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Stochastic response of a fractional vibroimpact system

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Abstract

The paper proposes a method to investigate the stochastic dynamics of a vibroimpact single-degree-of-freedom fractional system under a Gaussian white noise input. It is assumed that the system has a hard type impact against a one-sided motionless barrier, which is located at the system's equilibrium position; furthermore, the system under study is endowed with an element modeled with fractional derivative. The proposed method is based on stochastic averaging technique and overcome the particular difficulty due to the presence of fractional derivative of an absolute value function; particularly an analytical expression for the system's mean squared response amplitude is presented and compared with results obtained by numerical simulations.

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Keywords: vibroimpact systems; fractional derivative; stochastic averaging method.

1. Introduction

Investigating on dynamics of vibroimpact systems is a topic of great scientific interest, since systems which experience vibroimpact interactions can be found in different engineering applications [1,2]. Indeed, vibroimpact motion of a system can be thought up for specific aims, as for example for energy gathering [4,5], to improve a drilling process [8], or to attenuate the response of an other system making use of particle dampers [6,7].

Generally, vibroimpact systems work under random excitation and their main characteristic is the strong nonlinearity, which is caused by the vibroimpact interactions, and which makes these systems tough to treat from both a numerical and analytical point view. The main difficulties which are encountered derive mainly from the phase constraints and from the jump discontinuities in the velocity as the system impacts. These difficulties can be overcome in different way, depending on the amount of energy lose of the system during the impact. If the lose of energy is negligible, i.e. the impact is elastic, some positions on the displacement variable need to overcome the difficulties. On the contrary, if the amount of dissipated energy can not be neglected, other analytical approaches should be utilized,

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like energy balance method. However, various works can be found in literature that approach to the vibroimpact problems [1,2,10,11]. From a numerical point of view, a general integration method for a non-linear system driven by impulsive inputs requires a considerable reduction of the integration step when the impulse occurs, and then Monte Carlo simulations could become prohibitive; some of these aspects are discussed and overcome in [12,13].

It is underlined that all systems considered in the previous references, as well as in other works, involve viscoelastic devices modeled as Kelvin-Voigt behavior. In the last years, some researchers had the bright idea to express viscoelastic laws involving fractional operators, in order to better capture the behavior of viscoelastic materials [14–18]; particularly, they were inspired by Nutting's work [19], which found a more realistic description of creep and relaxation for this media in terms of a power law function with real order exponents. This was the basis that led to formulate the constitutive laws in terms of fractional operators; indeed, these operators are convolution integrals with power law kernel, and they are named "fractional" because they behave as derivatives and integrals of integer order [20,21]. Hereafter, the growing interest and need to study dynamics systems with single and multi degrees of freedom involving a fractional viscoelastic device [22–32].

This paper deals with the vibroimpact motion of a fractional single-degree-of-freedom (SDOF) system under a Gaussian zero-mean white-noise excitation. An analytical expression for a mean squared response amplitude is derived making use of the stochastic averaging approach, assuming that the impact is elastic. The paper is organized as follows. In Section 2 the problem is formulated and the proposed method is shown. In Section 3 a numerical example shows a comparison between results from proposed method and those from Monte Carlo simulations. Finally, conclusions are drawn.

2. Problem statement

Consider a single-degree-of-freedom system, endowed with a fractional derivative element and forced by a random broad band excitation $\zeta(t)$, modeled as a white noise. It is assumed that the system, during its motion, impacts against a motionless barrier located at distance $x = h$, where x is the system displacement. Denoting with $\dot{x}_-$ and $\dot{x}_+$ the velocities of the systems before and after the impact, the vibroimpact motion can be modeled as follows

$$\begin{aligned} &\ddot{x} + C_\alpha D^\alpha(x) + \Omega^2 x = \zeta(t), \qquad && x < h \\ &\dot{x}_+ = -r\dot{x}_-, && x = h \end{aligned} \tag{1}$$

where C_α is a coefficient that can be viewed as a damping coefficient (per unit mass) if $\alpha = 0$, or as a stiffness coefficient (per unit mass) if $\alpha = 1$, the symbol $D^\alpha(x)$ denotes the fractional derivative of order α for the displacement x with respect to time t, Ω is the natural frequency of the system, while r is the restitution coefficient, with $0 < r < 1$.

Fractional derivative $D^\alpha(x)$ is expressed as follows:

$$D^\alpha(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(t-\tau)}{(\tau)^\alpha} d\tau, 0 < \alpha < 1 \tag{2}$$

To find a solution of the vibroimpact problem (1), it is suitable to assume that $h = 0$ and to introduce a new variable, "y", defined as:

$$x = |y| \tag{3}$$

then, it follows that

$$\dot{x} = \dot{y}\mathrm{sign}(y), \ \ddot{x} = \ddot{y}\mathrm{sign}(y) \tag{4}$$

Substituting Eq.(3) and Eq.(4) in Eq.(1) gives the following relations

$$\begin{aligned} &\ddot{y} + \Omega^2 y = -C_\alpha D^\alpha(|y|)\mathrm{sign}(y) + \zeta(t)\mathrm{sign}(y) \\ &\dot{y}_+ = r\dot{y}_-, \end{aligned} \tag{5}$$

Assuming that the impact is elastic, i.e. $r = 1$, then the response is continuous and differentiable, and the problem can be addressed by the stochastic averaging technique. Hence, introducing a slowly varying amplitude A and phase θ of the process

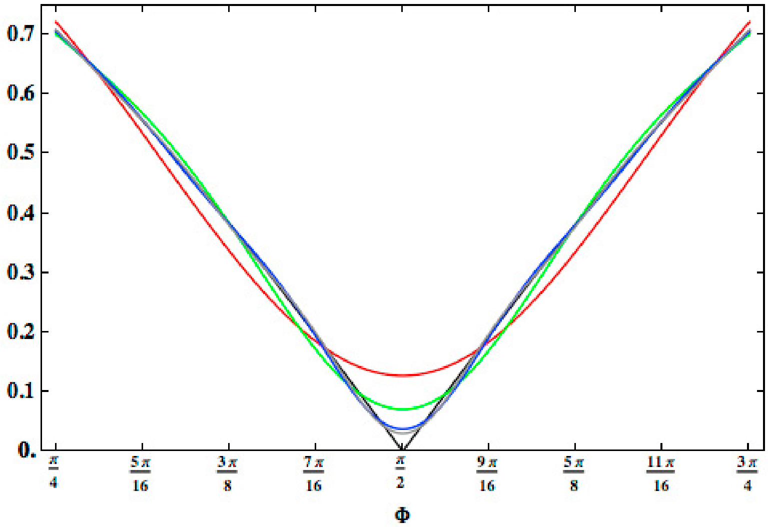

Fig. 1. Approximation of the absolute value function in Eq.(10) (black) assuming $n = 2$ - (red), $n = 4$ - (green), $n = 8$ (blue) and $n = 10$ - (gray)

$$y = A(t)\cos\theta(t),\ \dot{y} = -A(t)\Omega\sin\theta(t) \tag{6}$$

it is obtained

$$A' = -\frac{\sin\theta}{\Omega}\left[C_\alpha D^\alpha(|y|)\text{sign}(y) + \zeta(t)\right], \tag{7}$$

$$\theta' = \Omega - \frac{\cos\theta}{A\Omega}\left[C_\alpha D^\alpha(|y|) + \zeta(t)\right] \tag{8}$$

Under the assumption that $\theta(t-\tau)$ is a smooth function, then it can be assumed that $\theta(t-\tau) = \theta(t) - \Omega\tau$. Hence, the fractional derivative for $x = |y|$ can be expressed as follows:

$$D^\alpha(|y|) = D^\alpha(A|\cos[\theta(t)]|) = \frac{A}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{|\cos(\theta(t)-\Omega\tau)|}{\tau^\alpha}d\tau \tag{9}$$

Due to the analytically problems, deriving from presence of the absolute value function, it is proposed to replace it by its Fourier series. Then, the absolute value function for $\cos(\phi)$ can be expressed as follows:

$$|\cos(\phi)| = \frac{2}{\pi} - \sum_{m=1}^{\infty} A_m \cos(2m\phi);\ A_m = \frac{4}{\pi}\frac{(-1)^m}{(4m^2-1)} \tag{10}$$

A numerical convergence of the series for increasing value of m is shown in Fig.1, where results are compared with the original function. It can be observed that a good agreement is found for $n = 10$.

Then, inserting Eq.(10) into (9) leads to:

$$\int_0^t \frac{|\cos(\theta(t)-\Omega\tau)|}{\tau^\alpha}d\tau = \int_0^t \frac{2}{\pi\tau^\alpha}d\tau - \sum_{m=1}^{\infty} A_m \int_0^t \frac{\cos[2m(\theta(t)-\Omega\tau)]}{\tau^\alpha}d\tau \tag{11}$$

The first integral in the right hand of Eq.(11) is easy to solve, while the second one has to be expanded first:

$$\begin{aligned}\int_0^t \frac{\cos[2m(\theta(t)-\Omega\tau)]}{\tau^\alpha}d\tau &= \cos[2m\theta(t)]\int_0^t \frac{\cos[2m\Omega\tau]}{\tau^\alpha}d\tau + \\ &+ \sin[2m\theta(t)]\int_0^t \frac{\sin[2m\Omega\tau]}{\tau^\alpha}d\tau\end{aligned} \tag{12}$$

The two integrals in the right hand of Eq.(12) have been successfully solved in [31]:

$$\int_0^t \frac{\cos[2m\Omega\tau]}{\tau^\alpha} d\tau = (2m\Omega)^{\alpha-1} \int_0^t \frac{\cos u}{u^\alpha} du = \\ = (2m\Omega)^{\alpha-1} \left[\Gamma(1-\alpha) \sin\left(\frac{\alpha\pi}{2}\right) + \frac{\sin(2m\Omega t)}{(2m\Omega t)^\alpha} + O\left(\frac{1}{(2m\Omega t)^{\alpha+1}}\right) \right] \tag{13a}$$

$$\int_0^t \frac{\sin[2m\Omega\tau]}{\tau^\alpha} d\tau = (2m\Omega)^{\alpha-1} \int_0^t \frac{\sin u}{u^\alpha} du = \\ = (2m\Omega)^{\alpha-1} \left[\Gamma(1-\alpha) \cos\left(\frac{\alpha\pi}{2}\right) - \frac{\cos(2m\Omega t)}{(2m\Omega t)^\alpha} + O\left(\frac{1}{(2m\Omega t)^{\alpha+1}}\right) \right] \tag{13b}$$

where $O(z)$ defines terms of order z and higher. Combining Eqs.(11)-(12)-(13a)-(13b), the following expression is obtained:

$$D^\alpha(|y|) \approx \frac{d}{dt} \int_0^t \frac{|\cos(\theta(t) - \Omega\tau)|}{\tau^\alpha} d\tau = \frac{A}{\Gamma(1-\alpha)} \left\{ \frac{2}{\pi t^\alpha} - \right. \\ - A_m (2m\Omega)^{\alpha-1} \left[-2m \sin[2m\theta] \left\{ \Gamma(1-\alpha) \sin\left(\frac{\alpha\pi}{2}\right) + \right. \right. \\ \left. + \frac{\sin(2m\Omega t)}{(2m\Omega t)^\alpha} \right\} + 2m \cos[2m\theta] \left\{ \Gamma(1-\alpha) \cos\left(\frac{\alpha\pi}{2}\right) + \right. \\ \left. \left. \left. + \frac{\cos(2m\Omega t)}{(2m\Omega t)^\alpha} \right\} + G(\Omega t) \right] \right\} \tag{14}$$

being $G(\Omega t)$ the combined term comprising the time derivative of the second term in each expression in Eqs.(13a)-(13b). To finalize the derivations, it is required to average both Eqs.(7)-(8):

$$A' = -\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\sin\theta}{\Omega} \left[C_\alpha D^\alpha(|A\cos\theta|) sign(\cos\theta) + \zeta(t) \right] dt = \\ = -C_\alpha \frac{A A_m m (2m\Omega)^{\alpha-1}}{\Omega\pi} \int_0^{2\pi} \left\{ \cos[2m\theta] \sin\theta \cos\left(\frac{\alpha\pi}{2}\right) sign(\cos\theta) - \right. \\ \left. - \sin[2m\theta] \sin\theta \sin\left(\frac{\alpha\pi}{2}\right) sign(\cos\theta) \right\} d\theta + \frac{q}{2\Omega^2 A} \tag{15}$$

with $q = \sigma^2$ the white noise intensity. The first term in the right hand of Eq.(15), after integration, can be written as:

$$\int_0^{2\pi} \cos[2m\theta] \sin\theta \cos\left(\frac{\alpha\pi}{2}\right) sign(\cos\theta) d\theta = \\ = \cos\left(\frac{\alpha\pi}{2}\right) \frac{2}{2m^2 - 1} \left[2m \sin(m\pi) + 2m sin(3m\pi) + \cos^2(2m\pi) - 1 \right] = 0 \tag{16}$$

while the second term results in:

$$\int_0^{2\pi} \sin[2m\theta] \sin\theta \sin\left(\frac{\alpha\pi}{2}\right) sign(\cos\theta) d\theta = \\ = \sin\left(\frac{\alpha\pi}{2}\right) \frac{2}{4m^2 - 1} \left[2m\cos(m\pi) + 2m\cos(3m\pi) - \sin(2m\pi)\cos(2m\pi) \right] = \\ = (-1)^m \sin\left(\frac{\alpha\pi}{2}\right) \frac{8m}{4m^2 - 1} \tag{17}$$

Thus, substituting Eq.(16) and Eq.(17) in Eq.(15) led to the following expression for A'

$$A' = -C_\alpha \frac{32A}{\pi^2\Omega} \sin\left(\frac{\alpha\pi}{2}\right) \sum_{m=1}^{\infty} \frac{m^2 (2m\Omega)^{\alpha-1}}{(4m^2 - 1)^2} + \frac{q}{2\Omega^2 A} \tag{18}$$

Now, introducing a new variable $\mu =< A^2 >$, then:

$$\mu' = -C_\alpha \frac{64\mu}{\pi^2\Omega} \sin\left(\frac{\alpha\pi}{2}\right) \sum_{m=1}^{\infty} \frac{m^2(2m\Omega)^{\alpha-1}}{(4m^2-1)^2} + \frac{q}{\Omega^2} \tag{19}$$

so that the stationary mean square value of the response amplitude μ is given as follows:

$$\mu = \frac{q}{\Omega^2}\left[\frac{64C_\alpha}{\pi^2\Omega^{2-\alpha}} \sin\left(\frac{\alpha\pi}{2}\right) \sum_{m=1}^{\infty} \frac{m^2(2m)^{\alpha-1}}{(4m^2-1)^2}\right]^{-1} \tag{20}$$

3. Numerical Example

Consider a vibroimpact system governed by Eq.(1) with $C_\alpha = 1$ and with natural frequency $\Omega = 1$, forced by a white noise, whose intensity is indicated with q. It is stressed that the impact of the system is assumed to be elastic, i.e. there is no energy dissipation at the barrier. Mean square value of the response amplitude obtained by Eq.(20) is compared to that obtained from Monte-Carlo simulations. The simulations are led with a Grunwald-Letnikov discretization of the fractional derivative, choosing a time step $dt = 10^{-3}$s; in order to numerically model the interactions of the system with the barrier, placed at $h = 0$, it is assumed that the system impacts when the response of the system $x(t)$ falls within the interval $[h - \epsilon]$, with $\epsilon = 10^{-5}$. This interval is chosen because no significant differences in the sample response are found for smaller values of ϵ. A total number of 5000 samples are generated in the simulation. Figure 2 shows the mean square amplitude, as given by Eq.(20) and digital simulation, for different values of fractional order α, with $0 \leq \alpha \leq 1$; moreover, the results are given for different values of the noise intensity q.

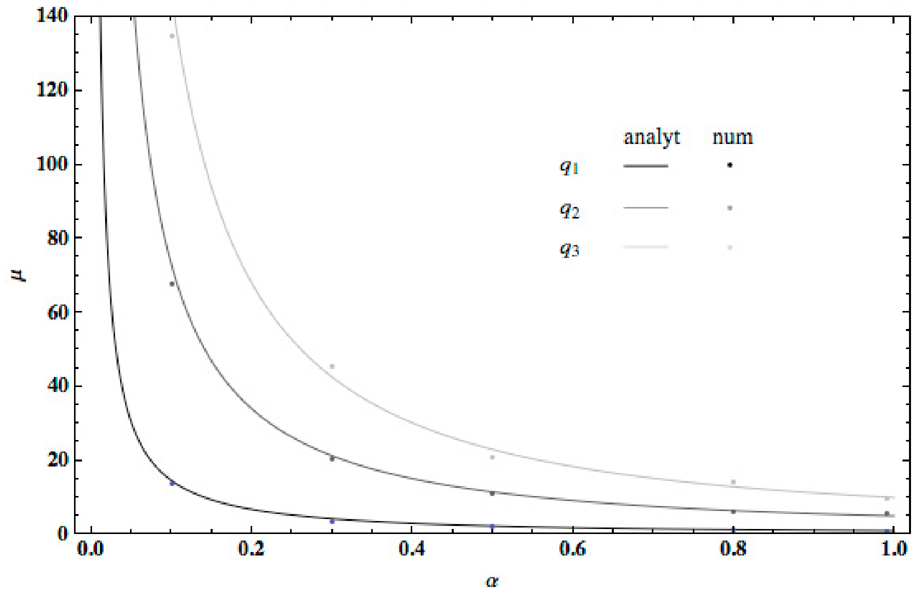

Fig. 2. Mean square value of the response amplitude μ for different values of the fractional order α and for three different values of the noise intensity q: $q_1 = 1$, $q_2 = 5$, $q_3 = 10$.

As can be observed from Figure 2, an excellent agreement is found between the two solutions when no dissipation at the barrier is assumed, for the whole domain of α.

4. Conclusions

An analytical expression for the mean square response amplitude is derived for a fractional vibroimpact single-degree-of-freedom under a Gaussian white noise input; the impact is assumed to be elastic, i.e. the restitution coefficient $r = 1$. The proposed method makes use of the stochastic averaging approach and overcomes the particular difficulty due to the presence of a fractional derivative of an absolute value function. For the considered case the proposed solutions are in excellent agreement with digital simulation data.

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