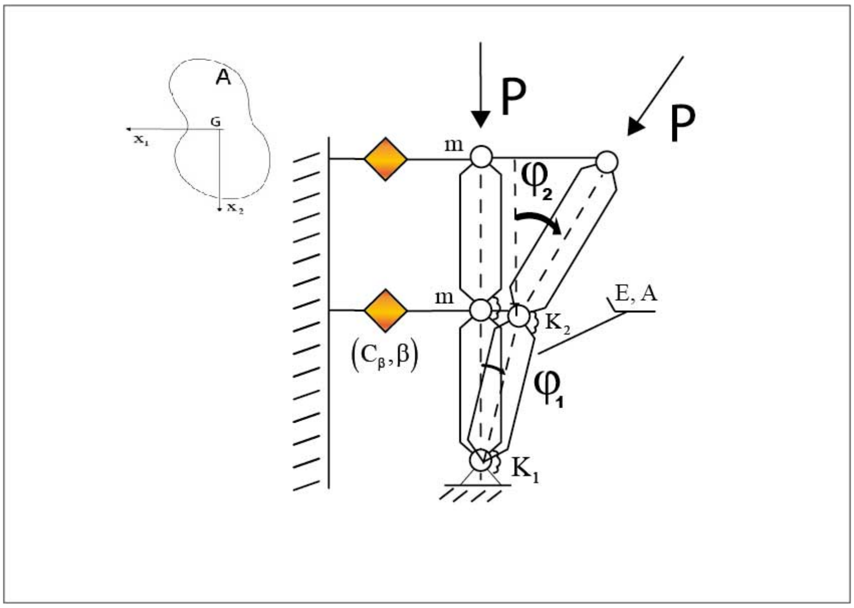

Figure 1: Onset of the column

Introducing the differentiation order $\overline{\beta} = \frac{1}{q}$ the system of FODE may be rewritten as:

$$\sum_{j=1}^{2q} B_j \left({}^C D_{0^+}^{j\overline{\beta}} \mathbf{y}\right)(t) + \mathbf{K}(P)\,\mathbf{y}(t) = \mathbf{0} \tag{9}$$

where

$$\mathbf{y(t)^T} = [\varphi_\mathbf{1}, \varphi_\mathbf{2}] \tag{10}$$

with $\mathbf{K}$ the stiffness matrix and B_j the coefficients of the viscosity matrix. At this system of differential equations must be added a set of 2q-1 identities:

$$\sum_{j=1}^{2q-1} B_{j+1} \left({}^C D_{0^+}^{\delta} \left({}^C D_{0^+}^{(1-j)\delta} \mathbf{y}\right)\right)(t) = \sum_{j=1}^{2q-1} B_{j+1} \left({}^C D_{0^+}^{j\delta} \mathbf{y}\right)(t) \tag{11}$$
$$\sum_{j=1}^{2q-2} B_{j+2} \left({}^C D_{0^+}^{\delta} \left({}^C D_{0^+}^{(1-j)\delta} \mathbf{y}\right)\right)(t) = \sum_{J=1}^{2q-2} B_{j+2} \left({}^C D_{0^+}^{j\delta} \mathbf{y}(t)\right)$$
$$\sum_{j=1}^{2q} B_j \left({}^C D_{0^+}^{j\delta} \mathbf{y}(t)\right) + \mathbf{K}(P)\,\mathbf{y}(\mathbf{t}) = \mathbf{0}$$

Introducing the state variable vectors $\mathbf{Z} \in \mathbb{R}$

$$\mathbf{Z}(t)^T = [\mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{y}_{2q-1}(t)] \tag{12}$$

with

$$\mathbf{y_1}(t) = \mathbf{y}(\mathbf{t}) \tag{13}$$
$$\mathbf{y_2}(t) = \left(D_{0^+}^{\overline{\beta}} \mathbf{y}\right)(t)$$

$$\mathbf{y_3}\left(t\right)=\left(D_{0+}^{\overline{\beta}}\mathbf{y_2}\right)\left(t\right)$$
$$\mathbf{y_{2q-1}}\left(t\right)=\left(D_{0+}^{\overline{\beta}}\mathbf{y_{2q-2}}\right)\left(t\right)$$

through this expansion of the state space the governing equation of the problem in the canonical form becomes:

$$\left({}^{C}D_{0+}^{\bar{\beta}}\mathbf{Z}\right)\left(t\right)=\mathbf{A}\left(P\right)\mathbf{Z}\left(t\right) \tag{14}$$

where the $\mathbf{A}$ are a dynamic matrix of the problem that contain coefficients linked to the physical model as K_j and B_j coefficients of stiffness matrix and dissipation matrix. Indeed, we study the problem written in this form:

$$\left(\mathbf{A}\left(P\right)-\chi\left(P\right)\mathbf{I}\right)\varphi\left(P\right)=\mathbf{0} \tag{15}$$

The solution in classical modal analysis as:

$$\mathbf{Z}\left(t\right)=\varphi\left(P\right)t^{\overline{\beta}-1}E_{\overline{\beta},1}\left(\bar{\omega}\left(P\right)t^{\overline{\beta}}\right) \tag{16}$$

where we introduced the generalization of the exponential function that is the Mittag-Leffler function:

$$E_{\overline{\beta},1}\left(t\right)=\sum_{k=1}^{\infty}\frac{\left(-1\right)^{k}t^{k}}{\Gamma\left(\overline{\beta}k+1\right)} \tag{17}$$

the eigenvalues of the system evaluated are determinate through the equation of characteristic polynomial as:

$$\det\left(\mathbf{A}\left(P\right)-\chi\left(P\right)\mathbf{I}\right)=0 \tag{18}$$

Except in some cases where the roots of the characteristic polynomial are available in closed form obtaining eigenvalues $\chi_1...\chi_{\mathrm{q}}\chi_{\mathrm{q}+1}...\chi_{4\mathrm{q}}$, as that the stability can be checked directly as the following inequalities is satisfied:

$$\arg\left[\left|\chi_j\right|\right]>\frac{\pi}{2}\beta \tag{19}$$

a general method that avoids the evaluation of eigenvalues as the the Routh-Hurwitz criterion is not available. In the following we introduce a generalization of the Routh-Hurwitz criterion to study the stability of FODE system.

4 APPLICATION OF ROUTH-HURWITZ THEOREM GENERALIZED AND NUMERICAL APPLICATION

A more efficient and popular method to assess the stability of an Ordinaries Differential Equation (ODE) system relies on the Routh-Hurwitz criterion predicting the number of roots with positive real part: $\mathrm{Re}\left(\chi_j\right)>0$. The method has been widely applied in several engineering and mathematical problems and it allow for the evaluation of the Cauchy Index of the polynomial yielding number of roots in the negative real half-problem. In order to obtain a generalization of Routh-Hurwitz theorem we introduce a one-to-one complex map as:

$$\overline{\chi}=\chi e^{i\left(\frac{1-\beta}{2}\right)\pi} \tag{20}$$

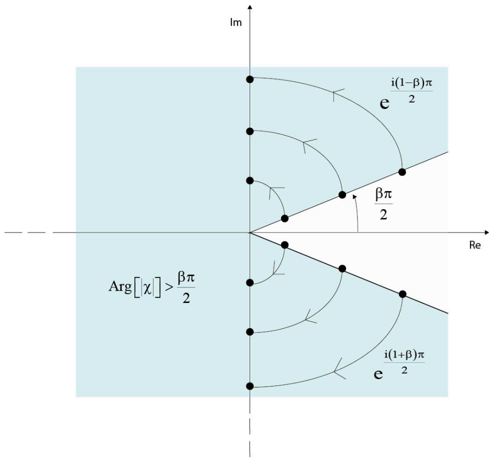

Figure 2: complex Map

so that complex number $\overline{\chi} \in$ along the edge $e^{i\frac{\pi\beta}{2}}$ like show in (fig.2).

As we substitute the eq. (20) in eq. (18) the following secular equation is obtained as:

$$
\begin{aligned}
&C_5\bar{\chi}^{4p} + C_4 C_\beta{}^{\frac{(p-q)}{p}}\bar{\chi}^{(2p+1)} + \left(\alpha_4 + \alpha_3 P^2\right)\bar{\chi}^{2p} + C_3 C_\beta{}^{\frac{(2p-2q)}{p}}\bar{\chi}^{2q} \\
&+ \left(C_2 C_\beta{}^{\left(1-\bar{\beta}\right)} - C_1 C_\beta{}^{\left(1-\bar{\beta}\right)} P^2\right)\bar{\chi}^{q} + \left(\alpha_2 + \alpha_1 P^2 + \alpha_0 P^4\right) = 0
\end{aligned}
\tag{21}
$$

in which C_j with j=1,2...5$\in \mathbb{C}$ and α_k with k=0,1,..,4 $\in \mathbb{R}$

The authors have validate the method comparing the value of critical load obtained in same specific case with known eigenvalues with the value of the stability load obtained with Routh-Hurwitz criterion applied to the complex-order polynomial in eq.(21). The case correspondent to an external dashpots foundation [4] has been also obtained for $\beta = 1$. The parametric analysis showing the influence of parameter C_β and the order of the derivation β on the critical load has been explained in (fig.3). After that, this method, has been applied for intermediate $0 < \beta < 1$ yielding the stability regions in (fig.3).

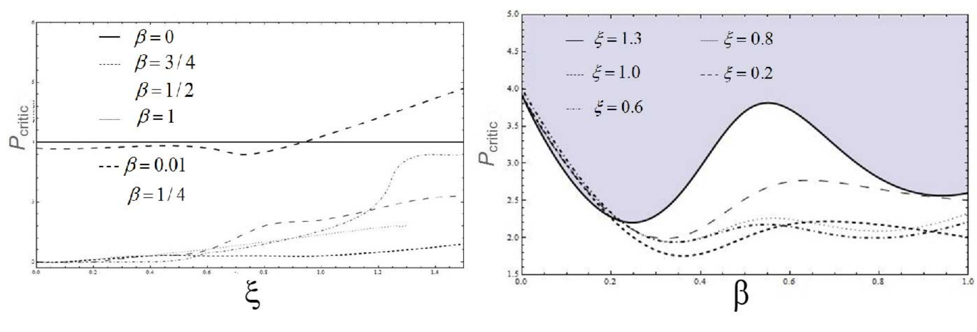

Figure 3: Effect of the parameters $\xi = C_\beta/m$ and β on the critical load

5 CONCLUSION

The generalization of the Routh-Hurwitz criterion has been formulated in the paper for a two-degree of freedom column resting a fractional-order external restraint. The analysis has validated the method confronting with some known cases and the stability regions have been obtained.

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