AIMETA 2017
XXIII Conference
The Italian Association of Theoretical and Applied Mechanics
Luigi Ascione, Valentino Berardi, Luciano Feo, Fernando Fraternali and Antonio Michele Tralli (eds.)
Salerno, Italy, 4–7 September 2017

HELLINGER-REISSNER VARIATIONAL PRINCIPLE FOR STRESS GRADIENT ELASTIC BODIES WITH EMBEDDED COHERENT INTERFACES

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Keywords: HR Variational Principle, Stress gradient elasticity, coherent interfaces

Abstract. *An Hellinger-Reissner (H-R) variational principle is proposed for stress gradient elasticity material models. Stress gradient elasticity is an emerging branch of non-simple constitutive elastic models where the infinitesimal strain tensor is linearly related to the Cauchy stress tensor and to its Laplacian. The H-R principle here proposed is particularized for a solid composed by several sub-domains connected by coherent interfaces, that is interfaces across the which both displacement and traction vectors are continuous. In view of possible stress-based finite element applications, a reduced form of the H-R principle is also proposed in which the field linear momentum balance equations are satisfied a-priori, the continuity condition of the displacements across the interfaces is relaxed and the analogous continuity condition of the traction is enforced as a side condition.*

1 INTRODUCTION

In the recent scientific literature, stress gradient elasticity theory is emerging as a valuable model [1], alternative to the most traditional strain gradient, which can handle long distance cohesive forces in real structured materials typically employed in micromechanics structures .

The very reason to employ such higher order material models is rooted on the fact they can provide significant improvements to specific shortcomings displayed by classic local continuum theories, such as crack tip stress singularity or stiffness size effects.

Stress gradient theory can be considered related to the original approach of Eringen [2], which replaced the nonlocal stress integral relation, $\boldsymbol{\sigma}(\boldsymbol{x}) = \int_V \alpha(|\boldsymbol{x}' - \boldsymbol{x}|)\, \boldsymbol{\sigma}^H(\boldsymbol{x}')\, \mathrm{d}V(\boldsymbol{x}')$ with a differential equation of the type $\boldsymbol{\sigma}^H(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x}) - \ell^2\nabla^2\boldsymbol{\sigma}(\boldsymbol{x})$ where ∇^2 is the Laplace differential operator, ℓ is a positive internal length material parameter, $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^H$ are respectively the Cauchy and the Hookean stress tensors. The two approaches are mathematically linked if the integral kernel function $\alpha(|\boldsymbol{x}|)$ is the Green function of the differential operator $\mathcal{L} := 1 - \ell^2\nabla^2$ or alternatively applying the differential operator $\mathcal{L}$ to the kernel function, the following relation holds: $\mathcal{L}[\alpha(|\boldsymbol{x}' - \boldsymbol{x}|)] = \delta(\boldsymbol{x}' - \boldsymbol{x})$, where $\delta(\boldsymbol{x})$ is the Dirac delta function [3, 4].

The real complete equivalence of the two approaches is still under debate, since the strong (integral) nonlocal elasticity and the stress gradient elasticity are indeed two different constitutive models. [5, 6]. Moreover, the most remarkable difference is in the gradient induced extra boundary conditions, namely boundary conditions which are produced by the differential nature of the material model.

The stress gradient approach has a number of theoretical and applicative advantages with respect to the most popular strain gradient elasticity [1]. In an earlier attempt to explore these advantages, a slightly different formulation, called *implicit gradient elasticity* model, was independently proposed and its Finite Element implementation carefully analyzed [7]. The stress gradient model has been also utilized as an effective regularization procedure which allows to smooth stress singularity that arise at the crack tip for linear elastic fracture mechanics problems [8].

The most recent and effective contributions for a rational framing of this new branch of the continuum mechanics is indeed rooted in the research work by Forest and co-workers [9, 10] and the one by Polizzotto [1, 11, 12].

The two above mentioned autonomous stress gradient formulations share the same mechanical background and both are developed in a consistent variational environment. Nevertheless they arrive to a boundary-value problem which has the same field differential equations, but surprisingly the boundary conditions proposed are different, not only for the type of equations, but also for the number of boundary conditions. The understanding of these differences is a quite important topic, which has been partially discussed in [12], but it deserve a more deep analysis. In this paper we adopt the boundary conditions proposed in [1, 11].

One of the reasons that seem to justify the better performances of stress gradient models with respect to the strain gradient ones [13, 14] is due to the fact that the latter approach requires an higher grade of continuity for the displacement field. Consequently, rather complex displacement based finite elements are necessary and, moreover, complex higher order boundary conditions are demanded. Stress gradient elasticity models can be approached from a static point of view, so that continuity required to stress field is less expensive and it also results easier to enforce constitutive boundary conditions.

The paper is organized as follow. In the next Section the stress gradient elasticity boundary-value problem is briefly recalled for a solid body with embedded coherent interfaces. Section 3 is devoted to propose a specific form of the Hellinger-Reissner (H-R) variational principle, showing that the stationarity conditions represent the equations of the boundary value problem exposed in Sect. 2. In Section 4 a reduced form of the H-R variational principle is reported which is developed relaxing some of the strong local conditions in a weak form. This reduced form of the H-R principle opens to a possible stress based Finite Elements applications, of a form similar to the one adopted in [18]. Finally in Section 5 some closing remarks and possible future developments are discussed. Along the paper standard Cartesian index notation is adopted.

2 THE STRESS GRADIENT ELASTICITY BOUNDARY VALUE PROBLEM

In this Section the relations governing the stress gradient elasticity boundary-value problem are briefly collected. A continuous solid body made up of stress gradient elastic material is considered. The body is referred to Cartesian orthogonal frame with axes x_i $(i = 1, 2, 3)$; it occupies the open domain $V \subset \mathbb{R}^3$ (see Figure 1); which is subdivided in m-subdomains $V_{(1)}, V_{(2)}, \dots V_{(m)}$. The interface separating the subdomains, called Γ, is geometrically characterized by the the outward normal vector $\boldsymbol{n}$ (with arbitrary chosen orientation); it in general contains line intersections and point junctions. The complete internal domain is denoted $\bar{V} = V \cup \Gamma$. The external boundary of $\bar{V}$, denoted as $\bar{S} = \partial\bar{V}$, is divided in the constrained part $\bar{S}_c$, where the displacement vector $\bar{\boldsymbol{u}}$ is assigned, and in the free part $\bar{S}_f$, where the traction vector $\bar{\boldsymbol{t}}$ is assigned. In order that after the deformation the subdomains $V_{(r)}$ fit to form a continuum, it is required that both the displacement $\boldsymbol{u}$ and traction $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$ are continuous across the interfaces.

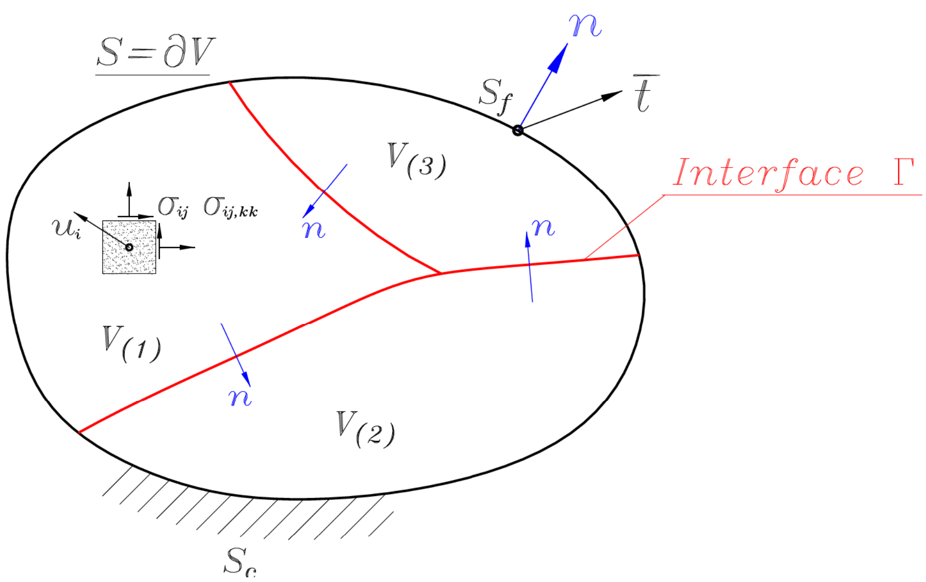

Figure 1: Sketch of a stress gradient elastic solid body formed by $m = 3$ subdomains $V_{(1)}, V_{(2)}, \dots V_{(m)}$ with interfaces Γ.

The complete set of equations that characterize the stress gradient elasticity problem are reported in the following Subsections.

2.1 Compatibility conditions

Since the problem is developed under the hypothesis of small displacements and small strains the usual linearized compatibility conditions have to be enforced at every point inside the regions $V_{(r)}$; moreover, the displacement vector has to take the value assigned on $\bar{S}_c$. Therefore the compatibility conditions read as

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad \text{in } V_{(r)}, (r = 1, \ldots m); \qquad u_i = \bar{u}_i \quad \text{on } \bar{S}_c \tag{1}$$

We notice that for stress gradient elasticy the compatibility relations are just the same relations as for classical elasticity.

2.2 Equilibrium conditions

In the assumed hypothesis of small displacements the field equilibrium equations can be written in their classical linearized format, whereas on the free boundary surface $\bar{S}_f$, with outward normal vector $\boldsymbol{n}$, the equilibrium condition with the external assigned traction must be enforced. Therefore the equilibrium equations read as

$$\sigma_{ij,j} + b_i = 0 \quad \text{in } V_{(r)}, (r = 1, \ldots m); \qquad \sigma_{ij} n_j = \bar{t}_i \quad \text{on } \bar{S}_f \tag{2}$$

Again we observe that for stress gradient elasticity the equilibrium equations have the same mathematical structure as for classical elasticity.

2.3 Stress gradient constitutive equations

The essential feature of stress gradient elasticity are incorporated within the constitutive relations. Let us introduce a standard Hookean stress, say σ^H_{ij}, which is related to the compatible strain ε_{ij} defined in eq. $(1)_1$, through the Hooke law, that is,

$$\sigma^H_{ij} = D_{ijhk}\varepsilon_{hk} \quad \text{in } V_{(r)}, (r = 1, \ldots m) \tag{3}$$

the latter stress $\boldsymbol{\sigma}^H$ is related to the equilibrium stress tensor $\boldsymbol{\sigma}$ by the following (Helmholtz) partial differential equations, with associated boundary conditions

$$\sigma_{ij} - \ell^2 \sigma_{ij,kk} = \sigma^H_{ij} \quad \text{in } V_{(r)}, (r = 1, \ldots m); \qquad \sigma_{ij,k} n_k = 0 \quad \text{on } S = \bar{S} \cup \Gamma^+ \cup \Gamma^- \tag{4}$$

For $\boldsymbol{\sigma}^H$ assigned within V, the above Helmoholz problem permits one to evaluate, at least in principle, the corresponding Cauchy stresses $\boldsymbol{\sigma}$. Equation $(4)_1$ introduces an internal length scale parameter ℓ, which is directly related to the long range microstructure interactions and serves ad a scaling factor for the Laplacian of the Cauchy stress tensor $\boldsymbol{\sigma}$

2.4 Interface conditions

To complete the problem equations it is necessary to address kinematic and static continuity conditions across the interface Γ, see Figure 2. Following a typical nomenclature of mechanical interfaces, the interface Γ is supposed to split after the deformation mechanism in a positive Γ^+ and in a negative surface Γ^-. As a consequence, a point $\boldsymbol{x} \in \Gamma$ splits in a point $\boldsymbol{x}^+ \in \Gamma^+$ and in a point $\boldsymbol{x}^- \in \Gamma^-$, with a concomitant jump of the displacement vector across the interface that is, $[\![u_i]\!] := u_i^+ - u_i^- = x_i^+ - x_i^-$. Taking the normals $\boldsymbol{n}^+$ to Γ^+ and $\boldsymbol{n}^-$ to Γ^- with outward directions from the respective adjacent subdomain, due to the small displacement

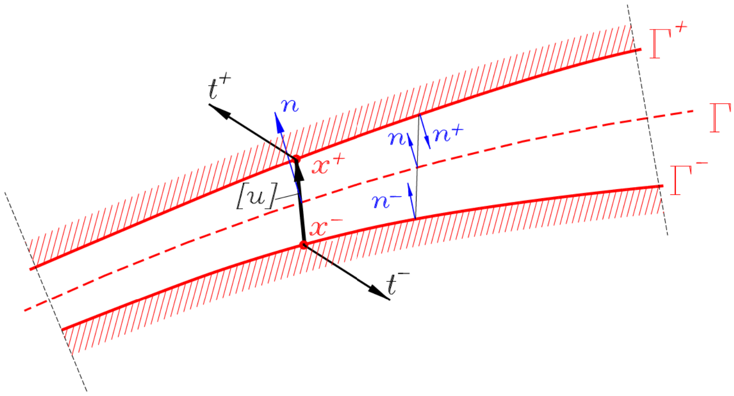

Figure 2: Sketch showing an interface element Γ with its positive (Γ^+) and negative (Γ^-) parts, the normal vector $n = n^- = -n^+$, the two traction vectors t^+ and t^- and the displacement jump $[\![u_i]\!] := u^+ - u^-$. For coherent interfaces $[\![u_i]\!] = 0$ and $[\![t_i]\!] = 0$.

hypothesis, the following condition holds $n_i := n_i^- = -n_i^+$. Since the interfaces considered here are coherent interfaces, in the sense that they present neither displacement, nor traction jumps across the interface, the following continuity relations hold

$$[\![u_i]\!] = 0, \text{ and } [\![\sigma_{ij} n_j]\!] = 0 \quad \text{on } \Gamma \tag{5}$$

We note that the stress gradient boundary conditions

$$\sigma_{ij,k} n_k = 0 \quad \text{on } \Gamma^+ \text{ and on } \Gamma^- \tag{6}$$

which are incorporated in eq.(4_2) (where $S = \bar{S} \cup \Gamma^+ \cup \Gamma^-$) do not pertain to the set of interface conditions, but they instead constitute the boundary conditions associated to the adjacent subdomains.

For coherent interfaces only the kinematic and static relations (5) and (6) must be considered. A different kind of interface has been often used for driving dechoesion or fracture processes along predefined potential surfaces. For these last cases cohesive interfaces are employed, in which, the traction continuity is still enforced, but the displacement jump can be different from zero, since it is a measure of the crack opening. In case of cohesive interfaces, beside static and kinematic relations, is also necessary to introduce specific interface constitutive equations (see e.g. [15, 16, 17]). In what follows reference is done only to coherent interfaces, leaving for future research works the extension to cohesive interfaces.

Equations (1)-(6) are then the governing equations of stress gradient elastic boundary value problem. In the next Section a stress based variational approach will be examined.

3 HELLINGER-RESSNER VARIATIONAL PRINCIPLE

In absence of interfaces Γ, the Hellinger-Reissner stationarity principle for stress gradient elasticity [1, 11] reads

$$H = \int_V \{\sigma_{ij} u_{i,j} - G(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) - b_i u_i\} \, \mathrm{d}V - \int_{S_f} \bar{t}_i u_i \, \mathrm{d}s - \int_{S_c} (u_i - \bar{u}_i) p_i \, \mathrm{d}s \tag{7}$$

where $\sigma_{ij}=\sigma_{ji}$, p_i and u_i are free variables and G is the (Gibbs) complementary stress energy potential, a convex function given in the following simplest quadratic form

$$G(\boldsymbol{\sigma},\nabla\boldsymbol{\sigma})=\frac{1}{2}A_{ijkl}\left(\sigma_{ij}\sigma_{kl}+\ell^2\sigma_{ij,m}\sigma_{kl,m}\right) \tag{8}$$

In the case of embedded interfaces, as depicted in Fig. 1, the functional (7), has to be modified as follows

$$\begin{aligned} H_1(\sigma_{ij},p_i,u_i,q_i)=&\int_V\{\sigma_{ij}u_{i,j}-G(\sigma_{ij},\sigma_{ij,m})-b_iu_i\}\,\mathrm{d}V\\ &-\int_{\bar{S}_f}\bar{t}_iu_i\,\mathrm{ds}-\int_{\bar{S}_c}(u_i-\bar{u}_i)p_i\,\mathrm{ds}+\int_\Gamma q_i\,[\![u_i]\!]\,\mathrm{d}\Gamma \end{aligned} \tag{9}$$

where the last integral is appended to (7) in order to enforce the interface condition $[\![u_i]\!]=0$ on Γ by means of a traction-like Lagrange multiplier q_i. Note that the variable p_i, in place of $\sigma_{ij}\,n_j$, is mandatory since p_i is *independent* of σ_{ij}. Rewriting explicitly the functional H_1 by inserting the Gibbs stress function (8)

$$\begin{aligned} H_1(\sigma_{ij},p_i,u_i,q_i)=&\int_V\left\{\sigma_{ij}u_{i,j}-\frac{1}{2}A_{ijkl}\left(\sigma_{ij}\sigma_{kl}+\ell^2\sigma_{ij,m}\sigma_{kl,m}\right)-b_iu_i\right\}\mathrm{d}V\\ &-\int_{\bar{S}_f}\bar{t}_iu_i\,\mathrm{ds}-\int_{\bar{S}_c}(u_i-\bar{u}_i)p_i\,\mathrm{ds}+\int_\Gamma q_i\,[\![u_i]\!]\,\mathrm{d}\Gamma \end{aligned} \tag{10}$$

taking the first variation

$$\begin{aligned} \delta H_1=&\int_V\left\{(u_{i,j}-A_{ijkl}\sigma_{kl})\,\delta\sigma_{ij}-\ell^2A_{ijkl}\sigma_{kl,m}\,\delta\sigma_{ij,m}\right\}\mathrm{d}V\\ &+\int_V\{\sigma_{ij}\,\delta u_{i,j}-b_i\,\delta u_i\}\,\mathrm{d}V-\int_{\bar{S}_f}\bar{t}_i\,\delta u_i\,\mathrm{ds}-\int_{\bar{S}_c}p_i\,\delta u_i\,\mathrm{ds}+\int_\Gamma q_i\,[\![\delta\,u_i]\!]\,\mathrm{d}\Gamma\\ &-\int_{\bar{S}_c}(u_i-\bar{u}_i)\,\delta p_i\,\mathrm{ds}+\int_\Gamma[\![u_i]\!]\,\delta q_i\,\mathrm{d}\Gamma=0 \end{aligned} \tag{11}$$

Applying the Gauss theorem at the two terms with $\delta u_{i,j}$ and $\delta\sigma_{ij,m}$, and considering also the following two identities

$$[\![\sigma_{ij}n_j\delta u_i]\!]\equiv[\![\sigma_{ij}n_j]\!]\delta u_i^+ + \sigma_{ij}^-n_j[\![\delta u_i]\!]=[\![\sigma_{ij}n_j]\!]\delta u_i^- + \sigma_{ij}^+n_j[\![\delta u_i]\!] \tag{12}$$

eq.(11) becomes

$$\begin{aligned} \delta H_1=&\int_V\left\{u_{(i,j)}-A_{ijkl}\left(\sigma_{kl}-\ell^2\sigma_{kl,mm}\right)\right\}\delta\sigma_{ij}\mathrm{d}V-\int_V(\sigma_{ij,j}+b_i)\,\delta u_i\,\mathrm{d}V\\ &+\int_{\bar{S}_f}(\sigma_{ij}n_j-\bar{t}_i)\,\delta u_i\,\mathrm{ds}+\int_{\bar{S}_c}(\sigma_{ij}n_j-p_i)\,\delta u_i\,\mathrm{ds}-\int_{\bar{S}_c}(u_i-\bar{u}_i)\,\delta p_i\,\mathrm{ds}\\ &-\int_{\bar{S}}\ell^2A_{ijkl}\,\sigma_{kl,m}\,n_m\,\delta\sigma_{ij}\,\mathrm{ds}+\int_\Gamma\ell^2A_{ijkl}[\![\sigma_{kl,m}\,n_m\,\delta\sigma_{ij}]\!]\,\mathrm{d}\Gamma\\ &+\underbrace{\int_\Gamma\left(q_i-\sigma_{ij}^-n_j\right)\,[\![\delta u_i]\!]\,\mathrm{d}\Gamma-\int_\Gamma[\![\sigma_{ij}\,n_j]\!]\,\delta u_i^+\,\mathrm{d}\Gamma}_{(*)}+\int_\Gamma[\![u_i]\!]\,\delta q_i\,\mathrm{d}\Gamma=0 \end{aligned} \tag{13}$$

Note that the term (*) can be alternatively written as $\int_\Gamma \left(q_i - \sigma^+_{ij} n_j\right) \; [\![\delta u_i]\!] \, \mathrm{d}\Gamma - \int_\Gamma [\![\sigma_{ij}\, n_j]\!] \, \delta u^-_i \, \mathrm{d}\Gamma$ by means of the identity (12).

The Euler-Lagrange equations at the statinarity condition of the Hellinger-Reissner principle then read

$$
\begin{aligned}
& u_{(i,j)} = \underbrace{A_{ijkl}\left(\sigma_{kl} - \ell^2 \sigma_{kl,mm}\right)}_{\varepsilon_{ij}} \quad \text{in } V \\
& \sigma_{ij,j} + b_i = 0 \quad \text{in } V; \qquad \sigma_{ij} n_j = \bar{t}_i \quad \text{on } \bar{S}_f \\
& \sigma_{ij} n_j = p_i \quad \text{on } \bar{S}_c; \qquad u_i = \bar{u}_i \quad \text{on } \bar{S}_c \\
& [\![u_i]\!] \; = 0, \quad [\![\sigma_{ij}\, n_j]\!] = 0, \quad q_i = \sigma^-_{ij} n_j (= \sigma^+_{ij} n_j), \quad \text{on } \Gamma \\
& \ell^2 A_{ijkl}\, \sigma_{kl,m}\, n_m = 0 \quad \Rightarrow \quad \sigma_{ij,m}\, n_m = 0 \quad \text{on } \bar{S} \\
& \ell^2 A_{ijkl}\, \sigma_{kl,m}\, n_m = 0 \quad \Rightarrow \quad \sigma_{ij,m}\, n_m = 0 \quad \text{on } \Gamma^+ \text{and on } \Gamma^-
\end{aligned}
\tag{14}
$$

It is easy to verify that the Euler-Lagrange equations reported above coincide with the boundary-value problem defined by all the field and boundary equations of eqs.(1)–(6)

4 A REDUCED FORM OF THE HELLINGER-RESSNER PRINCIPLE

A suitably reduced-form of the H-R principle of the preceding Section can be developed with the intent to derive Finite Elements applications. For this purpose let us assume

i) The field equilibrium equation are satisfied a-priori: $\sigma_{ij,j} + b_i = 0$;

ii) The continuity condition of the displacement u_i across Γ is relaxed: $q_i = 0$;

iii) The continuity condition of the traction $t_i = \sigma_{ij} n_j$ across Γ is enforced as a side condition by writing: $[\![\sigma_{ij}\, n_j\, u_i]\!] = \sigma_{ij}\, n_j\, [\![u_i]\!]$, provided that $[\![\sigma_{ij}\, n_j]\!] = 0$, see eq. (12).

As a consequence, the last integral of eq.(10) drops out. Also, applying the divergence theorem, the first integral of (10) transform as

$$
\begin{aligned}
H_1 = & -\underbrace{\int_V \left(\sigma_{ij,j} + b_i\right) u_i \, \mathrm{d}V}_{=0} - \int_V G(\sigma_{ij}, \sigma_{ij,m}) \, \mathrm{d}V + \int_{\bar{S}_c} (\sigma_{ij} n_j - p_i) u_i \, \mathrm{d}s \\
& + \int_{\bar{S}_c} p_i \bar{u}_i \, \mathrm{d}s + \int_{\bar{S}_f} (\sigma_{ij} n_j - \bar{t}_i) u_i \, \mathrm{d}s - \int_\Gamma \sigma_{ij}\, n_j \, [\![u_i]\!] \, \mathrm{d}\Gamma
\end{aligned}
\tag{15}
$$

This, introducing the modified functional H_r, can be rewritten in the form

$$
\begin{aligned}
H_r := & \int_V G(\sigma_{ij}, \sigma_{ij,m}) \, \mathrm{d}V - \int_{\bar{S}_f} (\sigma_{ij} n_j - \bar{t}_i) u_i \, \mathrm{d}s - \int_{\bar{S}_c} p_i \bar{u}_i \, \mathrm{d}s \\
& - \int_{\bar{S}_c} (\sigma_{ij} n_j - p_i) u_i \, \mathrm{d}s + \int_\Gamma \sigma_{ij}\, n_j \, [\![u_i]\!] \mathrm{d}\Gamma
\end{aligned}
\tag{16}
$$

we remark that the stationarity condition

$$\begin{aligned} & H_r(\sigma_{ij}, p_i, u_i) \rightarrow \text{stationary} \\ & \text{s.t.} \begin{cases} \sigma_{ij,j} + b_i = 0 & \text{in } V \\ [\![u_i]\!] = 0 & \text{on } \Gamma \end{cases} \end{aligned} \tag{17}$$

would lead to the same Euler-Lagrange equations like those obtained for the original H-R principle with interface, that is (14). In particular, at the stationarity condition, it would result that $p_i = t_i \equiv \sigma_{ij}\, n_j$ on $\bar{S}$. This implies that for FE discretization purposes, it is possible to replace p_i with t_i without errors, because after discretization, the variation will be taken with respect to the inherent discrete variables (node value of the unknown fields), not with respect the latter (distributed) unknowns. Additionally, since $u_i = \bar{u}_i$ on $\bar{S}_c$ at the stationarity limit, it is also possible to replace $u_i = \bar{u}_i$ on $\bar{S}_c$ in (16). Therefore, for discretization purposes, the functional (16) can be written in the simpler form:

$$\begin{aligned} H_r^* := & \int_V G(\sigma_{ij}, \sigma_{ij,m})\, \mathrm{d}V - \int_{\bar{S}_f} (\sigma_{ij} n_j - \bar{t}_i) u_i \, \mathrm{d}s - \int_{\bar{S}_c} \sigma_{ij}\, n_j\, \bar{u}_i \, \mathrm{d}s \\ & + \int_\Gamma \sigma_{ij}\, n_j\, [\![u_i]\!] \mathrm{d}\Gamma \end{aligned} \tag{18}$$

or also, since $\int_\Gamma \sigma_{ij}\, n_j\, [\![u_i]\!] \mathrm{d}\Gamma = - \int_{\partial V / S} \sigma_{ij}\, n_j\, u_i \mathrm{d}\Gamma$, we obtain

$$H_r^* := \int_V G(\sigma_{ij}, \sigma_{ij,m})\, \mathrm{d}V - \int_{\bar{S}_f} (\sigma_{ij} n_j - \bar{t}_i) u_i \, \mathrm{d}s - \int_{\bar{S}_c} \sigma_{ij}\, n_j\, \bar{u}_i \, \mathrm{d}s + \int_{\partial V / \bar{S}_c} \sigma_{ij}\, n_j \, \mathrm{d}s \tag{19}$$

Finally, the discretization by FEs starts by subdividing the domain in m subdomains, each of which is a finite element, whereas the interface Γ represents the inter-elements surface (or line for 2D problems). Inside each element the following modelling can be adopted

$$u_i(\boldsymbol{x}) \approx \boldsymbol{N}_i^u(\boldsymbol{x})\, \boldsymbol{U}; \qquad \sigma_{ij}(\boldsymbol{x}) \approx \boldsymbol{N}_{ij}^\sigma(\boldsymbol{x})\, \boldsymbol{S}; \tag{20}$$

where $\boldsymbol{U}$ and $\boldsymbol{S}$ are respectively discrete (or nodal) displacements and stresses. Details concerning FE discretization and element technology are not reported here, but it seems possible to extend recent stress approach adopted for standard elasticity problems [18] to the present stress gradient elasticity context.

5 CONCLUSIONS

In the present paper an ad-hoc Hellinger-Reissner variational principle for stress gradient elastic material structures has been presented. The H-R principle has been formulated considering that the continuum has embedded a number of coherent interfaces, that transform the usual continuum in an assemblage of subdomains interconnected in a way to ensure, to some degree, the original behavior of the monolithic continuum. The main purpose is to set up a consistent variational environment for the application of stress based finite elements for the structural analysis of stress gradient material structures. The specific development of a stress gradient Finite Element, with a deep analysis of characteristics and performances, is out of the targets of the present paper and will be the subject of future studies.

Acknowledgment

A grant from the Italian Ministry for University and Research (MIUR) for PRIN-2015, project No.2015LYYXA8, Multiscale mechanical models for the design and optimization of microstructured smart materials and metamaterials is gratefully acknowledged

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