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Proceedings / Actes

CIEAEM 67

Aosta (Italy)

July, 20 - 24 2015

TEACHING AND LEARNING MATHEMATICS : RESOURCES AND OBSTACLES

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ENSEIGNER ET APPRENDRE LES MATHEMATIQUES : RESSOURCES ET OBSTACLES

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Information about CIEAEM 67 and presentation of the Volume
Informations sur la CIEAEM 67 et présentation du Volume

The CIEAEM 67 conference was held in Aosta, Italy, from 21st to 24th July 2015 and successfully involved more than 120 participants from all over the world.

Researchers, teachers, educators, and students from 20 countries (10 non-European) met to discuss, in a collaborative and inspiring environment, the most prominent problems, obstacles and resources in mathematics education; they also presented their latest research findings in the several conference activities: plenary talks, a round table, working groups, workshops, and poster presentations (forum of ideas).

As in previous CIEAEM meetings, Working Groups constituted the beating heart of the conference, allowing the participants to fruitfully discuss in critical and constructive ways, in the true CIEAEM spirit, research studies and approaches from different perspectives on the conference theme: *Teaching and learning mathematics: resources and obstacles*. There were five Working Groups, each discussing between 11 and 16 papers, and addressing the conference theme from complementary viewpoints (see the Discussion Paper), under the guidance of the group animators. The conference schedule allowed time also to deepen the plenary talks in the dedicated "Meet the plenary speaker" sessions, and to engage participants in workshops, where actual dialogue between research and practice could be fostered.

This volume contains the final versions of the 85 papers presented during the conference, and revised by the authors, including the suggestions which emerged in the intense discussions in Aosta. Specifically, the volume chapters contain

- 3 Plenary talks,
- 4 contributions related to a Round Table on assessment,
- 66 papers presented and discussed in the 5 Working Groups in parallel sessions,
- 5 papers on the Workshops organized during the conference,
- 6 papers presenting Posters.

We thank all the contributors and the participants to the conference, because they made it such a unique experience, in which we had the good fortune to take part. We are grateful to the International Program Committee chaired by Luciana Bazzini, and the Local Organizing Committee chaired by Elisabetta Robotti, that made possible the realization of the conference in every detail with great care. Also to the University of Valle d'Aosta and to the Regione Valle d'Aosta, for the precious support and collaboration to the conference. And to the Working Group animators, who organized each day the sessions in inclusive as well high quality ways. A special thanks to all the people who contributed to the realization of the conference, and to Francesco Rossi, who helped in editing this volume.

As a result, the CIEAEM67 Proceedings offer a wide overview on national and international studies on the conference theme *Teaching and learning mathematics: resources and obstacles*. We hope that it can constitute an inspiring resource for the research community and stakeholders in Mathematics Education. From this perspective, the possibility of free downloading offers to CIEAEM67 participants, and also to interested people who could not take part in the Conference in Aosta, the possibility of developing a fruitful network of contacts that year after year is becoming richer and wider.

29th December 2015

Cristina Sabena

Benedetto Di Paola

Information about CIEAEM 67 and presentation of the Volume
Informations sur la CIEAEM 67 et présentation du Volume

La conférence CIEAEM 67 s'est déroulé à Aoste (Italie) du 21 au 24 juillet 2015 et a compté plus de 120 participants venus du monde entier.

Chercheurs, enseignants, formateurs et étudiants de 20 pays (10 non européens) se sont rencontrés pour discuter dans une ambiance collaborative et stimulante sur les principaux problèmes, obstacles et ressources de la didactique des mathématiques, et en même temps pour présenter leurs derniers résultats de recherche dans les différentes activités de la conférence : plénières, table ronde, groupes de travail, ateliers et sessions d'affiches (forum aux idées).

Comme dans les dernières éditions de la CIEAEM, les groupes de travail ont constitué le cœur de la conférence; en permettant aux participants de discuter de manière critique et constructive, dans l'esprit de la CIEAEM, ces groupes de travail ont tiré profit des études et des différentes approches pour élargir les perspectives sur le thème de la conférence : *Enseigner et apprendre les mathématiques : ressources et obstacles*. Cinq groupes de travail ont été organisés, pour discuter de 11 à 16 articles chacun et pour aborder le thème de la conférence sous plusieurs points de vues complémentaires (voir le Document de discussion), guidés par les animateurs des groupes. Le programme de la conférence a permis aussi de réserver du temps pour l'approfondissement des plénières dans les séances spéciales "Rencontre avec les conférenciers", et d'engager les participants dans des ateliers, où un véritable dialogue entre recherche et pratique a pu être encouragé.

Ce volume contient les versions finales des 85 articles présentés lors de la conférence, et révisés par les auteurs pour inclure les suggestions mises en avant dans les discussions intenses à Aoste. Plus précisément, les chapitres du volume contiennent

- 3 Plénières,
- 4 contributions liées à la Table Ronde sur l'évaluation,
- 66 articles présentés et discutés dans les 5 Groupes de Travail en séances parallèles,
- 5 articles sur les Ateliers organisés au sein de la conférence,
- 6 articles qui présentent les Posters.

Nous remercions toutes les personnes qui ont contribué et participé à la conférence, car ils en ont fait une expérience unique, que nous avons eu la chance de partager. Nous sommes reconnaissants au Comité International de Programme présidé par Luciana Bazzini, et au Comité Local d'Organisation présidé par Elisabetta Robotti, qui ont rendu possible la tenue de cette conférence et son déroulement harmonieux. De même nous adressons nos remerciements à l'Université de la Vallée d'Aoste et à la Région Vallée d'Aoste, pour leur soutien et leur collaboration précieuce. Nous remercions également les animateurs des groupes de travail, qui ont organisé chaque jour les séances d'une façon inclusive en maintenant une grande qualité scientifique. Un remerciement spécial va à tous ceux qui ont contribué à la réalisation de la conférence, et à Francesco Rossi, qui a aidé à l'édition de ce volume.

Comme résultat, les actes de la CIEAEM67 offrent un vaste panorama sur les études nationales et internationales autour du thème de la conférence *Enseigner et apprendre les mathématiques : ressources et obstacles*. Nous espérons que ce volume pourra constituer une ressource stimulante pour la communauté de recherche et pour les parties prenantes en didactique des mathématiques. Dans cette perspective, la possibilité de télécharger gratuitement les actes permet aux participants de la CIEAEM67, mais aussi à toute personne intéressée qui n'a pas pu participer à la conférence à Aoste, de développer un réseau fructueux de contacts qui d'une année à l'autre s'enrichit et s'élargit.

29th Décembre 2015

Cristina Sabena

Benedetto Di Paola

Discussion Paper

Conference Theme: ***Teaching and learning mathematics: resources and obstacles***

Introduction

Teaching and learning mathematics is a complex system, involving a plurality of factors and components, ranging from the epistemology of the discipline to cognitive psychology, socio-cultural environments, affective elements, and technological devices. At the very core of the system, making sense in doing mathematics is widely considered as a basic requisite for constructing knowledge. In this regard, it is worth analyzing mutual relationships between real objects and mathematical constructions, the role of thinking processes and languages (often related to embodied experiences), and the influence of beliefs and emotions. All factors can be double-faced, i.e., they can provide resources and/or obstacles for the development of mathematical knowledge. In this regard, the professional expertise of the teacher is of crucial importance: in fact the teacher is responsible for being up to date not only about the content aspects of the discipline, but also about those factors that interact (and interfere) with the teaching-learning processes. It is necessary for the mathematics teacher to be aware of these issues, both in designing classroom activities and in managing them with the students.]

The four subthemes (and related questions) we propose in the following are to be considered as a means to promote investigation and facilitate discussion. All the subthemes are closely interrelated: their distinction is purely functional to assist the organization of the working groups during the conference.

Subtheme 1. Mathematical content and curriculum development

The relationship between mathematics as a discipline and the mathematical content to be taught reminds us of the dialectic between theory and practice, which has received increasing emphasis since the 1990s (see, e.g., Brown & Cooney, 1991; Burton, 1991; Godino & Batanero, 1997; Wittmann, 1991). In the search of boundary conditions to mediate knowledge between the two poles, there is evidence that any conception which assigns to "theory" the role of instructing "practice" is doomed to fail and, consequently, there is a growing need for developing interaction between the two poles, and for co-operation between the actors involved in the education system (Bartolini Bussi & Bazzini, 2003).

Since the 1980s, an important contribution in the debate was given by Chevallard, who studied the didactical transposition phenomena, producing elements of knowledge about didactical systems and the content for mathematics teaching. This led to the development of the theory of didactic transposition as well as its practical realization (Chevallard, 1985). This idea has been further developed, in the 1990s and beyond, into a more general study within which mathematics is practised in terms of different praxeologies (combining praxis and logos).

Focusing on the epistemology of mathematics, and noticing persistent students' difficulties related to specific concepts, Brousseau (1997) discussed the notion of *epistemological obstacle* in mathematics. This idea has inspired research in mathematics education, opening the way to the search for other kinds of obstacles, related to didactical and cognitive aspects, as well as critique of the idea of epistemological obstacle, on the basis of historical-cultural discussion (Radford, 1997).

The dialectical interaction between theory and practice grounds the work of curriculum developers, mainly when different actors (researchers, teachers, school managers) are asked to work together. In such cases, curriculum development can be a great opportunity for co-operation and mutual enrichment, and make a positive contribution to the school (Bazzini, 1991). This theme will be also discussed in Subtheme 2 (see below).

The choice of content to be included in the curriculum is an important issue requiring attentive investigation in any context. Along with traditional topics, such as arithmetic, algebra, and geometry, relatively new topics need to be included in the curricula: Probabilistic and stochastic thinking constitute one striking example.

In recent years, most countries have introduced or developed statistical content in primary and secondary mathematics. The reasons are many: taking into account the rise of stochastical power in the discipline of mathematics, the will to develop other teaching approaches based on modelling from real situations, and interdisciplinarity, as a societal demand.

In Higher Education, more and more courses are incorporating statistics at the Bachelor level as in Doctoral programs. At this level, the sectorial variations are multiple (statistics for biology, management, psychology, etc.) with, as noted by Jeanne Fine (2010), in the words of Bourdieu, a high risk of hyperspecialisation and a weakening of the identity of the discipline. The foundations are supposedly acquired during previous schooling, and teaching of statistics is reduced to the presentation by non-specialists implementing techniques using specialized software. The operational dimension of knowledge is privileged at the expense of systematic and historical dimensions (see Fabre 2010), with the risk that students do not master basic statistical concepts, as highlighted in numerous research studies (see, in particular, Batanero et al, 1994; Delmas et al, 2007). The multiple epistemologies, most often not clarified, are a source of difficulty for students who do not identify where the professor or teacher is coming from, epistemologically (Armatte 2010).

It is true that statistics is a discipline whose epistemology is complex. However, it is important that this discipline is taught by specialists in higher education and is integrated into mathematics lessons in secondary school such that it is not diluted in the host disciplines (Gattuso, 2011). But there are many differences with mathematics, differences which must be made explicit in the context of the training of mathematics teachers. In statistics, students should be led to give up their deterministic worldview and to consider the lack of certainty as a feature of reality (Meletiou-Mavrotheris & Lee, 2002). A fundamental difference between statistics and mathematics is that, in statistics, the context has a special status: it is an integral part of the problem. The risk of misunderstanding between teacher and students, linked to the different representations, then becomes greater (Hahn, 2014). In statistics students should jointly master inductive and deductive reasoning (Fine, 2010), and combine the two perspectives: the data-centric approach and the more formal modeling (Armatte, 2010; Peters, 2011). This is not only to master the concepts but also to develop a statistical way of thinking (Gattuso, 2011), integrating the use of technology, which is essential in Statistics (Serrado et al, 2014).

The previous discussion opens the way for contributions to the subtheme 1 of the CIEAEM67 Conference, which focuses on issues related to the epistemological aspects of mathematics relevant to educational aims, and frames them in terms of the obstacle/resource dialectic. Subtheme 1 will focus on the following questions:

- *Which obstacles may interfere with teaching? What is their nature? What could be possible strategies to avoid/overcome them?*
- *Which obstacles interfere with learning? What is their nature? What could be possible strategies to avoid/overcome them?*
- *What are the resources and obstacles in different national curricula?*

- *What professional expertise is needed for developing and implementing curriculum?*
- *Is there any specific content in need of special attention?*
- *Should statistics be introduced in the primary school? How should we think about the preparation of teachers who will teach statistics at each level (primary, secondary, higher education)? What are the differences/complementarities between mathematics and statistics?*

Subtheme 2. Teacher education

Mathematics teacher education has been receiving increasing attention in research over the last decade (Clark-Wilson et al., 2014; Even & Ball (Eds.), 2009; Wood (Ed.), 2008). This 'emerging field' (Adler et al., 2005) has its roots in previous research on classroom teaching-learning processes. With the progressive diffusion of new learning and teaching models since the 1960s, the role of the teacher in the classroom has changed radically. In fact, new approaches to learning also require new approaches to teaching: this change is not spontaneous; on the contrary, in order to take place it needs to be fostered by suitable teacher education initiatives.

Research has pointed out different aspects with respect to mathematics teacher education: from the specificities of the knowledge needed by teachers to affective factors, from the inclusion of new technologies to systemic analyses.

Reflection on the kind of knowledge that characterises the mathematics teacher in his/her professional work has been carried out in the seminal work of Shulman (1986). Ponte et al. (1994) support the idea of blending mathematical content with pedagogical knowledge, drawing on different components of current knowledge to produce a restructuring of the teacher's craft knowledge. This pedagogical content knowledge has a much broader scope than just the representation of the subject matter: it must include "a comprehensive body of images, principles, and rules for action, some general, some more specific, organized with a clear rationale, bearing on the specific nature of the underlying content and powerful enough to guide the action of the teacher" (p. 358). Steinbring (1998) explores a specific component of professional knowledge for mathematics teachers, namely "epistemological knowledge of mathematics in social learning settings (p. 160)". He claims that "teachers surely need mathematical content knowledge and pedagogical knowledge; and, within the domain of pedagogical content knowledge, they also need epistemological knowledge, so that they are able to assess the epistemological constraints of mathematical knowledge in different social settings of teaching, learning, and communicating mathematics. This important component of epistemological knowledge of mathematics in social learning settings is not a systematized, canonical knowledge corpus, which could be taught to future teachers by way of a fixed curriculum. Rather, the epistemological knowledge consists of exemplary knowledge elements, as it refers to case studies of analysis of teaching episodes or of interviews with students, and comprises historical, philosophical, and epistemological conceptual ideas" (p. 160). Ball and Bass (2003) frame the typical features of mathematics that are involved in teaching within the Mathematical Knowledge for Teachers model, identifying the Specialized Content Knowledge as an important sub-domain of mathematical knowledge, strictly connected to the work of teaching. Specialized content knowledge intertwines often with knowledge and competences related to digital technologies, which have also gained increasing relevance in the teacher education context (Bairral & Powell, 2013; Drijvers et al., 2010).

On the other hand, several studies have investigated the social aspects of teacher education programs, especially the involvement of teachers in joint analysis and reflection together with researchers. Within the research literature we find important notions such as *community of practice* (Wenger, 1998) and *communities of inquiry* (Jaworski, 2006); the cornerstone of these studies being the notion of critical reflection, conceived not only as a fundamental attitude to be developed by teachers, but also as a professional responsibility. This idea is strictly interrelated with that of joint collaboration between teachers and researchers, as Krainer (2011) stresses when he suggests looking at researchers as "key stakeholders in practice" and teachers as "key stakeholders in research."

Besides epistemological and social dimensions, the affective dimension comes to play an important role in teacher work and in teacher education as well. It includes studying the influence of teachers' beliefs and emotions on their mathematics teaching. In fact, as Zembylas (2005) underlines:

> teacher knowledge is located in 'the lived lives of teachers, in the values, beliefs, and deep convictions enacted in practice, in the social context that encloses such practices, and in the social relationships that enliven the teaching and learning encounter'. These values, beliefs and emotions come into play as teachers make decisions, act and reflect on the different purposes, methods and meanings of teaching. (p. 467)

This is particularly relevant, especially concerning primary teachers, who are generalist teachers and sometimes have to teach mathematics despite their personal dispositions towards mathematics. Hence, teachers' beliefs and emotions towards mathematics can constitute obstacles to effective teaching practice. The study of the conditions under which this hypothesis is true remains an open problem. On the other hand, personal negative experiences and emotions may also become resources for teachers, as suggested by Coppola et al. (2013), focusing in particular on future teachers.

Finally, mathematics teacher education processes also need to be considered from a systemic point of view, with a focus on the relationships and dynamics between the several "variables" included in such complex processes as: teachers' knowledge and practices, results from research, institutional constraints (national curricula in particular), traditions, cultural aspects, and so on. Considering this complexity, teachers' development can be considered as a meta-didactical transposition process evolving over time (Arzarello et al., 2014).

Starting from this discussion, and from the contributions of the accepted papers, subtheme 2 in CIEAEM67 aims at rethinking the complexity of teacher education in terms of resources and obstacles for teaching and learning mathematics. The following questions may further guide the discussion:

- *How is it possible to support teachers to develop suitable knowledge and competences in digital technologies, so that they are effective in their mathematics teaching?*
- *What are the main obstacles for mathematics teacher development?*
- *How can the social dimension become a resource for teacher education? What are the challenges of programs strongly based on social interaction in communities of practice/enquiry?*
- *How can the affective dimension become a resource for teacher education?*

Subtheme 3. Classroom practices and other learning spaces

Mathematical thinking arises and develops in a complex interplay of languages and representations, through reference to intuitions, metaphors, and analogies, and by making use of various artefacts and tools, which interact with our bodily nature. All these components are crucial for teaching and learning activities within the classroom context, as well as within other learning spaces: in light of the Conference theme, they can constitute possible resources or, on the contrary, obstacles for the mathematics learning.

Whereas there has been a focus on language and written representations since the 1980s, more recently attention has also been given to embodied forms of representation and thinking, such as gestures, considered mainly as resources for teaching and for learning (Arzarello, 2008; Arzarello et al., 2009; Radford, 2002, 2014). Other studies have investigated the role of new technologies and ICT as possible mediators for learning (Drijvers et al., 2010). Thus, Subtheme 3 includes the discussion on the possible uses of new technologies as resources for the learning of mathematics, but also on the possible obstacles that the introduction of new technologies could produce at several levels (cognitive, didactic, communicative, etc.).

Concerning classroom practices, the role of the teacher comes to the fore. Even from possibly different theoretical positions, the teacher is usually intended as a resource for students' learning. In this regard, teachers need to deal with different cognitive demands, in particular with those of students having learning difficulties in mathematics, as widely discussed in literature (Dehaene, 1997, Landy & Goldstone, 2010). A conscious use of specific teaching strategies suitable for students diagnosed with learning disorders, in particular with developmental dyscalculia (Butterworth, 2005; Dehaene, 1997), is also important for those students who are not officially diagnosed, but have learning difficulty profiles very similar to those of dyscalculic students. Therefore, the development of innovative teaching support looks like an ever more necessary goal for research in mathematics education in general, and for teachers in particular.

Although school is the most important institution for learning, we know that it is not the only place where we learn. But, what do we mean by learning? It is common to find teachers with a restricted view concerning what it means to learn mathematics. Often learning is associated with the reproduction of counting procedures and calculation formulas. Although this idea has been overtaken, at least for research within mathematics education, it seems, unfortunately, that some teaching or training practices are still restrictive and do not acknowledge that learning can be observed through different lenses. We learn in formal and non-formal spaces (museums, distance learning programs, game playing, etc.), in face-to-face or online dynamic environments. We believe that teaching mathematics in any context should promote the development of thinking that offers potential for the student in their present and their future, regardless of their of future occupations or professional work. Processes such as developing curiosity, critical thinking, reasoning, and motivation to learn, as well as developing modes of verification, refutation, and deduction should all be leveraged both in the classroom and also in non-formal learning spaces.

Subtheme 3 includes the discussion about:

- *What are the features that characterize the teacher's practices as resources for students and how is it possible to foster these features?*
- *A provocative question: Can a teacher be an obstacle to the students' learning? Why and how does it happen? How could it be prevented?*
- *How can technologies and ICT be possible mediators for inclusive teaching and learning?*
- *How can embodied forms of representation and thinking, such as gestures, or other different registers of representation, such as visual-verbal, visual-non-verbal, auditory, and kinaesthetic, be considered as resources for inclusive teaching and learning ?*
- *Which resources or teaching strategies are being used to enhance the learning potential of all students, particularly those with learning difficulties?*
- *Which new aspects of mathematics learning can be improved in formal learning spaces or in non-formal environments?*
- *What the advantages or restrictions of ICT or more conventional resources (e.g., the manipulative ones) in promoting mathematical learning within formal or informal contexts?*

Subtheme 4. Cultural, political, and social issues

Since the 1980s at least, there have been challenges to assumptions that mathematics is culture- and value-free (Bishop, 1988; D'Ambrosio, 1985; Ellerton & Clements, 1989). There is also a developing awareness that mathematics education itself was not only portrayed as culture- and value-free, but also was effectively excluding or alienating many girls and women as well as boys and men who did not conform to the stereotypes found in classroom and textbook examples, or the choices of abstract, highly theoretical curricula. To epitomize this shift of research in mathematics education, the terms 'social turn' and 'sociopolitical turn' (Gutiérrez, 2010; Lerman, 2000) have appeared. Now, it has become broadly accepted that we can no longer think of mathematics and mathematics education as far removed from cultural, social and political issues when studying and trying to improve mathematics education.

Cultural, political, and social contexts can be considered as obstacles and/or as resources for students' success in mathematics. On the one hand, we can consider these as obstacles for students' access to, and their achievement in, mathematics education. Although less prevalent in Western countries, but nevertheless of fundamental importance, the physical access to schooling and mathematics classrooms has received attention (e.g., Kazima & Mussa, 2011). On a second level, curricular reforms and counter-reforms have often transformed the obstacles for mathematics learning that some social groups face (e.g., Jablonka & Gellert, 2011; Vithal & Skovsmose, 1997). This second level is concerned with the distribution of different forms of mathematical knowledge: Who gets access to which forms of mathematical knowledge? On a third level, the question has been raised as to how instructional and educational strategies complicate or impede access to, and participation in, institutionally and socially valued forms of mathematical activities for particular groups of students (e.g., Straehler-Pohl et al., 2014). Cultural (e.g., the culture-specific importance of orality), political (e.g., policies for integration of migrants), and social (e.g., relative poverty) conditions, taken separately, but mostly combined, often translate into obstacles for the teaching and learning of mathematics.

On the other hand, cultural, political, and social conditions can be regarded as resources. This is quite obvious in the case of privilege, where students' backgrounds and foregrounds easily prove beneficial for the acquisition of the school subjects' dominant registers and orientations to meaning (e.g., cultural capital and middle-class codes). The crucial point is if, and if so, how, not-yet-valued experiences and activities of underprivileged students can be used as resources for the teaching and learning of mathematics. As an example, Barton and Frank (2001) reflecting on minority cultures ask: "What are the conditions under which" (...) children, for whom the (conventionally) 'basic' mathematical concepts are not readily available because of incommensurable concepts powerfully present in their own cultural-linguistic heritage, "have a cognitive advantage in mathematics, and what is the nature of that advantage?" (p. 147). Healy and Powell (2013), examining multiple resources for mathematics learning, conclude that there is a wealth of studies showing how being multilingual relates positively to cognitive development. These studies also call for more invitation and encouragement of students to use their linguistic resources within mathematical activities. Bringing these two perspectives together, understanding the cultural, political, and social conditions that create obstacles for mathematics teaching and learning, might lead us to understand the micro- and the macro-social processes that disadvantage individuals.

As a matter of fact, diversity is an essential part of what it is to be human. Even within the same family unit there are differences between the children in terms of their interests and aptitudes. Within classrooms where students apparently share the same social and cultural backgrounds there is no uniformity. Particularly in recent times of global flows of people, many classrooms are likely to comprise students with diverse social, cultural, and linguistic backgrounds, and these offer both a resource and a challenge to teachers who may lack systemic support, as well as being expected to work under increasing pressures of time and accountability. This is in the face of mission statements and policy documents that state that each child or learner is an individual and should receive personalised attention from his/her provider of education.

Finally, understanding how cultural, political, and social conditions can become resources for learners might require us to analyse how curriculum, teaching strategies and learning scenarios can be more finely tuned to the backgrounds and foregrounds of particular groups of students.

Questions:

- *How do cultural, political, and social contexts restrict access to, and participation in, valuable forms of learning mathematics? How can these restrictions be overcome?*
- *How can underprivileged students' backgrounds and foregrounds be used as resources for the teaching and learning of mathematics?*
- *How could we rethink theories and practices of mathematics teaching to improve cognitive and affective outcomes for bilingual/multilingual students?*
- *How could we foster the inclusion of students from different cultural backgrounds within the mathematics classroom and in the broader society?*
- *How could we deal with challenges of gender stereotypes and other gender-related issues and the inequalities they create?*
- *How do policy designers take into consideration any kind of diversity and inequality in your country or region (e.g., the EU)?*

REFERENCES

Adler J., Ball D., Krainer K., Lin F. L., & Novotna J. (2005). Reflections on an emerging field: Researching mathematics teacher education. *Educational Studies in Mathematics*, *60*, 359-381.

Armatte, M. (2010). Le rôle de l'histoire dans l'enseignement de la statistique. *Statistique et Enseignement, 1*(2), 23-47.

Arzarello, F., Robutti, O., Sabena, C., Cusi, A., Garuti, R., Malara, N., & Martignone, F. (2014). Meta-didactical transposition: A theoretical model for teacher education programmes. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era. An international perspective on technology focused professional development* (pp. 347-372). Dordrecht: Springer.

Arzarello, F. (2008). Mathematical landscapes and their inhabitants: Perceptions, languages, theories. In E. Emborg & M. Niss (Eds), *Proceedings of the 10th International Congress of Mathematical Education* (ICME 10) (pp. 158-181). Copenhagen, Denmark: Roskilde University, IMFUFA.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, *70*(2), 97-109.

Bairral, M. A., & Powell, A. B. (2013). Interlocution among problem solvers collaborating online: A case study with prospective teachers *Pro-Posições, 24*(1), 1-16. doi: http://dx.doi.org/10.1590/S0103-73072013000100005.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group Edmonton* (pp. 3-14). AB: CMESG/GDEDM.

Bartolini Bussi M., & Bazzini L. (2003). Research, practice and theory in didactics of mathematics: Towards dialogue between different fields. *Educational Studies in Mathematics*, *54*, 203-223.

Barton, B., & Frank, R. (2001). Mathematical ideas and indigenous languages. In B. Atweh, H. Forgasz, & B. Nebres (Eds.), *Sociocultural research on mathematics education: An international perspective* (pp. 135–149). Mahwah, NJ: Lawrence Erlbaum

Batanero, C., Godino, J., Vallecillos, A., Green, D., & Holmes, P. (1994). Errors and difficulties in understanding elementary statistical concepts. *International Journal in Mathematics, Science and Technology, 25*(4), 527-547.

Bazzini, L. (1991). Curriculum development as a meeting point for research and practice. *Zentralblatt für Didaktik der Mathematik, 91/4,* 128-131.

Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective in mathematics education*. Dordrecht: Kluwer.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Brown, S., & Cooney, T.(1991). Stalking the dualism between theory and practice. *Zentralblatt für Didaktik der Mathematik, 91/4*, 112-117.

Burton, L. (1991). Models of systematic co-operation between theory and practice. *Zentralblatt für Didaktik der Mathematik, 91/4*, 118-121.

Butterworth B. (2005). Developmental dyscalculia. In J. I. E. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 455-467). Hove, UK: Psychology Press.

Chevallard, Y. (1985). *La transposition didactique. Du savoir savant au savoir einsegné*. Grenoble: La Pensée Sauvage.

Clark-Wilson, A., Robutti, O., & Sinclair, N. (Eds.) (2014). *The mathematics teacher in the digital era. An international perspective on technology focused professional development* (pp. 347-372). Dordrecht: Springer.

Coppola, C., Di Martino, P., Pacelli, T., & Sabena, C. (2013). Primary teachers' beliefs and emotional disposition towards mathematics and its teaching. In B. Di Paola (Ed), *Proceedings of CIEAEM 65. Quaderni di ricerca in didattica (mathematics), Issue 23*(1), 217-226.

D'Ambrosio, U. (1985). *Socio-cultural bases for mathematics education*. Campinas, Brazil: UNICAMP.

Dehaene S. (1997). *The number sense: How the mind creates mathematics*. New York, Oxford University Press.

DelMas, R., Garfield, J., Ooms A., & Chance, B. (2007). Assessing items students' conceptual understanding after a first course in statistics. *Statistics Education Research Journal, 6*(2), 28-58.

Drijvers, P., Kieran, C., & Mariotti, M.-A. (2010). Integrating technology into mathematics education: Theoretical perspectives. In C. Hoyles, & J.-B. Lagrange (Eds.), *Mathematics education and technology —Rethinking the terrain: The 17th ICMI study* (pp. 89-132). New York: Springer.

Ellerton, N. F., & Clements, M. A. (1989). *Teaching post-secondary mathematics at a distance: A report to the Commonwealth Secretariat*. Geelong, Vic: Deakin University.

Even, R., & Ball, D. L. (Eds.) (2009). *The professional education and development of teachers of mathematics: The 15th ICMI study.* New York: Springer.

Fabre, M. (2010). Problématisation des savoirs. In A. Van Zanten (Ed.), *Dictionnaire pédagogique* (pp. 539–541). Paris: Presses Universitaires de France.

Fine J. (2010). Probabilités et statistique inférentielle. Approche sondage versus approche modèle. *Statistique et Enseignement, 1*(2), 5-21.

Gattuso, L. (2011). L'enseignement de la statistique: Où, quand, comment pourquoi pas? *Statistique et Enseignement, 2*(1), 5-30.

Godino, J. D., & Batanero C. (1997). Clarifying the meaning of mathematical objects as a priority area of research in mathematics education. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity. The 8th ICMI study* (pp. 177-195). Dordrecht: Kluwer.

Gutiérrez, R. (2010). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, *44*(1), 37–68.

Hahn, C. (2014). Linking academic knowledge and professional experience in using statistics: a design experiment for business school students. *Educational Studies in Mathematics, 86*(2), 239-251.

Healy, L., & Powell, A. B. (2013). Understanding and overcoming "disadvantage" in learning mathematics. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 69–100). New York: Springer.

Jablonka, E., & Gellert, U. (2011). Potentials, pitfalls, and discriminations: Curriculum conceptions revisited. In B. Greer & O. Skovsmose (Eds.), *Critique and politics of mathematics education* (pp. 287–308). Rotterdam: Sense.

Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, *9*, 187-211.

Kazima, M., & Mussa, C. (2011). Equity and quality in mathematics education in Malawi schools. In B. Atweh, M. Graven, W. Secada, & P. Valero (Eds.), *Mapping equity and quality in mathematics education* (pp. 163–176). Dordrecht: Springer.

Krainer, K. (2011). Teachers as stakeholders in mathematics education research. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol.1), (pp. 47-62). Ankara, Turkey: PME.

Lagrange, J.-B., Artigue, M., Laborde C., & Trouche, L. (2003). Technology and mathematics education: A multidimensional study of the evolution of research and innovation. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second International Handbook of Mathematics Education* (pp. 239–271). Dordrecht: Kluwer.

Landy D., & Goldstone R. L. (2010). Proximity and precedence in arithmetic. *Quarterly Journal of Experimental Psychology, 63*, 1953-1968.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19–44). Westport, CT: Ablex.

Meletiou-Mavrotheris, M., & Lee, C. (2002). Teaching students the stochastic nature of statistical concepts in a introductory statistics course, *Statistics Education Research Journal, 1*(2)*,* 22-37.

Peters, S. (2011). Robust understanding of statistical variation, *Statistics Education Research Journal, 10*(1), 52-88.

Ponte, J. P., Matos, J. F., Guimarães, H. M., Leal, L. C., & Canavarro, A. P. (1994). Teacher's and students' view and attitude towards a new mathematical curriculum: A case study. *Educational Studies in Mathematics, 26*, 347-365.

Radford, L. (1997). On psychology, historical epistemology and the teaching of mathematics: Towards a socio-cultural history of mathematics. *For the Learning of Mathematics*, *17*(1), 26-30.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics, 22*(2), 14–23.

Radford. L. (2014). On the role of representations and artefacts in knowing and learning. *Educational Studies in Mathematics, 85*, 405–422.

Serrado A., Meletiou-Mavrotheris, M., & Paparitodemou, E. (2014). Early statistics: A course for developing teachers' statistics technological and pedagogical content. *Statistique et Enseignement, 5*(1), 5-29.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Steinbring H. (1998). Elements of epistemological knowledge for mathematics teachers. *Journal of Mathematics Teacher Education, 1,* 157-189.

Straehler-Pohl, H., Fernández, S., Gellert, U., & Figueiras, L. (2014). School mathematics registers in a context of low academic expectations. *Educational Studies in Mathematics*, *85*(2), 175–199.

Sullivan, P. (2008). Knowledge for teaching mathematics: An introduction. In P. Sullivan & T. Wood (Eds.), *Knowledge and beliefs in mathematics teaching and teaching development. The international handbook of mathematics teacher education* (Vol. 1), (pp. 1-12). Rotterdam: Sense.

Vithal, R., & Skovsmose, O. (1997). The end of innocence: A critique of 'ethnomathematics'. *Educational Studies in Mathematics, 34*(2), 131–157.

Wenger, E. (1998). *Communities of practice: Learning, meaning and identities*. Cambridge, UK: Cambridge University Press.

Wittmann E. C. (1991). From inservice-courses to systematic cooperation between teachers and researchers. *Zentralblatt für Didaktik der Mathematik, 91/5*, 158-160.

Wood, T. (Series Ed.). (2008). *The international handbook of mathematics teacher education* (Vols. 1-4). Rotterdam: Sense.

Zembylas, M. (2005). Beyond teacher cognition and teacher beliefs: The value of the ethnography of emotions in teaching. *International Journal of Qualitative Studies in Education*, *18*(4), 465-487.

Document de Discussion

Thème de la conférence: **Enseignement et apprentissage des mathématiques : ressources et obstacles**

Introduction

L'enseignement et l'apprentissage des mathématiques sont un système complexe, impliquant une multitude de facteurs et de composantes, allant de l'épistémologie de la discipline à la psychologie cognitive, aux environnements sociaux-culturels, à des éléments affectifs et aux systèmes technologiques. Au cœur du système se trouve l'idée selon laquelle donner du sens aux mathématiques est un prérequis à la construction de la connaissance. A cet égard, il convient d'analyser les interactions qui existent entre les objets réels et les constructions mathématiques, le rôle des processus de pensée et les langages (souvent en relation avec la réalisation d'expériences), ainsi que l'influence des croyances et des émotions. Tous ces facteurs peuvent être trompeurs, i.e., ils peuvent à la fois fournir des ressources et/ou des obstacles pour le développement de la connaissance mathématique. En ce sens, l'expertise professionnelle du professeur a une importance majeure : en réalité le professeur a la responsabilité d'être en mesure de faire faire face, non seulement aux aspects relatifs aux contenus de la discipline, mais également aux facteurs qui interagissent (et interfèrent) dans le processus liant l'enseignement à l'apprentissage. Il est nécessaire pour le professeur de mathématiques d'être conscient de ces enjeux, à la fois dans l'élaboration des activités en classe mais aussi dans la réalisation de ces activités avec les élèves et les étudiants.

Les quatre sous-thèmes (et questions affiliées) que nous proposons dans les lignes suivantes doivent être considérés comme des moyens visant à promouvoir l'investigation et faciliter le débat. Tous les sous-points sont fortement corrélés : leur différenciation est purement fonctionnelle dans la simplification de l'organisation des groupes de travail pendant la conférence.

Sous-thème 1. Contenu mathématique et développement du curriculum

La relation entre les mathématiques en tant que discipline et le contenu mathématique qui doit être enseigné nous rappelle la dialectique théorie / pratique qui s'est vu accorder un intérêt grandissant depuis les années 90 (voir, e.g., Brown & Cooney, 1991 ; Burton, 1991 ; Godino & Batanero, 1997 ; Wittmann, 1991). Dans la recherche des conditions pour séparer la connaissance entre les deux pôles, on retrouve notamment la preuve qu'une conception qui assigne à la « théorie » le rôle d'instruire la « pratique » est vouée à l'échec et, par conséquent, rend compte d'un besoin croissant de développer les interactions qui existent entre les deux pôles ainsi que dans la coopération qui existe entre les acteurs impliqués dans le système éducatif (Bartoli Bussi & Bazzini, 2003).

Depuis les années 80, une contribution significative dans le débat a été fournie par Chevallard, qui a étudié le phénomène de transposition didactique, produisant ainsi des éléments de connaissance à propos des systèmes didactiques des contenus pour l'enseignement des mathématiques. Cela conduisit au développement de la théorie de la transposition didactique ainsi que dans sa réalisation pratique (Chevallard, 1985). Cette idée a été davantage développée, dans les années 90 et ensuite, dans une étude plus générale à l'intérieur de laquelle on retrouve une pratique des mathématiques qui prend la forme de différentes praxéologies (combinant praxis et logos).

En se concentrant sur l'épistémologie des mathématiques et en remarquant la persistance des difficultés des étudiants lorsqu'il s'agit de concepts spécifiques, Brousseau (1997) s'est intéressé à la notion « d'obstacle épistémologique » dans les mathématiques. Cette idée a inspiré des recherches sur l'éducation mathématique, ouvrant ainsi la voie dans la recherche d'autres sortes d'obstacles, liés aux aspects didactiques et aux aspects cognitifs, ainsi qu'à la critique de la notion d'obstacle épistémologique basée sur l'histoire et la culture (Radford, 1997).

Les interactions dialectiques entre la théorie et la pratique ont constitué l'essentiel du travail des développeurs de curriculum, principalement quand les différents acteurs (chercheurs, enseignants, directeur d'école) se voient dans l'obligation de travailler ensemble. Dans de telles situations, le développement de curriculum peut être une belle opportunité pour la coopération et l'enrichissement mutuel, et se traduit par une contribution pour l'école (Bazzini, 1991). Ce thème sera également abordé dans le deuxième point (voir ci-dessous).

Le choix du contenu à inclure dans le curriculum est un enjeu important qui requiert des investigations attentives dans tous les contextes. A côté de contenus traditionnels tels que l'arithmétique, l'algèbre et la géométrie, de nouveaux contenus doivent être inclus dans les curricula : la pensée probabiliste et stochastique en sont un exemple.

Depuis ces dernières années, la plupart des pays ont introduit ou développé les contenus statistiques dans les cours de mathématique en primaire et dans le secondaire. Les raisons sont nombreuses : prendre en compte l'augmentation du pouvoir stochastique dans le domaine des mathématiques, la volonté de développer de nouvelles approches pédagogiques fondées sur la modélisation de situations réelles et de l'interdisciplinarité qui sont devenus une exigence sociétale.

Dans l'apprentissage supérieur, de plus en plus de cours incorporent les statistiques au niveau Licence comme dans les programmes Doctoraux. A ce niveau les variations sectorielles sont multiples (statistique pour la biologie, management, psychologie, etc.) avec, comme il l'a été remarqué par Jeanne Fine (2010), selon les mots de Bourdieu, un risque important d'hyperspécialisation et un affaiblissement identitaire de la discipline. Les fondations sont supposément acquises durant la période qui précède les études et l'enseignement de la statistique se résume à la présentation par des non-spécialistes implémentant des techniques reposant sur l'utilisation de logiciels spécialisés. La dimension opérationnelle de la connaissance est privilégiée aux dépens des dimensions systématiques et historiques (voir Fabre 2010), avec le risque que les étudiants ne maîtrisent pas les concepts statistiques de base, comme cela a été souligné dans de nombreuses études et recherches (voir , en particulier, Batanero et al , 1994 ; Delmas et al,2007). Les épistémologies multiples, le plus souvent non clarifiées, sont une source de difficulté pour les étudiants qui ne sont pas en mesure d'identifier clairement d'où l'enseignant parle d'un point de vue épistémologique (Armatte 2010).

Il est vrai que la statistique est une discipline dont l'épistémologie est complexe. Cependant, il est important que cette discipline soit enseignée par des spécialistes dans l'enseignement supérieur et soit intégrée dans les cours de mathématiques dans le secondaire afin qu'elle ne soit pas diluée dans les disciplines hôtes (Gatuso, 2011). Mais il existe beaucoup de différences avec les mathématiques, des différences qui doivent être explicites dans le contexte de la formation des professeurs de mathématique. En statistiques, les étudiants devraient être amenés à abandonner leur vision du monde déterministe et à considérer le manque de certitude comme un trait à part entière de la réalité (Meletiou-Mavrotheris & Lee, 2002). La différence fondamentale entre les statistiques et les mathématiques est que, dans les statistiques, le contexte bénéficie d'un statut particulier : il fait partie intégrante du problème. Le risque de malentendu entre les étudiants et les professeurs, lié aux différences de représentations, est donc plus grand (Hahn, 2014). En statistiques les étudiants devraient à la fois être en mesure de maîtriser les raisonnements inductifs et déductifs (Fine, 2010), et combiner ainsi deux perspectives : l'approche centrée sur les données et l'approche modélisation, plus formelle (Armatte, 2010 ; Peters, 2011). Cela n'est pas seulement fait pour maîtriser les concepts mais aussi pour développer une logique statistique (Gattuso, 2011), intégrant l'utilisation de la technologie, ce qui est essentiel en statistique (Serrado et al, 2014).

Le débat précédent ouvre la voie pour des contributions au sous-thème 1 de la conférence CIEAEM67, qui se concentre sur les enjeux liés aux aspects épistémologiques de l'enseignement des mathématiques et se situe dans une dialectique obstacle/ressource. Le sous-thème 1 mettra l'accent sur les questions suivantes :

- *Quels obstacles pourraient interférer avec l'enseignement ? Quelle est leur nature ? Quelles seraient les stratégies possibles pour les surmonter ?*
- *Quels obstacles pourraient interférer avec l'apprentissage ? Quelle est leur nature ? Quelles seraient les stratégies possibles pour les surmonter ?*
- *Quels sont les obstacles et les ressources dont on dispose dans les différents curriculums à l'échelle nationale ?*
- *Quelle expertise professionnelle est nécessaire pour développer et implémenter un curriculum?*
- *Faut-il accorder une attention particulière à un contenu spécifique ?*
- *La statistique devrait-elle être présentée à l'école primaire ? Comment devrions-nous penser la préparation des professeurs qui seront en charge d'enseigner la statistique à chacun des différents niveaux d'étude (primaire, secondaire, supérieur) ? Quelles sont les différences/complémentarités entre les mathématiques et la statistique ?*

Sous-thème 2. La formation des enseignants

La formation des professeurs s'est vu octroyer une attention plus importante au cours de la dernière décennie (Clark-Wilson et al.,2014 ; Even & Ball (Eds.), 2009 ; Wood (Ed.), 2008). Ce « domaine émergent » (Adler et al., 2005) trouve ses racines dans les recherches précédentes faites autour des processus apprentissage/enseignement des salles de classes. Avec la diffusion progressive des nouveaux modèles d'enseignement et d'apprentissage depuis les années 60, le rôle du professeur dans la salle de classe a changé radicalement. En réalité, les nouvelles façons d'apprendre requièrent également des nouvelles façons d'enseigner : ce changement n'est pas spontané ; au contraire, afin de le mettre en place il faut encourager des initiatives appropriées dans la formation des enseignants.

Les recherches ont mis en évidence les différents aspects ayant à voir avec la formation des professeurs de mathématiques : de la spécificité des connaissances nécessaires aux enseignants jusqu'aux aspects affectifs, de l'inclusion des nouvelles technologies aux analyses systémiques.

Les réflexions sur le type de connaissances qui caractérisent le professeur de mathématique dans son travail professionnel a été établi dans le travail précurseur et fondamental de Dhulman (1986). Ponte et al. (1994) soutient l'idée selon laquelle il convient d'allier le contenu mathématique aux connaissances pédagogiques, en faisant appel aux différentes composantes de la connaissance actuelle afin de produire une restructuration des compétences et des savoir faire des professeurs. Ce contenu des connaissances pédagogiques va bien au-delà d'une simple représentation du sujet en question : il doit inclure « un corps intelligible d'images, de principes, et de règles pour l'action, de la généralité, davantage de spécificité, architecturé autour d'une rationalité transparente, prenant en considération la nature particulière du contenu sous-jacent et suffisamment puissant pour guider l'action de l'enseignant » (p. 358). Steinbring (1998) explore une composante spécifique de la connaissance professionnelle pour les professeurs de mathématique, à savoir « une connaissance épistémologique dans des contextes sociaux d'apprentissage social (p 160) ». Il affirme que « les enseignants ont certainement besoin d'un contenu de connaissances mathématiques et pédagogiques ; et, au sein du domaine des connaissances pédagogiques, ils auraient également besoin d'une connaissance épistémologique, ainsi ils seraient en mesure de jauger des limites de la connaissance mathématique dans les différents domaines sociaux d'enseignement, d'apprentissage et de communication mathématique. Cette composante importante qu'est la connaissance épistémologique des mathématiques dans des contextes sociaux d'apprentissages n'est pas, un corpus de connaissances systématisé, qui pourrait être enseigné aux futurs enseignants par le biais d'un programme fixe. Il conviendrait plutôt de dire que la connaissance épistémologique se compose d'éléments de connaissances exemplaires, car elle se réfère à des études de cas relatives à l'analyse de composantes propres à l'enseignement ou à des entrevues avec des étudiants, et comprend idées conceptuelles historiques, philosophiques et épistémologiques » (p. 160). Ball et Bass (2003) formalisent les caractéristiques des mathématiques qui sont impliquées dans l'enseignement dans le Modèle du Savoir Mathématique des Enseignants. Ils identifient le savoir des contenus spécialisés comme un sous-domaine important de la connaissance mathématique, strictement lié à l'activité de l'enseignant. La connaissance du contenu spécialisé s'apparente souvent avec les connaissances et les compétences liées aux technologies numériques, qui ont également acquis une importance croissante dans le contexte de la formation des enseignants (Bairral & Powell, 2013; Drijvers et al., 2010).

D'autre part, plusieurs études ont portées sur la dimension sociale des programmes de formation des enseignants, en particulier l'implication des enseignants dans l'analyse et la réflexion commune avec des chercheurs. Dans la littérature de recherche, nous trouvons des notions importantes comme celles de « communauté de pratique » (Wenger, 1998) et de « communautés d'enquête » (Jaworski, 2006); la pierre angulaire de ces études étant la notion de réflexion critique, conçu non seulement comme une attitude fondamentale qui doit être développée par les enseignants, mais aussi comme une responsabilité professionnelle. Cette idée est liée à celle selon laquelle il doit exister une collaboration entre les enseignants et les chercheurs, comme Krainer (2011) le souligne lorsqu'il préconise de considérer les chercheurs comme « les principales parties prenantes dans la pratique » et les enseignants comme « les principales parties prenantes dans la recherche. »

Outre les dimensions épistémologiques et sociales, la dimension affective se voit octroyer un rôle important dans le travail des enseignants ainsi que dans la formation de ces derniers. Cela inclut l'étude de l'influence des croyances et des émotions des enseignants sur leurs façons d'enseigner les mathématiques. En fait, comme le souligne Zembylas (2005),les connaissances des enseignants se trouvent dans « les vies vécues par les enseignants, dans les valeurs, les croyances et les convictions profondes adoptées dans la pratique, dans le contexte social qui entoure ces pratiques, et dans les relations sociales qui animent l'enseignement et la rencontre d'apprentissage ». Ces valeurs, les croyances et les émotions entrent en jeu à mesure que les enseignants prennent des décisions, d'agir et de réfléchir sur les différents objectifs, sur les méthodes et sur les significations de l'enseignement. (p. 467)

Ceci est particulièrement édifiant lorsqu'il s'agit des enseignants du primaire, qui sont des enseignants généralistes et qui doivent parfois enseigner les mathématiques en dépit de leurs dispositions personnelles envers les mathématiques. Par conséquent, les croyances et les émotions des enseignants envers les mathématiques peuvent constituer des obstacles à la pratique d'un enseignement efficace. L'étude des conditions dans lesquelles cette hypothèse est vraie reste un problème ouvert. D'autre part, des expériences et des émotions personnelles négatives peuvent également s'avérer être des ressources pour les enseignants, comme cela l'a été suggéré par Coppola et al. (2013) lors, plus particulièrement, de sa réflexion sur les futurs enseignants.

Enfin, les processus de formations des enseignants de mathématiques doivent également être considérés selon une approche systémique, en mettant l'accent sur les relations et les dynamiques qui existent entre les différentes « variables » inclues dans ces processus complexes: les connaissances et les pratiques des enseignants, les aboutissements des recherches, les contraintes institutionnelles (contraintes de programmes nationaux en particulier), les traditions, les aspects culturels, et ainsi de suite. Compte tenu de cette complexité, le développement des enseignants peut être considéré comme un processus de transposition didactique méta-évolution au fil du temps (Arzarello et al., 2014).

A partir de ce développement, et à partir des contributions faites par cette dernière, le sous-thème 2 dans CIEAEM67 vise à repenser la complexité de la formation des enseignants en termes de ressources ainsi que les éléments qui font obstacle à l'enseignement et l'apprentissage des mathématiques. Les questions suivantes peuvent, en outre, orienter la discussion :

• Comment est-il possible d'aider les enseignants à développer les connaissances et les compétences appropriées dans les technologies numériques afin qu'ils soient efficaces dans leur enseignement des mathématiques ?

• Quels sont les principaux obstacles au développement de l'enseignement des mathématiques ?

• Comment la dimension sociale peut-elle devenir une ressource pour la formation des enseignants ? Quels sont les défis auxquels les programmes fortement basés sur l'interaction sociale dans les communautés de pratique / enquête sont confrontés?

• Comment la dimension affective peut-elle devenir une ressource pour la formation des enseignants ?

Sous-thème 3. Pratiques en classe et autres espaces d'apprentissage

La pensée mathématique est issue et se développe dans une interaction complexe de langues et de représentations, au travers de références à des intuitions, des métaphores et des analogies, et en faisant usage de divers objets et outils, qui interagissent avec notre nature corporelle. Tous ces éléments sont essentiels pour l'enseignement et pour les activités d'apprentissage qui sont réalisées dans le contexte de la classe, ainsi que dans d'autres espaces d'apprentissage : à la lumière du thème de la conférence, ils peuvent constituer des ressources possibles ou, au contraire, des obstacles pour l'apprentissage des mathématiques.

Alors que, depuis les années 1980, l'attention a d'abord porté sur la langue et les représentations écrites, plus récemment, l'attention a également été accordée aux formes plus matérialisées de la pensée et de la représentation, comme les gestes qui sont considérés, principalement, comme des ressources pour l'enseignement ainsi que pour l'apprentissage (Arzarello, 2008; et Arzarello al, 2009;. Radford, 2002, 2014). D'autres études se sont intéressées au rôle des nouvelles technologies et des TIC en tant que médiateurs possibles pour l'apprentissage (Drijvers et al., 2010). Ainsi, le sous-thème 3 fait aussi référence aux utilisations possibles des nouvelles technologies comme des ressources pour l'apprentissage des mathématiques, mais aussi sur les éventuels obstacles que l'introduction de nouvelles technologies pourrait produire à plusieurs niveaux (cognitif, didactique, communication, etc.).

En ce qui concerne les pratiques dans la classe, le rôle de l'enseignant doit aussi être abordé. Même à partir, éventuellement, de différentes positions théoriques, l'enseignant est généralement vu comme une ressource pour l'apprentissage des élèves. À cet égard, les enseignants doivent faire face à différentes exigences cognitives, en particulier à celles des élèves ayant des difficultés d'apprentissage en mathématiques, comme largement discuté dans la littérature (Dehaene, 1997, Landy et Goldstone, 2010). Une utilisation consciente de stratégies d'enseignement appropriées et spécifiques pour les élèves diagnostiqués comme ayant des troubles d'apprentissage, en particulier avec la dyscalculie développementale (Butterworth, 2005; Dehaene, 1997), est également important pour les étudiants qui ne sont pas officiellement diagnostiqués, mais présentent des difficultés d'apprentissage enclin aux difficultés très similaire à celui des élèves dyscalculiques. Par conséquent, le développement de pratiques innovantes dans l'enseignement est un objectif de plus en plus nécessaire pour la recherche en didactique des mathématiques en général, et pour les enseignants en particulier.

Bien que l'école soit l'institution la plus importante pour l'apprentissage, nous savons que ce n'est pas le seul endroit où l'on apprend. Mais, qu'entendons-nous par l'apprentissage ? Il est fréquent de trouver des enseignants qui ont une vue restreinte de ce que peut signifier apprendre les mathématiques. Souvent, l'apprentissage est associé à la reproduction des méthodes de comptage et à des formules de calcul. Bien que cette idée soit dépassée, au moins pour la recherche dans l'enseignement des mathématiques, il semble, malheureusement, que certaines pratiques d'enseignement ou de formation soient encore restrictives et ne reconnaissent pas que l'apprentissage peut être observé à travers un prisme différent. Nous apprenons, en effet, également dans les espaces formels et non formels (musées, programmes d'apprentissage à distance, jeu , etc.), dans les environnements dynamiques, en face-à-face ou en ligne. Nous croyons que l'enseignement des mathématiques en contexte devrait favoriser le développement d'une pensée qui offre un potentiel pour l'étudiant dans son présent et son avenir, indépendamment de lses futures occupations ou activités professionnelles. Des procédés tels que le développement de la curiosité, la pensée critique, le raisonnement et la motivation à apprendre, ainsi que les moyens de développer la vérification, la réfutation, et la déduction devraient tous être mis à profit à la fois dans la classe et aussi dans des espaces d'apprentissages non formels.
Le sous-thème 3 mettra l'emphase sur les questions suivantes :

• Quels sont les éléments qui caractérisent les pratiques de l'enseignant en tant que ressources pour les étudiants et comment est-il possible de favoriser ces éléments ?

• Une question provocatrice: un enseignant peut-il être un obstacle à l'apprentissage des élèves ? Pourquoi et comment cela serait-il possible ? Comment cela pourrait-il être évité ?

• Comment les technologies et les TIC peuvent-ils être des médiateurs possibles pour l'enseignement et l'apprentissage inclusif ?

• Comment les différentes formes incorporées de la représentation et de la pensée, comme les gestes, ou d'autres registres de représentation, tels que Visuel-verbal, visuel non-verbal, auditif, kinesthésique soient considérés comme des ressources pour l'enseignement et l'apprentissage inclusif ?

• Quelles sont les ressources ou les stratégies d'enseignement qui sont utilisées pour améliorer le potentiel d'apprentissage de tous les élèves, en particulier ceux qui ont des difficultés d'apprentissage ?

• Quels sont les nouveaux aspects de l'apprentissage des mathématiques qui pourraient-être améliorés dans des espaces d'apprentissages formels ou dans des environnements non formels ?

• Quels sont les avantages ou les restrictions des TIC ou celles des ressources plus classiques (par exemple, ceux de la manipulation) dans la promotion de l'apprentissage des mathématiques dans des contextes formels ou informels?

Sous-thème 4. Les questions culturelles, politiques et sociales

Depuis les années 1980, au moins, on a contesté l'hypothèse selon laquelle les mathématiques ne seraient pas influencées par la culture (Bishop, 1988; D'Ambrosio, 1985; Ellerton & Clements, 1989). Il y a aussi une prise de conscience croissante du fait que l'enseignement des mathématiques est non seulement dépeint comme n'étant pas influencé par la culture, mais que par ailleurs il excluait ou aliénait de nombreuses jeunes filles et femmes ainsi que les garçons et les hommes qui ne se conforment pas aux stéréotypes présents dans la classe ou dans les manuels scolaires, ou encore au choix de programme trop abstrait et très théorique. Pour incarner ce changement de la recherche dans l'enseignement des mathématiques, les termes de « changement social »et de « changement sociopolitique » (Gutiérrez, 2010; Lerman, 2000) ont fait leur apparition. Maintenant, il est devenu courant de penser que les mathématiques ainsi que leur enseignement ne peuvent être éloignés des questions culturelles, sociales et politiques. C'est pourquoi tout est mis en œuvre pour mener à bien des études visant à améliorer l'enseignement des mathématiques.

Les contextes culturels, politiques et sociaux peuvent être considérés comme des obstacles et / ou comme des ressources jouant dans la réussite des élèves en mathématiques. D'une part, nous pouvons les considérer comme des obstacles pour des étudiants en termes d'accès et en termes de compréhension vis-à-vis de l'enseignement des mathématiques. Bien que moins répandu dans les pays occidentaux, mais néanmoins d'une importance fondamentale, l'accès physique aux écoles et aux cours de mathématiques a reçu de l'attention (par exemple, Kazima & Mussa, 2011). À un second niveau, les réformes des programmes et les contre-réformes ont souvent transformées les obstacles à l'apprentissage des mathématiques auxquels certains groupes sociaux sont confrontés (par exemple, Jablonka & Gellert, 2011; Vithal & Skovsmose, 1997). Ce deuxième niveau concerne la répartition des différentes formes de la connaissance mathématique: Qui a accès à quelles formes de connaissances mathématiques ? À un troisième niveau, la question a été posée sur la façon dont les stratégies d'enseignement et d'éducation compliquent ou empêchent l'accès et la participation dans des formes institutionnellement et socialement reconnues des activités mathématiques pour des groupes particuliers d'étudiants (par exemple, Straehler-Pohl et al., 2014). Les conditions culturelles (par exemple, l'importance spécifique de la culture de l'oralité), politiques (par exemple, les politiques d'intégration des migrants), et sociales (par exemple, la pauvreté relative), prises séparément, mais surtout combinées, traduisent souvent des obstacles en termes d'enseignement et d'apprentissage des mathématiques.

D'autre part, les déterminismes culturels, politiques et sociaux peuvent être considérés comme des ressources. Cela est tout à fait évident dans le cas des privilèges, où le milieu social des étudiants et leurs horizons se mettent en avant facilement et peuvent s'avérer bénéfiques lorsqu'il s'agira d'assimiler les enseignements scolaires (par exemple, le capital culturel et les codes de la classe moyenne). Le point crucial est de savoir si, et si oui, comment, les expériences et les activités des étudiants défavorisés, qui ne sont pas évalués, peuvent être utilisées en tant que ressources pour l'enseignement et l'apprentissage des mathématiques. A titre d'exemple, Barton et Frank (2001) en s'interrogeant sur les cultures minoritaires demandent: « Quelles sont les conditions dans lesquelles » (...) les enfants, pour lesquels les concepts mathématiques « de base » ne sont pas facilement disponibles en raison de concepts incommensurables puissamment présents dans leur propre patrimoine culturel-linguistique, peuvent « avoir un avantage cognitif en mathématiques, et quelle est la nature de cet avantage? » (p. 147). Healy et Powell (2013), après avoir examiné plusieurs ressources relatives à l'apprentissage des mathématiques, concluent qu'il existe un grand nombre d'études montrant comment le fait d'être polyglotte est positivement lié au développement cognitif. Ces études appellent également à davantage encourager les élèves afin d'utiliser leurs ressources linguistiques dans les activités mathématiques. Rapprocher ces deux points de vues, comprendre les conditions culturelles, politiques et sociales qui créent des obstacles à l'enseignement des mathématiques et à l'apprentissage, pourraient nous amener à comprendre le micro et les macro-processus-sociaux des individus désavantagés.

En fait, la diversité est un élément essentiel de ce que c'est que d'être humain. Même au sein de la même unité familiale, il y a des différences entre les enfants en fonction de leurs intérêts et de leurs aptitudes. Dans les salles de classe où les élèves partagent apparemment les mêmes origines sociales et culturelles il n'y a pas d'uniformité. Plus particulièrement de nos jours, avec les flux mondiaux de personnes, de nombreuses classes sont susceptibles de comporter des étudiants venant de milieux sociaux, culturels et linguistiques divers, ce qui constitue à la fois une ressource et un défi pour les enseignants. Ils peuvent cependant manquer de soutien systémique, mais également à avoir à faire travailler de plus en plus souvent avec des contraintes de temps et de responsabilité. C'est dans le cœur même des énoncés des missions et des politiques d'éducation qui stipulent que chaque enfant ou étudiant est un individu et devrait recevoir une attention personnalisée de la personne qui fait son éducation.

Enfin, comprendre comment les conditions culturelles, politiques et sociales peuvent devenir des ressources pour les apprenants pourraient nous obliger à analyser comment le curriculum, les stratégies d'enseignement et les scénarios d'apprentissage pourraient être plus finement réglés par et pour les arrière-plans et avant-plans de groupes d'élèves.

Questions:

• Comment les contextes culturels, politiques et sociaux limitent-ils l'accès et la participation à des formes de valeur de l'apprentissage des mathématiques ? Comment ces restrictions peuvent-elles être surmontées ?

• Comment les *backgrounds* et les *foregrounds* des élèves défavorisés peuvent-ils être utilisés comme ressources pour l'enseignement et l'apprentissage des mathématiques ?

• Comment pourrions-nous repenser les théories et les pratiques de l'enseignement des mathématiques pour améliorer les résultats cognitifs et affectifs des élèves bilingues / multilingues ?

• Comment pourrions-nous favoriser l'intégration des élèves et des étudiants de différentes origines culturelles dans la classe et dans la société en général ?

• Comment pourrions-nous faire face aux défis de stéréotypes de genre et d'autres questions liées au genre et aux inégalités qu'ils créent ?

• Comment les concepteurs de politiques prennent-ils en considération tous types de diversités et d'inégalités dans votre pays ou région (par exemple, l'UE)?

REFERENCES

Adler J., Ball D., Krainer K., Lin F. L., & Novotna J. (2005). Reflections on an emerging field: Researching mathematics teacher education. *Educational Studies in Mathematics*, *60*, 359-381.

Armatte, M. (2010). Le rôle de l'histoire dans l'enseignement de la statistique. *Statistique et Enseignement, 1*(2), 23-47.

Arzarello, F., Robutti, O., Sabena, C., Cusi, A., Garuti, R., Malara, N., & Martignone, F. (2014). Meta-didactical transposition: A theoretical model for teacher education programmes. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era. An international perspective on technology focused professional development* (pp. 347-372). Dordrecht: Springer.

Arzarello, F. (2008). Mathematical landscapes and their inhabitants: Perceptions, languages, theories. In E. Emborg & M. Niss (Eds), *Proceedings of the 10th International Congress of Mathematical Education* (ICME 10) (pp. 158-181). Copenhagen, Denmark: Roskilde University, IMFUFA.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, *70*(2), 97-109.

Bairral, M. A., & Powell, A. B. (2013). Interlocution among problem solvers collaborating online: A case study with prospective teachers *Pro-Posições, 24*(1), 1-16. doi: http://dx.doi.org/10.1590/S0103-73072013000100005.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group Edmonton* (pp. 3-14). AB: CMESG/GDEDM.

Bartolini Bussi M., & Bazzini L. (2003). Research, practice and theory in didactics of mathematics: Towards dialogue between different fields. *Educational Studies in Mathematics*, *54*, 203-223.

Barton, B., & Frank, R. (2001). Mathematical ideas and indigenous languages. In B. Atweh, H. Forgasz, & B. Nebres (Eds.), *Sociocultural research on mathematics education: An international perspective* (pp. 135–149). Mahwah, NJ: Lawrence Erlbaum

Batanero, C., Godino, J., Vallecillos, A., Green, D., & Holmes, P. (1994). Errors and difficulties in understanding elementary statistical concepts. *International Journal in Mathematics, Science and Technology, 25*(4), 527-547.

Bazzini, L. (1991). Curriculum development as a meeting point for research and practice. *Zentralblatt für Didaktik der Mathematik, 91/4,* 128-131.

Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective in mathematics education*. Dordrecht: Kluwer.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Brown, S., & Cooney, T.(1991). Stalking the dualism between theory and practice. *Zentralblatt für Didaktik der Mathematik, 91/4*, 112-117.

Burton, L. (1991). Models of systematic co-operation between theory and practice. *Zentralblatt für Didaktik der Mathematik, 91/4*, 118-121.

Butterworth B. (2005). Developmental dyscalculia. In J. I. E. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 455-467). Hove, UK: Psychology Press.

Chevallard, Y. (1985). *La transposition didactique. Du savoir savant au savoir einsegné*. Grenoble: La Pensée Sauvage.

Clark-Wilson, A., Robutti, O., & Sinclair, N. (Eds.) (2014). *The mathematics teacher in the digital era. An international perspective on technology focused professional development* (pp. 347-372). Dordrecht: Springer.

Coppola, C., Di Martino, P., Pacelli, T., & Sabena, C. (2013). Primary teachers' beliefs and emotional disposition towards mathematics and its teaching. In B. Di Paola (Ed), *Proceedings of CIEAEM 65. Quaderni di ricerca in didattica (mathematics), Issue 23*(1), 217-226.

D'Ambrosio, U. (1985). *Socio-cultural bases for mathematics education*. Campinas, Brazil: UNICAMP.

Dehaene S. (1997). *The number sense: How the mind creates mathematics.* New York, Oxford University Press.

DelMas, R., Garfield, J., Ooms A., & Chance, B. (2007). Assessing items students' conceptual understanding after a first course in statistics. *Statistics Education Research Journal, 6*(2), 28-58.

Drijvers, P., Kieran, C., & Mariotti, M.-A. (2010). Integrating technology into mathematics education: Theoretical perspectives. In C. Hoyles, & J.-B. Lagrange (Eds.), *Mathematics* *education and technology —Rethinking the terrain: The 17th ICMI study* (pp. 89-132). New York: Springer.

Ellerton, N. F., & Clements, M. A. (1989). *Teaching post-secondary mathematics at a distance: A report to the Commonwealth Secretariat*. Geelong, Vic: Deakin University.

Even, R., & Ball, D. L. (Eds.) (2009). *The professional education and development of teachers of mathematics: The 15th ICMI study.* New York: Springer.

Fabre, M. (2010). Problématisation des savoirs. In A. Van Zanten (Ed.), *Dictionnaire pédagogique* (pp. 539–541). Paris: Presses Universitaires de France.

Fine J. (2010). Probabilités et statistique inférentielle. Approche sondage versus approche modèle. *Statistique et Enseignement, 1*(2), 5-21.

Gattuso, L. (2011). L'enseignement de la statistique: Où, quand, comment pourquoi pas? *Statistique et Enseignement, 2*(1), 5-30.

Godino, J. D., & Batanero C. (1997). Clarifying the meaning of mathematical objects as a priority area of research in mathematics education. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity. The 8th ICMI study* (pp. 177-195). Dordrecht: Kluwer.

Gutiérrez, R. (2010). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, *44*(1), 37–68.

Hahn, C. (2014). Linking academic knowledge and professional experience in using statistics: a design experiment for business school students. *Educational Studies in Mathematics, 86*(2), 239-251.

Healy, L., & Powell, A. B. (2013). Understanding and overcoming "disadvantage" in learning mathematics. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 69–100). New York: Springer.

Jablonka, E., & Gellert, U. (2011). Potentials, pitfalls, and discriminations: Curriculum conceptions revisited. In B. Greer & O. Skovsmose (Eds.), *Critique and politics of mathematics education* (pp. 287–308). Rotterdam: Sense.

Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, *9*, 187-211.

Kazima, M., & Mussa, C. (2011). Equity and quality in mathematics education in Malawi schools. In B. Atweh, M. Graven, W. Secada, & P. Valero (Eds.), *Mapping equity and quality in mathematics education* (pp. 163–176). Dordrecht: Springer.

Krainer, K. (2011). Teachers as stakeholders in mathematics education research. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol.1), (pp. 47-62). Ankara, Turkey: PME.

Lagrange, J.-B., Artigue, M., Laborde C., & Trouche, L. (2003). Technology and mathematics education: A multidimensional study of the evolution of research and innovation. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second International Handbook of Mathematics Education* (pp. 239–271). Dordrecht: Kluwer.

Landy D., & Goldstone R. L. (2010). Proximity and precedence in arithmetic. *Quarterly Journal of Experimental Psychology, 63*, 1953-1968.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19–44). Westport, CT: Ablex.

Meletiou-Mavrotheris, M., & Lee, C. (2002). Teaching students the stochastic nature of statistical concepts in a introductory statistics course, *Statistics Education Research Journal, 1*(2), 22-37.

Peters, S. (2011). Robust understanding of statistical variation, *Statistics Education Research Journal, 10*(1), 52-88.

Ponte, J. P., Matos, J. F., Guimarães, H. M., Leal, L. C., & Canavarro, A. P. (1994). Teacher's and students' view and attitude towards a new mathematical curriculum: A case study. *Educational Studies in Mathematics, 26*, 347-365.

Radford, L. (1997). On psychology, historical epistemology and the teaching of mathematics: Towards a socio-cultural history of mathematics. *For the Learning of Mathematics*, *17*(1), 26-30.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics, 22*(2), 14–23.

Radford. L. (2014). On the role of representations and artefacts in knowing and learning. *Educational Studies in Mathematics, 85*, 405–422.

Serrado A., Meletiou-Mavrotheris, M., & Paparitodemou, E. (2014). Early statistics: A course for developing teachers' statistics technological and pedagogical content. *Statistique et Enseignement, 5*(1), 5-29.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Steinbring H. (1998). Elements of epistemological knowledge for mathematics teachers. *Journal of Mathematics Teacher Education, 1,* 157-189.

Straehler-Pohl, H., Fernández, S., Gellert, U., & Figueiras, L. (2014). School mathematics registers in a context of low academic expectations. *Educational Studies in Mathematics*, *85*(2), 175–199.

Sullivan, P. (2008). Knowledge for teaching mathematics: An introduction. In P. Sullivan & T. Wood (Eds.), ***Knowledge and beliefs in mathematics teaching and teaching development.*** *The international handbook of mathematics teacher education* (Vol. 1), (pp. 1-12). Rotterdam: Sense.

Vithal, R., & Skovsmose, O. (1997). The end of innocence: A critique of 'ethnomathematics'. *Educational Studies in Mathematics, 34*(2), 131–157.

Wenger, E. (1998). *Communities of practice: Learning, meaning and identities*. Cambridge, UK: Cambridge University Press.

Wittmann E. C. (1991). From inservice-courses to systematic cooperation between teachers and researchers. *Zentralblatt für Didaktik der Mathematik, 91/5*, 158-160.

Wood, T. (Series Ed.). (2008). *The international handbook of mathematics teacher education* (Vols. 1-4). Rotterdam: Sense.

Zembylas, M. (2005). Beyond teacher cognition and teacher beliefs: The value of the ethnography of emotions in teaching. *International Journal of Qualitative Studies in Education*, *18*(4), 465-487.

PLENARIES

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"Quaderni di Ricerca in Didattica (Mathematics)", n. 25, Supplemento n.2, 2015
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Un enseignement fondé sur des situations didactiques de recherche

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Résumé : Le travail présenté dans cette conférence s'appuie sur les travaux de recherche du groupe DREAM (Démarche de Recherche pour l'Enseignement et l'Apprentissage des Mathématiques). Ce groupe étudie la possibilité de fonder un enseignement sur la recherche de problème et questionne les pratiques enseignantes autour de la résolution de problèmes. Les questions qui se posent alors concernent les savoirs construits lors des situations de recherche et la façon dont ils prennent leur place dans la progression du temps didactique, mais aussi la possibilité de fonder un enseignement sur des situations de recherche en classe.
Le cadre théorique et la méthodologie sont construits sur une idée de co-construction entre chercheurs en didactique des mathématiques et en mathématiques et enseignants.
En partant de considérations épistémologiques et en exemplifiant à la fois les situations didactiques de recherche de problèmes et la construction des ingénieries nous mettons à l'épreuve les cadres théoriques et donnons des éléments de réponses aux questions posées.

Abstract: The work which will be presented in this conference is built on the work of the DREAM[1] research group (Research Approach to Teaching and Learning of Mathematics). This group is studying the possibility of basing teaching practice on the issue of research problems in the classroom and the questioning of teaching practices around problem solving. The questions that arise are related to the knowledge constructed during didactical situations of problem research and its place in the progression of the didactical time but also the possibility of an annual progression based on the implementation of classroom research situations possible in the regular classroom.
The theoretical framework and the methodology adopted are based on the idea of a co-construction between researchers in mathematics, researchers in mathematics education, and teachers.
Starting from theoretical considerations, the presentation will exemplify both the didactical situations of the research problems and the process of designing such lessons. It will also be an occasion to put to the test the theoretical frameworks described above.

Introduction

Le document de discussion de la conférence indique qu'il « convient d'analyser les interactions qui existent entre les objets réels et les constructions mathématiques, le rôle des processus de pensée et les langages (souvent en relation avec les réalisation d'expériences) ainsi que l'influence des croyances et des émotions. » (document de discussion, CIEAEM 67[2]) ; cette phrase pose les problèmes fondamentaux qui sont à la base d'une réflexion sur les apports des problèmes de recherche en classe et pointe les éléments importants liés à la dimension expérimentale des mathématiques et à leurs relations avec l'enseignement des mathématiques. Dans cette conférence, je souhaite tisser les liens existant entre *faire des mathématiques*, ce qui est du ressort des mathématiciens (incluant les professeurs de mathématiques lorsqu'ils *font* des mathématiques) et *faire faire des mathématiques*, ce qui relève des enseignant(e)s ou des formateurs (trices) d'enseignants. Comment tisser des liens entre ces deux mondes pour que, finalement, les élèves et les étudiants fassent l'expérience dans leur apprentissage d'une rencontre avec les mathématiques dans leur dimension la plus créative ? Dans la première partie j'explore les concepts de problèmes et leurs potentiels didactiques et dans la seconde j'exemplifie mes propos à partir d'expérimentations conduites dans le cadre d'une recherche menée à l'Institut Français de l'Éducation[3].

1 Démarche de Recherche pour l'Enseignement et l'Apprentissage des Mathématiques (DREAM)
2 http://cieaem67.perladidattica.it/doc/C%2067%20final%20(12%20Jan)%20Disc%20Paper_FR.pdf
3 IFÉ, Ecole Normale Supérieure de Lyon

Les problèmes, quelques points de vue didactiques

Les problèmes sont au cœur de l'enseignement des mathématiques depuis bien longtemps comme en témoigne, par exemple, les paragraphes que Ferdinand Buisson y consacre dans son dictionnaire à la rubrique Mathématiques (Buisson, 1911). Plus près de nous et dans la tradition de John Dewey (1938), il est difficile de parler de problèmes et de résolution de problèmes sans faire référence à Polya (1945). Il propose dans son ouvrage *« How to solve it »* des heuristiques devant faciliter la recherche et la résolution de problèmes, heuristiques qu'il a construites sur l'observation de son activité propre de mathématicien :

« Studying the methods of solving problems, we perceive another face of mathematics. Yes, mathematics has two faces; it is the rigorous science of Euclid, but it is also something else. Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears to be an experimental, inductive science. » (Id. p.VII)

Cette autre face des mathématiques nécessite quelques réflexions et la dimension expérimentale citée par Polya se doit d'être précisée ce qui sera l'objet du paragraphe suivant. Ce travail fondamental a été à la base de développements importants pour mettre en relation, à travers les problèmes, le « faire des mathématiques » au « faire faire des mathématiques ». Les évolutions et les développements se sont faites en intégrant les critiques qui peuvent être apportées aux thèses défendues par Polya, la plupart des auteurs qui ont évoqués la résolution de problèmes dans l'enseignement des mathématiques se positionnant par rapport à son travail. Je relèverai, parmi d'autres deux objections qui me semblent faire avancer la compréhension du rôle des problème dans l'enseignement. La première porte sur la contextualisation de la recherche d'un problème et les liens avec les notions et les concepts mathématiques en jeu. Elle pointe la difficulté à relier la résolution d'un problème particulier avec des règles générales dé-contextualisées :

« Teaching general problem solving does not lead to mathematical skills or knowledge » (Sweller & al., 2011).

La seconde objection, relevés déjà par Schoenfeld (1994), est l'inclusion des problèmes dans le curriculum :

« In the standard curriculum such contexts might be used as "cover stories" to motivate a unit, and then one would get down to the "real math," as traditionally organized. But here, the solutions to the problems, in context, are the large part of the mathematics studied. That is, the mathematics often appears in a particular context, and aspects of it are worked out in that context; the more extended, formal presentation and decontextualization of the mathematics is not undertaken. » (Ib. Page 73)

Bien que des situations de recherche de problèmes continuent à vivre en classe, et bien que de nombreux travaux montrent les apports des problèmes pour l'enseignement et l'apprentissage des mathématiques, elles ne se sont pas généralisées. Les deux objections précédentes constituent des freins importants pour cette intégration et l'accent mis principalement dans l'approche des problèmes de recherche sur le développement de compétences métamathématiques est en opposition avec les contraintes institutionnelles qui pèsent sur les professeurs.

Par ailleurs, les *problèmes réels* dans la tradition des « realistic mathematics » contextualisent les notions mathématiques pour leur donner du sens. La question du transfert en lien avec la construction ou la réinvention du concept dans un contexte particulier pointe la tension d'un point de vue didactique entre cette réinvention et le nécessaire guidage par le professeur comme le mettait en évidence Paul Drijvers dans sa conférence lors de la CIEAEM 66 à Lyon[4]. Comment et pourquoi, telle notion perçue comme pertinente dans un contexte particulier pourra atteindre un statut de notion mathématique universelle ? Pour prendre l'exemple de l'algèbre, dans quelles conditions didactiques, la résolution d'un problème réaliste menant à la résolution d'une ou de

4 http://www.cieaem.org/?q=fr/node/39

plusieurs équations pourra mener à la construction du concept d'équation et à son caractère universel ? Ces questions conduisent à considérer le rôle des problèmes dans l'enseignement des mathématiques comme un lieu d'expérience sur les objets mathématiques à enseigner. Et, avant de développer des réponses possibles, je voudrais approfondir un peu la place de l'expérience dans la création des mathématiques et les relations existantes entre les perceptions sensibles des objets et leur théorisation.

Point de vue épistémologique

La notion d'expérience peut être regardée à la fois dans le domaine de la philosophie des sciences et dans celui de la philosophie de la connaissance. La subjectivité de l'expérience a été largement mise en évidence dans l'histoire des sciences et l'immédiateté des perceptions sensibles ne peut impliquer un caractère scientifique aux résultats de l'expérience. De nombreux exemples peuvent être développés dans les sciences expérimentales mais aussi en mathématiques comme je peux l'illustrer par les deux exemples suivants :

1- Construire un carré inscrit dans un cercle de rayon 1. Sur chacun de ses côtés construire le triangle isocèle dont le sommet appartient au cercle : on obtient alors un octogone régulier inscrit dans le cercle. Recommencer. A la nieme étape, le polygone obtenu est un polygone régulier à 2^{n+2} côtés qui se rapproche du cercle et dont le périmètre est une approximation du périmètre du cercle, peut-on ainsi calculer une approximation de Pi ?

2- Construire un demi-cercle de rayon 1. Construire sur le diamètre deux demi-cercles de rayon ½. Recommencer le processus. A chaque étape, la longueur de la ligne est invariante, en effet, on remplace un demi cercle de rayon R par deux demi-cercles de rayon R/2. En itérant le processus on obtient une ligne qui se rapproche du diamètre ; peut-on alors en déduire que la longueur de la ligne est égale à la longueur du diamètre, c'est à dire Pi=2 ?

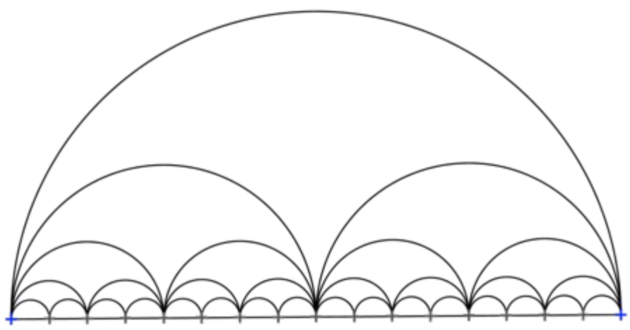


Fig. 1 Pi = 2 ?

Ce paradoxe (apparent) montre bien cette subjectivité et la nécessité de dépasser la seule expérience pour la relier à la théorie : l'expérience dans les deux cas semble la même mais le fait que l' « écart » entre la ligne brisée et le segment (au sens de distance maximum ou au sens d'aire) ne suffit pas à faire converger les longueurs. Le calcul de la longueur d'une courbe fait intervenir des dérivées ; dans la deuxième construction, les pentes infinies de la ligne aux points de contact avec le segment lèvent le paradoxe.

L'empirisme classique conduit au scepticisme (Hume, 1946) parce que la justification objective d'un fait par l'expérience ne peut être déduite d'une expérience subjective. Sans vouloir entrer dans une description exhaustive du rôle de l'expérience dans les sciences, les liens entre la théorie et l'expérience ont toujours amené à considérer les relations entre le domaine sensible et sa formalisation théorique dans un langage particulier. Toute expérience est directement reliée à la théorie (aux hypothèses) sous-jacentes ; Kuhn (1962) soutient que chaque théorie porte en elle les interprétations des termes qu'elle emploie si bien qu'une même expérience et ses résultats pourront conduire à des interprétations différentes suivant les hypothèses théoriques sous-jacentes.

« My remarks on incommensurability and its consequences for scientists debating the choice between successive theories, In Sections X and XIII have argued that the parties to such debates inevitably see differently certain of the experimental or observational situations to which both have recourse. Since the vocabularies in which they discuss such situations consist, however, predominantly of the same terms, they must be attaching some of those terms to nature differently, and their communication is inevitably only partial. As a result, the superiority of one theory to another is something that cannot be proved in the debate. » (Ibid. page 198)

L'observation, la manipulation en mettant en relation l'action (la relation au sensible) et la réflexion (la relation au théorique) constituent un fondement de l'expérience qu'il s'agit de transposer d'une part vers les mathématiques et d'autre part vers l'enseignement. Une première question qui peut se poser est la nature des objets qu'une expérience mathématique peut mettre en jeu. Le sensible en mathématiques peut être vu à travers les objets concrets manipulés (figure, objets matériels, artefacts tangibles,...) ou à travers les objets mathématiques naturalisés, c'est à dire suffisamment familiers pour pouvoir être considérés comme « concrets » :

2015

$2^{11} - 3 \times 11$

MMXV

« Le concret c'est l'abstrait rendu familier par l'usage » (Langevin, 1950)

Fig. 2 Trois écritures d'un même nombre.

Ainsi les objets mathématiques objets des expériences peuvent être dialectiquement perçus d'un point de vue sensible par la manipulation directe de certaines de leurs représentations et d'un point de vue théorique par leurs mises en relation dans des structures abstraites à travers des systèmes de signes. Manipuler des objets mathématiques revient donc à s'approprier des systèmes de signes pour rendre les objets familiers, maîtrisables dans leurs relations aux théories sous-jacentes. Les trois écritures d'un même nombre (Figure 2) illustrent bien cette relation dialectique qu'on les considère dans des systèmes de numération (ici la numération romaine et la numération décimale de position actuelle) ou dans une écriture mettant en jeu des opérations, c'est à dire des relations entre objets de même nature. L'objet lui-même se construit à travers cette familiarisation avec ses représentations et la capacité à saisir les propriétés spécifiques mises en exergue dans chacune de ses représentations. « Il faut concevoir ces nombres comme des unités intentionnelles, l'intentionnalité étant un renvoi de quelque chose à quelque chose d'autre qui le transcende. » (Descaves, 2011 page 11). De la même façon, observer les trajectoires des étoiles autour de l'étoile polaire constitue une expérience sensible permettant de mettre en évidence une perception de l'idée du cercle, insuffisante pour agir, mais constituant une étape vers une définition théorique et sa traduction dans différents systèmes de signes (Fig. 3). La rupture épistémologique entre la perception sensible de l'objet et la manipulation effective passe à travers la référence à la théorie dans une construction des objets constitutifs de la théorie. Ces quelques considérations amènent donc à considérer l'expérience en mathématiques comme une synthèse des manipulations sur les représentations des objets mathématiques et des références théoriques à travers des systèmes de signes.

$$(x-x_0)^2+(y-y_0)^2=R^2$$

Fig. 3 Un cercle, objet sensible, objet mathématique.

Bien sûr, réfléchir à la façon dont nous faisons des mathématiques fait aussi réfléchir à la façon de transmettre les mathématiques et donc de *faire faire des mathématiques*. La question est alors essentiellement didactique et s'intéresse aux finalités de l'enseignement des mathématiques. Faut-il considérer les mathématiques comme une école de la discipline, de la rigueur et de l'obéissance à un ensemble de règles ou bien les voir comme un espace de créativité ? Les réponses à ces questions déterminent fondamentalement le type d'enseignement et le rôle des problèmes dans cet enseignement et rejoignent les considérations didactiques du paragraphe précédent.

Une expérimentation en classe

L'équipe de recherche DREAM (Démarches de recherche pour l'enseignement et l'apprentissage des mathématiques) considère comme objet de recherche les situations de classe mettant en jeu des problèmes et s'intéresse à la fois aux questions relatives à la gestion de la classe qu'aux questions de constructions de connaissances, dont, en particulier, les questions d'évaluation. Les « problèmes pour chercher » sont une façon différente d'envisager l'apprentissage et l'enseignement des mathématiques dans le cours ordinaire de la classe. Ils permettent de mettre en évidence et en pratique les ressorts fournis par la dimension expérimentale de l'activité mathématique sur des connaissances mathématiques en lien avec les programmes à différents niveaux d'enseignement. Ce travail s'inspire et s'appuie sur des travaux de recherche (Aldon & al., 2010, Gardes, 2012, 2013, Front, 2010, 2012) et cherche à répondre aux questions suivantes :

1. Quelles sont les connaissances, les compétences transversales et méta-mathématiques qu'il est possible d'évaluer dans une pratique de recherche de problème ? Et quels sont les indicateurs qu'il est possible de mettre en place ?
2. La créativité et l'invention mathématique développées dans les problèmes de recherche modifient elles l'image des mathématiques chez les élèves (et leur envie de faire des mathématiques). Et chez les professeurs ?
3. Les problèmes de recherche qui développent une forme d'acquisition des savoirs font ils progresser les élèves dans les autres domaines de l'activité mathématique ? Comment les élèves réinvestissent-ils dans d'autres cadres les compétences et les connaissances développées ?

Ce travail nous a conduit à proposer une définition des situations didactiques de recherche de problèmes (SDRP) mises en œuvre en classe en nous appuyant sur la théorie des situations didactiques (TSD) de Brousseau (1986) et à expérimenter un enseignement fondé sur ces situations dans une classe de troisième (grade 9) en France.

Situations didactiques de recherche de problèmes

Construire un enseignement sur des problèmes conduit à s'intéresser plus précisément au type de situation introduite dans la classe, à leur gestion par les enseignants et aux évaluations des connaissances acquises. Les situations didactiques de recherche de problèmes sont des situations au sens de Brousseau :

« Le maître cherche à faire dévolution à l'élève d'une situation adidactique qui provoque chez celui-ci l'interaction la plus indépendante et la plus féconde possible. Pour cela, il communique ou s'abstient de communiquer, selon le cas, des informations, des questions, des méthodes d'apprentissage, des heuristiques, etc... L'enseignant est donc impliqué dans un jeu avec le système des interactions de l'élève avec les problèmes qu'il lui pose. Ce jeu ou cette situation plus vaste est la situation didactique. » (Brousseau, 2011, page 2)

Mais ce sont des situations qui ne visent pas une connaissance précise et en ce sens se démarquent des situations fondamentales de la TSD. Elles sont aussi reliées au courant du « problem solving » en ce sens que l'enrôlement des élèves les amènent à découvrir une petite partie des mathématiques mais s'en démarquent parce que au delà des compétences meta-mathématiques et des heuristiques visées, des connaissances mathématiques repérées constituent un objectif d'enseignement. Il s'agit donc, à travers des expériences sur les objets en jeu d'affermir les connaissances de ces objets ou d'en explorer de nouveaux nécessaires à la résolution du problème. La dimension expérimentale telle que décrite précédemment est une composante essentielle des SDRP en ce sens que les liens entre les objets mathématiques en jeu se construisent dans la mise en œuvre d'expériences qui peuvent utiliser des artefacts concrets ou des objets mathématiques naturalisés ou en cours d'acquisition.

Un point particulièrement développé dans l'expérimentation en cours est l'intégration des SDRP dans l'enseignement non pas comme une activité ponctuelle mais comme une structure d'une progression permettant de fonder un enseignement dans un curriculum donné. Cette utilisation repose sur une analyse *a priori* des situations mathématiques et sur leurs transformations en des situations didactiques. L'analyse mathématique de chaque problème permet de dégager les différents objets potentiellement en jeu dans la situation mathématique, fondements des constructions de connaissances. L'exemple du paragraphe suivant montre une telle analyse et les potentialités didactiques afférentes.

Un exemple de problème

Cette analyse est reprise du travail réalisé par l'équipe DREAM et présente dans Aldon & al (2010).

Problème

Trouver tous les nombres entiers qui sont la somme d'au moins deux nombres entiers naturels consécutifs.

Les méthodes proposées vont de la plus élémentaire à la plus experte. Certaines ne donnent qu'une solution partielle au problème posé. Pour chacune les savoirs mathématiques utilisés, ainsi que les compétences transversales mises en œuvre sont décrites.

Méthodes	**savoirs mathématiques utilisés**	**compétences transversales mises en œuvre**
Expérimentation numérique : En essayant des sommes de deux, ou trois ou quatre entiers consécutifs, on arrive assez vite à la conjecture que tous les entiers peuvent être décomposés en somme d'entiers consécutifs, sauf les puissances de 2 (différentes de 1). **Remarque : cette méthode est très féconde pour trouver la conjecture, mais il reste à la démontrer !** On peut faire des essais inorganisés, et parvenir à la conjecture de façon assez difficile. On peut par contre mener une recherche très organisée. On se prête à une expérimentation numérique avec des sommes de deux entiers (0+1=**1**; 1+2=**3**; 2+3=**5**; …), de trois entiers (0+1+2=**3**; 1+2+3=**6**; 2+3+4=**9**; …) puis de quatre entiers consécutifs (0+1+2+3=**6**; 1+2+3+4=**10**; 2+3+4+5=**14**; …), etc. On remarque qu'en considérant une somme de deux entiers consécutifs, on trouve tous les entiers solutions en allant de deux en deux à partir du premier entier trouvé, c'est-à-dire 1. De même si on considère une somme de trois (respectivement quatre) entiers consécutifs, on trouve tous les entiers solutions en allant de trois en trois (respectivement quatre en quatre) à partir de 3 (respectivement 6). Ceci permet de trouver, de proche en proche, toutes les solutions et, à l'aide d'un crible, d'arriver à la conjecture que seuls 0 et les puissances de 2 différentes de 1 ne conviennent pas:	*Calcul de sommes d'entiers, calcul mental et réfléchi.* *Ou bien utilisation de la calculatrice, éventuellement pour vérifier des calculs faits mentalement.*	*- expérimenter sur des valeurs numériques à la main, à la calculatrice* *- conjecturer* *- rédiger une conjecture*

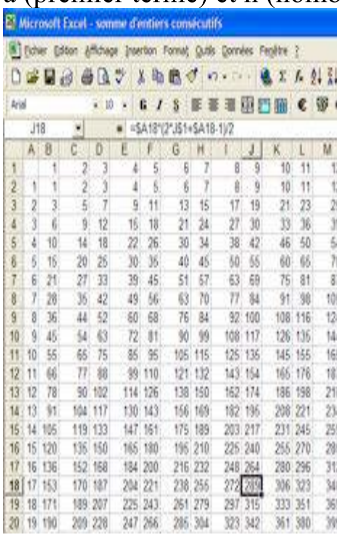
~~1~~	**2**	~~3~~	**4**	~~5~~	~~6~~	~~7~~	**8**
~~11~~	~~12~~	~~13~~	~~14~~	~~15~~	**16**	~~17~~	~~18~~
~~21~~	~~22~~	~~23~~	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~
~~31~~	**32**	~~33~~	~~34~~	~~35~~	~~36~~	~~37~~	~~38~~
~~41~~	~~42~~	~~43~~	44	~~45~~	~~46~~	~~47~~	~~48~~
~~51~~	~~52~~	~~53~~	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~
~~61~~	62	~~63~~	**64**	~~65~~	~~66~~	~~67~~	~~68~~
~~71~~	~~72~~	~~73~~	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~

organiser des calculs et présenter les résultats sous forme de tableau

Résolution algébrique « à la recherche d'une formule explicite pour les entiers cherchés »: on appelle n le plus petit de ces nombres entiers. On réduit l'expression de la somme de deux entiers consécutifs, de trois entiers consécutifs, de 4, …. Ceci donne lieu à la résolution de sous-problèmes intéressants : • « tous les entiers impairs conviennent ». La démonstration utilise le calcul algébrique : Soit n un nombre entier naturel. n+ (n+1) = 2n +1, ce qui démontre que tout entier impair est la somme de deux entiers consécutifs. • « Peut-on trouver une méthode par récurrence pour décrire les entiers cherchés ? » n+ (n+1) = 2n +1 « les impairs » n+ (n+1) +(n+2) = 3n +3 « les multiples de 3 » n+ (n+1) +(n+2)+(n+3) = 4n +6 n+ (n+1) +(n+2)+(n+3)+(n+4)= 5n +10 On trouve alors expérimentalement une façon de déterminer les coefficients « rouges » et les coefficients « bleus » : les rouges augmentent de 1 à chaque ligne ; les bleus sont égaux à la somme des deux coefficients (le rouge+ le bleu) de la ligne précédente. Ceci permet, en y mettant le prix, de trouver tous les entiers solutions…. En raisonnant de manière plus astucieuse dans le cas de sommes "impaires" d'entiers consécutifs, on démontre que tous les multiples d'un nombre impair conviennent. En effet, si au lieu d'écrire n + (n+1) + (n+2), on pose N = n+1, la somme de trois entiers consécutifs s'écrit (N-1) + N + (N+1) ce qui est égal à 3N. Les multiples de 3 s'écrivent donc comme la somme de trois entiers consécutifs. Exemple: 27 = 3 x 9 = (9-1) + 9 + (9+1) = 8 + 9 + 10 De la même façon, si au lieu d'écrire n + (n+1) + (n+2) + (n+3) + (n+4), on pose N = n+2, la somme de cinq entiers consécutifs devient (N-2) + (N-1) + N + (N+1) + (N+2) ce qui est égal à 5N. Les multiples de 5 s'écrivent donc comme la somme de cinq entiers consécutifs.	*Utilisation de la lettre pour désigner un nombre* *Calcul algébrique* *- nombres entiers naturels* *- entiers pairs, impairs (caractérisation « algébrique »)* *- calcul algébrique* *- arithmétique : divisibilité (par 2, ici)* *- nombres entiers naturels* *- entiers pairs, impairs (caractérisation « algébrique »)* *- calcul algébrique*	*- dégager des sous-problèmes, que l'on s'attache à démontrer* *- se poser le problème de la démonstration, de la preuve* *- observer des invariants et/ou des relations de récurrence*

Recherche d'une preuve de la conjecture en utilisant des exemples que l'on va « faire parler » 12 = 3+4+5 12 = 4+4+4 40 = 8+8+8+8+8 (5 fois 8) 40 = 6+7+8+9+10 Si l'on essaye de généraliser cette idée issue de l'expérimentation sur des exemples, on obtient : Si un nombre N s'écrit (2k+1)n, avec k et n entiers naturels, alors : N = (2k+1)n N =n+ n+…+n+…+n+n *(2k+1) termes égaux à n* N = (n-k)+(n-k+1)+…(n-1)+n+(n+1)+…+(n+k – 1)+(n+k) Les termes se regroupent deux à deux, avec pour somme 2n : (n-k)+(n+k) = 2n (n-k+1)+(n+k –1)= 2n , etc. On obtient donc k fois 2n, auquel il faut ajouter n, le terme central, donc on retrouve bien N = (2k+1)n La condition pour que les termes de la somme soient tous entiers naturels est : n ³ k. Il semble donc, à cette étape, que cette démonstration ne marche pas dans tous les cas, par exemple : 10 = 5 ´ 2 = 0+1+2+3+4 correct, car ici n ³ k 14 = 7 ´ 2 = –1+0+1+2+3+4+5 ne convient pas car il y a un nombre entier négatif ! ici n < k Ce qui est très fort, c'est que de la dernière égalité on peut tirer une autre égalité qui va convenir à notre problème , à savoir, comme –1+0+1 = 0, on obtient : 14 = 2+3+4+5. Mais là, il n'est pas facile de rédiger une démonstration « générale ». On comprend pourtant aisément que ce calcul va pouvoir être possible dans tous les cas où n < k. On peut parler d'un exemple générique, qui à lui seul convainc.	*- nombres entiers naturels* *- calcul algébrique*	- distinguer conjecture et propriété démontrée - se poser le problème de la démonstration, de la preuve - revenir à des exemples pour en déduire une preuve (des exemples « génériques », qui ne donnent pas une démonstration *- expérimenter sur des valeurs numériques à la main, à la calculatrice, au tableur* *- observer des invariants* *- critiquer une démonstration, en percevoir les limites ;* *- analyser les conditions de validité d'un calcul*

Démonstration de la conjecture : Elle utilise le résultat suivant : Si n est un entier naturel, $1+2+3+\ldots+n =$. On peut noter S_n cette somme. (Cette démonstration permet de revoir ou d'introduire ce résultat, comme outil de résolution de problème) N étant un entier naturel, on cherche s'il existe deux entiers naturels a et p tels que $N = a+(a+1)+(a+2)+\ldots+(a+n-1)$ $N = S_{a+n-1} - S_{a-1}$ Soit: $2N = (a+n)^2 - a - n - a^2 + a$ $\mathbf{2N = n(2a+n-1)}$ On peut alors raisonner sur la parité de l'entier n. Si n est pair : 2a+n-1 est impair Si n est impair : 2a+n-1 est pair • Par conséquent, des deux entiers n et 2a+n-1, l'un est pair et l'autre impair : leur produit étant égal à 2N, cela entraîne que N possède un facteur premier impair : N n'est pas une puissance de 2. • Autre raisonnement, par l'absurde : si $N = 2^m$ alors on cherche a et n tels que $2^{m+1} = n(2a+n-1)$, ceci est impossible car l'un des deux facteurs du second membre est impair. Il reste encore à démontrer que tout nombre N qui n'est pas une puissance de 2 peut s'écrire comme somme d'entiers consécutifs. 2N est donc le produit d'un nombre impair i par un nombre pair p. $2N = ip$ et $2N = n(2a+n-1)$ si $i < p$ alors il suffit de poser $n= i$ et $p = 2a+n-1$, soit $a = (p-n+1)/2$ si $i > p$ alors il suffit de poser $n = p$ et $i = 2a+n-1$, soit $a = (i-n+1)/2$ La conjecture est ainsi complètement démontrée, et cette démonstration donne un procédé pratique pour déterminer a et n entiers naturels tels que $N = a+(a+1)+(a+2)+\ldots+(a+n-1)$. b) Exemple : N = 168 • 2 x 168 = 21 x 16. Or 21>16 donc on aura : n =16 et a =3. **168 = 3 + 4 + 5+…+ 18** • Par la méthode numérique donnée sur des exemples génériques, on obtient : 168 = 21 x 8 = 8+8+8+…….+8 168 = (8-10) +(8-9) +(8-8)+(8-7)+….+(8+7)+(8+8)+(8+9)+(8+10). 168 = –2 –1 +0+1+2+3+……+15+16+17+18 **168 = 3 + 4+…….+ 18** • On peut aussi remarquer que 168 est un multiple de 3, donc **168 = 55 + 56 + 57** • En utilisant la méthode « par récurrence » : 168 est multiple de 7 ; or la somme a+a+1+a+2+….+a+6 = 7a + 21 7a+21 = 168 équivaut à a = 21, ce qui donne : 168	*- nombres entiers naturels* *- entiers pairs, impairs (caractérisation « algébrique »)* *- calcul algébrique* *- somme des n premiers entiers naturels (ou suite arithmétique) et à ce propos, anecdote sur Gauss…* *- fonction de deux variables et tableau de valeurs de cette fonction sur un tableur* *- utiliser le tableur (adresses absolues et relatives)* *- arithmétique : divisibilité (par 2, ici)*	*- raisonnement par analyse-synthèse : partie « analyse »* *- raisonnement « par le pair, et l'impair » : raisonnement par exhaustion* *- raisonnement par l'absurde* *- raisonnement par analyse-synthèse : partie « synthèse »*

= 21+22+23+24+25+26+27. **Remarque :** La décomposition en somme d'entiers consécutifs n'est pas unique. Il semble que des méthodes différentes donnent des résultats différents, sauf la méthode « exemples génériques » et la démonstration par le pair et l'impair, qui donnent la même décomposition en somme d'entiers consécutifs.		
Expérimentation sur tableur, visant à obtenir toutes les décompositions pour un nombre entier donné (qui n'est pas une puissance de 2) On peut aussi envisager de construire une feuille de calcul sur tableur, pour y faire figurer les nombres entiers solutions, en fonction des entiers a (premier terme) et n (nombre de termes).  premier terme : a sur la ligne 1 nombre de termes de la somme : n dans la colonne A Ce tableau Excel permet de conjecturer quels sont les entiers solutions du problème et aussi de déterminer, pour un nombre donné N du tableau, quels sont les sommes d'entiers qui lui sont égales. Il peut permettre de travailler expérimentalement sur le problème suivant : pour un entier N qui n'est pas une puissance de 2, trouver toutes les décompositions de N en sommes d'entiers consécutifs. On « voit » en augmentant à droite la taille de ce tableau que 135 = 67+68	*- utiliser le tableur (adresses absolues et relatives)*	*- penser à utiliser un tableur pour construire le tableau de valeurs d'une fonction de deux variables entières* *conjecturer*

135 = 44+45+46 135 = 25+26+27+28+29 135 = 20+21+22+23+24+25 135 = 11+12+…+19 135 = 9+10+…+18 135 = 2+3+4+…+16 Et c'est tout ! sept sommes possibles pour N = 135.		*se poser un nouveau problème*

Table 1. Analyse d'un problème

Tous les éléments de cette analyse permettent de construire une situation didactique de recherche de problème et de l'insérer dans une progression en lien avec des connaissances visées. L'expérimentation conduite en classe[5] a permis de mettre en évidence les connaissances effectivement mises en œuvre par les élèves dans la situation :

Bilan dans une première classe :

Phase de recherche, mise en commun et validation des conjectures émises (Voir figure 4)

• Les notions mathématiques travaillées sont ici les notions de multiples, de nombres pairs ou impairs,

Etude 1 - Validation des conjectures n°1 et 2 (figure 4)

• Les notions mathématiques travaillées portent sur le calcul littéral comme outil de preuve, la caractérisation des nombres pairs et impairs sous forme littérale, la caractérisation des multiples de 3 (5, 7, …) sous forme littérale, la formule de distributivité et les techniques de développement et de factorisation.

Etude 2 - Validation de la conjecture n°3 (figure 4)

• Les notions mathématiques travaillées correspondent ici à la notion de puissance (puissances de 2), mais aussi se réfèrent à un autre domaine des mathématiques, le calcul de l'aire d'un trapèze

Etude 3 - Approfondissement sur l'utilisation de la distributivité (calcul littéral)

• Les notions mathématiques travaillées portent encore sur les techniques de développement, de simplification et de réduction d'une expression littérale, de factorisation, en s'appliquant à des programmes de calculs, et enfin débouchent sur la notion de distributivité double.

5 Deux classes de 3eme du collège Emile Zola de Belleville (France). Le professeur, Antoine Guise, avait en charge les deux classes et a construit sa progression à partir des SDRP. Dans l'expérimentation, il a rempli au fur et à mesure de l'année un journal de bord duquel sont extraits les bilans des deux classes.

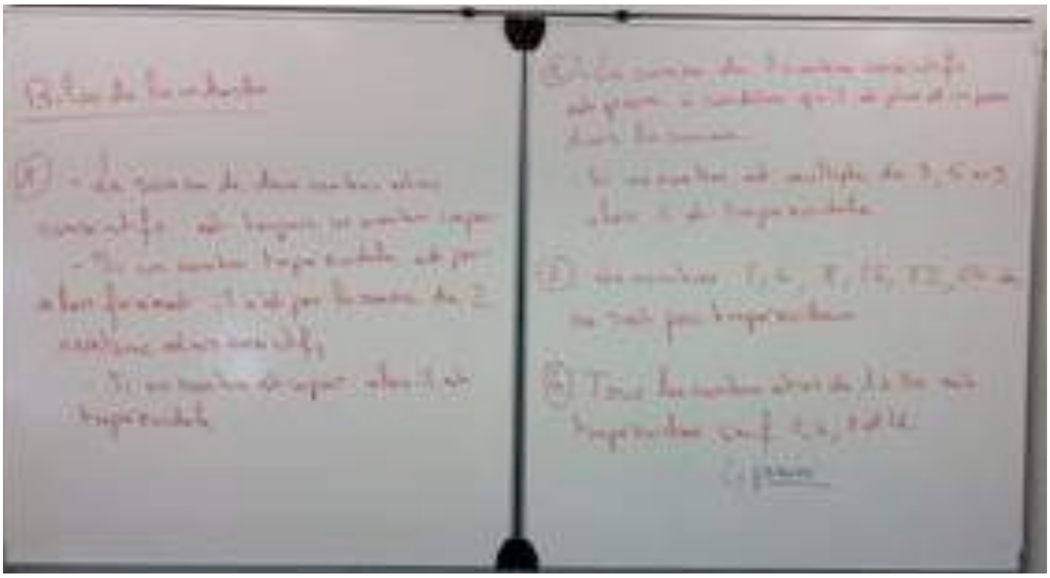


Fig. 4 Bilan de la recherche dans la classe 2

Bilan dans une deuxième classe :

Phase de recherche, mise en commun et validation des conjectures émises (Voir Fig. 5)

• Les notions mathématiques travaillées reliées à cette conjecture portent ici aussi sur le concept de multiples, de nombres pairs ou impairs,

Etude 1 - Validation des conjectures émises (n°1 à 5) (figure 5)

Les notions mathématiques travaillées portent sur le calcul littéral comme outil de preuve, la caractérisation des nombres pairs et impairs sous forme littérale, la caractérisation des multiples de 3 (5, 7, …) sous forme littérale, et les techniques de distributivité, développement et factorisation.

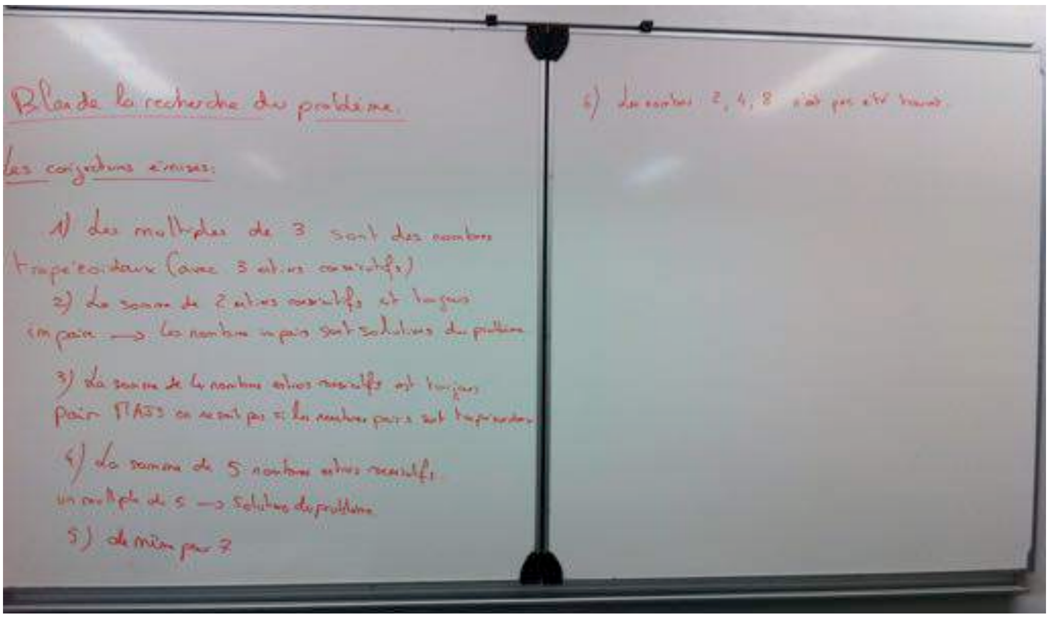


Fig. 5 Bilan de la recherche dans la classe 2

Etude 2 - Validation des conjectures émises (n°6 et 7) (figure 5)

• Les notions mathématiques travaillées rejoignent le concept de puissance à travers la manipulation des puissances de 2 et, comme précédemment, la référence à la géométrie à travers un lien vers la détermination de l'aire d'un trapèze.

Etude 3 - Approfondissement sur l'utilisation de la distributivité (calcul littéral)

• Les notions mathématiques travaillées portent encore sur les techniques de développement, de simplification et de réduction d'une expression littérale, de factorisation, dans une application de programmes de calculs et débouchent encore sur la notion de double distributivité.

Il est intéressant de noter que les recherches des élèves résumés dans les affiches qu'ils ont produit, conduisent à des résultats et des conjectures différents mais que l'analyse du problème est suffisamment robuste pour que les connaissances potentielles apparaissent dans les deux classes et permettent au professeur de construire des institutionnalisations de connaissances compatibles avec les éléments du curriculum tout en tenant compte des niveaux de connaissance mis en œuvre dans les recherches. Cet exemple s'insère dans une construction globale de la progression s'appuyant sur différents problèmes dont certains seulement sont construits et gérés comme des SDRP. La progression (Figure 6) est construite à partir des problèmes et les connaissances mathématiques liées sont institutionnalisées dans une logique d'enseignement spiralé, nécessaire à la souplesse de la gestion relative au travail des élèves et aux résultats auxquels ils arrivent dans leurs recherches.

Conclusion

La négociation du contrat dans lequel les élèves acceptent d'apprendre des mathématiques en les construisant est une phase fondamentale et délicate de la mise en œuvre de telles situations. Elle repose sur un équilibre entre les phases de recherche, les entraînements à des tâches techniques et les apports du maître lorsqu'ils sont nécessaires. La progression de la figure 5 montre bien qu'en parallèle des SDRP proposées en classe, le professeur construit son cours en s'appuyant sur des évaluations formatives permettant de voir où les élèves en sont et de construire son enseignement en s'appuyant sur ces constats pour emmener les élèves la où les injonctions institutionnelles le stipulent. D'autres questions, notamment relatives au rôle des technologies dans la dimension expérimentale des mathématiques, sont encore à explorer.

Le travail se poursuit et la dissémination de cette approche du curriculum dans des classes ordinaires constitue la deuxième phase de cette recherche. La question de la formation des enseignants et de leur accompagnement est ainsi posée et peut être mise en relation avec les questions du document de discussion de la CIEAEM 67 dans lequel le rôle du professeur est interrogé, notamment avec cette question provocatrice :

Un enseignant peut-il être un obstacle à l'apprentissage des élèves ?

Mais d'une façon plus générale, l'expertise professionnelle nécessaire pour développer et implémenter un curriculum se pose et le rôle de la recherche dans le développement professionnel des enseignants apparaît comme crucial :

Comment est-il possible d'aider les enseignants à développer les connaissances et les compétences appropriées dans les technologies numériques afin qu'ils soient efficaces dans leur enseignement des mathématiques ? (Document de discussion CIEAEM 67)
Nul doute que les travaux de cette rencontre et les actes apporteront des éléments de réponses et probablement affineront les questions qui ont été posées.

Progression classe de 3ème - 2014/2015

Sem	Chapitre	Durée approx	Partie rituelle	évaluat° formatives	Eval° sommatives	prépa DNB
36	Rentrée scolaire	≈ 12h	Op° sur les fractions	Test n°1: calcul mental sur les entiers relatifs et les fractions relatives (sauf ÷)		
37	Le problème du billard					
38				Test n°2: Div; mul; Fract°; PGCD		
39					Interro n°1	2h: sujets arithmétiques
40	Le pb de la distance maison/lac	≈ 10h	Calculs de puissances et écriture scientifique	DM Euclide - tableur		
41					DS n°1	
42				Test n°3: Thales		2h: sujets Géo (Thales)
43	Vacances Toussaints					
44						
45	Les nombres trapézoidaux	≈ 15h	Puissances & écriture sci	DMR n°1		
46				Test n°4: écrt sci + puissances		
47	Absence					
48			Pythagore		DS n°2	
49				Test n°5: Calcul littéral		1h: Programme calculs + 1h: Pb de Géo plane
50	Congés					
51						
52	Vacances de Noël					
1						
2	Le problème du JT	≈ 6h	Calcul littéral	DMR n°2	Interro n°2	
3				Test n°6: fact° et %		
4	Equations	≈ 6h	Identités remarquables		BB n°1	
5						2h: Prgm de calcl & Equ°
6	Pb de l'endos	≈ 6h			Interro n°3	
7	Vacances d'hiver					
8						
9			Pgm de calcul : calcul litt / Id Rem / équations	DMR n°3		
10	Initiation à la probabilité	≈ 6h			DS n°4	
11			Fonction: boites noires , calculs d'antécédents et d'images			
12	Le pb de la boite sans couvercle	≈ 5h		Test n°7: Probabilités		
13					Interro n°4	2h: Notion fct° & calcul litt
14	Statistiques	≈ 4h	Inéquations	DM (fct° et proba)		
15					BB n°2	1h: Satistiques
16	Vacances de printemps					
17						
18	Trigonométrie	5h	Inéquations et équations + Not° fct° (familles, courbes et volumes, boites noires)	DMR n°4		
19				Test n°8: Notion de fct° + inéq°		1h: trigonométrie
20	Le pb de la photocopieuse	5h			Interro n°5	
21			Opérations sur les racines carrées + eq° x^2=a			
22	Fonctions affines et linéaires	4h			DS n°6	2h: fct° affines & linéaires
23	Les pavages du plan	6h	Grandeurs composées + systèmes d'équations			
24						
25	Géométrie dans l'espace	4h				3h: Pondichéry 2015, Amérique du nord 2015
26	Brevet des collèges					
27	Correction DNB - Fin d'année					
28	Vacances d'été					

Fig. 6 Progression d'une classe expérimentale

REFERENCES

Aldon, G., Cahuet, P.-Y., Durand-Guerrier, V., Front, M., Krieger, D., Mizony, M., Tardy, C. (2010). *Expérimenter des problèmes de recherche innovants en mathématiques à l'école.* Cédérom, INRP.

Brousseau (1986). Théorisation des phénomènes d'enseignement des mathématiques, *Thèse d'état de l'Université de Bordeaux*

Brousseau, G. (2011). Éléments pour une ingénierie didactique, http://guy-brousseau.com/wp-content/uploads/2011/04/92-1-El%C3%A9ments-pour-une-ing%C3%A9nierie-didactique.pdf

Buisson, F. (1911). *Nouveau dictionnaire de pédagogie et d'instruction primaire,* Paris, Hachette

Descaves, A (2011). *Comprendre des énoncés, résoudre des problèmes*, Paris, Hachette

Dewey, J. (1938). *Experience and education*, New York, NY: Kappa Delta Pi.

Front, M. (2010) Pavages semi-réguliers du plan. Élaboration d'une situation favorable à la dialectique théorie-objets, Master HPDS, *Université Lyon 1*.

Front, M. (2012). Pavages semi-réguliers du plan, une exploration favorable aux élaborations mathématiques. *Repères IREM 89*, 5-37.

Gardes ML (2012). Démarche d'investigation en théorie des nombres : un exemple avec la conjecture d'Erdös-Straus. In Dorier J.-L., Coutat S. (Eds.) *Enseignement des mathématiques et contrat social : enjeux et défis pour le 21è siècle – Actes du colloque EMF2012* (GT10, pp.1342-1353). http://www.emf2012.unige.ch/index.php/actes-emf-2012

Gardes M-L. (2013). Étude de processus de recherche de chercheurs, élèves et étudiants, engagés dans la recherche d'un problème non résolu en théorie des nombres. Thèse de doctorat. *Université de Lyon 1*.

Hume, D. (1946).*Treatise of human nature,* Selby-Bigge, Clarendon-Press Oxford

Kuhn, T. (1962). *The structure of scientific revolutions,* Chicago University Press

Langevin, P. (1950). *La pensée et l'action*, textes recueillis par Paul Labérenne, préfaces de Frédéric Joliot-Curie et Georges Cogniot, Paris, Les Éditeurs Français Réunis

Polya, G. (1945). *How to solve it?,* Princeton University Press

Schoenfeld, A. H. (1994). What do we know about mathematics curricula? *Journal of Mathematical Behavior*, 13(1), 55-80.

Sweller, J., Clark, R. E., Kirschner, P.A. (2011). Teaching general problem solving does not lead to mathematical skills or knowledge, *Notices of the American Mathematical Society*, 57, 1303-1304.

Mathematics teacher education in the institutions: new frontiers and challenges from research

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Abstract: The paper deals with teacher education in the institutional context, using the Meta-Didactical Transposition framework (based on Chevallard theory of transposition) for analysing the processes of researchers and teachers working together in designing tasks of an educational programme, training teachers in these tasks, and following them while experimenting the same tasks with their students. The research aim of the paper is to describe the features of the practices of these communities, to identify them as community of design and community of experimentation, according to the praxeologies they put into action in the processes of designing, training, experimenting. Particularly, this study is referred to a research and educational programme on MERLO (Meaning Equivalence Reusable Learning Objects) items, a didactical tool for involving students in group/individual activities that involve deep understanding in mathematics and assessing them.

Résumé: Cet article concerne la formation des enseignants dans le contexte institutionnel, en utilisant le cadre de la transposition Meta-didactique (fondée sur la théorie de transposition de Chevallard) pour analyser les processus des chercheurs et des enseignants qui travaillent ensemble à la conception des tâches d'un programme de formation, à la formation des enseignants en utilisant ces tâches, et au suivi des expérimentations de ces mêmes tâches avec leurs élèves. Le but de cet article est de décrire les caractéristiques des pratiques de ces communautés, afin de les identifier comme communautés de conception et d'expérimentation, selon les praxéologies qu'elles ont misées en place dans les processus de conception, de formation, d'expérimentation. En particulier, cette étude s'inscrit dans un programme de recherche et de formation sur les tâches MERLO (Meaning Equivalence Reusable Learning Objects), qui est un outil didactique pour engager et évaluer les élèves dans des activités de groupe ou individuelles qui impliquent la compréhension profonde des mathématiques.

This is the era of the teacher

Mathematics teacher education raised to the attention of the community of researchers in the last years in a very deep way and with many facets: educational programmes, use of digital technologies, teaching practices and methodologies, … The variety of articles and books published shows the evidence of a great complexity in the teaching processes, both in terms of teaching practices in the classroom and of teacher education, which involve the use of new technologies, new methodologies of teaching, new kind of students, new claims from the institutions. This complexity can be interpreted using different frameworks, and can be showed in either a more static or more dynamic way, or in a mixt approach, according to the framework chosen and the research aims (see for example respectively Ball & Bass, 2003; Arzarello et al., 2014; Kaiser et al., 2014). With these three kinds of perspective (not to be intended in a rigid way, of course, but as a sort of lens through which teacher processes can be observed and interpreted), we can intend: as static, highlighting features, indicators and elements of the context of teaching that seems important to describe and classify; as dynamic, the complexity of teacher profession in terms of evolution in teaching processes and in professional development; and as mixt approach all the intermediate nuances from the static to dynamic, where different elements of the two approaches are used more or less intensively.

The current research interest in teacher education is a relatively new domain when compared to other research themes relating to mathematics content, curriculum, students, learning, cognitive processes, policy and equity. This era marks an important milestone in the evolution of mathematics education, toward the teacher having an important role to play within the classroom. Sfard (2005) stresses that we are living in the *era of the teacher* and that the advent of this era has brought about a re-conceptualisation of the relationship between the teacher and the researcher, arguing that in most of the international research studies, the question is not *what* is taught in classrooms, but *how* it is taught. This paper is aimed at directing attention to the role of the teacher, and showing some activities, from the point of view of professional development and of daily work in the classroom. The role of the teacher, according to the context in which operates, has many facets. It can be the role of a teacher-researcher, if works with researchers in designing a project, a teaching experiment, an activity of teacher education. Or it can be the role of a teacher-experimenter, if is available to experiment new kind of activities and teaching practices in her/his class, observing the students' processes. It can be the role of teacher-trainer, involved in an education program as teacher of teachers, or broker/tutor of a group of teachers. Or it can be the role of teacher-learner, involved in an education program with other colleagues. These roles are not to be intended rigidly separated: they can be partially or totally overlapped, according to the grade of involvement of the teacher in the life of school, training, and research. From their viewpoint, researchers have choices to make with respect to the 'grain size' of their focus and analysis, not only with respect to the more static or dynamic descriptive approach, but also with the choice of variables to take into consideration, which can range from the individual analyses of teachers/lessons, to studies of the evolution of teaching/learning processes over time, or the observation of communities of teachers, or the introduction of new methodological tools (Clark-Wilson et al., 2014). I would like to stress the necessity of observing mathematics teacher education processes from a systemic point of view, with a focus on the relationships and dynamics between the several "variables" altogether, included in such complex processes as: teachers' knowledge and practices, results from research, institutional constraints (national curricula in particular), traditions, cultural aspects, and so on.

There are communities, and communities…

The term community is actually present in literature, in many declinations and meanings. After a brief review of the use of this term in literature, I will introduce the concept of *community of design* and *community of experimentation*, which are particularly referred to the research reported in this paper.

The introduction of the term *communities of practice* is due to Wenger (1998), who says that they are formed by people who engage in a process of collective learning in a shared domain of human endeavour: for example a tribe learning to survive, a band of artists seeking new forms of expression, a group of engineers working on similar problems, a clique of pupils defining their identity in the school, a network of surgeons exploring novel techniques, a gathering of first-time managers helping each other cope. For Wenger, communities of practice are groups of people who share a concern or a passion for something they do and learn how to do it better as they interact regularly (Wenger, 1998).

Communities of practice are groups of people who share an interest, a scope, a challenge, and so forth: they group spontaneously and they meet and work together. The characteristics of such communities of practice can vary, according to the context and the interests and tasks of participants. Some communities of practice are quite formal in organisation, others informal.

Communities of learners are less spontaneous than communities of practice (Bos-Ciussi et al., 2008). A class of students is a community of learners, in that students take part in the construction of consensual domains, and "participate in the negotiation and institutionalisation of ... meaning" (Roth & Lee, 2006). In a learning community in fact, the educational goal is to let collective knowledge advance, in a way that supports the growth of individual knowledge (Bielaczyc & Collins, 1999). Students are actively engaged in the social process of construction of meanings, and in doing so, they learn as individuals. Teacher is part of the community of course, and takes place in the process of construction of knowledge, designing and guiding it, or coaching from one community to another one (Rasmussen et al., 2009).

In their book, Borba and Villarreal (2006) focus on communities of learners with the tools they use: *humans-with-media* is their paradigm. This point of view overcomes the traditional deep-rooted dichotomy between humans and technology, because it considers learning as a process of interaction amongst humans as a community that includes tools. Media interact with humans in a double sense, namely technologies transform and modify humans' reasoning, as well as humans are continuously transforming technologies according to their purposes.

Yerushalmy and Elikan (2010) describe the unique relationship that must evolve in a class that seeks to function as an *inquiring mathematical community*, in which students' explorations are often based on use of software tools and where students constantly convey their observations, eliciting arguments from peers and from themselves. The assigned tasks required measurements and construction of a model, then the examination of the model itself. The authors claim that the students advanced in knowledge as a community of learners, learned the mathematics of function throughout the elaboration of ideas connected to the modelling actions, and developed new norms of discourse.

The experience of learning together (learning to be with others in mathematics, as written by Radford, 2006) with the use of a technological tool, can be described also in a frame that takes the multimodal production of the students, the teacher and the technology itself into account. In this semiotic-cultural approach learning mathematics is a matter of *being-in-mathematics* (Radford 2006), living in a classroom as a community, working together and sharing activities and results.

Although communities of practice have "insiders" and "outsiders", there are various ways in which communities may be connected across the boundaries that define them. Wenger (1998) discusses several types of connection, including boundary encounters — discrete events that give people a sense of how meaning is negotiated within another practice. The briefest of these encounters is the one-on-one conversation between individuals from two communities to help advance the boundary relationship. For example, after participating in a one-off professional development workshop, a mathematics teacher might have a conversation with the presenter about some aspect of the workshop that could inform the teacher's practice. A more enriching instance of the boundary encounter involves immersion in another practice through a site visit. This may occur when a mathematics educator visits a school over a period of weeks or months to collect classroom data, such as lesson observations or interviews with teachers and students, for a research project. Particularly important in boundary encounters are the brokers (Rassmussen et al., 2009), who mediate the passage of practices and knowledge from one community to another, and possibly are part of both of them. Their role is fundamental in teacher education. For example, Goodchild (2007) extends Wenger's theory of community of practice to conceptualise teacher and didactician learning as taking place reflexively within *communities of inquiry*. Teachers and didacticians together form a *project community*. Through participation in established communities of practice (school or university) teachers and didacticians use inquiry as a critical tool to promote learning within the project. Inquiry results in critical alignment with the norms of established practice, allowing teachers and didacticians to act within their practice while at the same time questioning its dynamics and exploring new ideas (Jaworski, 2006).

Jaworski (2008) proposed the term *community of inquiry* from Wenger's work: "In terms of Wenger's (1998) theory, that *belonging to a community of practice* involves *engagement, imagination* and *alignment*, we might see the normal desirable state as *engaging* students and teachers in forms of practice and ways of being in practice with which they *align* their actions and conform to expectations...

In an inquiry community, we are not satisfied with the normal (desirable) state, but we approach our practice with a questioning attitude, not to change everything overnight, but to start to explore what else is possible; to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to provide answers to them (Wells, 1999). In this activity, if our questioning is systematic and we set out purposefully to inquire into our practices, we become researchers." Jaworski (2008, pp. 313-314). Moreover, the asking of questions is a developmental tool in drawing students, teachers and didacticians into a deeper awareness of their own actions, motives and goals.

This is an excellent way to teach students to become curious, to questioning, to conjecturing and arguing, just as in a laboratory: the *mathematics laboratory* (Arzarello et al., 2006) is the name Italian research has given to this kind of methodology to be established and used in the classroom. Anyway, it is not automatic that a teacher shifts from a traditional way of teaching (mostly based on lecturing) to an inquiring way, mostly based on laboratory activities. For that reason, it is quite important that teachers learn to be and to work in communities of inquiry of colleagues in order to transfer these practices in the class with their students. So, the term community is particularly useful in considering and treating teachers as learners, in pre-service or in-service educational programmes. The term community in fact has been used also in professional learning of practising teachers (Llinares & Krainer, 2006). Speaking of communities of teachers, we have also to consider the larger social system in which the community is nested. For example, teaching mathematics in a particular secondary school is linked to the larger social system of secondary school education in a region or a country. Communities of practice have a common cultural and historical heritage, and it is through the sharing and re-construction of this repertoire of resources that individuals come to define their relationships with each other in the context of the community. Based on this description, Goos (2014) argue that mathematics teachers and mathematics education researchers are members of distinct, but related, communities of professional practice. Goos (2014) examines ways in which teachers and university-based mathematics educators might work together to develop theoretical and practical knowledge, using an analytical frame (Novotna & Goos, 2007) raised from questions and issues identified by participants to Psychology and Mathematics education (PME) working session in discussing their own experiences in research and development work with teachers, being particularly interested on how the researcher–teacher partnership begins (e.g., a university-based researcher seeks out teachers to participate in a project that has already been planned), and on which are the practices developed by the two communities.

Also Arzarello et al. (2014), Aldon et al. (2013) consider the two distinct communities: the *community of teachers* and the *community of researchers*, interacting in a research and educational programme. With the Meta-didactical transposition, these communities are described and analised in terms of their praxeologies and their possible evolution over time, giving data on the process that takes place and gives changes in both communities. The community of researchers generally reflects upon the nature of, and reasons for, the changes produced by the teachers' education programme, and possibly shares such reflections with the community of teachers. This can result in new researcher praxeologies. Also the teacher praxeologies may change, and develop into new teacher praxeologies, a process that can repeat and further refine itself. As Goos said, there are two distinct communities, but during the process they may become nearer, sharing praxeologies or some components of them, thanks to the interaction. The model of Meta-didactical transposition is useful in describing the evolution to these shared praxeologies.

In this paper, I will introduce the terms *community of design* to indicate the community of researchers and of teacher-researchers working at the task design of activities for classes, and the *community of experimentation* to designate the community of teachers who carry out the teaching experiments involving the tasks designed by the previous community.

Teacher education in the institutions

One of the themes in CIEAEM67 aims exactly at rethinking the complexity of teacher education in terms of resources and obstacles for teaching and learning mathematics.

Taking for granted this complexity, I will use the *Meta-Didactical Transposition* model (MDT), to describe teachers' activities in a dynamic way, namely as processes evolving over time (Arzarello et al., 2014). This framework is properly built to highlight the need to take the complexity of teacher education into account with respect to the institutions in which the teachers operate, alongside the relationships that teachers must have with these institutions. To address this need, the framework is constructed from Chevallard's (1985) *Anthropological Theory of Didactics* (ATD), which is mainly centred on the transposition of mathematics managed by the teacher with the students in the classroom. In particular the model refers to the notions of didactical transposition and praxeology. According to Chevallard, a praxeology is made of a task, a technique, a technology and a theory: the first two are the pragmatic side of it, while the other two are the theoretical counterpart, which justify the first two. Chevallard defines didactical transposition as the transition from knowledge regarded from an epistemological point of view by the community of experts, to knowledge as something to be taught and learnt. Since the aim of the MDT model is to frame and reflect on teacher education programmes, the term "didactical" has been substituted with "meta-didactical" (Aldon et al., 2013; Arzarello et al., 2014), to stress that the processes under scrutiny are, in this case, the practices and the theoretical reflections developed within teacher education activities. In other words, in the case of teacher education programmes, fundamental issues related to the didactical transposition of knowledge are faced at a meta-level, the level of teachers as professional figures and as learners in communities. The complexity of teaching processes can be interpreted particularly well with this framework, because it gives the possibility to describe these processes in a dynamic way, taking into account modification of practices over time, changes in teaching, using materials, and introducing technologies, not only teachers' knowledge at a certain stage.

Actually, the MDT framework is particularly useful in the description of the evolution of teachers' processes over time, because it gives a model for analysing the different variables involved: *dialectic interactions* between the communities of teachers and researchers; *components of teachers/researchers praxeologies* that change from external to internal or viceversa; and *brokers,* who support teachers, interacting together.

The framework is essential in situations such us national/regional teachers' educational programmes, contextualised in the institutions, where researchers fix a research project in which the educational programme is inserted, then design the programme with its activities and actions, in collaboration with teacher-researchers, and carry it out, involving teachers from schools as learners in communities with colleagues. The involvement of teachers can be done at different geographical levels and institutional modalities: through a national educational programme, or according to professional development needs at local level (region, province, city, net of schools…), or based on a curriculum or assessment change introduced by the Ministry, etc.. In any case, I will use the framework of meta-didactical transposition for describing evolutions in practices of the communities involved in working together, namely:

A. The *community of design*: researchers and teacher-researchers involved in the process of task design (phase 1 below) that guides to a final product in terms of tasks for students;

B. The *community of experimentation*: teachers and trainers involved in the professional development (phase 2 below) and in the teaching experiments in the classrooms (phase 3 below).

The analysis of the mutual interactions between the communities involved (researchers, teacher-researchers – who are also teacher-trainers – and teachers) during the two processes (design and experiments in the classes) can evidence the role of each community, the relationships with the others, and the possible change of the praxeologies or of parts of them.

In this way, we can give ideas on the CIEAEM theme to which I referred at the beginning of this section. One of the main questions of this theme is:

> *How can the social dimension become a resource for teacher education? What are the challenges of programs strongly based on social interaction in communities of practice/enquiry?*

This question is properly fitting with my research interests, on different phases of the project:

1. The process of task design (community A above) in a community composed by researchers and teacher-researchers, working strictly together;
2. The process of teachers' training, where the community of teachers as learners is guided by the community trainers, who make their action of mentoring and brokering and often are the same teacher-researchers who take part of the design phase (community B above);
3. The process of experimentation of the tasks – produced in phase 1 and solved by teachers in phase 2 – by teachers in their classrooms, with the tutoring action of trainers (community B above).

I will examine praxeologies related to the communities involved in the same research and teacher training programme (Master for prospective mathematics teacher trainers) at the Department of Mathematics, University of Turin:

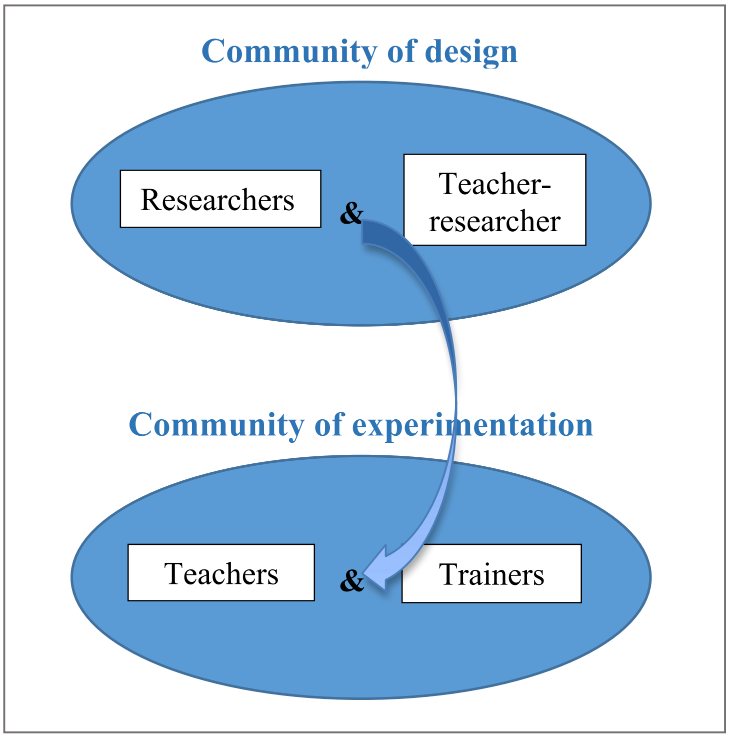

Figure 1: communities of design and experimentation (trainers in the second are often teacher-researcher of the first community)

New frontiers from research to teacher education: MERLO items

MERLO is the acronym for Meaning Equivalence Reusable Learning Objects; it is a didactical and methodological tool developed and tested since the 1990s by Uri Shafrir and Masha Etkind at Ontario Institute for Studies in Education (OISE) of University of Toronto, and Ryerson University in Toronto, Canada (Shafrir & Etkind, 2010). MERLO is a very adaptable tool, suitable for several subjects and based on equivalence of meaning across different kinds of representation. It is particularly recommended in mathematics: as it is well known, mathematical objects are sophisticated cultural products that are accessible only by means of representations (Duval, 2006), that is through suitable semiotic representations. As Duval points out "there is no noesis without semiosis" (Duval, 1995, p. 5, 22: Noesis is the intentional act of intellect, and can be defined as the action and the effect of understanding). This is the main reason why semiotic systems are central to mathematical activities and understanding, as pointed out by many scholars, such as Duval himself, Johnson-Laird (1983), Peirce (1931-1958, 1977), Sfard (2000) and Leung, Graf and Lopez-Real (2006). A first consequence of such a situation is what Duval calls the "cognitive paradox" of access to mathematical objects: "How can [students] distinguish the represented object from the semiotic representation used if they cannot get access to the mathematical object apart from the semiotic representations?" (Duval, 2006, italics in the original). A second consequence is that for grasping the meaning of mathematical objects one must cope with the multiple semiotic representations of the same mathematical object in more than one semiotic register, and with their mutual relationships. These capabilities are fundamental for understanding mathematics and consequently crucial for its effective learning (Duval, 2006). The ability to shift from one representation of an object to another representation of the same object is a competence that students should acquire in order to access the underlying meaning (Duval, 2006). The ability to shift between representations is evaluated in international (PISA, TIMSS) and national assessment tests (INVALSI, in Italy). MERLO approach is in line with these directives. Moreover, it is also a didactical tool for avoiding or overcoming the so called "duplication obstacle" (Duval, 1983), a real source of difficulty in mathematical learning as reported in literature (Fishbein, 1987), sometimes ignored or underestimated by teachers and currently difficult to be solved in the practice of the didactics of mathematics. This kind of obstacle leads students to the consideration of two representations of the same mathematical object as two different mathematical objects, but also, conversely, it may represent students' inability in grasping two different meanings of a mathematical object in only one representation.

MERLO (Arzarello et al., in press; Etkind, Kenett, Shafrir, 2010) is a collection of items, which allows the sorting and mapping of relevant concepts within a discipline, through multi-semiotic representations in multiple sign systems (numeric, graphical, symbolic, verbal, …). Specifically, each element in the MERLO database is a structured item, anchored to a target statement that describes a conceptual situation and encodes different features of an important concept; each element also includes other statements that may or may not share the meaning with the target. In a mathematical context, for example, an element of MERLO database could be about "parabola": then this element could include a target statement with the definition of parabola, and other statements in different kinds of representations (symbolic notation, Cartesian graph, table) that share or not share the meaning with the definition of parabola. The figure below is a template for constructing an item anchored to a single target statement.

Figure 2: template for constructing a MERLO item

Statements in the four quadrants of the template - Q1, Q2, Q3 and Q4 - are thematically sorted by their relations to the target statement that anchors the particular item. They are classified by two sorting criteria: *Surface Similarity* and *Meaning Equivalence* with respect to the target. The term *Meaning Equivalence* designates a commonality of meaning across several representations. The term *Surface Similarity* means that representations "look similar": they are similar only in appearance, sharing the same sign system, but not the meaning. Hence, for example, at an intuitive level the statements "two plus one" and "two plus three" share Surface Similarity but not Meaning Equivalence, while the statements "two pair" and "2+2" share Meaning Equivalence but not Surface Similarity.

A typical MERLO activity contains five statements: a target statement plus four additional statements of type Q2, Q3 and Q4; they can be in a variable number, provided that at least one Q2 statement is present, in addition to the TS. The inclusion of Q1 statements is avoided, because creators' experience shows that inclusion of this kind of statement makes the activity too easy (Etkind M., Kenett R.S., Shafrir U., 2010), for their equivalence both in appearance and in meaning with TS.

Here is an example of MERLO activity designed by a group of teacher-researchers enrolled to a Master programme for Italian in-service teachers (future teacher educators), held in the University of Turin, Department of Mathematics. This MERLO activity is inspired by a question asked in a test of INVALSI, the National Evaluation Institute for the School System (INVALSI, 2012) and requires recognition of relations and functions in different semiotic systems.

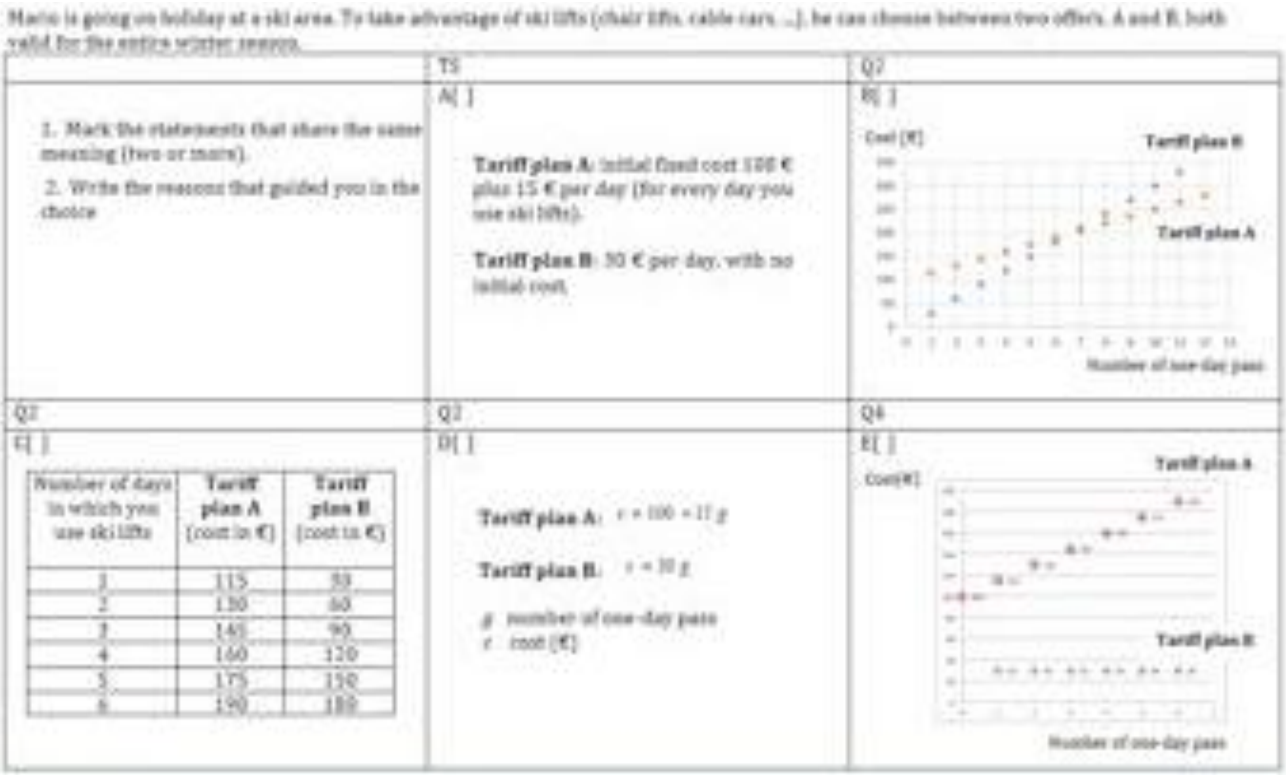

Figure 3: an example of MERLO activity (teacher's papersheet)

As we can see from the figure the activity is linked with a real life context and shows:

- A natural language description of two tariff plans, chosen as target statement TS;
- The same tariff plans represented in a different way (Cartesian graph, table and formal language) as Q2 statements, that share meaning, but do not share surface similarity with TS;
- Another Cartesian graph, chosen as Q4 statement, that does not share neither meaning, nor

surface similarity with TS.

In the version of MERLO activity for students, obviously, the type of each statement is not revealed and the position of statements can be changed by the teacher. The task for students (in a student's paper-sheet, without labels Q, TS) is to recognize the statements in multiple representations that share the same meaning and to write the reasons for the choice. In this way MERLO activity combines multiple-choice (recognition) and short argumentation answer (production). The correct solution of a MERLO activity gives a feedback on students with two main scores: recognition score and production score. The first score comes from the recognition of statements with shared meaning among the given 5 statements, while the second score comes from the writing production of reasons for the decisions. This feedback is useful also to the teacher, for getting information about the level of understanding (the so called "deep understanding") of their students on a particular conceptual knowledge, and for that reason MERLO can be used in formative assessment.

These items are completely new in Italy, both in the community of researchers and that of teachers at national level. For that reason, their introduction with teachers is particularly delicate and has to be discussed not only in the value of the mathematics involved, but particularly for what deals with their use with students. They can be used as well as a tool for an activity aimed at discussing and arguing, and as a tool for formative assessment (at this moment, we do not have data about the use of MERLO for summative assessment).

If interested in deepening the research, the teacher education, and the kind of collaboration between researchers and teachers in practical activities (of design-training-experimentation of MERLO items), one can read in these Proceedings the paper (Arzarello et al., 2016).

Community of design and community of experimentation in the context of MERLO project

The work of the communities and their dialectic interactions are specifically described introducing their tasks, and the theoretical reasons of them. The communities involved have been described above (A and B) and are respectively a community of design and a community of experimentation. The components of praxeologies involved in the work and interaction of communities are referred to the tool studied, designed, and applied in the teaching experiment: What the design community do (phase 1), is to produce MERLO items to be used at school by teachers, who are involved in a teaching education programme (Piano Lauree Scientifiche), aimed at professional development and introduction of teaching experiments in the classes of the teachers participating at it. What the experimentation community do (phase 2 and 3) is to use the MERLO items in the educational programme, then to experiment them in their classes, observing students' processes.

Stating from the experience of working on design of MERLO items, we can have a general description of the steps to go through in designing an activity of this type. First, the concept and the clusters are part of the choice of the conceptual node on which you plan to concentrate the didactical activity. It is a decision to be taken before the definition of TS, according to the curricular reference given by the institutions. For example, the design community chose to create an item MERLO starting from an INVALSI question (namely, a question taken from the national school assessment test). This experience of the community allowed to face with several theoretical aspects, which are critical and important for the design phase, and related to the experimentation phase: to touch a recent mathematical aspect from the point of view of its definition, representations, or properties, or to direct questions of consolidated work made in the past, soliciting a deep understanding of studied concepts. The design community worked at preparing a wide set of MERLO items, following the most important concepts of the National Curriculum (Indicazioni Nazionali). The community prepared the items according to an organisation of concepts in grapes:

in such a way, the teacher may have more than one item linked to the same concept (or concepts strictly related to it), to be used in individual/group activities, in class or at home, and in assessment (formative/summative). This phase of design is particularly delicate, for the choice of the different statements of MERLO items within a boundary of meaning, according to each grape. The boundary of meaning represents the set of representations that share the same meaning with a representation chosen as TS: all of them are meaning equivalent.

The design of an item passes through some steps, in order to obtain different forms of meaning equivalence: different semiotic representations of a concept, or situations requiring the application of the concept, or logical consequence of a statement. So, preparing the items, the community of design encountered some situations where to make choices, obtaining then a methodological procedure in the design, and - accordingly - developing and applying some praxeologies typical of the design. Some of them are described in the following:

a. In preparing the statements TS and Q2, the designers choose a concept or a grape of concepts, and write a statement as TS, then find one or more Q2 (statements that share the same meaning with TS, in various ways as written above – different representations, or logical consequence, and so on). In this phase, the designer made the choice to consider a Q2 statement more or less close to the TS, according to the kind of meaning sharing (very close if only a translation of representation in another register, or far if necessary to make some passages to obtain the Q2 from the TS, with other nuances in the middle). The three distances from TS - very close, in the middle or far – are what the designers called distance from the target.

b. Willing to prepare a Q3, the designers have to use surface similarity to represent a concept in the same semiotic register as TS, but with different meaning: these Q3 do not belong to the same boundary of meaning as TS and Q2.

c. For Q4, the work is different, because it does not have meaning equivalence, nor surface similarity with TS and Q2, so, it is out of the boundary of meaning, but also out of the same register of TS. A choice that the community did, is to have Q4 that do not share meaning with Q3, to avoid the risk that the students have in front two sets (one set: TS and the Q2, the other set: Q3 and Q4) with separate boundaries of meaning, and to avoid the risk of choosing Q3 as TS and Q4 as Q2. Since the students does not know which is the Target Statement TS (they have to choose only all the statements in the item that share a meaning), the designer have to put attention in preparing some statements that share a meaning, and the others completely not related among them.

d. Having prepared a set of five statements, they group together to obtain a MERLO item (as in Figure 3), and have in this way the teacher's paper-sheet (with labels TS, Qi), and the student's paper-sheet (without labels).

e. Generally, the MERLO item is composed of two or more TS and Q2 altogether, and at least one Q3, because it allows investigating the capability of the students to go beyond the outward appearance of a mathematical object. In addition or alternatively there may be a Q4 (or more than one), which plays a similar role. The quantity of Q2 is not determined a priori, but depends on the production of equivalent representations that you can do.

The design of MERLO items and of their modalities of implementation at school has been described in details with the previous points, which can be seen as praxeologies of the design communities of researchers and teacher-researchers: in term of task, the construction of MERLO items; in terms of technique, the specific issues described in the previous points, in relation to the TS and Qi compositions and connections (see the points above, particularly b and c); in terms of technology, the mathematical (epistemological) and didactical/institutional (related to the curriculum) reasons to have a coherent MERLO item, and in terms of theory the mathematical meaning of a concept, its semiotic representations and all the references (Duval, and so on) described before.

The training phase took place after the design phase, and involved those teacher-researchers who participated to the design, now with the role of teachers' educators in the Piano Lauree Scientifiche programme. The praxeologies of the trainers, who have an important involvement as brokers - because members of different communities – can be described briefly in the following. They have the task of educate teachers in using MERLO items in their classes, and their technique is the presentation of the MERLO activities to teachers, giving them examples to solve and to discuss on (Arzarello et al., 2016); and the theoretical reasons (technology and theory) lie on the corresponding of the praxeologies of design community, on the didactical side as dissemination of the research side (the use of MERLO in mathematics laboratory, with discussions, group work). For example, the trainers make the teachers aware of the design phase, with the connections and implications and choice of TS and Qi, and on the sharing of meaning in different representations. Particularly useful for teachers in the training phase is to be available in experimenting new didactical methodologies, with the use of new praxeologies or components of them, and to discuss with brokers about the pedagogical implications of the use of MERLO items in class.

As described in another paper of these Proceedings (Arzarello et al., 2016), MERLO pedagogy as carried out in our teaching experiments, is made of:

- An *individual phase*, where students, have to identify in a MERLO item those representations that share the same meaning and to write the reasons for their choice.
- A *group phase*, where students in in groups have to compare their personal choices with those of their group-mates, discussing for arriving at the ultimate goal of a shared answer.
- A *class discussion*, coordinated by the teacher with the aim to clarify and reflect on the deep understanding of the concept in relation to its representations that share the same meanings.

In this experimentation phase of MERLO items, the teachers do what they have learnt in the training phase in terms of MERLO pedagogy, and those teachers who feel not sure to do it by themselves, are supported by a trainer in the class. Their praxeologies have as task and technique the application of MERLO items with their students, using MERLO pedagogy (individual-group-class phase described above), while as technology and theory the elements that justify task and technique: the equivalence of meaning, the boundary of meaning, the use of mathematics laboratory, instead of lecturing and assessing in a traditional way.

In the related paper in these Proceedings (Arzarello et al., 2016) there are several examples referring to the design of MERLO items, their implementations in teaching experiments and data on the results.

Discussion

From the research point of view, what is important is to study these praxeologies or even only one or more components of them, which at the beginning of the training phase are absolutely external to teachers' community, and at the end of this phase and the experimentation phase become internal. This means that in the teacher's professional life something has changed, for example passing from a traditional lecturing to the use of MERLO items in individual and group work among students, or in the formative assessment. Not all the teachers can do it in the same time and in the same ways, but everyone follows her professional history, motivation, and aims. And this is the main result observed through the lens of meta-didactical transposition, in this experience but also in other experience of teacher education. When a teacher says (Arzarello et al., 2016):

> Using MERLO in oral questions in class, it is easier for me to know students mental processes. Because some of them make a choice but do not write anything about arguing, for several reasons…

we have reached a twofold aim: one didactical, as an improvement of her pedagogy according to theoretical constructs from mathematics education, and one of research, because the teacher has changed something in her approach to teaching, with respect to the past.

Moreover, we have the possibility to observe the birth and growth of new professional figures:

- The teacher trained in MERLO pedagogy and involved in her professional change, who becomes a leader in her school, organising meeting with colleagues for comparing results, discussing about MERLO pedagogy, and connecting this novelties with the institutional constraints (curriculum, assessment, books, …);
- The teacher trainer who not only applies in educational programmes for teachers, but also wants to jump in the research context, and participate to national/international seminar, congresses not only for learning, but also for presenting her experience of designer of MERLO items, of broker in the teachers' community, of researcher herself.

These new professional profiles are a product of the process of meta-didactical transposition not only for their roles, but also for all the competencies and praxeologies they are carriers and witnesses in other communities. Therefore, a challenge for future research in teachers' professional development could be the study of these professional profiles in details.

And also for teacher education and teaching to students there can be new challenges: the introduction and application of MERLO items in the institutions, namely for all the curricular themes and concepts, according to the institutional guidelines on curriculum and assessment.

REFERENCES

Aldon, G., Arzarello, F., Cusi, A., Garuti, R., Martignone, F., Robutti, O., Sabena, C., & Soury-Lavergne, S. (2013). *The meta-didactical transposition: a model for analysing teachers education programs*. In: A.M. Lindmeier & A. Heinze (Eds.), *Proceedings of PME 37* (Vol. 1, pp. 97-124). Kiel, Germany: PME.

Arzarello, F., Carante, P., Kenett, R.S., Prodromou, T., Robutti, O. & Shafrir, U. (in press). Meaning equivalence: a methodological tool for assessing deep understanding, *Proceedings of INTED 2016, 10th annual International Technology, Education and Development Conference*, Valencia (Spain), 7th - 9th of March, 2016.

Arzarello, F., Kenett, R.S., Robutti, O., Carante, P., Abbati, S., Cena, A., Coviello, A., Fratti, S., Genoni, L., Trinchero, G., Turiano, F. (2016). Teaching and learning with MERLO: a new challenge for teachers and an opportunity for students. *Proceedings of CIEAEM 67 Conference*, Aosta, Italy, 20-24 July 2015.

Arzarello, F., Paola, D., & Robutti, O. (2006). Curricular innovation: An example of a learning environment integrated with technology. Proceedings of 17th ICMI Study Conference, Hanoi Institute of Technology. Published on CD Rom.

Arzarello, F., Robutti, O., Sabena, C., Cusi, A., Garuti, R., Malara, N., & Martignone, F.. (2014). Meta-Didactical Transposition: A Theoretical Model for Teacher Education Programs. In A. Clark-Wilson, O. Robutti & N. Sinclair (Eds.), *The Mathematics Teacher in the Digital Era: An International Perspective on Technology Focused Professional Development*. Dordrecht: Springer, 347-372.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis and E. Simmt (Eds.) Proc. of the 2002 annual meeting of the Canadian Mathematics Education Study Group Edmonton, AB: CMESG/GDEDM, 3-14.

Bielaczyc, K., & Collins, A. (1999). Learning communities in classrooms: A reconceptualization of educational practice. In C. M. Reigeluth (Ed.), *Instructional-design theories and models. A new paradigm of instructional theory*. Mahwah: Lawrence Erlbaum Associates, 269–292.

Borba, M. C., & Villarreal, M. E. (2005). Humans-with-media and the reorganization of mathematical thinking: Information and com- munication technologies, modeling, visualization and experi- mentation. USA: Springer.

Bos-Ciussi, M., Augier, M., & Rosner, G. (2008). Learning Communities Are Not Mushrooms - or - How to Cultivate Learning Communities in Higher Education. In Kimble, C., Hildreth, P. & Bourdon, I. (Eds.), *Communities of Practice: Creating Learning Environments for Educators*. Information Age Publishing, 287-308.

Chevallard, Y. (1985). *Transposition Didactique du Savoir Savant au Savoir Enseigné*. Grenoble: La Pensée Sauvage Éditions.

Clark-Wilson, A., Aldon, G., Cusi, A., Goos, M., Haspekian, M., Robutti, O., & Thomas, M. (2014). The challenges of teaching mathematics with digital technologies – the evolving role of the teacher. In P. Liljedahl, C. Nicol, S. Oesterle & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 1, pp. 87-116). Vancouver, Canada: PME.

Duval, R. (1983). L’obstacle du dedoublement des objets mathematiques, *Educational Studies in Mathematics*, vol. 14 pp. 385-414.

Duval, R. (1995). *Semiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels.* Bern: Peter Lang.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103-131.

Etkind, M., Kenett, R.S., Shafrir, U. (2010). The evidence based management of learning: diagnosis and development of conceptual thinking with meaning equivalence reusable learning objects (MERLO). In: *8th International conference on teaching statistics (ICOTS)*, Ljubljana, Slovenia.

Fischbein, E. (1987). *Intuition in Science and Mathematics, an Educational Approach*, D. Reidel Publishing Company: Dordrecht.

Goodchild, S. (2007). Inside the outside: Seeking evidence of didacticians’ learning by expansion. In B. Jaworski, A. B. Fuglestad, R. Bjuland, T. Breiteig, S. Goodchild., & B. Grevholm (2007). *Learning communities in mathematics*, pp. 189-204. Bergen, Norway: Caspar Forlag.

Goos, M. (2014). Technology integration in secondary school mathematics: The Development of teachers’ professional identities. In A. Clark-Wilson, O. Robutti & N. Sinclair (Eds.), *The Mathematics Teacher in the Digital Era: An International Perspective on Technology Focused Professional Development* (pp. 139-161). Dordrecht: Springer.

INVALSI (2012). Rilevazione degli apprendimenti, Anno Scolastico 2011-2012, Prova di Matematica Scuola Secondaria di II grado Classe Seconda.

Jaworski, B. (2006). Theory and practice in mathematics teaching development: critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher*, 9, 187-211.

Jaworski, B. (2008). Building and sustaining inquiry communities in mathematics teaching development. In K. Krainer and T. Wood (Eds.), *Participants in Mathematics Teacher Education*, 309–330.

Johnson-Laird, P. N. (1983). *Mental models: Towards a cognitive science of language, interference and consciousness*. Cambridge, UK: Cambridge University Press.

Kaiser, G., Blömeke, S., Busse, A., Döhrmann, M. & König, J. (2014). Professional knowledge of (prospective) mathematics teachers – its structure and development. In:

Liljedahl, P., Nicol, C., Oesterle, S., & Allan, D. (Eds.). *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 1, pp. 35-54). Vancouver, Canada: PME.

Leung, F.K.S., Graf, K., and Lopez-Real, F. (2006). Mathematics Education in Different Cultural Traditions - A Comparative Study of East Asia and the West: *The 13th ICMI Study*. Springer.

Llinares, S., & Krainer, K. (2006). Mathematics (student) teachers and teacher educators as learners. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future*. Rotterdam: Sense Publishers, 429–459.

Novotná, J., & Goos, M. (2007). Teachers researching with university academics. Working Session 2. In J. Woo, H. Lew, K. Park & D. Seo (Eds.), *Proceedings of PME 31* (Vol. 1, p. 188). Seoul: PME.

Peirce, C.S. (1931-1958). *Collected Papers (Volumes I-XIII).* Eds. Harthstorne, P.W.C., and Weiss, P. Cambridge (Ma): Harvard University Press.

Peirce, C.S. (1977). *Semiotics and Significs*. Ed. Hardwick C., Bloomington (IN): Indiana University Press.

Radford, L. (2006). The Anthropology of Meaning, *Educational Studies in Mathematics*, *61*, 39-65.

Rasmussen, C., Zandieh, M., & Wawro, M. (2009). How do you know which way the arrows go? The emergence and brokering of a classroom mathematics practice. In W.-M. Roth (Ed.), *Mathematical representations at the interface of the body and culture* (pp. 171-218). Charlotte, NC: Information Age Publishing.

Roth, W-M. & Lee, Y-J. (2006) Contradictions in theorising and implementing communities in education. *Educational Research Review*, 1, (1), 27-40.

Sfard, A. (2000). Symbolizing mathematical reality into being: How mathematical discourse and mathematical objects create each other. In P. Cobb, K. E. Yackel, & K. McClain (Eds), *Symbolizing and communicating: perspectives on Mathematical Discourse, Tools, and Instructional Design* (pp. 37-98). Mahwah, NJ: Erlbaum.

Sfard, A. (2005). What could be more practical than good research? On mutual relations between research and practice of mathematics education. *Educational Studies in Mathematics*, 58(3), 393-413.

Shafrir, U. and Etkind, M. (2010). *Concept Science: Content and Structure of Labeled Patterns in Human Experience*. Version 32.0.

Wells, G. (1999). *Dialogic inquiry: Toward a sociocultural practice and theory of education*. Cambridge, UK: Cambridge University Press.

Wenger, E. (1998). *Communities of Practice: Learning, Meaning, and Identity*. Cambridge: Cambridge University Press.

Yerushalmy, M. & Elikan, S. (2010). Exploring reform ideas for teaching algebra: Analysis of videotaped episodes and of conversations about them. In R. Leikin & R. Zazkis (Eds.). *Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice*. Springer, 191-207.

Math That Matters

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Abstract: As a consultant in the educational field, it is my job to coach teachers in helping them to educate pupils in a way that not only meets their own standards, but also the expectations of educational stakeholders about general accepted standards for good education. Daily work as a consultant is to train and coach many teachers. In this role I attend many lessons to give feedback to teachers, school leaders and sometimes also students for an improvement of the learning process. Also consultancy for the improvement of results of learning/education (tests), and/or the improvement of motivation of students to learn mathematics is part of my interest.

As organiser and tutor of Quality class[1] (since 1996), which is mostly related to CIEAEM-conferences, I've experienced the differences in the national discourse in math education and also the struggle for student teachers to find out their personal way of teaching mathematics, in such a way that it fits to their own expectations and beliefs.

From this background new ideas arise for a mathematics education that matters for students and teachers, also for teacher trainers, researchers or other stakeholders in the educational field.

In many places and different cases, teachers make use of the theoretical framework which is called OBIT-model to clarify and distinguish different learning activities. The model is helpful to determine the underlying reasons for successful and challenging education. Also the OBIT-model can be helpful to discuss the problems of low-motivation for students at the moment they face new knowledge without seeing at least a glance of its application or usefulness. Many examples can be given to meet ideas, opportunities and challenges for math-lessons of tomorrow where students love to learn, admire mathematics, and prepare themselves for their own future which starts today (!) Consequences for education and daily classroom practice are discussed too.

Résumé: Comme consultant dans le domaine de l'éducation, mon travail consiste à aider des enseignants à éduquer des élèves en m'appuyant non seulement sur leurs conceptions, mais aussi sur les attentes des politiques d'éducation et des standards reconnus dans la communauté comme participant à une bonne éducation. Le travail quotidien du consultant est d'entraîner et de former de nombreux professeurs. De ce fait, j'assiste à de nombreuses leçons pour renvoyer des remarques, des réactions aux professeurs, aux chefs d'établissement et quelques fois aussi aux élèves pour améliorer leur façon d'apprendre. De la même façon, des conseils concernant l'amélioration de résultats des apprentissages (tests) et/ou l'amélioration de la motivation des élèves pour apprendre les mathématiques font partie de mes préoccupations.

Comme organisateur et responsable de la "Quality class"[1] (depuis 1996), qui est très liée aux conférences de la CIEAEM, j'ai pu faire l'expérience des différences existantes dans les discours officiels concernant l'éducation mathématique mais aussi la lutte que doivent mener les élèves-professeurs pour trouver leur voie personnelle pour enseigner les mathématiques de telle façon que leurs propres conceptions et croyances puissent être respectées.

Dans ce contexte, de nouvelles idées émergent pour une éducation mathématique qui conviennent aux élèves et aux professeurs mais aussi aux formateurs d'enseignants, aux chercheurs et aux responsables dans le domaine de l'éducation.

Dans de nombreux endroits et dans différents cas, les professeurs utilisent le cadre théorique nommé « le modèle OBIT », modèle qui clarifie et distingue différentes activités d'enseignement. Ce modèle aide à déterminer les raisons sous-jacentes pour une éducation exigeante et promouvant la réussite. De même le modèle OBIT peut aider à discuter les problèmes de démotivations des élèves lorsqu'ils sont confrontés à des

1 Quality class is a 10 days exchange program for teacher-students. During quality class different countries, different

cultures and different opinion about mathematics education meet. Part of the program is a visit to an international conference about didactics of mathematics education. Members of the quality class change every year. The program is made for students of teacher training college and teaching-starters.

nouvelles connaissances sans qu'ils ne soient capables d'en comprendre les tenants et les aboutissants. De nombreux exemples peuvent être donnés permettant de faire se rencontrer de nouvelles idées, des opportunités et des défis pour les leçons mathématiques du futur, futur qui commence aujourd'hui (!)

Les conséquences pour l'éducation et pour les pratiques quotidiennes de la classe sont également discutées.

General accepted standards for good education?

It is hard to formulate what is seen as general standards for good education. It depends on the vision of the teacher and the school what is seen in daily classroom practice. Nevertheless in many countries we can notice a lot of similarities in the way the learning is organised in schools. Students are sitting in front of a central point (black/whiteboard and/or screen, with a teacher…) and are supposed to notice what is going on. In these ways students are invited to learn and create their own process of mastering the presented knowledge.

In this kind of setting the duty of teachers is to maximise the outcomes of the learning process, by means of didactical en pedagogical tools and organising their classroom. Because this is a personal process, many mathematics lessons are not equal, but differ in the way students admire mathematics, love to learn and discover what mathematics can bring them for their future life. In this personal process it is important to learn as a teacher too. Student's needs are different and that makes it for the teacher a challenge every day to adept to the individual learning processes.

Training and coaching will be the way to provide teachers with modern insights of learning and teaching, with new ideas and evidence based learning concepts in order to help the teacher to perform in the role of tutor for students learning.

Own expectations and beliefs of starters in teaching

For starters in the educational field it seems most important to know what to do at the moment you're standing in front of your classroom with all eyes focused on you. It's your turn! Your pupils expect not only you to have enough knowledge, but also have variety of ideas to organise the setting of learning. How starters perform is highly related to their personal beliefs of what they expect a teacher to do. Mostly their own (math) teacher is an important identification for their similar role. But not only the ways students were educated themselves should be repeated, because modern insights of learning and teaching will find their ways to daily classroom practice, by means of newly incoming fresh starters in teaching. Teaching is a personal job and as teacher you're invited to put in your best ideas to educate the new generation, to build the future society.

Depending on the way teacher training colleges in different cultures are organised the starters in teaching feel more secure, better prepared (or not) to fulfil the new job. Licences to teach students in the age up to 18 should not only be given to people who are a professional in the subject (masters in mathematics), because also knowledge and experience of how learning processes take place are necessary to be able to know what to do in (front of) your classroom. Pupils in class need masters in mathematics education.

OBIT-model for learning*

This model distinquises two types of learning called surface approach and deep approach (Smith & Colby, 2007). In the Netherlands the different approaches are formulated in the so called 'OBIT-model'. The OBIT-model was introduced by Ebbens and Ettekoven (2004) based on the research of learning by Boekaerts and Simons (2003) and is used for observing teaching and learning in class.The OBIT-model distinguishes four different learning activities: 'Onthouden' =

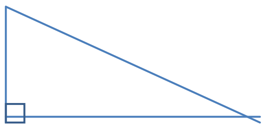
remember/remind, 'Begrijpen' = understand/able to reproduce, 'Integreren' = relate new concepts to existing concepts (former knowledge) and see the similarities and differences, and 'Toepassen' = creative application in new situations.
We can recognise these different learning activities in every learning situation, in the mathematics classroom as well as in tests (Spijkerboer at al. 2007).

Examples

Examples of mathematical exercises which focus on one of the different learning activities are given below. Important to stress is that the learning activity, used by the pupils in solving mathematical tasks, is highly related to the lessons given before and for that reason can be managed by the teacher. So it depends on the focus of the teacher and targets of the curriculum to be sure what learning activity is taken into account. Furthermore sometimes more than one of the learning activities mentioned in the OBIT-model is necessary to use in solving problems.

Examples of regular exercises:

1. What is the name of this triangle?



2. Which of these equations are quadratic equations?
   - $(x-4)^2 = 16$
   - $5x + 6 = x - 4$
   - $5x + 14 = x^2 - 3x + 5$
   - $x^2 = 0$

3. What is wrong with: $\sqrt{(x^3 - 2x^2)} = x\sqrt{(x-2)}$?

4. Find the first derivative:
   a. $f(x) = \sin x$
   b. ….

These tasks invites the student to remember the words or recognise the classification and the rules (like $\sqrt{x^2} = |x|$). This is what you can learn by heart, without understanding i.e. you just know the first derivative of sinus. This is why these tasks are examples of 'Onthouden'.

*** OBIT-model, lower stages** (Ebbens & Ettekoven, 2004)
Onthouden = Remembering
For doing Remembering activities it is not necessary to grasp what you are learning exactly, it is only a copy-paste activity. You learn by heart Pythagoras theorem or the formula for $(a + b)^2 = a^2 + 2ab + b^2$. Remembering is based on reproduction of the knowledge you know, not necessarily understand.

Begrijpen = Understanding
De learning activity understanding is shown when a student is able to explain in his own words what has to be done in a certain situation, learned during the lessons. He is able to copy the way of working, the way of solving problems the same ways as was educated by the teacher in class or with help of the explanation in the book. Understanding is a straight forward activity, and can be done by students who do their homework and are mentally present in class. The thinking steps are given, not to be made by the student themselves. All activities were done before by their teachers, so it is a rehearsal.

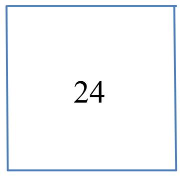
OBIT-model, higher stages
Integreren = Relate
Using the learning activity 'Integreren' the focus is on the relation between different parts of knowledge. Relate means you use your insight in the situation. You make use of your knowledge and connect it to the new information achieved. Compared to understanding, during integration activities there are more thinking steps in discussion, and the learner add something themselves, it is a productive activity, not only reproductive.
Especially for this learning activity you need to ague, explain, try, search, etc. and can best be done in communication with classmates.

Toepassen =creative application
In the learning activity creative application the thinking steps are not given anymore, like in 'Integreren', but the student have to build a - sometimes creative - thinking process, to solve the problem; "What do I know?", "What can I use?", "How to connect the different know-how I have or I want to gain to make sure the solution of the problem is correct?". In this learning activity the student really make use of his own skills, as well as (subject) knowledge. There is a design process going on, which is stimulated by a new situation, never faced before in context. Students have to apply their knowledge and explore.

You need to remember/remind former learned knowledge and give it back literally.

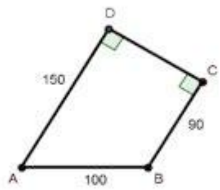
5. Find x by solving these equations:
    - $(x - 4)^2 = 16$
    - $5x + 6 = x - 4$
    - $5x + 14 = x^2 - 3x + 5$

6. Give the first derivative of $f(x) = x^4$



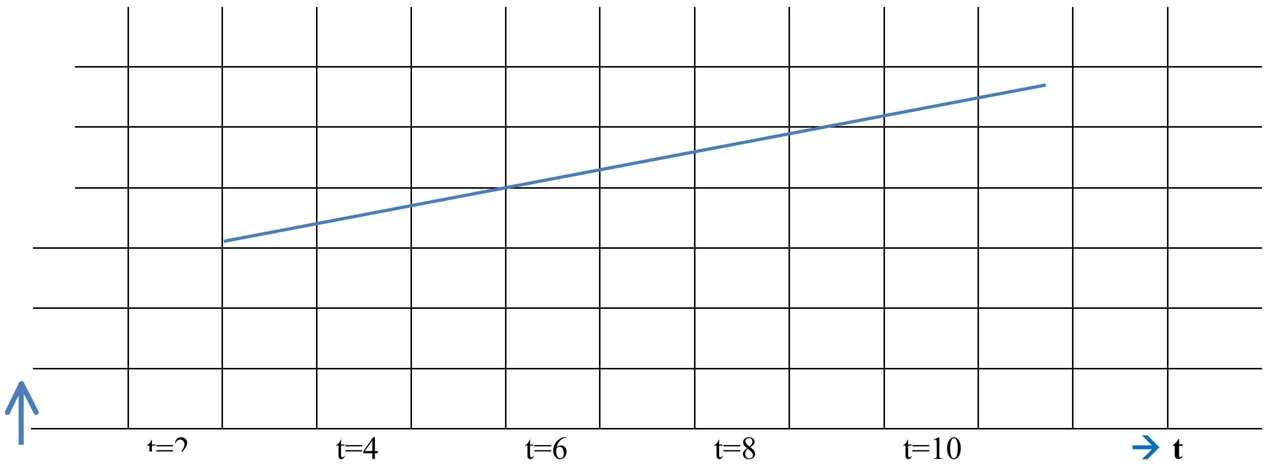
7. Calculate the circumference of a square, with area 24.

Tasks 5-7 are not carried out by heart, students have to reproduce a certain procedure with other numbers and forms. Many mathematicslessons are given to master the algoritm and procedure, to make use of and reproduce in such cases. Students learn to rehearse the procedure in many exercises of the same kind. Important is that the procedure is given in former education. The input is deliverd by the book, teacher, computer, … and we ask the student to give the input back in his own words. These tasks belongs to the learning activity called 'Begrijpen', which is best translated by understanding how to perform.



8. Calculate the circumference of this figure ABCD.

9. Design the formula related to this graph.



t=time in months (t=1 is January 1).

In these tasks the students are invited to think themselves. Of course they make use of their knowledge, but the student has to dicide which knowlegde he wants to make use of. In task 8. the student has to find out how he can be able to calculate DC. Drawing a line from point B, perpendicular to AD is not given. The decision to make use of the right angles and apply Pythagoras theorem here is the students' own choice. Also when the procedure to solve the problem can be different - as in task 9. - the student have to make (more than one) thinking steps by himself. If this is the case we call it 'Integreren' (relate/connect) because the connection between the new problem and former knowledge should be restored by the students own decission.

10. Make a good guess of the height of the school building. Explain exactly how you use your mathematical knowledge for this task.

In task 10 less information is given for how to produce an answer, or how to solve the problem. The task is providing students with learning activity 'Toepassen'. The explanation of how they worked exactly and formulate the mathematical procedures they choose, is a creative proces. Therefore the best translation for the word 'Toepassen' is creative application.

The OBIT-model is based upon Bloom's Taxonomy, but with other words and meanings. Especially the word Application has different meanings in OBIT and Bloom. Another difference is that Blooms model is a taxonomy while OBIT is not.

Problems of low-motivation

The learning activities 'Onthouden' and 'Begrijpen' are mainly reproductive. With these learning activities the knowledge is build up in the short term memory of the brains. This is called surface approach because after some days or weeks, this knowledge can vanish if it is not connected to some other experiences and/or emotions. Knowledge in the short term memory should be repeated continuously to get it into the long term memory, for later use.

Problems with low motivation of pupils in mathematics classrooms are related to the type of exercises and problems they have to solve. If students face new knowledge without seeing at least a glance of its application or usefulness it is an invitation for low motivation. If they only have to know for the teachers sake, students are invited to master the curriculum in a surface approach. After some weeks (days of years), they can start all over again, if they still need to know. Many problems of low motivation are organised by education focussing on knowledge only by using learning activities from the lower stages of the OBIT-model.

The learning activities 'Integreren' and 'Toepassen' both belong to the deep approach and are mainly productive. Tasks focussing on these learning activities build up experience and knowledge what is stored in the long term memory and will last longer. The knowledge is connected to other knowledge and is connected to memories like emotions and information got by other senses. For that reason the results of this learning are not easily forgotten, that is why this type of knowledge building is called deep approach. Also this knowledge should be saved by rehearsals.

Students with lower mathematical performance are more easily motivated by doing tasks, in which they are invited to perform on the higher stages of the OBIT - model. Than they will see some of the probable applications of the mathematical concepts in focus. It is the teachers challenge to present tasks not too easy, not too complex, but with the focus on usefulness. Motivational lessons contain inviting tasks and active participation of students in a challenging environment with teachers who are prepared to guide the learning process of the pupils themselves. (Hattie, 2002). Many mathematics teachers are busy to explain how the tasks in the math-book should be carried out and students perform in classroom situation by listening and trying to understand. Meanwhile doing their duties; making exercises. What is not finished during the lessons should be done later at home. At times, it seems such a system limits students' prospects for moving beyond superficial thinking (Kohn, 2000).

OBIT in the mathematics classroom

Findings from a study examining the teaching practices and student learning outcomes of sixty four teachers in seventeen different states in USA indicated that most of the learning in these classrooms was characterized by reproduction, categorizing of information, or replication of a simple procedure (Smith et al., 2005). Also daily practice in mathematics classrooms in other countries is much more focused on the lower stages than higher stages of the OBIT-model. Main focus seems to be knowledge transmission. Knowledge transmission is focusing on learning activities like remembering (onthouden) and understanding (begrijpen).

In an everyday math lesson, making exercises is mostly done part of the time. After a (short?) instruction of the teacher, students are supposed to practice their skills in making (a lot of) exercises of the same kind. Having enough practice is one way to master the theory, and be able to pass exams with equal exercises as were proposed in the education before. In spite of the desire of mathematics teachers in general, the daily practices will be much more challenging in doing 'Integreren' and 'Toepassen' in class. Unfortunately this will not be seen so easily, while you step into a regular classroom on a regular day.

Starters in teaching recognise the potential of mathematics in contributing for making judgements or reaching consensus, but they complain there is not enough participation of students in mathematics classrooms for that. (Serradó et al., 2015)

Not only the kind of exercises can change the pupil's behaviour, also the way the exercises are supposed to be handled and discussed with classmates will invite student for active participation. Different invitations for learning are guided by different ways of working. There are many different ways of working, for exploring mathematical skills, for rehearsal of routine tasks and for explanation of the concepts behind the posed problems. In different ways of working the change of teachers' role should also be taken into account. The challenge is to change teachers' behaviour with small changes of the way they handle lesson time, the math book, and the grouping of students. Small changes can cause big effects. There are many different ways to follow. In this paper three approaches for acting of teachers are discussed:

I. Continuation with the same.
II. Level raising questions.
III. Research assignments.

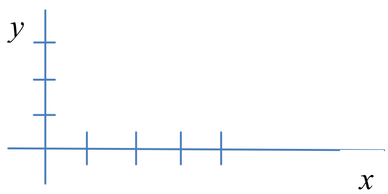
I. Continuation with the same.

The first way to reach the learning activities in the higher stages of the OBIT model, is the advice to teachers in the situation when the exercise from the textbook is solved: don't stop, move on; what if…..! Based on the exercise there are many possibilities for other questions to reach a rich understanding of the concept in focus.

As an example; students in school are dealing with graphs, i.e. learn to find a y-value, which belongs to a given x-value, or the other way around, like in the task below.

11. Given the equation: $y = \frac{1}{4}x + 1$, fill in the table and draw the graph of this relation.

x	0	..			
y	..	..			



How to carry out this task is demonstrated by the teacher and students copy that way of working. They make (many) exercises and get good marks for their test, if they rehearse enough. No problem. They know what to do. To continue, the next step to do in this situation is to find a relation between graph and situation, which has some kind of a meaning to the student. For example like a relation between height and time of an air balloon.

12. The way a balloon is rising from ground is given by the formula: $h = 1 + 0{,}25\ t$ (h in meter, t in sec)
    a. At what time the balloon will reach a height of 100 m.?
    b. At what height the balloon will be after 1 minute?
    c. Explain why $h \neq 0$, when $t = 0$.

And some chapters later in the math-book or sometimes years later, the same questions are given in a formal way:

13. Given the relation: $f(x) = 0{,}25\ x + 1$.
    a. Find the intersection point of the graph of f with the line $x=4$
    b. At what value of x the graph of this function will intersect with the line $y=11$

Here the higher stages of the OBIT-model can be reached, by going on asking questions about the same situation. I.e. in task 12. with the air balloon:

d. What do you guess; at what time this formula doesn't give answers for the realistic situation anymore?
e. In what way the formula should be changed to adept to the realistic situation?
f. If the balloon is rising faster/slower, what changes should be made to the formula?
g. What can be the meaning of +1 in the formula?
h. …

The learning for the long-term memory (for the higher stages of the OBIT-model) takes place when students are invited to communicate with others to reflect and argue, to negotiate and building consensus. For this, different solutions and different strategies, ideas or proposals are helpful for the learning process of every participant in classroom. As a consequence students in mathematics classrooms doing this kind of activities are better off in cooperative work, with a teacher using ways of working that invites the pupils to communicate among each other and use mathematical tools to express their ideas. Mathematical tools like graphs can be very helpful for that, because graphs are telling a story. To exchange the stories in relation to the graph is an 'integreren'-task and will invite also low performers to use their creative skills. Higher stages of the OBIT - model are not only for the gifted students!

The switch between graph, formula, table and words describing relations can be a very strong and helpful tool for pupils to understand (and use) the concept of a relation from different perspectives. Not even the formal way to draw a graph by a given formula, like in situation 13. The higher stages of the OBIT - model can be reached here also to move on asking challenging questions like:

c. Will there be a line where the graph intersects twice? How do you know?
d. Find another straight line graph which intersect the line $y = 11$ at the same point as the graph of f. and give the formula of this graph.
e. Express the formula of a graph, which is a mirror image of the graph of f in the line x=4.
f. …

Important is that the new added questions focus on the same concept and are not necessary more complex and only suitable for the gifted students in mathematics. It's obvious that more complex exercises are easily designed with the higher stages of the OBIT-model. Like in the examples above, the questions are still dealing with linear functions. The concept is that one variable belongs to another, or generates another. The added questions are asked to help students understand the same concept but in the higher stages of the OBIT-model.

II. Level raising questions

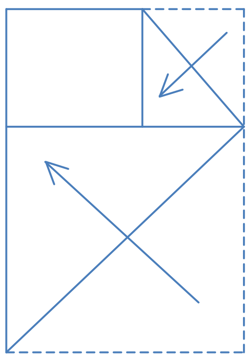
Using level raising questions is another way to help students develop learning activities in higher stages of the OBIT-model. The mathematics teacher dealing with a part of the curriculum, knows (or could easily find out to know) what are the connections to other parts of the math-curriculum in focus. The tasks of the lesson today have something to do with other tasks in the same chapter, or in a curriculum part of tomorrow, next semester, next year, … This means that connected to every task carried out in classroom the teacher will be able to find other questions, which are beyond the focus of this particular task, but has the opportunity for smarter students to carry on. Doing level raising questions, students get more insight in the connection between the tasks of one lesson with the tasks proposed in the next step of their development. Some students are motivated by means of this way to challenge them. Level raising questions are not suitable for everyone to learn in the higher stages of the OBIT-model.

Asking level raising questions is something a teacher can prepare himself. The experience is that teachers know the questions, because of their overview in the mathematics curriculum. But it takes some preparation time, because you don't know the level raising question in the split second you have to decide during lesson situations. Nevertheless it is what most mathematics teachers like to do and they are challenged by the task of the textbook to design some level raising questions in addition.

Important to take notice is, that you never ask level raising questions in plenary situations, because the level raising questions are not suitable for every student in class, only for those who have time and interest in the next step in their mathematics performance. If presented in plenary discussion with class, what easily can happen is that the weaker students switch off, and feel their incompetency, which can be harmful. For that reason level raising questions are used in the classroom situation where students work independently in the not-plenary part of the lessons. Dealing with differences in level of performance in mathematics, is the main reason to use level raising questions in class.

Asking level raising questions is another way as was mentioned above, by I. *Continuation with the same*, because in this situation we ask the questions to those students which have potential for higher performance in mathematics. They are provided, not only with different questions, but also a higher level if thinking.

Examples of level raising questions



Preparation: Fold an A4-sheet of paper like this:

1st level questions:

- How many corners does the new figure have?
- Do you know special mathematical figures with 4 corners?
- Is this figure one of the special figures you mentioned?
- How do you know?

2nd level questions:

- Try to prove this figure is a kite.
- What are sufficient and necessary conditions to prove this is a kite?

3rd level questions:

- Becomes every sheet of paper what is folded this way, a figure like a kite?
- What are the conditions for the length-width ratio of the sheet of paper, in order to get out a kite of it?
- Why is chosen for this length-width ratio in the case of an A4-sheet of paper?

4th level question:

- Can you prove for the case of an A4-sheet of paper this becomes a kite, while folding the way like shown in the figure?

5th level questions:

- Is the rectangle in the left upper corner of the figure probably also an A-shaped piece of paper? In other words: can you fold out a kite of this piece of paper as well?
- And what is the number of the A-paper connected to this rectangle? A4, A5, A6…?

6th level question:

- Design a formula, giving the relation between the number of A-paper and the length of the baseline of the paper (in cm)
-

In this example the following sequence of mathematical knowledge is recognised, connected to this particular problem:

- Right angle.
- Geometric figures, with 4 vertices.
- The conditions for a kite.
- Necessary and sufficient conditions for a proof.
- Symmetry and axis of symmetry
- Pythagoras Theorem
- Dealing with brackets
- Dealing with forms with square roots
- Dealing with second power expressions, like $(a + b)^2$
- Using ratio's calculations
- Carry out a geometrical proof
- Dealing with exponential expressions like $A_n = ½^n$
- Solving logarithm equations, like n = k • log A
- Equation-manipulation
- …

It depends on the performance of the pupils, to what level the teacher can go on with asking level raising questions.

III. Research assignments

A third type of action teachers can take to invite students for the higher stages of the OBIT - model is to design research assignments for their students. Research assignments are mostly seen in laboratories in or outside the classroom. Students have to find out certain phenomena and try to explain or describe the phenomena with mathematical tools. Very often connection between real life and mathematics has to be made. Making mathematics walks around the school with well-defined tasks and assignments, is a strong and – if carried out the proper way – helpful educational tool for learning mathematics in context. (Spijkerboer, 2004)

The teacher has to take care of the complexity of research assignments, because easily the demands are complex and unusual for the students. Research assignments mostly invite students for creative application. That can cause uncertainty and fear, because the tasks are not familiar to them and can confuse students. It is important that students approach the tasks open minded, and with some feelings of confidence. Without being prepared to make a lot of calculations, the focus is really to find out what is the connection between the situation and the related mathematical concepts in discussion.

Because students can learn so much from each other in this case, especially research assignments are creative application activities that can be carried out in groups very well. Cooperative work gives space for different approaches, different tasks and different learning styles. With the teacher as a coach, students can overcome their uncertainty and fear. (Bellanca J. & Fogarty R.,1994).

Reflection

Because there are many different ways to bring students to the higher stages of the OBIT - model, every teacher can choose his/her own way to perform in class. Teachers differ as students are different, so the choices for any situation, any class, and any level can be different for any teacher. And that is an advantage for the students. The more variation in classroom experience, the more students are motivated, challenged and served for their own interest, level and learning style. And teachers make a difference! (Hattie, 2002). The approaches differ from easy to carry out to advanced educational design, which adepts to the level of educational development of the teacher. In other words, the OBIT-model invites teachers to continue educational development all the time. An exchange of ideas among teachers in their team will give raise to suggestions for more questions, level raising questions and research assignments related to the curriculum. By the way it will help teachers to reflect upon the concepts found in the curriculum rather than carry out the pile of tasks in the math book only. Evidence has shown that teachers can adopt a surface or deep approach to teaching, which has consequential effects on what and how students learn (Boulton-Lewis et al. 2001).

Also in teacher training the main focus is to teach the student-teachers how to organise your class and explain mathematical procedures, besides the mathematical content and learning theories. The OBIT - model can make a direct link between learning theory and daily practise, so for student-teachers it can be helpful to see the straight forward consequences of learning with the gain of mathematical knowledge and skills.

Not only how to make exercises is the task for teacher trainers to learn the student-teachers, but also to work on how to build mathematical concepts in students mind, in a way they kept motivated. The reason why mathematics is taught is because of the higher stages of the OBIT - model. That is exactly what student-teachers answer to the question of their desire when starting in this job. Besides the desire of student-teachers, also an important reason for the invitation to students to perform on the higher stages of the OBIT model is that high-quality learning outcomes are associated with deep approaches whereas low-quality outcomes are associated with surface approaches (Biggs 1987; Entwistle 2001; Marton and Säljö 1984).

Researchers in the educational field can contribute to education if they come up with findings how to challenge and motivate the new generation in mathematics education. Youngsters of tomorrow need mathematical skills to be able to face new situations, in an ever changing world. The role of knowledge will be of less importance than before, but the competence for being prepared for the unknown is even more valuable.

REFERENCES

Bellanca J. & Fogarty R. (1994). *Blueprints for Thinking in the Cooperative Classroom*. Australia: Hawker Brownlow Education.

Biggs, J. (1987). *Student approaches to learning and studying*. Melbourne: Australian Council for Educational Research.

Boekaerts, M. & Simons, P.R.J. (2003). *Leren en instructie, psychologie van de leerling en het leerproces.* Assen: van Gorcum.

Boulton-Lewis, G. M., D. Smith, A. R. McCrindle, P. C. Burnett, and K. J. Campbell. (2001). *Secondary teachers' conceptions of teaching and learning. Learning and Instruction* 11 (1): 35–51.

Ebbens, S. & Ettekoven, S. (2004). *Actief leren*. Groningen: Wolters Noordhoff.

Entwistle, N. (2001). *Conceptions, styles and approaches within higher education: Analytic abstractions and everyday experience*. In *Perspectives on cognitive, learning, and thinking styles*, ed. R. Sternberg and L. F. Zhang, 103–36. Mahwah, NJ: Erlbaum.

Hattie, J. A. C.. (2002). *What are the attributes of excellent teachers?* In *Teachers make a difference: What is the research evidence?*, ed. Bev Webber, 1–17. Wellington: New Zealand Council for Educational Research.

Hill, S. & Hill, T. (1990). *The Collaborative Classroom: A Guide to Cooperative Learning*. South Yarra, Victoria: Eleanor Curtain.

Kohn, A. (2000). *Standardized testing and its victims*. Education Week. (http://www.alfiekohn.org/teaching/edweek/staiv.htm)

Marton, F., and R. Säljö. (1976). *On qualitative differences in learning: Outcome as a function of learners' conception of task.* British Journal of Educational Psychology 46: 115–27.

Marton, F., and R. Säljö. (1984). *Approaches to learning*. In *The experience of learning*, ed. F. Marton, D. Hounsell, and D. N. Entwistle, 39–58. Edinburgh: Scottish Academic Press.

Munzenmaier, C & Rubin, N. (2013). *Bloom's Taxonomy: What's old Is new again, The eLearning Guild* http://onlineteachered.mit.edu/edc-pakistan/files/best-practices/session-2/Pre-Session-Munzenmaier-Rubin-2013.pdf

Reid, J. (2002). *Managing small group Learning*. Newtown, NSW: Primary English Teaching Association (PETA).

Serradó, A.,Vanegas, Y., Giménez, J. (2015) *Facilitating deliberate dialogue in mathematics classroom*. In *Educational paths to mathematics, a CIEAEM source book.* Ed. Gellert, U. et al. Springer.

Smith, T. W., B. Gordon, S. A. Colby, and J. Wang. (2005). *An examination of the relationship between depth of student learning and national board certification status.* (http://www.nbpts.org/UserFiles/File/Applachian_State_study_D_-_Smith.pdf)

Smith, T.W. and Colby, A.C. (2007). *Teaching for deep learning*, *Clearing House: A Journal of Educational Strategies, Issues and Ideas*, v80 n5 p205-210

Spijkerboer, L. 2004. *Wiskundewandelingen ontwerpen.* Utrecht: APS (http://www.aps.nl)

Spijkerboer, L. & Santos, L (2015). *Organizing dialogue and enquiry, A commentary*. In *Educational paths to mathematics, a CIEAEM source book.* Ed. Gellert, U. et al. Springer.

ROUND TABLE / TABLE RONDE

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Gilles Aldon, ENS de Lyon (France)
Cristina Sabena, Università di Torino (Italy)

Lisa Björklund Boistrup, Stockholm University (Sweden)

Rossella Garuti INVALSI (Italy)
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G.R.I.M. (Departimento di Matematica e Informatica, University of Palermo, Italy)

Introduction to the Round Table on 'Assessment in Mathematics Education: Resource or Obstacle'

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Abstract : In the format of a Round Table, we discuss the question of how assessment can support teaching and learning activities in the mathematics classroom from various perspectives. Our aim is to understand under which conditions and in which ways assessment is, or can be, a resource and not an obstacle for the teaching and learning of mathematics.

Résumé : Dans le format d'une Table Ronde, nous discutons de diverses perspectives la question de comment l'évaluation peut soutenir des activités d'enseignement et d'apprentissage dans la classe de mathématiques. Notre objectif est de comprendre dans quelles conditions et comment l'évaluation est, ou peut être, une ressource et non un obstacle pour l'enseignement et l'apprentissage des mathématiques.

Introduction

In 1993, the 45th conference of CIEAEM had been organised around the theme of *Assessment Focussed on the Student*. In its opening plenary, the at that time vice-president of CIEAEM, Lucia Grugnetti, stated:

> The way mathematics instruction functions, as well as the entire spirit in which it takes place, is strongly influenced by assessment methods. Assessment is not just a separate appendix to mathematics instruction; it is one of its crucial components. (Grugnetti, 1994, p. 3)

Grugnetti's view harmonises well with the ways in which many students experience the teaching and learning of mathematics in school. For many students the question whether assessment in mathematics education is a resource or rather an obstacle does not make much sense. Assessment in the mathematics classroom is so natural that school mathematics *without* assessment can hardly be imagined. However, as Clarke (1996, p. 327) points to, assessment is not a neutral element, but "a powerful mechanism for the social construction of mathematical competence". Assessment serves the institutional functions of schooling of qualification, of cultural reproduction, and of allocation. It informs the students about the criteria for legitimate participation in social settings such as the classroom itself, but also in everyday contexts. It gives feedback on relative achievement in mathematics in the face of the expected learning outcomes; and it provides grounds for orienting the students towards the diverse vocational fields.

At the same conference in 1993, Leonor Cunha Leal and Paulo Abrantes distinguished four facets of assessment: (1) summative assessment, in which assessment is a measurement aiming at a score, (2) diagnostic assessment, the focus of which is on the preparedness of the student(s) for the coming mathematical topics, (3) formative assessment, focussing on the teacher's control of the teaching-learning-process, and (4) assessment as a dynamic interpretation of student performance, in which problems are characterised and hypotheses generated. In the fourth sense, assessment is an "integral part of the learning process" (1994, p. 49). From the first towards the fourth facet, it seems to be obvious that assessment can increasingly be considered a resource for mathematical instruction in the classroom.

Cunha Leal and Abrantes's clarification of the concept of assessment is located in the context of a developmental project in mathematics education. Beyond this particular context, and from more micro and more macro perspectives, further meanings of assessment can be identified. For instance, studies of mathematical classroom practice from a conversation analysis perspective clarify in which way assessment is an intrinsic element of the classroom discourse. Within the sequential structure of the classroom discourse, continuous assessment is obligatory for the discourse to continue. If the teacher does not confirm the legitimate character of a student's utterance, its conditional relevance remains open. It is the teacher's reaction, which is based on her assessment of the pertinence of the student's utterance, that decides whether the propositional content becomes included in the taken-as-shared knowledge of the class. That a teacher 'assesses' a student's utterance by saying "yes", "alright", by nodding or else is first of all functional for the continuation of the classroom discourse. Only on a second plane, the micro assessment can also express praise (cf. Mehan, 1979; Streeck, 1979).

Another, more recent and more macro, aspect of assessment can be seen in the establishment of international comparative studies of student achievement and student competence which, originally, had been intended to measure the effectiveness and efficiency of national school systems (e.g., the PISA). Although it may be doubted if the methods for measuring achievement at system level can easily be transferred to the level of the learning individual, or if the test items used for the first purpose should have any influence on the arrangements for teaching and learning in the mathematics classroom, in many countries a considerable impact of PISA on mathematical education practices on the classroom level as well as on the curricular level has been noted (cf., Stacey & Turner, 2015). Policy makers and practitioner seem to experience this impact differently.

Against this background, the Round Table on assessment in mathematics education focuses on the question of how assessment, on its various levels, can support teaching and learning activities in the mathematics classroom. The aim is to discuss under which conditions and in which ways assessment is, or can be, a resource and not an obstacle for the teaching and learning of mathematics. The three contributions cover a broad spectrum of perspectives, some of them sketched above. Rossella Garuti and Francesca Martignone draw on their work in the context of the Italian assessment system. They discuss what seems to be necessary to convert the items and results of national standardised testing into resources for schools and teachers on the one hand, and for researchers on the other hand. Gilles Aldon and Cristina Sabena report on the European project *Improving Progress through Formative Assessment in Science and Mathematics Education* and its aim to use technology for formative assessment. They reflect on their classroom experience in contexts of low achieving students and show how formative assessment can serve as a didactical resource for teachers with the potential to foster meta-cognitive approaches to the learning of mathematics. Finally, Lisa Björklund Boistrup summarises her empirical classroom research with teachers in Swedish schools in which she identified four dominant assessment discourses of teachers. She discusses which of the four tend to build obstacles for students' learning and which of them constitute resources. Such a categorisation can be useful for involving practitioners in reflections about possible effects of assessment practices.

The experience of the Round Table shows how fruitful it results to juxtapose perspectives on assessment in mathematics education that differ substantially in the educational strategies they facilitate, while pulling in the same direction: to make assessment a resource for mathematics teachers' teaching and for students' learning of mathematics.

REFERENCES

Clarke, D. (1996). Assessment. In A.J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 327–370). Dordrecht: Kluwer.

Cunha Leal, L., & Abrantes, P. (1994). Is it possible to integrate learning and assessment? In L. Grugnetti (Ed.), L'évaluation centre sur l'élève – Assessment focussed on the student. Actes de la 45e rencontre de la CIEAEM – Proceedings of the 45th CIEAEM meeting (pp. 46–55). Cagliari (Italy), 4–10 July 1993.

Grugnetti, L. (1994). De la recherche à la pratique scolaire: la problématique de l'évaluation. In L. Grugnetti (Ed.), L'évaluation centre sur l'élève – Assessment focussed on the student.

Actes de la 45e rencontre de la CIEAEM – Proceedings of the 45th CIEAEM meeting (pp. 3–9). Cagliari (Italy), 4–10 July 1993.

Mehan, H. (1979). *Learning lessons*. Cambridge: Harvard University Press.

Stacey, K., & Turner, R. (Eds.) (2015). *Assessing mathematical literacy*. Cham: Springer.

Streeck, J. (1979). "Sandwich. Good for you." Zur pragmatischen und konversationellen Analyse von Bewertungen im institutionellen Diskurs der Schule. In J. Dittman (Ed.), *Arbeiten zur Konversationsanalyse* (pp. 235–257). Tübingen: Niemeyer.

Formative assessment in the FaSMEd Project: reflections from classroom experiences

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Mathematical objects can be seen both as *objects* and as *tools* (Douady, 1986). For example, knowing a rotation is both understanding a formal definition of the transformation and its main properties of conservation of distances and angles, but also being able to use a rotation in solving a problem and/or in a proving process. Therefore, assessing the deep understanding of a mathematical object must take into account these different and complementary aspects.

But which kind of assessment do we speak of? The function of different assessments within an institution can be roughly split into two different roles: a role in certification of acquisitions of knowledge or competencies and a role in learning accompanying. The former refers to *summative assessment* and we will not discuss deeply this aspect of assessment here; the latter can be included in the process of *formative assessment* (FA) and appears to be a tool (a resource) for teachers in order to enhance mathematical students' learning. If teaching and learning are driven by summative assessment, the relationship to knowledge and to the learning process risks to be modified: instead of learning for understanding knowledge at stake, learning becomes understanding the way of succeeding in typical tests. Even if summative assessment and exams have a great importance in educational systems, this kind of assessment cannot be considered as a resource for acquiring knowledge. On the contrary, formative assessment can be seen as a resource for teachers and for students in the teaching and learning process.

Our contribution to this Round Table on assessment stems from our joint participation to the FaSMEd Project. FaSMEd (“Improving Progress through Formative Assessment in Science and Mathematics Education”) is a European Project (FPVII, 2013-15) aiming at investigating

> *the use of technology in formative assessment classroom practices in ways that allow teachers to respond to the emerging needs of low achieving learners in mathematics and science so that they are better motivated in their learning of these important subjects.* (FaSMEd Project Document, p. 2)

Three main polarities can be identified within the project, considering the teaching-learning of mathematics and science: (i) formative assessment practices; (ii) the role of technology; (iii) attention to raise attainment, especially of low-achieving students.

Within FaSMEd, we share the definition or *formative assessment* given by Black & Wiliam (2009) on pragmatic basis, according to which

> *evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited* (Black & Wiliam, 2009, p. 7).

This definition takes into account three key processes in learning and teaching (Ramaprasad, 1983):

- Establishing where the learners are in their learning;

- Establishing where they are going;
- Establishing what needs to be done to get them there.

It is also stressed that it is not only the teacher to be responsible for these processes, but the learners, both as individuals and as groups, play a crucial role as well.

Within this view, assessment is no longer considered as an object, but as a process that may change entirely the way teachers are organizing their lessons, and the way learners are managing their learning path.

The table of William & Thompson (2007; the table is in the annex) crosses the fundamental questions of formative assessment and the actors: the teacher, the class and the (generic) student. For example in the first cell of the table, the learning intentions and the criteria of success are pointed out. Overall, five key strategies are identified:

1. "Clarifying and sharing learning intentions and criteria for success;

2. Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding;

3. Providing feedback that moves learners forward;

4. Activating students as instructional resources for one another; and

5. Activating students as the owners of their own learning". (Black & William, p. 8)

Drawing on two examples from the classroom context, one in France and one in Italy[1], we will now point out the relationships between FA and the mathematical knowledge at stake, and some potentialities of new technologies in the teacher's hand for formative assessment processes.

The first example (France) relates to a formative lesson about fractions. The teachers begins with a list of mathematical competencies the students have to acquire in the lesson:

- to read and to write a fraction,
- to code and to decode a fraction,
- to give equal fractions,
- to represent a fraction on a numeric line,
- to read a fraction represented on a numeric line,
- to compare a fraction to the unit and to another fraction with the same denominator.

The teacher proposes in the classroom a quiz focusing on these competencies and collects the results using a student response system. In the next lesson, the class results are given individually to the students: for each of these competencies, the students have the representations of their own capabilities using a representation of fractions that corresponds to the "Understanding learning intentions and criteria for success" of the table. The next lesson allows them to write themselves on the sheet their own result and to have a visual representation of their progression. For the teacher, the FA strategy corresponds to the "Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding" because she organised the classroom depending on the success or the difficulty of each student relatively to the targeted competencies.

[1] The French team is formed by Gilles Aldon and Monica Panero. The Italian team is formed by Annalisa Cusi, Francesca Morselli and Cristina Sabena. See also the workshop by these authors in this volume.

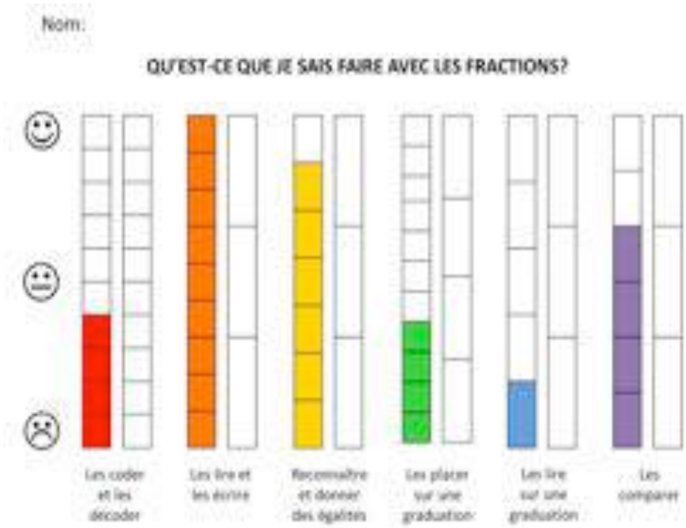

Fig. 1. Representations of a student's capabilities about fractions, using fractions representations.

The second example (Italy) illustrates some potentialities of connected-classroom technologies for formative assessment, when suitably exploited by the teacher as a didactical resource. In a classroom where the students have difficulties in writing their reasoning (IV grade), the use of instant polls has worked as a means for supporting all the students to express their answer, and then explain them during a classroom discussion. For instance, faced with the students’ difficulties in correlating formulas to verbal expressions describing a given situation, the teacher shows two different representations (a word sentence like “adding always 5” and a formula, like “k=n*7”) and asks: what is written in the formula does correspond to what is written in the text? Three answers are given to choose from: yes, no (correct answer), and I don’t know.

Working in pairs with connected tablets, the students give their answers, which are then visualised on a whiteboard and showed through a bar diagram (Fig. 2). The grouped answers allow the teacher to get an immediate grasp on the overall situation of the classroom, so to realize that even with a task considered (by the teacher) relatively simple, some students either failed or did not dare to choose a yes/no answer. At the same time, the bar diagram allows the students to see the answers given by the other classmates, so to get a first feedback on one’s own answer (it is or it is not in the main stream).

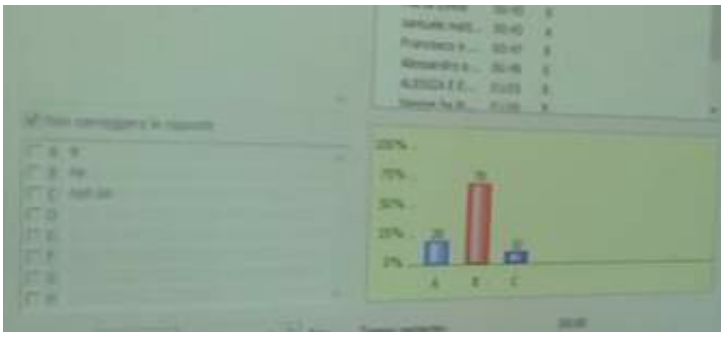

Fig 2. The results of the instant poll on the meaning of text and formula, shared in a whiteboard so to foster discussion

By means of the class discussion (strategy 2, “Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding”), the students are asked to express their reasoning behind their choice, and this constitutes a first reflective moment for them. Students may also profit from hearing their mates arguments, and so compare their own reasoning to others' (strategy 4, "Activating students as instructional resources for one another").

These two short examples illustrate how it is possible to apply the FA framework to give a picture at a certain moment of a technology-based formative assessment lesson and to realize the dynamics that occurs when the teacher and the students become aware of their work.

The last cell of the table crossing the student's viewpoint and the questions "Where the learner is right now?" and "How to get there?" concerns the possibilities to activating students as the owners of their own learning, which can be related to an approach of meta cognition.

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REFERENCES

Black, P., & Williams, D. (2009). Developing the theory of formative assessment. *Educational Assessment, Evaluation and Accountability,* 21(1), 5-31. [n/a]. 10.1007/s11092-008-9068-5.

Douady, R. (1986). Jeux de cadres et dialectique outil-objet. *Recherches en didactique des mathématiques*, *7*(0), 2.

Ramaprasad, A. (1983). On the definition of feedback. Behavioral Science, 28, 4–13.

Wiliam, D., & Thompson, M. (2007). Integrating assessment with instruction: what will it take to make it work? In C. A. Dwyer (Ed.) *The future of assessment: shaping teaching and learning,* 53-82. Mahwah, NJ: Lawrence Erlbaum Associates.

Annex

	Where the learner is going?	Where the learner is right now?	How to get there?
Teacher	Clarifying learning intentions and criteria for success	Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding	Providing feedback that moves learners forward
Peer	Understanding and sharing learning intentions and criteria for success	Activating students as instructional resources for one another	
Learner	Understanding learning intentions and criteria for success	Activating students as the owners of their own learning	

Aspects of Formative assessment (Wiliam & Thompson, 2007, as in Black & Wiliam, 2009, p. 8)

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Assessment in mathematics education: Inevitable, but resource or obstacle? Different assessment discourses in mathematics

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Abstract: In this paper, four assessment cultures (discourses) are presented where the assessment may provide with obstacles, or, resources, for the learning and engagement in mathematics. Assessment is in this text understood as broadly encompassing tests as well as teacher feedback in day-to-day mathematics teaching and learning, The four presented discourses were construed and adopted on a total in research in 30 mathematics classrooms with students in the ages of 7-16 years. The four discourses are « do it quick and do it right », « Anything goes », Openness to mathematics », and « Reasoning takes time ». The first two assessment discourses hold various obstacles for students' learning and engagement in mathematics, whereas the two last ones may constitute resources for students learning and active agency in mathematics. A claim made in the paper is that the four presented discourses offer teachers, students, and decision makers means to grasp essential aspects of assessment practices in mathematics classrooms including testing practices.

Résumé : Dans cet article, quatre cultures d'évaluation (les discours) sont présentés lorsque l'évaluation peut fournir des obstacles, ou, les ressources pour l'apprentissage et l'engagement en mathématiques. L'évaluation est dans ce texte comprend aussi les tests ainsi que des évaluations des enseignants dans l'enseignement et l'apprentissage des mathématiques au jour le jour. Les quatre discours présentés ont été interprété et adopté sur un total de recherche dans 30 classes de mathématiques avec les élèves dans les âges de 7 -16 ans. Les quatre discours sont «faire rapidement et le faire bien», «Tout va», « l'ouverture aux mathématiques », et« Raisonnement prend du temps ». Les deux premiers discours d'évaluation tiennent divers obstacles pour l'apprentissage et l'engagement des élèves en mathématiques, tandis que les deux derniers peuvent constituer des ressources pour les étudiants comme acteur en mathématiques. Les quatre discours présentés offrent aux enseignants, les étudiants et les décideurs les moyens de saisir les aspects essentiels de pratiques d'évaluation dans les cours de mathématiques, y compris les pratiques d'evaluation.

Assessment and feedback in mathematics classrooms

In this contribution to the round table at CIEAEM67 the focus is on assessment as a key notion relevant for problematizing all students' possibilities to getting the opportunity to learn mathematics. When doing this I point at assessment cultures (discourses) where the assessment may provide with obstacles, or, resources, for the learning and engagement in mathematics. Assessment is in this text understood as broadly encompassing tests as well as teacher feedback in day-to-day mathematics teaching and learning (Björklund Boistrup, 2015b; Tunstall & Gipps, 1996). Feedback is then interpreted as conveying the teacher's assessments to the student, through words (for example "Well done") and/or body movements (for example a nod with a smile). Drawing on Ball et al (2012) and Foucault (2003; 2008) I discuss assessment as part of governings within the system of school. On one hand any mathematics classroom itself is immersed in a political context with decisions made on local and national levels, which have a direct, and governing, impact on the teaching and learning (Valero, 2004). On the other hand there are structural, often implicit, factors that contribute to how students are governed, and hence invited, or not invited, into mathematics (Straehler-Pohl & Gellert, 2013).

One example of a discussion of assessment and its consequences within the field of mathematics education is Morgan's (2000) critique noting mainstream traditions of mathematics assessment research. Morgan emphasised research that adopts a social perspective, arguing that a main concern of research from a social perspective is to understand how assessment works in mathematics classrooms and more broadly in education systems. As I see it, one consequence of this reasoning is that it is essential to discuss the effects of assessments within all kinds of mathematics education, both from the perspective of teacher-student communications but to also bring in the broader political context in the analysis. Today, many years after Morgan's text, there still seem to be a rather modest interest in assessment in mathematics education research addressing policy and political matters as part of the research (Björklund Boistrup, 2015a). Words like assessment and evaluation may be mentioned but often these practices are taken for granted as a non-problematic part of education. In this paper I present assessment discourses which can constitute analytical tools for investigating how students are assessed, and invited – or not –, into mathematics education. The four presented discourses were firstly construed from five mathematics classrooms in grade four (Björklund Boistrup, 2010). After that they have been interpreted from more than 25 mathematics classrooms with students in the ages 7-16, and have been adopted in action research projects with mathematics teachers (Björklund Boistrup & Samuelsson, in preparation).

Construal of assessment discourses in mathematics

The term discourse is adopted drawing on Foucault (1993, 2003). A discourse is viewed as part of the institution of school. It goes beyond a particular mathematics classroom communication (taken here in a multimodal sense according to Van Leeuwen, 2005) and, consequently, the discourses presented in this text are possible to interpret from many mathematics classrooms. For the people who are part of a discursive practice, like teachers and students, the "rules" of the discourses affect how it is possible to act and what is possible to communicate (Foucault, 1993; 2003). An inspiration for adopting discourse as something smaller than entire disciplines is Walkerdine (1988) who construed a "testing discourse" where the teacher posed questions to which she already knew the answer.

All discourses were construed drawing on three initial analyses:

- Analysis of assessment acts (feedback) in mathematics classroom communication between teachers and students (described in Björklund Boistrup, 2010, chapter 5). In what direction is the feedback – from teacher to student, and/or vice versa? What directions – (dis)approving, (dis)agreeing/recognising, (dis)interest/(dis)engagement, checking, guiding, challenging – are mainly present?
- Analysis of the focuses of the feedback (Björklund Boistrup, 2010, chapter 6). Is it about the student as a person, no-mathematical procedures or mathematical processes? What processes are present, for example knowing mathematical facts, practicing/routine, reasoning/arguing, defining/describing, inquiring/problem-solving?
- Analysis of communicative resources including artefacts part of the assessment acts (Björklund Boistrup, 2010, chapter 7). What roles do different resources play in the assessment acts? How are communicative resources promoted or restricted? How are open questions and/or silences present in teacher-student communication?

Four assessment discourses in mathematics education

The first discourse, "Do it quick and do it right" has similarities to a traditional discourse of mathematics education described in the literature where the main "rule" is that the work should be done quickly and what is counted is whether an answer is right or not. The teacher's feedback focuses on procedures with non or limited mathematical content. Feedback in this discourse typically focuses on whether an answer is mathematically correct or not, instead of why and how the answer may be counted as mathematically relevant. Another typical feedback focus concerns how many items from the textbook the student has accomplished. The affordances for students to be invited to learn mathematics are limited since they are not really invited to engage in any aspect of mathematics through the feedback. Looking at the discourse from a multimodal approach, it may be possible to construe in writings when a teacher's feedback on a test is focused on the number of correct answers, for example when a teacher writes 11/21 (11 points out of 21). Here it is important to keep in mind that the items on the test may well be mathematically rich and also inviting to the students. What is analysed here is mainly the subsequent feedback. In speech, teacher's feedback where this discourse is construed can be really short, along with body movements, and describe whether the student's work is correct or not or whether the student is doing the "right" thing. The affordances for students' learning in this discourse are low.

The second discourse, "Anything goes", is more of the opposite to the first discourse and a discourse where students' performances, which can be regarded as mathematically inappropriate, are left unchallenged. There is not much articulated feedback apart from general approval. There is a presence of open questions, but challenges are not common. There are no critical discussions about students' solutions, and wrong answers can be left unchallenged. The students are invited by the teacher to use whatever communicative resources they want, without any considerations by the teacher or the students on what resources that have most affordances for their learning at that specific occasion. Because the teacher values the students' performance so often, the teacher, at the same time, takes the role as the main agent, as "the one that is evaluating". Sometimes the teacher takes a more passive role in the discourse. S/he then does not interfere with students' reasoning even though something wrong is demonstrated. The affordances for students' learning in this discourse are low.

The third discourse, "Openness with mathematics", has more of an open focus on mathematical processes. In this discourse the feedback goes both in the direction from teacher to student and vice versa. Occasionally, goals for the learning are present. Quite often the questions posed are open. The teacher and student often show interest in mathematical processes, and there is also an awareness of students' alternative interpretations of tasks. Sometimes the student is challenged with respect to her/his continued learning. The focus is mostly on mathematical processes and sometimes on the student's own reflection of her/his own learning. "Wrong" answers are here used as starting points for discussions, and it is always clear what can be considered mathematically correct. Various kinds of feedback from teacher to student are often communicated through questions. Different semiotic resources are acknowledged and at times the teacher promotes, whilst at other times restricts, the use of semiotic resources dependent upon the meaning making and learning process demonstrated by the student(s). This seems to be in order to serve the continuing learning process. The teacher and students communicate in longer utterances, but not more than a few utterances each time. In this discourse, there are considered to be affordances for students' active agency and learning of mathematics.

Finally, the fourth discourse, "Reasoning takes time", takes the characteristics of "Openness with mathematics" one step further with a slower pace and an emphasis on mathematics processes such as reasoning/arguing, inquiring/problem-solving, and defining/describing. In this discourse assessments take place in both directions between teacher and student. There are often instances of recognition of the students' demonstrated knowing, which are sometimes in relation to stated goals, and the questions posed are mostly open ones. At times feedback as interest and engagement are communicated by the teacher to the student and vice versa. The students are often challenged towards new learning with the focus mainly on mathematical processes and the students' reflections on her/his own learning. Here, most emphasis is on the processes inquiring/problem-solving, reasoning/arguing, defining/describing, and, occasionally, constructing/creating. Different communicative resources are acknowledged, and the use of semiotic resources can also be promoted or restricted when serving a certain process. In this discourse, silences in teacher-student interactions are common, and the possibility (for both teacher and student) to be silent serves the mathematics focus. Various kinds of feedback from teacher to student are often communicated, sometimes through open questions. Both the teacher and student can be active for longer periods of time. In this discourse as well, the affordances for students to take active agency are high. The possibility to be quiet and think for a while promotes this potential agency. Similarly, the affordances for students' learning of mathematics are high and include a wide range of mathematics processes.

Concluding discussion

To summarise, the first two assessment discourses hold various obstacles for students' learning and engagement in mathematics, whereas the two last ones may constitute resources for students learning and active agency in mathematics. A point I want to make is that teaching without any assessment is not possible, since there always will be feedback taking place, and the kind of assessment that different students encounter may serve as a gatekeeper to the subject of mathematics.

The discourses presented here, with the connections between assessment acts, focuses in the mathematics classroom, and roles of communicative resources, offer teachers, students, and decision makers means to grasp essential aspects of assessment practices in mathematics classrooms including testing practices. There is positive power in an increased awareness of discourses like these, not the least since different students most likely encounter different assessment discourses in school, and hence different learning opportunity. For teachers, the discourses can be a starting point for identifying how various assessment discourse practices take place in the classroom. In such an activity, the implicitness of assessment practices is made more explicit. One example here is how the discourse “Do it quick and do it right” is possible to construe in the classroom, and possibly contrary to the teacher's original plan. The reason for this can be governings within the system of school, for example, through a strong tradition within mathematics teaching and/or through demands from municipalities, where a dominant discourse such as “Do it quick and do it right” can be construed. Discourses like these can be a starting point for discussions about assessment practices and what kind of governings they hold among teachers and school heads, and among people responsible on the municipal and national level.

REFERENCES

Ball, S., Maguire, M., Braun, A., Perryman,. J., & Hoskins, K. (2012). Assessment technologies in schools: ‘deliverology’ and the ‘play of dominations’. *Research Papers in Education. 27*(5). 513-533.

Björklund Boistrup, L. (2010). *Assessment discourses in mathematics education: A multimodal social semiotic study*. PhD thesis. Stockholm University: Stockholm.

Björklund Boistrup, L. (2015a). Assessment, governmentality, and school mathematics. Paper presented at *The disorder of mathematics education. Working towards research of social and political relevance. Conference by invitation at Freie Universität. Berlin. January 15-18, 2015*.

Björklund Boistrup, L. (2015b). Governing through implicit and explicit assessment acts: Multimodality in mathematics classrooms. I M. Hamilton, R. Heydon, K. Hibbert, & R. Stooke (Red.), *Negotiating spaces for literacy learning: Multimodality and Governmentality*. London: Bloomsbury Publishing.

Foucault, M. (1993). *Diskursen ordning* [The order of the discourse]. Stockholm, Sweden: Brutus Östlings Bokförlag Symposium.

Foucault, M. (2003). *Övervakning och straff* [Discipline and Punish]. Lund, Sweden: Arkiv förlag.

Foucault, M. (2008). Regementalitet [Governmentality]. In T. Götselius & U. Olsson (Eds.), *Diskursernas kamp* [The battle between discources] (pp. xx-xx). Stockholm, Sweden: Brutus

Östlings Bokförlag Symposium.

Morgan, C. (2000). Better assessment in mathematics education? A social perspective. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp 225-242). Westport, CT: Ablex Publishing.

Straehler-Pohl, H., & Gellert, U. (2013). Towards a Bernsteinian language of description for mathematics classroom discourse. *British Journal of Sociology of Education. 34*(3). 313-332.

Tunstall, P., & Gipps, C. (1996). Teacher feedback to young children in formative assessment: a typology. *British Educational Research Journal, 22*(4), 389-404.

Valero, P. (2004). Socio-political perspectives on mathematics education. In P. Valero & R. Zevenbergen (Eds.), *Researching the socio-political dimensions of mathematics education: Issues of power in theory and methodology* (pp. 5-24). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Van Leeuwen, T. (2005). *Introducing social semiotics.* London, UK: Routledge.

Walkerdine, V. (1988). *The Mastery of reason. Cognitive development and the production of rationality.* London: Routledge.

The SNV (INVALSI) experience

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Abstract : In this paper we would like to show how the items and the results of the Italian national standardized tests can be considered a resource, not only for the Ministry, but also for schools, teachers and researchers. After a brief description of some peculiar aspects of the Italian Assessment System, we show an example of SNV items. Through the discussion of the item goals, we show how the standardized tests could be considered a resource for teachers and researchers.

Résumé: Dans cet article, nous voulons montrer comment les éléments et les résultats des tests standardisés nationaux italiens peuvent être considérés comme une ressource, non seulement pour le ministère, mais aussi pour les écoles, les enseignants et les chercheurs. Après une brève description de certains aspects particuliers du système d'évaluation italienne, nous montrons un exemple des éléments de la SNV. Grâce à la discussion des objectifs de l'exemple, nous montrons comment les tests standardisés pourraient être considérés comme une ressource pour les enseignants et les chercheurs.

The Italian assessment System (SNV)

The Italian Assessment System (SNV: Servizio Nazionale di Valutazione[2]) started its work in 2008 through annual surveys conducted by the National Evaluation Institute for the School System (INVALSI) at different school grades. The INVALSI[3]develops standardised national tests to assess pupils' reading comprehension, grammatical knowledge and mathematics competency. Tests are administered at the end of the school year in grades 2-5-8-10. The results of a national sample are annually reported and they are public as well as the test items. From 2008, only for grade 8, the standardised SNV test is part of the national final examination, which is carried out at the end of middle school and it is organised by the school. Therefore, the SNV test contributes to the final assessment of the students.

SNV Framework

SNV investigations aim at taking a snapshot of schooling as a whole: in other words, it is an evaluation of the effectiveness of education provided by Italian schools. Currently, SNV tests are administered every year to all students in grade 2-5-8-10 (grade 6 was involved until 2010). The results of a national sample are annually reported, stratified by regions and disaggregated by gender, citizenship and regularity of schooling. The results of each school are delivered to the principal at the beginning of the following schoolyear. The preparation of the SNV items is performed in two steps. A first set of items is prepared by in-service teachers of all grades, who also classify them according to the question intent and the links with the National Guidelines[4]. Subsequently, the SNV National Working Group builds the test by selecting items, so that the test can assess the different aspects reported in the National Guidelines. The framework adopted by SNV assessment is strictly connected to the National Guidelines and includes aspects of mathematical modelling as in PISA research (Arzarello et al., 2015), and aspects of mathematics as a body of knowledge logically consistent and systematically structured, characterised by a strong cultural unity (Anichini et al.

[2] http://www.invalsi.it/areaprove/index.php
[3] http://www.invalsi.it/invalsi/index.php
[4] http://www.indicazioninazionali.it/

2004).Another important reference is the UMI-CIIM curriculum "Mathematics for the citizen"[5], which is based on results of mathematics educational research and has deeply influenced the last formulation of the National Guidelines.

The SNV Framework defines what type of mathematics is assessed with the SNV tests and how it is evaluated. It identifies two dimensions along which the questions are built:

- the mathematical content, divided into four major areas: Numbers, Space and Figures, Relations and Functions, Data and Forecasts;
- the link with the National Guidelines: each item is linked to a specific final objective.

This subdivision of content into four main areas is now shared at the international level: in PISA there are four content categories (Quantity, Space and Shape, Change and Relationships, Uncertainty and Data) and in TIMSS there are four content domains (Number, Geometry, Algebra, Data and Chance).

The SNV tests differ from PISA or TIMSS surveys not only for the frequency (annual vs. triennial), for the type of tested population (census vs. sample), for the chosen population (grade-based vs. age-based students for PISA), for the links with a National curriculum, but above all for the goals. As a matter of fact, the SNV test results aim at providing the Ministry with a national benchmark for the assessment of the Italian students at different grade levels, taking into account the national guidelines.

An Example from the SNV test in grade 8

The following example highlights the peculiar features of the items in the SNV tests.

The teacher asks: "Can an even number greater than 2 always be written as the sum of two different odd numbers?"

Below are the answers of four students.

Who has given the correct answer, justifying it properly?

Antonio:	Yes, because the sum of two odd numbers is an even number.	(44.0%)
Barbara:	No, because 6 = 4 + 2.	(6.4%)
Carlo:	Yes, because I can write it as the odd number that precedes it, plus 1.	(34%)*
Daniela:	No, because every even number can be written as a sum of two equal numbers.	(14.0%)

*correct

Figure 1: Item from SNV (2011-2012) in Grade 8 (indicating the percentage for each option)

The item in Figure 1 would like to assess a competence identified as a goal in the National Guidelines (i.e. to recognize and produce correct arguments based on theoretical knowledge). It arises in the context of the latest Italian research in mathematics education (Mariotti 2006; Boero et al., 2007). It somehow condenses the results of a wide research about the approach to argumentation and proof in mathematics, even with young students (Garuti & Boero, 1994). In this item, Grade 8 students are required to select arguments about the validity or non-validity of a statement: they must

[5] http://www.umi-ciim.it/materiali-umi-ciim/primo-ciclo/ ; http://www.umi-ciim.it/materiali-umi-ciim/secondo-ciclo/

choose the right answer with the correct justification. This item requires that the student understands that every even number can be written as (2n −1) +1. When it comes to number 2 the formula still holds, but the sum is between two equal odd numbers.

The chosen options correspond to the more frequently observed behaviours of students. They all involve students' understanding and exploration of the statement. In particular, option A, which had 44% of the answers, highlights a typical mistake. To answer the question it is not relevant that the sum of two odd numbers is always even. We consider questions of this type very important because:

- within a standardized test, they assess mathematical skills that are typical of the cultural aspect of mathematics (argumentation and proof);
- this kind of item shows the possibility of using algebra as a tool for supporting reasoning and consequently they push teachers towards a change of their practices as a result of the discussions they have in their schools about the nature of the SVN tests.

This type of item could be an important stimulus for teachers to reflect on, to consider a new approach to the culture of theorems at school, and to challenge standard teaching practices. Usually, in Italy (and possibly also in other countries) the teacher asks the students to understand and repeat proofs of statements rather than to produce conjectures themselves or to proof on their own. Arguing and proving activities are not generally common in Italy in the first years of secondary school (lower and upper), but we think that an early approach to theoretical thinking is important (Garuti & Boero, 1994). This item and its results may represent a kind of script for the construction of classroom activities on these aspects of mathematics education. In fact.

SNV tests as a resource for teachers and researchers

Being aware that the national assessment tests are a tool for the Ministry to give comparative information at the national level on students' learning, in this Round Table we would like to highlight how the analysis of SNV tests can also be a resource for teachers and for researchers. The example analysed before could be an interesting subject of study and reflection for teachers and researchers also within teacher education programs. Regarding the research, we quote an example from the project "Ideas for the Research"[6] funded by INVALSI. Starting in 2014 an educational study in mathematics on the SNV test was carried out. Some results of this study were presented at CERME9[7] (Branchetti et al, in press), at PME39 (Ferretti et al. 2014) and also at CIEAM (Lemmo et al, in these proceedings). The study proposes an integrated analysis, qualitative and quantitative, which can provide input for reflection on national assessment tests. The goal of the study is to build analytical tools to select "chains of items" (i.e. questions administered in successive levels that can be connected by qualitative and quantitative analysis) that could identify situations of difficulties related to specific topics. For example, the comparison and ordering of rational numbers, the management of their different representations, etc. The theoretical lenses developed in the first part of the research were shared with secondary school teachers. The focus was on structures and contents of the items and on the students' possible answers.

In working with teachers, the researchers try to answer these questions: Which mathematical content knowledge is involved? What do the items assess? How do the students answer? Where do they make mistakes? Which could be the reasons for their behaviours ?

This is an ongoing research, but it seems that the SNV items can become object of educational activities both in the classroom and in teacher education programs. It is clear that SNV tests cannot assess many important processes that can be mastered in a classroom by teachers, but teachers can use the information given by these standardized tests in order to identify possible students' mistakes

6 http://www.invalsi.it/invalsi/ri/sis/app_met.php
7 http://www.cerme9.org/

and misconceptions in tasks designed by taking into account the National Guidelines. The students' possible strategies and mistakes can be object of discussion among teachers and researchers in a teacher education activity, arguing why and how these items and the answers may be mathematically relevant.

REFERENCES

Anichini, G., Arzarello F., Ciarrapico, L., Robutti, O. EDS. (2004). New mathematical standards for the school from 5 through 18 years,. *The curriculum of mathematics from 6 to 19 years, on behalf of UMI-CIIM, MIUR* (edition for ICME 10). BOLOGNA: UMI (ITALY).

Arzarello, F., Garuti, R., Ricci, R., (2015). The impact of PISA Studies on the Italian National Assessment System. In K. Stacey & R. Turner (Eds.), *Assessing Mathematica Literacy. The PISA experience*. Springer International Publishing Switzerland.

Boero, P., Garuti, R. & Lemut, E. (2007). Approaching theorems in grade VIII. In P. Boero (Ed.), *Theorems in school*, (pp. 250–264). Rotterdam, The Netherlands: Sense Publisher.

Branchetti, L., Ferretti, F, Lemmo, L., Maffia, A., Martignone. F., Matteucci. M. & Mignani, F. (in press). A longitudinal analysis of the Italian national standardized mathematics tests. *Proceedings of the 9th Conference of European Research in Mathematics Education*, 4-8 February, 2015, Prague.

Ferretti F., Lemmo, A. & Maffia, A. (2015). "Half of something": how students talk about rationals. *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education*, (vol. 1, p.159). Hobart, Australia: PME.

Garuti, R. & Boero, P. (1994). Mathematical modelling of the elongation of a spring: Given a double length spring. In J. P. da Ponte, & J. F. Matos (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education,* (vol. II, pp.384–391). Lisbon, Portugal: Department of Education, University of Lisbon.

Lemmo, A. Branchetti, L. Ferretti, F. Maffia, A. & Martignone, F.(in press). Students' difficulties dealing with number line: a qualitative analysis of a question from national standardized assessment. *Proceeding of CIEAEM67*, 20-24 July 2015, Aosta.

Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutierrez, & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education,* (pp. 173–204). Rotterdam, The Netherlands: Sense Publisher.

WORKING GROUP 1 / GROUP DE TRAVAIL 1

Mathematical content and curriculum development /
Contenu mathématique et développement du curriculum

Working Group 1 / Group de Travail 1

Mathematical content and curriculum development / Contenu mathématique et développement du curriculum

Marcelo Bairral, Sixto Romero and Ana Serradó

Discussion document challenged presenters of the Working Group 1 to analyse the complexity of the teaching and learning of mathematics through the double-face of resources and obstacles of the curriculum development of mathematical knowledge.
Although we propose an initial analysis of the arguments presented through the traditional topics to understand if there is any specific content that needs a special attention, these were only a resource for the discussants of the group to establish as a proposition the need of making cross-curricular connections.

In the topic of arithmetic, three papers complemented the view of number line, fraction and rational number. The analysis of the Italian national standardized assessment in Primary Education led to reflect on the difficulties students face dealing with the number line (Lemmo et al.). Those difficulties could come from the relationship that exists between the concepts of number line and measurement. Mengual et al. analysed the epistemological obstacles related to the arithmetization of measure in textbooks. Becoming these textbooks also a didactical obstacle in the teaching and learning process. Robotti recognised these difficulties and proposed the use of the number line to construct the notion of fraction through the use of colours. There were recognised by the assistants the main ideas of Cousenaire (CIEAEM1957), when presenting her rules identified with colours as a resource for constructing natural numbers.

On the analysis of the difficulties of the number line and the different conceptualizations of the notions of fraction, always remained the idea of how those fractions can become an obstacle to construct the notion of rational number. Rottoli (Alessando et al.) presented theoretically these reflections, and he proposed a new paradigm to understand the mathematization process.

New proposals to understand the mathematization process were presented, exemplified and discussed by Romero, when introducing the problem solving as a tool for mathematical modelling. The modelling was presented as a resource to grow from the real life, although some obstacles can emerge on these grow. Serradó presented a categorization of the epistemological, didactical and ontogenic obstacles, as described by Brousseau (1997), that can emerge in a probabilistic modelling process. In particular, the didactical obstacle of the need of circularity between the theory-driven approach and data-driven approach can also be understood as a resource when designing classroom activities. In these sense, Ginovart presented a classroom activity for tertiary Byosystems Engineering students to build diverse models with the help of the computer. In this case the use of the technology become a tool to improve the growth of students understanding of modelling process.

This was not the only paper that presented the use of digital technology as a resource to enhance the learning of students' skills and content knowledge: variable (Zimoch), geometric foci (Ferrarello et al.), rotation (Bairral, et al.), similarity, homothety and Thales Theorem (Gualdron). Different technological tools were presented Scratch, Mathlab, GeoGebra, GeoGebraTouch and Maple, that were contrasted with outdate digital software such as Logo. The word "use" of technology had a broad sense that needed to be clarified in different senses. For instance, touching is more that an action; it is also an aim for communicating, reasoning and thinking. Programing is more than a process; it is a resource to reason on mathematical concepts and to develop algorithmic thinking. Geometrical transformation is more than a tool; it can be seen as a mathematical object to grow on. This grow was categorized through Van Hiele model of geometric reasoning, concluding that GeoGebra help to conjecture but it does not really aims improving proving skills.

As, consequence, these technologies were considered as a resource to enhance teaching and learning of participant teachers during all the schooling. Although discussions on the group made emerge different unsolved questions as: what to do first-paper or technology? What could be different in each approach? When it is an obstacle or a resource? What opportunities does the technology give to introduce, teach or enhance the reason and thinking of concepts?
The duality pen and paper or technology can not be considered as a dichotomy, it has to be analysed through the view that one enhances the stimulating use of the other when promoting student thinking and teacher professional development. Furthermore, its different uses promote the student integration of some concepts and processes, and developing new conceptual structures.
This is not a neutral proposition recognised by the participants for new curriculums that should aim to integrate technology, make cross-curricular connections through different levels of schooling, though different topics, and to real-life. And, it means a new opportunity for Geometry, Statistics and Probability in the intended, enacted and learned curriculum.

As a final proposition of the working group 1 was that between researchers and teachers have to be developed new forms of communication to reflect on the teachers use of technology to enhance the relationships between mathematical content that make sense of the world.

High school students rotating shapes in geogebra with touchscreen

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Abstract: This research investigates aspects of students' cognition during the process of solving tasks using GeoGebra with single touch. We designed teaching experiments with Brazilian High School students. In this paper we focused on strategies used by students to solve the proposed tasks where they applied the concept of rotation or other manipulation related with plan transformation. Based on previous research we identified students using one or two fingers to solve the tasks. Even without previous instruction concerning rotation and reflection students applied these concepts naturally, sometimes even doing composition between them.

Résumé: Cette recherche s'intéresse aux connaissances des élèves dans un processus de résolution de tâches utilisant GeoGebra avec "simple touche". Nous avons élaboré des expérimentations avec des élèves brésiliens de lycée. Dans cet article, nous nous intéressons plus particulièrement aux stratégies utilisées par les élèves pour résoudre les tâches proposées, dans lesquelles ils utilisent les concepts de rotation ou d'autres transformations du plan. En nous appuyant sur des recherches précédentes nous avons repéré les élèves qui utilisaient un ou deux doigts pour résoudre ces tâches. Même sans instructions préalables sur les rotations et les réflexions, les élèves appliquent spontanément ces concepts, parfois même en les composant.

Introduction

The emergence of multi-touch devices is providing new insights and challenges in mathematics learning and instruction. For instance, rotating and other kind of gyrating movements on screen often take place due the freedom of handling on touchscreen device. Is this paper we will illustrate some strategies used by Brazilian High School students applying rotation concept to solve task on GeoGebra with touch.

Since students and teachers are becoming increasingly familiar with multi-touch technology and manipulation, this kind of research addresses issues on CIEAEM67 subtheme 3 (Classroom practices and other learning spaces). We believe that looking for types of manipulation can provide new epistemological insights for geometrical conceptualizing in touchscreen devices. Particularly, identifying in which geometric construction the manipulation with more than 2 fingers occurs may be an interesting issue on plan transformation.

Interaction and performing rotation on touchscreen devices

Interaction through current mobile touchscreens basically occurs with the computer recognizing and tracking the location of the user's input within the display area. Most current tabletop interaction techniques rely on a three state model: contact-down, contact-move, and contact-up—more akin to mouse dragging (Tang et al., 2010).

We assume that touchscreen manipulation on mobile device is not cognitively the same as mouse clicks, those we often do in dynamic geometry environment (Arzarello et al. 2014), for instance, due to the simultaneity of motion in different elements (points, sides, angles, areas etc.) from one picture (Bairral et al. 2015). Regarding the usage of single or multi touch fingers in previous research we observed that students had been manipulating the figures using mainly one or two fingers only (Tang et al. 2010). Since they worked in pairs sometimes they also shared fingers (for instance, 1 finger each) or hands to manipulate some figure, especially when the shape had more geometric objects or constructions.

Although rotating appeared few times, those appearances allow us to observe three different ways

1 Research granted by Capes (Ministry of Education, Brazil), Observatório da Educação Program.

of rotating on Geometric Constructer (GC) multitouch device (Arzarello et al. 2013, 2014): rotation using one finger; rotation using two fingers, but with one fixed finger, and rotation with two fingers, with both in movement, as we illustrate in the next Chart.

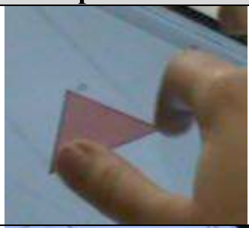
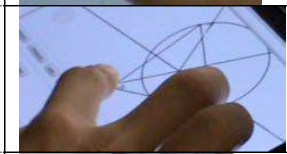
Rotation types using GC	**Example**	**Geometric process**
Rotate using one finger		Student making construction and moving the selected point with one finger.
Rotate using two fingers but with one fixed finger		Student keeps one finger fixed (from the left), moves the middle and observes what happens.
Rotate with two fingers (both in movement)		*S*tudent selects and rotates the shape in two points.

Chart 1: Example of students' rotating on GC (Arzarello et al. 2014, p. 46)

Although the first two types seem the same mathematically, we think cognitively they can provide different insights in terms of the use of the fingers. Conceptually, in order to rotate one shape we need to determine before in each point (the center of rotation) and with the use of two fingers the decision could have not been done beforehand. Or, at least this was not explicit for touch users. In that sense it can bring new conceptual aspects for the way we deal with rotation and, so far, in the same direction, the last observed way of rotating (two fingers in movement) we agree with Sinclair and Pimm (2014) that these types of manipulations involve contact with a screen and they perform an action. Finally, since mobile touchscreen devices provide more freedom on manipulation, that particular way of rotation may serve as an important function of grounding mathematical ideas in bodily form and they may also communicate spatial and relational concepts (Boncoddo et al. (2013) in the field of plan transformation.
Manipulation on screen with more than 2 fingers may be an interesting and challenging issue in future mathematic education research with touchscreen mobile devices. As we said before, due to the nature of software GC (multi-touch) and of the geometrical task (Varignon Theorem) previously proposed we identified that rotate manipulation occurred few times. To solve the task students haven't applied the rotation concept or other concept related with plan transformations. In our current analysis we are providing tasks where students have to apply the concept of rotation. In this paper we address results from students dealing with GeoGebra touch to solve the proposed task.

Methodological aspects of the study

We are conducting teaching experiments with High School students (15-17 years old) at *Instituto de Educação Rangel Pestana* (Nova Iguaçu, Rio de Janeiro, Brazil). All of them had no previous experience with dynamic geometry environment (DGE) and had no lesson concerning plan transformation. In each session the students worked out on proposed activities with GeoGebra app as describe below.

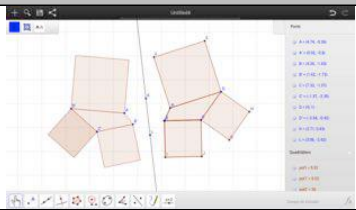
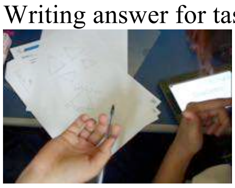
Software	Interface	Device features
GeoGebra touch		-Runs and allows save constructions *off-line* -Version used on the analyzed task in this paper: 4.3 -Single touch only

Chart 2: GeoGebra touchscreen features

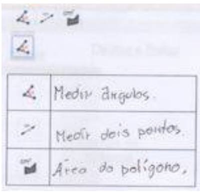
Each session was 2 hours long and in each one the students did three activities like the one illustrated above.

The analysis process was mainly based on the (1) videotapes of students working on the software, (2) written answers for each task and (3) the use of the shift of icon.

Writing answer for tasks



Sheet of icon[2]



Video[3]

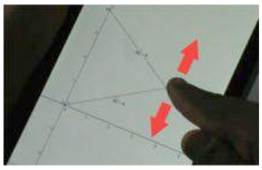
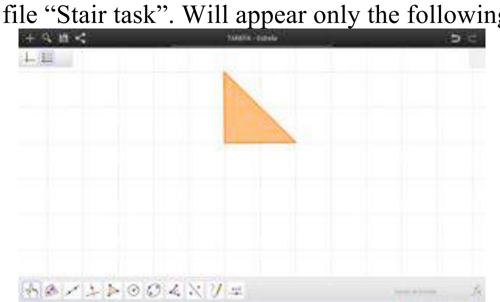
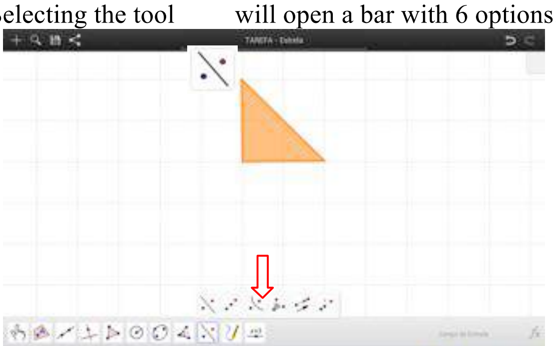


Chart 3: Data collection (three main sources)

Open the file "Stair task". Will appear only the following triangle:



Selecting the tool will open a bar with 6 options:



Elaborate a strategy for construct the following picture using only the tools

[2] Sheet containing all GeoGebra icons. Each student had his/she own sheet and every TE they filled it and reviewed it as they wanted.

[3] The red arrows indicate some motion on screen done by students.

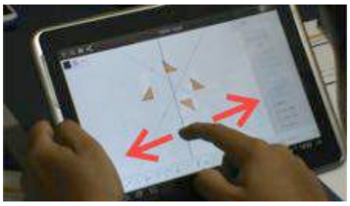
Chart 4: Stair task

We observed all the students' manipulations on the screen and identified the type of actions (tap, hold, drag, flick, free and rotate). In this paper we will focus on the student's strategies to solve the tasks, which applies the rotation concept, as we illustrate below.

Results

In the following pictures and timing interval we illustrate and describe student Adriano dealing with the task on GeoGebra by using single touch. He starts (12:14) constructing lines and reflecting triangles relating with them. Moving the line (27:34) he tries to locate the triangle to become coincident, but since he has no success he decides to restart the construction.

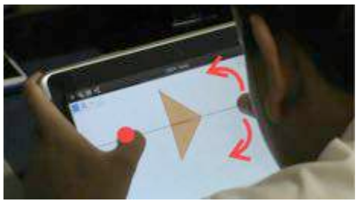
12:14



Using reflection tool and moving the line trying to adjust the reflected triangle

27:34

28:14



Restarting the construction, observing and adjusting

While observing and adjusting it's interesting to highlight how he keeps his left finger under some point on the line and makes the rotation of the line using his right finger. In the next figures we observe Adriano constructing lines and using reflection to move the triangles.

28:28-28:33

Constructing line and using reflection tool

35:51

Using reflection tool and line by two points afterword reflects the triangle

38:16

Applying rotation motion

Student constructed line (28:28) and used reflection tool (28:33) to move the triangle. Afterword constructed other lines and repeated the process of reflection the triangles (35:51). In the next three pictures we illustrated Adriano applying rotation motion keeping on finger on the line. Particularly, at 38:17 he makes a rotate motion with his finger to move the triangle and complete the shape (38:18).

Rotate touch

38:16

38:17

38:18

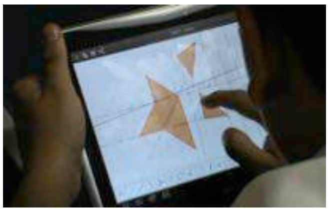
The next following pictures show how Adriano was dealing with his constructions to put the triangle (38:18) in a right position according to the task.

38:19

38:22

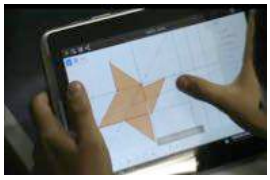
After motion realize that triangle is inverted

49:48



Reflecting triangle

51:39



Adjusting and finishing the construction

Using the same finger that Adriano was working with before he selects the line (38:19) and translate it in a way that the triangles become coincident. He created one more line and reflected the triangle (49:48). Afterwards he adjusted and finished the construction according the task statement.

Final remarks and future analysis

For solving a task, which involved the concept of rotation and using a device with single touch, we observed that students used their fingers – no more than two (Tang et al. 2010) – in the similar way that students did when dealing with software GC in a open task which did not apply the referred concept (Arzarello et al. 2014).

Although GC it is a device with multitouch, we decided to use, at this moment of our research, GeoGebra, which provides only single touch, due its stability and its possibility for working and saving constructions off-line. Our prior assumption was that single touch provided by GeoGebra would be a restriction on us to observe different ways of rotate manipulation on screen. However, even students without previous lessons concerning rotation or reflection they used those concepts naturally, sometimes isolated, or even doing composition between them.

Finally, since students were unacquainted with DGE the sheet of icon was didactically helpful for them. During each teaching experiment they had the opportunity to remember the functionality of the tool, review it and do new adding on the sheet. Throughout the sessions we observed they resorted the sheet as a source to realize the best tool to use in some task. As we observed students doing rotation and reflection into some shape we believe that looking for the types of manipulation can provide new epistemological insights for geometrical conceptualizing in classroom touchscreen devices. Our next step will be to use a multitouch device as GC.

REFERENCES

Arzarello, F., et al. (2014). Moving from dragging to touchscreen: geometrical learning with geometric dynamic software. Teaching Mathematics and its Applications 33(1): 39-51. doi: 10.1093/teamat/hru002

Arzarello, F., et al. (2013). Ways of manipulation touchscreen in one geometrical dynamic software. International Conference on Technology in Mathematics Teaching (ICTMT11), *Bari, Itália.*

Bairral, M., Assis, A. R., & Silva, B. C. da. (2015). Mãos em ação em dispositivos touchscreen na educação matemática. Seropédica: Edur.

Boncoddo, R., Williams, C., Pier, E., Walkington, C., Alibali, M., Nathan, M., Dogan, M.F. & Waala, J. (2013). Gesture as a Window to Justification and Proof. Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. M. C. S. Martinez, A. Chicago, IL, University of Illinois at Chicago: 229-236.

Hostetter, A. B. and M. W. Alibali (2008). Visible embodiment: Gestures as simulated action. Psychonomic Bulletin & Review 15(3): 495-514.

Sinclair, N., & Pimm, D. (2014). Number's subtle touch: expanding finger gnosis in the era of multi-touch technologies. Paper presented at the Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education, Vancouver, Canada.

Tang, A., Pahud, M., Carpendale, S., & Buxton, B. (2010). VisTACO: Visualizing Tabletop Collaboration. Paper presented at International Conference on Interactive Tabletops and Surfaces (ITS '10), Saarbrücken, Germany.

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Familiariser avec les nombres fractionnaires: ressources et obstacles

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Abstract: The results of teaching the fractions are not generally satisfactory. Among the many obstacles there are: the complex structure of the concept of rational number; the uniqueness of the "action scheme" that underlays teaching and learning fractions in classroom practice; the "bias" of the whole number. The three points that characterize proposal that we are experiencing in the third year of primary school, are the following ones: the universe of fractional numbers is a new universe, different from the universe of natural numbers already well-known to children; the comparison between two homogeneous quantities gives rise to an elementary and fundamental process of mathematization; the measure, defined as the comparison between quantity and whole, is an ordered pair of numbers that shows how many times the quantity and the whole contain the common unit respectively.

Résumé: Les résultats de l'enseignement des fractions ne sont pas généralement satisfaisants. Parmi les nombreux obstacles, il y a: la structure complexe de la notion de nombre rationnel; l'unicité du "schéma d'action" qui fonde l'enseignement et l'apprentissage des fractions dans la pratique en salle de classe; le «rappel» du nombre entier. Les trois points qui caractérisent notre proposition, que nous expérimentons dans la troisième année de l'école primaire, sont les suivantes: l'univers des nombres fractionnaires est un nouvel univers, différent de l'univers des nombres naturels déjà bien connus des enfants; la comparaison entre deux quantités homogènes engendre un processus élémentaire et fondamental de mathématisation; la mesure, définie comme la comparaison entre la quantité et l'entier, est un couple ordonné de nombres qui indique combien de fois la quantité et l'entier contiennent respectivement l'unité commune.

Introduction

Dans cette présentation, nous proposons une réflexion sur quelques activités qui actuellement sont en progression dans deux troisièmes classes de l'école primaire (enfants de 8/9 ans) et qui concernent l'introduction du concept de nombre fractionnaire.

A l'origine de notre proposition il y a cette constatation. Malgré les efforts depuis plus d'un demi-siècle dans la recherche et dans la pratique, les résultats de l'enseignement des nombres fractionnaires ne sont pas satisfaisants et les difficultés sont très répandues et persistantes.

Obstacles

Les obstacles qui sont à l'origine de ces difficultés sont nombreux. Nous en énumérons trois, selon nous, particulièrement significatifs.

1. La structure complexe des concepts de nombre rationnel et de nombre fractionnaire est certainement un obstacle : « Le nombre rationnel est un méga concept dans lequel nombreux filaments s'entrelacent » [Wagner (1976) dans Kieren]. Cette complexité est bien

représentée dans le schéma des cinq « sous-constructions de la construction du nombre rationnel » qui a été proposé par Kieren.

2. Autre obstacle : la recherche et la pratique didactique ont proposé des processus d'enseignement-apprentissage qui sont souvent fondés sur un unique « schéma d'action ». Cette approche implique des difficultés dans le transfert vers autres types de situation des connaissances acquises. Dans la pratique didactique, la situation utilisée dans la plupart des cas est celle de division associée à la sous-construction part-entier ; une situation dont désormais la littérature scientifique a largement confirmée l'efficacité limitée[Nunez & Bryant].

3. Un obstacle crucial est représenté par le choix très répandu d'introduire l'ensemble des nombres fractionnaires comme une extension de l'ensemble des nombres naturels. A notre avis, l'inefficacité de ce choix réside dans le fait que les trois caractéristiques « structurantes » (l'équivalence, l'ordre et les opérations) sont de natures entièrement différentes dans les deux ensembles.

   - Nunez et Bryant parlent de l'équivalence dans les rationnels comme extension de l'équivalence dans les naturels. Mais l'équivalence dans les naturels est reliée à l'équivalence entre les ensembles et cette dernière n'a rien à voir avec l'équivalence dans les nombres rationnels.
   - De l'autre coté, tandis que l'ordre des naturels est conséquence de la consécutivité, l'ordre des nombres fractionnaires est lié à la densité ; consécutivité et densité sont deux concepts très lointains et pas associables dans un processus d'extension immédiat.
   - En ce qui concerne les opérations, tout le monde sait que quelques-unes des erreurs les plus communes que les élèves commettent dans les opérations avec les nombres fractionnaires, résultent de l'extension inappropriée des procédures relatives aux opérations entre les nombres naturels.

Nous croyons que le chemin à parcourir pour éliminer, au moins partiellement, ces obstacles est de repenser l'approche de l'enseignement et de l'apprentissage des nombres fractionnaires. La nouvelle approche produirait avant tout un changement dans la façon dans laquelle les enseignants conçoivent les nombres fractionnaires ; seulement après ce changement, l'approche peut être transférée à la classe.

Notre proposition a une valeur générale ; elle se concrétise en activités concernant les enfants de l'école primaire. Précisément pour cette raison, elle n'est pas structurée dans un chemin spécifique d'enseignement / apprentissage mais plutôt elle vise à un processus de familiarisation visant à façonner l'environnement d'apprentissage.

Ressources

Dans notre proposition, il y a trois éléments principaux qui devraient fournir des ressources utiles pour surmonter les obstacles ci-dessus :

- l'univers des nombres fractionnaires comme nouvel univers ;
- une mathématisation élémentaire et fondamentale ;
- le « dialogisme » entre les activités et les situations.

L'univers des nombres fractionnaires est un nouvel univers

L'univers des nombres fractionnaires appartient à l'univers des couples de nombres entiers : « Les fractions sont utilisées à l'école primaire pour représenter des quantités qui ne peuvent pas être représentées par des nombres uniques. » [Nunez & Bryant, 2007] Nous proposons d'acheminer dès le début les élèves le long d'un processus d'exploration et de découverte de cet univers ; un nouvel univers libre des contraintes que les connaissances procédurales antérieures peuvent générer. Dans ce processus d'exploration et de découverte, les élèves utilisent essentiellement les mêmes matériaux didactiques qu'ils avaient utilisé dans l'exploration et la découvert de l'univers des nombres naturels. Mais maintenant, ils sont guidés vers la découverte des objects, de propriétés, d'opérations nouvelles ; c'est-à-dire, ils sont sollicités à changer la manière de lire les caractéristiques de ces matériaux didactiques et la manière d'opérer avec eux. Ils s'apprêtent à entrer dans un univers numérique nouveau, autre que l'univers familier des nombres naturels ; un univers dont ils ne connaissent aucune propriété et dont ils commencent maintenant l'exploration.

Mathématisation élémentaire et fondamentale

Dans notre projet, l'introduction du nouvel univers est fondée sur un type particulier de mathématisation qui se réfère à la façon dont les pythagoriciens comparaient deux quantités homogènes. Il étend la proposition de Davydov, dérivée de mathématiciens éminents tels que Klein, Lebesgue et Kolmogorov, selon laquelle l'origine véritable du concept de fraction réside dans la mesure de quantités. Nous la transformons dans la forme suivante : l'origine véritable du concept de fraction réside dans la comparaison de quantités homogènes, avant que dans leur mesure. Cette extension vise à donner un sens plus immédiat au formalisme, qui, dans la manière présentée par Davydov, risque d'apparaître injustifié.

Notre point de départ est la formalisation de l'acte de comparer deux quantités homogènes par un couple de nombres naturels. Ceci est un acte « élémentaire », qui peut être appliqué d'une manière simple aux situations «ordinaires» de comparaison de quantités homogènes ; en même temps il est un acte « fondamental », par le fait qu'il permet de gérer un chemin unitaire d'exploration et de découverte à travers toutes les sous-constructions du nombre rationnel.

Exemples d'activités en salle de classe : la comparaison entre quantités

Dans les activités dans la salle de classe, les élèves ont travaillé soit avec des quantités discrètes soit avec des quantités continues. En ce qui concerne les activités avec quantités discrètes, le choix décisif de l'enseignante a été d'associer la comparaison à un jeu, le jeu des tables de multiplication4 : le contexte du jeu a permis de créer une motivation pour l'acte de la comparaison.

La comparaison des quantités continues a mis à profit les activités avec l'eau, proposées par Davydov et utilisées par nous pendant les années précédentes pour introduire les concepts de quantité et de multiplication.

Les élèves ont achevé ces activités avec :

- la description de la procédure suivie, dans la quelle l'existence de l'unité commune est soulignée ;
- la représentation par des carrés ou des segments ;
- la désignation de la comparaison soit avec le couple de lettres indiquant les grandeurs comparées, soit avec le couple des nombres obtenues :

$$A;B = 13;8 :$$

c'est-à-dire : « la comparaison entre les quantités A et B est le couple de nombres 13 ; 8. »

[4] Le jeu des tables de multiplication: l'enseignante prépare un paquet de carte. Sur chacune est écrite une multiplication. La classe est divisée en deux groupes. L'enseignante joue une carte et lit la multiplication. Le groupe qui le premier donne le produit gagne un bonbon qui est mis dans le panier du groupe. Si les réponses des deux groupes sont presque simultanées, chaque groupe remporte un bonbon. A la fin il y a le processus de comparaison des bonbons remportés par chaque groupe.

« Dialogisme » entre les activités et les situations

Pour indiquer la troisième ressource qui, dans notre projet, peut aider à surmonter les obstacles à l'apprentissage des nombres fractionnaires, nous empruntons le terme de Bakhtine « dialogisme », en le paraphrasant de la manière suivante : notre objectif est de créer collaboration et intégration entre les différentes activités et situations, de sorte que chacune reçoive sens des autres et donne sens à eux-mêmes. Tous les matériaux didactiques offrent la possibilité de mieux mettre en évidence des propriétés particulières ; le « dialogisme » est l'acte de mettre en jeu des matériaux différents en fonction des propriétés que l'on veut présenter. C'est grâce au « dialogisme » que nous essayons de donner forme à l'exigence, souvent exprimée dans la littérature didactique, de donner aux élèves une large base d'expériences soit pratiques soit linguistiques, concernant le nombre fractionnaire.

Exemples d'activités en salle de classe : la mesure « composite »

Le noyau de la deuxième partie de notre proposition concerne l'introduction de la mesure « composite », c'est-à-dire le comparaison entre la quantité et l'entier ; quantité et entier qui partagent l'unité commune.

Par rapport aux activités de comparaison faites jusqu'ici en classe, la mesure « composite » présente nouvelles caractéristiques importantes qui devraient être mises en évidence avec soin et attention.

- Tandis que la comparaison précédente produisait un couple non ordonné de nombres, dans la comparaison-mesure les couples deviennent ordonnés grâce au rôle d'entier possédé par la deuxième quantité.
- L'enseignante a utilisé plusieurs stratégies pour mettre en évidence le rôle spécial de l'entier : la position de l'entier dans la représentation, la couleur, l'utilisation constante de la lettre « W » pour l'indiquer.
- La comparaison-mesure est indiquée avec une forme différente: A / W.
- Aussi le couple des nombres représentant la comparaison-mesure est écrit d'une manière différente : 13/8. Voilà la « fraction ».
- La lecture de la formule A/W = 13/8 est la suivante : « la mesure de la quantité A par rapport à l'entier W est la fraction 13/8 ».

Dans l'exécution des activités de cette deuxième partie, la classe a utilisé plusieurs types de matériaux didactiques, soit discrets soit continus : les coquetiers, les lego, les bonbons, l'eau, la tarte, les bandes de papier etc.. Les matériaux didactiques sont disposés sur un plan de travail et les activités avec les différents matériaux sont alternés selon l'idée que les différents matériaux fournissent l'occasion de souligner plus efficacement des propriétés particulières des nombres fractionnaires. Voici quelques exemples.

- Les coquetiers ont semblé efficaces pour mettre en évidence l'unification du double compte (le compte du nombre total d'œufs et le compte du nombre d'œufs contenus dans un coquetier) dans le nombre de paquets qui peuvent être susceptibles de se former.
- Les activités avec les lego permettent d'introduire dans une manière plus incisive les fractions avec numérateur plus petit et égal au dénominateur et de commencer les premières réflexions sur la comparaison et sur la fraction entière.
- Les sachets de bonbons, qui contiennent traditionnellement un nombre impair de bonbons, attirent l'attention sur les fractions avec le dénominateur impair. Également ils permettent de faire face à un dépaysement que la manque d'une explicite organisation des unités dans l'entier introduit dans les activités concernant la fraction « entière », la fraction « unité » et la fraction « nulle ».

- Alors que les matériaux discrets sont caractérisés par la détermination physique de l'unité commune, les activités avec l'eau abordent des situations dans lesquelles l'unité commune n'est pas déterminée a priori. Dans ces activités, la question du choix de l'unité commune est entrelacée avec le problème du reste.
- Dans les activités avec les tartes, l'entier est déterminé physiquement, tandis que la détermination de l'unité introduit les élèves à différentes situations problématiques, soit de partition, soit de relation [Nunez & Bryant, 2007]. Ces situations problématiques aident à attirer l'attention sur la spécificité de quelques caractéristiques des opérations d'addition et de soustraction de fractions.
- Enfin les mesures de longueur avec les bandes de papier amènent à discuter la division en dix parties égales au cours de la recherche de l'unité commune.

Réflexions

Nous concluons avec quelques réflexions sur trois concepts significatifs de cette présentation : la familiarisation, le paradigme, la trace.

La familiarisation

Notre proposition didactique n'est pas organisée dans un cadre structuré d'enseignement / apprentissage. En s'inspirant de Davydov, elle est structurée en un processus de familiarisation ; c'est-à-dire, elle ne vise pas à la connaissance systématique et opérationnelle des propriétés et des opérations entre fractions ; plutôt, elle a pour but de « donner aux élèves une large base d'expériences soit concrètes soit linguistiques, liées aux nombres fractionnaires ». Cette base devrait être l'environnement dans lequel ensuite développer des activités efficaces d'enseignement/apprentissage. Ce choix implique d'une part de privilégier tous les éléments qui favorisent le développement de l'intérêt, de la curiosité et de la participation active des élèves ; d'autre part de diriger le processus d'évaluation à ce que les élèves arrivent à faire et comment ils se conduisent lorsqu'ils sont correctement guidés.

Peut-être que la familiarisation avant l'enseignement / apprentissage peut influencer positivement non seulement la didactique des nombres fractionnaires, mais aussi la didactique d'autres sujets.

Le Paradigme

L'approche des nombres fractionnaires, que nous avons proposée et expérimentée dans la salle de classe, oblige les enseignants à déconstruire leur façon de penser ces nombres. Les nœuds qui affectent cette déconstruction sont :

- Dans ces activités, les enseignants utilisent principalement les mêmes matériaux de manipulation couramment utilisés jusqu'ici. Toutefois ils doivent changer leur façon de voir et de penser, pour conduire les élèves à agir différemment, à la découverte d'objets, de propriétés, d'opérations qui appartiennent à un nouvel univers. En empruntant les termes de Khun, on demande aux enseignants un changement de « paradigme », une « révolution » qui déconstruit leur manière habituelle de voir et de penser les nombres fractionnaires.
- Cette révolution est très profonde parce que le nombre fractionnaire est un « méga-concept dans lequel de nombreux filaments s'entrelacent ». Alors que la «simplicité» du nombre naturel permet une libre action de découverte des propriétés, en capitalisant une quantité riche d'activités également efficaces, la structure complexe du nombre fractionnaire exige que parmi les activités sur les fractions existe « un dialogisme ». Comme mentionné ci-dessus, chaque activité doit dialoguer avec les autres dans le but de créer une polyphonie entre les différents filaments du méga-concept.

La trace

Ce qui caractérise notre proposition est qu'elle est basée sur l'acte de mathématisation, élémentaire et fondamental, d'identifier la comparaison entre deux quantités homogène avec un couple de nombres entiers. C'est l'extension de la recherche par Davydov de « la véritable signification de la notion de fraction » ; une extension en direction de la détermination d'un point de vue originaire. Le fait d'identifier une valeur originaire dans l'acte de mathématisation, s'oppose à la recherche de la « construction mathématique plus large », qui a souvent caractérisé la recherche du XXe siècle soit dans les mathématiques (et la physique), soit dans leur enseignement.

La mathématisation élémentaire et fondamentale constitue la « trace » autour de laquelle se développe le processus de la didactique des nombres fractionnaire ; une trace qui ne doit pas être comprise dans le sens d'un chemin prédéfini à suivre, mais plutôt dans le sens de Lévinas, avec ses caractéristiques de « présence et absence ». Dans la complexité du « méga-concept » de fraction, l'acte mathématique élémentaire et fondamental déforme les lignes d'action de « l'espace-temps didactique » [Mercier, 2012] , en devenant le focus autour duquel les diverses activités et situations gravitent : un foyer vers lequel toutes celles-là convergent et d'où elles émergent. Dans cet acte mathématique les différentes suos-constructions du nombre fractionnaire trouvent unité.

REFERENCES

Les références sont écrites dans le style References. Assurez vous de ne rien oublier (numéros des pages si nécessiare, lieu,...). Les références sont dans le style APA (http://www.tandf.co.uk/journals/authors/style/reference/tf_A.pdf).

V.V. Davydov, ZH. TSvetkovich (1991) *On the Objective Origin of the Concept of Fractions*. Focus on Learning Problems in Mathematics, vol. 13 n. 1.

T.E. Kieren (1980) *Research on Number Learning*. ERIC Clearinghouse for Science, Mathematics, and Environmental Education, Columbus, Ohio. National Inst. of Education (ED), Washington, D.C.

(http://files.eric.ed.gov/fulltext/ED212463.pdf)

E. Lévinas (1982) La trace de l'autre. Dans *En découvrant l'existence avec Husserl et Heidegger*, Vrin, Paris, pp. 187-202

A. Mercier (2012) *Sur l'espace-temps didactique études du didactique, en sciences de l'éducation*. (https://tel.archives-ouvertes.fr/tel-00688585)

T. Nunes and P. Bryant (2007). Understanding rational numbers and intensive quantities. In *Key understandings in mathematics learning*. Paper 3. Nuffield Foundation. (http://www.nuffieldfoundation.org/sites/default/files/P3_amended_FB2).

Studying geometric loci

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Abstract: In this paper we present an experimental activity that we conducted with 10 teachers in several high school in Sicily. The activity deals with the study of some geometric loci and is based on the use of a Dynamic Geometry System on the one hand and on properties of geometric transformations on the other hand. The results of the experiment are presented.

Résumé: Dans cet article, nous présentons une activité expérimentale que nous avons menée avec 10 enseignants dans plusieurs lycée en Sicile. L'activité traite de l'étude de certains lieux géométriques et est basé sur l'utilisation d'un logiciel de géométrie dynamique d'une part et sur les propriétés de transformations géométriques, d'autre part. Les résultats de l'expérience sont présentés.

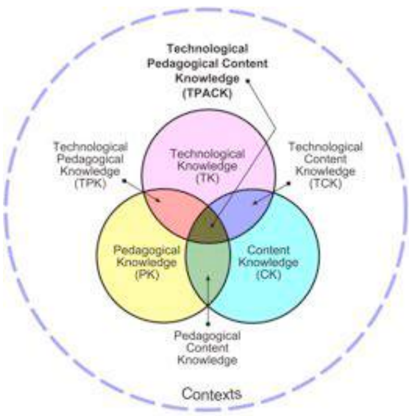
Introduction

We present an activity we carried out with teachers and students in several high schools in Sicily and the results of the experimentation. The whole activity is centred on the concept of geometric locus: several loci are introduced and some properties are explored and proved.

The concept of locus is usually introduced in school studying the axes of a segment, the circumference first and the other conics afterwards. It remains a quite hard topic to understand for students (Pech, 2012), and teachers quite often do not go into details of it. The idea of the activity is to present new geometric loci so to deepen the concept. We based the activity on the use of a Dynamic Geometry Software (DGS) on the one hand and some properties of the geometric transformations on the other. The software helps students in the "investigation" process, the geometrical transformations in the proving part.

Theoretical framework

Koehler and Mishara highlight in the TPACK framework the interplay of the three components, Technological, Pedagogical and Content Knowledge, in the learning and teaching process. "Good teaching is not simply adding technology to the existing teaching and content domain. Rather, the introduction of technology causes the representation of new concepts and requires the development of sensitivity to the dynamic and transactional relationship between all three components suggested by the TPACK framework" [Koehler & Mishara, 2008].



TPACK image (from http://tpack.org/)

The activity we present fits this framework, conjugated as it follows:

- As for the Technological Knowledge, (TK) “Knowledge about certain ways of thinking about, and working with technology, tools and resources” (Koehler & Mishra, 2009), we use the DGS to discover, conjecture and verify properties. Students are guided by worksheets and teachers.
- As for the Pedagogical Knowledge, (PK) “Teachers’ deep knowledge about the processes and practices or methods of teaching and learning” (Koehler & Mishra, 2009), we refer to the zone of proximal development of Vygotsky, Learning by doing of Dewey and Enactivism. In fact, students face new problems to work on, that they solve together with classmates, so to expand their zone of proximal development (Vygotsky, 2006). In this process the students are the principal actors (Rossi, 2011): they “get their hands dirty” (Dewey, 1950) interfacing with the DGS, indispensable tool for understanding and using the worksheets. In fact, using the DGS, peer confronting, supported by the careful presence of the teacher, students discover, understand and master a new mathematical concept. Many factors take part in the action: involvement, enthusiasm, communication skills in showing the contents, accepting and enhancing others’ interventions. These factors do not descend automatically by the chosen methodology, but rather by how the teacher uses them "in action" during class practice (Shulman, 1987).
- As for the Content knowledge, (CK) “Teachers’ knowledge about the subject matter to be learned or taught” (Koehler & Mishra, 2009), we refer to some recent studies of Ferrarello et al, (Ferrarello, Mammana, Pennisi, 2014) that deals with the study of some geometric loci.

In this methodological context, it is very important to pose problems, that is watching mathematics, school, world with a critical sense, to become a citizen who uses mathematics as an aware person. Acquiring critical skills that will be useful well beyond math’s or school’s environment, the student dominates the techniques and is not dominated by them.

Among students, but also between students and teacher, space to mathematics discussion is given: this allows students to check the accuracy and richness of the proposed solutions, their consistency and the reliability and their level of adopted generalization. This phase leads to the construction of meanings that go beyond those directly involved in the solution of the task, to enable students to know new aspects of mathematical culture, enhancing in particular, a gradual but systematic approach to theoretical thinking. In the mathematics discussion teacher has a leading role: he/she influences the discussion in a decisive way, with proper and effective interventions, because he/she has in mind both general and specific targets of the activity.

The students become a major player in the construction of his knowledge, in accordance with the Enactivism principles (Rossi, 2011): they overcome difficulties by themselves, but also working with each other (collaborative learning) or with the teacher who does not tell answers, but rather gives some hints and to make them think and own the topic (favouring the growth of the zone of proximal development). Mathematics is discovered and handled (learning by doing), also, but not only, with the use of computers, which facilitate learning. All within a "mathematics laboratory," according to the indications of Curricula UMI (Anichini, Arzarello, Ciarrapico & Robutti, 2004).

Contents

The activity is inspired by a recent work of Ferrarello et al, (Ferrarello, Mammana & Pennisi, 2014) and deals with the study of some geometric loci. The students’ work is based on the use of a DGS (Geogebra) for the discovery of properties and the use of the geometric transformation for their proof. In this way we provide examples of applications of geometric transformations: this topic is often mistreated and relegated to a separate chapter, while it often simplify demonstrations otherwise very long and hard to understand. We get, then, elegant proofs, allowing students to reason in a simple way, without getting lost in the calculations and focusing on the concepts.

In particular, we studied loci generated as it follows: let γ be a circle with centre O and radius r; fix n points on γ, with n=1, 2, 3 and consider a generic point P of γ; we define a point L that depends on P and study the locus λ described by L when P moves on γ.

In the following table we report the cases we examined.

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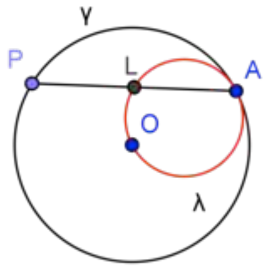
n=1



Let A be a fixed point of γ and P a generic point of γ. Let L be the middle point of the segment AP. The locus λ described by L when P moves on γ is a circle with radius r/2 and centre the middle point of AO

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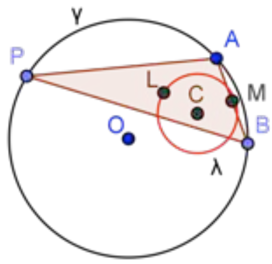
n=2, a



Let A and B be two distinct points of γ, P be a generic point of γ. When P moves on γ, the locus λ described by the centroid L of ABP is the circle with radius r/3 and centre the point C of the segment OM such that CM=1/3OM, where M is the middle point of AB.

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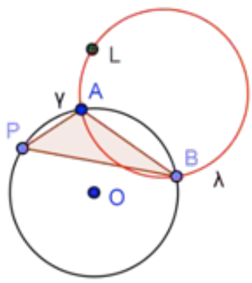
n=2, b



Let A and B be two distinct points of γ, P be a generic point of γ. When P moves on γ, the locus λ described by the orthocentre L of ABP is the circle that is symmetric of γ with respect to AB.

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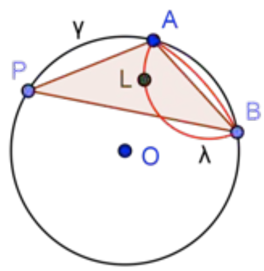
n=2, c



Let A and B be two distinct points of γ, P be a generic point of γ. When P moves on γ, the locus λ described by the incentre L of ABP is the done by two arcs of the circles through A, B and with centres the common points to γ and to the axes of AB.

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n=2, d

Let A and B be two distinct points of γ, P be a generic point of γ. When P moves on γ, the locus λ described by the circumcentre L of ABP is one point, the centre O of γ.

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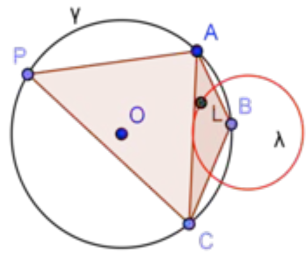
n=3, a



Let A, B and C be three distinct points of γ. Let P be a generic point of γ. When P moves on γ, the locus λ described by the anticentre L of ABCP is the Feuerbach circle of the triangle ABC.

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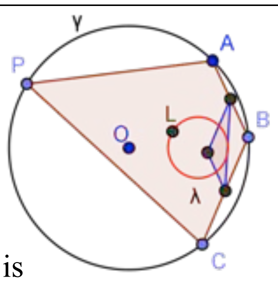
n=3, b



Let A, B and C be three distinct points of γ. Let P be a generic point of γ. When P moves on γ, the locus λ described by the centroid L of ABCP is the Feuerbach circle of the medial triangle of ABC.

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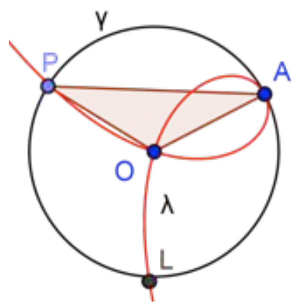
Some more loci, a



Let A and P be points of γ. The locus λ described by the orthocentre L of the triangle AOP, when P moves on γ, is a cubic with a node in O (*Strophoid*).

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</table>

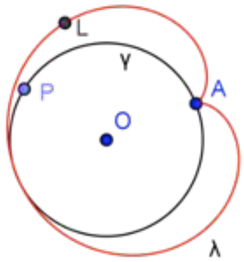
<table>
<tr><td>Some more loci, b</td><td>Some more loci, c</td></tr>
<tr><td>Let A and P be points of γ, t the tangent line to γ in P and L the foot of the perpendicular to t from A. The locus λ described by L when P moves on γ is a bicircular quartic with one cusp in A (Cardioid).

</td><td>Let AB be a diameter of γ. Let r be a perpendicular line to AB. Let P a point of γ and R the point in which AP meets r. Let L the midpoint of PR. The locus λ described by L when P moves on γ depends on r, but it is in general a cubic.</td></tr>
</table>

Table 1 - Topics

Teaching experiment and activity

The activity was carried out by using the method, widely experienced by the authors, of a double laboratory (Ferrarello, Mammana & Pennisi, 2013): in the first laboratory (teachers lab) teachers design teaching materials and in the second one (students lab) students benefit the produced material.

This is in accordance with the "Learning by doing" of Dewey, and the in the Vygotskijan practical intelligence perspective, which supports the creation of mathematical concepts through a mediated relationship, through the use of artifacts. The activity is based on the idea that "the best way for students to learn is to touch and build ... [and] for teachers the best way to learn to teach is to experience first hand, touch and build on their own teaching materials" (Ferrarello, Mammana & Pennisi, 2013).

Then, the construction of the classroom worksheets are the heart of the teachers' lab. The tutors (the authors of the paper together with two more high school teachers) plan a preliminary worksheet for the teachers. The worksheets have usually two columns: in the left column an action is indicated, in the right column the action is explicitly made. In such a way students are perfectly aware of what they are doing (Ferrarello & Mammana, 2012). Some parts of the worksheet presents construction of figures, exploration, and the student has to follow the instruction and verify the properties; other parts of the worksheet have to be filled by students, for example he/she has to write the conjecture or the proof of the theorem. In preparing the worksheets particular attention was given to the zone of proximal development related to the individual competence, with the aim of collocate the teaching experiment in the zone of proximal development and to organize good hints and the metacognitive reflection. The teachers, guided and supported by tutors, build other worksheets comparing each others ideas and practicing "first-hand" how to run a laboratory. In this way the teachers acquire skills useful for the conduction of the students lab, which has different protagonists (students), but the same methods of the Teachers lab.

Common factor to the two laboratories is the *modus operandi* of the mathematics laboratory (Chiappini, 2007) in a collaborative learning environment, since working with peers, with an experienced guide (tutor and teacher respectively) facilitates the socialized learning in the zone of proximal development in a positive atmosphere, as advocated in the theory of Vygotsky: given a problem that is slightly above his/her current capacity, the student, thanks to the collaboration between peers, acquires new skills that allow its zone of proximal development to expand more than how it would expand if the student was working alone.

The choice of writing the worksheets with teachers is due to several fact: above all because, "The integration of technology in mathematics education is not a panacea that reduces the importance of the teacher. Rather, the teacher has to orchestrate learning. To be able to do so, a process of professional development is required" (Drijvers, 2012). Furthermore, the possibility of using an artifact like the DGS lets the learners discover the properties independently. Nowadays the educational software are widespread, but not always the teachers know how to use them from an educational point of view (Doğan 2011). So, it is very useful to make teachers able to effectively manage such tools, in order to disseminate good practices of teaching (and learning).

The teaching materials built up by teachers consists of worksheets, according to the scheme described in (Ferrarello & Mammana 2012), leaving the student the opportunity to fully use the "explore-discover-test-conjecture-proof" model. In this way the student is not a passive listener of a lecture, but an active subject in the first hand experience of discovery.

10 worksheets were written, according to the various cases described in Table 1. Each one is divided into two parts: the first part guides discovery's activities with the use of the software, the second part guides the proofs.

Students, as mentioned, work on their own and together: they work on their own because each student has a worksheet to be elaborated and they also work together, comparing ideas and insights, and having at their disposal a space (the worksheet and the working environment of the software) and a time, suitable to their own learning. So each one builds his/her own knowledge according to his/her need of time and space, in an inclusive education perspective, which fosters both "less smart" students in motivation (see paragraph on the results) and "more smart" students in deepening the concepts. Students work independently, but they are never left completely alone: the teacher supports, encourages, helps them. Moreover, at the end of each worksheet, the teacher, by means of a final discussion with all the students, remarks the obtained results, giving strength and clarity to the concepts just discovered, so that they are clear to all.

In the final discussion the students reflect together or alone on the difficulties they encountered, about what they did to overcome them, on the hints that have been decisive and what was misleading, thus developing the metacognitive awareness, which allows them to assimilate new skills and knowledge to add to those they already hold in the long-term memory.

Results

The experimentation involved 210 high school students from different kinds of secondary schools: scientific oriented high schools, foreign language high schools and a human science high school.

At the end of the activity teachers logbooks and students worksheets were collected, together with a questionnaire that was given to teachers and a questionnaire given to students.

In the logbooks, teachers described the development of activity in class (faithful to the protocol), reporting the management for the whole task, the behavior of students and the results, both on the motivational point of view (attitude, interest, engagement) and on the cognitive point of view (learning development).

The teachers questionnaire consisted of 10 open questions, asking them to highlight remarks on the effectiveness of the teaching proposal, on the mood created in class during the activity, and some other questions about the possibility to share such a methodology with colleagues or to participate to similar activities.

The students questionnaire, aimed at verifying if the goals of the activity were achieved and evaluate the satisfaction level, consists of two three sections:

- Overall assessment of the course;

- Self-assessment of skills that the student believes to have developed during the course;

- personal comments.

In the first section there were 9 closed questions, structured using ordinal scales, with these answers: Definitely not, More no than yes, More yes than no, Definitely yes.

In the second section there were 6 closed questions, structured by ordinal scales at intervals whose parameters answers vary from a minimum of 1 to a maximum of 4 points.

In the third section there are 7 open questions.

In the following we report some considerations arising from logbooks, teachers questionnaires and students questionnaires (sentences of the teachers or of the students are written with italic font).

As a general remark, the activity was appreciated both by teachers and students.

In particular, teachers underlined the importance to create and write the class worksheets together with a team of teachers from University (the tutors), because "*not only we focus on the targets to be reached, but also we did not overlook the difficulties pupils could meet in learning*"; moreover, during our meetings teachers not only mastered the use of Geogebra, but they also comprehended "*the methods and techniques to offer the activity to students and to get in such a way the best results*". Then teachers understood the interplay of "what" *(CK)*, "how" *(PK)* and "by what" (TK) of the TPACK framework.

All the teachers unanimously stated that it is desirable to integrate dynamic geometry systems in everyday teaching, because "*not only it is able to make immediately visualize the geometric locus, but also it is a valid help to verify the results obtained in analytical way*", perceiving the role of practical intelligence and semiotic mediation of tools.

The same is strengthened by their students, that said, for instance "*the use of Geogebra is useful to better understand studied topics, because by the construction you get the definition. Step by step construction of geometric loci through Geogebra surely helped to facilitate the understanding of the concept of locus*" and "*Geogebra is a very interesting program because it let me to verify with my eyes all the properties I just studied in the books*", highlighting the connection between Technlogy and Content.

As for the use of class worksheets, it came to light that "*the activity worksheets were a valid support: each student, with the help of the worksheet, succeeded to work peacefully and in autonomy*" being "*teacher of him/her self*". All the teachers highlighted enthusiasm in his/her own pupils, that "*faced the task with a certain level of autonomy, sometimes giving some original contribution*: […] the enthusiasm is due to *feeling of the activity as cool and easy, almost a geometric game. A game of construction e visual verify, free by the struggle of computations*".

Moreover a collaborative way of working seems to be effective in students' learning. They stated that "*among us (classmates) there was a collaborative climate, that let us carry out the assigned task in the best way, we learned several topics in a easy way, collaborating in group*" (in a social learning mode).

Working in the Laboratory of Mathematics, all students tested their knowledge, used artifacts and tools, made explorations, formulated conjectures, acquired concepts and skills: "*mathematics does not belong to another planet and it is not just for someone who has some special skills, but it is accessible to all"*. Some teachers pointed out that "*the activity allowed us to stimulate the interest of those pupils more fragile and less inclined to discipline, being personally committed to work, those students had fun discovering and building geometry through Geogebra*".

On the other hand the teacher was always ready, if necessary, to correct their conjectures with appropriate suggestions; to ask questions to make them guess something could be useful and/or necessary to discover something else; to encourage them to continue; to praise them for any significant obtained result, in the Vygotskijan perspective (ZPD and positive mood).

Students had no problems in approaching the proposed loci. Some of them even considered easy to apply the encountered properties and to use them. We should underline that worksheets are sometimes too guided for some students, but we took this choice, so that all students would feel equally involved.

Finally, students knew that they were participating to an experimental activity and that they will not be evaluated by the teacher. This allowed also fearful or less able students to feel involved, the wrong answers were not evaluated negatively, indeed those answers served as a basis to trigger collective discussions to clarify the problem: The error is seen as a resource, rather than as something to be condemned, according to the Enactivism's perspective.

The following general overview concludes the analysis of the student questionnaire, with regard to the first and the second part. An overview of the Questionnaire is given in the end.

For most of the pupils involved in the activity (51%) the topics are interesting (Figure 1), also confirmed by 55% that said they had actively participated to it (Figure 2).

More than half of the students found the activity is not too demanding, but they worked hard into it (Figure 3). Students appreciated working in groups, in fact 58% of the statistical units is favourable to the work group (Figure 4), and the interaction established between the members of each group (Figure 5) were positively judged by more than half of the sample (65%).

It is interesting to report the claims made by some students who confessed their disaffection to mathematics, "I hate math, for me it's boring because I do not understand it. But I liked this experience with the computer because math seemed to me more interesting."

In any case, the 45% of students admits that it was is worth to participate to the activity (Figure 6); moreover some students said that such an activity allowed them to see the math from another point of view; they would like to repeat a similar experience and hope the same for the students of lower classes "because maybe everyone could like more mathematics, if you start on the right foot."

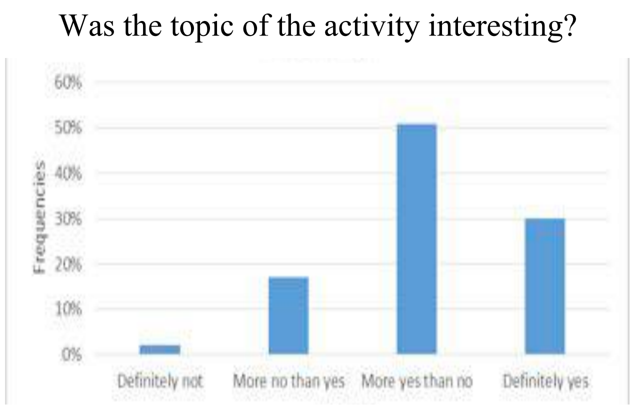


Figure 1

Figure 2

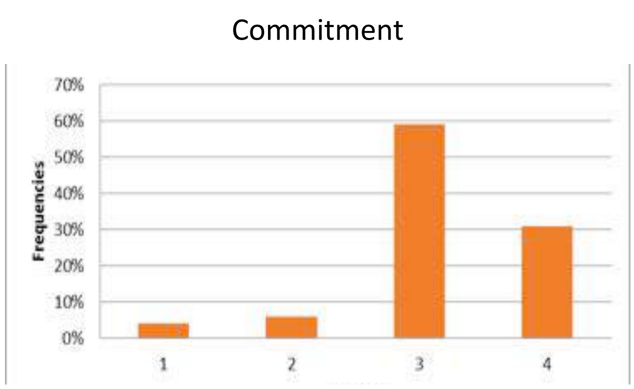


Figure 3

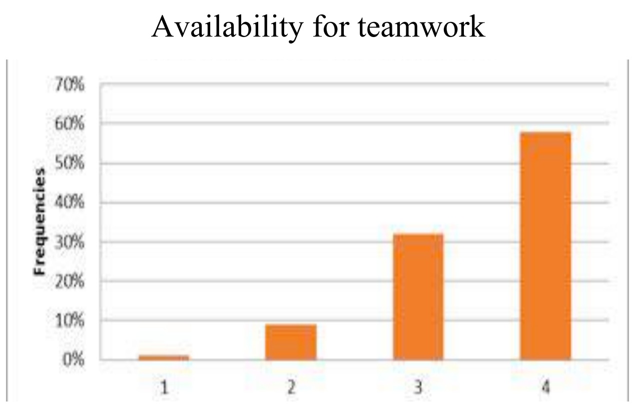


Figure 4

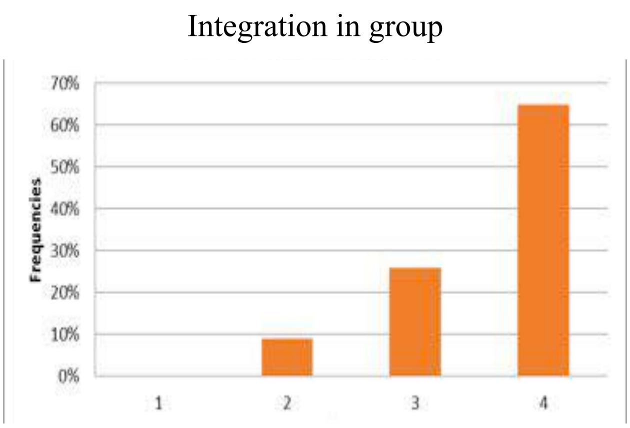


Figure 5

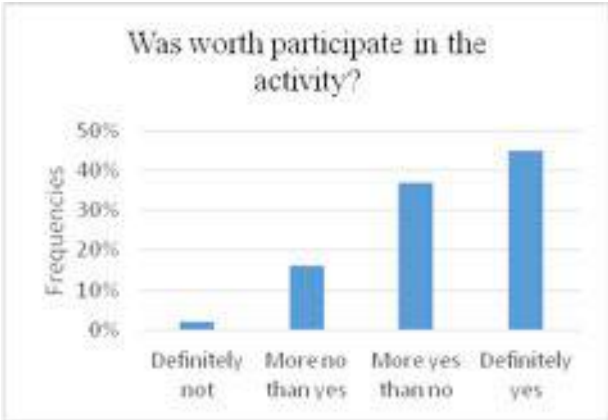


Figure 6

Students questionnaire

Overall assessment of the course	**Definitely not**	**More no than yes**	**More yes than no**	**Definitely yes**
Was the topic of the activity interesting?	2%	17%	51%	30%
Was the activity difficult?	7%	42%	36%	15%
Was your school knowledge enough to attend the activity?	0%	65%	54%	40%
Were the activity worksheets clear?	2%	10%	38%	50%
Was interesting and surprising studying loci?	4%	19%	47%	30%
Were the teachers clear?	0%	4%	34%	62%
After this course has your motivation to study mathematics increased?	6%	32%	40%	22%
Was your participation in the course active?	1%	4%	40%	55%
Was it worth to participate in the activity?	2%	16%	37%	45%

Self-assessment of skills that the student feels to have developed during the course:	**1(min)-4(max)**			
	1	**2**	**3**	**4**
Commitment	4%	6%	59%	31%
Desire to explore the topics	3%	32%	45%	20%
Availability for teamwork	10%	28%	47%	15%
Ability to speak in public	0%	15%	64%	21%
Ability to think about things	1%	9%	32%	58%
Integration in group	0%	9%	26%	65%

What would you like your math teacher would give more attention to?	
- practical and applicative aspects	56%
- theoretical aspects	10%
- history	3%
- relationship with other subjects	31%

REFERENCES

Anichini, G., Arzarello, F., Ciarrapico, L. & Robutti, O. (eds.)(2004). Matematica 2003. Attività didattiche e prove di verifica per un nuovo curriculo di matematica. Ciclo secondario. Lucca. *Matteoni stampatore.*

Chiappini, G. (2007). Il laboratorio didattico di matematica: riferimenti teorici per la costruzione. *Innovazione educativa*, Inserto allegato al numero 8, 9-12.

Dewey, J. (1916). Democracy and education: an introduction to the philosophy of education. *Free Press; Reprint edition (February 1, 1997)*

Doğan, M. (2011). Trainee teachers' attitudes about materials and technology use in mathematics education. In W. Yang, M. Majewski, T. de Alwis, & E. Karakirk (Eds.), *Proc. of ATCM 2011*. Mathematics and Technology, LLC.

Drijvers, P. (2012). Digital technology in mathematics education: Why it works (or doesn't). *In Proceedings of ICME 12*. Seoul, Korea. 9-12 July 2012.

Ferrarello, D. & Mammana, M.F. (2012). Training teachers to teach with technologies. Proceedings of La didactique des mathématiques: approches et enjeux. Paris, France. 31mai, 1e et 2 juin 2012.

Ferrarello, D., Mammana,M.F. & Pennisi,M. (2013). Teaching by doing. *Quaderni di ricerca in didattica / mathematics (QRDM). 23, 1. 466-475.*

Ferrarello, D., Mammana, M.F., & Pennisi, M. (2014a). Teaching/learning geometric transformations in high-school with DGS. *IJTME vol 21 n1 p11-17*

Ferrarello, D., Mammana, M.F., & Pennisi, M. (2014b). Geometric loci and homothetic transformations. *JMEST* Vol 45, Issue 2

Koehler, M.J., Mishra, P. (2008). Introducing Technological Pedagogical Knowledge, In *the Handbook of Technological Pedagogical Content Knowledge for Educators, Edited by AACTE Committee on Innovation and technology, AACTE and Routledge, NY.*

Koehler, M. J., & Mishra, P. (2009). What is technological pedagogical content knowledge? *Contemporary Issues in Technology and Teacher Education, 9*(1), 60-70.

Pech, P (2012). How integration of DGS and CAS helps to solve problems in geometry. In Procedings of ATCM2012.

Rossi, P.G. (2011) Didattica Enattiva. Complessità, teorie dell'azione, professionalità docente. *Franco Angeli, Milano*

Shulman, L. (1987). Knowledge and Teaching: Foundations of the New Reform. *Harvard Educational Review.*

Vygotskij, L.S. Thought and Language (1986). *The MIT Press; revised edition.*

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A classroom activity to work with real data and diverse strategies in order to build diverse models with the help of the computer

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Abstract: Taking into account that the majority of the models with more tradition in the mathematical curriculum to represent temporal evolutions of the number of individuals are continuous models in the class of empirical models, it has an added value to deal with these models, but where the parameters involved are claimed to have a biological meaning. These types of models are not completely mechanistic o heuristics, and therefore they are called pseudo-mechanistic models. It is a challenge to link mathematical tools and concepts with biological ideas, and also a chance to use the help that computers provide in this context. The aim of this study was to design a set of rich tasks to be performed with the help of the computer and implement them in the classroom in order to investigate a real data set to deal with empirical models and pseudo-mechanistic models. One of the main purposes in the designing of this set of tasks was to configure a framework showing different strategies to deal with the data and also, how each of these approaches could generate a variety of responses to the problem in hand. The sequence and structure of these tasks, according to the students' perceptions collected, enhanced the understanding about the construction and use of these primary growth models.

Résumé: Étant donné que la plupart des modèles programmes de mathématiques les plus traditionnels pour représenter l'évolution temporelle du nombre de gens sont des modèles continus appartenant à la classe des modèles empiriques, a ajouté la valeur de travailler avec ces modèles, mais où les paramètres impliqués Ils ont une signification biologique. Ces types de modèles ne sont pas totalement mécaniste ou heuristiques, et sont donc appelés modèles pseudo-mécaniste. Il est un défi de relier les outils et concepts mathématiques avec des idées biologiques, et aussi une chance d'utiliser les ordinateurs d'assistance fournies dans ce contexte. Le but de cette étude était de concevoir un riche ensemble de tâches à accomplir avec l'aide de l'ordinateur et de les mettre en œuvre dans la salle de classe afin d'étudier et de décrire un ensemble de données réelles avec des modèles empiriques et les modèles de pseudo-mécaniste. L'un des principaux objectifs dans la conception de cet ensemble de tâches a été de mettre en place un cadre pour montrer différentes stratégies pour traiter les données et aussi comment chacune de ces approches pourrait générer une variété de réponses au problème à la main. La séquence et la structure de ces tâches, selon les perceptions recueillies auprès des étudiants, sont susceptibles d'améliorer la compréhension de la construction et l'utilisation de ces modèles de croissance primaires.

Introduction

Actually, in the twenty-first century, computation is more than an assistant support of scientific activity, computation is changing the fundamental way that science is practised and also how this it is being learnt and taught (Shiflet and Shiflet, 2014).

Computation allows us to obtain and analyse big data, consider and solve problems inaccessible until now, build sophisticated models, visualize phenomena, conduct experiments difficult or impossible in laboratories, among other options. Teaching and learning mathematics in any context should promote the development of thinking and the possibility of an appropriate use of the technology available nowadays. Processes such as developing curiosity, critical thinking, reasoning, as well as developing modes of verification, refutation, and deduction should be found in the activities proposed to our students in classroom, and for some of these processes the help of a computer can be very valuable. In this context, for instance, analysis of real data and the building of different kinds of models for this data could be good opportunities to train in those processes. In the teaching of applied mathematics, it would be desirable to design and develop profitable strategies to tackle real data and deal with models that require thinking at multiple levels of abstraction or understanding, plus to know how to use the computational resources offered by general or specific software accessible nowadays. The computer must be seen as a convenient work “companion” and an attractive resource, and never as an obstacle for the mathematics learning. The potential of the software present in the majority of the computers of our laboratories cannot remain unexplored and unexploited when activities related to quantitative modelling and numerical methods are carried out in the classroom. In this line, teachers need to have (or develop) suitable knowledge and competences in digital technologies, otherwise their teaching will not be so effective (Bennison & Goos, 2010).

It is widely accepted that mathematical thinking arises and develops in a complex interplay of languages and representations. There is a relatively new term or idea that is “Computational Thinking” (Papert, 1996; Wing, 2006), that although its definition is still under discussion, it appears to focus on computer science concepts in relation to processes of problem solving such as: pattern recognition, pattern generation, abstraction (composition, de-composition, generalization, and specialization), modelling, algorithm design (sequence, iteration, and selection), data analysis and visualization (Caspersen & Nowack, 2014). Thus, it is also accepted that “Computational Thinking” can be envisaged as a fundamental skill for everyone and in particular that it is very attractive for anyone who is involved in teaching and learning mathematics. In the context of the Millennium Mathematics Project (a maths education and outreach initiative) can be found the NRICH website (http://nrich.maths.org/) containing a list with some characteristics that make a task rich, highlighting the fact that it is the way in which the task is planned and used in the classroom that makes it rich. Some of those characteristics can be common to the way in which “Computational thinking” can be practised and trained.

In the field of mathematical biology, the capability of designing classroom activities that encompass quantitative modelling with mathematical concepts and tools to deal with biosystems is much appreciated (de Vries, Hillen, Lewis, Müller & Schönfisch, 2006). In addition, these activities can justify and give room for the introduction and workout of complementary computational methods (Ginovart, 2014). Mechanistic or heuristic models are those whose development comes from the understanding of the underlying biochemical or biological processes governing populational phenomenon and their parameters have biological meaning, while empirical models are mathematical functions simply describing observations of the phenomenon. Taking into account that the majority of the models with more tradition in the mathematical curriculum to represent temporal evolutions of the number of individuals that configure the populations in certain environments reported in periods of time are continues models in the class of empirical models, it has and added value to deal with these models but in which the parameters involved are claimed to have a biological meaning. These types of models are not completely mechanistic o heuristics, and therefore they are called pseudo-mechanistic models (Perez-Rodriguez, 2014), and it is a challenge to link mathematical tools and concepts with biological ideas, and also a chance to use the help that the computers can provide.

The aim of this study was to design a set of rich tasks to be performed with the help of the computer and implement them in the classroom in order to investigate a real data set (a temporal evolution of a microbial population grown in a specific environment) to deal with empirical models and pseudo-mechanistic models. One of the main purposes in the designing of this set of tasks was to configure a framework showing different strategies to deal with the data and also, how each of these approaches could generate a variety of responses to the problem in hand. The students’ perceptions about if the sequence and structure of these tasks have enhanced their understanding about the construction and use of these primary growth models were collected and analysed.

Material and methods

The participants in this study were a group of 50, third-year students of a Bachelor's degree in the field of Biosystems Engineering at the Universitat Politècnica de Catalunya (Barcelona, Spain). The prior coursework for these students was related with the following compulsory subjects: Mathematics I and II, Physics I and II, Chemistry I and II, General Biology, Microbiology, Statistics, among others. This previous preparation guarantees a good knowledge of some biological systems (microbial systems in particular) and basic mathematical concepts and tools.

Students’ responses regarding analyses and modelling of the growth population data and the distinct methodologies applied sequentially for the study of the different phases observed in the pattern of this temporal evolution, were collected via commented spreadsheets, outputs of statistical software, open-ended questionnaires, and face-to face dialogues during the development of the sessions in the computer lab. The students' perceptions about the set of tasks conducted were explicitly questioned and collected at the end.

A set of 17 observations corresponding to the size of a population (numbers of microbes) grown in a liquid medium of 1 mL with an initial quantity of sugar and no further addition of nutrient during a period of 48 hours is the data to be analysed (Table 1), and used to build empirical models and pseudo-mechanistic models. Each student had a computer with access to spreadsheet (Excel), mathematical software (Maple) and statistical software (Minitab and R) utilized in previous subjects, and with a free connexion to Internet when looking for specific information. The activities were designed to be carried out individually, however, with lab sessions and small groups of students, comments, suggestions, and interactions with the teacher were always held whenever required or considered appropriate.

Time (h)	Number of microbes
0	145349
3	146217
6	139333
9	143620
12	168557
15	287768
18	972270
21	2996236
24	4444266
27	5953756
30	7245644
33	7614686
36	8187928
39	10214427
42	11842517
45	13650985
48	12837014

Table 1. The experimental data to be analysed.

Some preliminary questions were answered without reading the guide to the activity, as a first task, so that the students had the possibility to reflect on what they have studied and learned up to now, as well as what was convenient to apply to solve this kind of problem. These preliminary questions were: *Taking into account that the set of data in Table 1,*

- *Which strategies or resources that already know can be used to analyse this data*?
- *Which type of function or model can be adjusted*?
- *Which programs can be utilised for that purpose?*

The outline of the set of tasks to be performed with the help of the computer

Part A. To perform an exploratory analyse of the data and decide the best way to represent it, performing or not some nonlinear transformations of this data. It is important to point out that the context application of this data is microbiology and we are dealing with the growth of huge number of microbes.

Part B. To deal with the use of polynomial functions to describe or fit the data and highlighting some advantages and disadvantages of this type of approach.

Part C. To identify the three different phases in the temporal evolution and use straight lines to describe each of these phases. The succession of phases may be conveniently distinguished because they are characterized by variations of the growth rate: first, the lag phase with growth rate null, second, the exponential phase with a constant rate, and third, the stationary phase with no clear growth. The reading of the paper of Buchannan and coauthors (Buchanan, Whiting & Damert, 1997), where the so-called three-phase linear model is presented, completes this part where the students can recognize their own work.

Part D. To build, step by step, a discrete logistic model by means of set of calculations and linear approximations of the transformed data. After this, the comparison of the built model with the observed data is carried out with a simulation in a simple way in a spreadsheet assessing the agreement between them.

Part E. To introduce the family of mathematical models known as sigmoid functions and their relation to the growth models, being the continuous logistic model one of them. The use of the mathematical software to manipulate their mathematical expressions and graphical representations with the purpose of identifying the role of the set of parameters involved in their definitions. The identification of the meaning of these parameters is followed by a convenient reparameterization of them, allowing us to have parameters with a clearer biological meaning and models utilized in the prediction of microbial growth in real applications.

Part F. To explore the options that R, a free statistical software, and the packages developed to manage growth curves have to deal with this data (http://www.inside-r.org/packages/cran/nlstools/docs/growthmodels).

At the end of the activity, and after the work was carried out through the six parts describe above, the questions posed at the start were presented again to the students in a new context: *If you have to analyse another set of microbial growth data, what would your answers be to those preliminary questions?,* and one more: *What would your priority be in the set of tasks to be carried out in order to model the new data set?*

Results and discussion

Students' responses regarding the analyses of the data and the various microbial growth models were prepared individually and collected by means of file texts, giving answers to the questions posed, spread sheets with comments, and outputs of mathematical and statistical programs inserted in an explicative text. In addition, interviews and personal communication during the sessions in the computer lab made it possible to collect further information on the development of the activity.

This activity has been divided into a set of tasks which have been grouped in six different parts (A, B,..., F) facilitating the organization of the work to be performed by the students. The various parts were carried out in three lab sessions of two hours each, so the extension and diversity of the results obtained by the students were remarkable.

The answers to the preliminary questions before beginning the tasks of the activity ("*Which strategies or resources already known can be used to analyse this data*?", "*Which type of function or model can be adjusted?*, and *Which programs can be utilised for that purpose?*") were rather disappointing and poor. The majority of these responses were about the use of a spreadsheet to achieve a graphical representation, but no references to the use of mathematical programs like Maple or statistical programs like Minitab or R, programs that had been used in the previous mathematical and statistical subjects. Undoubtedly, it was also evident that no relationships or links between the previous knowledge of microbiology and different phases of the growth of the population were made or identified, in spite of these phases being linked with the numerical derivatives of the data by means of growth rates. The possibility of connecting the two disciplines involved in the modelling process, microbiology and mathematics, did not appear in those first answers. Only the 10% approximately of the students mentioned the fact that the transformation of the data by means of the logarithm function could be helpful in this microbial context, due to the magnitude of the numbers and the rate in the population growth. It was really a discouragement to see the lack of association with other subjects in these students' answers. Maybe, this is something not really surprising in the teaching in general, taking into account that subjects and teachers have their own specific areas, and on very few occasions do they allow interference or collaboration in sharing activities that involve simultaneously diverse areas of knowledge, and it that moment the students were in a mathematical-computing context.

The set of results that the students obtained, analysed and discussed in connection with their knowledge of biology were graphic representations of the temporal evolutions of the number of individuals in the population and the transformations performed with the data, definitions and manipulations of mathematical functions to construct and formulate different models, together with the calculation of the corresponding parameters involved in those models, plus the assignment of biological meaning to them in the case of pseudo-mechanistic models. Only some aspects or parts of the processes involved in the tasks designed will be presented in this section, accompanied by some computer screenshots to illustrate them.

Regarding the tasks in Part A which were in some way conducted with questions like:

- *Is it better to work with the original data or can you imagine a linear or nonlinear transformation of this data?*
- *Are the magnitudes of the observed values (and their changes) at the beginning of growth very different from those obtained at the end of the experiment?*
- *What mathematical function can be used to modify or modulate this behaviour?*

the students completed an exploratory analysis of the data by deciding which was the best way to represent it, considering the nature of the experimental observations that they handled and remembering the way to present data in a microbiology field. The graphical representation of the original data at different scales and through some nonlinear transformations were options examined by the students. At the beginning of the temporal evolution the original data had values of about 10^5 and values of around 10^7 were reached at the end of the evolution, so the magnitude of the range to be represented was considerable. The increases (in absolute values) of one time to another time showed very different magnitudes depending on the time sampled. At this point, the students appreciated why in the context of microbial communities, the populations are usually expressed in logarithmic units (base 10). Logarithm transformations with different bases were one of the strategies tested by students in the spreadsheet (Figure 1).

In Part B, and after verifying that neither linear nor exponential growth were detected with this data, the fittings with polynomials of diverse degrees to describe the original data were examined and discussed by students, focusing this discussion on some advantages of this empirical modelling and emphasis on some of its disadvantages. In particular, the concept of the model or what a model should be, together with its purpose was debated, and the conclusion in this part was that this type of model (polynomial functions) did not exemplify it very well. They discussed what it was like to work with empirical models (fittings of functions) and what the limitations were, and applauded the possibility of providing a pseudo-mechanistic model in this context. The coefficients of the fitted polynomials showed no opportunity to incorporate any of the ideas behind the microbial growth phenomenon. The purpose would be to try the description of this phenomenon: a population located in a space, in a new environment with nutrient, using the energy resources found in it, thus enabling the reproduction of individuals in the population, increasing its size; nevertheless, with this utilization of nutrient from the environment without its replacement, leads to an unfavourable situation, which is no longer possible for the population to continue to grow. The inability of these empirical models to explain the biological phenomenon, added to the fact that some of the fitted polynomials reached negative values, made it evident that they are not appropriate and these mathematical expressions had lost all meaning to represent this data.

In Part C of the activity, the use of the logarithm transformation of the number of microbes observed over time, along with the increases observed in each of the sample time (Figure 2), allowed the identification of various growth phases that occurred during the temporal evolution of the population size. A first approach for the construction of a simple model with the capability to pick up the features or major trends in this type of growth goes through consideration or recognition of the three main phases of the growth, lag phase, exponential phase and stationary phase, and it uses piecewise linear functions for its formulation. The students studied this option with the spreadsheet using linear regressions to describe each of these phases which are characterized by significant variations in growth: first, the lag phase with a zero growth rate, after the exponential phase with a constant growth rate, and finally no growth for the stationary phase (Figure 3). Reading the companion document of Buchannan and others [2] to which the students could access, entitled "When good enough is simple: a comparison of the Gompertz, Baranyi, and three-phase linear fitting models for bacterial growth curves", illustrated and revealed that the piecewise linear model built by the students was in tune with one of the possible solutions that researchers presented and accepted in the environment of predictive microbiology. The three-phase linear model was the first pseudo-mechanistic model obtained, as in its formulation parameters with biological significance were recognized, such as the logarithm of the initial population and the final population, duration of the lag phase (intersection of the two first straight lines) and the maximum growth rate (slope of the second line), start of the stationary phase (intersection of the second and third lines). Students were able to recognize their own work in a spreadsheet as a simple modelling choice but well placed in a broader context of interest in biotechnology and predictive microbiology.

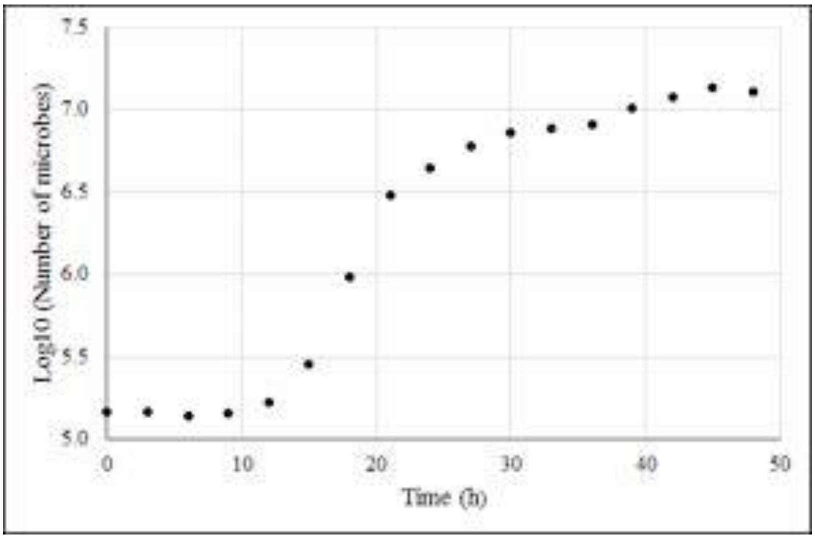

Figure 1. Temporal evolution of the logarithm of the number of microbes.

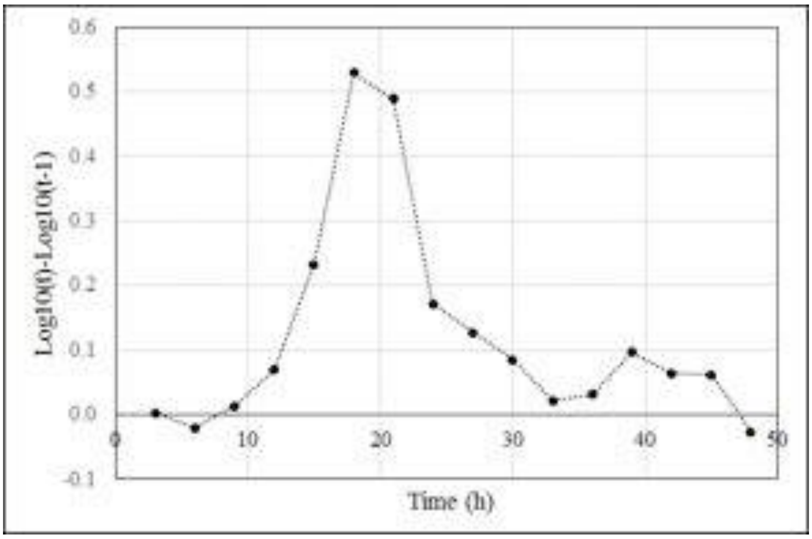

Figure 2. Variations in the logarithm of the number of microbes during the temporal evolution.

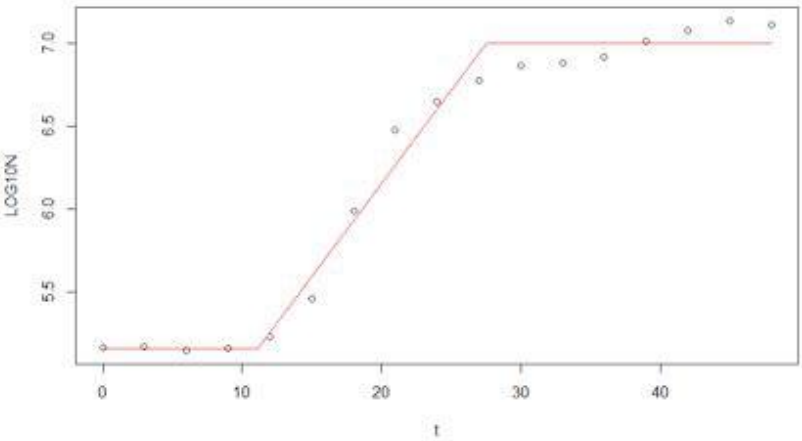

Figure 3: The three-phase linear fitting model for microbial growth, a simple pseudo-mechanistic model.

In relation to the possible use of the logistic function and other functions of the sigmoid family (corresponding to the Parts D and E), only some results will be presented and discussed here, those connected with the process of assigning biological meaning to the parameters involved in the definitions of those functions. Figure 4 shows some parameters illustrating or denoting important features of the microbial population growth. The logistic function is one of the most frequently studied, but in this study we focus on the use of Gompertz function, another sigmoidal function also widely used for certain temporal evolutions, and defined as follows:

$y(t) = A\exp(-\exp(B - C\cdot t)) = Ae^{-e^{B-Ct}}$

where A, B and C are the parameters involved. The process allowing reassignment of these parameters to others with biological meaning makes it possible to connect and use previous mathematical knowledge. As we know that microorganisms can grow exponentially over a period of time, it is useful to apply the logarithmic transformation on population size (N = N (t)) as has been mention before. Thus, working with the natural logarithm (or logarithm) in this case, the relative size of the population can be considered a new variable as follows:

$$y(t) \quad Ln\left(\frac{N(t)}{N_0}\right)$$

The three main phases of the microbial growth curve can be described by three parameters (Figure 4):

1) The maximum specific growth rate, μ_m, which is defined as the tangent at the inflection point of the curve.

2) The latency or lag time, λ, which is defined as the value of the intersection of this tangent with the abscissa.

3) The asymptote determined by the maximum value that the population can reach, Max.

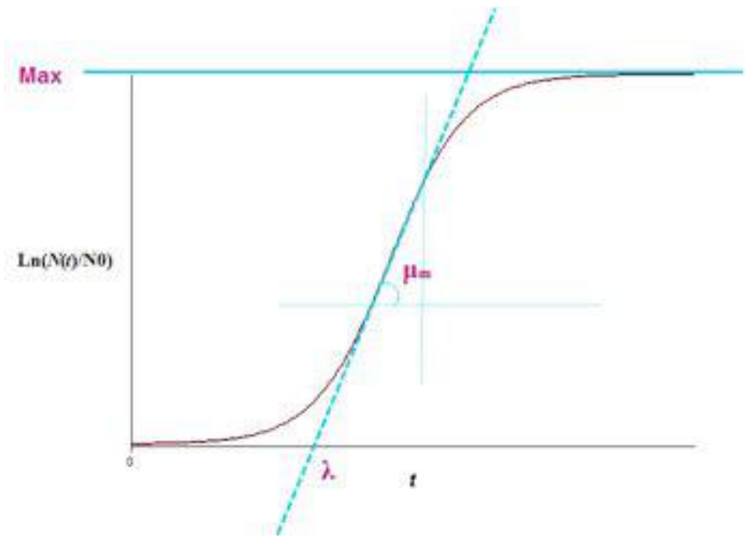

Figure 4. Generic microbial growth curve parameters: maximum growth rate (μ_m), lag time (λ), and the maximum possible value that can be achieved (Max).

A study of the first derivative and second derivative of the function allows us to identify the role played by the parameters involved in the definitions. Identifying the meaning of the parameters of these Gompertz functions, followed by a suitable reassignment, enables us to have a function with meaningful biological parameters in the context of microbial growth, and consequently obtaining a new pseudo-mechanistic model for this context [12].

The purpose of this task is to rewrite the Gompertz function using the Maple program, a new expression replacing the parameters A, B, and C given in the above expression (with no biological significance) by the parameters μ, λ, and Max (with biological significance) as Figure 4 illustrates. Whereas this could be solved by hand (with pencil and paper), and because one of the purposes is the use of the computer, the calculations involved in this process were carried out with Maple, avoiding errors in transcription or errors in simplification, and making these calculations faster. The students' responses to this part of parameters' reassignment to obtain the pseudo-mechanistic model were surprisingly interesting. It allowed students to use the incessantly repeated knowledge during their previous stages in Mathematics: first derivative, second derivative, inflection point, slope of the tangent line at a point, asymptote ..., but this time in a context that gave meaning to everything they had learned in both mathematics and microbiology.

A screenshot of the sequence of steps performed with Maple to get this pseudo-mechanistic model with μ, λ, and Max is as follows:

\> $y(t) := A\cdot e^{-e^{(B-C\cdot t)}}$

$$y := t \rightarrow A\, e^{-e^{B - C t}}$$

> $\frac{d}{dt} y(t)$

$$A\, C\, e^{-C t + B}\, e^{-e^{-C t + B}}$$

> $\frac{d}{dt} \frac{d}{dt} y(t)$

$$-A\, C^2\, e^{-C t + B}\, e^{-e^{-C t + B}} + A\, C^2 \left(e^{-C t + B}\right)^2 e^{-e^{-C t + B}}$$

> $solve\left(\frac{d}{dt} \frac{d}{dt} y(t) = 0, t\right)$

$$\frac{B}{C}$$

> $eval\left(\frac{d}{dt} y(t), t = \frac{B}{C}\right)$

$$A\, C\, e^{-1}$$

> $\boldsymbol{mumax = A\, C\, e^{-1}}$

$$mumax = A\, C\, e^{-1}$$

> $solve\left(mumax = A\, C\, e^{-1}, C\right)$

$$\frac{mumax}{A\, e^{-1}}$$

> $\boldsymbol{C := \frac{mumax}{e^{-1} A}}$

$$C := \frac{mumax}{A\, e^{-1}}$$

> $0 = y\left(\frac{B}{C}\right) + mumax \cdot \left(t - \frac{B}{C}\right)$

$$0 = A\, e^{-1} + mumax \left(t - \frac{B\, A\, e^{-1}}{mumax}\right)$$

> $solve\left(0 = y\left(\frac{B}{C}\right) + mumax \cdot \left(t - \frac{B}{C}\right), t\right)$

$$\frac{A\, e^{-1}\, (B - 1)}{mumax}$$

> $\boldsymbol{lag = \frac{e^{-1} A\, (B - 1)}{mumax}}$

$$lag = \frac{A\, e^{-1}\, (B - 1)}{mumax}$$

> $solve\left(lag = \frac{e^{-1} A\, (B - 1)}{mumax}, B\right)$

$$\frac{A\, e^{-1} + lag\ mumax}{A\, e^{-1}}$$

> $\boldsymbol{B := \frac{e^{-1} A + lag\ mumax}{e^{-1} A}}$

$$B := \frac{A\, e^{-1} + lag\ mumax}{A\, e^{-1}}$$

> $simplify\left(\frac{e^{-1} A + lag\ mumax}{e^{-1} A}\right)$

$$\frac{lag\ mumax\ e + A}{A}$$

> $B := \frac{lag\ mumax\ e + A}{A}$

$$B := \frac{lag\ mumax\ \mathrm{e} + A}{A}$$

> $\boldsymbol{A := Max;}$

$$A := Max$$

> $y(t)$

$$y := t \rightarrow Max\ \mathrm{e}^{-\mathrm{e}^{\frac{-mumax\ (t - lag)\ \mathrm{e} + Max}{Max}}}$$

Now considering that $y(t) = Ln\left(\frac{N(t)}{N_0}\right)$, $Max = Ln\left(\frac{N_{Max}}{N_0}\right)$, $mumax = \mu_{Max}$, $lag = \lambda$, we obtain

$$Ln\left(\frac{N(t)}{N_0}\right) = Ln\left(\frac{N_{Max}}{N_0}\right) e^{-e^{\left(\frac{-\mu_{max}(t-\lambda)e}{Ln\left(\frac{N_{Max}}{N_0}\right)}+1\right)}}$$

$$Ln(N(t)) = Ln(N_0) + (Ln(N_{Max}) - Ln(N_0))e^{-e^{\left(\frac{\mu_{max}(\lambda-t)e}{Ln(N_{Max})-Ln(N_0)}+1\right)}}.$$

This last expression, which is the formulation of a pseudo-mechanistic model of the population growth from Gompertz function, is appropriate for microorganisms when population sizes are expressed in logarithmic units, and it is one of the models that the program R uses. This model has been carried out with our set of data (Table 1) and the results achieved are shown in Figure 5, where the punctual estimations of the parameters μ_m, λ, Log10(N_0) and Log10(N_{max}) were 12.6, 0.35, 5.15 and 7.03 respectively.

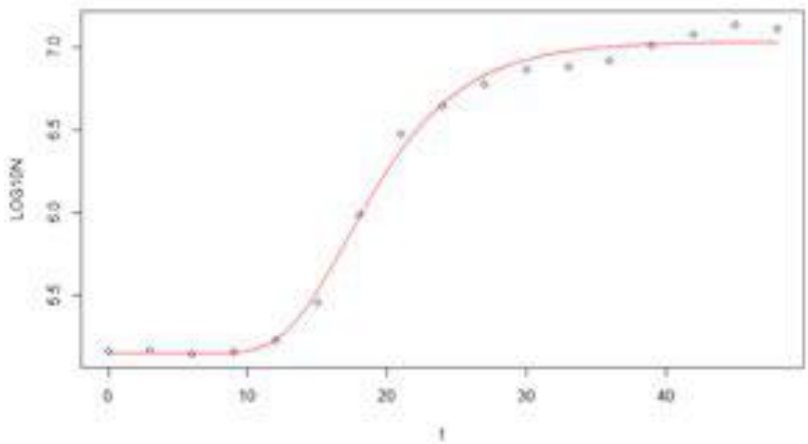

Figure 5. The Gompertz model adjusted to the data with the free statistical software R.

The perceptions of the students carrying out these tasks were collected by means of the following question: "*Please, indicate any positive aspect of the activity performed*", and some of the answers were:

- ✓ Tracking the activity in parts I think it is good and helps you to get a pretty solid idea of the different ways to build a model from the same data and compare these models.
- ✓ One of the positive aspects of the activity is the use of software tools for processing data. In order to improve the development of this practice, I think it would be necessary to have prior explanations or recall programs used in previous years.
- ✓ The previous analysis of the data that has been done with spreadsheets helped me a lot. I think the section corresponding to the use of R is a higher level because you have to

understand the program and run it properly. You should also have a very clear idea of what you are analysing and what you want to achieve, because R gives you more information (graphs and tables) than what is absolutely necessary for the analysis or which for me is not completely understandable.

- ✓ We have built and represented models which are easily understood in a particular case like the growth of microorganisms.
- ✓ It is important to learn to manage helpful data processing programs, and this practice gives us an idea of how to do this. Plot some functions using Maple and compare the differences between Excel solver and the R program has helped me to better understand the data analysis and modelling. However, I think prior oral explanation of the practice would have been helpful to refresh some mathematical concepts and programming.
- ✓ On my part, the entire process, applying different ways of analysing the same data, trying to discover which is the best fit for the data, testing what happened when increasing, or decreasing or increasing the parameters of functions, was a good idea. So, the whole process of going slowly and answering the questions has helped me learn a way to try using some experimental data, and also the fact that you can apply many models and some of them will adjust better or worse. Nevertheless, it is really important to try to understand the situation and not just end up having results or numerical values of certain parameters, for example. If you have not made the effort to think about the data, may not be able to interpret these results correctly.
- ✓ One positive aspect of this activity could be the use of mathematical programs such as Maple, because it is a very useful when solving differential equations, representing data, drawing graphs of functions, and manipulating expressions.
- ✓ It helps you understand that in order to solve the problem with the data available you can choose different methods. My opinion is that the practice has been very good. The fact that it is structured in straightforward steps makes it possible to follow without getting lost, while improving learning. At the same time it encourages an order and an explanation of the steps. It teaches us that every experiment fits a certain model and goes through a series of steps to obtain the best result.
- ✓ I have seen many different models to fit the initial data and that means that I will be able to analyse data using methods which I did not know before this activity.
- ✓ I learned various methods to address a situation that may recur in my studies such as the growth of microorganisms, and also the fact that what I did can be extrapolated to other populations.
- ✓ I found it very positive to understand the fundamentals of a solid growth model used in the R program. Testing other available options and how these can help us to solve our problems as biological engineers, using our current knowledge without advanced programming skills, have been profitable.

Final remarks

A rich context for exploring mathematical ideas and developing mathematical skills is biology. In this context, the models have been used in order to analyse and understand phenomena and to design and construct instruments that make a virtual "experimentation" possible, improving in an iterative way our representation of reality. The tasks designed have encouraged some imaginative applications using previous mathematical knowledge that students already had in order to build new models, starting with initial and not complicated approximations and relatively simple models to go deeper into the mathematical understanding of more sophisticated models, where the help of the computer has been revealed as indispensable. Its use was justified and integrated efficiently, training and improving the digital literacy of the students, that is, the general ability to use computers to tackle real problems.

REFERENCES

Papert, S. (1996). An exploration in the space of mathematics educations. *International Journal of Computers for Mathematical Learning, 1*, 95-123.

Bennison, A., & Goos, M. (2010). Learning to Teach Mathematics with Technology: A Survey of Professional Development Needs, Experiences and Impacts. *Mathematics Education Research Journal*, *22*, 31-56.

Buchanan, R.L., Whiting, R.C., Damert, W.C. (1997). When is simple good enough: a comparison of the Gompertz, Baranyi, and three-phase linear models for fitting bacterial growth curves. *Food Microbiology*, *14*, 313-326.

Caspersen, M.E., & Nowack, P. (2014), Model-based thinking and practice: a top-down approach to computational thinking. In: Koli Calling '14 *Proceedings of the 14th Koli Calling International Conference on Computing Education Research*. ACM New York, pp.147-15.

de Vries, G., Hillen, T., Lewis, M., Müller, J. & Schönfisch, B. (2006). *A course in mathematical biology: quantitative modeling with mathematical & computational methods*. Philadelphia: Society for Industrial and Applied Mathematics.

Ginovart, M. (2014). Discovering the power of individual-based modelling in teaching and learning: the study of a predator-prey system. *Journal of Science Education and Technology*, *23*, 496-513.

Perez-Rodriguez, F. (2014). Development and application of predictive microbiology models in food. In: D. Granato & G. Ares (Eds.) *Mathematical and statistical methods in food science and technology* (321-361). IFT Pess, Wiley Blackwell.

Shiflet, A.B, & Shiflet, G.W. (2014). *Introduction to computational science: Modelling and simulation for the science*. Princenton: Princenton University Press.

Wing, J.M. (2006). Computational thinking. *Communications of the ACM*, *49*, 33-35.

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Similarity, Homothety and Thales theorem together for an effective teaching

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Abstract: In this paper we show partial results from a research aimed to analyse the ways of reasoning and evolution of some students when they carry out homework for a teaching unit related to similarity, homothety and Thales theorem. The analysis is based on Lemonidis (1991) and the Van Hiele model of reasoning (Gutiérrez and Jaime, 1998). The sample for this study was a group of 9th grade students (14-15 years old) in a school from Floridablanca (Santander, Colombia). A preliminary analysis from the data collected shows interesting ways of solving certain tasks in which the participants used a rich language and showed a variety of ways of reasoning.

Résumé: Dans cet article, nous montrons des résultats partiels d'une recherche visant à analyser les modes de raisonnement et de l'évolution de certains étudiants quand ils effectuent des devoirs pour une unité d'enseignement liés à la similitude, homothétie et théorème de Thalès. L'analyse est basée sur Lemonidis (1991) et le modèle de raisonnement de Van Hiele (Gutiérrez et Jaime, 1998). L'échantillon de cette étude était un groupe d'élèves de 9e année (14-15 ans) dans une école de Floridablanca (Santander, la Colombie). Une analyse préliminaire des données recueilliés montre façons intéressantes de résoudre certaines tâches dans lesquelles les participants ont utilisé un langage riche et ont montré une variété de modes de raisonnement.

Introduction

Several studies conclude that teaching of similarity is very precarious and, thus, it generates weak learning in students, which does not contribute to the effective development of students' geometric thinking (Gualdrón, 2006).
We present here some results of a teaching experiment based on an original teaching unit that integrates similarity, homothety, and the Thales Theorem. The teaching unit was designed to improve the ways of geometric reasoning of a group of students who participated in the study. The main difference between this instructional proposal and others is that the activities we have designed present in a connected way the concepts and properties of similarity, homothety, and the Thales Theorem. This helps student to better understand the topic and to learn tools useful to solve problems and to improve their geometrical reasoning.

Theoretical Framework

To design a teaching unit for similarity, homothety and the Thales theorem, and then to analyse students' performance in the unit, we have taken into account the analysis of teaching homothety by Lemonidis (1991) and the Van Hiele model of reasoning (Gutiérrez and Jaime, 1998; Gualdrón, 2007).
Lemonidis (1991) characterized three different approaches to similarity to be considered when teaching this topic:

a) *Intrafigural relationship*: when the correspondence between elements of a figure and elements of a similar figure is highlighted, but without considering the idea of transforming a figure in the other one. Within this approximation, we may distinguish:
    - When the figures are part of a Thales configuration, in which the homothety components are considered, with adequate reasons.

- When the figures appear apart one from the other.

b) *Geometrical transformation seen as a tool*: the geometrical transformation is perceived as an application from the set of points in the plane into itself. This approach to similarity is useful to solve problems, for instance in trigonometry and calculus.

c) *Geometrical transformation seen as a mathematical object*: the geometrical transformation is characterized by an algebraic approach in which the objective is to find the transformation resulting from the composition (product) of two or more transformations.

The Van Hiele model of geometric reasoning is proving to be an excellent theoretical reference to organize and assess teaching and learning of geometry. However, similarity is a geometry topic in which research about the application of the Van Hiele model is very limited (Gualdrón, 2006). In this study we used the descriptors of level identified by Gualdrón (2006) and we have extended them to the specific contents of the teaching unit.
The activities in the teaching unit are mainly focused to students in Van Hiele level 2, although some students could be reasoning at level 1 or at level 3. Then, we include below the main descriptors of levels 1 to 3 for the topic of similarity, based on Gualdrón (2006):

- Descriptors of *Van Hiele level 1* for similarity:
  - Students recognize similar shapes based on their appearance that is based only on visual characteristics.
  - Students describe and compare shapes with terms like bigger, smaller, longer, etc.
  - Some students can use mathematical characteristics of similarity, but they do not do it in a consistent way. For instance, they may measure some corresponding angles and note that they are congruent.
  - Students identify the similarity of shapes in Thales configurations, but their arguments are visual.
  - Students can build or draw shapes being similar to a give one, but they do it visually, without taking into consideration mathematical properties like measure of angles or length of sides.
- Descriptors of *Van Hiele level 2* for similarity:
  - Students recognize similar shapes based on mathematical characteristics like measure of angles or length of sides.
  - Students may identify and generalize properties of similar shapes, like proportionality of corresponding sides, parallelism of sides when they are in a Thales configuration, etc.
  - Students assume that relative positions of similar shapes are irrelevant. They also assume that congruence is a particular case of similarity.
  - Students can build or draw shapes being similar to a give one taking into consideration mathematical properties like measure of angles, length of sides, or ratio, and making arithmetic calculations.
  - Students can use Thales configurations to prove that two shapes are similar.
  - Students may discover or induce some properties of similarity, for instance the cases of similarity of triangles.
  - Students can use the definition of similar shapes.
- Descriptors of *Van Hiele level 3* for similarity:
  - Students identify empirically the necessary and sufficient conditions for two triangles to be similar, and prove their truth deductively.
  - Students understand and use relationships among properties of similarity. For instance, they relate the characteristics of homothety to the different Thales configurations to prove that such configurations are cases of homotheties.

- Students identify necessary and sufficient properties of specific similar shapes. For instance, they acknowledge that having equal pairs of angles is necessary and sufficient for two triangles to be similar, but it is not sufficient for other polygons.
- Students can write informal deductive proofs of properties involving similarity of plane shapes.

Methodology

The sample for this study was a group of 27 grade 10 students (14-15 years old) in a secondary school in Floridablanca (Santander, Colombia). These students had studied in different grades the Thales theorem and homothety, but they had not studied the general concept of similarity before. Then, we planned the teaching unit to integrate the contents of similarity, homothety and Thales theorem, aiming to create on students a network of knowledge.

The teaching unit was designed taking into account the phases and levels of the Van Hiele model (Gutiérrez and Jaime, 1998) and the results of Lemonidis (1991). The activities were posed to students one after the other in the order determined by the consideration of Van Hiele phases. Each activity was presented to students in a sheet of paper, where students should justify the steps to the answer (numerically, graphically or verbally). Students could use ruler, square, compass, and other auxiliary drawing elements.

The teaching experiment took place in the ordinary classroom during the scheduled mathematics classes; the physical characteristics of the classroom allowed working in groups of three students. The teacher and a researcher (the first author) participated in all the experimental teaching sessions. The teacher was the responsible for the organization of the classes. The researcher was a participant observer, observing the students activity and, at the same time, cooperating with the teacher in tutoring and orientation to students during the classes.

The experiment was developed in nine sessions of 100 minutes, during 8 weeks. The students were organized in groups of three students, who discussed the possible solutions and then, each student wrote her own answer and conclusions.

The data collected consists on students' worksheets, video tapes of the classes showing the work made by the groups of students, interviews to some students, and researcher's field notes. The videotapes were done from. Some interviews were conducted in order to ask for clarifications and extra justifications, to ask students to explain what they have done, or to pose them other analogue tasks.

Analysis and Discussion

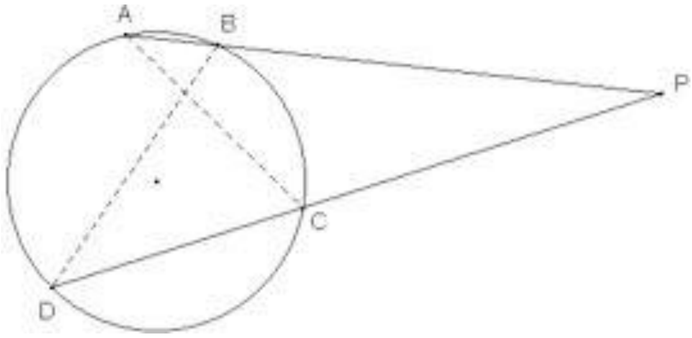
We present here the most relevant aspects of the data we have collected, with some representative examples of students' ways of reasoning and comments about them. We have focused on one activity (activity 24), shown in Figure 1. Each part of this activity asked students to prove that two triangles are similar.

The first part of the activity shows a intrafigural relationship among triangles APC and DPB, since the statement focus students' attention to corresponding sides of the triangles.

The second part of the activity presents the geometrical transformation of homothety as a tool to solve the problem, since identifying an homothety relating both triangles is the key to solve it.

Activity 24.

(1) Segments AP and DP belong to lines that intersect in point P and, at the same time, cut the circumference as shown in the diagram below. Justify that triangles APC and DPB are similar.



(2) Draw any triangle and, for each vertex, draw a line containing the vertex and parallel to the opposite side. In this way, you get a bigger triangle. Justify that this triangle is similar to the first one.

Figure 1. Activity 24.

One of our students, named Carlos (pseudonymous), used the homothety to prove the similarity of triangles in part 1 of the activity 24. In his answer, he first justified that there is a homothety, and then he used this homothety to justify the similarity of the triangles:

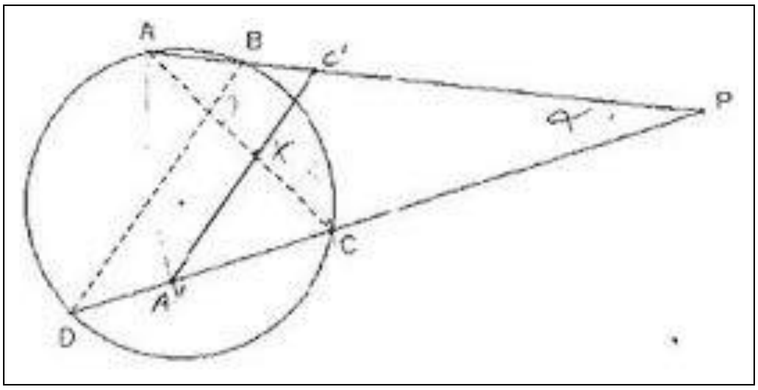


Figure 2. Carlos' answer to activity 24(1).

Carlos: *Translating* [copying] *distance PC over line PA and distance PA over line PD, we have triangle PA'C', which is similar to triangle DPB, since this process is just like tumbling the figure* [PAC]. *With this, it is possible to establish a homothety with centre P, since the correspondent vertices and P are collinear, PC'||PA and PA'||PD. And to prove that A'C'||BD, I used that, as <BAC and <BDC comprise the same arc, they are congruent, and as I have line PD, with two different lines cutting it forming the same angle, so the two lines are parallel and as these lines are BD and C'A', we have proved the last condition necessary to have a homothety.*

The previous answer shows a consistent use of Van Hiele level 3 reasoning, since the student did not use any specific measurement, but organized different properties of triangles and of the angles among parallel lines cut by another line to deductively prove that triangles APC and DPB are similar. The style of the proof is far from the formal proofs typical of level 4.

In his answer to activity 24(2), Carlos also used the homothety to justify that triangles ABC and A'B'C' are similar:

Carlos: *As the sides of the big triangle* [A'B'C'] *are parallel to the sides of the small one* [ABC] *(the correspondents), and knowing that, if the correspondent sides are parallel then they can be aligned to make an homothety and confirm the similarity, then we have a complete proof* *of the similarity* [of A’B’C’ and ABC].

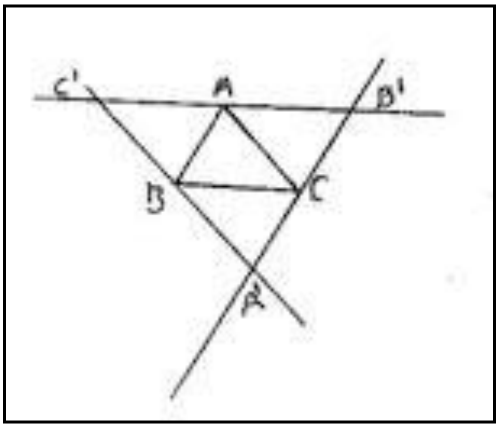


Figure 3. Carlos’ answer to activity 24(2).

Carlos’ answer to the second part of the activity is consistent with his previous answer, since he also produced a deductive proof. However, Carlos still has not fully acquired the ability of proof typical of Van Hiele level 3 since, in this answer, he did not feel the need to make explicit mention of Thales theorem nor to provide details to identify the focus of the homothety. One of the objectives of the teaching unit was to provide students with opportunities to practice and improve their proving abilities.

The results of the analysis seem to indicate that this way of teaching similarity (linking it to homothety and Thales theorem) was a factor highly positive for the acquisition by students of more and better abilities of reasoning.

Conclusions

Traditionally, the teaching of similarity is limited to the presentation of conditions for two geometrical figures to be similar, and then to the presentation of some graphic examples. Diverse studies (for example, Gualdrón, 2006) have shown that this way of teaching the subject limits extremely the geometric reasoning of students.

In this study we have analysed the arguments that a group of students stated to justify similarity among polygons, and we have found that students had more and better reasoning tools respect to results presented by previous studies (Gualdrón, 2006) where direct connections between similarity, homothety and the Thales Theorem were not established.

The results of this study contribute to the literature about effective ways of improving geometric reasoning, specifically in tasks related to similarity, by showing successful ways of connection among geometric subjects that, in most of cases, are taught in an isolated way.

REFERENCES

Gualdrón, É. (2006). *Los procesos de aprendizaje de la semejanza por estudiantes de 9º grado* (master thesis). Valencia, Spain: Departamento de Didáctica de las Matemáticas, Universidad de Valencia.

Gualdrón, É., Gutiérrez, Á. (2007). Una Aproximación a los descriptores de los niveles de razonamiento de Van Hiele para la semejanza. In Camacho, M., Flores, P., and Bolea, P. (eds.), *Investigación en Educación Matemática XI* (pp. 369-380). Tenerife, Spain: SEIEM.

Gutiérrez, Á., and Jaime, A. (1998). On the assessment of the Van Hiele levels of reasoning, *Focus on Learning Problems in Mathematics, 20*(2/3), 27-46.

Lemonidis, C. (1991). Analyse et réalisation d´une expérience d´enseignement de l´homothétie. *Recherches en Didactique des Mathématiques 11*(2/3), 295-324.

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Students' difficulties dealing with number line: a qualitative analysis of a question from national standardized assessment

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Abstract: In this paper we analyse some peculiar students' mistakes in a task selected from the Italian National Mathematics Standardized Tests. In particular we show different types of errors identified in the solutions of grade 6 students who faced an item that involved the management of the number line. We use both the statistical results of the national sample and the qualitative analysis of answers in a smaller sample of 181 students. Our study draws on previous research on difficulties that students face placing rational numbers on the number line. We use an intertwinement of different theoretical lenses to explain the possible causes of failure. We show how some students' answers can be interpreted as results of different misconceptions. The identified mistakes are related both to the management of the rational numbers representations (i.e. decimal representation and fraction) and to the manipulation of the graduate scale of the number line.

Résumé: Dans cet article, nous analysons certains particulières erreurs des élèves dans une tâche sélectionnée à partir des tests standardisés national italien en mathématique. En particulier, nous montrons différents types d'erreurs relevées dans les solutions des élèves de 6e année qui ont repondu à une question qui a impliqué la gestion de la ligne graduée. Nous utilisons à la fois les résultats statistiques de l'échantillon national et l'analyse qualitative des réponses dans un plus petit échantillon de 181 étudiants. Notre étude se fonde sur des recherches antérieures sur les difficultés que les étudiants ont à placer des nombres rationnels sur la ligne graduée. Nous utilisons un combinaison des différentes lentilles théoriques pour expliquer les causes possibles de l'échec. Nous montrons comment les réponses de certains élèves peuvent être interprétés comme des résultats de différentes misconception. Les erreurs identifiées sont liés à la gestion des représentations des numéros rationnels (i.e. représentation décimale et fraction) et à la manipulation de la ligne graduée.

Introduction

This paper presents an ongoing research developed in the *Ideas for the Research* project, funded by the Italian National Institute for the Educational Evaluation of Instruction (INVALSI). The aim of the project is the analysis of the outcomes of the Italian mathematics standardized tests that were collected by INVALSI from 2010 to 2013 in grades 6 and 8. Here we discuss only one of the stages of this research: the identification and analysis of students' errors in answering to an item that will be part of a vertical study over different grades. Our research group is composed by researchers in Mathematics Education (authors of this paper) and Statistics.

In the first part of the research, qualitative and quantitative analyses were intertwined. The quantitative analysis of the national sample was carried out by Mariagiulia Matteucci and Stefania Mignani: the two researchers in Statistics of our group. They used some INVALSI results to investigate possible trends in the data. They implemented a multilevel latent class analysis (Vermunt, 2003, 2008) to classify students and to judge the item characteristics. This Statistical method has given a classification of students in a fixed number of groups characterized by different levels of performance. The researchers identified some items in which the students with low performances had a higher probability to give a wrong answer in comparison with the other students (Branchetti et al., in press). Starting from these results, we carried out a qualitative analysis of the solution strategies developed by students when facing the identified items. We made a longitudinal analysis of the outcomes of the Italian national standardized mathematics tests in grade 6 and 8 (ibidem): in a subsample of the national one, students' answers were analysed to identify the possible mistakes that occurred in linked items. We focused on the mathematical concepts (in the sense of Vergnaud, 2009) involved and the possible difficulties arising from conversion among different semiotic representations (Duval, 2003).

We conjecture that the grade 6 items selected by our study could have a predictive power for the outcomes in the linked items identified in grade 8. Therefore, in the second step of the research, we are carrying out some classroom activities in order to collect new data about the possible students' answers to the items analysed in the first part of the research. For example, we have already re-administered some items to a sample of 181 students from different Italian cities (both from North and South Italy) and from schools with different socio-economical backgrounds. The data collected are coherent with the national sample results.

In this paper we discuss this last part of our ongoing research on the analysis of students' answers and identification of their errors over different grades.. In particular, we show the analysis of the solutions of some grade 6 students who faced a task that involves the management of rational numbers' different representations and the number line. We selected this topic because literature shows how ability to perform well on this task with fractions is highly predictive of later performance in mathematics (Jordan et al., 2013). Our analysis is based on previous research on students' difficulties with natural or rational numbers on the number line.

After a brief overview of the theoretical lenses used to look at students' answers, we describe and analyse the data collected, showing our interpretation of students' difficulties.

The number line

The international Research highlights the crucial role of the number line in mathematics education. For example, in Skoumpourdi (2010) the number line is presented as a didactical tool with high potential, especially since it provides a simple way to picture mathematical concepts: as a matter of fact, the number line is used for counting, for estimations and for representing time, but also for the representation of different number sets. Moreover, number line can be used for providing geometric models of the arithmetical operations, for measuring, and comparing quantities. In the same work it is also pointed out that many studies report difficulties and limitations in the use of the number line and propose educational activities to overcome these difficulties.

> The potential of the number line for organizing thinking about numbers and operations on the one hand, and the difficulties that arise from its use on the other hand, lead many researchers to propose learning strands and teaching sequences for its use in the teaching/learning process of mathematics. (Skoumpourdi, 2010, p. 2)

Skoumpourdi stresses that the number line can be presented in different versions: structured or semi structured, with or without numbers and other symbols, but also empty. For each of these representations, students may use different approaches in finding solutions.

In this paper we will focus on the difficulties that students could have facing tasks in which they have to place rational numbers on the number line.

In the following, we briefly present some results from the educational research about this specific topic.

Concerning the placing of rational numbers on the number line, we initially have to distinguish the difficulties about the management of decimals and fractions. According to Iuculano and Butterworth (2011) both adults and children are more accurate when performing this task with decimals rather than fractions because "decimals afford direct mapping onto a mental number line and, therefore, allow for easier magnitude assessment than do fractions" (De Wolf et al., 2014, p.2136).

Students may also have difficulties in conversion between representations in different semiotic registers (Duval, 1993): they could have troubles in finding strategies to pinpoint numbers on the number line because the number line is a hybrid representation (a line with a scale on it). Every geometric operation can be translated into an arithmetic operation and carried out algorithmically and vice versa (Gagatsis, et al., 2003). Some studies on the number line and fractions highlight the distinction between making partitions and reading pre-marked partitions (Mitchell & Horne, 2008). In fact, the identification of the unit in number lines seems to be problematic; in particular students' strategies may change whether the line is partitioned or not, since marks may act as perceptual distractors (Lesh et al, 1982). Students may have difficulties if one unit of length is divided into parts (Behr & Bright 1984): for instance some students, in order to determine the fraction denominator, ignore the endpoint and count only internal hash marks. If intervals between points already drawn have unequal lengths, students can count the number of points ignoring the distance. Other difficulties may stem from an over-generalization of part-whole partitioning strategies in measurement contexts. For example, Saxe and colleagues (2007) show that a student can progressively divide one unit by 2 and then an half by 2 and find ¾, but this strategy does not work in general: e.g. 2/7 . Therefore many mistakes can be generated by the interlacement of misconception about rational numbers and number line management. Other studies (e.g. Hartnett & Gelman,1998) show the conflict between ordering natural numbers on the number line (when numbers get bigger as values increase) and fractions (when the denominator gets bigger the fraction is smaller). A common error is to put the fraction close to the value of numerator or of the denominator: this can be explained in terms of the "whole number bias", that means considering fractions as two separated whole numbers (Ni & Zhou, 2005) and comparing them separately (Stafylidou & Vosniadou, 2004).

In the next paragraph we analyse many of the difficulties identified in the quoted literature. Moreover we show how the same mistakes can be interpreted as the product of different misconceptions.

Data analysis

The item that we analyse in this paper was administered at national level to grade 6 italian students in 2011. This item has been selected, according to statistical methods, because it has a high discriminating power for low achievers: i.e. in the Italian national sample the students with low performances had a higher probability to give a wrong answer in comparison with the other students (Branchetti et al., in press).

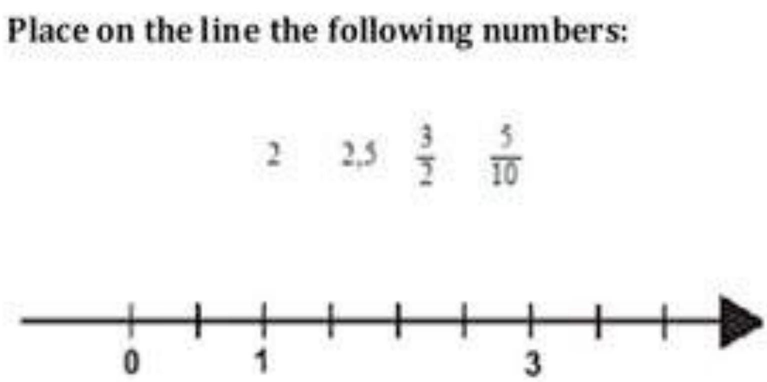
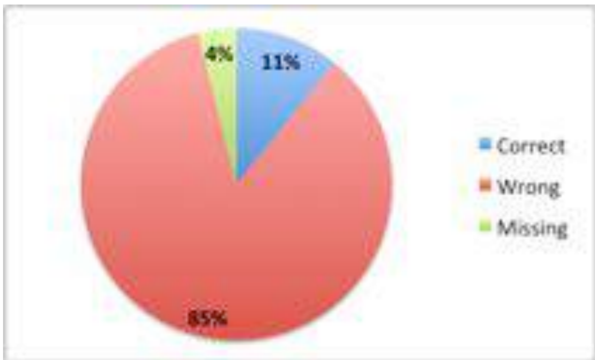


Figure 1: Item D8 from INVALSI test administered in grade 6 (2011)

Item D8 (Figure 1) concerns the placement of rational numbers on an oriented line. The rational numbers are presented in both decimal and fraction representations. The hash marks drawn on the line, refer to a specific unit of measure: the distance between two consecutive hash marks is 0.5.

On national level, only 11% of students places all the numbers in the correct positions (Fig. 2a). In order to identify possible students' difficulties, we administered this item to a sample of 181 students from different Italian cities (both from North and South Italy) and from schools with different socio-economical backgrounds. The data collected are coherent with the national sample results (Fig. 2b).

2a: National results



2b: Sample results

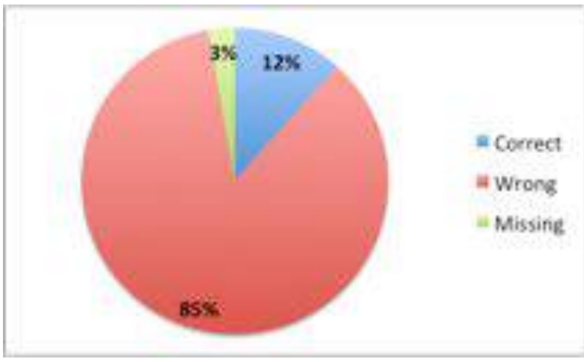


Figure 2: Results of Item D8

Drawing on the literature results described in the previous paragraph, it sounds reasonable to suppose that students, who faced the D8 item, should:

- identify the correct unit of measure referring to the hash marks (Saxe et al, 2007; Behr & Bright, 1984);
- manage different number representations (Duval, 1993; Gagatsis et al. 2003);
- find out the order relation between these numbers (Hartnett & Gelman, 1998).

First of all, we focus on some difficulties already noticed in putting decimal numbers on the line: identify the unit of measure and the order relation between numbers.

Students' answers analysis allows us to recognize the following possible wrong behaviours.

a) Students do not correctly manage the unit of measure.

Among students who give a wrong answer, almost half of them (46%) fails in the management of the unit of measurement. Students do not consider the size of the intervals (the distance between hash marks): e.g. 2 and 2.5 are placed in two consecutive hash marks, subsequently to 1 (Fig. 3).

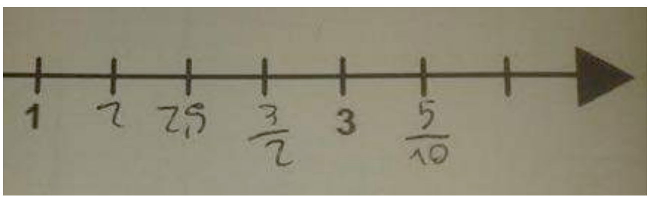


Figure 3: Students who pinpoint 2 and 2.5 subsequent to 1.

b) Students identify correctly just the natural number (2).

The 15% of students correctly uses the hash marks only for the natural number: the most common mistake is to put the 2.5 between the hash mark of 2 and the next one (Fig. 4). The other types of errors are less than 10% frequent.

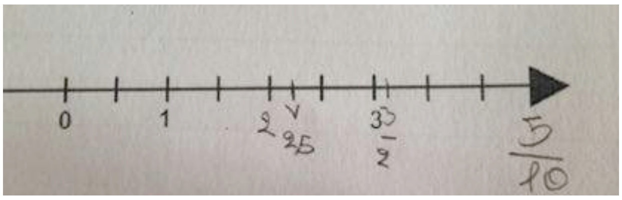


Figure 4: Students who place 2.5 between two consecutive hash marks.

c) Students have many difficulties in placing fractions on the number line

In our sample, 80% of the students correctly places the numbers in decimal form. The main students' difficulty is to identify the correct order between the numbers in different representations: decimal and fraction. In this case, the most frequent approach for numbers comparison is the conversion between registers (Duval, 1993): in particular from fractions to decimals, a conversion that frequently results in errors. Analysing students' errors, we define some common difficulties observed in converting fractions to decimals. Some students try to convert numbers from fractions to decimals dividing numerator and denominator, but they invert the order calculating the division between denominator and numerator. For example, a group of students (8%) places 5/10 as if it were 2 (Fig. 5), probably because they divide 10 by 5.

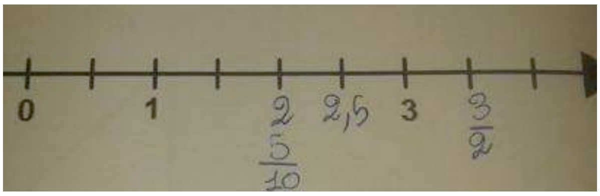


Figure 5: Students place 5/10 as if it were 2.

Other students (41%) pinpoint 3/2 near the hash mark referred to the value 3, probably because they convert 3/2 as if it were 3.2 (Fig. 6).

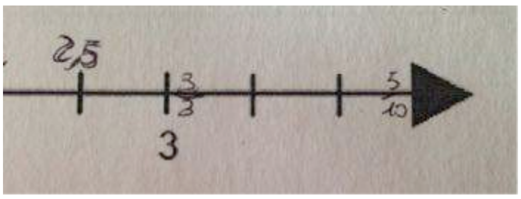


Figure 6: Students place 3/2 near 3.

We can also provide an alternativeinterpretation of this phenomenon:students may consider 3/2 equivalent to 3+½, so they put the number in the hash mark next to 3 (Fig. 7).

Similarly, some students place 5/10 over the end of the line (7%), probably because they interpret 5/10 as 5.10, 5+1/10 or something else greater than 5 (Fig. 7).

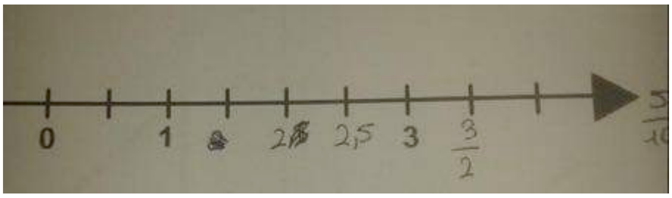


Figure 7: Students place 3/2 as it were 3+1/2 and 5/10 as it were more than 5.

Other errors are less frequent (6% of the students). For example, in Figure 8 we can see one of these unusual answers. A possible interpretation of this excerpt is that the student tries to represent the numbers with brackets that, in two cases (5/10 and 2), are long as the number that they would like to identify.

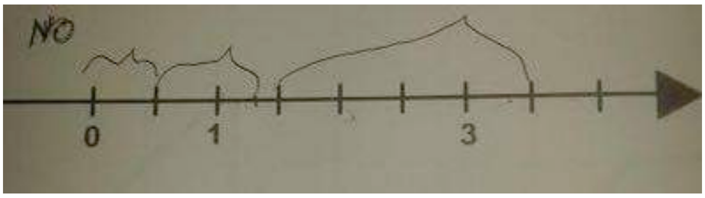


Figure 8: Unusual approach.

Conclusions

In our research we identify and analyse items from Italian national mathematics standardized tests over different years in different grades. We have identified items in which the students with low performances in the tests had a higher probability to give a wrong answer in comparison with the other students and, among those, we have selected items that deals with longitudinal topics. For this reason in this paper we argue an items on rational numbers and number line. Jordan et al (2013) also argue that tasks involving the placing of fractions on a number line, can be high predictive for future outcomes in mathematics.

The data analysis presented in this paper shows some difficulties that students face while dealing with the number line. These difficulties depend both on the number line scale's management and the specific number set that is considered. In particular, according to literature (Iuculano & Butterworth, 2011; Saxe et al., 2007), we observe that most of errors concern the pinpointing of fractions.

Our data confirm that students have fewer difficulties with natural numbers, while errors are more frequent when rational numbers are involved. Students' mistakes concerning decimal representation are related to the management of the placement on hash marks. The analysis carried out in our study show that some students seem to interpret the number line as a list of ordered numbers. They do not take into account that the distance between the hash marks represents the difference between numbers: e.g. many students have difficulty in positioning 2.5 correctly (Fig.3). Furthermore, we find many examples of excerpts as the one shown in Figure 4. These students' answers suggest a kind of *implicit model* (in the sense of Fischbein et al., 1985): hash marks can be employed just for integers.

As already emphasized by many studies, more frequently students show difficulties with fraction and, in particular, in the conversion from fraction to decimals. The explanation of the fact that many students try to convert fractions to decimals can be both didactical and psychological. DeWolf et al. (2014) highlight that in many textbooks decimals are used mostly to refer to continuous models (as the line) while fractions generally represent the discrete ones. Such a result suggests that pupils may try to convert fractions to decimals because decimals are more familiar in the context of a continuous model as the number line is. The same authors suggest, quoting Iuculano & Butterworth (2011), that decimals afford direct mapping onto a mental number line and, therefore, allow easier magnitude assessment than fractions do. The analysis presented in the previous paragraph suggests that most of the errors are not related specifically to the number line representation, but they are connected to the manipulation of fractions in general. For example, students who place 5/10 as if it were 2 can be influenced by what Fischbein et al. (1985) define as "implicit model's rule that the dividend must be larger than the divisor". According to these authors the violation of such rule shows in reversing the order of terms (Fig.5). Moreover, Markovits & Sowder (1991) show that there are students who convert the fraction *a/b* as the decimal *a.b*, as we observe in the answers given by pupils who put *a/b* near the numerator *a* (Fig. 6).

Other errors with fractions could be related to the "whole number bias" (Ni & Zhou, 2005) as, for example, the placing of 5/10 over the position of 5 on the number line. There could be different interpretations about the possible mistakes identified:

- To put 5/10 after 5 because it is transformed in 5.10 or because is seen only as something greater then 5 (5+1/10)

- To put 3/2 after 3 because it is transformed in 3.2 or because is seen only as something greater then 3 (3+1/2)

There are also mistakes that depend on the particular fraction that is considered as, for instance, the identification of 5/10 with 2 (the greater number divided by the smaller one if the bigger is a multiple).

As the difficulties that students face with decimals and fractions seem to be different, it can be questioned if there is a relationship between them. By combining the analysis of the different errors observed in our sample, we have found out an interesting fact: almost all students who place 2.5 wrongly on the number line show difficulties also in placing the two fractions. This evidence leads to conjecture that the inability to pinpoint decimal numbers can be a predictor of the inability of placing fraction, but the question remains still open. In our further research we will deeply investigate this aspect.

This analysis can be useful in teacher education: as a matter of fact, it contains suggestions concerning the most significant students' wrong strategies in a typical item of the Italian national assessment (the management of number line).

Students' strategies can be further investigated in "vertical chains" made of questions on different levels that involve same contents. This kind of analysis may give insights about the predictive power of these tasks.

REFERENCES

Behr, M., & Bright, G. (1984). Identifying fractions on number lines. *Paper presented at the meeting of the American Educational Association*. New Orleans, LA.

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. (1982). Rational-number concepts. In R. Lesh & M. Landau (Eds.), Acquisition of mathematical concepts and processes (pp. 91-126). New York: Academic Press.

Branchetti, L., Ferretti, F., Lemmo, A., Maffia, A., Martignone, F., Matteucci, M., & Mignani, S. (in press). A longitudinal analysis of the Italian standardized mathematics test. *Proceedings of the 9th CERME*. Prague: ERME.

DeWolf, M., Rapp, M., Bassok, M., & Holyoak, K. J. (2014). Semantic alignment of fractions and decimals with discrete versus continuous entities: A textbook analysis. In B. Bello, M. Guarini, M. McShane, & B. Scassellati (Eds.), *Proceedings of the 36th Annual Conference of the Cognitive Science* Society, 2133-2138. Austin: Cognitive Science Society.

Duval, R. (1993). Registres de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de didactique et de sciences cognitives*, 5, 37-65.

Fischbein, E., Deri, M., Sainati Nello, M., & Sciolis Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16(1)*, 3-17.

Gagatsis, A., Shiakalli, M., & Panaoura, A. (2003). La droite arithmétique comme modèle géométrique de l'addition et de la soustraction des nombres entiers. *Annales de didactique et de sciences cognitives, 8*, 95-112.

Hartnett, P.M., & Gelman, R. (1998). Early understandings of numbers: Paths or barriers to the construction of new understandings? *Learning and Instruction: The Journal of the European Association for Research in Learning and Instruction*, *8(4)*, 341-374.

Iuculano, T., & Butterworth, B. (2011). Understanding the real value of fractions and decimals. *Quarterly Journal of Experimental Psychology, 64,* 2088-2098.

Jordan, N.C., Hansen, N., Fuchs, L.S., Siegler, R.S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fractions concepts and procedures. *Journal of experimental psychology, 116*, 45-58.

Markovits, Z., & Sowder, J.T. (1991). Students' understanding of the relationship between fractions and decimals. *Focus on learning problems in mathematics, 13*, 3-11.

Mitchell, A., & Horne, M. (2008). Fraction Number Line Tasks and the Additivity Concept of Length Measurement. *Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia.* M. Goos, R. Brown, & K. Makar (Eds.)

Ni, Y., & Zhou, Y.D. (2005). Teaching and learning fractions and rational numbers: the origins and implications of whole numbers bias. *Educational psychologist, 40(1*), 27-52.

Saxe, G.B., Shaughnessy, M.M., Shannon, A., Langer-Osuna, J., Chinn, R., & Gearhart, M. (2007). Learning about fractions as points on a number line. In P. Elliot, W.G. Martin, & M.E. Strutchens (Eds.), *NCTM Sixty-ninth Yearbook*. Reston: NCTM.

Skoumpourdi, C. (2010). The number line: an auxiliary means or an obstacle? *International Journal for Mathematics Teaching and Learning*.

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction, 14*, 503-518.

Vermunt, J.K. (2003). Multilevel latent class models. *Sociological Methodology, 33*, 213–239.

Vermunt, J.K. (2008). Latent class and finite mixture models for multilevel data sets. *Statistical Methods in Medical Research, 17(1)*, 33–51.

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What teachers think about mathematical proof?

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Abstract: This paper presents a quantitative study, initial part of a larger work that also involves a qualitative component, which aims to study the conceptions of mathematics teachers in 5th to 9th grades (n=115) about mathematical proof. The results, that are based on the application of a questionnaire, show that teachers, despite their different academic backgrounds (all of them with a background in mathematics, but some performed courses with a strong pedagogical component and others with a predominant mathematical component), recognize the nature of proof and its importance in student learning, showing awareness of the need to adapt proof to students capabilities.

Résumé: Cet article présente une étude quantitative, première partie d'un plus grand travail qui implique également une component qualitative, qui vise à étudier les conceptions des enseignants de mathématiques de la 5ème à la 9ème année (n = 115) au sujet de la preuve mathématique. Les résultats, qui sont fondées sur l'application d'un questionnaire, montrent que les enseignants, en dépit de leurs différents formations académiques (chacun d'eux avec une formation en mathématiques, mais certains ont effectué cours avec une forte component pédagogique et les autres avec une component mathématique prédominante) , reconnaître la nature de la preuve et de son importance dans l'apprentissage des élèves, montrant prise de conscience de la nécessité d'adapter la preuve aux capacités des étudiants.

Introduction

The influence of knowledge, conceptions and beliefs of mathematics teachers in their professional practices has been widely documented by research (Ponte & Chapman, 2006; Thompson, 1992). This influence can result in opportunities for successful teaching practices, and, consequently, in rich student learning, or, on the contrary, in obstacles to the complex teaching-learning process. An example of this is the way the teacher outlooks and integrates mathematical proof in classroom activities. The mathematical proof is an important element in mathematics and mathematics teacher's activity, both in training and teaching practices. Recent curriculum changes in Portuguese mathematics programs (ME, 2007; MEC, 2013) have given more emphasis to mathematical proof. This is the context from which this work arises, that. We are interested in understanding how Portuguese Mathematics teachers look to mathematical proof and how they integrate it in their practices and also the goals they seek to achieve through it, having on the horizon, the current math programs.

In this study, proof is understood in a wide manner, and not strictly formal, covering reasoning and communication mathematical processes which allow to sign the veracity of certain mathematical statements by the force of the reason and not based on authority criteria, whatever it is. The broader research, combining quantitative with qualitative data, integrating survey of teachers (questionnaire and interview) and classroom observation, starts with a macro look at mathematics' teachers. It is that more general look that we present here, that is, we seek to an approach to conceptions of mathematics teachers in 5th to 9t^{h} grades by applying a structured questionnaire.

Background

To prove is an activity present in various fields of human action, whenever you want to ensure a certain statement, by its intrinsic value, and not by any other powers associated with who says or where says. This activity, although it is present in our everyday life, it is particularly relevant in contexts of genesis and communication of knowledge, such as the scientific production, in particular mathematics, and the case of the teaching-learning of mathematics in the classroom. In any of these contexts, prove is a condition of freedom and an affirmation of the "force of reason", being connected to the argumentation capacity (Boavida, 2005).

Proof, understood as the result of the proving action, assumes in various fields of science different forms that result from the nature of the knowledge that is concerned and especially from the practices followed by the communities in which that is developed. In mathematics, proof assumes proper contours, leading some authors (Dreyfus, 2000; Hanna, 2000, 2002; Knuth, 2002) to consider that proof is what distinguishes mathematics from other sciences.

Definitions for mathematical proof are usually related to the functions assigned to it. De Villiers (2003) considers six proof functions: i) verification; ii) explanation; iii) systematization; iv) discovery; v) intellectual challenge; and vi) communication. These proof functions are related to the context in which they are performed. While verification is more common in mathematician's activity, the explanation appears more linked to educational activity (CadwalladerOlsker, 2011). In turn, the communication function may either arise in the mathematician's or school mathematics activity. In this work, considering the educational context in which the research takes place, we have adopted a definition of mathematical proof as a process of argumentation, emphasizing it explicative and communicative functions, aimed learning of mathematical topics and transversal capabilities (Boavida, 2005).

This didactic value of mathematical proof has been stated by several authors and professional organizations (Hanna, 2000, 2002; NCTM, 2000). In this regard, the *Principles and Standards for School* Mathematics (NCTM, 2000) consider that must be provided to all students the opportunity to "recognize reasoning and proof as fundamental aspects of mathematics; make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs; select and use various types of reasoning and methods of proof" (p. 56).

In this sense, Portuguese programs of Mathematics (5th to 9th grades) suggest student work with proof, first informally and then with a progressive degree of formalization. In 5th and 6th grades it is suggested in the program, for example, that "to the sum of the amplitudes of internal and external angles of a triangle resort to informal proof" (ME, 2007, p. 38). For students of 7th to 9th grades, mathematics program advocates that students should "understand the notion of demonstration and be able to do deductive reasoning" (ME, 2007, p. 51). The current mathematics programs of basic education (MEC, 2013) also gave greater prominence to proof, although in order to give it greater level of formalization.

In this context of curricular recommendation, and because of their potential to influence practices, it seemed pertinent to study what teachers think about mathematical proof and how they conceive proof in student's activity and in their own professional activity.

Methodology

In this paper, we focus on conceptions of mathematics teachers of 5th to 6th grades (2nd cycle) (n=43) and 7th to 9th grades (3rd cycle) (n=72) on the mathematical proof, adopting a quantitative approach in the treatment of information resulting from the application a questionnaire (Gall, Gall & Borg, 2003). In the sample selection, we sent questionnaires to elementary schools of the northern of Portugal, where the majority of the schools stand. There two districts were chosen, one inland and the other from the coast. Doing so, we pretend to cover a diversity of schools. The sampling method was by convenience (Hill & Hill, 2012), since the questionnaires were distributed in various schools by some teachers who had contact with the Project team.

This methodological approach arises from the fact that, initially, we want to meet in a broad way teachers' conceptions about mathematical proof, with no intention to generalize to all the mathematics teachers. The sample is formed by all questionnaires received. Among participant teachers (n=115), the female gender is prevalent (85), the age is 41 years old (ranging between 30 and 62 and the average age is 44) and 15 is the mode of years of service (ranging from 5 to 35 years).

The questionnaire consists of five parts: the first part includes four questions about age, gender, school year and years of service; the second part consists of 14 closed questions about proof in mathematics; the third part includes 11 closed questions about the proof in the student activity (5th to 9th grades); the fourth part has nine closed questions about proof in teacher activity; and the fifth and final part includes eight questions about proof in mathematics curricula.

In data analysis, responses were organized and processed using the SPSS software. Analysis was guided by the following dimensions: (i) proof in mathematics; (ii) proof in elementary school student's activity; and (iii) proof in teacher's activity. In these dimensions, the answers to the items of the questions relate to the selection of a frequency option, according to the scale: Strongly Disagree (SD); Disagree (D); Neither Agree Nor Disagree (NAND); Agree (A) and From the answers we have determined average and standard deviations for all options after the coded options SD, D, NAND, A and TA with the values 1, 2, 3, 4 and 5, respectively.

From the numerical values, we applied the Student's t-test for independent samples, considering the group of the 2nd cycle teachers (5^{th} to 6^{th} grades) and the group of teachers of the 3rd cycle (7^{th} to 9^{th} grades). In the statistical analysis performed adopted the level of significance of 0.05.

Results

To understand what teachers think about mathematical proof, we organize information from their responses to a questionnaire according to the dimensions already mentioned.

Proof in Mathematics

The underlying formality to logical argument of proofing the veracity of a mathematical statement leads, in general, teachers to distinguish this method from experimental ones that are used in other areas of knowledge. In addition to the deductive method, teachers recognize other proof methods. Whichever proof method to which they use to prove the veracity of a mathematical result, to most teachers this activity is evidenced by being the basis for mathematical knowledge construction (Table 1).

	2^{nd} cycle		3^{rd} cycle	
	$\bar{x}$	s	$\bar{x}$	s
The proof in maths has different nature of the proof in other sciences.	3,5	1,07	3,7	1,05
The deductive method is the only method that proves mathematical results.	2,1	0,84	1,9	0,88
The proof is essential for the construction of mathematical knowledge.	3,8	0,99	3,9	0,83

Table 1. Nature of mathematical proof.

Comparing the average of the two groups of teachers (see Table 1), we find that there are no statistically significant differences between these groups. Teachers' conceptions about proof are associated with the functions they give to it.

For all teachers, proof performs several functions, including the verification and explanation of a mathematical statement's veracity. Already the discovery/ invention function of new mathematical results gathers indecision among teachers, which may be due to the absence of this proof function in the school context. With respect to the function of systematizing mathematical statement, observation of Table 2 show there are considerable differences between the means of two groups of teachers (Table 2).

	2nd cycle		3rd cycle	
	$\bar{x}$	s	$\bar{x}$	s
The proof has the function of verification the mathematical statement.	4,1	0,92	4,1	0,99
The proof has the function of explanation the mathematical statement.	4,1	0,83	3,8	0,96
The proof has the function of discovery / invention of new results.	3,1	1,01	2,9	1,16
The proof has the function of systematization a mathematical statement.	3,6	0,85	3,1	1,09

Table 2. Functions of mathematical proof.

Comparing the average of the two groups, the application of T-Test determines statistically significant differences in the item, "The proof has the function of systematization of a mathematical statement" ($p=0.007$), which is more emphasized by 5th/6th grades teachers than by 7th to 9th grades teachers.

The proof in elementary school student's activity

The consideration of mathematical proof in school context leads us to investigate the role that students play in this activity. Teachers of both cycles agree with the involvement of students in the proof of mathematical results, which, in their perspective, leads students to understand the nature of this activity. In this involvement, they disagree that the proof should be reserved for the best students (Table 3).

	2nd cycle		3rd cycle	
	$\bar{x}$	s	$\bar{x}$	s
Students must participate in the proof of mathematical results.	3,8	0,85	3,8	0,86
Proofs should be made only by the best students.	2,1	0,92	2,4	1,08
Proving leads students to understand the nature of mathematical activity.	3,7	0,85	3,9	0,83
Students should use the mathematical results without proving them.	2,7	0,85	3,0	0,93

Table 3. The student's activity in mathematical proof.

The comparison of the means of teachers' answers of each school cycles reflects that there are no statistically significant differences between the groups (Table 3). Concerning the use by the students of mathematical results without being proved, teachers expressed indecision. The involvement of students on proving mathematical results gathers the agreement of teachers of both cycles in the development of students understanding of mathematical concepts and capabilities to reason logically and to communicate mathematically (Table 4).

	2nd cycle		3rd cycle	
	$\bar{x}$	s	$\bar{x}$	s
Proving increases understanding of mathematical concepts by students.	3,9	0,87	3,6	0,88
Proving develops student's ability to reason logically.	3,9	0,83	4,1	0,69
Proving develops student's mathematical communication capability.	3,9	0,92	3,9	0,76

Table 4. The mathematical proof in student learning.

Comparing the average of the two groups of teachers (see Table 4), we observe that there are no statistically significant differences between these groups.

The proof in the teacher's activity

The systematization of knowledge and the abstract nature involved in the proof of mathematical results are factors that increase the complexity of this activity (de Villiers, 1990). The complexity inherent of the activity of proving mathematical results seems to be the reason why teachers did not express their agreement to the difficulty of integrating the proof in their teaching strategies, and developing this activity in their lessons and engage students in activities that lead to conjecture and proving mathematical results. Despite this hesitation, teachers tend to disagree that proof does need not be offered to students of elementary school (Table 5).

	2nd cycle		3rd cycle	
	$\bar{x}$	s	$\bar{x}$	s
I have difficulty integrating the proof in my classroom.	2,9	1,07	3,2	1,15
I often prove the mathematical results in my classroom.	3,2	0,92	3,2	0,95
In my classroom, I challenge students to formulate and prove conjectures.	3,3	0,91	3,3	0,89
I believe that it is not necessary teach elementary school students to prove.	2,5	1,10	2,4	1,06

Table 5. Proof in the teaching practice of the mathematics teacher.

The comparison of means of two groups of teachers (see Table 5) does not highlight significant differences between these groups.

Final considerations

Mathematics teachers recognize the specificity of mathematical proof distinguishing the nature of this activity of experimental methods. In addition to the deductive method, they recognize other proof methods of mathematical results. This might be connected to the conceptual framework of how they organize mathematical knowledge.

Teachers from both cycles of teaching identify multiple functions of proof, such as verification and explanation of a mathematical statement. The systematization function is prevalent in teachers of the 2nd cycle (5th to 6th grades), which is understandable considering the student's school grade. Teachers in both cycles agree with the participation of students, and not just the top ones, on the proof of mathematical results, because it favors the development of mathematical concepts and reasoning and mathematical communication, with special emphasis to argumentation capacity, as advocated by Boavida (2005). This result shows that teachers recognize the didactical recommendations concerning the mathematical proof (Hanna, 2000, 2002; NCTM, 2000). Nevertheless, mathematics teachers have difficulties integrating proof situations in their classes, which can be derived from the fact that this is an activity that is conceptually demanding or this is a practice that needs teacher training, as pointed by Boavida (2005). Despite the difficulties, teachers consider that it is necessary to involve students on the proof of mathematical results. In summary, teachers of the two cycles, although with different backgrounds (all of them with a background in mathematics, but ones performed courses with a strong pedagogical component and the others with a predominant mathematical component), reveal similar conceptions about the mathematical proof. Considering not just the difficulties but also the desire that teachers expressed in integrating mathematical proof in their teaching strategies it makes sense to promote formation dynamics, such as training activities, to encourage this integration.

REFERENCES

Boavida, A.M. (2005). *A argumentação em Matemática: Investigando o trabalho de duas professoras em contexto de colaboração* (Dissertação de doutoramento). Lisboa: APM.

CadwalladerOlsker, T. (2011). What do we mean by mathematical proof? *Journal of Humanistic Mathematics*, 1(1), 33-60.

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, 24, 17-24.

De Villiers, M. (2003). *Rethinking proof with The Geometer´s Sketchpad*. Emeryville: Key Curriculum Press.

Dreyfus, T. (2000). Some views on proofs by teachers and mathematicians. In A. Gagatsis (Ed.), *Proceedings of the 2nd Mediterranean Conference on Mathematics Education* (Vol. 1, pp. 11-25). Nikosia. Cyprus: The University of Cyprus.

Gall, M., Gall, P., & Borg, W. (2003). *Educational research: An introduction*. Boston: Allyn and Bacon.

Hanna, G. (2000). Proof and its classroom role: a survey. In *Atas do Encontro de Investigação em Educação Matemática-IX EIEM* (pp. 75-104). Fundão: SPCESEM.

Hanna, G. (2002). Mathematical Proof. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 54-62). New York: Kluwer Academic Publishers.

Hill, M. M., & Hill, A. (2012). *Investigação por questionário*. Lisboa: Edições Sílabo.

Knuth, E. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, 5, 61–88.

Ministério da Educação (2007). *Programa de Matemática do Ensino Básico*. Lisboa: Ministério da Educação.

Ministério da Educação e Ciência (2013). *Programa de Matemática do Ensino Básico*. Lisboa: Ministério da Educação e Ciência.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Ponte, J. P., & Chapman, O. (2006). Mathematics teachers' knowledge and practices. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 461-494).Rotterdam: Sense.

Thompson, A. (1992). Teacher´ beliefs and conceptions: A synthesis of the research. In D. A. Grows (Ed.), *Handbook of research in Mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

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The Mathematical Textbook as an obstacle in the learning of measure

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Abstract : The following study discloses how textbooks may potentially become obstacles in the treatment of certain topics in the field of measure, as presented in the educational curriculum. A qualitative methodology is employed herein to describe the content of textbook activities. In addition, the frequency with which the latter appear throughout the chapters is quantified.

Résumé: L'étude ici présent montre comment le libre de texte peut supposer un obstacle per traiter certains aspects de la branche de mesure présents dans le programme d'études. C'est pour ça qu'on emploie la méthodologie qualitative au moment de décrire le contenu des activités du livre de texte et qu'on quantifie la fréquence dont ils apparaissent tout le long des activités.

Introduction

From the beginning of compulsory education, the implementation and use of a textbook in the mathematics classroom has been a given fact. Moreover, most teachers use the resource of a textbook most of the time (Pepin, Gueudet and Trouche, 2013; Vincent and Stacey, 2008). Likewise, the TIMMS study, centred on the measurement of mathematical and scientifical knowledge of fourth and eighth grade students, brings forward that most mathematics teachers use the textbook as the main written source of teaching material (Alajmi, 2012; Plianram and Inprasitha, 2012). Even in the time of digitalisation, when the impact of technology is an obvious reality, textbooks still play a central role amongst teaching resources in order to accomplish the educational and learning process (Pepin, et al., 2013).

Attention has been recently drawn towards textbooks due to the acknowledgement of problems in teaching quality (Azcárate and Serrado, 2006). The width and depth of teachers' knowledge of school-level mathematics is one of the most influential elements to the quality of the teaching and learning process (Schoenfeld and Kilpatrick, 2008).

It therefore seems clear that textbooks used to and still do exert a great deal of influence on what is taught in the mathematics classroom. Furthermore, the mathematical knowledge that is brought into play in the classroom is conveyed through textbooks (Bromme and Brophy, 1986). The state of research on mathematical textbooks has changed considerably in the last three decades, during which the international research community has focussed growing attention on this subject. However, the study and analysis of mathematical textbooks as a field of research in itself has not been developed as much as other fields in mathematics education (Fan, 2013).

We have directed our attention to the sections of textbooks that are devoted to 'measure'. The reason for our choice resides in the fact that, since the creation of elementary school as an institution, this branch of mathematics takes up an important role within mathematics education due to its multiple applications in both social and scientific fields (Chamorro, 1996).

Our interest is centred upon determining the adequacy of learning contents regarding measure that are present in mathematics textbooks with regard to the teaching objectives of the curricula. For this purpose, we analysed the measure-related tasks proposed in textbook activities and compare them to current curricular guidelines.

The textbook's approach to "measure"

Measure constitutes one of the main human activities from which mathematics is developed. It is present in all cultures, from the most ancient, due to the fact that it allows comparison, order, estimation or calculation with more or less precision at different magnitudes (Luelmo, 2001; Bishop, 1999). The Cockcroft report (1982) justifies the importance of the study of measure both regarding the necessities of adult life – due to the large amount of measurements we perform on a daily basis – and with respect to the requirements of the working world. Moreover, an annual publication of the National Council of Teachers of Mathematics (NCTM), specifically the Yearbook of 1976, dedicates a whole chapter to highlighting the technological advances achieved thanks to the precision of measurements.

An education excluding measure would be inconceivable given that it is essential for students to face up to everyday life's needs. In addition, different features converge on the subject of measure; geometrical, arithmetic and problem-solving elements, as well as a wide range of developmental abilities and skills such as creativity and the ability to think (Del Olmo, Moreno and Gil, 1989).

Another sample of the relevance of measure is found in the content of international educational evaluation tests. The PISA tests of mathematical competence evaluation cover problems of quantity, amongst other types. In addition, these tests regard the notion of quantity as an essential part of mathematical literacy and may be considered the "most dominant mathematical element that is essential to functioning and taking part in the world" (OCDE, 2003). This involvement with the world is understood with regard to the quantification of the latter, which requires a comprehension of measure, recounting, magnitudes, units, indicators, relative size, numerical tendencies and patterns. In this field, mathematical literacy allows us to apply our knowledge of numbers and operations to a wide range of situations and environments (Caraballo, Rico and Lupiáñez, 2013).

Chamorro (2001) highlighted the phenomenon of *arithmetisation* of measure in textbooks, in other words, a colonisation of measure on the part of arithmetics. Textbooks specifically provide students with values for the majority of measures used in measure activities, rendering these activities useful to practice elementary arithmetic operations or ordering exercises. Indeed, the measure problems found in textbooks do not raise questions that may conceptually be related to measure in itself.

Given these circumstances, students reach secondary school with serious difficulties, since they lack in basic measure-related concepts and procedures required in other situations. It is no wonder that, as Chamorro (1988) points out, the teaching of magnitudes and their measure are associated to mastering the metric system. In addition, students are considered to have reached the set objectives when they are able to make conversions confidently and rapidly enough (Chamorro, 1988).

In addition, practical activities such as measurements are almost non-existent and are often undergone with a considerable lack of resources and inadequate classroom management (Chamorro, 2003). With regard to surface area, activities that require paving of an area continue to be overlooked; only grids are used for the purpose of solving activities that are too advanced for the students' capacity (Chamorro, 2001). Furthermore, the latter are used in contexts that are far apart from everyday life, since the students are not asked to measure the surface of objects familiar to them (Chamorro, 2001). There is currently no agreement on which teaching process is most suited to measure. Even though teaching staff and researchers theoretically recognise the advantages of certain approaches, classroom practices and textbooks reflect very different positions.

However, Alsina (2000) detects four types of curricula, developed in different contexts and locations:

- The official curriculum, which corresponds to the official documents proposed by educational authorities who elaborate the programmes of each subject, highlighting the contents, objectives, evaluation criteria, etc.

- The potential curriculum, which is determined in several teaching publications and in resources such as textbooks.
- The taught curriculum, which is developed by the teacher throughout the length of the school year.
- The learned curriculum, which is the curriculum the students have acquired.

The potential curriculum keeps to the official curriculum by building upon it from a theoretical and practical point of view (Alsina, 2000). In the following study we ask ourselves how far apart the official and potential curriculums actually are in regard to the teaching of measure portrayed in mathematics textbooks of the third stage of primary education.

Hierarchical organisation of tasks

In Gairín, Muñoz and Oller's contribution (2012), the grading of PAU mathematics exams (A-levels equivalent) is done by the hierarchical organisation of the different tasks required for solving the mathematical exercises in question. This classification divides the necessary mathematical steps to accomplish an exercise as follows:

A. Main tasks: steps that clearly constitute the main objective of the activity.

B. Auxiliary tasks, where:

> B1. Specific auxiliary tasks: Tasks that play an instrumental role in reaching the solution of a problem or exercise that includes main tasks concerning specific content items.
>
> B2. General auxiliary tasks: Mathematical tasks that the student has carried out throughout his/her previous mathematical training.

This hierarchical organisation is used in our analysis in order to classify the different tasks that come up in the resolution of measure activities present in mathematics textbooks of 5^{th} and 6^{th} grade. Given the length of this communication, we will limit our analysis to the main tasks promoted in the textbooks.

Goals of the study

Envisioning the aforementioned situation, we intend to determine the type of mathematical knowledge on measure that is addressed in textbook exercises for 5th and 6th grade of elementary school (i.e. ages 10-12) and how this relates to the official curriculum. For this purpose we have defined the following objectives:

- To characterise the measurement activities from textbooks by analysing their main aims, requiring the solution of tasks.
- To determine the hidden curriculum on measure proposed in textbooks and to compare the treatment given to each of the different tasks with that of the official curriculum.

Methodology

In order to carry out the following study, we aim to conduct a qualitative research study from an interpretative perspective (Bisquerra, 2004). We will focus on describing the main mathematical task promoted in each activity of the textbook on the subject of measure. The selected textbooks (Fraile, J., 2012; 2013) were chosen for being the most sold in Catalonia and Spain (MEC, 2014);

hence textbooks from this publishing house are amongst the most widely used in primary school classrooms.

In our first step we describe the tasks required to reach the solution to each exercise. In other words, we elaborated a transcription of the information in the form of tasks following the hierarchical organisation proposed by Gairín, Muñoz and Oller (2012). Therefore, we obtained a first description of the information collected in the textbooks.

We observed that the richest and most interesting knowledge content was found in the higher stage of primary education. Consequently, we decided to start with the textbooks of 5th and 6th grade, focussing on the high-level mathematical knowledge that the primary school teacher is required to have for a minimal use of the textbook.

Analysis

The analysis of textbook activities is carried out from the characterisation of the mathematical tasks posed within them. Figure 1 displays an example based on the calculation of the addition of complex time quantities.

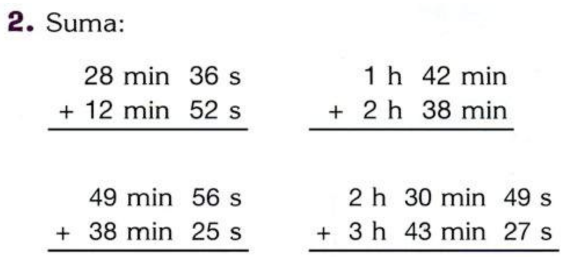

Figure 1. Exercise on time quantity sums.

The process is initiated by the comprehension of the activity, leading to the determination of the main task, which entails adding up amounts of time expressed in a complex way and requires knowledge of the relationship between sexagesimal units. The specific auxiliary tasks required to solve this activity are the following: the knowledge of measure units for time magnitudes, the knowledge of the relation between units, what this relation is, the concept of sexagesimal base change and application of the algorithm of the sum in decimal base between the same units. The general auxiliary tasks of this activity involve writing vertical sums, the sum of two numbers and the algorithm of the sum. In some cases, the activity may lead to situations such as the following: when adding up two numbers we obtain their exact equivalence and 0 units remain, we can remove them according to the context and carry one if the sum obtained in the same units is greater than 1. A detail of the final sketch of the hierarchical task organisation is displayed in Figure 2.

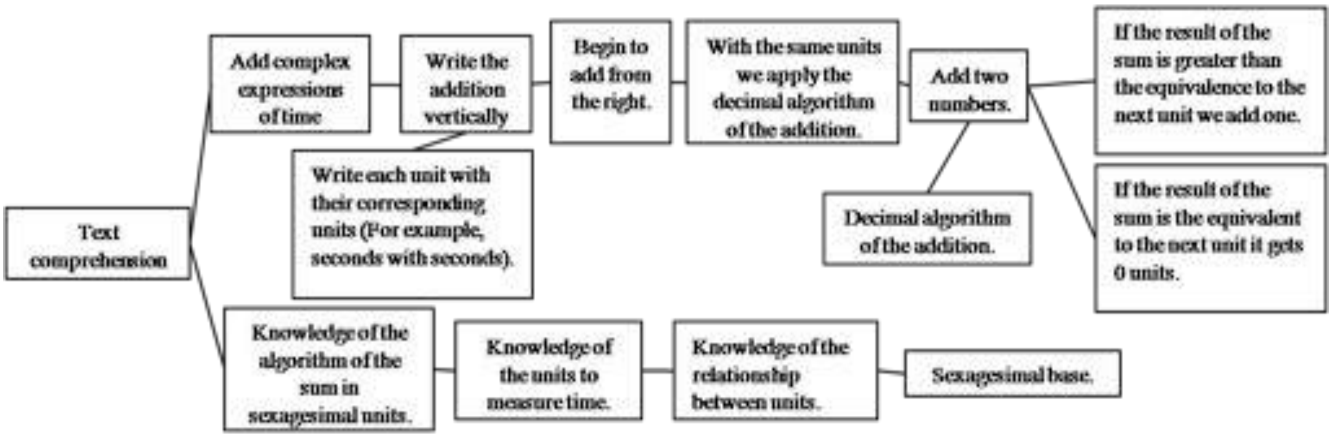

Figure 2. Diagram of the task analysis for the exercise in Figure 1.

Once this type of analysis has been carried out for all the measure activities from the textbooks, we employed the software "NVivo" to code the different tasks identified and group them into different categories according to their principal task, i.e. the objective of the activity in itself. This allowed us to obtain a description of the mathematical processes and sub-processes required for each type of activity analysed, as shown in the following results section.

Analysis

We decided to group the different tasks described in the former into sections with the purpose of addressing processes and concepts that are directly interrelated and to promote result visibility. The aforementioned sections are outlined as follows:

Section related to	Main tasks
Basic arithmetic operations (BAO)	Addition, subtraction (directly or calculating an added to fulfil an equation), multiplication, division and verification of the result of the operations. Rounding up. Writing a mixed fraction such as a decimal number. Choice of objects that add up to an established measurement.
Complex and non-complex expressions (EX)	Change the expression of a measurement from complex to non-complex and vice versa.
Operations with complex time expressions (OP_EX)	Addition and subtraction of complex time expressions. Interpretation of time zones.
Relations between measurements (REL)	Ordering of different measurements. Comparison of different measurements. Relating expressions with different units.
Magnitudes and units of measure (MU)	Identification of magnitudes and measurements in a body of text. Choice of the most appropriate unit or measure.
Measurement of geometrical magnitudes (MGM)	Volume, surface area or perimeter calculation by element recounting, measuring or applying a formula. Calculation of the radius/diameter of a circumference.
Unit change (UC)	Unit change process.
Algebra (ALG)	Tasks related to the introduction of algebraic procedures.
Measure (MEA)	Measurement, estimation of error in a measurement, reading off a measurement on an instrument as presented in the activity.
Draw (DRAW)	Depiction of a measure with the appropriate tools.
Proportionality and scale (PS)	Use of proportions and scale between numerical values of measures.

Table 3.

The resolution of activities classified into the section of basic arithmetic operations require the execution of one of the tasks proposed in the latter taking the context of measure into account. For instance:

Activity 1. A medicine pill contains 25 mg of sugar. Calculate the grammes of sugar needed to manufacture 6,000 pills of the same medicine.

Activity 2. A person weighs 68,5 kg. If when weighing him/herself while carrying a backpack, the scales indicate 70.15 kg, how much does the backpack weigh?

Even though these two activities belong to and are contextualised as measure activities, we must note that the main objective of these activities is to multiply and subtract, respectively, therefore, the main task associated to this activity is related to the basic operation of multiplication and subtraction.

We considered the frequency of occurrence of each task, as represented in the following graph:

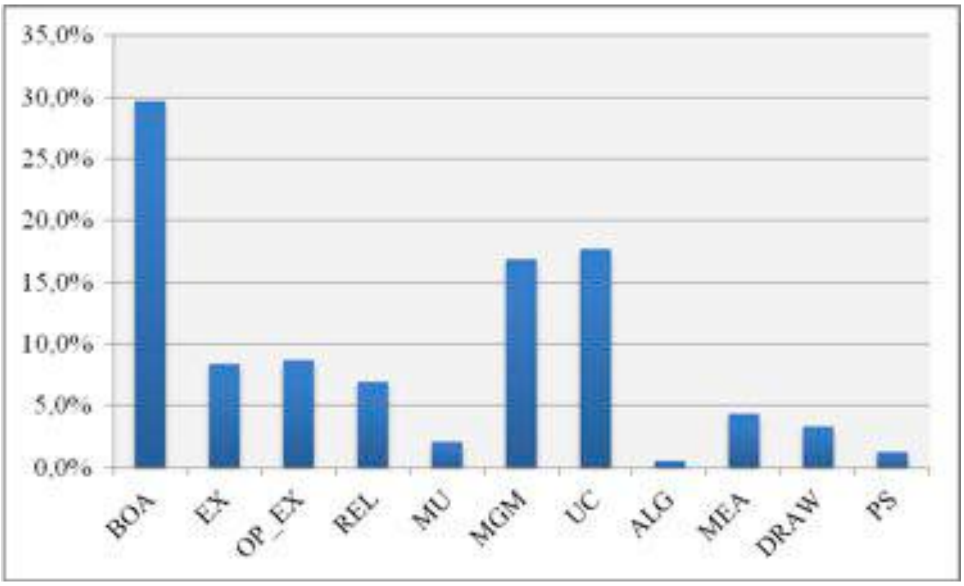

Figure 4. Percentage of occurrence of each of the task sections

As can be observed in the previous figure, the main tasks that occur most frequently in the analysed textbooks are those related to the execution of basic operations, to unit change and to the calculation of volumes, surface areas, perimeters and radiuses or diameters.

If we notice the different objectives and techniques that appear in the autonomic curriculum of Catalonia for the section on measure of the higher stage of primary school and we compare the latter to the hidden curriculum revealed in our results, we observe an excessive treatment of some elements and, in turn, the complete absence of certain curricular objectives.

Given the current treatment of measure in textbooks, it is no wonder that, as pointed out by Balbuena (2002), some measuring instruments are still genuinely unknown to students. In the autonomic curriculum, one of the techniques required is to critically select the appropriate tools and techniques to measure with a certain precision, as well as to be able to produce an oral, graphic and written description of the measure of different magnitudes to contrast and analyse different measuring strategies. As we can observe in the displayed results, the analysed textbook does not promote the design of measuring strategies for performing a measurement in a significant context.

Within the section of measure-related tasks, the activities that demand the student to perform measurements as principal task add up to only 1% of the total measure activities proposed. In addition, the anticipation and interpretation of the error of a measure only amount to 0.3% of the tasks analysed and are only treated from the approach of obtained error interpretation. Given that the error of a measurement is inherent to the measuring process, we may suspect that the measurement action is presented in an unreal form. This treatment is presented in a skewed and incomplete way, since it provides the false belief that measuring processes are always exact.

The work related to another curricular objective, that of comparison and ordering of different magnitudes, is almost always carried out numerically, with previously-assigned measures in order to speed up the comparison and ordering process. We therefore observe that empirical work related to measure is not promoted.

A curricular objective which receives greater attention in the textbook analysed is comprehension and use of the international measure system and time units, the equivalence between units and the use of equivalence, mainly numerical, applied to the measuring process. The activities concerning this objective account for 17.7% of total activities and account for the main body of measure-related assignments proposed in the textbook. Not to be overlooked is the fact that the continuous execution of conversion activities favours the devastation of order of magnitude (Chamorro, 2001). As noted by Chamorro (2001), this may be due to the fact that society demands a practical measure of magnitudes that underlies the idea of repeated and algorithmic practice.

Activities related to measure estimation are reduced to the choice of an appropriate measure unit or of the value of a previously performed measurement that is more suited to the context. Therefore, estimation strategies are neither developed by using common referents nor are measurements carried out to be contrasted with the corresponding estimations made. The other great section considered is that of the calculation of the surface area of flat figures, volume, and the relation between the surface area and the volume of a figure.

Discussion

The review of curricular objectives covered by textbooks shows us a very skewed approach to the middle school teaching of measure. Textbooks scarcely promote the usage of measuring tools and when they do, it is in very exceptional cases. This may be due to the fact that this type of activity requires a greater control and management of the classroom on the part of the teacher. However, the execution of manipulative activities and measuring exercises with suitable tools favours students’ conceptualisation from their own experience (Burgués, 2000; Chamorro, 2001). Throughout the stage of primary education it is rare to find manipulative resources in classrooms (Burgués, 2000). Different arguments regarding the scarce presence of these resources are usually involve lack of time and discipline and limited abstraction level of the activities (Burgués, 2000). In addition, the use of the aforementioned resources allows the students to learn actively and feel more motivated towards such an important subject of the mathematics curriculum (Casas, Luengo and Sánchez, 1997).

The different approaches to working on magnitudes and measure in the classroom vary from the centrality of the Metric Decimal System to the most recent, in which a more complete working method is put forward, based on the construction of a magnitude, its measurement and the possibility of carrying out estimations about it (Del Olmo, Moreno y Gil, 1989).

Estimation has again become one of the most forgotten subjects in the textbook’s treatment of measure. As reasoned by Callís, Fiol, De Luca and Callís (2006), estimation of measure has not received the attention it deserves in research of mathematics didactics, amongst other reasons, due to the scarce presence of resources to work on it. The limited number of estimation activities is added to teachers’ general lack of knowledge of methods of promoting its development in the classroom. Joram, Gabriele, Bertheau, Gelman, and Subrahmanyam (2005) show that when a teacher is asked to propose measure estimation activities to his pupils, the requested tasks turn out to be riddles instead of measure estimations, since the teaching staff do not provide them with the context or information necessary to create the appropriate working environment. Therefore, it would be desirable that the mathematics textbook would provide samples of estimation activities to complement the knowledge of the teacher.

Therefore, we can state that textbooks, as the main source of mathematical activities in primary school classrooms, have a didactical approach to measure that is oriented towards a spectrum of measure activities that are unrelated to the act of performing measurements or to putting the latter into context. For this reason, we understand that the strict use of the textbook provides incomplete learning options with regard to measure and poses an obstacle for students in the way of grasping the concepts of measure that are required both for their academic life and in their personal development.

In conclusion, it is necessary to point out that the current arithmetisation of measure as proposed by Chamorro continues to be a fact. Following this style of teaching, one might ask what happens to the knowledge of measure acquired in this manner.

REFERENCES

Azcárate, P. y Serradó, A. (2006). Tendencias didácticas en los libros de texto de matemáticas para la ESO. *Revista de Educación*, *340*, 341-378.

Alajmi, A. H. (2012). How do elementary textbooks address fractions? A review of mathematics textbooks in the USA, Japan, and Kuwait. *Educational Studies in Mathematics, 79* (2) , 239–261.

Alsina, C. (2000). Mañana será otro día: un reto matemático llamado futuro. In J.M., Goñi (Coord.) *El currículum de matemáticas en los inicios del siglo XXI* (pp.13-20). Barcelona: Grao.

Balbuena, L. (2002). La medida del tiempo a través del tiempo. *Uno: Revista de Didáctica de las Matemáticas*, *8*(30), 31-40.

Bisquerra, R. (Coord.). (2004). *Metodología de la investigación educativa.* Madrid: La Muralla.

Bromme, R., & Brophy, J. (1986). Teachers' cognitive activities. In *Perspectives on mathematics education* (pp. 99-139). Springer Netherlands.

Burgués, C. (2000). El currículum de primaria. In J.M. Goñi (Coord.), *El currículum de matemáticas en los inicios del siglo XXI* (pp.59-66). Barcelona: Grao.

Callís, J., Fiol, M. L., Luca, C., y Callís, C. (2006). Estimación métrica longitudinal en la educación primaria. factores implícitos en la capacidad estimativa métrica. *Uno: Revista De Didáctica De Las Matemáticas*, *43*, 91-110.

Caraballo, R.M., Rico, L., Lupiáñez, J.L. (2013). Cambios conceptuaales en el marco competencial de pisa : El caso de las Matemáticas. *Revista de curriculum y formación del profesorado, 17*(2), 225-241.

Casas, L.M. , Luengo, R., Sánchez, C. (1997). Recuperación de instrumentos y unidades de medida tradicionales en Extremadura como motivación al estudio de la medida. *Suma : Revista sobre Enseñanza y Aprendizaje de las matemáticas*, *25*, 97-112.

Chamorro, M. C. y Belmonte, J. M. (1988). *El problema de la medida*. Madrid: Síntesis.

Chamorro, M.C. (1996). El currículum de medida en educación primaria y ESO y las capacidades escolares. *Uno: Revista de didáctica de las matemáticas*, (10), 43-62.

Chamorro, M.C. (2001). Las dificultades en la enseñanza-aprendizaje de las magnitudes en Educación Primaria y ESO. In *Dificultades del aprendizaje de las matemáticas* (pp. 79-122). Madrid: Secretaría General Técnica.

Chamorro, M. C. (2003). El tratamiento escolar de las magnitudes y su medida. En M.C. Chamorro (Ed.), *Didáctica de las matemáticas* (pp.221-272). Madrid: Pearson.

Cockcroft, W.H. (1985). *Las matemáticas sí cuentan*. Madrid: MEC.

Del Olmo, M. A., Moreno, F. & Gil, F. (1989). *Superficie y volumen*. Madrid: Síntesis.

Fan, L. (2013). Textbook research as scientific research: towards a common ground on issues and methods of research on mathematics textbooks. *ZDM*,*45*(5), 765-777.

Fraile, J. (2013). *Matemàtiques 5*. España: Vincens Vives.

Fraile, J. (2012). *Matemàtiques 6*. España: Vincens Vives.

Gairín, J.M., Muñoz, J.M., & Oller, A.M. (2012). Propuesta de un modelo para la calificación de exámenes de matemáticas. In A. Estepa, A. Contreras, J. Deulofeu, M. C. Penalva, F. J. García y L. Ordóñez (Eds.), *Investigación en Educación Matemática XVI* (pp. 261-274). Baeza: SEIEM.

Joram, E., Gabriele, A. J., Bertheau, M., Gelman, R., ySubrahmanyam, K. (2005). Children's use of the reference point strategy for measurement estimation. *Journal for Research in Mathematics Education*, *36*(1), 4-23

Luelmo, M.J. (2001). Medir en secundaria: algo más que fórmulas. *Actas del X JAEM*. (pp. 727-737) España.

MEC (2013). *Panorámica de la Edición Española de Libros 2013.* Madrid: MEC.

OCDE (2003). *El programa pisa de la OCDE qué es y para qué sirve*. Descargado el 6 de Mayo de 2015 de http://www.oecd.org/pisa/39730818.pdf

Pepin, B., Gueudet, G., & Trouche, L. (2013). Investigating textbooks as crucial interfaces between culture, policy and teacher curricular practice: two contrasted case studies in France and Norway. *ZDM*, *45*(5), 685-698.

Plianram, S., & Inprasitha, M. (2012). Exploring Elementary Thai Teachers’ Use of Mathematics Textbook. *Creative Education, 3*(6) , 692–695.

Schoenfeld, A. H. y Kilpatrick, J. (2008). Towards a theory of profiency in teaching mathematics. En D. Tirosh & T. Wood (eds.), *Tools and Processes in Mathematics Teacher Education* (pp. 321-354). Rotterdam: Sense Publishers.

Vicent, J. & Stacey, K. (2008). Do mathematics textbooks cultivate shallow teaching? Applying the TIMMS video study criteria to Australian eight-grade mathematics textbooks. *Mathematics Education Research Journal*, *20*(1), 82-107.

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Constructing meanings of fraction with MLD5 students

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Abstract: There is consensus among researches that fractions are among the most complex mathematical concepts that children encounter in their years in primary education. Factors contributing to the complexities of teaching and learning of fractions lies in the fact that they comprise a multifaceted construct (Charalambous and Pitta-Pantazi, 2005) encompassing five interrelated sub-constructs: part-whole, ratio, operator, quotient and measure. However, while part-whole sub-construct is strongly linked to ratio and operator, it is weakly linked to the measure and quotient sub-construct. Our aim is to present a didactical sequence that fosters the development of meanings of fractions related to the relationship between part-whole and measure and between part-whole and ratio, in order to conceive fractions as numbers that can be placed on the number line. The didactical sequence is addressed to elementary school classes that are "inclusive" with respect to students with mathematical learning disorders (MLD).

Résumé: Il y a un consensus parmi les recherches que les fractions sont parmi les concepts mathématiques les plus complexes que les enfants rencontrent dans l'enseignement primaire. Les facteurs qui contribuent à la complexité de l'enseignement et de l'apprentissage des fractions réside dans le fait qu'ils comprennent une construction à multiples facettes (Charalambous et Pitta-Pantazi, 2005) comprennent cinq interdépendants sous-constructions: partie-tout, rapport, opérateur, quotients et mesure. Néanmoins, alors que la sous-construction de partie-tout est fortement liée à pourcentage et l'opérateur, elle est faiblement liée aux sous-constructions de mesure et quotient. Notre objectif est de présenter une séquence didactique qui favorise le développement du sens de fraction lié à la relation entre partie-tout et mesure et entre partie-tout et rapport, afin de concevoir les fractions en tant que nombres rationnelles qui peuvent être placés sur la ligne de nombres. La séquence didactique est adressée aux élèves de l'école primaire y compris les élèves ayant des troubles d'apprentissage mathématiques.

Conceptual framework

This research is based on a range of different perspectives, from mathematics education to neuroscience and cognitive psychology. I discuss how such perspectives can be combined and provide the theoretical bases to design the didactical sequence, which allows implementing inclusive education. The main ideas I taken into account from cognitive psychology is that mathematical achievement depends on short-term memory (STM) and working memory (WM) (Raghubar et al. 2010). Moreover, it depends on non-verbal intelligence, addressed to general cognition without reference to the language ability (DeThorne & Schaefer, 2004). These findings suggest that non-verbal intelligence may partially depend on spatial skills (Rourke & Conway, 1997) and, these last, can be potentially important in mathematical performances, where explicit or implicit visualization is required. Moreover, research in cognitive science (Stella & Grandi, 2012) has identified specific and preferential channels of access and elaboration of information. For students with MLD these are the visual non-verbal, the kinaesthetic-tactile and/or the auditory channels at the expense of the verbal channel. Students who come to prefer the visual non-verbal channel, tend to appreciate and elaborate visual-spatial representations, and are best at memorizing images, symbols, graphs, diagrams (Miller, 1987). Studies in mathematics education as well, although with different conceptual frameworks, have highlighted how sensory-motor, perceptive, and kinaesthetic experiences are fundamental for the formation of mathematical concepts – even highly abstract ones (Arzarello, 2006; Radford, 2003). For example, Arzarello (2006) points to how

[5] Mathematical Learning Disorders: a discrepancy between low arithmetical abilities and overall intelligence level and chronological age; difficulty in acquiring formal arithmetic operations and arithmetic facts (Classification systems of developmental disorders: the ICD-10 and theDSM-IV)

recent research in math education underline that the construction of mathematical knowledge, as cognitive activity, is supported by the sensori-motor system activated in suitable contexts. Also Radford (2006) highlights that the understanding of the relationship among body, actions carried out through artefacts (objects, technological tools, etc.), and linguistic and symbolic activity is essential in order to get the human cognition and mathematical thinking in particular.

Main difficulties in teaching-learning fractions

As highlighted by Hannula (2003), the multiplicity of interpretations and applications of fractions is generally not reflected in a corresponding variety of representations and problems presented within educational activities. The most used approaches are: regularly shaped regions divided into equal parts of which some are distinguished, and the number line. This highlight that the sub-constructs linked to the notion of fraction are not really interrelated and there are some of them more developed of the other (for instance, part-whole sub-construct) as well as some stereotype representations (such as shaped regions divided into equal parts). As we will show in the following, this kind of representation can be a useful starting point to develop the notion of fractions but it cannot be considered the only one representation of fraction. For this reason we will take into account other kinds of representations and other sub-construct linked to part-whole sub-construct and to measure. As matter of fact, a limited set of representations can lead to difficulties. Hunnula (2003) found that the number line was more problematic then other kind of representation and postulated that the earlier experience of students led them to look for something divided into m parts of which they could take n, but without having a clear idea of what the appropriate "whole" would be. Moreover, "Many students indiscriminately identify a presentation of m indicated out of n parts of a region as a representation of m/n. Such a limited "expertise" is manifest in widely documented behaviors such as accepting a representation of m out of n unequal parts (Newstead, Olivier, 1999), not accepting that m out of n equal parts can also represent any equivalent fraction (Carraher, Schluemann, 1991), failure to grasp the interpretation of fractions as numbers (Amato 2005)" (Verschaffel, Green and Torbeyns, 2006, pag 76).

This leads students to have difficulties concerning the ordering of fractions on number line. For instance, assuming that the properties of ordering natural numbers can be extended to ordering fractions (e.g. assuming that the product/quotient of two fractions makes a greater/smaller fraction), or positioning fractions on the number line using the pattern of whole numbers (Iuculano & Butterworth, 2011). As matter of fact, frequently, at least in Italian education, the conception of fraction is not explicitly identified as a rational number. Only when it is transformed into a decimal number is it placed on the number line. Fractions constitute an important leap within domain of arithmetic because they represent a first approach towards the idea of extension of the set of Natural Numbers. In this sense, fractions need to assume a specific position on the number line (Bobis et al., 2013).

Methodology, sequence of activities and main results

The sequence of activities was designed by 22 primary school teachers and 1 supervisor (the author) composing a study group. The activities were carried out during a pilot experimentation, which involved 22 classes (nine 5th grade classes, six 4th grade classes and seven 3rd grade classes), before being revised for an upcoming full-blown study. In this paper I will report on the pilot experimentation carried out in the 3rd grade classes. The activities asked to work with different artefacts: A4 sheets of paper, squared-paper strips or represented squared strips in notebooks, the number line represented in notebook and a string on the wall. As described below, the teacher guided the use of these artefacts by focused tasks. The activities are clustered in three main groups: partitioning of the A4 sheet of paper, partitioning of a strip of squared paper, and placing fractions on the number line. I focus here on the first and second activities.

Activity 1: Partitioning of the A4 sheet of paper

The aim of this activity was the introduction of “equivalent fractions” as equivalent surfaces, and of “sum of unit fractions” for obtaining the whole (the chosen unit, that is, the A4 sheet).

Teacher asks students to:

- Partitioning colored A4 sheets in equal parts by folding and using the ruler. Each color corresponds to a unit fraction (Fig. 1);

Fig. 1 Partitioning A4 sheets in equal parts by folding and using the ruler. In these images, the A4 sheet is not a coloured sheet. This educational choice is adopted in the educational activities of other classes.

- Put in a box marked with the unit fraction’s label (box of the 1/2 unit fractions, box of the 1/4 unit fractions,…) the unit fractions obtained by each student;

- Compare the different shapes of each unit fraction and verifying the equivalence, introducing the “equivalent fractions” as equivalent surfaces, by a "cutting and recomposing" strategy. This activity allows students to overcome the idea that regions congruent are the only representations of equivalent fractions considering also representations of equivalent regions (Fig. 2).

Fig. 2 Equivalent regions represent equivalent unit fractions.

- Cover a A4 white sheet with different unit of fractions (Fig.3) taken from the unit fractions’ boxes. The task requires the use of a procedure in which the fraction is conceived as part-whole, where the “whole” is the A4 sheet of paper and the part is the unit fraction.

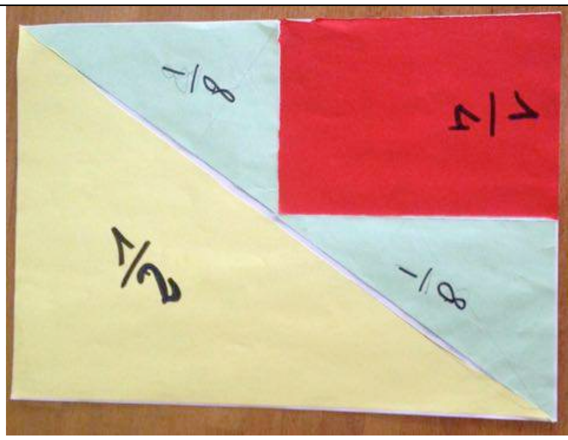

Fig. 3 The A4 sheets of paper is covered by different unit of fractions. Thus, summing unit fractions, students obtain the whole. Note that *equivalent fractions* are *equivalent surfaces*

The “sum of unit fractions” for obtaining the whole (A4 sheet) is a crucial task both to introduce the meaning of sum of unit measures, and to foster the conceptualization of unit fractions as independent to their shape (as congruent regions). Note that the artifacts used in this activity exploit the preferential channels of access to information for MLD students: visual non-verbal and kinaesthetic-tactile.

Activity 2: Partitioning of a strip of squared paper

These activities are clustered into three sessions.

In *Session A*, the aim was comparing unit fractions by representing them on different squared strips. Thus, given a certain unit of measure, the teacher asks students to position it on different strips (concrete strips, Figure 3, or represented on the notebook, Figure 4) and to position the unit fractions 1/4, 1/2, 1/8 each on a strip. The task requires the use of a procedure in which the fraction is conceived as measure (distance from zero): considered the unit of measure, it is divided into 2 or 4 or 8 equall parts; each of them is considered as unit fraction. The manipulation of these artifacts is prevalently a perceptive experience, developed by kinaesthetic-tactile and non-verbal visual channel.

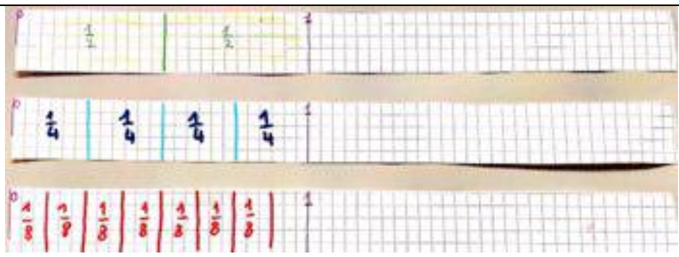

Fig. 3 Three unit fractions 1/4, 1/2, 1/8, represented each one on a different concrete strips of squared paper, are compared in visual and perceptive way.

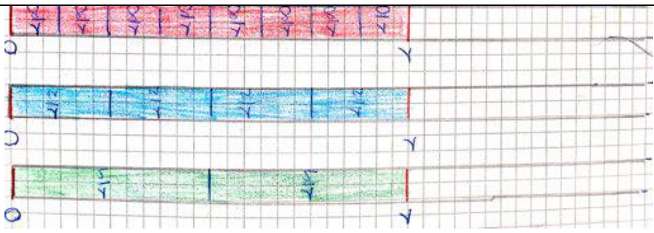

Fig. 4 Three unit fractions 1/4, 1/2, 1/8, represented each one on a different strips drawn on notebook, are compared in visual way.

Students produce linguistic signs associated to the name of the fraction expressed in verbal language (“Un mezzo” – tr. “One half”), in verbal visual language (the writing “un mezzo”- tr. One haf) and arithmetical language (1/2). The teacher institutionalizes the relationship between the different signs (partitions of the strips, visual verbal, visual non verbal, and arithmetical signs) in terms of rational numbers. Note that the task was completed by all groups of students

The aim of the *Session B* was to introduce the dependence of the fraction on the unit of measure. The students are clustered in groups and the teacher asks them to choose a unit of measure and to reproduce it on their own strip. Then, she asks to place the fraction ½ on the strip; Students observe the dependence of the fraction on the unit of measure by comparing the results of the different groups (Figure 5). Note that the previous kinaestetyc-tactile approach is no longer an effective strategy in order to compare the results. Now is necessary managing the meaning of fraction as measure.

Fig. 5 Strips of squared paper where students have choosen different unit of measure and designed the unit fraction ½.

In the *Session C* the main aim was to introduce *lcm* of denominators and ordering unit fractions on the same strip. Thus, chosen an appropriate unit of measure (in this case, 24 squares), the teacher asks students to position it on the strip and to represent the following unit fractions 1/3, 1/6, 1/8, 1/2, 1/4 (Figure 6). We observed that students did not simply looked for the unit of measure spontaneously, generally using trial and error methods (cm), but they also checked the efficiency of their choice (lcm). Moreover, positioning on a single strip different fractions, makes the ordering of fractions exactly like that of the other perceptively evident numbers.

The task supports the overcoming of the epistemological and cognitive obstacle concerning the positioning of fractions on the number line using the pattern of whole numbers (Iuculano & Butterworth, 2011, Bartolini Bussi et al., 2013). Note that here the labels are referred to points on the strip and not to area as before. The color becomes a tool supporting working memory and possibly also long term memory, through which the meanings developed can be recalled and used.

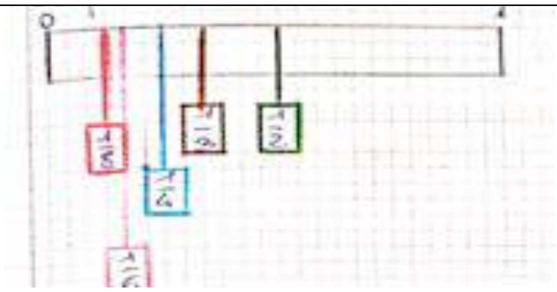

Fig. 6 Strip drown by students choosing the appropriate unit of measure in order to represent on it the unit of fractions 1/3, 1/6, 1/8, 1/2, 1/4

This activity is functional to the ordering of fractions on the number line, as we can observe in the following figure (Figure 7).

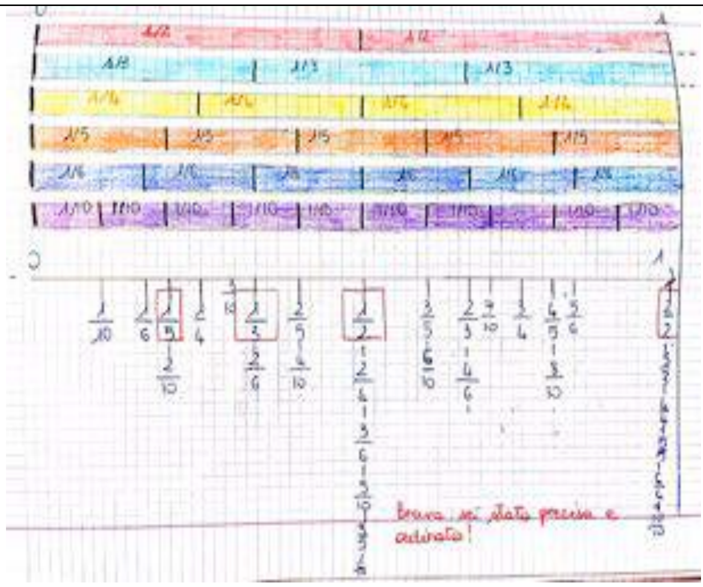

Fig. 7 The ordering of fractions on the strips (on top of the sheet) is functional to the positioning and, consequently, to the ordering of fractions on the number line (on the bottom of the sheet).

In order to overcome the unit of measure (the whole) and comparing fractions greater than 1, teacher asks students to consider four strips of paper. The strips are hanged on the wall, putting them one over the other. Thus, given a unit of measure, teacher asks to position it on each strip and to represent the fractions 4/5, 2/3, 5/3, 7/5 each on a strip (Figure 8).

Fig. 8 Four strips are hanged on the wall, putting them one over the other. In each of them there is indicated the same unit of measure and there are represented different fractions. Among them the fractions 5/3 and 7/5 that are greater than 1.

Students can compare fractions supported by the colour of fractions. Indeed, we can observe that both the green fraction (5/3) and the blue fraction (7/5) overcome the unit of measure 1 and 5/3 is greater than 7/5. The need to overcome the unit of measure is essential in order to visualize fractions on the strip (and not only unit fractions) and, then, on the number line, as visualized in the following figure (Figure 9).

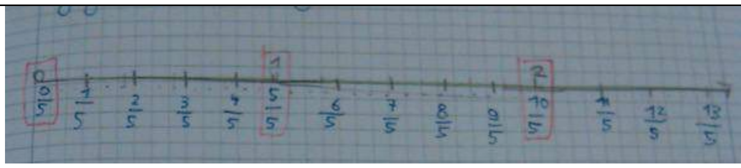

Fig. 9 Different fractions are positioned on the number line.

Conclusion

I have described a sequence of activities, designed on the base of a range of different perspectives, from mathematics education to cognitive psychology, which allow implementing inclusive education. I have outlined particularly significant and relevant passages of the sequence of activities, showing how different sub-constructs of the concept of fraction are activated and how the transition among them was guided. Exploiting preferential channels of access and elaboration of information for students with MLD, described by Miller (1987) and Stella & Grandi (2012), we have designed educational activities which, starting to the use of visual non-verbal, kinaesthetic-tactile and the auditory channels, allow student to access sub-construct of fraction concerning part-whole, as equivalent surfaces, and measure. In the second part of the educational sequence, the transition towards fraction as rational number on the number line is performed exploiting visual verbal channel as well. The analysis of the teaching intervention has shown that students have elaborated personal meanings consistent with the mathematical meanings related to fractions and they have overcome main difficulties highlighted by the educational research. In particular, partitioning colored A4 sheets in equal parts allows student to compare the different shapes of each unit fraction and verifying the equivalence, introducing the "equivalent fractions" as equivalent surfaces, by a "cutting and recomposing" strategy. This activity allows students to overcome the idea that regions congruent are the only representations of equivalent fractions considering also representations of equivalent regions.

Moreover, covering a A4 white sheet with different unit of fractions taken from the unit fractions' boxes, requires the use of a procedure in which the fraction is conceived as part-whole, where the "whole" is the A4 sheet of paper and the part is the unit fraction. The "sum of unit fractions" for obtaining the whole (A4 sheet) is a crucial task both to introduce the meaning of sum of unit measures, and to foster the conceptualization of unit fractions as independent to their shape (as congruent regions). Afterwards, the strip was used as instrument to develop the meanings related to fractions as operators on unit of measure and, then, to the ordering of fractions, to equivalent fractions and finally to equivalence classes. The use of the strip, the string and color (for a certain period of time), has had a key role in favoring the construction of the number line as a mathematical object. On the number line fractions, associated with points, could assume the role of rational numbers being representatives of equivalence classes. Finally, it is possible that this kind of construction of meanings related to fractions might also support the management of procedural aspects involved in operations with fractions, as various researches both in mathematical education and in cognitive science have already suggested (Siegler, 2013; Robotti, Antonini, Baccaglini Frank, 2015, Robotti, 2013). Further studies are needed to explore and to confirm this hypothesis that we consider significant both for research and for teaching.

We would like to greatly thank all the teachers of the "Questione di numeri: mediatori e didattica della matematica efficace"[6] project who have realized, together with the author, this research study.

REFERENCES

Amato, S. A. (2005). Developing students' understanding of the concept of fractions as numbers. In H.L., Chick & J. L. Vincent (Eds.), *Proceedings of the 29th PME International Conference, 2,* 49-56.

Arzarello, F. (2006). Semiosis as a multimodal process. *Relime*, Numero Especial, 267-299.

Bartolini, M.G, Baccaglini-Frank, A., & Ramploud, A. (2013). Intercultural dialogue and the geography and history of thought. *For the Learning of Mathematics, 34*(1), 31-33.

Bobis, J., Mulligan, J, & Lowrie, T. (2013). *Mathatics for children challenging children to think mathematically*, 4th edition. Malaysia: Pearson Australia.

Carraher, D. W., Schliemann, A. D. (1991). Children's understanding of fractions as expressions of relative magnitude. In F. Furinghetti (Ed.), *Proceedings of the 15th PME International Conference, 1,* 184-191.

Charalambous, C. Y., & Pitta-Pantaz, D. (2005). Revisiting a theoretical model on fractions: Implications for teaching and research. In H. L. Chick & J. L. Vincent (Eds.) *Proceedings of the 29th PME International Conference,* 2, *233-240.*

DeThorne L.S. & Schaefer B.A. (2004). A guide to child nonverbal IQ measures. *American Journal of Speech-Language Pathology*, *13*, 275–290.

Hannula M.S. (2003). Locating fractionon number line. In N.A. Paterman, B. J. Dougherty, J. T. Ziliox (eds), *Proceedings of the 27th PME International Conference,* 3, 17-24.

Iuculano, T. & Butterworth, B. (2011). Understanding the real value of fractions and decimals. *The Quarterly Journal of Experimental Psychology*, *64*(11), 2088–2098. DOI: 10.1080/17470218.2011.604785

[6] "Questione di numeri: mediatori e didattica della matematica efficace", project coordinated by Università della Valle d'Aosta- Université de la Valle d'Aoste, Assessorato Istruzione e Cultura - Sovraintendenza agli studi Ufficio Supporto all'Autonomia Scolastica, Région Autonome de la Vallée d'Aoste.

Miller A. (1987), *Cognitive Style: An Integrated Model.* In Educational Psychology, vol.7 (4), pp. 251-68.

Mewstead, K., Olivier, A. (1999). Addressing students' conceptions of common fractions. In O. Zaslavsky (Ed.), *Proceedings of the 23rd PME International Conference, 3,* 329-336.

Robotti, E. (2013). Dynamic representations for algebraic objects available in AlNuSet: how develop meanings of the notions involved in the equation solution. In C. Margolinas (Ed.), *Task Design in Mathematics Education. Proceedings of ICMI Study 22*: Vol. 1, (pp. 101-110). Oxford: ICMI.

Robotti, E., Antonini, S., Baccaglini Frank, A. E., (2015). Coming to see fractions on the number line. *Proceedings of the 9th CERME European Conference* (in press).

Radford, L. (2003). Gestures, speech, and the sprouting of signs: a semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5*(1), 37–70. DOI: 10.1207/S15327833MTL0501_02

Radford, L (2006). The anthropology of meaning. *Educational Studies in Mathematics*, *61*, 39-65.

Raghubar, K.P., Barnes, M.A., & Hecht, S.A. (2010). Working memory and mathematics: a review of developmental, individual difference, and cognitive approaches. *Learning and Individual Differences, 20*, 110–122. DOI: 10.1016/j.indif.2009.10.005

Rourke, B.P., & Conway, J.A. (1997). Disabilities of arithmetic and mathematical reasoning: perspectives from neurology and neuropsychology. *Journal of Learning Disabilities, 30,* 34–46. DOI: 10.1177/002221949703000103

Stella G., Grandi L., (2011). *Come leggere la dislessia e i DSA*, Milano: Giunti Editore.

Verschaffel, L., Green, B., and Torbeyns, J. (2006) Numerical thinking. In A. Gutiérrez, P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education. Past, Present and Future,* 51-82.

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Problem solving as tools for mathematical modeling: case study for real life.

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Abstract: Not many years ago that researchers in mathematics education have focused on designing activities based on mathematical modeling of real situations with the conviction collateral for greater profit, by our students, the mathematical learning, and hence in teaching by teachers. The present paper aims to use the Problem Solving as a useful and necessary to arrive at the concept of Mathematical Modeling.

Résumé: Non il y a plusieurs années que les chercheurs dans l'enseignement des mathématiques ont mis l'accent sur la conception des activités basées sur la modélisation mathématique des situations réelles avec la garantie de condamnation pour un plus grand profit, par nos étudiants, l'apprentissage des mathématiques, et donc dans l'enseignement par les enseignants. Le présent travail vise à utiliser la résolution de problèmes comme utile et nécessaire pour arriver au concept de modélisation mathématique.

Introduction

Math is for many a somewhat unfriendly science. Most people have and / or have had contact with them, even if only in the school years, and live with them every day, we look with suspicion because they are sometimes cryptic, arid and so abstract that we would not know where to undertake them if we try. Throughout history, mathematics have occupied a prominent place in the school curriculum. They have achieved this role not by the importance they have in themselves and for reasons of cultural and social. Such is the importance achieved practically taught in all schools in the world.

Traditionally there have been two basic reasons for weighting Mathematics:
a) "***His ability to develop the capacity of thought***." Juan Luis Vives (1492-1540) and he said "are a subject to express the sharpness of mind".
b) "***Its usefulness for both daily life and to learn from other disciplines necessary for personal and professional development***."

We will use the proposal from the Problem Solving as a management tool to ensure that our students deepen with varying degrees of mathematization and get directions to the concept of Mathematical Modeling. But of course, a doubt arises us

How to link mathematics to other areas of knowledge?

Of the problems that exist worldwide in the secondary stage, in the area of knowledge of mathematics, which is currently attracting the attention of Mathematics Education professionals have regarding how to flirt and structure the curriculum in their relation to other areas of knowledge and even mathematics itself: most issues are disconnected from the real world and, why not say, of science, which has the handicap that students do not conceive the true usefulness of mathematics necessary for their formation.

Case studies

Since not many years ago, researchers in mathematics education have focused on designing activities based on mathematical modeling of real situations with the conviction for greater security in profit, by our students, the mathematical learning, and thus in teaching by teachers.

To arrive at model mathematically, from the point of view of attitudes and conceptions, the role of Problem Solving should, in our view, the vehicle of mathematical learning through:
a) The development of an open attitude.

b) Give examples that lead to a dynamic conception of the evolution of knowledge.
c) In an integrated vision of Mathematics.
d) Facilitating the significant introduction of a new concept.
e) The highlighting of the inductive and deductive processes rigorously.
f) From the sample of the utility of mathematics in life.
g) The development of strategies for becoming a "good citizen".

With the subsequent presentation of several examples of modeling for different levels of education, each illustrative of different levels of complexity that may arise in the process of mathematization, it becomes explicit theoretical framework underlying globally each steps taken, the approaches or results. Consequently, if we get our students try from mathematical modeling, we would get a greater tendency and encouragement to the understanding of the concepts and methods, thus allowing a more comprehensive overview of mathematics.

Therefore, in today's society must endow the paper facing problem solving, estimation, decision making, ..., ultimately face a mathematization of the culture.

Numerical Black Holes

The field of Physics, Mathematics some repetitive processes give rise to results that do not vary in successive iterations. This allows to present such processes as effects of pseudo-mentalism with numbers. Just as a black hole is a body with gravity so strong that nothing can escape it, not even light, there are numbers that attract others to perform certain operations.

Fig.1. Black Holes

A black hole: the number 123

The President Club Football, Recreativo de Huelva, is concerned about the low influx of public to the football field, "Nuevo Colombino", after the last path consecutive losing games. He has designed a strategy to convince people to go to the field.
He will give away half the revenue of the football match to one of the fans attending the soccer field.
The lottery system is as follows:

* *Each person, when you enter the stadium, choose a number between 0 and 9 (*inclusive), *with which it will form a number.*

* *Thus, if the first spectator entering the stadium chooses 1, the following 4,* **14389561112345**……….*the following 3, ... go forming number*:
This number will have as many digits as spectators in the stadium.
Once you have entered all the fans, and we already have the full number, proceed as follows:
*Will form a new number whose first digits shall be the amount of even numbers containing our numbers, the following digits shall be the amount of odd numbers containing the number, and finally add the total number of digits (*odd *+ even).*
For example

1) Write any number of the number of digits that is, **13245673477695683201 84.**
2) We count the even numbers, odd and total numbers and with these 3 issues another form: this case has **11 even numbers, 11 odd numbers** and **22** numbers in total. The new number is **111122**.
3) We again the even numbers an odd numbers and for this number, obtaining: **246.**
4) Above and obtain: **303**
5) And so we go.

The award is to give half of the proceeds to the person whose seat number matches the number resulting from this process. ("Nuevo Colombino" all seats of 1 to 22.670).
With this course of action the President got a few weeks to return to the stadium filled. Fans were delighted with the opportunity to take home a great prize. But gradually the audience started to decrease again.

What do you think is the reason?

It seems that fans will only return when the team "Recreativo de Huelva" regain your fitness level a few years ago. Because it is clear that the financial incentive was not enough to regain the level of attendance at the Nuevo Colombino. Especially when it has released the list of the winners for the draw, and it is that all prizes have so far been delivered to the same spectator: THE PRESIDENT OF THE RECREATIVO DE HUELVA

How is this possible?

Consider what once filled the stadium happens.
We have a number of 22.670 digits corresponding to all partners that have come:
718281828459045235360287471352662497757247093699959574966967627724076630353547594571382178525166427427466391932003059921817413596629043572900334295260595630738132328627943490763233829880753195251019011573834187930702154089149934884167509244761460668082264800168477411853742345442437107539077744992069551702761838606261331384583000752044933826560297606737113200709328709127443747047230696977209310141692836819025515108657463772111252389784425056953696770785449969967946864454905987931636889230098793127736178215424999 2............

There are various possibilities, which can be

a) That all people have chosen an odd number: **022670**22670.
b) That there are less people who choose odd numbers, for example: **13258941222670.**
c) That there are less people who choose even numbers, for example: **8573140922670.**
d) That the odd and even numbers are the same or very similar: **113351133522670**.
e) That everybody has chosen a even number: **22670022670**.

So we see that the number that we get after the first process will take between 11 and 15 digits. In the case that this number has the greatest number of figures, that is, 15 figures, and following the same reasoning, after the second process will get a total of 4 or 5 digits: **01515, 11415, 21315, ..., 7815, ..., 15015**

Suppose the case that has the greatest possible number of digits, in this case, 5. We apply again the procedure and the possible results will now be 3 digits: **055, 145, 235, 325, 415, 505.**

After several iterations, you will reach a 3 digit that can only be **303** (3 numbers are even), **213** (2 are even and 1 odd), **123** (1 is even and 2 odd) or **033** (3 digits are odd). In these 4 cases to redo the calculation is inevitably get the **123.**
¡That curiously coincides with the number of seats of the President of Club Recreativo de Huelva!

Number 6174: another black hole?

The number 6174 is known as the constant Kaprekar in honor of its discoverer the Indian mathematician Kaprekar. Dattatreya Ramachandra Kaprekar (1905- 1986) was born in Dahanu, near Bombay.

Fig.2. Dattatreya Ramachandra Kaprekar

He became interested in the numbers remain very small. From 1930 until his retirement in 1962, he worked as a school teacher in Devlali, India. Kaprekar discovered many interesting properties in recreational number theory.
Let us consider the number **6174**, ordain their digits with them to build as many as possible. Then we ordered to build the fewest possible and make the difference. We get this:
7641-1467 = 6174, which is the number you started.
b) Consider another number, for example **4959.**
Step 1
9954-4599 = 5355
So far there seems to be nothing interesting happened.
Step 2
Do the same with the difference **5355**
5553-3555 = 1998.
Nothing special
Step 3
We continue with **1998.**
9981-1899 = 8082
Step 4
8820-0288 = 8532
Step 5
8532-2358 = 6174.
Again the happy number!
As we move into the resolution may arise questions like the following:
1. Always we arrived at this number?
2. If this ever occurs what is the maximum number of steps required to get the number 6174?
Very important question: Is **6174** the only number with this property**?**
Not, but look at what occurs with other numbers of different length mystery that sheds more light to the subject.
a) If testing with two-digit numbers never reaches a fixed number, but a cyclical loop type **09, 81, 63, 27, 45, 09**
b) With three digits to reach **495**
c) For four digit number it is the mysterious **6174**
d) For five digits, no fixed number, but three cycles (also of different lengths)
e) For six digits, you can be reached at **549945**, to **631764** or a seven numbers

Mathematics and lacing

Just like physics or chemistry, the truth is that math is everywhere in our daily life and there are many ways to lose them fear. There are many math behind the universal mounting system for shoes.

Fig.3. Shoes

It is a cord passing through a series of holes in various combinations. How many are there? And how are they calculated? Is not this maths?

In a medium shoe with six pairs of loops there are almost two billion different ways of passing a string through all of the eyelets. Of course, in practice, when tying a shoe, not all such combinations are practical and comfortable!

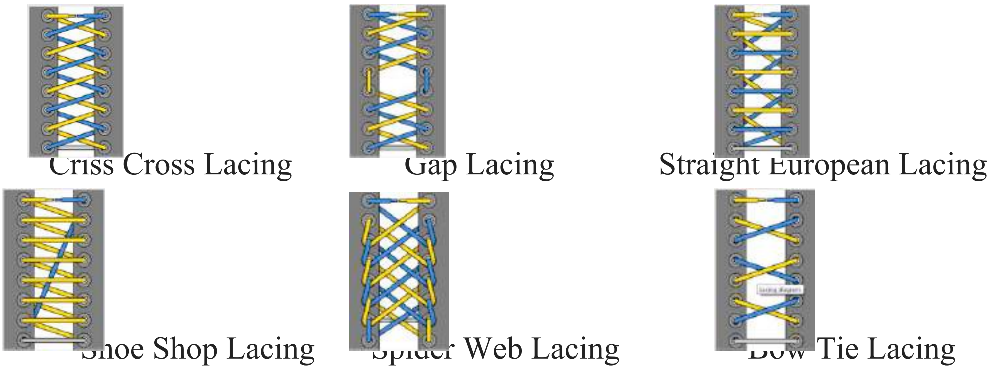


Fig.4. Different Types of Lacing

Question we can ask:

a) Esthetics showing different styles (Optical buyer)?

b) More relevant would be that kind of lacing cords requires shorter and therefore cheaper (Manufacturer?.

c) What lacing pattern among all possible, it requires shorter cords?

Idealization of the problem

We idealize the problem and we contribute some mathematical concepts creates a model of the situation.

Let´s focus on the length of the cord to the two eyelets at the top. The amount of extra cord is required basically to the effective knot, and since it is the same for all methods, we can ignore it.

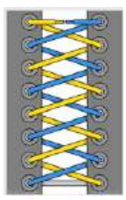
Based on an approach to rough, the cord length can be calculated in terms of the three parameters of the problem:

The number "**n**": pairs of eyelets.

The distance "**d** : between successive eyelets.

The space "**r**": between the left and right corresponding eyelets.

With the aid of the Pythagorean Theorem (What would have thought of this particular application) we have the length of the cord:

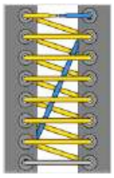


1. **Criss Cross Lacing**

$$L = r + 2(n-1)\sqrt{d^2 + r^2}$$

2. **Shoe Shop Lacing**

$$L = (n-1)r + (n-1)\sqrt{d^2 + r^2} + \sqrt{(n-1)^2 d^2 + r^2}$$



We can ask what lengths is smaller.
To simplify, for example: If n = 8, d = 1 r = 2:

a) **For *Criss Cross Lacing***

$$L = 2 + 14\sqrt{5}$$
$$L = 32,30492$$

b) **For *Shoe Shop Lacing***

$$L = 14 + 7\sqrt{5} + \sqrt{53}$$
$$L = 36,9324$$

Conclusion: It is noted that the shorter length is provided by what American style lacing.
But, *can we be sure that this will always be so or otherwise*, *is possible that the result depends on the values of* "n", "d" *and* "r"?
Only a mathematician would worry about the different cases that appear giving values other than "n", "d" and "r!

Fibonacci stairs

A special case study: how many ways we can up the stairs?

a) Stair with 1 step

Fig.5. Stair (1)

1 One way to up

b) Stair with 2 step

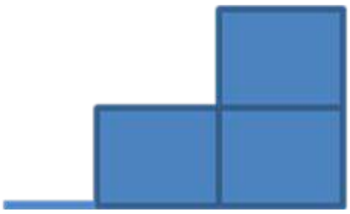


Fig.6. Stair (2)

2 way to up: *steps directly, One on One*

c) Stair with 3 step

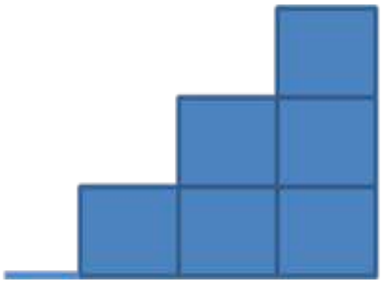


Fig.7. Stair (3)

3 Ways: *One by one, Two steps and then one, A step and then two.*

d) Stairs with 4 step

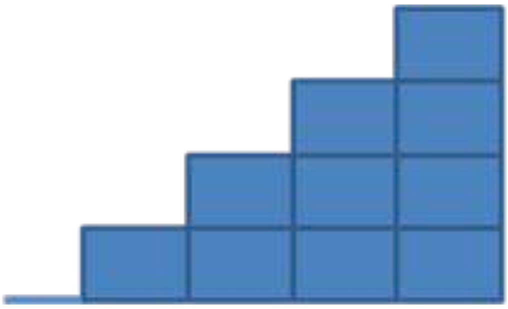


Fig.8. Stair (4)

5 Ways: One by one, Two by two, One-one-two, Two-one-one, One-two-one

e) Stairs with 5 step

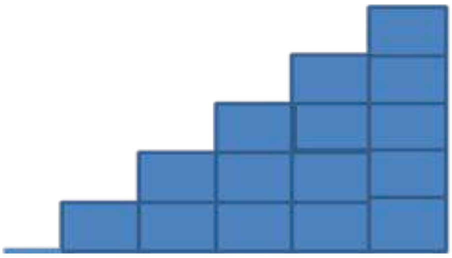


Fig. 9. Stair (5)

8 ways: decided to go up the first step, and we know that there are 5 ways to up 4 steps. Decided to go up two steps, and we know that to go up the remaining three are 3 ways to up. Therefore, in total will be: 5 + 3 = 8 ways

Question.

We can ask: ***How many different ways can we climb a ladder out of 6, 10, 20, 30, 40, 50 steps?***
For a stairs with 6 Steps will be: 8 + 5 = 13
For a stairs with 7: 13 + 8 = 21
With this procedure the following table is obtained:

Stairs	1	2	3	4	5	6	7	8	9	10	11
Ways	**1**	**2**	**3**	**5**	**8**	**13**	**21**	**34**	**55**	**89**	**144**

Table1. Fibonacci Stair

This table produces, the called Fibonacci serie:
1,1,2,3,5,8,13,21,34,55,89,144,233.....where the general term for a stair of n-steps the different ways to upload is expressed recurrently:

$$f_n = f_{n-1} + f_{n-2}$$

Where

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This sequence was described in Europe by Leonardo of Pisa, the thirteenth century Italian mathematician also known as Fibonacci. It has numerous applications in computer science, mathematics and game theory. Also appears in the rabbit, biological settings, such as in the branches of trees, arrangement of leaves on a stem, inflorescences of broccoli romanescu in the flora of the artichoke.

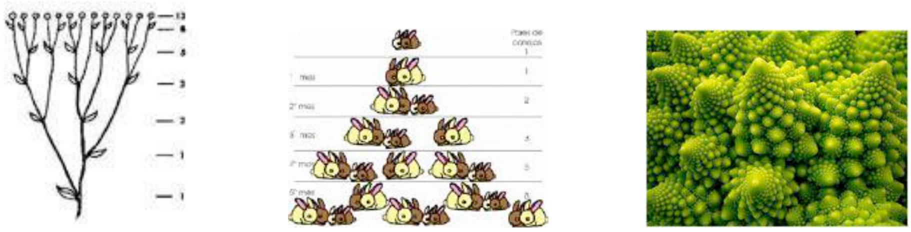


Fig. 10. Fibonacci serie

CONCLUSION

This work presented aims to show, based on the experience in investigations, problematized and modeled the mathematics teaching allows us to realize: *"... in principle, there is a modeling process behind all mathematical model. This means that someone implicitly or explicitly has come a process of establishing a relationship between a mathematical idea and a real situation. In other words, in order to create and use a mathematical model it is necessary, in principle, go all the way to a process of modeling* ... ". (Morten Blomhøj, 2004). We have found in different levels of education in several primary schools, secondary and university that: "... *make a" small or large mathematization "represents the core part in the teaching/learning of mathematics ... model*."

a) Modeling with mathematics is reaching out to our everyday acts we perform in the surrounding environment. Thus also we matematization culture through school, institutional actions, ... ultimately social.

b) The use of a problematized and modeled attitude of today's society, using ICTs to support, makes much stronger individual, when faced with problem solving, and consequently more dynamic and secure.

REFERENCES

Bender, E.A. (1978). " *An introduction mathematical modeling*". New York. Ed. Wiley. Págs.4-8

Blomhøj, M. (2010). "*Introduction - Applications and Modeling*". Proceeding Pedagogique from CERME 6: The 6th Congress of the European Society for Research in Mathematics Education. Ed. Lyon: Institut National de Recherche. Págs. 2042-2045.

De lange, J., Keitel, C.; Huntley, I. Y., Niss, M. (1993). "*Innovations in maths education by modeling and applications.*" London: Ellis Horwood.

Fieggen,I. (2015): http://www.fieggen.com/shoelace/lacing.htm

Kaprekar, D. R.(2015) https://en.wikipedia.org/wiki/D._R._Kaprekar

Kapur, J.N. (2005)." *Mathematical Modelling*".Ed. New Age International Publishers. First Edition, 1998. Págs. 7-9. New Delhi.

Kieldsen, T., BlomhøJ, M.(2013)." *Students´mathematical learning in modeling activities". Teaching Mathematical Modeling: Connecting to Research and Practice.* Ed. Stillman, G et al. New York: Springer Publising Company. Págs. 141-151. (International Perspective on the Teaching and learning of Mathematical Modelling).

O'connor, J.; Robertson, Edmund F. (2007). *"D. R. Kaprekar"*, *MacTutor History of Mathematics archive*, University of St Andrews.Scorland.

Romero, S.; Castro, F. (2008). *Modelización matemática en Secundaria desde un punto de vista superior. El problema de Dobogókó*. Vol.1. pp.11-23. Modelling in Science Education and Learning. Valencia. Doi: http://dx.doi.org/10.4995/msel.2008.3129.

Stillman, G.A., Kaiser, G., Blum, W., BrowN, J.P. (Eds). (2013). *"Teaching Mathematical Modelling: Connecting to Research and Practice"*. International Perspecives on the Teaching and Learning of mathematical Modelling. Springer-Science+Business Media Dordrecht.

Romero, S., Romero, J., Benítez, R., Rodríguez, I.M., Salas, I.M. (2015). "*La resolución de problemas como instrumentos para la modelización matemática: ejemplos para la vida real*"**.** MSEL. Vol. 8. Núm. 2. pp. 51-66. http://dx.doi.org/10.4995/msel.2015.3962 .Valencia (España).

Romero, S., Romero, J. (2015) *"¿Por una enseñanza problematizada y modelizada de las matemáticas? estudio de casos.* Rev. Uno. Número 69. Modelización. Barcelona.

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Obstacles on a Modelling Perspective on Probability

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Abstract: We consider the challenge of a modelling perspective on teaching and learning probability. This challenge can occasion new or the same obstacles in students learning than the traditional classical approach. Following Brousseau (1997), we use the term obstacle as a previous piece of knowledge, which was once interesting and successful but which is now revealed as false or simply unadapted. We identify theoretically the obstacles of ontogenic, didactical and epistemological origin. Firstly, the epistemological obstacles are identified through the analysis of the historical and philosophical perspective of the frequentist approach of probability to prove the "Law(s) of Large Numbers". Secondly, the analysis of the probability modelling learning approaches gives us information about the possible didactical obstacles. And finally, the reinterpretation of the biases, heuristics, paradoxes and fallacies that emerge in the probabilistic frequentist approach is the basis to describe the ontogenic obstacles that intercept with the epistemological evolution of the notion and the cognitively integrating prior conceptual structures, modelling approaches and theories.

Résumé: Nous considérons le défi d'une perspective posant en enseignement et l'apprentissage de la probabilité. Ce défi peut l'occasion nouvelle ou les mêmes obstacles dans des étudiants apprenant que l'approche classique traditionnelle. Après Brousseau (1997), nous utilisons le terme de l'obstacle comme une pièce précédent de connaissance, qui était une fois intéressante et fructueuse, mais qui est maintenant révélé que faux ou simplement inadapté. Nous identifions théoriquement les obstacles d'ontogenic, l'origine didactique et épistémologique. Premièrement, les obstacles épistémologiques sont identifiés par l'analyse de la perspective historique et philosophique de l'approche de frequentist de probabilité pour prouver "la Loi (s) de Grand nombre". Deuxièmement, l'analyse du modelage de probabilité apprenant des approches nous donne des informations sur les obstacles didactiques possibles. Et finalement, la réinterprétation des biais, l'heuristique, des paradoxes et les erreurs qui apparaissent dans l'approche de frequentist probabiliste est la base pour décrire les obstacles ontogenic qui interceptent avec l'évolution épistémologique de la notion et les structures conceptuelles antérieures cognitivement intégrantes, modelant des approches et des théories.

Introduction

Researchers, curriculum designers and tertiary teachers have begun to consider why and how modelling can help students to reason about formal and informal statistical inference. Some of the opportunities that models and modelling can provide to teaching statistics have been explored by researchers. For example: (a) providing the basis for introducing estimation an hypothesis testing (Garfield and Ben-Zvi, 2008); (b) fostering students' statistical thinking (Wild and Pfannkuch, 1999); (c) steering probability learning (Batanero, Henry and Parzysz, 2005); (d) providing a choice of whether to access real world data (Graham, 2006); (e) using technological tools to integrate exploratory data analysis approaches and probabilistic models through simulations and visualization (Eichler & Vogel, 2014); and (f) for developing new learning theories and proposed learning progresions to inform future standards and curriculum efforts in mathematics and science education (Lee, 2013).

The development of new learning theories and proposed learning progressions are new opportunities in mind when teaching and learning statistics and probability (Lee, 2013; (Zieffler et al., 2014). However, this challenge can occasion new or the same obstacles in students learning than the traditional classical approach. And, with this aim, we identify theoretically some of the possible obstacles on a modelling perspective on probability.

Epistemological, didactical and ontogenic obstacles

Following Brousseau (1997), we use the term obstacle as: *"a previous piece of knowledge which was once interesting and successful but which is now revealed as false or simply unadapted"* (p. 82). He identifies obstacles of epistemological, didactical and ontogenic origin. Epistemological obstacles are usually identified from the historical analysis, since they coincide with difficulties that arose in the development of the subject. Didactic obstacles, which are related to the way a topic is taught, are seen as obstacles for the students because of ill through out presentation of subject matter, or the result of narrow or faulty instruction. Ontogenic obstacles are related to children's cognitive development, due to lack of prior knowledge, students' limitations and developmental limitations.

Research about obstacles will inform about the usefulness of its identification in designing instructional approaches, in which both ontogenic and didactic obstacles should be avoided and the epistemological should not be avoided because it is clue in the construction of the knowledge (e.g. Brousseau, 1997). We consider that the recognition of the obstacles is crucial in a design based research methodology to develop coherent hypothetical learning trajectories and tasks, and to understand the difficulties that emerge during the teaching and learning process. And, with this aim, in this paper, we identify theoretically some of the possible obstacles of epistemological, didactical and ontogenic origin on a modelling perspective on probability.

Epistemological obstacles on a modelling perspective on probability

The first mathematician that synthesized the ideas of probability was Huygens. For him, probability was not yet a number; it was a collection of arguments (pros and cons) that could be used to weigh arguments. From these arguments, he derived the expected value of an event. It was Bernoulli, who introduced the probability as a number instead of an estimation of possibilities. Bernoulli derived a mathematical relation (a kind of convergence) between equal probabilities and the observed frequencies in the repetition of such games, which comprised the very first version of the "Law of Large Numbers". The assumption of the subjective equiprobability of the equally likely outcomes allowed Bernoulli considering the convergence of individual events. Von Mises' disquisitions around the repeated tossing of coin, a pair of dices or the record of the sex of newborn children in a population, allowed him to establish that the relative frequencies of certain attributes become more and more stable as the number of observations is increased. The contrast between the experimental values obtained through the analysis of the stability of the relative frequency and the classical theoretical value, lead him to introduce the relative frequency model. Von Mises' ideas culminated in the first attempt to give an axiomatic approach to the discipline based on idealized properties of random sequences (Borovnik and Kapadia, 2014). Von Mises understood the obstacle of distinguishing between model and reality as unavoidable in mathematical science, arguing that: *"the transition from observation to theoretical concepts cannot be completely mathematized"* (Von Mises, 1964), pp. 45 conferring a new and crucial role to models and modelling.

This historical and philosophical perspective on the evolution of the frequentist notion of probability has allowed identifying different difficulties that can emerge in a frequentist approach to probability to prove the Law of Large Numbers (e.g. Batanero et al., 2005; Borovnik and Kapadia, 2014). Now, in this paper we categorize them through the lenses of the obstacles and their epistemological nature. We summarise them as: (a) considering the probability as the value expected instead of the theoretical value, (b) conceptualizing the convergence on probability when establishing the Law(s) of Large Numbers, (c) considering individual events instead of an aggregate view of the data to conceptualize the distribution of probability, and (d) undistinguishing between the experimental value, the modelled value and the theoretical value of the probability of an outcome.

Furthermore, the epistemological distinction of the relationship between the theoretical world (of probabilities) and the empirical world (of data) has allowed illustrating three modelling perspectives: the classical, frequentist and subjectivist (Eichler and Vogel, 2014). The modelling structure with regard to the classical approach is conceived as a unidirectional way between the theoretical world, where the theoretical model is built, and the empirical world, where the model is validated. The modelling structure with regard to the frequentist approach is bidirectional. Beginning in the empirical world, where the analysis of the available empirical data and the detection of patterns take place, then building a theoretical model base, and finally returning to the empirical world to validate the model. For the modelling structure with regard to the subjectivist approach, iterative cycles of getting information in the empirical world and rebuilding a theoretical model are suggested. An obstacle can emerge when conceptualizing models and modelling, if it is not discussed the nature of the relationship between the empirical world and theoretical world in each modelling approach.

Didactical obstacles on a modelling perspective on probability

On the research agenda on statistical and probabilistic thinking, the didactical analysis of models and modelling have become a goal to understand the possibilities that they could give to mimic some aspect of random behaviour in the real world.

If we focus our attention on the human action related to the practice of models defined as the modelling process, we can find in the literature on probabilistic research mainly two probability modelling learning approaches: the theory-driven approach and data-driving approach. In the theory-driving approach, the problem and "data" are given, from which students are able to recognise the underlying theoretical model, proposed by the teacher. Students use it by "its goodness" to predict future outcomes in the real world system, although it is not built or tested. In this case, students work within the enclosed world of the model, asking questions about the model, looking at the consequences of the model, and using the model to choose between different actions to improve a situation (Borovnick and Kapadia, 2011). However, two obstacles can emerge associated to the developing of this theory-driving approach. On one hand, the fact that it is theoretically presented by its goodness, it can promote that the students do not reason about its "goodness" in relation to its accuracy to the real context presented. On the other hand, to think that the finality of using the model is to work within the enclosed world of the model, instead of considering the purpose of the model to predict future outcomes in the real world system.

In contrast, the data-driving approach students experience models for which there is no theoretical probability model or the presumed theoretical model is inadequate. The introduction to modelling is done through measurement activities that help them to reflect about the variability of the data, getting a sense of distribution of the measurement data and appreciate the types of measurement errors. This, in turn, can lead them to construct the model observed measurement as an initial theoretical view of the real world system for their probability model, fit the probability model to the data, check whether the model is an adequate model of real world or not, and then adjust the model until obtaining a working model that adequately reflects the actual data (Konold and Kazak, 2008). Attending to students' learning, (Pfannkuch and Zledins, 2014) argue that independent experiences using data-driven or theory-driven approaches are not enough. Students need to appreciate the circularity between theory-driven and data-driven probability modelling. If students do not percibe this circularity, it can appear another didactical obstacle.

If instead of focusing the attention on the human action, we analyse the structural complexity of the transfer between the empirical world (of data) and the theoretical world (of probabilities), we have to carefully analyse the relationship between model and reality through the lenses of the problems at hand. Eichler and Vogel (2014) diferentiate three kinds of problems situations: (a) a virtual problem situation, which contain all necessary information and represent a stochastic content; (b) a virtual real world problem situation, which demand analysing a situation's context that is more "authentic" and provide a "narrative anchor"; and (c) real world problems, which are situations that include the aim to reproduce real societal problems. If the problems proposed to modelize are circunscrite to "virtual problem situations" or "real world problem situation", it can emerge an obstacle associated to the confusion between model and reality, instead of conceptualizing model as an approximation to reality.

To sump up, some obstacles can emerge when conceptualizing both notions of model and modelling and the problem at hand: (a) the confusion between model and reality, instead of model as an approximation to reality, due to the kind of problem situation provided to the student, (b) do not consider the purpose of the model to analyse its "goodness" in relation to its accuracy to the real context presented, (c) do not consider the purpose of the model to predict future outcomes in the real world system, (d) the lack of circularity between the theory-driven and data-driven probability modelling (Pfannkuch and Zledins, 2014), and (e) do not percieve that modelling is an iterative cycle, which leads to more insights step by step (Borovnick and Kapadia, 2011).

Ontogenic obstacles on a modelling perspective on probability

Since probability is a theoretical concept, its estimated value depends on numerous factors, such as the observer's knowledge, the observation conditions or the data that he is able to collect. The selection of the real situations, contexts, scenarios in which data is collected can cause ontogenic obstacles. If the context selected is games of chance, the reasoning about the random generators is not free from subjective judgements or theoretical classical argumentation about the equidistribution of probability that can reinforce the equiprobability bias. Moreover, we cannot give a frequentist interpretation to the probability of an event, which only occurs on time under the same conditions, such is often found when using historical data. From a psychological point of view, these contexts are not sufficient cognizant of random sampling fluctuation and the effect of sample size on sampling variability (Batanero et al., 2005). In consequence, they could reinforce on students the sense of accuracy of the representativeness of the sample misconception, the gambler's fallacy or the "outcome approach" (Serrado et al, 2005).

In relation to the observation conditions, Watson (2005) summarises the difficulties that can emerge thinking on the random behaviour and the independence of the events. Firstly, an epistemological obstacle emerges when trying to establish the link between the intuitive idea of independence related to the analysis of the data in real contexts, and its formal stochastic definition emptied of its intuitive content. This epistemological obstacle might become ontogenic when students will be involved in the construction of complex models of compound experiments based upon simpler ones. Secondly, an ontogenic obstacle due to the credence of deterministic behaviour of the events can emerge when developing the first intuitions about the measurements, the proportional reasoning and the notion of density in a data-driving approach.

The ontogenic obstacle that emerges from considering deterministic the data with random behaviour can be recognised in the first steps of the cognitive development of the hypothetical thinking of the student when involved in a task based on a modelling perspective on probability. We think that the developing of the hypothetical thinking should be encompassed with the broad maturation of proof structures (Serradó, 2014), considering that with the interplay of expecting, theoretically defining and modelling probability underlie the "Laws of Large Numbers" and the need of proving it. And, in coherence with this idea, we take in this paper a cognitivist point of view of the broad maturation of proof structures and the obstacles that could emerge on the construction of increasingly sophisticated knowledge structures (Tall et al., 2011).

Ontogenic obstacles could emerge of an inadequate and encompassed maturation of three knowledge structures: distribution, sampling and variability. Considering that the relationship between data (individual value) and distribution (conceptual entity) should provide a bridge between relative frequency distribution and the probability. In the data-driven approach, if students do not develop an aggregate view of the data they can have an obstacle on constructing the notion of distribution (Bakker and Gravemeijer, 2004), in general, and sampling or probability distribution, in particular. In the case of samples, Ben-Zvi, Bakker and Makar (2015) express that one of the key ideas for effective reasoning about samples and sampling is the need to balance two related ideas, sample representativeness and sample variability. An ontogenic obstacle could emerge when students over-relied on sample representativeness, believing that a random sample has to be representative of the population, and not randomness but some other mechanism must have caused sampling variability. Furthermore, it can appear ontogenic obstacles if they do not encompass the increasing sophistication conceptualizing sample size, selection method, resulting representativeness and appreciation for variation in the population.

Meanwhile, in the theoretical-driven approach ontogenic obstacles can emerge if students do not encompass and integrate three notions: (a) the variability of the results obtained when repeating an experiment, (b) the stability of the frequencies of the observed outcomes, and (c) the relation between the value of the limit of frequencies, the distribution of possible outcomes, and the theoretical value of the probability (Yánez and Jaimes, 2013).

Conclusions

We have presented the epistemological obstacles identified through the analysis of the historical and philosophical analysis. This analysis has allowed understanding the obstacle that can emerge when conceptualizing models and modelling process from the relationship between the empirical world and theoretical world from the different approaches: frequentist, classical and subjectivist. In particular, the historical analysis of the evolution of the frequentist approach has provided information of how it has been surpassed the distinction between estimated value, modelled value and theoretical value.

From a didactical point of view, to modelling approaches the data-driven and theoretical-driven have been analysing, concluding that obstacles can emerge if there is not circularity between both models, or the selection of the problems at hand are disconnected from the real world, becoming basically virtual or virtual real problem situations. Furthermore, from a didactical point of view, obstacles can emerge if students are not asked to formalize to process of the modelling process, which are: the analysis of "its goodness" and "its predictive nature".

The analysis of the probability modelling learning approaches has given us information about the possible didactical obstacles. The ontogenic obstacles have been analysing taking in consideration a cognitivist point of view. The first obstacles emerge from the students' perceptions due to biases, misconception, and fallacies of the frequentist probabilistic approach. Other, ontogenic obstacles can emerge due to a lack of encompassed evolution of three integrated notions, distribution, sampling and variability.

Research about obstacles informs us about the usefulness of its identification in designing instructional modelling approaches, such as data-driven or theoretically driven. We consider that the recognition of the obstacles is crucial in a design based research methodology to develop coherent hypothetical learning trajectories and tasks, and to understand the difficulties that emerge during the teaching and learning process. In consequence, further research should be developed in order to design hypothetical learning trajectories to overcome the epistemological obstacles and avoid the ontogenic and didactical ones.

REFERENCES

Bakker, A., & Gravemeijer, K. P. (2004). Learning to reason about distributions. In D. Ben-Zvi, & J. Garfiel, The Challenge of Developing Statistical Literacy, Reasoning and Thinking (págs. 147-168). Dordrerch, The Netherlands: Kluwer Academic Publishers.

Batanero, C., Henry, M., & Parzysz, B. (2005). The Nature of Chance and Probability. In G. Jones (Ed.), Exploring probability in school: challenges for teaching and learning (págs. 16-42). Dordrecht: Kluwer.

Ben-Zvi, D., Bakker, A., & Makar, K. (2015). Learning to reason from samples. Educational Studies on Mathematics (88), 291-303.

Borovnick, M., & Kapadia, R. (2011). Modelling in probability and statistic. In J. Maasz, & J. O'Donoghue (Eds.), Real-world-problems for secondary school mathematics students: case studies (págs. 1-43). Rotterdam: Sense Publishers.

Borovnik, M., & Kapadia, R. (2014). A Historical and Philosophical Perspective on Probability. In E. J. Chernoff, & B. Sriraman (Eds.), Probabilistic Thinking. Presenting Plural Perspectives (Advances in mathematics Education ed., págs. 7-34). Dordrecht: Springer.

Brousseau, G. (1997). Theory of Didactical Situations in Mathematics. Dordrecht, Netherlands: Kluwer.

Eichler, A., & Vogel, M. (2014). Three Approaches for Modelling Situations with Randomness. In E. J. Chernoff, & B. Sriraman (Eds.), Probabilistics thinking (págs. 75-100). Dordrecht: Springer.

Garfield, J., & Ben-Zvi, D. (2008). Developing students' statistical reasoning: Connecting research and teaching practice. New York: Springer.

Graham, A. (2006). Developing thinking in statistics. London: Paul Chapman Publishing.

Konold, C., & Kazak, S. (2008). Reconnecting Data and Chance. Technology Innovations in Statistics Education , 2 (1).

Lee, H. (2013). Quantitative reasoning in digital world: laying the pebbles for future research frontiers. Quantitative reasoning in mathematics and science education: paper from an international STEAM research symposium, WISDOM e Mongraph # 3 (págs. 65-82). Laramie, Wyoming: University of Wyoming College of Education.

Mises, R. (1964). Mathematical Theory of Probability and Statistics. New York: John Wiley.

Pfannkuch, M., & Zledins, I. (2014). A Modelling Perspective on Probability. In E. J. Chernoff, & B. Sriraman (Eds.), Probabilistic Thinking. Advances in Mathematics Education. (págs. 101-116). Dordrecht: Springer Science+Busines Media.

Serradó, A. (2015). Developing hypothetical thinking through four cycles of informal stochastical modelling. (G. Aldon, B. Di Paola, & C. Fazio, Eds.) Quaderni di Didattica , 24 (1), 173-176.

Serradó, A., Cardeñoso, J. M., & Azcárate, P. (2005). Obstacles in the learning of probabilistic knowledge: influence from the textbooks. Statistics Education Research Journal , 4 (2), 59-81.

Tall, D., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y.-H. (2011). Cognitive development of Proof. In M. De Villiers, & G. Hanna (Eds.), Proof and proving in mathematics education (págs. 13-49). Dordrecht: Springer.

Yánez, C. G., & Jaimes, E. (2013). Efectos de la simulación en la comprensión de la ley de los grandes números. Revista Integración, 31 (1), 69-86.

Watson, J. (2005). The probabilistic reasoning of middle school students. In G. Jones (Ed.), Exploring probability in school: challenges for teaching and learning (págs. 145-170). New York: Springer.

Wild, C., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry (with discussion). International Statistical Review , 67 (3), 223-265.

Zieffler, A., Del Mas, R., Garfield, J., & Brown, E. (2014). The symbiotic, mutualistic relationship between modeling and simulation in developing students' statistical reasoning about inference and uncertainty. In K. Makar, B. de Sousa, & R. Gould (Eds.), Sustainability in statistics education. Proceedings of the Ninth International Conference on Teaching Statistics (ICOTS9, July 2014). Flagstaff, Arizona, USA: Voorburg, The Netherlands: International Statistical Institute.

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In how far does programming foster the mathematical understanding of variables? - a case study with scratch

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Abstract : The recent development and growing popularity of Scratch, a visual programming environment, has led many students to programming, especially those who attend Computer Science classes. Several concepts of programming are closely related to mathematical objects and algorithmic thinking. Within my master thesis, I focused on the role of variables. In a case study involving 8th graders as well as university students, I examined the understanding of variables of the participants in the context of programming with Scratch. Fostering the understanding of variables is a key task for all secondary math teachers. Since variables are closely related to the field of Algebra, even though not limited to it, I used Malle's model of "aspects of variables" to identify more specifically which underlying ideas and perspectives on variables can get activated within the process of programming.

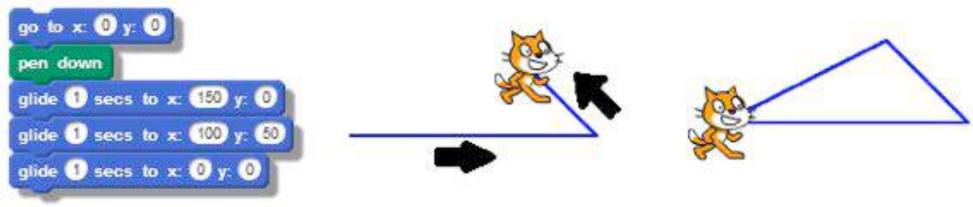
Résumé : Le développement récent et l'augmentation de la popularité de Scratch, un environnement de programmation visuel, a conduit beaucoup d'étudiants à la programmation, particulièrement ceux qui assistent aux cours d'informatique. Plusieurs concepts de programmation sont étroitement reliés à des objets mathématiques et aux réflexions algorithmiques. Dans ma thèse de master, j'ai focalisé mon attention sur le rôle des variables. À l'aide d'une étude de cas comprenant des élèves de la 4ème ainsi que des étudiants de l'université, j'ai examiné la compréhension des variables des participants dans le contexte de la programmation avec Scratch. L'encouragement de la compréhension des variables est une tâche clé pour tous les professeurs de mathématique de l'enseignement secondaire. Puisque les variables sont étroitement liées au champ de l'algèbre, bien qu'elles n'y soient pas limitées, j'ai utilisé le modèle « les aspects des variables » de Malle pour identifier plus spécifiquement quels idées et points de vue sous-jacentes sur les variables peuvent être activés dans le processus de la programmation.

Introduction

The purpose of this article is to report about a case study that focused on analyzing different aspects of variables that can be fostered through visual programming. The main research question was: What benefit do math students gain from programming considering the understanding of variables? Since Computer Science gains more and more importance as a middle school and high school subject, it is an interesting question to examine synergetic effects on Math, taking interdisciplinary learning and its benefits into account. It is well known that activities in both subjects require algorithmic thinking, structuring, analyzing and modeling problems. I focused on variables since they are basic objects for both algebraic activities and implementation of algorithms.

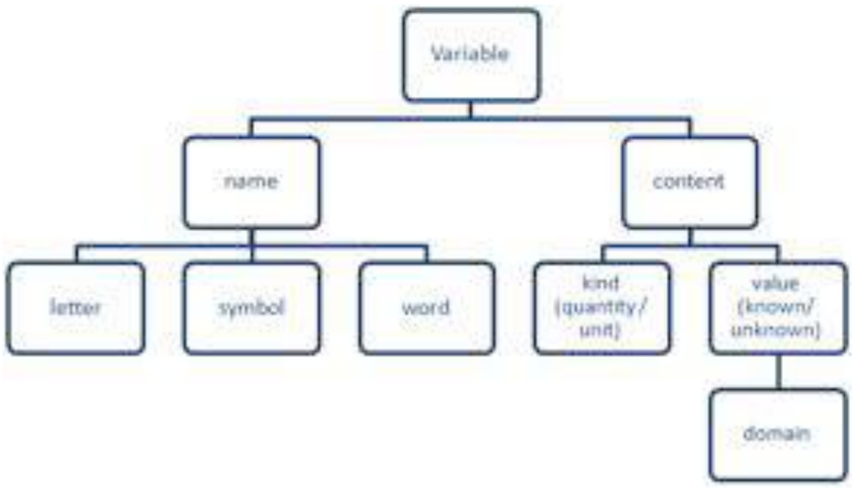
As a programming environment, the popular visual programming language Scratch (over 5.6 million users, see http://scratch.mit.edu/statistics) was chosen because it allows learners to *easily* implement algorithms in an experimental way. Scratch, which is developed by the Lifelong Kindergarten Group at the MIT, builds on basic concepts of Logo, a widely-known programming language for children, developed by Papert in the 1960s and 1970s (see Papert, 1980). The purpose of both languages is to learn programming. Scratch offers a block-based concept of programming, so there are practically no syntactical mistakes possible. Therefore, Scratch facilitates a positive motivational self-concept of the learners when they are experimenting with algorithms. Scratch

allows students to program games and animations, so learners are motivated to discover algorithms and programming structures to realize their ideas. As the developers point out: "Keeping score in a game is a frequent motivator for young designers to explore variables" (Brennan & Resnick, 2012, p. 6).
Moreover, the integration of Bruner's modes of representation seems to make Scratch suitable for many learners. The blocks are symbolic elements that can, when executed, create an iconic representation. If the execution is slow enough, another sort of representation, that can be described as 'virtual-enactive' (by Hole; see Weigang & Weth, 2002) is observable for the programmer, e. g. the cat is drawing the triangle (see figure). For the learner, as Jones (2010) concludes, change within and between modes of representation is effective in terms of both promoting understanding and disclosing misconceptions.

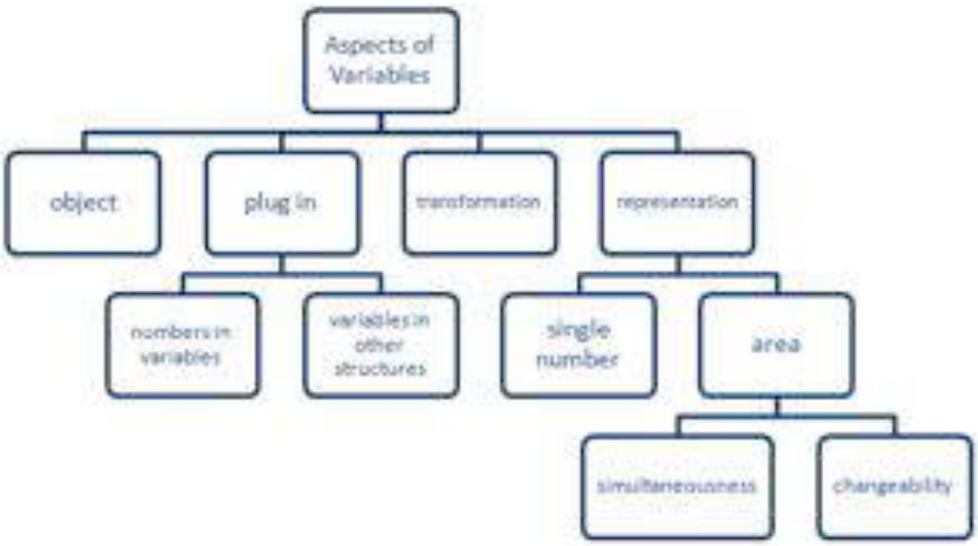


Theoretical Background

The following figure shows a generic model of variables. When students learn about variables they have to differentiate name and content of the variable. The name can be represented by a single character, a word, which is not handy for calculation but emphasizes the connection between meaning and symbolization, or another kind of symbol, like a circle or square. A variable has a known or unknown value that has a specific meaning, often connected to a unit or it represents just a quantity. The value is an element of a domain set, which is depending on the interpretation.



When mathematics is performed, several typical understandings of variables are activated. Malle (1993) denotes these understandings as "aspects of variables". This model was used in the present case study to identify in detail which concepts of variables are activated when learners work with Scratch. The most fundamental is the **object aspect** of variables. Variables represent an unspecified object of thinking or, in the words of the generic model, an unknown number. Especially exercises which are embedded in a context foster the object aspect rather than transformational tasks. The **plug in aspect** describes the fact that variables are placeholders that can be substituted by numbers. This aspect can be generalized to a basic idea of mathematics, the principle of substitution. Therefore, I divide between Malle's original idea of plugging numbers in variables and the idea of plugging variables into other structures. For example, variables can be plugged in functions or can be used as a 'helper' to simplify larger algebraic expressions. The **transformation aspect** of variables gets active when it comes to manipulation of expressions and equations. The variable is reduced to a meaningless symbol on which the specific rules of algebraic transformations can be carried out. Examples are simplification of expressions, conversions and solving of equations.



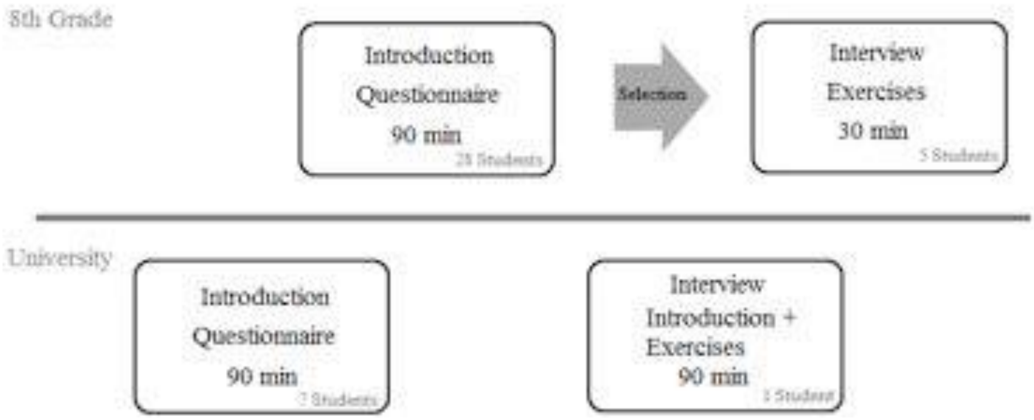
These three aspects can be identified with concepts of other theories, too. Steinweg (2013) summarizes and compares different models of variables and points out that these three aspects are crucial and also occur, in a different terminology, in other theories, e. g. those of Freudenthal and Ursiskin (for a detailed comparison see Steinweg, 2013). Taking Kieran's model of algebraic activities (2004) into account, it can be recognized that aspects such as the transformation aspect are closely related to the transformational activities. Also the object aspect typically gets activated during generational or global/meta-level activities.

Three other aspects of variables are mentioned by Malle in the context of formulas and functions. Here, variables appear as representatives. He divides between the **single number aspect** and the range aspect. The first one considers variables as certain unknown numbers of a set. For example, it can be a constant parameter of a family of functions. Within the range aspect, in which the variable represents every number from the domain, Malle differentiates between **simultaneity** - all numbers are represented at the same time - and **changeability** - numbers are seen with regard to a chronological order. Characteristics of these six introduced aspects, especially changeability, range, plug in and object aspect, also overlap with the roles of parameters (see Drijvers, 2003, p. 315).

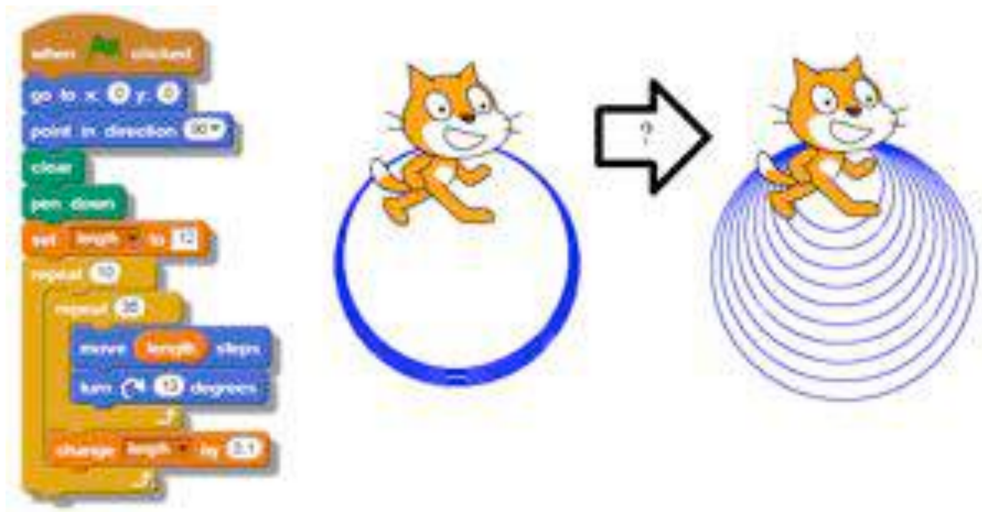
Certain exercises can emphasize specific aspects of variables but typically, these aspects work together in mathematical activities. For example, if a student solves an equation and carries out a division by a variable (transformation aspect), she/he has to consider whether the divisor can be zero for a certain number that can be plugged in the variable. The domain depends on the meaning of the variable, so the object aspect is also important to consider, especially in contextualized tasks. The object aspect will also be activated when the learner checks the result of the calculation in the context of the exercise.

Methodology

An expert-novice-study was chosen as a design scheme for this qualitative approach to validate results and examine differences that are related to a rich or small foreknowledge about variables. Two groups of students have been analyzed with questionnaires and guided interviews after and while working in typical programming situations. Both groups, one group consisted of 26 8th graders, the other of six university students at master level, got an introduction to the Scratch programming environment.



The following figure shows a task from the study. In this example, the students had to answer questions like "What is the meaning of the 'go to x:0 y:0'-block?" or "How do you have to change the code to get the cat to draw the shown 'circle tower'?". A larger exercise within the interview session was to improve a given program, which allows the user to control a little mars robot with the arrow keys, in a way that it measures the driven distance and the fuel consumption. Of course, the key competency to all these tasks is the usage and the understanding of variables.



Results

During the analytical process, the students' and experts' statements of the questionnaire and the interview have been matched with the six aspects of variables. Sometimes answers were hard to relate to a distinct aspect. In general, the interview delivered more reliable results than the questionnaire, since the interviewer could ask the students to explain their answers more detailed. It can be shown that both experts and 8th graders used several aspects of variables when they solved the tasks, but not all aspects have been activated to the same extent.

After working with Scratch, as a task of the questionnaire, students were asked to give an explanation, what a variable is. Most high school students explained variables by focusing one aspect. Answers were, for example: "A variable is a line, to which one can give a length" (object aspect), "It is an unknown number" (single number), "Variables are letters, so in mathematics, a number you can calculate with" (single number, transformation). The experts' answers were linguistically elaborated and closer to the scientific standard: "A variable is a placeholder, behind it, there can be different numbers. They can change." (single number, changeability), "Basically all numbers can be plugged in a placeholder…" (simultaneity, plug in). Moreover, all experts covered two or more different aspects.

Taking both questionnaire and interview into account, it seems that there is a stress on the object and plug in aspect while learners work with Scratch, for example in the circle task above. Also the changeability aspect gets activated in the scenario of the 'growing circle tower'. Even the high school students recognized that the 'change length by _'-block is crucial because otherwise only one circle will be drawn. This indicates that they have a mental image of the idea that the variable is used to represent different (continuously changing) numbers at the same time, which is part of the simultaneity and changeability aspect.

Both changeability and object aspect have been activated when the students recognized that the mars robot should not be able to drive after the variable 'fuel' has dropped to zero. For example, one student said, "He had not enough fuel to keep driving […] !" which is a strong indicator that he interprets the variable as an entity which has to satisfy real-life conditions. From Malle's point of view, this is typical for the object aspect and its activation in the student's mind. Another student answered, when he was asked to explain what a variable is, directly after this task, "A variable is something. One can plug in numbers and it is something that one has to search [solving an equation for a variable to get a numeric value; author's note], I mean, you can use it to calculate and also to count". This is a sophisticated student statement since it integrates the three basic aspects, object (something, to count), plug in and transformation (to search, to calculate).

In addition to that, the idea of a suitable domain is promoted through a non-realistic negative amount of fuel. The single number aspect was promoted passively since every variable has always a known, certain value that is shown on-screen. A few students described the meaning of the used variable with the words "length stands for the number 12", which is an indicator for a dominating single number aspect. The transformation aspect was activated mostly in specific, artificial tasks, that needed the decision whether a condition holds or not.
It was observable that activated aspects strongly depend on the task. For example, contextualized settings usually foster an object view on the variable. Differences between experts and novices could be shown through a different, elaborated form of expression, the number of mentioned aspects of variables in open questions and the higher speed of the expert group. However, most tasks activated the same aspects of variables - on a different level - in both groups, university and high school students.

Conclusions

Similar results have been found in a 7th graders math class, as Ginaidi describes (2013). The findings in this teaching sequence are comparable with regard to the three basic aspects of variables, object, plug in and transformation aspect. In Ginaidi's teaching unit, like in the present case study, the emphasis lies on object and plug in aspect. As a possible solution to this observation, the underrepresented transformation aspect could be implemented through a clever design of tasks, for example a converter for Fahrenheit to Celsius degrees and vice versa or a solver program for linear or quadratic equations. Even if the transformational activities are not carried out directly with Scratch, they are necessary to solve the tasks and perhaps students are motivated through the setting to work on them in a constructive way. In the expected case, the product of the transformation process (with pen and paper) is just plugged into the 'set variable to _'-blocks.

To summarize the results, math classes can benefit from Scratch on at least three different levels. There is an **inherent level** when students work with variables in the context of programming games or realizing other projects. In addition to that, Scratch can be used as a **tool** to work on mathematical contents (e. g. development of a function plotter or an algorithm that decides whether a number is prime or not). The third level of influence might affect **classroom culture and teaching concept** as a whole since programming activities have the potential to foster creative learning and a more experiment-focused way of teaching and learning. Especially the younger students were very engaged in working with Scratch. As regards mathematical content, especially probability theory might benefit from Scratch since a lot of random experiments can be executed automatically, for example to calculate pi with the Monte Carlo method or to visualize the law of large numbers.

A positive reinforcement for the students during the work with Scratch was a phenomenon that can be described as 'product pride'. The program that was constructed by the students had in most cases valuable meaning to them. This was observable when they were showing their program to others. Moreover, students were able to work and experiment with the code even if they did not how to realize a specific task. The instant feedback of the executed program, as well as the visibility of all blocks in the command categories, often helped the students getting constructive ideas.

To come to a conclusion and with regard to interdisciplinary learning, Scratch seems to offer a large potential for mutual benefits in teaching and learning in both areas, Math and Computer Science.

REFERENCES

Brennan, K. & Resnick, M. (2012). *New frameworks for studying and assessing the development of computational thinking.* Retrieved from: http://web.media.mit.edu/~kbrennan/files/Brennan_Resnick_AERA2012_CT.pdf

Drijvers, P. (2003). *Learning algebra in a computer algebra environment. Design research on the understanding of the concept of parameter*. Dissertation. Universiteit Utrecht.

Ginaidi, S. (2013). Variablen in der Sekundarstufe I - ein Zugang über das Programm Scratch. *Praxis der Mathematik*, 52/55.

Jones, M. (2010). ICT and the Mathematics Curriculum. *Mathematics in School, March*, 3 - 8.

Kieran, C. (2004). The Core of Algebra: Reflections on its Main Activities. In Chick, H., Kendal, M., & Kaye, S. (Eds.), *The Future of the Teaching and Learning of Algebra. The 12th ICMI Study* (pp. 21-33). New York: Kluwer.

Malle, G. (1993). *Didaktische Probleme der elementaren Algebra.* Wiesbaden: Vieweg.

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas.* New York: Basic Books.

Steinweg, A. (2013). *Algebra in der Grundschule. Muster und Strukturen - Gleichungen - Funktionale Beziehungen*. Berlin: Springer.

Weigand, H.-G. & Weth, T. (2002). *Computer im Mathematikunterricht. Neue Wege zu alten Zielen.* Berlin: Spektrum Akademischer Verlag.

WORKING GROUP 2 / GROUP DE TRAVAIL 2

Teacher education / La formation des enseignants

Working Group 2 / Group de Travail 2

Teacher education / La formation des enseignants

Animators: Daniela Ferrarello, Ruhal Floris and Joaquin Gimenez Rodriguez

It is impossible to talk about obstacles in mathematics learning and possible resources to overall them, without talking also about obstacles and resources in mathematics teaching. Different aspects of mathematics teacher education are studied by researcher: affective problems, lived not only by students who dislike mathematics, but also by teachers (think about primary teachers who sometimes have not personal disposition towards mathematics), problems about an effective inclusion of technology in mathematics teaching, links between teachers' beliefs and their teaching styles, and many others.
Our group worked on teachers' education problems, reflecting on the following questions:

- How is it possible to support teachers to develop suitable knowledge and competences in digital technologies, so that they are effective in their mathematics teaching?
- What are the main obstacles for mathematics teacher development?
- How can the social dimension become a resource for teacher education? What are the challenges of programs strongly based on social interaction in communities of practice/enquiry?
- How can the affective dimension become a resource for teacher education?

The discussion started with papers on teachers' beliefs and word problems and the following papers were presented: "Investigating future primary teachers' grasping of situations related to unequal partition word problems" (Samková Ticha), "L'orientation des enseignants de mathématiques et sciences sur les modèles constructivistes et transmissivistes d'enseignement. Les résultats de la recherche Prisma sur les enseignants valdôtains des niveaux primaire et secondair" (Zanetti et al.) and "Do teacher's beliefs regarding the pupil's mistake influence willingness of pupils to solve difficult word problems?" (Bruna).
The second session was especially on mathematical knowledge requested to teachers, with the contribution of "Is this a proof? Future teachers' conceptions of proof" (Gomes et al.), "Study about the knowledge required from teachers to teach probability notions in early school years" (Pietropaolo et al.) and "A Pedagogical Coaching Design Focused on The Pedagogy of Questioning in Teaching Mathematics" (Mulat, Berman).
In the third session we focused on problems of pre-service teachers by discussing on the following themes: "Additive conceptual knowledge for admission to the degree in primary education: an ongoing research" (Castro et al.), "Collaborative study groups in teacher development: a university - school project" (Galvão), "Pre-service teacher conceptualisation of mathematics" (Cooke), "Math trails a rich context for problem posing - an experience with pre-service teachers" (Vale et al.), "Pre-service Teachers' Informal Inferential Reasoning" (Orta Amaro et al.) and "Sociocultural contexts as difficult resources being incorporated by prospective mathematics teachers" (Vanegas et al.).
The last day the discussion was about technology and its appropriate integration in the teaching/learning process. Papers also dealt with experiments in school, especially in secondary level, and were focused on "Pedagogical use of tablet in Mathematics Teachers Continued Education" (Prado et al.), on the integration of digital environments in the teaching of mathematics, "Un dispositif de formation initiale pour l'integration d'environnements numeriques dans l'enseignement des mathematiques au secondaire" (Floris), on "Instrumentation didactique des futurs enseignants de mathématiques. Exemple de la co-variation" (Venant), on "Mathematics

Teaching and Digital Technologies: a challenge to the teacher's everyday school life" (Lobo de Costa et al.), on the possibility to take into account learning styles to raise up low-performed students "Rescuing casualties of mathematics" (Ferrarello).

As a final discussion, taking into account all the themes, we reflected on obstacles and resources in teaching/learning mathematics, finding out that everything could be an obstacle, if it is unsuitably handled or a resource if it is suitably handled. We analyzed some components of the teaching/learning process and identified them as obstacles or resources, depending on their unfitting or fitting handling:
An a-priori analysis or other didactical tools can be obstacles in case of a mismatch between teacher and students paces, or resources in case of a match of the paces.
The delay in the answers, after a question posed by the teacher, can be an obstacle if the teacher does not give space to questioning and argumentation, or a resource if he/she stimulates questions, making students think.
Collaboration with colleagues and researchers and recourse to books and other teaching material can be obstacles if done in the classical "theory/practice"-model, or resources if collaboration is open and if there is space for special activities based on game-problems rather than on pure technique and on concrete models. Real-world problems also have a double-face: they are good resources if they are well linked with mathematical topics, but sometimes they rise problems not solvable by students.
Students learning styles can be an obstacle, because teachers often have one teaching style, which does not fit with students' learning styles. Instead, when they are considered, they can help the whole class to get different perspectives.
Technology, in the end, is not a panacea to solve every problem and can be an obstacle or a resource if it seen as "instrumentalisation" or "instrumentation", respectively.

And finally we concluded that a good tool for a teacher to pass from an unfitting handling of a component of teaching/learning process to an effective handling of it, is awareness. To be aware of their teaching processes teachers have to be supported by research and researchers in their pre-service and continuous education.

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Math trails a rich context for problem posing - an experience with pre-service teachers

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Abstract: This paper presents a study about the potential of the construction of creative math trails as a non-formal context in the teaching and learning of mathematics. This research is of qualitative nature and was developed with future teachers of basic education. Preliminary results suggest that despite the construction of the trail not being easy, including the process of designing the tasks, it was possible to identify traces of originality and involvement on the part of future teachers.

Résumé: Cet article présente une étude sur le potentiel de la construction d'un sentier mathématique créatif en tant que contexte non formel dans l'enseignement et l'apprentissage des mathématiques. Cette étude qualitative a été développée avec des futurs enseignants en éducation primaire. Les résultats préliminaires suggèrent que, quoique la construction du sentier ne soit pas facile, comprenant le processus de création des tâches, il a été possible d'identifier des traces d'originalité et d'engagement de la part des futurs enseignants.

Introduction

There are many students who dislike mathematics, or don't understand the purpose of studying it, because they never had the chance to enjoy it or maybe they didn't have the opportunity to be exposed to an adequate teaching. This can lead to demotivation and poor results on the assessment of this subject. In this sense, as teachers have a key role on what is going on in the classroom, teacher education should promote a new vision about mathematics knowledge and teaching, allowing future teachers to experience the same tasks that it's expected they will use with their own students.

In recent decades, problem solving has played an important role a bit around the world, as an organizing axis of the mathematics curriculum. Students' mathematics learning should include more than routine tasks, it should be enriched with challenging tasks, such as problem solving and posing. This is of great importance, not only for students but also for teachers, especially if these tasks lead to structural understanding of mathematical concepts and encourage fluency, flexibility and originality as essential components of creative thinking. If the teacher does not provide moments in which students are creative it will deny them any opportunity to develop their skills in mathematics, but also to appreciate this subject. Teachers have a determinant role in the teaching process, so, according to that perspective, teacher education should promote a new vision about mathematics knowledge and its teaching, experiencing the same tasks that we expected they will use with their own pupils.

To overcome some of the referred shortcomings, we developed a project named *Mathematical Trails* outside the classroom. With this project, we intended to promote the contact with a contextualized mathematics, starting from the daily life features, walking through and analyzing the city where we live in, connecting some of its details with exploration and investigation tasks in school mathematics. Our aim is to study the impact of mathematical trails in the teaching and learning of mathematics, as non-formal contexts outside of the classroom. In order to do this, the following questions were considered: (1) In what way the construction of the trails can contribute to the promotion of creativity in mathematics?; (2) Which mathematical contents may emerge from the formulation of the tasks based on the local environment?; (3) Which difficulties are experienced by the participants in the construction of the trails?; (4) How do future teachers relate with non-formal contexts in the learning of mathematics?

Theoretical Framework

Problem solving, problem posing and creativity

It is essential to invest in innovative educational initiatives aimed at student motivation for learning mathematics and at the development of higher order cognitive skills, such as problem solving, communication and reasoning. Creativity is also a transversal ability that should be highlighted in these experiences, since it involves curiosity and raises imagination and originality, being directly related to problem posing and solving. In fact, research findings show that mathematical problem solving and posing are closely related to creativity (e.g. Leikin, 2009; Silver, 1997). Environments where students have the opportunity to solve problems with multiple resolutions and create their own problems, allow them to be engaged and motivated, to think divergently, hence to be creative.

Analyzing this relation with more depth we can say that, in order to trigger creativity, the tasks used must be open-ended and ill structured, allowing students to exhibit the previously mentioned dimensions of creative thinking, fluency (ability to generate a great number of ideas and refers to the continuity of those ideas, flow of associations, and use of basic knowledge), flexibility (ability to produce different categories or perceptions whereby there is a variety of different ideas about the same problem or thing) and originality (ability to create fresh, unique, unusual, totally new, or extremely different ideas or products. It refers to a unique way of thinking) (e.g. Leikin, 2009; Silver, 1997).

As we said before, creativity has strong connections with problems and the process of creating problems has been defined in various ways and with different terms like invent, create, pose, formulate. Silver (1997) considers problem posing either being the generation (creation) of new problems or the reformulation of a given problem. Stoyanova (1998) considers problem posing as the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems. The problem posing activity involves for the student to problematize situations using his/her own language, experiences and knowledge. Brown and Walter (2005) discuss two problem posing strategies. The first strategy is *Accepting the given*, which starts with a static situation that can be an expression, a table, a condition, a picture, a diagram, a phrase, a calculation or simply a set of data, from which the student poses questions to have a problem, without changing the given. The second consists of extending the task by changing the given using the *What-If-Not* strategy. From the information of a particular problem, we identify what is the problem, what is known, what is in demand and the constraints that the answer to the problem involves. Modifying one or more of these issues and questions that are formulated in turn, may generate more questions.

So, in the frame of problem solving we are talking about tasks that enable different approaches to find a solution, hence promoting divergent thinking. As for problem posing, either by reformulating a given situation or creating something new, the creativity relies on the relational nature of the mathematical knowledge used. It's important to state that these tasks shouldn't be considered separately, since the creative activity results from the interplay of reformulating, attempting to solve, and eventually solving a problem.

Teachers have a critical role since they have the power to unlock students' creative potential. So it's fundamental to offer pre-service teachers diverse experiences, in order for them to develop a new vision about mathematical knowledge and teaching, allowing them to experience the same tasks that we expect them to use with their pupils.

Math Trail

Bolden, Harries and Newton (2010) consider important to discuss with (future) teachers their beliefs about creativity in mathematics, trying to perceive how these ideas impact their teaching strategies and translate into classroom practice. In this sense, it is not enough that teachers know the general meaning of creativity, but understand that the dimensions or characteristics of creativity can vary with the subject and the context they are dealing with. It's crucial that professional development promotes reflection about these issues (Vale, Barbosa & Pimentel, 2014).

However, very often students don't develop such abilities, aren't able to make connections among different topics and use diversified tools to approach the same problem, since curriculum features and extension leads teachers to avoid this type of exploration. In this context we must stress the importance of complementing learning in other environments, like non-formal contexts. Normally completion-like environments, clubs, journals, lectures, projects, can give students the chance to enjoy mathematics, that, due to several factors, could never experience its beauty (Kenderov et al., 2009). For some students, the simple fact of participation is a great success (Pimentel & Vale, 2014).

The classroom is just one of the "homes" where education takes place (Kenderov et al., 2009). The process of acquiring information and the development of knowledge by students can occur in many ways and in many places. Whereas the stimulus for an affective environment can influence the initial expectations and motivations of students, the use of the surroundings as an educational context can promote positive attitudes and additional motivation for the study of mathematics, allowing them to understand its applicability.

The math trails arise in this context. They are considered as a sequence of stops along a pre-planned route by which students can learn mathematics in the environment (Cross, 1997) and offer concrete learning experiences for any of the mathematics concepts taught in the school curriculum. It also offers huge potential for learning experiences at all ages. This type of activity facilitates the creation of a non-formal meeting space, focused on learning, and also the approach to problem posing and solving, the establishment of connections and the encouragement of communication, applying these skills in a meaningful context. A bounty of opportunities exist to utilize the outdoors in orchestrating learning experiences, not only in mathematics, but also through the integration of knowledge with outcomes stated in other learning areas. Because it takes place outside the classroom, a math trail creates an atmosphere of adventure and exploration, giving students the opportunity to solve problems (in real life context) and pose problems. By learning to solve problems and by learning through problem solving, students are given numerous opportunities to connect mathematical ideas and to develop conceptual understanding, having also opportunities to develop their creative thinking. In this sense, students are effectively motivated to learn mathematics, discovering its role in the environment, and simultaneously mobilize fundamental abilities and attitudes.

Encouraging teachers to propose problems to their students and supervising their work can increase their professionalism and confidence in these activities, developing their competence and enthusiasm in future teaching/learning actions in contexts outside of the classroom. Teachers have a key role here, being highly relevant to study their knowledge and perceptions, particularly in innovative initiatives.

Methodology

Based on the goals of this study we adopted a qualitative methodology of exploratory nature. The participants were 70 future teachers of basic education (3-12 years old) that attended a unit course of Didactics of Mathematics.

Throughout the classes of this subject they were provided with diversified experiences, distributed in curricular modules, focusing on: problem posing and solving (Silver, 1997); creativity in mathematics (e.g. Leikin, 2009); the establishment of connections, particularly those involving mathematics and daily life; and other mathematical processes (e.g. communication, reasoning, representations). In addition to these aspects, some examples of math trails were explored in this unit course in order to clarify its structure and allow these future teachers to perceive the presence of the previously analysed abilities (problem posing and solving, creativity, connections). After these teaching modules, the participants had to build a math trail in small groups, based in the city of Viana do Castelo, posing tasks centred on elements of the local environment, aimed at basic education students (3-12 years old) school.

First they had to choose an artery of the city that would constitute the route to be explored in the math trail. Then, along that route, the future teachers took photographs of elements that had potential for mathematical exploration. These photographs would be the basis to design the tasks in the trail. During the lessons of this unit course, the participants shared and discussed the photographs taken along the trail they selected, and they also presented some hypothesis of tasks formulated, based on those elements. Mostly they used as problem posing strategy *accepting the data* (Brown & Walter, 2005), since they started with static situations, the photographs (e.g. windows, buildings, monuments, gardens, doors, wrought iron, tiles), on which they formulated problems without changing what was given.

Data was collected in a holistic, descriptive and interpretative way and included classroom observations and document analysis, mainly focusing on written records of the math trails and on a questionnaire centred in the opinion of the participants about this type of work (e.g. difficulties, potential, impact). In the data analysis the criteria used were: creativity, diversity and rigor of the mathematical contents.

Results

To clarify the results we start by presenting some examples of the work produced by these future teachers.

The different groups chose diversified structures for the visual presentation of the trails. The majority presented the trail in the form of a flyer, containing the route and the tasks (Figure 1). Some of them included maps for the students to read and interpret, since it's a content of the curriculum. In a few cases the trail assumed the form of a game with several stations, corresponding to the stops, where the students would receive points for each task solved.

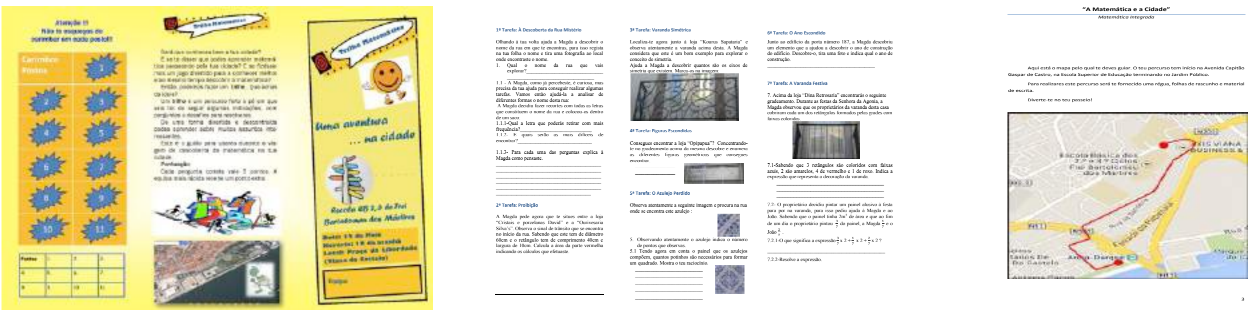

Figure 1. Examples of the visual presentation of the trails

Other structures were presented, that we considered to more original, since only a few participants chose to do it. In this group we include, for example, the structure of a treasure map, a book in the shape of a heart (symbol of the city), a book with riddles representing the elements students had to identify (Figure 2).

Figure 2. Examples of the visual presentation of the trails

Some of the future teachers also organized, alongside the math trail, a kit with materials to be used along the route (e.g. ruler, measuring tape, rope, pencil, eraser, notebook, calculator, train schedule) (Figure 3).

Figure 3. Examples of the visual presentation of the trails

The future teachers participating in this study, as previously mentioned, designed the tasks included in the math trails. They had to organized them in a sequence that would allow students to execute the trail in context, having a starting and a finishing point and also a diversity of stops on which they had to solve a task. The tasks create, by the futures teachers, in the trail were mainly problems for pupils to solve. They also involved elementary mathematical concepts and can be applied in different contexts of the classroom, in the 1st and/or 2nd cycles of basic education (6-12 years old). In figure 4 we present some examples of problems formulated by these future teachers.

 You are in *Avenida Capitão Gaspar de Castro*. If you turn your back to *Escola Superior de Educação* what building do you see? In this hotel you can see that the 1st floor is oriented to the left, the 2nd to the right, the 3rd to the left and the 4th to the right. Imagine that this building would have 20 floors. What would be the orientation of the 16th floor?	The photograph shows some details of the Riverside Garden where we can see a set of four equal flower beds. - Classify the geometrical figure represented by each flower bed.  - Identify, if existing, the axis of symmetry of the mentioned figure. And of the figure composed by the four flower beds? - Use two threads to mark the diagonals of the figure and count the number of different triangles that you can identify. - Considering the arrangement of the plants, how can you count, in two different ways, the number of plants in each flower bed?

Can you find a pattern?	
Walk down the Manuel Espregueira street till you find Olivenza street. Continue down this street and on the right stop at the door with the number 37. Observe the wrought iron door and its structure.  Count all the triangles that can you see.	In the *Marginal Garden* you can find many plants and flower beds. Look at the one in the picture. How do you think the gardener constructed it? Explain the process. 

Figure 4. Some examples of problem posing tasks

After finishing this project the participants were given a questionnaire in order for us to get to know their main difficulties, the positive aspects of this work and overall the impact it had on their perspective about mathematics teaching and learning.

The design of the tasks was not always an easy process for the participants, which can be understood because it was a new experience and also because of the fact that problem posing is a higher order ability, which implies a regular work. Overall they showed a clear tendency to involve concepts of elementary geometry, since the elements involved in the trail were of a more visual nature. We will present the content of some of the problems posed in the context of the photos that were taken in town, and which were later analysed with detail in order to construct rich problems. As we can see, in Figure 4, the second, the third and the fourth problems deal with geometric figures while the first is based on numerical features. However, in most of them we can observe connections among several topics, namely patterns, visual countings and functions. Geometry (e.g. figures, area, perimeter, volume) and Patterns were the easiest contents to approach. The most difficult was Statistics. Perhaps this relates with the former mathematical experiences of these students in the topic of Patterns and also with the geometrical nature of most of the observations, while it is not so natural a connection with statistics.

Another weakness which was reflected in the final work concerns the ignorance of the measures of the buildings/monuments, and the difficulty in making estimations. Overall we noted that for the great majority of the students it was not easy to pose problems based on the local environment. We as teachers wanted students to use diverse elements of the environment, as well as diversify the questions posed and this is not easy because this competence also relates with previous knowledge and mathematical experiences of the students. The discussions generated in the classes provided clarification on some confusing aspects of tasks, allowing students to do some refinement.

In the words of these future teachers this project had a positive impact on their perspective about mathematics, allowing them to perceive things like: *This project changed my perspective about Mathematics because I always explored it in the classroom*; *I started to look to everything around me with math eyes*; *I knew we could connect math to daily life but this project showed me that there is much more than I imagined and we can do spectacular things in math*; *I loved to walk through the city trying to discover situations that could lead to questions, measuring, testing, …*; *Students often ask "what is math for?" and this project helps find the answer*; *The formal work in the classroom can be related to these experiences exploring the contents in a more practical way. We observe math in the real world*; *This project helps with creativity and allows us to know better our city*; *With this type of work we can motivate the interest and taste for mathematics contributing to students learning.*

Discussion

With this study it was possible to conclude that the future teachers showed a more positive attitude and appreciation towards mathematics and can be a natural extension of the classroom and the work developed in extending their perspective about the possible connections that can be established outside the classroom, in particular with the local environment.

The trails, provided a better knowledge of the environment where it was built using a mathematical eye, but also focusing on the culture and heritage of the city. By organizing a math trail (future) teachers improve their problem posing skills and their critical sense, having the opportunity to: be creative (in particular, be original); choose the contents to be approached; show a contextualized and engaging mathematics to their students. Being challenging, based on collaborative work, a math trail can be a way of reaching students of all levels of achievement and also of different grade levels.

It was possible to identify traces of creativity in the tasks, particularly regarding the originality dimension. In general, it can be said that these future teachers showed will and motivation to overcome the obstacles they encountered and the tasks presented in the various trails indicated that this type of work has the potential to promote creativity in mathematics.

REFERENCES

Bolden, D., Harries, & Newton, D. (2010). Preservice primary teachers' conceptions of creativity in mathematics. *Educational Studies in Mathematics*, *73*(2), 143-157.

Brown, S. & Walter, M. (2005). *The art of problem posing.* Mahwah, NJ: Erlbaum.

Cross, R. (1997). Developing Math Trails. *Mathematics Teaching*, *158*, 38–39.

Kenderov, P., Rejali, A., Bartolini Bussi, M., Pandelieva, V., Richter, K., Maschietto, M., Kadijevich, D., & Taylor, P. (2009). Challenges Beyond the Classroom—Sources and Organizational Issues. In E. Barbeau & P. Taylor (Eds.), *Challenging Mathematics In and Beyond the Classroom – New ICMI Study Series 12* (pp. 53-96). Springer.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A., Berman & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129-145). Rotterdam, Netherlands: Sense Publishers.

Pimentel, T. & Vale, I. (2014). A Mathematical congress: a window to affect in problem solving. In S., Carreira, N., Amado, K., Jones, & H., Jacinto (Eds.), *Proceedings of the Problem@Web International Conference: Technology, creativity and affect in mathematical problem solving,* pp. 308-320. Faro, Portugal: Universidade do Algarve.

Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM*, *3*, 75-80.

Stoyanova, E. (1998). Problem posing in mathematics classrooms. In A. McIntosh & N. Ellerton (Eds.), *Research in Mathematics Education: a contemporary perspective* (pp. 164-185). Edith Cowan University: MASTEC

Vale, I., Barbosa, A. & Pimentel, T. (2014). Teaching and learning mathematics for creativity through challenging tasks. In S., Carreira, N., Amado, K., Jones, & H., Jacinto (Eds.), *Proceedings of the Problem@Web International Conference: Technology, creativity and affect in mathematical problem solving,* p. 335. Faro, Portugal: Universidade do Algarve.

Do teacher's beliefs regarding the pupil's mistake influence willingness of pupils to solve difficult word problems?

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Abstract : This paper presents a didactic experiment which focuses on finding possible links between teacher's beliefs and teaching style on the one hand and strategies pupils use to solve complex word problems on the other hand. In this experiment two Czech mathematics teachers and three of their classes of seven graders – one for the first and two for the second teacher – were included. The data were collected from questionnaires for teachers and sheets with pupils' solutions of selected word problems. The outcomes of the experiment suggest that there may be a link between teacher's beliefs regarding the performance in mathematics of a good pupil and willingness of pupils to solve complex, unfamiliar and non-standard (CUN) word problems. Although the results cannot be generalised they show areas of interest for further research into classroom practices and may in the future inform teacher-training as well.

Résumé : Cet article présente une expérience didactique qui a pour but d'observer comment le style d'enseignement et la conviction de l'enseignant influencent les stratégies utilisées par les élèves cherchant la solution d'un problème complexe. Cette expérience analyse deux enseignants de mathématiques tchèques et leurs trois classes du niveau du collège (7e année). Les données ont été recueillies à partir des questionnaires pour des enseignants et aussi à partir des fiches de travail remplies par les élèves. À la lumière des données, la conviction de l'enseignant peut etre en rapport avec la bonne capacité de résoudre les problèmes complexes, inconnus et non standards. Les résultats ne peuvent pas être généralisés mais ils signalent des domaines qui peuvent être encore plus examinés dans les recherches futures et qui pourraient être utilisés dans la formation des enseignants.

General Introduction

This paper aims to introduce a didactical experiment which focuses on linking teacher's beliefs and teaching style to the strategies pupils utilise in order to solve complex word problems. It was conducted as a part of a Ph.D. programme course at the Charles University in Prague. The didactical experiment was conducted at two Czech basic schools, specifically in three classes of seven-graders (approximately twelve years of age) taught by two different teachers.

Any research that attempts to establish connections between teaching and learning is necessarily faced with severe obstacles both theoretical and methodological (Hiebert, Grouws, 2007). Because the university course may be considered introductory and the resources, most notably the suitable teachers available, were limited, the main aim of the didactical experiment was not to arrive at conclusive and well-established links. Much rather the main aim was to find potentially significant aspects of teacher's beliefs and teaching practices which may have an impact on solving strategies on the part of their pupils, in the sense that collected data are suggestive of such a link. These findings may be relevant to other researchers covering related areas as well as inform teacher training.

As will be shown below the overall framework of the didactical experiment was not so much grounded in theoretical background apart form the article mentioned above. However, there was an attempt to provide links for the findings to other mathematics education research in related areas.

Methodology

The didactical experiment in question relies on qualitative approach and was conducted in several stages. In the first stage two word problems were chosen. The criterion for selection was the richness of possible solving strategies. This was necessary for individual differences between students to manifest. This approach was inspired by the article on promoting creativity (Hershkovitz, Peled, Littler, 2009). Furthermore, the word problems were modified in terms of language and clarity so that undesired misunderstandings on the part of pupils were minimised. Both word problems could be classified as complex, unfamiliar, non-routine problems (CUN) (Mevarech, Kramarski, 2014). The following lines present English translations of the two word problems.

Word problem 1
The typist was asked to write down numbers from 1 to 500 one by one. How many times does he have to type the digit “1”, provided that he does not make a mistake?

Word problem 2
How many times a day is the sum of the digits on the display of a digital alarm clock, which shows time values from 00:00 to 23:59, equal to seven? (For example, the sum of the digits in the time value 02:45 is 0 + 2 + 4 + 5 = 11.)

In the second stage the questionnaire for teachers was created and was distributed to two selected teachers. For the English version of the questionnaire see the *Appendix* at the end of the paper. The only requirement for including the teacher was that she/he teaches seven-graders at a basic school. Teachers of pupils from grammar schools were excluded due to a concern that given the pupils' selection after the fifth-grade this would render the samples of pupils from both kinds of school mutually incomparable. In agreement with the aim of the didactical experiment the questionnaire attempted to cover various areas of teaching practice, most notably interaction patterns and organizing work in the classroom, teacher error correction and pupil's autonomy in the problem solving process, as well as teacher's beliefs on the nature of mathematics, criteria of good pupil's performance and the nature of the solution to the mathematical problem.

As far as the structure of the questionnaire is concerned there were three parts. In the first one the scale from one to six was utilised. The second part included one open question regarding the organization of the work in the classroom. In the third part the teacher's task was to select the view of mathematics that influences her/his teaching practice the most.

After the questionnaires were collected the didactical experiment continued in the classrooms. In their mathematics lessons the teachers distributed the word problems to their pupils to solve them on pre-prepared sheets of paper with the printed word problem (answer sheets). The teachers were instructed to let pupils solve the first word problem first then conduct a whole-class discussion of the solution, then let the pupils solve the second problem and have a discussion again. The pupils were asked to work individually, although this was not followed in every case. The pupils were not allowed to modify their solution during the discussion. The answer sheets were then collected. The lessons were also video-recorded, however, the main sources of data were the questionnaires for the teachers and the answer sheets.

Although the original intention was to include one class per teacher in the didactical experiment, one of the teachers volunteered to let another of her classes solve the word problems as well. The data obtained from this class helped refine the conclusions. In particular it ruled out the differences between solving strategies which would be otherwise attributed to the difference of the teachers although they appear in between the classes of the same teacher as well.

In the next stage the questionnaires were analysed and the instances of substantially different answers were found. In order to do this the scale was utilised in the first part. The substantially different answers were defined as the answers to the same question which differ at least by three on the scale. In the second part the answers were compared in terms of the ratio of individual work to pair-work/group-work to the whole-class discussion they describe. In the third part the chosen beliefs were compared.

Finally, the answer sheets were analysed in terms of strategies pupils used to approach the word problem. First every answer sheet was analysed separately, later strategies sharing the same underlying principles while only varying in details were grouped into general strategies. The outcome of the analysis of the answer sheets was the list of general strategies used by pupils. It was sorted by class and the word problem and included the number representing the number of pupils who used the particular solving strategy in the particular class to tackle the particular word problem. It is important to point out that even the instance of a pupil not attempting to solve a word problem or her/his stating that she/he does not know how to solve the problem without any attempt to indulge in the problem was considered to be a case of a solving strategy as well.

In order to make sense of the situation the data from the teacher questionnaire and the analysis of solving strategies were put together. The key assumption was that the differences in beliefs and teaching styles of the teachers will account for differences in solving strategies between the respective classes.

Conclusions

As detailed in the section on methodology the output data were two-fold. As far as the analysis of teacher questionnaires is concerned there were differences in all three parts. However, this paper focuses mainly on those detected in the first part for two reasons. Firstly, the teachers are in sharper disagreement about the selected statements (see below) compared to the second part of the questionnaire. Secondly, the statements in the first part are very straight-forward and isolated pieces of beliefs and practices. Therefore further research can deal with them more easily compared to the more complex pieces of belief in the third part.

The answers of teachers in the first part differed mostly in the area of practices connected with error correction and beliefs regarding the view of pupil's good performance in solving problems. The statements in question are:

I try to correct pupil's mistakes immediately.

Good students solve mathematical problems easily and immediately.

The responses given by teachers show a certain kind of dichotomy. While the first teacher (teacher A) tries to correct pupils' mistakes immediately he at the same time does not agree that a good student solves problems easily and immediately. The second teacher (teacher B), on the other hand, shows the opposite preference. She does not try to correct pupils' mistakes immediately yet she believes that a good student solves the problems quickly and without difficulty.

The analysis of the strategies pupils used to solve the word problems produced rich outcomes. In total pupils used 11 general strategies for the first word problem and 17 for the second word problem. There were as many as seven general strategies per class for the first word problem and as many as nine general strategies per class for the second problem. Of particular interest are the results obtained while analysing strategies employed to solve the second word problem because they show a pattern described below. Other patterns may still emerge after more complex analysis.

Generally speaking, the set of data from the other class taught by the same teacher (teacher B) proved useful in suggesting that contrary to the key assumption the differences in teacher's beliefs and practices cannot entirely account for the differences in pupils' strategies. Not only do the dominant solving strategies differ when classes of different teachers are contrasted. They also differ when the two classes of the same teacher are compared.

Nevertheless, a pattern emerged that holds for both of the classes taught by teacher B and does not appear in the class of teacher A. In the classes of teacher B not a negligible number of pupils appear who refuse to solve the problem. These pupils often state that they do not know how to approach the problem or express the belief that the solution would be too long and/or demanding. As mentioned above, this approach to solving the word problem is treated as a solving strategy in this paper. The question now arises if beliefs and practices of the teacher may influence the willingness of pupils to solve word problems which they think of as too difficult.

To put it more explicitly, the main finding, as of now, is that there are two suspected links between teacher's beliefs and practices and the strategies her/his pupils use to solve word problems:

(A) The way the teacher corrects mistakes (immediate/delayed correction) has an impact on the willingness of pupils to solve difficult problems.

(B) If the teacher believes that good pupils solve mathematical problems easily and immediately it has a negative impact on willingness of some pupils to solve difficult problems.

Discussion and concluding remarks

It has to be explicitly stated that the main value of this research does not lie with establishing firm links. It is clear that given the size of the sample and the complexity of the teaching learning relationship (Hiebert and Grouws, 2007) the results cannot be generalized. Nevertheless, conclusions made here can help direct further research by hinting at the areas of interest. The subsequent research could, among other things, interview pupils who are not willing to solve difficult word problems to map other factors possibly contributing to this behaviour. This was not done in this research mainly due to its format.

The following lines provide a brief discussion of the results based on the research literature concerned with related areas to support the existence of these links, most notably link (B). The paper of Santagata (2005) supports the existence of link (B) in two ways. Firstly, it asserts that through classroom practice, more specifically public mistake-handling, pupils may experience a range of ideas on the part of the teacher concerning mistakes. Secondly, it says that the ways teacher frame mistake-handling activities shape the experience of pupils itself and can have an impact on their willingness to take on new and/or complex tasks. Connected to the data from the didactical experiment this may suggest that the teachers' beliefs regarding the good student and whether this student makes mistakes or not while solving problems may indeed be transferred to pupils and influence their strategies of solving word problems.

Hejný and Kuřina (2009) talk not only about the sources of the views of errors linking them to cultural traditions but also illustrate the anxiety of failure on the example of one pupil. One possible interpretation of the data with respect to this phenomenon is that pupils of teacher A have little reason to feel discouraged even if they are experiencing difficulty in the solving process. In other words this supports the existence of link (B) even further. With respect to link (A) the data suggest, on the other hand, that the immediate correction is not a big obstacle in terms of pupils' willingness to solve difficult word problems because the pupils of teacher A, who tries to use immediate correction, seems willing to take on difficult tasks. In other words it is possible that immediate correction does not result in the anxiety of failure.

The conclusions can be also interpreted in the light of what Hejný (2004) says. He states that the issue with guiding lower-performing pupils is not as much cognitive as it is volitional. He further claims that main aim of such guidance is not to teach them something but to ensure that the pupils believe the learning is meaningful. Although the didactical experiment does not provide observational evidence to show that teacher A is acting towards lower-performing pupils in this way, assuming that it is the case the pupils of this teacher may be further encouraged to try to solve difficult word problems through their belief that such a behaviour is meaningful, in agreement with link (B).

Furthermore the collected data may serve as particular example of teacher beliefs that influence error-handling practices of teachers in the classroom. The connection between beliefs and error-handling practices is established for example in Bray (2011). As such the findings of this experiment may be of interest to researchers interested in teacher development and teacher trainers.

REFERENCES

Bray, W. S. (2011). A Collective Case Study of the Influence of Teachers' Beliefs and Knowledge on Error-Handling Practices During Class Discussion of Mathematics. *Journal for Research in Mathematics Education*, 42(1), 2-38.

Hejný, M. (2004). Chyba Jako Prvek Edukační Strategie Učitele. *Dvacet Pět Kapitol Z Didaktiky Matematiky*. Praha: Univerzita Karlova v Praze - Pedagogická Fakulta, 2004. 63-80.

Hejný, M., & Kuřina, F. (2009). *Dítě, škola a matematika: Konstruktivistické přístupy k vyučování*. Praha: Portál.

Hershkovitz, S., Peled, I., & Littler, G. (2009). Mathematical creativity and giftedness in elementary school: Task and teacher promoting creativity for all. *Creativity in mathematics and the education of gifted students*, 255-269.

Hiebert, J., & Grouws, D. A. (2007). The Effects of Classroom Mathematics Teaching on Students' Learning. *Second Handbook of Research on Mathematics Teaching and Learning a Project of the National Council of Teachers of Mathematics*. Charlotte, NC: Information Age Pub., 371-404.

Mevarech, Z., & Kramarski, B. (2014). *Critical Maths for Innovative Societies: The Role of Metacognitive Pedagogies*. Paris, France: OECD.

Santagata, R. (2005). Practices and Beliefs in Mistake-handling Activities: A Video Study of Italian and US Mathematics Lessons. *Teaching and Teacher Education, 21*, 491-508.

Appendix – The Teacher Questionnaire

State an extent to which you agree with the following statements concerning your teaching style. Use the attached scale (1 – I completely agree; 6 – I do not agree at all).

I put emphasis on discovery-based learning.

1 2 3 4 5 6

I try to correct pupils' mistakes immediately.

1 2 3 4 5 6

Pupils work individually during the lessons.

1 2 3 4 5 6

I encourage students to come up with their own solving strategies.

1 2 3 4 5 6

I include class discussions on a regular basis.

1 2 3 4 5 6

I include non-standard and/or challenging problems on a regular basis.

1 2 3 4 5 6

There is a space for pupils' self-reflection in my lessons.

1 2 3 4 5 6

Pairwork and/or groupwork is an important part of my lessons.

1 2 3 4 5 6

I frequently include exposition on the subject matter at least ten minutes long.

1 2 3 4 5 6

For every problem there is the best way to solve it.

1 2 3 4 5 6

Good students solve math problems easily and immediately.

1 2 3 4 5 6

Briefly describe the typical ratio between individual work, pairwork/groupwork and whole class discussions in your lessons. (For example which of the forms is the most frequent and which is the least etc.)

Indicate which of the following views on mathematics influences your teaching style the most.

I understand mathematics as a tool for solving problems.

I understand mathematics as a specific way of thinking.

I understand mathematics as a body of knowledge.

I understand mathematics as a supporting science for other fields, for instance physics or chemistry.

I understand mathematics as an entertaining activity.

Different view (please specify):

Additive conceptual knowledge for admission to the degree in primary education: an ongoing research[1]

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Abstract: It is desirable for students starting their Degree in Primary Education to possess some preliminary disciplinary knowledge. The results presented herein are part of an ongoing study that aims to establish and evaluate the extent of Basic Mathematical Knowledge, in its conceptual sense, required to initiate didactics of addition and subtraction (from now on referred to as CBMK-A). For this purpose, it is invaluable to identify additive knowledge profiles in students of the Degree in Primary Education, being of great assistance when planning the class module "didactics of arithmetic". Specifically, we present the 4 components which make up this CBMK-A, including some of the tools for their evaluation.
Résumé : Il y a savoirs disciplinaires en mathématiques qu'ils sont souhaitable que les étudiants de la Maîtrise en Éducation Primaire (MEP) doivent avoir aux commencer leur formation. Nous présentons le cadre d'une étude en cours, dont l'objectif est établir et évaluer la connaissance mathématique fondamentale (dans son aspect conceptuel) nécessaire pour commencer à enseigner addition et la soustraction (CMFC-A). L'identification des profils de connaissances additifs d'étudiants de la MEP, est une information précieuse pour guider la planification des cours à la didactique de l'arithmétique. Plus précisément, nous présentons les 4 éléments qui composent cette CMFC-A, avec une partie de l'instrument pour les évaluer.

Introduction

Disciplinary knowledge in mathematics is a necessary and even fundamental element in the development of teacher students. We consider that a certain extent of mathematical knowledge – both conceptual and procedural – is a necessary requirement for students at the start of their Degree in Primary Education (DPE).
The first stage of the study presented herein reviews the different theories of teacher's knowledge in relation to the teaching of mathematics. Based on this review and considering the requirements of professional practice and mathematical competences in Primary School, we establish the concept of Basic Mathematical Knowledge (BMK) (Castro, Mengual, Prat, Albarracín and Gorgorió, 2014).
Arithmetic is a key component of the Primary School Mathematics; therefore it has to be also a key component in the mathematical teacher training of future teachers' education. Our study aims to determine the fundamental mathematical knowledge that will allow future teachers to successfully construct the pedagogical knowledge of content related to the "numeration system and arithmetic operations". We seek to establish: (i) BMK from its conceptual point of view the students should have of the DPE when starting didactics of addition and subtraction (CBMK-A); and (ii) additive conceptual knowledge profiles in students of the DPE.
After the literature review on conceptual and procedural knowledge in mathematics, we centred our attention on previous research done on the addition principles, conducted on children as well as adults. By focussing on conceptual aspects we established CBMK-A around 4 components. We herein present part of the instrument to evaluate the conceptual aspects.

[1] This research is under the project *Caracterización del conocimiento disciplinar en matemáticas para el grado de educación primaria: matemáticas para maestros, I+D, RETOS, Dirección General de Investigación* (ref. EDU2013-4683-R).
[2] The authors are members of the Research Group *Educació Matemàtica i Context: Competència Matemàtica (EMiC:CoM)* , ref. 2014SGR 00723.

Theories of teacher’s knowledge and bmk

Disciplinary knowledge in mathematics has been recognised as a fundamental component (Ma, 1999) and necessary for the development of other types of knowledge. Based on the work of Shulman (1986, 1987), different theoretical views have been developed with the intention of adapting the concept to the needs of mathematics teaching (KQ – Knowlege Quartet – de Rowland, Huckstep and Thwaites, 2003; MKT – Mathematical Knowledege for Teaching –Ball, Thames and Phelps, 2008; and the development of the MTSK – Mathematical Teacher Specialized Knowledge – Montes, Contreras and Carrillo, 2013). These studies are mainly centred on knowledge for the teaching of mathematics of in-service teachers, as well as on aspects regarding their training. We agree with Linsen and Anakin (2012; 2013) on the fact that descriptions of the mathematical teacher’s knowledge found in the literature contain teaching characteristics that have been identified and associated to expert teachers, and are therefore unsuitable to describe the nature of the knowledge required by initial education teachers at the beginning of their programmes.

In the search for the elements that will allow us to describe the BMK that the future teacher should have at the beginning of his/her initial training, we noted that we could not find a model of teacher’s knowledge that could effectively encompass the extent of knowledge we were looking for. As an example, the definition of content knowledge proposed by Shulman (1986, 1987) is centred on the idea that professors should critically comprehend the entirety of ideas he/she is going to teach, since without this comprehension of the subject they will not be able to transform these ideas for the better understanding of their students. Thus according to this author, the teacher not only needs to be familiar with the procedures, but is also required to understand the concepts underlying them, that is, to know why things are the way they are. However, during their mathematical training at school, students starting the DPE have not necessarily learned the reason “why” that leads to deep understanding.

In the same way, in the foundation component of Rowland and contributor’s KQ, that involves the knowledge of the content, amongst other aspects, we may implicitly find some of the features of BMK. This is due to the fact that the foundation component of KQ refers explicitly to elements of the knowledge of mathematical content that future teachers should develop during their training and that is demonstrated during classroom practice. Regarding the MKT proposed by Ball and contributors, which divides the content knowledge into the sub-domains: common content knowledge, specialised content knowledge and knowledge of the mathematical horizon. We can place partially the BMK within the domains of common knowledge and knowledge of the mathematical horizon. We consider that, as a result of their mathematical education, these future teachers should have acquired the common knowledge that any mathematically educated adult possesses at the end of their schooling. On the other hand, they may also partly fit the category of knowledge of the horizon, since their training has allowed them to get to know more mathematical content that what they are going to teach.

In regard to the view on MTSK proposed by Carrillo and others, mathematical knowledge includes three sub-domains: knowledge of topics, knowledge of mathematical structure and knowledge of mathematical practice. This outlook refers to elements that are associated to and identified from the practice of expert teachers. Following this theory, we could identify BMK as the elements that offer a solid foundation for them to develop successfully their training and the practice of this knowledge.

When trying to relate BMK to the aforementioned theories, we find they prove unfitting to our purpose because these theories are associated to knowledge in professional practice. From this point, in Castro et al. (2014) we take the definitions of: foundation content knowledge of content proposed by Linsell and Anakin (2012; 2013), elementary mathematics proposed by Ma (1999), and also different theories of teacher’s knowledge. We define the BMK as the basic mathematical knowledge necessary for being the future teacher able to achieve the pedagogical content knowledge. Including the knowledge of concepts, procedures and problem-solving processes that students of the DPE have learned during their school years and need to carry with them to start their training.

Conceptual knowledge and cbmk-a

Several research studies have highlighted the important role played by conceptual knowledge in learning mathematics. However, after decades of research there does not seem to be a consensus on the notion of conceptual knowledge, nor on what the best way of measuring it is (Baroody, Feil & Johnson, 2007; Crooks & Alibali, 2014). The wide variety of existing characterisations and the nature of the tasks used to measure this type of knowledge are not always in tune with their definition and they may represent an obstacle to the understanding of the main findings in this field (Crooks & Alibali, 2014).
The different characterisations of the conceptual knowledge in mathematics suggest that this knowledge can be equate with a deep knowledge, well connected, flexible and associated to significant knowledge. Along these lines, possibly one of the most renowned and employed characterisations is that suggested by Hiebert and Lefevre (1986). These authors define conceptual knowledge as a complex network of relationships between pieces of information that allow for flexibility in the access and use of information –knowing how and why. This characterisation has been widely reproduced and interpreted over the years. For instance, conceptual knowledge has been defined in terms of the interrelations between different items of knowledge, of the comprehension of basic concepts or regarding the principles that govern a domain, being explicit or not (Rittle-Johnson & Alibali, 1999; Rittle- Johnson, Siegler & Alibali, 2001), being generalised and expressed verbally or not.

Conceptual additive knowledge has been widely studied over the years. After reviewing the literature on conceptual and procedural knowledge in mathematics, we focussed on those studies that dealt with addition and its principles, being centred upon those aspects of a predominantly conceptual nature. We established CBMK-A around 4 components that represented 4 types of conceptual mathematical knowledge that we consider fundamental. The first of these is the knowledge of the decimal numeration system and the positional value. It should be noted that the understanding of the decimal numeration system and of the concept of positional value is essential to the development of a numerical sense. In addition, it is the basis of the comprehension of fundamental operations involving numbers, fractions and decimals. In the context of teacher training, studies by Montes, Liñan, Contreras, Climent & Carrillo (2015) and Salinas (2007), amongst others, have revealed that future teachers lack a solid understanding of the numeration system. According to these authors, future teachers have a merely technical and limited command of the numeration system and with conceptual gaps in the understanding of significant concepts at the start of their training. Moreover, these studies have highlighted how important it is for future teachers to have a sound knowledge of certain concepts at the beginning of their training. These concepts are often assumed as known in the DPE but, without reassuring their acknowledgement, teacher students may face difficulties when teaching related subjects, leading to negative consequences for the education of their students.

The second component of the CMF-A is centred on the knowledge of the meanings of addition and subtraction. A key conceptual advance in conceptual additive knowledge is to understand that addition and subtraction may be defined as unitary or binary operations (Baroody and Ginsbung, 1986; Cañadas and Castro, 2011). In addition, the subjects’ view on this type of operations may be reflected in the formulation of verbal elementary arithmetic problems, the verbal explanations of addition and subtraction, and in the use and perception of keywords found in the formulation of these problems. For instance, Castro, Gorgorió & Prat (2014) suggest that teacher students have a limited view on addition and subtraction. They observe that future teachers essentially pose additive problems with keywords that coincide with the operation needed, largely involving change structures with increases and decreases. It reflects a unitary view on addition (Baroody & Ginsbung, 1986).

Thirdly, we consider the comprehension of part-whole relationships. Most of the research done in this field has focussed, to a large extent, on the study of additive principles such as additive composition, commutativity, associativity, complementary addition and subtraction, and inversion (Canobi, 2005; 2009; Gilmore and Bryant, 2006; 2008; amongst others). Conceptual change is vital in the sense of acknowledging how a set is made up of different additive parts. This involves the ability to perform a calculation and to use the principles underlying mathematical relations (Gilmore & Bryant, 2006).

Finally, we regard domain and the use of addition and subtraction algorithms to be essential features of the additive structure (Cañadas & Castro, 2011; Dickson, Brown & Gibson, 1991). In order to understand the formal algorithms of addition and subtraction to a symbolic level, knowledge of the structure of the decimal numeration system is required, as well as an idea of how objects are counted. With regard to the comprehension of addition, the knowledge of basic sums, of addition tables and of the commutative and associative properties is also required. However, in the case of subtraction, the command of descendant counting as well as of double-simultaneous counting, ascendant and descendant needs to be added to the skills required for addition.

After establishing the 4 components of CBMK-A: (1) the knowledge of the decimal numeration system and positional value; (2) the meaning of addition and subtraction; (3) part-whole relationships; and (4) algorithms, we consider Crooks and Alibali's (2014) suggestion to organise conceptual knowledge. These authors arrange conceptual knowledge into: (i) knowledge of general principles, and (ii) knowledge of principles underlying procedures.

Given that it is unclear how to measure conceptual knowledge independently from methodological knowledge effectively, we follow the proposal of Crooks and Alibali (2014) to establish evaluation indicators for this purpose. These tasks include the usage of conceptual knowledge indicators, both that of explicit and that of implicit nature. We specifically use tasks that measure explicit knowledge as indicators of the knowledge of general principles. An example of the latter is the explanation of concepts (definitions, elements of the structure of a domain and norms or rules) and the evaluation of example tasks that deal with implicit knowledge (recognise examples, definitions or statements of principle). In order to evaluate the second dimension of conceptual knowledge, the knowledge of principles underlying methods, we consider the use and justification of procedural tasks. In addition to this, we also include the evaluation of these tasks as correct or incorrect and why, reasoning whether the methods used are adequate to certain situations.

Instrument and data collection

Two questionnaires were elaborated for the data collection, in order to evaluate the 4 components of CBMK-A defined above. The following themes are included as sections in the first questionnaire: (1) knowledge of the decimal numeration system and positional value; and (2) the meanings of addition and subtraction. The second questionnaire features the sections involving: (3) part-whole relations; and (4) algorithms.

We elaborated 4 types of questions for each section considering the following three conditions: (i) the content to be evaluated; (ii) the type of conceptual knowledge that involves; and (iii) indicators of conceptual knowledge. Once both questionnaires were elaborated, a group of experts validated each group of questions. The latter were strategically distributed within the questionnaire before determining its final version. Each questionnaire has questions of different sections. The aim of this is to triangulate the students' answers, with the intention of avoid a wrong interpretation of the results. Question sheets were handed out in two different sessions of 1 hour each, in which students answered the questions individually without calculator.

Given the length of both questionnaires, as an example we have only presented 2 of the 4 sections of questions that evaluate CBMK-A. We particularly present the group of questions used to evaluate CBMK-A in relation to (i) the decimal numeration system and positional value, and (ii) the knowledge of part-whole relationships.

Section: CBMK-A of the decimal numeration system and positional value

As evaluation content for this section, we chose some conceptual features instrumental to the comprehension of the decimal numeration system and the concept of positional value. The following table displays the questions that include the knowledge of general principles or the knowledge of principles underlying procedures, with their respective indicators (see Table 1).

Element to be evaluated	Type of knowledge and indicator	Question
Comprehension of the multiplicative recursive structure of base-10 of the decimal numeration system.	Principles underlying procedures (*Type*) Use and justification of procedural tasks *(Indicator)*	1) Complete and explain why. Version 1: a) 5 hundred is __units = _thousand. b) 7 thousand is ___tens = __units. Form 2: a) 50 hundred is ___units= ___thousand. b)70 thousand is ___tens = _units.
Rounding for the part concerning closest value. Forward and backward counting for positional value. Relative and positional value. Read and write a number with letters and figures.		2) Express your answer in tens. What is the group of ten that is closest to the following amounts and explain why: a)43 b)36 c)68 d)65
		3) How many hundreds are there in the number 130,025? How would the number 130,025 be expressed verbally?
Comparison of structured amounts. Acknowledgement and use of equivalent representations of the same number. Compose, decompose, combine and transform structured amounts.	General principles (*Type*) Evaluation of example tasks *(Indicator)*	4) Which of the following decompositions correspond to number 342? Enclose them in a circle. a)3C + 4D + 2U b)30D + 42U c)2C + 14D + 2U d)1C + 2D + 42U e)34D + 2U

Table 1. Questions to evaluate the CBMK-A of the decimal numeration system and positional value

Section: CMFC- A of part-whole relationships

In this case, we considered additive composition, commutativity, associativity and inversion as content to be evaluated. This choice was based on the fact that they are fundamental principles and properties of addition and reflect different elements of part-whole relationships as understood by the subjects. Secondly, we elaborated the questions about knowledge of general principles, that include verbal explanations of concepts; and those concerning knowledge of the principles underlying procedures, involving the use of direct access strategies on a conceptual basis (see Table 2).

Aspect to be evaluated	Type of knowledge and indicator	Question
Understanding of associativity and inversion. Use of shortcuts	Principles underlying procedures (*Type*) Application and justification of procedural tasks *(Indicator)*	Solve each of the following expressions. Indicate how you did it in each case. ( ) + 27 – 27 = 17 13 + 38 + 42 = 28 + 17 + 23 = 12 + 8 – 8 = ( ) + 12 – 8 = 22 18 + 36 – 27 = 13 – 5 – 8 + 12 = 18 + 9 – ( ) = 13 16 + 28 + 14 = 28 – 7 – 15 + ( ) = 12 7 + ( ) – 7 = 14 5 + 27 – 23 = 8 + 56 – 12 = 28 + 14 – ( ) = 28 ( ) + 11 – 3 – 3 = 23 16 + 16 – 8 = 13 + ( ) – 13 = 18 15 + 37 + 45 15 + 34 – 25 = 22 – 4 –18 + ( ) = 12 ( ) + 12 – 9 – 3 = 17 17 + 16 + 18 = 26 + 14 – 14= 24 + 19 – 17= 13 + ( ) – 23 = 23 28 + 11 – 6 – 5 =
	Principles underlying procedures (*Type*) Evaluation of procedural tasks *(Indicator)*	Observe the resolution of the following three expressions. Which resolution method do you prefer? Why wouldn't you choose any of the others? Case: 19 + 9 – ( )= 19 a) 19 + 9 – x = 19 19 + 9 – 19 = x 28 – 19 = x 9 = x b) 19 + 9 – () = 19 Adding 19 and 9, is 28. Then 28 minus 19, it's 9. 9 is the result. c) 19 + 9 - () = 19 Because 19 is in both sides of the equal sign, then I cancel them. Thus equality holds if the answer is 9.
		Observe the resolution of the following two expressions. Which resolution method do you prefer? Why? Case: 27 + 19 – 7= a) 27 + 19 – 7 = 46 – 7 39 b) 27+ 19 – 7= 27 + 12 39
Understanding of commutativity and additive composition	General principles (*Type*) Application and justification of procedural tasks *(Indicator)*	Group the following operations. How did you do it? Can they be grouped in a different way? If so, how? 7 + 3 1 + 6 6 + 4 4 + 6 5 + 2 3 + 5 2 + 7 5 + 3 3 + 7 4 + 4 6 + 1
	Explanation of conceptual tasks	What does a commutative operation mean?
		Are there any operations in which changing the order of their elements will cause to obtain a different result? Why? Give an example.

Table 2. Questions section that evaluates the CMFC- A of part-whole relationships

Expected results

Our study is currently at the last stage of data collection. We expect to give the questionnaire to 200 students of the first and second year of the DPE that have not yet begun their studies in didactics of arithmetic. As far as data analysis is concerned, we will apply a mixed approach. The identification of additive knowledge profiles will be carried out through conglomerate analysis with the software SPSS, version 15. On the other hand, those questions that may include the students' verbal explanations or justifications will be analysed qualitatively and will also be categorised, numerically coded and included within the conglomerate analysis.

At the starting point of our investigation we hold no preconceptions on what type of mathematical knowledge DPE students actually have when initiating their studies as teachers. We are however aware of the theoretical consideration that teacher students have a limited command of teaching content in general and specifically of the Primary Education curriculum at the start of their training, as well as a lack of basic knowledge of elementary mathematics (Castro et al., 2014; Montes et al., 2015). In sight of this issue we expect our results to, on one hand, provide a closer approach to the reality of students of the DPE at the start of their training in didactics of mathematics. On the other hand, we hope the characterisation of CBMK-A turns out to be a useful tool to identify additive knowledge profiles in DPE students, in order to guide the planning of subjects on arithmetic didactics.

REFERENCES

Ball, D.L., Thames, M.H. y Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59(5),* 389-407.

Baroody, A., Feil, Y., & Johnson, A. (2007). An alternative reconceptualization of procedural and conceptual knowledge. *Journal for Research in Mathematics Education, 38*, 115–131.

Baroody, A., & Ginsburg, P. (1986): The relationship between initial meaningful and mechanical knowledge of arithmetic. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: the case of mathematics* (pp.75-112). Hillsdale, NJ: Lawrence Associate.

Cañadas, M.C., y Castro E. (2011). Aritmética de los números naturales. Estructura aditiva. En Segovia y Rico (coord.). *Matemáticas para maestros de Educación Primaria (pp. 75-78).* Madrid: Pirámide.

Canobi, K. (2005). Children's profiles of addition and subtraction understanding. *Journal of Experimental Child Psychology, 92*, 220-246.

Canobi, K. (2009). Concept–procedure interactions in children's addition and subtraction. *Journal of Experimental Child Psychology, 102*, 131–149.

Castro, A., Gorgorió, N., & Prat, M. (2014). Indicios verbales en los PAEV aditivos planteados por estudiantes para maestro. En M. T. González, M. Codes, D. Arnau y T. Ortega (Eds.), *Investigación en Educación Matemática XVIII* (pp. 217-226). Salamanca: SEIEM.

Castro, A., Mengual, E., Prat, M., Albarracín, L., y Gorgorió, N. (2014). Conocimiento matemático fundamental para el grado de educación primaria: inicio de una línea de investigación. En M. T. González, M. Codes, D. Arnau y T. Ortega (Eds.), *Investigación en Educación Matemática XVIII* (pp. 227-236). Salamanca: SEIEM.

Crooks, N., Alibali, M. W. (2014) Defining and measuring conceptual knowledge in mathematics. *Developmental Review. 34(4), 344-377.*

Dickson, L., Brown, M. & Gibson, O. (1991). El aprendizaje de las matemáticas. Barcelona: Labor.

Gilmore, C., & Bryant, P. (2006). Individual differences in children's understanding of inversion and arithmetical skill. *British Journal of Educational Psychology, 76*, 309-331.

Gilmore, C., & Bryant, P. (2008). Can children construct inverse relations in arithmetic? Evidence for individual differences in the development of conceptual understanding and computational skill. British Journal of Developmental Psychology, 26(3), 301-316.

Hiebert, J., & Lefevre, P. (1986): Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), Conceptual and procedural knowledge: the case of mathematics (pp.1-27). Hillsdale. NJ:Lawrence Associates.

Linsell, C., y Anakin, M. (2012). Diagnostic Assessment of Pre-Service Teachers' Mathematical Content Knowledge. Mathematics Teacher Education and Development, 14(2), 4–27.

Linsell, C., y Anakin, M. (2013). Foundation Content Knowledge: What do pre-service teachers need to know? En V. Steinle, L. Ball y C. Bardini (Eds.), Mathematics Eduction: Yesterday, today and tomorrow (Proceedings f the 36th annual conference of MERGA) (pp. 442-449). Melbourne, VIC: MERGA.

Ma, L. (1999). Knowing and teaching elementary mathematics: teachers’ understanding of fundamental mathematics in China and the United States. Hillsdale, NJ: Erlbaum.

Montes, M. A., Contreras, L. C. y Carrillo., J. (2013). Conocimiento del profesor de matemáticas: Enfoques del MKT y del MTSK. En A. Berciano, G. Gutiérrez, A. Estepa y N. Climent (Eds.), Investigación en Educación Matemática XVII (pp. 403-410). Bilbao: SEIEM.

Montes, M., Contreras, L., Liñán, M., Muñoz-Catalán, M., Climent, N., y Carrillo, J. (2015). Conocimiento de aritmética de futuros maestros: debilidades y fortalezas. Revista de Educación, 367, 36-62.

Rittle-Johnson, B., & Alibali, M. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? Journal of educational psychology, 91(1), 175.

Rittle-Johnson, B., Siegler, R., & Alibali, M. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. Journal of educational psychology, 93(2), 346.

Rowland, T., Huckstep, P., & Thwaites, A. (2003). The Knowledge Quartet. Proceedings of the British Society for Research into Learning Mathematics, 23(3), 97-102.

Salinas M. (2007). Errores sobre el sistema de numeración decimal en estudiantes de Magisterio. En M. Camacho, P. Flores y P. Bolea (Eds.), Investigación en Educación Matemática XI (pp. 381- 390). La Laguna: SEIEM.

Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15 (2), 4-14.

Shulman, L.S. (1987). Knowledge and teaching: Foundations of the new reform. Harvard Educational Review, 57(1), 1-22.

Pre-service teacher conceptualisation of mathematics

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Abstract: Mathematics can be seen in a variety of ways which can impact how individuals interact with mathematics. Pre-service teacher conceptions about mathematics could potentially influence how they engage with mathematical activities during their degree, their development and understanding of mathematical ideas, and their interpretation of mathematical activities. These conceptions can then influence the mathematical experiences they create for their students. As a result, it is important that pre-service teacher education programs identify pre-service teacher conceptions of mathematics and provide experiences that can challenge and change these conceptions.

Résumé : Mathématiques peuvent être vues dans une variété de façons qui peuvent avoir une incidence sur la façon dont les individus interagissent avec les mathématiques. Les conceptions sur les mathématiques des enseignants en pré-service pourraient potentiellement influencer la façon dont ils s'engagent à des activités mathématiques au cours de leur diplôme, leur développement et leur compréhension des concepts mathématiques, et leur interprétation des activités mathématiques. Ces conceptions peuvent ensuite influencer les expériences en mathématiques qu'ils créent pour leurs élèves. En conséquence, il est important que les programmes de formation des enseignants en pré-service identifient les conceptions sur les mathématiques des enseignants en pré-service et fournissent des expériences qui peuvent mettre en cause et changer ces conceptions.

Introduction

Investigations of teacher beliefs have been conducted for more than two decades. Pajares (1992) considered research investigating teacher beliefs as necessary as teacher beliefs have the potential impact on teacher behaviour in their classrooms. Ernest (1989) connects teacher behaviours in the mathematics classroom to teacher knowledge, teacher beliefs and teacher attitudes towards mathematics. He considers teacher knowledge related to mathematics, teaching and learning starts with the teacher's views of what mathematics is, how it can be taught, and how children will learn. If teacher beliefs, attitude, and knowledge can have such an impact, it would indicate that investigations into these should be considered for teachers. Likewise, it would indicate that there is a need to include consideration of pre-service teachers' beliefs, attitudes, and knowledge. These components can also be included as elements of disposition towards mathematics. Disposition towards mathematics has been considered within a similar time frame as teacher beliefs. Anku (1996) explored pre-service teacher disposition towards mathematics, using the phrase to indicate the level of positivity towards and interest in learning mathematics. The year after, the Australian Association of Mathematics Teachers' [AAMT] (1997) considered the importance of disposition when discussing numeracy.

Cooke (in press) described disposition towards mathematics as incorporating the components of attitudes towards mathematics, mathematics anxiety, confidence with mathematics, and conceptualisation of mathematics. Justifications for why these components should be included were provided and discussed in detail. In addition, the measurement of the elements of these components were examined and deliberated. Attitudes towards mathematics were considered in terms everyday life, the classroom, and teacher education (Beswick, Ashman, Callingham, & McBain, 2011). Mathematics anxiety was considered in response to thinking about working with others on a mathematical task, taking a mathematics test, and teaching mathematics (Cooke, Cavanagh, Hurst, & Sparrow, 2011). Confidence with specific mathematics considered mathematical content in terms of the Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting and Authority [ACARA], 2010) to determine pre-service teacher confidence (Beswick et al., 2011). The final component, conceptualisation of mathematics, was described in detail in the paper, a summary of which is provided below.

The instrument to measure conceptualisation of mathematics comprised 20 statements. These statements addressed how mathematics could be perceived in terms of its use and its usefulness. Some statements addressed whether mathematics was useable beyond a mathematics classroom, such as in other subjects, with games, in conversations, and in everyday life. Several statements explored whether mathematics was concerned with just numbers, rules and procedures, or connected ideas, and whether it was possible to have more than one correct answer. The consideration of whether mathematics could be used to describe nature or whether mathematics was creative was included in statements. Other statements referred to whether attempts were made to understand how rules and procedures worked or if enjoyment was found when engaging with mathematical activities.

As the statements from the instrument were designed to investigate how mathematics is conceptualised, consideration of the work by Ernest (1989) enable the statements to be categorised in terms of the three philosophies of mathematics he outlined. As a result, the statements were grouped in terms of the three philosophies of mathematics – mathematics as a revisable problem solving field; mathematics as a static interconnecting set of truths; or mathematics as a collection of unrelated facts and skills – outlined by Ernest (1989). As shown in Figure 1, four statements are relevant to only one philosophical perspective, that is, mathematics as a revisable problem solving field. Half of the statements are relevant to two philosophical perspectives, that is, mathematics as a revisable problem solving field and mathematics as a static interconnecting set of truths. The final six statements are applicable to all three philosophical perspectives.

Figure 1. Using the three mathematical philosophies outlined by Ernest (1989) to categorise the statements in the instrument

The instrument was used as part of suite of eight instruments within an activity designed to enable pre-service teachers to interrogate their disposition towards mathematics. The aim of the activity was to provide pre-service teachers with tools of reflection that could enable meta-disposition, where pre-service teachers critically evaluate their disposition (Cooke, in press). This could enable pre-service teachers to identify elements they wished to change. This meta-approach mimics the metacognition approach outlined for teachers by Lin, Schwartz, and Hatano (2005). Specifically, their approach concerned how teachers could monitor their thinking in a variety of situations with the aim of producing performance at a higher level. This is relevant to pre-service teachers as they are constantly learning they are engaged in a continuously changing milieu regarding their thoughts concerning theory and how they will enact that theory when they are teaching. It is also crucial for pre-service teachers to consider their disposition towards mathematics as it has the capacity to impact on their teaching in a variety of ways, for example, negative beliefs about their abilities (Gresham, 2008), avoidance of any mathematical activities (Isiksal, Curran, Koc, & Askum, 2009), types of tasks provided to students (Choppin, 2011), or enthusiasm and enjoyment (Ernest, 1989). In addition, they could impact on the results of their students (Beilock, Gunderson, Ramirez, & Levine, 2009).

This paper reports on data obtained from the use of the instrument with a large cohort of pre-service teachers. The focus of the paper is the the percentage of students who agreed with each statement and the overall percentage agreement for the set of statements within each of the categories constructed from the philosophical perspectives outlined by Ernest (1989) and identified in Figure 1. Differences in students' agreement with individual statements are provided together with the percentage agreement with the statements identified within the three categories constructed by membership to the philosophical perspectives (that is, mathematics as a revisable field, the two philosophical perspectives of mathematics as a revisable field and as a static interconnected set of rules, and all three philosophical perspectives). Potential implications of the results for teacher education programs are outlined and discussed.

Methodology

This study is concerned with pre-service teacher perceptions about mathematics. These perceptions may have developed over time and through their experiences with the world. Their experiences, in turn, are impacted by their perceptions. This interrelationship of perceptions and experiences situates this research within the constructivism ontology and the social constructionist epistemology (Crotty, 1998).

Data collection

Data was collected from a compulsory first-year mathematics education unit is provided to students enrolled in either a Bachelor of Education (Primary) or a Bachelor of Education (Early Childhood) at an Australian metropolitan university. The unit was conducted fully on-line using a learning management system (LMS).

Participants

As part of the first assessment in their compulsory first-year mathematics education unit, students completed eight instruments addressing disposition towards mathematics. These eight instruments addressed attitudes towards mathematics (one instrument each to consider attitudes towards mathematics in everyday life, attitudes towards mathematics in the classroom, and attitudes towards mathematics in teacher education), mathematics anxiety (one instrument each to consider mathematics anxiety when thinking about working with others on a mathematical task, mathematics anxiety when thinking about taking a mathematics test, and mathematics anxiety when thinking about teaching mathematics), confidence to complete specific mathematics, and conceptualisation of mathematics. The last instrument, conceptualisation of mathematics, is the focus of this research. Of the total 851 students who accessed the LMS during the study period, 673 completed the instrument on the conceptualisation of mathematics (79%).

Questionnaires

The instrument addressing conceptualisation of mathematics contained 20 statements regarding how maths was viewed. Responses were provided using a 4-point Likert-style scale, where students could strongly disagree with the statement, disagree with the statement, agree with the statement, or strongly agree with the statement. All statements were worded positively, which negated the need to recode responses.

Procedure

The instruments were administered through the LMS via a link provided to students. All eight instruments were initially available for seven days during the first week of the study period, but due to technical issues, it made available for another seven days during the second week of the study period. During the second timeframe when the instruments were available, only students who had not completed the instruments were able to access the link. The data file was downloaded and the text responses were converted into numerical responses in a spreadsheet program. Strongly disagree responses were allocated a value of 1, disagree a value of 2, agree a value of 3, and strongly agree a value of 4. The final file was imported into SPSS for analysis.

Two forms of analysis were conducted. Mean values were calculated for each statement based on the numerical value allocated for the response to the statement. The calculation for the mean ignored missing cells, that is, the number of responses was the divisor for the summated responses.

The percentage of agreement for each statement was calculated to provide an indication of the proportion of students in agreement with each statement. This was achieved by creating a new variable with a value of 1 for responses of 1 or 2 to indicate disagreement and a value of 2 for responses of 3 or 4 to indicate agreement. Frequencies were then calculated for disagreement and agreement for each statement.

Results

The percentage of pre-service teachers who agreed with each statement and the mean response for each statement are provided in Table 1. Statements are ordered from those with lowest percentage agreement to those with highest percentage agreement.

Statement	N	% Agreement	Mean
5. Maths problems and questions can often have more than one correct answer.	671	45.5	2.45
16. I view maths as something I can use to explain the world.	672	52.8	2.58
18. Using maths to find out about other things is enjoyable.	672	64.1	2.72
14. I use maths in everyday conversations.	672	65.2	2.75
19. Maths is creative.	672	71.3	2.85
12. Maths can be used when describing nature.	672	77.5	2.94
1. I often use the maths I learnt at school.	673	79.0	2.97
7. The maths I did at school has been very useful to me.	672	79.3	3.00
17. I use maths to successfully play games.	672	79.6	2.99
11. Maths involves much more than following rules and procedures.	672	86.2	3.11
20. The maths I have learned in the classroom links and connects to what I do in the real world.	672	86.5	3.11
3. Maths learned in the classroom is widely used outside the classroom.	671	86.6	3.16
13. Maths involves many ideas that are easily and clearly connected to other ideas.	672	89.3	3.11
4. I can see how maths is related to games.	671	92.5	3.28
15. I try to understand maths I have to use.	672	95.7	3.32
9. It is important to know why mathematical rules and procedures work.	672	96.3	3.35
10. I can see how maths connects to the world.	672	96.7	3.37
2. Maths is about more than just numbers.	672	97.9	3.52
8. I see maths as useful in life.	672	97.9	3.48
6. Maths can be used in many subjects at school.	672	98.2	3.34

Table 1. Statements ranked by the percentage of student agreement.

Responses to statements within each of the three categories were grouped and the percentage of agreement and mean were calculated and provided in Table 2.

Category of statements	N	% Agreement	Mean
Relates to Mathematics as a revisable problem solving field	2685	73.97	2.94
Relates to both Mathematics as a revisable problem solving field and Mathematics as a static interconnecting set of truths	6720	81.70	3.06
Relates to Mathematics as a revisable problem solving field, Mathematics as a static interconnecting set of truths, and Mathematics as a collection of unrelated facts and skills	4033	87.63	3.19

Table 2. Percentage agreement for categories based on mathematical philosophies (Ernest, 1989).

Discussion and Conclusion

It is worthy noting that more than 75% of students agreed with 15 of the 20 statements and only one statement had less than half the cohort agreeing, *Maths problems and questions can often have more than one correct answer* (45.5% agreement). The low agreement may be due to pre-service teachers focusing on past mathematical experiences with closed maths problems with only one answer possible (Sullivan, Mousley, & Zevenbergen, 2006). Mathematical experiences may have focused on one "correct" answer, concentrating on the product and not the process, which would contrast with Ernest's (1989) philosophical description of mathematics as revisable problem solving. The statement with the next lowest percentage agreement was *I view maths as something I can use to explain the world*. This may reflect a perceived lack of skill in using maths to explain the world or a lack of recognition of mathematics in the world (Cooke, in press). Of the five statements with less than 75% of student agreement, two were in the first category and three were in the second category outlined in Table 2.

A lower percentage of students agreed overall with the set of statements that correspond only to the mathematics philosophy that considers maths as a revisable problem solving field (Ernest, 1989). This could be due to past experiences that presented mathematics as a set of procedures that were reinforced through the completion of exercises. The percentage of agreement for the second category overall (81.7%) may reflect experiences that present mathematics as more than a series of steps or procedures (Beswick, 2012). The final category incorporates statements that correspond to all philosophical approaches, making it difficult to determine experiences and reasons why pre-service teachers responded as they did. Future research to investigate the thinking behind the responses to these statements could illuminate why pre-service teachers agreed or disagreed with the statements.

If past experiences impact on beliefs (Sullivan et al., 2006) and these beliefs determine the responses to this instrument, teacher educators need to provide experiences that challenge past experiences. In doing this, it may be possible to enact points made by Beswick (2012) and Wilkins and Brand (2004), that education courses and mathematics education units should challenge and influence pre-service teachers' beliefs and attitudes about mathematics. However, the focus needs to be on new mathematical experiences that enable pre-service teachers to see mathematics differently than they did in the past. This could result in pre-service teachers moving more towards a consideration of mathematics as a revisable problem solving field (Beswick, 2012; Ernest, 1989). Changes in how mathematics is conceived could be beneficial beyond the individual pre-service teachers. The future students of these pre-service teachers may benefit as research over the last quarter of a century has shown that how the teacher conceives mathematics may determine what is experienced in their mathematics classrooms. Behaviours engaged in within the mathematics classroom (Ernest, 1989), how mathematics is taught and learnt (Ernest, 1989), what activities and experiences teachers use (Pajares, 1992), and how they reflect on their teaching experiences (Beswick, 2012) are all impacted by the teacher's conceptualisation of mathematics.

REFERENCES

Anku, S. E. (1996). Fostering students' disposition towards mathematics: A case from a Canadian university. *Education, 116*(4), 536-542. Retrieved from http://search.proquest.com.dbgw.lis.curtin.edu.au/docview/196420710/fulltextPDF?accountid=10382

Australian Association of Mathematics Teachers [AAMT]. (1997). Numeracy=everyone's business. Report of the Numeracy Education Strategy Development Conference. Adelaide, SA: Australian Association of Mathematics Teachers. Retrieved from http://www.aamt.edu.au/Professional-reading/Numeracy

Australian Curriculum, Assessment and Reporting and Authority [ACARA]. (2010). Foundation – year 10 curriculum. Retrieved from http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10

Beilock, S. L., Gunderson, E. A., Ramirez, G., & Levine, S. C. (2009). Female teachers' math anxiety affects girls' math achievement. Proceedings of the National Academy of Sciences of the United States of America [PNAS], 107(5), 1860-1863. Retrieved from http://www.jstor.org.dbgw.lis.curtin.edu.au/stable/40536499

Beswick, K. (2012). Teachers' beliefs about school mathematics and mathematicians' mathematics and their relationship to practice. Educational Studies in Mathematics, 79(1), 127-147. doi: 10.1007/s10649-011-9333-2

Beswick, K., Ashman, D., Callingham, R., & McBain, D. (2011). Teachers' and pre-service teachers' confidence to teach primary school mathematics. Paper presented at the 2011 AARE international Research Conference, Hobart, Australia, 27 November-1 December, 2011. Retrieved January 20, 2014, from http://www.aare.edu.au/data/publications/2011/aarefinal00535.pdf

Choppin, J. (2011). The role of local theories: Teacher knowledge and its impact on engaging students with challenging tasks. Mathematics Education research Journal, 23(5), 5-25. doi: 10.1007/s13394-011-0001-8

Cooke, A. (in press). Considering Pre-service Teacher Disposition Towards Mathematics. Manuscript submitted for publication. Retrieved from http://www.merga.net.au/ojs/index.php/mted/article/view/188/210

Cooke, A., Cavanagh, R., Hurst, C. & Sparrow, L. (2011, November-December). Situational effects of mathematics anxiety in pre-service teacher education. Paper presented at the 2011 AARE international Research Conference, Hobart, Australia, 27 November-1 December, 2011. Retrieved from http://www.aare.edu.au/data/publications/2011/aarefinal00501.pdf

Crotty, M. (1998). The foundations of social Research: Meaning and perspective in the research process. Crows Nest, Australia: Allen & Unwin.

Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: a model. Journal of Education for Teaching: International research and pedagogy, 15(1), 13-33. doi: 10.1080/0260747890150102

Gresham, G. (2008). Mathematics anxiety and mathematics teacher efficacy in elementary pre-service teachers. Teaching Education, 19(3), 171-184. doi:10.1080/10476210802250133

Isiksal, M., Curran, J. M., Koc, Y, & Askun, C. S. (2009). Mathematics anxiety and mathematical self-concept: Considerations in preparing elementary-school teachers. Social Behavior and Personality, 37(5), 631-644. DOI:10.2224/sbp.209.37.5.631

Lin, X., Schwartz, D. L., & Hatano, G. (2005). Toward teachers' adaptive metacognition. Educational Psychologist, 40(4), 245-255. doi: 10.1207/s15326985ep4004_6

McLeod, D. B. (1994). Research on affect and mathematics learning in the JRME: 1970 to the present. Journal for research in Mathematics Education, 25(6), 637-647. Retrieved from http://www.jstor.org/stable/749576

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. Review of Educational Research, 62(3), 307-332. doi: 10.3102/00346543062003307

Sullivan, P., Mousley, J., & Zevenbergen, R. (2006). Teacher actions to maximize mathematics learning opportunities in heterogeneous classrooms. International Journal of Science and Mathematics Education, 4(1), 117-143. doi: 10.1007/s10763-005-9002-y

Wilkins, J. L., & Brand, B. R. (2004). Change in preservice teachers' beliefs: An evaluation of a mathematics methods course. School Science and Mathematics, 104(5), 226-232. Retrieved from http://search.proquest.com

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Rescuing casualties of mathematics

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Abstract: In this paper we present an activity carried out inside a remedial course for 15 years-old students of a science high school, to recover their mathematical competencies. A Kolb test was given to the students to recognize their learning styles, in order to set the whole work in the respect of their own attitudes. Results of the activity are presented.

Résumé: Dans cet article nous présentons une activité réalisée dans un cours de rattrapage pour les élèves de 15 ans d'un lycée, de récupérer leurs compétences en mathématique. Les étudiants ont eu un test de Kolb à reconnaître leurs styles d'apprentissage, afin de mettre l'ensemble des travaux en fonction de leur aptitude. Nous présentons le résultats de l'activité.

Introduction

National and international studies highlight a strong difficulty of Italian students in mathematics, in particular in southern Italy. Their problems are both in mathematical knowledge and in using mathematical artefacts to interpret reality.

A branch of research studies psychoanalytic approaches to discuss mathematics as an object through which and with which the self is constructed. At the beginning, in pre-school children, mathematics is inside the self, (Vygotskij, 1986) rather than a foreign disclipline, like the one children are forced to learn at school. In that period of life their relationship with mathematics is not as problematic as it will become in their future. We usually keep on forgetting about that « primitive » relationship with mathematics, ignoring it in the typical pedagogical interaction with the classroom (Appelbaum, 1998).

Fig. 1

What is worst, after a few years many students fall into a vicious circle, resulting in a hate against mathematics, that turns students into casualties of mathematics.

Roughly we can divide the circle on Fig.1 in two main parts: the right part (referred to as « the heart » in the following) is related to affective problems with the discipline (Zan 2006), (Di Martino 2007), the left part, (« the brain » in the following) is related to study and diligence. It is very difficult to understand where the circle starts because "affect, far from being the « other » of thinking, is a part of it. Affect influences thinking, just as thinking influences affect. The two interact" (Brown, 2012). But at the end it is not so important to undertsand how the circle started. What is important is to break it.

Most of the teachers try to act on « the brain », by working on the "I don't care" and just asking their students to spend more time in studying. Nowadays, with the spread of technology, some teachers work also on the "I'm not able", because many students are good in using technologies, so they can learn by using technology as a language, in the sense of horizontal teaching (Ferrarello, Mammana, Pennisi 2014a, p.4) : Students, trought the ability in technology feel able to handle mathematics concepts, sometimes they feel this « technological mathematics » as another mathematics.

We believe that one should also act inside « the heart », on the "I don't like" and the "I'm not motivated". In this paper we discuss how we worked also on « the heart » part during an activity carried out in a 50 hours remedial course (in almost weekly meetings in the afternoon) in a science high school, with second class students, aged 15-16, studying algebra (equations and inequalities) and geometry (special point of triangles, Euclid's theorems, Pytagora's theorem, angles at the center and at the circumference, geometric transformations).

The first step was to understand which kind of students we had, in order to realize how we could support them. We strongly believe that affective problems with mathematics of a single student are related both with his/her proper learning attitude and with the usual style of teaching mathematics. When the two aspects (teaching and learning styles) do not coincide, students get a vision of mathematics that is different from their own idea of mathematics. So, in the first meeting with students, we proposed a learning test, the Kolb test, whose theory we discuss in the following paragraph, to study what vision of mathematics was appropriate to present to the students.

According to the results of the Kolb test, we organized the whole work with students according to their own learning attitude, i.e. based on practice and experience: we gave great importance to "learning by doing" (Dewey 1916), especially in geometry, that was conducted only in the lab with the use of both new and old artefacts (Maschietto, Trouche 2008), in particular DGS with guided worksheets, and mathematical machines, that we used for geometric transformations. We briefly discuss the results arised from the analysis of initial and final competencies of students and from the satisfaction questionnaire, that students filled anonymously at the end of the course.

Theoretical background: Kolb learning styles and maths lab

David Kolb (Kolb, Fry 1975) adopted as learning styles two thought types and two cognitive processes, namely the two thought types introduced by Hudson (Hudson, 1968): converging and diverging; and the two cognitive processes introduced by Piaget: assimilating and accomodating. In such a way Kolb distinguishes learning styles, described in the following, taking into account the four capabilities: Concrete Experience (feeling), Reflective Observation (watching), Abstract Conceptualisation (thinking) and Active Experimentation (doing).
Briefly:
A **converging** character is strongly talented to practical applications of ideas, he/she uses a lot of deductive reasoning, he/she is good in applying Abstract conceptualization in Active experimentation, so he/she is between thinking and doing. A **diverging** person, on the contrary, lives in ideas' world, he/she has a strong imaginative ability, he is gifted to pose problems, rather than to solve them, to produce new ideas and see things from different perspective, so he/she is suitable to pass from Concrete experience to Reflective observation, between feeling and watching.
A student who uses mainly the **assimilating** process is very able to see theoretical models and abstract concepts, he/she uses inductive reasoning, so he/she performs well when an Abstract conceptualization is required by a Reflective observation, passing from watching to thinking. Who uses **accomodating** thinking solves problem intuitively by practice; he/she does not care so much about theory, but he/she is a good reactor when the context changes suddenly. If the theory does not work anymore on a new circumstance, he/she is able to find new examples that afterwards can develop a new uploaded theory. His/her learning is between Active experimentation and Concrete experience, from doing to feeling.
At the end of the Kolb test, every student has four scores: for the capabilities CE (Concrete Experience), RO (Reflective Observation), AC (Abstract Conceptulization) and AE (Active Experimentation). It is very rare that a student has a pure attitude toward just one of the four capabilities, so the Kolb test provides combined profiles, by subtracting RO from AE and by subtracting CE from AC, namely comparing feeling vs thinking and watching vs doing. The result of the two subtractions gives the learning-style profiles: converging, diverging, assimilating and accommodating.

Of course there are "more gifted" and "less gifted" students, but in many cases when a student is not good in mathematics (when he/she has low grades) it is not only a matter of intelligence: evaluation is strongly related with learning styles, as we are going to explain.
Converging students are those highly-valued by the teachers, since they learn better by following fixed schemes, in particular they follow teachers' schemes. So the evaluation is high because the students tell to the teachers exactly what they want to listen to.
Assimilating students, those very gifted in pure thinking, are often evaluated good as well, but they are rare, you can find a few of them in a class of 20 students.
Accomodating students are those considered quite smart, because they are good in solving real problems, but they are considered unwilling, because they don't care so much about theory and don't study so much.
Diverging pupils are often seen as those with head in the clouds, their ability to pose problems is not appreciated because teachers poses the problems, or problems aren't posed at all.
So, it is not surprising that most of the students we handled in the remedial course, were diverging and accomodating, according to the graphic in Fig. 2. So, they were gifted for Concrete Experience.

As Felder claims in (Felder, 2010) "Both logic and published research suggest that students taught in a manner matched to their learning style preferences tend to learn more than students taught in a highly mismatched manner. It does not follow, however, that matching instruction to fit students' learning styles is the optimal way to teach. For one thing, it is impossible if more than one learning style is represented in a class." Our fortune was to handle students with similar learning style, in such a way our teaching style could match with students' attitude.

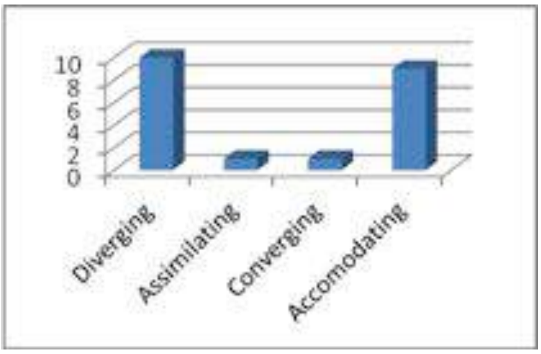

Fig. 2

This led us to decide to work according to the next strategies:

- Visual experience;
- no fixed scheme;
- maths lab;
- practice.

In particular we decided to make practice with many exercises on algebra topics, without any explanation, by means of traditional frontal lecture, of any geometric concept (it would have been useless, their teachers already did it during the morning class). Students worked only with maths lab for geometry topics, learning independently, but in a collaborative environment and under the guide of the teachers. In this paper we concentrate especially on the work done with geometry.

Mathematics to be seen and touched

The majority of students inclined to Concrete Experience led us to use "Learning by doing" as pedagogical context in the Vygotskian perspective of practical intelligence, supported by maths lab (Chiappini, 2007) for practice, using especially eyes and hands, as described in the following.

Geometry with DGS: seeing math with my eyes (with digital artefacts)

Nowadays the use of technology is widespread in class practice, and many researchers in mathematical education investigate the (good and not good) effects of technology (see, for instance, (Drijvers, 2012) and (Artigue, 2000)). In our case technology was a very good partner, because we had not a class, but a special selection of students: they were not so good in computations (but not stupid!). In order to grasp the concepts, they needed to be relieved from calculations and see

directly the core of a mathematical object, undressed by all formulas and symbols. Formulas come later. To do so we used a DGS (Geogebra above all) to easily represent properties: not only geometric ones as in (Ferrarello, Mammana, Pennisi 2014b), but also algebraic ones, as in (Ferrarello, Mammana, Pennisi 2013). Geometric properties investigated with a DGS were: special point of triangles, angles at the center and at the circumference, Euclid's theorems. Algebraic properties visualized by means of a DGS were some special products and inequalities.
Not a single geometric property was explained by the teachers, but they were discovered and verified by students. Most of activities were carried out in the lab and students were guided by worksheets; those worksheets contained some special tag (different font or pictures) for the name of the DGS tools involved on the construction or exploration requested. In such a way, students could complete the task without asking every single step to the teacher. They have a time and space to think alone.
The teachers went around the lab, helping students, if necessary, taking advantage also of gestures and glances. They took care of every student, interacting with each one personally, instead of talking to all. Every student had his/her own computer, but they often interacted with each other in a peer to peer relationship. Students who completed the task helped their mates, in a collaborative learning framework. At the end of the meeting the teachers asked students what they discovered, summarizing the results, in such a way to clarify the concepts for everyone.

Mathematical machines: touching math with my hands (with manual artefacts)

The best appreciated activity: handling maths. No one could think to teach dance only in theory, without practicing steps, but too many times it happens that mathematics is taught just in theory, as a set of preexistent rules, and rarely teachers gave the possibility to handle real object, like some artifacts. What happened with mathematical machines was something magic: some students of the remedial course got a higher score of their classmates in the written assignment concerning geometric transformations! Moreover, they enjoyed a lot those machines, and they also presented them to their classmates during a annual school event named "Mathematics'party"; usually the main character of this "party" are the best students; this time the "remedial students" were protagonist as well. That year they had an "Inclusive Mathematics' party". So this activity had effect both on "*the brain"* and on "*the heart"*. The machines we used, constructed by hand, are: Kempe's machine for translation, Sylvester's pantograph for rotation, machine for axial symmetry, machine for central symmetry, Scheiner's machine for homothety, Cavalieri's machine for parabola, plus a self-produced machine for composition of two axial symmetries. First the students guessed the transformation simply by trying, and then the teacher guided them, by reasoning about "why" that machine provides that transformation.

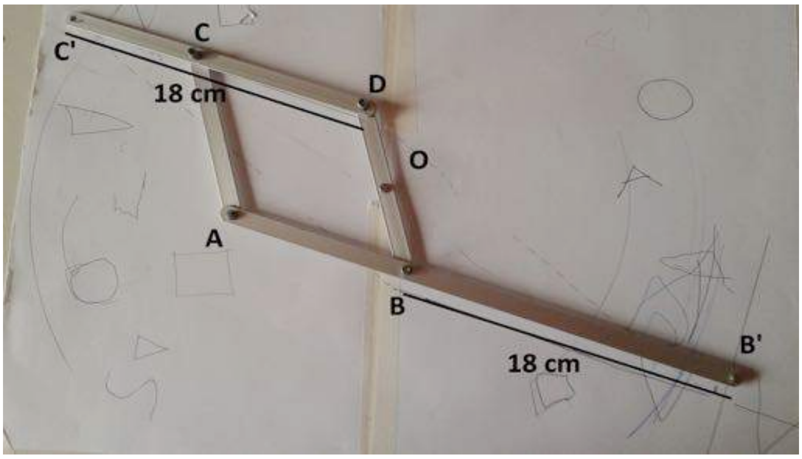

Fig. 3

For instance, we discuss a bit about the use of the central symmetry machine. We refer to the picture of the machine we really used in class (Fig 3.). It is realized by a parallelogram, ABDC. Both of two parallel edges AB and CD in the picture, are extended in such a way BB' = C'D. The middle point O of BD is signed and it is a fixed point (it is fixed to a wooden board).

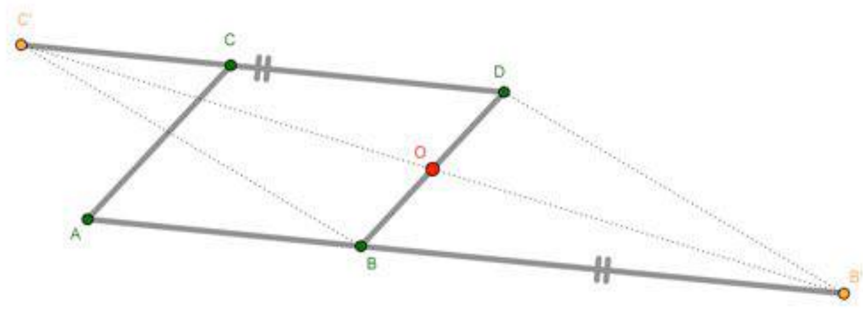

Fig. 4

As mentioned above, first of all students handled the object and discovered what happens by fixing the middle point O, following a picture with B' and putting a pen in C'. The picture is isometric, and it is symmetric with respect to O.

Then we recall the definition of central symmetry: points B' and C' have to be on the same line with O, and O has to be the middle point of B'C'. Is O the middle point of B'C'? … We have to think a little and see which mathematics is hidden inside the machine. We reasoned in terms of the mathematical objects involved and we drew step by step, with the pencil, the dotted segments in the picture (Fig 4.)

We noticed (by measuring) that, by construction, BB'= C'D. Moreover, they are parallel, so we can draw another parallelogram BB'DC'(whose real fixed edges are BB' and C'D, while BC' and B'D can vary).

BD is a diagonal of this new semi-unreal parallelogram, and O is the middle point of BD.

So, by the properties of parallelograms, we argue that the other diagonal B'C' (that now we can draw) is bisected by O, i.e. C' is the symmetric point of B', with respect to O.

We proceeded with similar strategies to discover and explain how the other machines work.

We obtained two important goals: first, pupils remembered better the definition of central symmetry (since they remembered the construction), and, by using visual intelligence, they remembered the whole figure; second, students perceived mathematics as a whole, and not separated in chapters. Many times, at least in Italian schools, geometric transformations are taught as a stand-alone topic, every concept dies after the teacher finishes that chapter, and they are not used anymore. Moreover, they use only concepts inside the chapter, so there are links neither with the "previous" mathematics nor with the "following" one.

We showed that there are, for instance, quadrilateral properties hidden in the geometric transformation chapter.

A similar work of "merging separated chapters" was done with Cavalieri's machine for parabola, that uses second Euclid theorem to draw a curve (while triangles and conics are definitively in different chapters of the book).

Conclusions

As for "*the brain"*, students who duly attended the course improved their performance in mathematics, as briefly summarized in Fig. 5[3].
It is worth to point out that the initial and final tests were on different topics. The final test was directed to verify the validity of the teaching style on the topics faced in the course. A test on those topics would have been meaningless at the beginning, before treating the tasks, so the initial test was a sort of entrance exam, aiming to take a picture of the starting situation, and the questions were on topics of the previous scholastic year. For this reason we cannot give quantitative results,

[3] The graph refers to the 14 on 20 students participating both to the initial and to the final test.

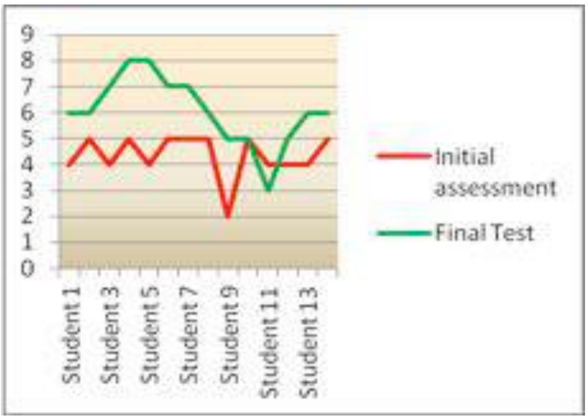

Fig. 5

but only qualitative considerations can be made. In Italy assessment is expressed as a number from 1 to 10. Being a remedial course, all the students had low grades: 5 or less. And in the initial test they reached no more than 5, as shown by the red polyline in Fig. 5. The final test consisted in 8 multiple choice questions, 4 open exercises and a geometry problem to be solved by using a DGS. In the geometry problem a property was asked to be verified and proved. The students could/should use the software both to verify the property and to prove it, as an help for reasoning. In the final test, only one student had a lower grade (student 11) and another student had the same grade (student 10). All the other students improved of 1.75 as average.

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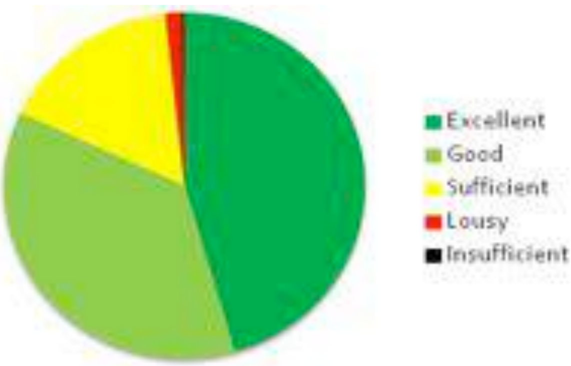
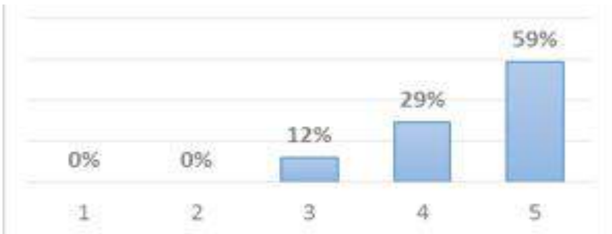
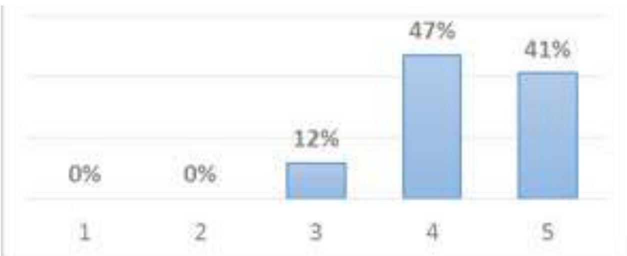


Fig. 6

As for "*the heart*", the anonymous satisfaction test, administered by an external evaluator tutor, highlights the result shown in the graphic on Fig. 6 with respect to the overall evaluation of the course. The 81.90% judged the course excellent or good, and only the 1.70% judged it lousy or insufficient.

We report some crucial questions and the relative graphics, with 1 = Definitively no, 5 = Definitively yes

1) Do you think that the use of a new methodology, based also on new technologies, was effective? (Fig. 7)	 *Fig. 7*
2) How much did the topics and the teacher's strategy rise your interest? (Fig. 8)	 *Fig. 8*

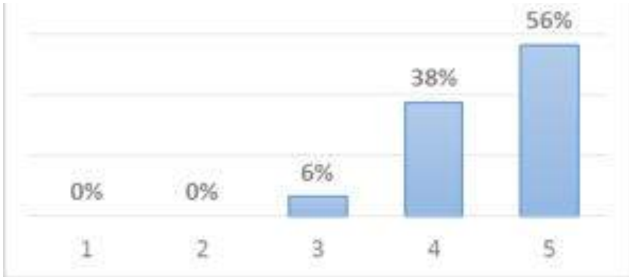
<table>
<tr><td>3) Had practical drills a suitable space? (Fig. 9)</td><td>

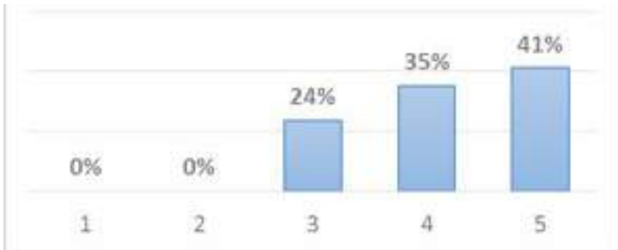
Fig. 9</td></tr>
<tr><td>4) Did the course fulfill your expectations and needs? (Fig. 10)</td><td>

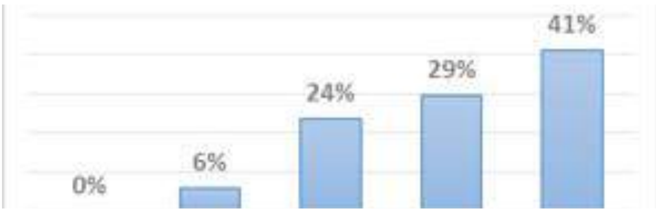
Fig. 10</td></tr>
<tr><td>5) Do you think the course allowed you to recover your competencies and skills? (Fig. 11)</td><td>

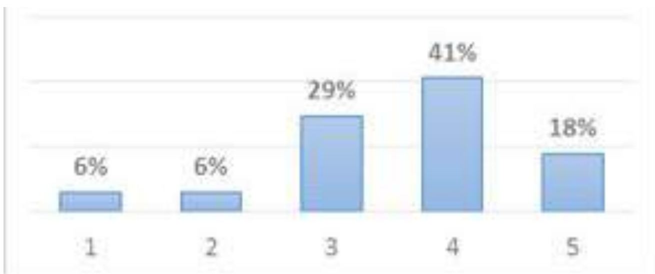
Fig. 11</td></tr>
<tr><td>6) Do you think the course helped to improve your attitude towards mathematics ? (Fig. 12)</td><td>

Fig. 12</td></tr>
</table>

Answers to question 1) and 2) highlight the effectiveness of the strategy based on practical activities (in suitable number for 94% of students as pointed by question 3)) and technology (above all the use of DGS), since 88% of students answered with a Yes or Definitively yes.
The 70% of pupils found the course useful to recover their competencies, as they need, and 76% of them thought the course fulfilled their expectations.
As for a positive attitude towards mathematics, 59% of students answered with a full Yes, but, as you can see by comparing graphic from Fig. 12 to the previous graphics, it is the only one with a scale of less than 40% on a Definitively yes, and positive (low, but positive) percentage on Definitively no. We are perfectly aware that is not immediate to change a well rooted attitude, and

there needs years of hard and everyday work of good teachers to reach a success. We met only once a week and only for a few months. But we firmly think that if teachers will continue in such a direction, students could change their mind (and hearth!) towards mathematics.

Open questions about the positive and negative aspects of the activity highlight what follows: the main positive elements were the helpfulness of the teachers, the use of new technologies and the use of mathematical machines. The only negative element was the too heavy scheduling.

Of course the activity worked because we had a homogeneous class and we could use Concrete Experience, a thing you cannot do in a regular class, otherwise you would disadvantage convergent and assimilating students. Kolb himself suggests how to reach all learning styles by "teaching around the cycle": teachers should work by using all the four capabilities, in a spiral. So, every topic should be divided into four phases. For instance you can start from Concrete Experience by introducing a real example of life, then you pass to Reflective Observation by posing questions on why those experience holds. Then you lead students to Abstract Conceptualization by answering to those questions posed in the previous phase and setting the theory. Finally, you act with Active Experimentation by putting theory into practical exercises. At the end of the cycle you should restart a new cycle for the next topic, with a new experience that does not work if one just uses the practical skills acquired in the previous phase, in a learning spiral.

By using such a teaching/learning practice the teacher brings:

- divergent students from Concrete Experience to Reflective Observation;
- assimilating students from Reflective Observation to Abstract Conceptualization;
- convergent students from Abstract Conceptualization to Active Experimentation;
- accommodating students from Active Experimentation to Concrete Experience.

All the students participate to the lectures by using their best abilities, all the learning styles are respected. More, all students are taught sometimes in their preferred mode, and they take advantage in the learning activity, and sometimes in their less preferred mode, so they can develop those skills they might never develop if the teaching style was perfectly matched to their preferences. (Felder, 2010)

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REFERENCES

Appelbaum, P. (1998). An Other Mathematics: Object Relations and the Clinical Interview. *Journal of Curriculum Theorizing 14, 2: 35-42.*

Artigue, M. (2000). Instrumentation issues and the integration of computer technologies into secondary mathematics teaching. *Proceedings of the Annual Meeting of GDM. Potsdam (Germany).*

Brown,T. (2012). Affective productions of mathematical experience. *Educational Studies in Mathematics, May 2012, Volume 80, Issue 1-2, pp 185-199*

Chiappini, G. (2007). Il laboratorio didattico di matematica: riferimenti teorici per la costruzione. *Innovazione educativa*, Inserto allegato al numero 8, 9-12.

Di Martino, P. (2007), L'atteggiamento verso la matematica: alcune riflessioni sul tema. *L'insegnamento della matematica e delle scienze integrate, Vol. 30A-B, n.6*

Dewey, J. (1916). Democracy and education: an introduction to the philosophy of education. *Free Press; Reprint edition (February 1, 1997)*

Drijvers, P. (2012). Digital technology in mathematics education: Why it works (or doesn't). In *Proceedings of ICME 12. Seoul, Korea. 9-12 July 2012*

Felder, R.M. (2010) Are Learning Styles Invalid? (Hint: No!). *On-Course Newsletter, September 27,* retrieved at http://www4.ncsu.edu/unity/lockers/users/f/felder/public/Papers/LS_Validity(On-Course).pdf

Ferrarello, D., Mammana, M.F., & Pennisi, M. (2014a). Teaching by doing. *QUADERNI DI RICERCA IN DIDATTICA / Mathematics (QRDM) Quaderno N.23 Supplemento n.1 - PALERMO 2013, pp.429-433 http://math.unipa.it/~grim/quaderno23_suppl_1.htm*

Ferrarello, D., Mammana, M.F., & Pennisi, M. (2014b). Teaching/learning geometric transformations in high-school with DGS. *IJTME vol 21 n1 p11-17*

Ferrarello, D., Mammana, M.F., Pennisi, M. (2013) Using DGS to introduce algebra properties. *ICTMT 11 Conference Proceedings. http://www.dm.uniba.it/ictmt11/download/ICTMT11_Proceedings.pdf*

Hudson, L. (1968). Contrary imaginations: A psychological study of the English schoolboy. *Harmondsworth: Penguin.*

Kolb, D.A., & Fry, R. (1975) Toward an applied theory of experiential learning. *C. Cooper (ed.), Theories of Group Process, London: John Wiley.*

Kolb, D.A. (1984). Experiential Learning: Experience as the Source of Learning and Development. *Englewood Cliffs, NJ: Prentice-Hall.*

Maschietto, M., & Trouche, L. (2008). ICT, new insights on old problems. *In Menghini M, Furighetti F, Giacardi L, Arzarello F. The first century of the International Commission on Mathematical Instruction (1908-2008). Reflecting and shaping the world of mathematics education, Roma: Istituto della Enciclopedia Italiana.* http://www.unige.ch/math/EnsMath/Rome2008/WG4/Papers/TROUMAS.pdf

Radford, L. (2013). On the role of representations and artefacts in knowing and learning. *Educ Stud Math. DOI 10.1007/s10649-013-9527-x*

Vygotsky, L. (1986) Thought and Language. *Cambridge, MA: MIT Press.*

Zan, R. (2006). Difficoltà in matematica. Osservare, interpretare, intervenire. *Springer, Milano.*

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Un dispositif de formation initiale pour l'integration d'environnements numeriques dans l'enseignement des mathematiques au secondaire

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Abstract : In this paper we present an preservice training workshop in Information Technologies and their use in the mathematics classroom. The theoretical framework is mainly that of the Theory of Situations of Brousseau (1986) in particular the concepts of milieu and adidacticity (learning feedback). We describe how this framework has structured the workshop and we analyze how students have integrated it in their projects. The workshop proposed to compare a question and answers software from the french group Sesamath with the feedbacks systems of Geogebra and Aplusix (dynamic algebra).

Résumé : Dans cette communication nous présentons un dispositif de formation initiale aux technologies de l'information et à leur usage en classe de mathématiques. Le cadre théorique est principalement celui de la Théorie des Situations de Brousseau, en particulier le concept de milieu (1986). Nous décrivons la façon dont ce cadre a structuré le dispositif et analysons la façon dont les étudiants l'ont intégré dans leurs projets. L'atelier a proposer la comparaison d'un exerciseur proposé par le groupe Sesamath avec les systèmes de rétroactions de Geogebra et Aplusix (algèbre dynamique).

Introduction

A l'Institut Universitaire de Formation des Enseignants (Université de Genève), lors de leur seconde et dernière année, les étudiants sont stagiaires à mi-temps dans une école secondaire. C'est dans ce cadre, qu'ils pour tâche de réaliser une séquence d'environ trois leçons intégrant l'utilisation de l'ordinateur. Les trois ateliers préalables leur permettent de prendre en main quelques logiciels en analysant le type de rétroaction fourni, plus particulièrement le milieu. Entre les ateliers mensuels, les stagiaires effectuent quelques expérimentations ou analyses consignées dans un tableau de bord. Par groupe de 2-3 étudiants, ils mettent en place et observe une séquence en se centrant sur le rôle du milieu spécifique (environnement numérique). Ils présentent et analyse ensuite leur travail lors d'un colloque réunissant tous les participants.

Questionnement

Quelles raisons peuvent conduire un enseignant à utiliser un environnement informatique dans l'enseignement des mathématiques ? Aujourd'hui encore la réponse n'est pas immédiate. Pour des enseignants en formation initiale, il est nécessaire d'y réfléchir et de construire une réponse. A première vue, certaines ressources (Labomep de Sesamath, par exemple) fournissent une aide utile à l'enseignant en prenant en charge sa tâche pour la gestion d'exercices répétitifs et en améliorant l'efficacité par rapport au cycle habituel : exercices individuels-correction collective. De nombreux logiciels sont également disponibles, dans le domaine de la géométrie dynamique par exemple. La recherche dans ce domaine a montré que l'intégration efficace de ces outils est soumise à des conditions parfois complexes à mettre en place (Floris & al. 2007). Une partie importante de ces recherches mettant en exergue le rôle des rétroactions pour l'utilisateur, nous en avons fait un des points essentiels du dispositif, en lien avec le concept de milieu adidactique, comme nous le développons ci-dessous

Cadre théorique

Concernant le concept de milieu, nous nous référons aux travaux de Guy Brousseau.

« En situation scolaire l'enseignant organise et constitue un milieu, par exemple un problème, qui révèle plus ou moins clairement son intention d'enseigner un certain savoir à l'élève mais qui dissimule suffisamment ce savoir et la réponse attendue pour que l'élève ne puisse les obtenir que par une adaptation personnelle au problème proposé » (Brousseau, 1989, p.325).

« Nous pouvons définir le milieu adidactique comme étant l'image dans la relation didactique du milieu "extérieur" à l'enseignement lui-même. En fait, cette image joue le rôle de lien avec des pratiques de référence dans lesquelles le système enseigné est supposé pouvoir fonctionner de manière non-didactique »

« Le milieu adidactique concerne également l'ensemble des savoirs (acquis à l'école ou ailleurs) supposés connus par l'élève, dont la mobilisation peut l'aider à réaliser la tâche demandée par l'enseignant. »

« On appelle rétroaction une réponse fournie par le milieu qui est reçue par l'élève comme une sanction, positive ou négative, relative à son action et qui lui permet d'ajuster cette action, d'accepter ou de rejeter une hypothèse, de choisir entre plusieurs solutions. (Brousseau, 1986).

On distingue deux types de rétroactions:

Une rétroaction de type didactique, pour laquelle l'intention d'enseigner est explicite, la réponse en juste ou faux et la procédure correcte est explicitée.

Une rétroaction de type adidactique, lors de laquelle l'intention d'enseigner est cachée. Avec ce type de réponse, il y a maintien du milieu dans lequel travaille l'élève.

Comme exemple de rétroaction adidactique, on peut se référer à la situation de l'agrandissement d'un puzzle, dans laquelle chaque élève d'un groupe élabore l'agrandissement d'une pièce. La rétroaction est fournie par l'assemblage final du puzzle : si ce dernier n'est pas possible, la procédure d'agrandissement, par exemple par addition d'une mesure, est invalidée. L'invalidation ne provient pas de l'enseignant, mais de la réussite dans le milieu mis en place. On remarquera que le milieu ne se limite pas à la situation extérieure, mais qu'il est également constitué par la consigne qui définit le but à atteindre. Ce point est à noter, car les étudiants se limitent souvent à analyser la rétroaction, sans tenir compte des contraintes de la situation didactique, en particulier des consignes données.

Exemples

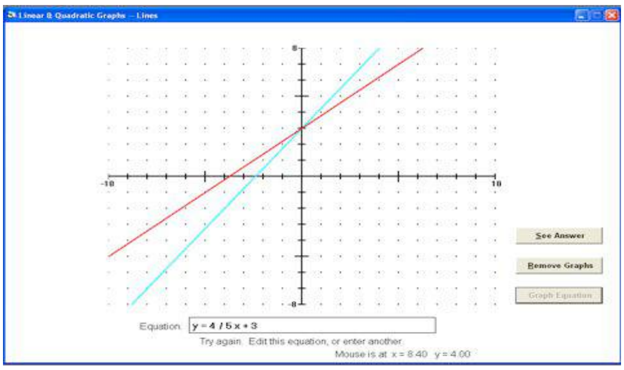
Rétroaction didactique

Ce que nous appelons exerciseur, tels les ressources proposées par labomep (labomep.net), on propose souvent une tâche précise, pour laquelle la connaissance d'une formule est nécessaire. La rétroaction est « juste ou faux » et les aides fournies récapitulent la formule à utiliser :

La réponse donnée et l'aide sont externes à la situation d'action de l'élève, c'est-à-dire aux gestes qu'il doit effectuer pour fournir cette réponse. Les informations fournies sont des énoncés (type déclaratif).

Retroaction adidactique

Les connaissances de l'élève lui permettent d'obtenir un résultat à analyser. Ce résultat est une rétroaction fournie dans le milieu dans lequel se situe l'action de l'élève. C'est ce dernier qui est responsable de l'interprétation, de l'évaluation en juste/faux. En guise d'exemple, choisissons le logiciel Green Globs (Dugdale, S & Kibbey, 2008) qui propose entre autres des graphiques de fonctions linéaires ou quadratiques dont il faut déterminer l'équation. La rétroaction consiste simplement en un tracé de l'équation entrée par l'utilisateur. L'examen de cet exemple permet d'introduire différents type de rétroaction adidactique. Ce logiciel donne en effet la possibilité de voir la réponse, mais on peut imaginer qu'un paramétrage ne fournisse cette possibilité qu'après un certain nombre d'essais. De plus, les rétroactions fournies permettent à l'élève de trouver la solution simplement par essais et erreurs (phénomène de pêche, Artigue, 1997) de sorte que l'on n'a pas de garanties sur les potentialités d'apprentissage. Suivant Bloch (2005) on dira que le milieu ici n'est guère antagoniste. Ceci permet alors de travailler le rôle de la situation didactique, celle que propose l'enseignant, qui peut par exemple introduire des phases de formulation (Brousseau, 1986) conduisant les élèves à expliciter leurs méthodes et à les justifier.



Deux ressources essentielles

La question du milieu est travaillée avec l'étude des potentialités de deux types de ressources. La géométrie dynamique, tout d'abord, qui offre une rétroaction spécifique, le déplacement qui permet de valider le caractère mathématique des constructions effectuées. Il s'agit là de rétroactions ne permettant pas la pêche (essais et erreurs). Le milieu proposé est antagoniste en ce sens que des connaissances mathématiques sont nécessaires pour réussir. La géométrie dynamique fait presque officiellement partie du programme scolaire en Suisse Romande. L'accès facile à la ressource Geogebra facilite aussi son utilisation. Une autre ressource intéressante pour l'étude du milieu est le logiciel Aplusix qui offre un milieu qui permet de contrôler l'équivalence des formules ou des expressions littérale. La rétroaction est là aussi à caractère adidactique, mais les essais-erreurs sont possibles. A charge de l'enseigner de paramétrer logiciel et consignes afin de favoriser l'antagonicité.

Ces deux ressources, sont prioritairement choisies par les enseignants pour leurs séquences.

Evolution des projets

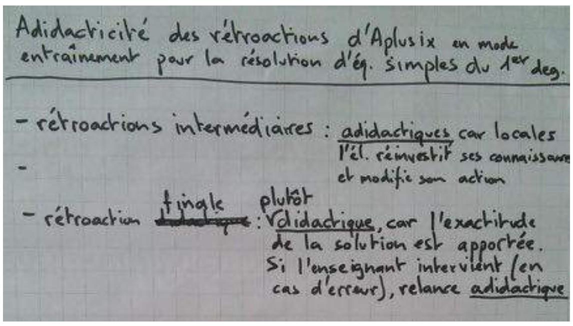
Le dispositif de formation a évolué au cours des années. A l'origine, de nombreux projets de séquences consistent en un enseignement balisé par des fiches de travail permettant un contrôle accru. Rares sont les exploitations des rétroactions fournies par les environnements informatiques. Les formateurs ont alors décidé de mettre l'accent sur la question du milieu et des retroactions en invitant les étudiants à étudier différentes situations d'utilisation de l'ordinateur et à consigner leurs commentaires dans un tableau de bord :

	Exerciseur	Geométrie /fonctions	Algèbre	En classe à tous les élèves	Analyse Scénario	Séquence principale avec partie en labo	Calculatrice
Quoi (lien ou nom précis)							
Date(s)							
Type (description/analyse ou passation avec élèves)				Passation	analyse	Passation ou observation	
Milieu et rétroactions	Milieu déclaratif (juste/faux)				Peu de rétroactions adidactiques	Hypothèses	
Annexes	Lien : extrait de réponses d'élève				Lien : détail de l'analyse	Projet de scénario	

Les surlignés jaunes sont des exemples
Un exemple de tableau de bord partiellement rempli

Conclusion

L'expérimentation décrite est actuellement en cours. Les observations d'ores et déjà effectuées mettent en évidence un questionnement autour de la notion de milieu adidactique. Pour certains étudiants, l'intégration en classe reste difficile, alors que d'autres élaborent des consignes et énoncés favorisant les rétroactions. Le colloque final a permis des discussions approfondies sur le concept d'adidacticité, mettant en évidence l'appropriation de ce concept par les étudiants (figure)



Lors de la conférence ces résultats seront ajournés par l'étude des séquences proposées par leurs étudiants ainsi que leur texte final de synthèse.

REFERENCES

Artigue M. (1997), Le logiciel DERIVE comme révélateur de phénomènes didactiques liés à l’utilisation d’environnements informatiques pour l’apprentissage, *Educational Studies in Mathematics, 33, 2*.

Bloch, I (2005) Quelques apports de la théorie des situations… , habilitation, Université Paris-Diderot - Paris VII. https://tel.archives-ouvertes.fr/tel-00012153

Brousseau, G. (1986). Fondements et méthodes de la didactique de mathématiques. *Recherches en didactique des mathématiques, 7.2*.

Dugdale, S & Kibbey, D (2008, 2013) Green Globs & Graphing Equations, www.greenglobs.net

Floris R. et Conne, F. (2007) Environnements informatiques, enjeux pour l’enseignement des mathématiques, De Boeck.

Rabardel, P. (1995). *Les hommes et les technologies, approche cognitive des instruments contemporains.* Paris : Armand Colin.

Collaborative study groups in teacher development: a university - school project

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Abstract : This paper aims to present the results of an investigation of a continuing education process conducted with school teachers organized into study groups to work collaboratively, based on participating schools. The participating teachers work at the early years of elementary education of São Paulo schools and had previously taken part of a continuing education process under the Brazilian Program "*Observatório da Educação*". This is a qualitative study in which data was collected through direct observations, recordings and testimonials collected in interviews and reflective reports. Initially, we present the research setting, references and methodological procedures, especially those relating to collaborative work. Llinares (1999) and Jaworski (2009) studies support our data analysis and discussions of issues related to teacher development and collaborative study groups. At the end, we present the analysis of information collected during and after the formative process.
Key Words: continuing education process, collaborative groups

Résumé : Cet article vise à présenter les résultats d'une enquête d'un processus de formation continue menée avec les enseignants de l'école, organisés en groupes d'étude pour travailler en collaboration, qui se déroule dans les écoles participantes. Les enseignants participants travaillent sur les premières années de l'enseignement primaire des écoles de São Paulo et avaient déjà pris partie d'un processus de formation continue dans le cadre du programme brésilien "Observatório da Educação". Ceci est une étude qualitative dans laquelle les données ont été recueillies par des observations directes, des enregistrements et des témoignages recueillis dans les entrevues et les rapports de réflexion. Initialement, nous présentons les cadre de la recherche, des références et des procédures méthodologiques, en particulier ceux concernant le travail collaboratif. Les études Llinares (1999) et Jaworski (2009) confirment notre analyse des données et des discussions sur les questions liées à la formation des enseignants et des groupes d'étude de collaboration. A la fin, nous présentons l'analyse des informations recueillies pendant et après le processus de formation.

Introduction

This research was developed by a Brazilian Post-graduate Program in Mathematical Education in the scope of the Education Observatory Program whose purpose was to establish groups in which researchers and mathematics teachers at the early years of elementary school work in collaboration. We sought to develop researches to analyze the changes in teaching practices and the professional development of teachers when they are deeply convinced to promote curricular innovations in their classes. This project was developed between 2011 and 2014 and is one of some other projects that occurred since 2008. The establishment of study groups at the schools on a permanent basis was led by the teachers participating in the continuing education process with the university professors during the formative period. Teacher development took place, as described in the original project, in alternate actions happening on-site and online. The project indicated that study groups would be "one alternative to be tested". Researchers interact with teachers and trainers, as managers, partakers or observers of the formation process or of the activities in the classroom. The researchers conduct the activities and subjects are chosen, usually from the group demand, the analysis of diagnostic activities and research results information. The researcher responsible for the activity seeks to motivate discussions and reflections on matters relating to educational processes and group interest in bringing mathematics learning situations related to teaching practice. For example, the elements and classifications of the Conceptual Fields Theory Additives emerged from the analysis made by the group of problem situations drawn up by the participating teachers regarding their work in the classroom. After the discussion of the problems brought to the group, teachers have developed activities that returned to their classroom. The group in the light of the knowledge of the theory acquired in the training process subsequently evaluated students' protocols. One of the groups analyzed here has already been the research object of one master thesis produced within the project scope. Miranda (2014) presented results concerning the process of (re)construction of knowledge required to teach the Additive Conceptual Field.

Theory

Regarding teacher development and study group formation, we based our support on Llinares (1999) and Jaworski (2009), among others. Such studies have pointed out that the development performed through the establishment of teacher study groups promotes a favorable environment for professional teacher development.
We also supported our analysis in Murphy and Lik (1998) studies, which highlight two possible ways to organize study groups: the one involving the entire school and the one defined as independent. The first type usually consists of the administrative school team (principal, coordinator) and they tend to solve school problems, which have already been identified by the group. The independent ones are, according to the authors, those who do not depend on the participants' organizational structure, "have common interests and consider themselves as a group" (Murphy and Lick, 1998, p.10).
In Jaworski's studies (2009), we looked for references to our belief that joint work between elementary schools teachers and university researchers is an important stance to promote reciprocal learning and the establishment of what the author calls *investigative communities*. Like the author, when she quotes Wells (1999), we believe that the teacher in such communities places himself in the role of investigator of his own practice. Therefore, we deem relevant to investigate the convergences noted in the establishment of study groups during the project development.

Methodology

This research has a qualitative nature, in the sense defined by Bogdan and Biklen (1999). We will present the analysis of collected information about the establishment of the four study groups that took place throughout the four years of the institutional project. Information was collected during the sessions, in the transcript of interviews recorded during and after the continuing education process, and in the analysis of the reflective logs handed to us. We will name the groups as follows: *G1; G2; G3* and *G4*.
Group *G1* was established at the school of one teacher participating in three development modules of the project. During her participation, this teacher enrolled in the post-graduate program and decided to analyze the (re)construction of knowledge by the participating teachers of the study group established at the same school where she taught. The group's creation was the result of a need observed by the teachers when they received a document from the state education agency (SEE, in Portuguese) presenting recommendations for a diagnostic test involving problem-situations in the Additive Conceptual Field. At that moment, the author states that

> (...) I noticed how close the proposal was to the one worked in the development process. In face of the difficulties reported by the group to apply such diagnostic activities, the teachers reinforced the idea of how important the studies about this theme were for them to be able to perform what the document proposed. (Miranda, 2014, p. 37).

Group *G1* was established by the enrollment of 15 teachers and allocated at the school created by the SEE called Collective Pedagogical Work Activities (ATPC, in Portuguese). Data presented here were collected in video recordings of the 24 sessions and interviews that were conducted by

Miranda.

Group *G2* was established at the school where teacher Santiago works. Studies took place during the ATPC and this teacher was invited by the school management to be the facilitator at the sessions. According to him, the time assigned to studies during the ATPC, up to that moment, did not work as

> (...) the theory in question is posed by the study tables at SEE and reaches the teachers who need to sit down once a week and study on their own, because the middle people (pedagogical coordinators) don't always have the conditions to supervise them (Santiago, teacher)

This group consisted of 9 teachers and Santiago and numbers and geometry themes were discussed. Data analysis presented here refer to this group's meetings and also to interviews and testimonials given by Santiago to the authors.

Study group *G3* already carried its studies during the ATPC time, but the focus on mathematics themes happened in the second and third modules of the development process when the school's principal, who has a degree in mathematics, the school coordinator and 12 more teachers started to participate. According to this school principal

> (...) when I'm at the school I don't have that much time to follow the meetings, here, out of my working hours I manage to discuss issues related to the teaching of maths with the teachers (...) (School Principal)

Information analyzed here was collected in interviews given to the authors of this article and during 16 sessions of development in which themes on plane and spatial figures and decimal numbers were discussed.

Study group *G4* had also been developing their studies during the ATPC time, but like group *G3,* it prioritized mathematics themes being discussed in the development process. The school coordinator stated that his participation in the development sessions was due to his need to broaden his development: "our participation here helps us understand what is written in the SEE's document "(School Coordinator). This group consisted of the coordinator and 14 teachers. Among them, there were 7 who participated in the development that discussed aspects related to area and perimeter, and to the multiplicative conceptual field. Information analyzed here was collected during the 20 sessions of development and in interviews given to the authors of this article.

Data Analysis and Discussion Convergences Synthesis

We will consider, according to Murphy and Lick (1998), *G1* as an 'independent' group and the remaining ones were classified as "entire school group" type, since they were established with the school administration participation. Analyzing the collected information we noticed a few convergences: the participating teachers, of the four groups, had the opportunity to discuss theories and research results which were not part of their knowledge basis. The group participants reported this fact in a recurrent manner:

> Participating in the Observatory made us discuss a little more about area and perimeter and about the children's difficulties. I never thought that children would have trouble to understand that the 10 little squares organized in a different way would have the same area. I thought it would be obvious for them and it is not, the same way we saw in that text (Participant 1 of G4)

In group *G1*, while analyzing the testimonials and practices of the participating teachers in the group which studied at the school premises, Miranda (2014) comes to the conclusion that:

> During the study sessions it became clear how much the experience exchange among peers and managers was helpful to rethink how they were developing their teaching practices and how essential it is to organize some time assigned to ATPC aiming to value the exchanging of experiences, to promote studies that meet the development needs of teachers and the knowledge about the curriculum. We noticed that the participation of these teachers in the study groups, besides helping the (re) construction of knowledge about the Additive Conceptual Field Theory, also strengthened the group's investigative spirit, and the feeling of belonging to the school group as a person who learns and develops continuously, triggering changes. (MIRANDA, 2014, p.197)

The awareness about the need to understand the new curriculum, learn about research results and change their practice were also observed in the four groups: "looking at everything we learned here [referring to the development sessions] and at the school [in the study groups] we noticed that we increasingly need to study the curriculum and see how my student learns, but now I can help my student better".

Final Remarks

As pointed by Jaworski (2009), we noticed that, mainly in groups *G1, G2* and *G3*, the developed dynamics placed the teacher in the role of investigator of his own practice. The information collected also enabled us to identify aspects that, according to Murphy and Lick (1998), allow for the promotion of professional development of teachers through the establishment of study groups formed at their own school; in other words, mutual support, planning and learning together, contributing for knowledge and practice, immersing into work based on ideas, materials and colleagues; testing ideas, sharing and reflecting together, constructing knowledge about the content.

REFERENCES

Bogdan, R.; Biklen, S. (1999). *Investigação qualitativa em educação*. Uma introdução à teoria e aos métodos. Porto: Porto Ed.

Jaworski, B. (2009). Building and sustaining inquiry communities in mathematics teaching development. Em K. Krainer, & T. Wood, *The international handbook on mathematics teacher education, Participants in mathematics teacher education: individuals, teams, communities and networks* (pp. 309-330). Rotterdam: The Nerthelands: Sense publisher, 2009. p.309-330.

Llinares, S. (1999). Conocimiento y práctica profesional del profesor de matemáticas. Características de una agenda de investigación. *Zetetiké, 7*.

Miranda, M. (2014). *Uma investigação sobre a (re)construção do conhecimento de professores participantes de um grupo que estuda o campo conceitual aditivo.* Dissertação de Mestrado , UNIAN , São Paulo-Brasil.

Murphy, C., & Lick, D. (1998). *Whole faculty study groups: A powerful way to change schools and enhance learning.* California, USA: Corwin.

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Is this a proof? Future teachers' conceptions of proof

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Abstract: In this paper we present part of an ongoing investigation that aims at disclosing the conceptions of proof held by future elementary school teachers. Using a qualitative and interpretative approach, we analyzed data from 66 questionnaires and results show that almost all participants recognize the formal aspect of an algebraic proof but they also accept some examples as proof.

Résumé: Dans cet article, nous présentons partie d'une enquête en cours qui vise à révéler les conceptions de la preuve détenue par les futurs enseignants du primaire. En utilisant une approche qualitative et interprétative, nous avons analysé les données de 66 questionnaires et les résultats montrent que presque tous les participants reconnaissent l'aspect formel d'une preuve algébrique mais ils acceptent aussi quelques exemples comme preuve.

Introduction

Proof plays a fundamental role in the construction of mathematical knowledge, having a different nature and assuming a different structure that in other sciences (Davis & Hersh, 1995; Dreyfus, 2000; Hanna, 2000; Knuth, 2002). To prove is intrinsic to mathematical activity because no result can be considered valid/acceptable until it is proven. Assuming a formal, logical view, proof can be defined as "a sequence of transformations of formal sentences, carried out according to the rules of the predicate calculus" (Hersh, 1993, p. 391). But proof can also be regarded in a more practical manner as "an argument that convinces qualified judges" (ibidem). Therefore, for the same result we can have different proofs. Some proofs have a geometric approach, some are more algebraic, and some have only words while others have only diagrams. Thus, what characteristics must a proof have to be considered as such?

Also important is to consider why proof is used. There are several functions attributed to proof (De Villiers, 2003). In teaching and learning mathematics two functions tend to be predominant: conviction and explanation (Hanna, 2000; Hersh, 1993). The importance of proof in the classroom, especially in the early years, has not always been recognized (Hanna, 2000). Proof appears more linked to secondary and higher education and its' understanding is sometimes referred to the good students only (Knuth, 2002). However, several authors have highlighted the role of proof in the construction of mathematical knowledge by students from the beginning of schooling (Bussi, 2009; Hanna, 2000; Knuth, 2002; Stylianides, 2007). Proof of mathematical results comes prized in the current reformulation of programs from the early grades, in Portugal (ME, 2007; MEC, 2013). From the early grades, children should start to learn and to deal with proof and proving. For this to happen it is important that teachers develop strategies to motivate and encourage students for the activity of proving showing them the power/importance of proof and do not reduce proof to a mere memorization of sequential meaningless steps. This process is clearly dependent on the perception that teachers have of proof and of what it means to prove. Therefore, teachers need to be prepared to deal with proof and to have a clear understanding about proof and proving. Consequently, during initial teacher training, future teachers should be involved in proof activities that enable them to deepen their knowledge on proof, and enhance their ability to validate, organize, justify and generalize acquired mathematical knowledge.

Methodology

The purpose of this study was to disclose the conceptions held by future elementary school teachers (grades 1 through 6) concerning the notion of proof. Nowadays, in Portugal, in order to become an elementary school teacher, one has to have a 3 years degree in Basic Education and then take a Masters degree in teaching. These teachers have to teach several subjects; therefore, the curriculum of these degrees is very wide and covers a great variety of topics.

The participants on this study were 66 students enrolled in the 3rd year of a Basic Education Degree. These participants had already taken 5 courses in mathematics during which they had some contact with proof, performing proofs of some mathematical results, such as the irrationality of the square root of two, Pythagoras theorem or the constant sum of the distances from a point inside a triangle.

Given the nature of the outlined goal, we adopted a qualitative and interpretative approach in order to understand the meaning that future teachers give to this activity (Bogdan & Biklen, 1994).

We designed a short questionnaire containing four mathematical results and, for each, we gave three proposals of proof. We then asked the students to say whether they considered each proposal to be a proof or not and to justify their choices. This questionnaire was completed individually, at the end of a regular class, taught by one of the authors. The participation was voluntary and could be anonymous, that is, students only identified themselves if they wanted to.

For this paper we selected the following two results: (1) Diagonals of a rhombus are perpendicular; (2) For $a, b \in R, (a+b)^2 = a^2 + 2ab + b^2$.

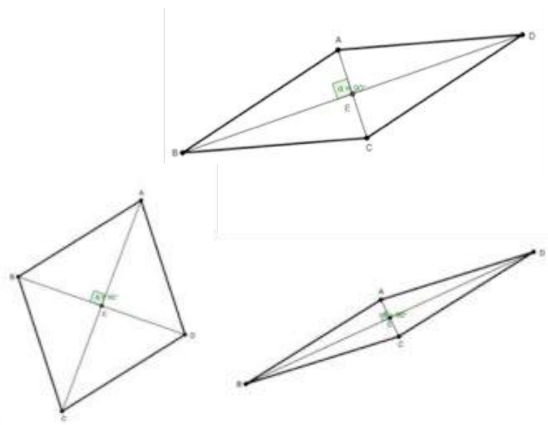
Some findings

Result 1

Diagonals of a rhombus are perpendicular. For this result, students were confronted with the 3 proposals of proof.

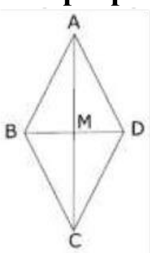
1st proposal of proof

Draw the diagonals [AC] and [BD] and consider the intersection point (E). Measuring one of the angles defined by the diagonals we easily conclude that the diagonals are perpendicular. Now we check that the same happens for other rhombus, in other positions and other shapes:



In these cases, measuring the angles, we also check that the diagonals are perpendicular. For many other different rhombus we would come to the same conclusion. Hence, we have proved that the diagonals of a rhombus are perpendicular.

2nd proposal of proof



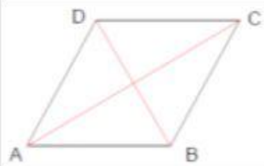
Let [ABCD] be an rhombus represented as follows:

[AC] and [BD] are the diagonals of [ABCD]. M is the midpoint of [AC] and of [BD].

[ABD] is an isosceles triangle, then [AM] is the height of the triangle relatively to the side [BD]. [AC] is perpendicular to [BD].

3rd proposal of proof

Let [ABCD] be an rhombus represented as follows:



Since $\overline{BC} = \overline{DC}$, we have that the point C belongs to the perpendicular bisector of [BD]. Since $\overline{AB} = \overline{AD}$, we have that the A belongs to the perpendicular bisector of [BD].

Hence, the line segment [AC] is contained in the perpendicular bisector of the line segment of [BD]. Therefore, [BD] and [AC] are perpendicular.

The choices made by the students were the following:

	Is proof	**Is not proof**	**No answer**	**Total**
Proposal 1: Examples	45	18	3	66
Proposal 2: Loci	23	40	3	66
Proposal 3: Measurement	32	31	3	66

Table 1. Frequency of student answers to each of the proposals.

For most students, the verification, in individual cases, of geometric results to prove is seen as a proof. The same students have difficulty following an argument that works only with loci and so, although it is a proof, the second proposal was not considered as such. Working with measures, there is a balance between the number of students that identifies the proposal as evidence and as no proof.

Analysing the reasons given for proposal 1, we found the following results:

	Is proof		**Is not proof**	
	Particular cases allow generalization	28	Particular cases don’t allow generalization	10
	Use the result	15	No justification	2
	Meaningless justification	2	Meaningless justification	6
Total		45		18

Table 2. Justifications provided in response to proposal 1.

Examining the reasons given for proposal 1, we found that the majority of students said that proposal 1 was a proof and justified by saying that particular cases could be generalized. We illustrate this with an example of a justification given by one of the students:

This case is a mathematical proof, because we started with a case and found a conjecture. After we used further examples to see if it works in order to generalize.

Interestingly, the justification used for saying that it was not a proof is opposite: particular cases don’t allow generalization, as another student referred:

It is not a mathematical proof, since it makes use of particular cases.

A significant number of students that said it was a proof, also justified using the result, as shown in this students’ answer:

In my opinion, this proposal 1 is according to the result since drawing two diagonals, and finding the intersection point on the rhombus, they are always perpendicular.

Looking at the reasons given for proposal 2, we found the following results:

	Is proof		Is not proof	
	Use the result	6	Diagram as particular case	28
	Identifies a correct reasoning	5	Insufficient data	5
	Meaningless justification	6	Meaningless justification	5
	No justification	2	No justification	2
	Validation of a step of reasoning	4		
Total		23		40

Table 3. Justifications provided in response to proposal 2.

The majority of the students said that it was not a proof and among these, more than half justified their opinion saying that they considered the drawing/diagram to be a particular case. An example of the justifications given is:

If [*AC*] *is perpendicular to* [*BD*] *then this rhombus has its diagonals perpendicular. However this is only a particular case, it does not occur for all rhombi.*

Some of those saying it was a proof, used the result itself as justification:

Proves because the diagonal pass through the point M and have the same angle.

Others said it was a correct reasoning and others justified the all proof identifying just a correct step, as the following example:

This is a proof since in an isosceles triangle the height of the triangle is perpendicular to its base.

As for proposal 3, the opinions are divided.

	Is proof		Is not proof	
	Identify a correct reasoning	8	Diagram as particular case	20
	Validation of a step of reasoning	7	Meaningless justification	6
	Use the result	3	Insufficient data	3
	Meaningless justification	9	No justification	2
	No justification	5		
Total		32		31

Table 4. Justifications provided in response to proposal 3.

As before, many of those who consider that it is not a proof assume the diagram as a particular case. One of the students wrote:

Does not prove, because we have not proven that we can generalize what is happening with this case.

Also, some of the students that said it was a proof, used the result as justification:

It is a proof, because [*AC*] *and* [*BD*] *intersected themselves and form a right angle.*

There are students that considered the proposal to be a proof since identified a correct reasoning:

This is a proof since it is obtained by a sequence of the mathematical statements that no one can refute.

Result 2

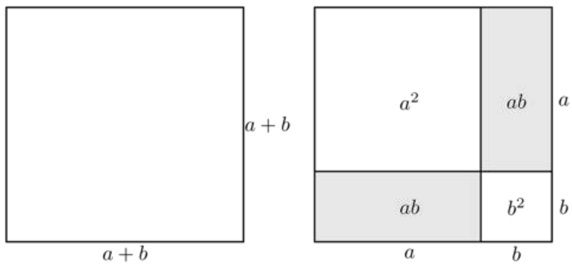
Considering the result: "$a, b \in R, (a+b)^2 = a^2 + 2ab + b^2$", we provided the following proof proposals:

1st proposal of proof

Consider, for example, $a = 2$ and $b = 1$. Then, $(a+b)^2 = 3^2 = 9$ and
$a^2 + 2ab + b^2 = 2^2 + 2 \times 2 \times 1 + 1^2 = 4 + 4 + 1 = 9$. If we consider $a = -1$ and $b = \frac{1}{2}$, we have $(a+b)^2 = (-\frac{1}{2})^2 = \frac{1}{4}$ and
$a^2 + 2ab + b^2 = (-1)^2 + 2 \times (-1) \times \frac{1}{2} + (\frac{1}{2})^2 = 1 - 1 + \frac{1}{4} = \frac{1}{4}$.

For any real values considered for a and b, we will obtain equal values for both members of the equality. Hence, the equality is true.

2nd proposal of proof



3rd proposal of proof

Let a and b be real numbers. Then

$(a+b)^2 = (a+b)(a+b)$	[Definition of powers]
$= a(a+b) + b(a+b)$	[Distributivity of $\times$ over $+$]
$= aa + ab + ba + bb$	[Distributivity of $\times$ over $+$]
$= a^2 + ab + ab + b^2$	[Definition of powers and commutativity of $\times$]
$= a^2 + 2ab + b^2$.	

Students chose whether each proposal was a proof or not, in the subsequent manner:

	Is proof	Is not proof	No answer
Proposal 1: Examples	35	30	1
Proposal 2: Without words	26	38	2
Proposal 3: Algebraic	60	4	2

Table 5. Frequency of student answers to each of the proposals(n=66).

In proposal 1, slightly more than half the students said it was a proof.

	Is proof		Is not proof	
	Formula is checked in particular cases	32	Particular cases don't allow generalization	24
	Use the result	2	No justification	1
	Meaningless justification	1	Meaningless justification	5
Total		35		30

Table 6. Justifications provided in response to proposal 1.

Almost all justifications were the same as in proposal 1 from the 1st result, that is, particular cases allow verifying the truth of the statement. One model of this was:
Yes, through these examples we can consider that the equality is always true.

For proposal 2 we have the following results:

	Is proof		**Is not proof**	
	Identify the figures to complete equality	14	Insufficient data	28
	Check with concrete values	2		
	Use the result	2		
	Meaningless justification	6	Diagram as particular case	8
	No justification	2	No justification	2
Total		26		38

Table 7. Justifications provided in response to proposal 2.

In this proposal, more students said it wasn't a proof since most of them considered that data was insufficient. An example of a justification given by students is:
It's not a proof, as is not accompanied with any explanation / justification.

Nevertheless, half of the students that considered this proposal to be a proof did it because they could read the figures to complete the equality:

It proves because both figures represent the same number.

Finally, almost all students consider proposal 3 a proof.

	Is proof		**Is not proof**	
	Identify a correct reasoning	29	Insufficient data	3
	Validation of a step of reasoning	8		
	Taken as a generalizable example	9		
	Meaningless justification	5	Meaningless justification	1
	No justification	4		
	Use the result	4		
	Verify each step with values	1		
Total		60		4

Table 8. Justifications provided in response to proposal 3.

The justification given by most of them is that they recognize a well-justified reasoning. For example:
It is a mathematical proof as it demonstrates the equality through mathematical properties and definitions for any real numbers a and b.

There are some students that see this proposal as a generalizable example. One such answer was:

Yes, through this mathematical proof it is possible to generalize.

As before, there are some students that only gave importance to one of the steps of the proof:

It is a proof because it uses the distributive law of the multiplication to adition for real numbers.

Conclusions and Implications

The majority of the students accept that a mathematical result may be proven using a few examples. They tend to evaluate specific cases to ascertain the truth of a result. This finding is corroborated by several researches (Harel & Sowder, 1998). Even though there is some sense that to prove, one needs to generalize, students fail to give a valid argumentation that supports the generalization or they fail to recognize that only giving some examples doesn't necessarily mean that the result is valid. They still think empirically/inductively. This result points to the need of providing opportunities for these students to evolve from this inductive stage.

Some students look at diagrams, in geometry, as particular cases and not as generic as they are intended to be. For them, diagrams seem to represent particular, concrete objects, so they haven't acquired the figural concept (Fischbein, 1993). On the other hand, this result may also be due to the lack of activities during the school trajectory of students that consider different representations of mathematical proof in the results. (Hanna, 2000)

Several students fail to see the need for proof since they simply use the result as a justification. This apparent lack of curiosity for why such result is true seems to indicate that students consider that proof is something irrelevant, meaningless that they just need to memorize.

Almost all students recognize the formal aspect of an algebraic proof. They seem to pay more attention to the formal aspect of a proof than to its' correctness.

These results point to the necessity of rethinking the way proof is approached in the initial teacher training courses. The training of these future teachers should emphasize the understanding and the process of proving rather than the memorization and replication of proofs. Future teachers should appreciate the role of proof and be challenged through appropriate tasks to evolve from the inductive stage. They also should be given opportunities to select and use various types of reasoning and methods of proof.

REFERENCES

Bogdan, R., & Biklen, S. (1994). *Investigação Qualitativa em Educação.* Porto: Porto Editora.

Bussi, M. (2009). Proof and proving in primary school: An experimental approach. In *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education* (Vol. 1, pp. 53-58).

Davis, P. J., & Hersh, R. (1995). *A experiência matemática*. Lisboa: Gradiva.

De Villiers, M. (2003). *Rethinking proof with The Geometer´s Sketchpad*. Emeryville: Key Curriculum Press.

Dreyfus T. (2000). Some views on proofs by teachers and mathematicians. In A. Gagatsis (Ed.), *Proceedings of the 2nd Mediterranean Conference on Mathematics Education* (Vol. 1, pp. 11-25). Nikosia. Cyprus: The University of Cyprus.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in*

Mathematics, 24(2), 139-162.

Hanna, G. (2000). Proof and its classroom role: a survey. In *Atas do Encontro de Investigação em Educação Matemática-IX EIEM* (pp. 75-104). Fundão: SPCESEM.

Harel, G. & Sowder, L. (1998). Students' proof schemes: results from exploratory studies. *CBMS Issues in Mathematics Education*, 7, 234-283.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, 24, 389-399.

Knuth, E. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, 5, 61–88.

Ministério da Educação (ME) (2007). *Programa de Matemática do Ensino Básico*. Lisboa: Ministério da Educação.

Ministério da Educação e Ciência (MEC) (2013). *Programa de Matemática do Ensino Básico*. Lisboa: Ministério da Educação e Ciência.

Stylianides, A. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education,* 38(3), 289-321.

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Mathematics teaching and digital technologies: a challenge to the teacher's everyday school life

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Abstract: This article aims to discuss and analyze the support to be offered to mathematics teachers in continuous formative processes, so as to help them develop the type of knowledge involved in the integration of Digital Information and Communication Technologies (DICT) in their classroom practice. The central idea is to discuss the theoretical frame in which digital technologies are acquired so as to enable the construction and reconstruction of knowledge represented in TPACK, which consists of the intersection of three types of knowledge - specific content, pedagogical and technological - and their connections, interactions and restraints. Based on theoretical references and on results from various studies, we highlight one event that involved the planning and the application of a teaching practice activity using digital technology. We came to the conclusion that it is not simple for the teacher to integrate the available digital technologies to mathematics teaching in order to help the student learn to think using digital technology and "do math" in the classroom. The latter requires him to have new learnings and reconstruct knowledge in a learning process throughout his life, in the professional development perspective.

Key words: Educational Technology, Mathematics Teaching, Professional knowledge, DICT, TPACK

Résumé: Cet article vise à discuter et analyser le soutien à offrir aux professeurs de mathématiques en un processus de formation continue, de manière à les aider à développer le type de connaissances impliquées dans l'intégration des technologies de l'information numérique et (DICT) dans la pratique en salle de classe de l'école secondaire de mathématiques. L'idée centrale est de discuter le cadre théorique dans lequel les technologies numériques sont acquises afin de permettre la construction et la reconstruction de la connaissance représentée dans TPACK, qui se compose de l'intersection de trois types de connaissances - un contenu spécifique, pédagogique et technologique - et de leurs connexions , les interactions et les contraintes. Basés sur des références théoriques et sur les résultats de diverses études, nous soulignons un événement qui implique la planification et l'application d'une activité pratique de l'enseignement en utilisant la technologie numérique. Nous sommes venus à la conclusion qu'il est pas simple pour l'enseignant d'intégrer les technologies numériques disponibles pour l'enseignement des mathématiques afin d'aider l'élève à apprendre à penser en utilisant la technologie numérique et "faire des mathématiques» dans la salle de classe. Ce dernier lui demande d'avoir des nouveaux apprentissages et de reconstruire des connaissances dans un processus d'apprentissage tout au long de sa vie, dans la perspective de développement professionnel.

Mots clés: Technologie Éducative, l'enseignement des mathématiques, connaissances professionnelles, DICT, TPACK

Introduction

The evolution of Digital Information and Communication Technologies (DICT) has stimulated researchers from different areas of knowledge to understand the potentialities of using technological resources in teaching and learning processes. However, many studies confirmed the difficulty experienced by teachers to make pedagogical use of computer resources integrated to syllabus contents (BORBA & PENTEADO, 2001; PRADO, 2005). Such difficulty stems from the fact that teachers already have an ingrained practice that has been built and consolidated, generally, without using DICTs. When these teachers are faced with multiple digital technological resources available in his school routine, they experience uncertainties, questioning and doubts; in other words, feelings that range from denial to desire to learn how to use DICTs in their classroom practice.

This learning requires that teachers appropriate DICT pedagogically not only by inserting them in their classroom, but also by integrating them to the syllabus and adequately exploring their potentialities regarding teaching and learning. Technological appropriation geared toward school teaching requires a process of knowledge construction and reconstruction. Moreover, other studies about technological appropriation conducted by Richt (2010), Vieira (2013), Prado & Lobo da Costa (2013) corroborate the existence of this gradual process, which the authors also point out as being laden by emotional factors. The appropriation depends on how the teacher deals with challenges and, as an adult professional, is willing to learn and reconstruct his knowledge to use technology in his teaching practice.

Artigue (2000), reflecting on the use of digital technology in the mathematics classroom, reminds us that the teacher must be aware of its double-function role in the teaching process. One function is pragmatic, which contributes to the production of answers, and the other is epistemic, which helps understand the mathematics objects involved in the process. It is from such awareness that the teacher can explore both functions of the digital technology use - pragmatic and epistemic.

We understand that in the process of technological appropriation and of the use of digital technology to teach, the teacher needs to construct other frames, which require re elaboration and reconstruction of knowledge, not only technological and/or mathematical, but also pedagogical knowledge in an integrated perspective.

If digital technology is used to teach math in a manner that enables the learner to build concepts, it is necessary that this technology, when used, provide conditions for the student to raise, test and manifest his conjectures, giving support to structuring thought in a display of "thinking with" and "thinking about thinking", as proposed by Papert (1980), towards problem solving and understanding concepts. Providing this kind of support implies knowing the specific characteristics of the chosen digital technology, whether it is software, simulators, learning objects, programing languages, or other kinds that need to be connected to the specific fields of mathematics. For instance, a dynamic geometry software such as *Cabri-géomètre*, *Wingeon* and others, can be suitable for the teaching of geometry, but not necessarily be the best choices to teach statistics, for example. For the math teacher there certainly is the need to know, for each mathematics field, the possibilities and limitations of available educational software. In order to explore didactic potentialities it is necessary to know the software structure (whether it uses programming, or it is iconic, or allows macros, or uses defined mathematical objects - such as triangle, square, circumference etc.), so as to create activities and develop pedagogical strategies that can lead the student to experience the founding ideas of mathematics and rich situations for learning, ensuring that such situations enable the learner to construct knowledge.

However, even with a variety of specific software for mathematics teaching, the teacher mediation is an essential aspect. Hence, it is increasingly more necessary to be concerned about how to support teachers in the continuous formative processes to develop suitable knowledge and competences in digital technologies, in order to improve their mathematical teaching.

Several researchers has conducted researches about theoretical aspects related to technology integration in teachers' formative and professional development. Among these researchers, one of the highlights have been on the ideas of Mishra and Koehler who created a model with three intercepting sets representing the knowledge base required for teaching: content - in our case, mathematics - pedagogy and technology.

The model was named TPACK[4](*Technological Pedagogical Content Knowledge*) and it is a theoretical structure aimed to help understand the nature of knowledge fields that are retrieved by teachers in their practice. In the intersection shared by the three fields is the pedagogical technological knowledge of content.

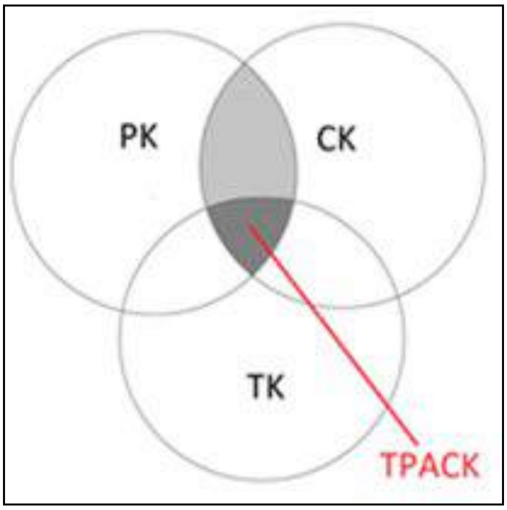


Figure 1: TPACK model
Source: Adapted from Mishra & Koehler (2006)

TPACK, which is in the intersection of three fields: Pedagogical Knowledge – PK, Content Knowledge – CK and Technological Knowledge – TK, symbolizes a blend that stands as an emerging form of knowledge, which goes beyond all its components (content - in our case, mathematics - pedagogy and technology). This is the kind of knowledge to be retrieved to teach with technology (KOEHLER & MISHRA, 2009).

How could we give support to teachers to help them construct TPACK?

In search of the answer to this question, we have conducted researches and we need theories to provide us support. We sought to understand the TPACK construction process by the teacher and, chiefly in this article; we focused our reflection in studies about Rabardel's instrumented activity, which specifically encompasses technology.

Artigue (2000) reminds us not to underestimate the complexity of instrumentation process (instrument adaptation by the user for specific purposes) and the instrumented activity (how the instrument models the user's strategies and knowledge) by teachers. She also emphasizes Rabardel's (1995) idea of instrumental genesis, that is, the process from which an artifact (the object - that is, a map, some software, a computer, a tablet etc.) becomes an instrument for the individual. When an individual starts to use an artifact, he constructs his own schemes of utilization and, by doing so, develops his mental schemes.

Technological appropriation is not simple, and from the TPACK's perspective, such knowledge cannot be seen in an isolation way. The integrated understanding of *pedagogical, technological and content* knowledge has been a great challenge both *in* and *for* teacher development, as they require new knowledge construction.

Considerations based on researches

From this theoretical discussion, we present what we have learned from our experience and

[4]Initially, the authors adopted the acronym TPCK and later renamed it as TPACK - pronounced "tee-pack", so it would sound like a "total package" (*total package*, that is, an integrated amalgam of the three kinds of knowledge: technological, pedagogical and content, which then produce a new kind of knowledge.

researches about what can help and/or hinder the TPACK development by teachers. In the researches mentioned here, the teachers participated in continued education development processes which all focused on the use and/or integration of technology in the teaching of mathematics. The research projects were supervised by each of us, and include: Muraca (2011), Castro (2011) and Vieira (2013), which provided support for our conclusions. All of them share at least one common aspect which is the need for the teacher to develop a specific kind of integrating-based knowledge defined as "pedagogical technological content knowledge" (TPACK) in a process of instrumented activity and technology appropriation to teach, in our case, mathematics.

In this article, we bring detailed considerations about the support to be offered to teachers based on aspects of the study conducted by Vieira (2013), who researched the professional knowledge of elementary school teachers and the appropriation of digital technology for geometry teaching. The process of digital technology appropriation by the teachers of the same school for teaching spatial and plane geometry and where the professional knowledge construction, in particular TPACK, was studied. Data collection was conducted through direct observation, audio and video recording of meetings, and logs of the teachers' production. The teachers had no experience with the use of software for teaching geometry and were involved in the research project during one school term, learning and using *SketchUp*[5].

The software *SketchUp* is free and makes it possible to create 3D models and post them and also to import shapes from the Internet for its interface. This software has not been conceived for specific educational purposes, although it can help students develop their spatial perception and, mainly, lead them to assimilate knowledge related to the spatial representation of shapes on a two-dimensional screen. In Figure 2 there is an example of a project developed within the research project using the application.

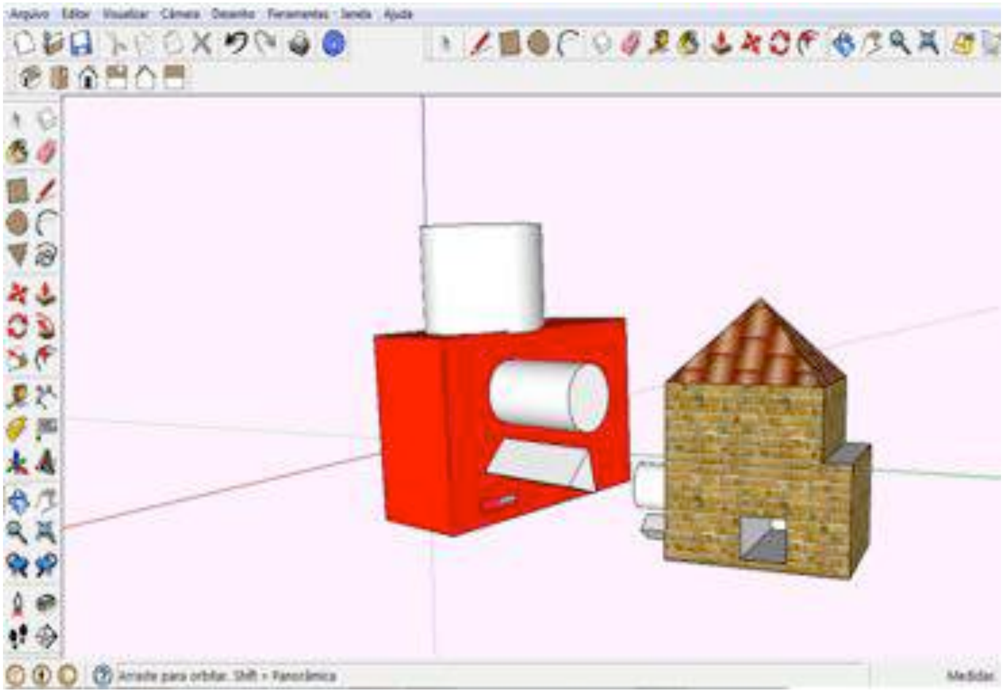


Figure 2: Screen showing spatial shapes created using *SketchUp*
Source: Vieira, 2013, p. 242.

At the geometry class, the use of this software enables students to explore geometric elements such as points, lines, planes, angles, parallel lines, perpendicular lines, plane shapes and geometric solids.

[5] *SketchUp* available for download at: <http://www.sketchup.com/intl/pt-BR/download/gsu.html>. Last accessed: February 28, 2015.

One of the software tools called Orbit allows for adding animation to created shapes to be viewed from different viewpoints. According to Vieira (2013) "*The student is able to change the spatial reference system, choose the perspective and change the viewpoint to observe and explore the spatial geometric shapes, hence achieving a better three-dimensional view of the object*" (p.75). Figure 2 brings the representation of a pentagonal prism and some viewings available when the solid is animated using the Orbit tool.

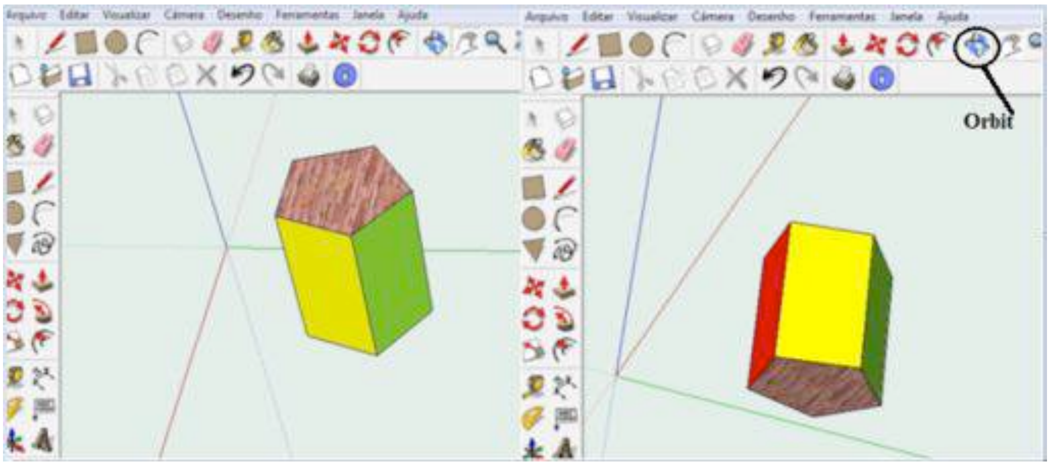


Figure 3: Two views of a pentagonal prism using the Orbit tool
Source: Adapted from Vieira (2013) p.77.

Next, we will discuss an episode that exemplify **the process of mobilization of knowledge** by one of the participants of the formative process (Teacher A). This involved the planning and development of classroom activities with students aged between 7 and 10 with Teacher-A.

In the lesson planning session, Teacher-A made decisions about the content - determining the type of solid to be explored, which software tools to use in the constructions and which pedagogical approach and strategy to apply. During the classroom activity planning section for Teacher-A's, dialogs like the one that follows were present, specifically when choosing between approaching prisms or pyramids, besides the teaching of terminology to be introduced to the students and the way to use the software tools in class.

Researcher: *Let's create a pentagonal prism. How?* Teacher-A: *Click on number 7.* Researcher: *Which faces?* Teacher-A: *Two five-sided ones. Two pentagons and five rectangles.* Teacher-A: *I think it's difficult for a child, because they have to know the number of edges and the type of face.* Researcher: *Use transparent!* [a software tool] ***Teacher-A:*** *How interesting! In the transparent mode I can see the vertices and the edges.*

Table 1: Dialogs among Researcher and Teacher-A

In the dialog above it is possible to notice reflections that point towards the construction/mobilization of technological, pedagogical and geometrical knowledge by Teacher-A. There is one attempt made by the researcher to give support to Teacher-A for TPACK construction, which is the kind of knowledge that supports decisions regarding the exploration of concepts in class.

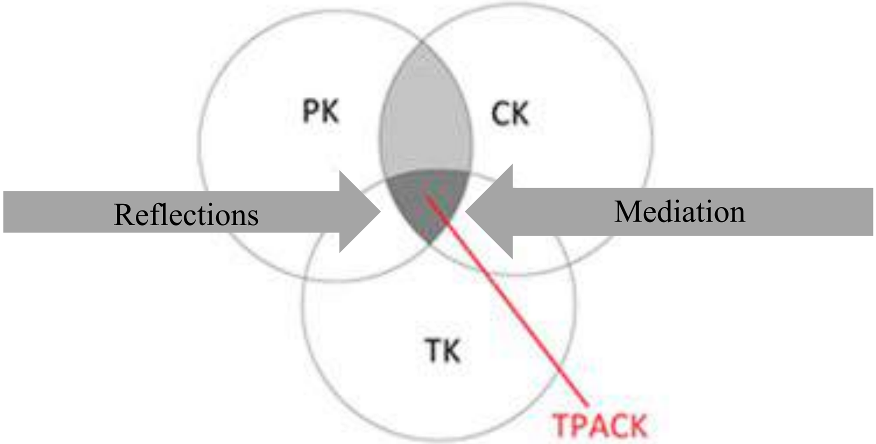
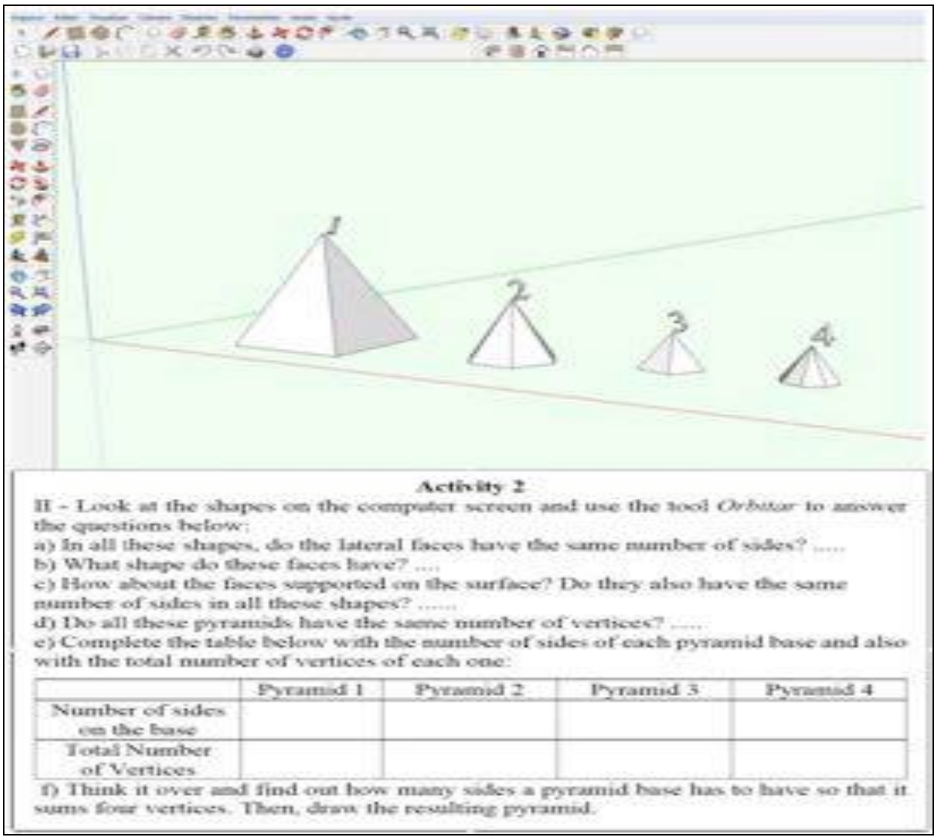


Figure 4: Mediations and Reflections

The option to plan the activity for her class was to explore pyramids (instead of prisms). There was a discussion about the possibility of using the software to teach students to investigate the shapes to identify differences and similarities between the pyramids, learn the terminology to classify the pyramid by the base polygon, identify the type of polygon that composes the lateral face of a pyramid and realize the relation between the number of sides on the base polygon (edges of the pyramid), the number of vertices and of lateral edges. It was also decided that wooden models of pyramids would be used together with their corresponding virtual representation through the software.

The planning resulted in a classroom activity in which the students should explore a *SketchUp* file containing various pyramids (see Figure 5) and then answer the questions in a protocol (Activity 2), presented in the same figure.



Activity 2

II - Look at the shapes on the computer screen and use the tool *Orbitar* to answer the questions below:

a) In all these shapes, do the lateral faces have the same number of sides? .....

b) What shape do these faces have? ....

c) How about the faces supported on the surface? Do they also have the same number of sides in all these shapes? ......

d) Do all these pyramids have the same number of vertices? .....

e) Complete the table below with the number of sides of each pyramid base and also with the total number of vertices of each one:

	Pyramid 1	Pyramid 2	Pyramid 3	Pyramid 4
Number of sides on the base				
Total Number of Vertices				

f) Think it over and find out how many sides a pyramid base has to have so that it sums four vertices. Then, draw the resulting pyramid.

Figure 4: File with pyramids and protocol of the proposed activity

Source: Adapted from Vieira (2013) p.173.

This activity was applied when the students were familiarized with the software and its basic tools (they already have learned about its basic tools, which involved the construction and exploration of various plane shapes and solids). The proposal was presented to explore the four pyramids in the file, analyze them and fill out the protocol.

The excerpt below was taken from Teacher-A classroom dialog during her class.

Student: *How many are there on the base?* [referring to the edges on the base] Teacher-A: *You have to count them.* Student: *The sides?* Teacher-A: *Orbit around it so you can see and count. In pyramid 1 the base has 4 edges and 4 vertices, with the top one it makes 5, doesn't it? Pyramid 2 has 6 on the base and with the top one, 7. Look, you have to keep on counting.* Teacher-A: *Now, tell me, how many does pyramid 3 have on its base? And the total?* Student: *5 and 6.* Teacher-A: *How about pyramid 4?* Student: *8 and 9.* Teacher-A: *What will be the pyramid base with a total of 4 vertices? Consider using the Polygon tool. What do you have to enter?* Student: *Three.*

Table 2: Dialogs among Student and Teacher-A

Teacher-A instigates her student with questions to stimulate and guide him in the exploration and investigation of the proposal situation. It can be seen that Teacher-A mobilizes pedagogical, technological and content knowledge to guide the student's thinking.

She chooses not correct imperfect nomenclature used by the student in order to focus on what considered essential in the activity and was talking with the student and urging him to explore the figures, investigate and reason with the software tools

The discussions and reflections about geometric solid properties that happened between Teacher-A and the researcher during the joint lesson planning were key for her guidance, and her experience as a learner, when she developed activities in the software. The construction and reconstruction of concepts were made possible, as well as the re-signification of the teaching of these concepts.

Conclusion

The challenge of knowing how to use technology to teach mathematics in order to attend the specificities of each mathematical field (numeric, algebraic, geometric), considering the educational context of practice; so as to help the learner to construct syllabus-established mathematical knowledge and achieve goals set by us, researchers, is a challenge that requires the teacher to develop knowledge of the type discussed here as TPACK, which takes place through gradual appropriation and instrumented activities.

To help and provide support to teachers construct TPACK in formative processes, it is necessary to propose situations with technology so that the participants can experience them.

Such situations should have as their starting point the specific content knowledge and involve both pedagogical discussions and the ones referring to implications of the use of technological resources in teaching as a means of structuring thought. The actions into the formative process should prioritize strategies that help the teacher in his process of appropriation of digital technology for the teaching of mathematics, always taking into account that such process is gradual and experience needs to take place and be discussed to reconstruct practices.

Finally, we reiterate that it is not simple for the teacher to integrate the available digital technologies to mathematics teaching in order to help the student learn to think using digital technology and "do math" in the classroom. The process requires, mainly from teachers, that they experience new learnings and knowledge reconstruction in a life-long learning process, under the professional development perspective.

Acknowledgements

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REFERENCES

Artigue, M. (2000). Instrumentation issues and the integration of computer technologies into secondary mathematics teaching. *Proceedings of the Annual Conference on Didactics of Mathematics.* Potsdam. Acesso em 28 de February de 2015.

Borba, M. D. & Penteado, M. G. (2001). *Informática e educação matemática.* Autêntica.

Castro, A. L. (2011). *Tecnologias Digitais de Informação e Comunicação e o Ensino de Funções Quadráticas: contribuições para a compreensão das diferentes representações.* Dissertação de Mestrado, Universidade Bandeirante de São Paulo, São Paulo, Brasil. Available for download at: <http://www.matematicaepraticadocente.net.br/pdf/teses_dissertacoes/dissertacao_AnnaLuisadeCastro_2011_Bette.pdf> Last accessed: February 28, 2015.

Koehler, M. J. & Mishra, P. (2009). What is technological pedagogical content knowledge? *Contemporary Issues in Technology and Teacher Education, 9*. Available for download at:<http://goo.gl/zXfvrr>. Last accessed: February 28, 2015.

Mishra, P. & Koehler, M. (2006). Technological Pedagogical Content Knowledge: A framework for teacher knowledge. *Teachers College Record*, 108(6), 1017-1054.

Muraca, F. (2011). *Educação Continuada do Professor de Matemática: um contexto de problematização desenvolvido por meio de Atividades Exploratório-Investigativas envolvendo Geometria Espacial de Posição.* Dissertação de Mestrado, Universidade Bandeirante de São Paulo, São Paulo, Brasil. Available for download at:<http://www.matematicaepraticadocente.net.br/pdf/teses_dissertacoes/dissertacao_FernandoSpadiniMuraca_2011_Nielce.pdf> Last accessed: February 28, 2015.

Papert, S. (1980). *Mindstorms: Children, Computers, and Powerful Ideas*. New York: Basic Books.

Prado, M. E. (2005). Integração de mídias e a reconstrução da prática pedagógica. In: Almeida, M.E.B. & Moran, J.M. (orgs.). *Integração de Tecnologias, Linguagens e Representações.* Brasília, Brasil: Secretaria de Educação a Distância, Ministérios da Educação. Available for download at:<http://salto.acerp.org.br/saltotvescola/livros.asp> Last accessed: February 28, 2015.

Prado, M. E. & Lobo da Costa, N. M. (2013). O processo de apropriação das TIC e a reconstrução de novas práticas no ensino de Matemática. *Actas del VII CIBEM - Congreso Iberoamericano de Educación Matemática*, *1*, pp. 6973-6970. Montevidéu.

Rabardel, P. (1995). *Les hommes et les technologies une approche cognitive des instruments contemporains.*Paris: Armand Colin.

Vieira, E. R. (2013). *Grupo de Estudos de professores e a apropriação de tecnologia digital no ensino de geometria: caminhos para o conhecimento profissional.* Tese de Doutorado, Universidade Bandeirante de São Paulo, São Paulo-Brasil.

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Pre-service Teachers' Informal Inferential Reasoning

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Abstract: The aim of this research is to explore the informal inferential reasoning (IIR) of pre-service teachers. We pose the question: to which extent do pre-service teachers depend on prior statistics knowledge and on day-to-day knowledge or experience? The proposal of Zieffler et. al. about the RII has been used to guide the theoretical approach of this research. Since results show the absence of statistical concepts in rational decision making, actions carried out in the classroom have to be set up in order to balance the pre-service teachers' statistical and day-to-day knowledge.

Abstract: Le but de la recherche est d'explorer le raisonnement déductif informel (IIR) des enseignants en formation. Nous posons la question: dans quelle mesure les enseignants en formation dépendent des connaissances préalables en statistiques et des connaissances acquises au long de la vie quotidienne? La caractérisation de l'IIR dû à Zieffler et al. a été utilisé pour guider l'approche théorique de cette recherche. Puisque les résultats montrent une absence des concepts statistiques dans la prise des décisions rationnelles, il a fallu mettre en classe une série des actions afin d'équilibrer les connaissances statistiques des futurs enseignants avec les connaissances qu'ils ont acquis au long de sa vie quotidienne.

Introduction

The study of statistics provides tools and ideas to analyze and interpret information; statistical inference enables people to read, understand and interpret conclusions derived from data analysis. The teachers, as part of their daily exercise, need to understand statistical information as charts, averages, and other concepts (Estrada, 2007). Make efficient use of this information is useful when they prepare their classes, or if they are part of a research team. This knowledge coupled with appropriate analysis tools lead them to make decisions in a changing society. Mexican kindergarten teachers (teaching students between 3 and 5 years of age) must be able to collect, organize, present and analyze data to take on problem solving in the educational context, aside from applying those concepts and procedures in research projects (SEP, 2012). However, students and teachers make mistakes for instance, in the concepts of sampling and distribution (Castro-Sotos, et al. 2007). This has led to a greater interest to study IIR, which is reasoning between the exploratory data analysis and the formal statistical inference. This research seeks to explore pre-service teachers' IIR and, based on that, to know the way they reason, the way they become the model to plan the teaching during their training. Zieffler, Garfield, delMas and Reading (2008) ask the question that guides this exploration: how heavily the student depends on prior knowledge of statistics (previously learned concepts) and how much the student depends on his or her knowledge of the world (or experience) (p.53).

Background and conceptual framework

Several publications in recent years deal with IIR, but there is no agreement on its meaning.

Pfannkuch (2006) described the term informal inference as the drawing of conclusions from data that is based mainly on looking at, comparing, and reasoning from distributions of data in an empirical enquiry cycle. Ben-Zvi (2006) compares inferential reasoning to argumentation, and emphasizes the need for this type of reasoning to include data-based evidence. Zieffler et al. (2008) in an attempt to combine these perspectives, define IIR as the way in which students use their informal statistical knowledge to make arguments to support inferences about unknown populations based on observed samples. In this exploration, IIR is used to describe the way in which pre-service teachers formulate conclusions.

A type of tasks that allows to studying the IIR and the arguments of the students are the comparisons of data sets, which have been used by different authors (Gal, I., Rothschild, K. & Wagner, D.A. 1989; Watzon & Moritz, 1999). Garfield and Ben-Zvi (2008) indicate the following advantages of comparing data sets: this activity can be structured as an informal and early version of statistical inference, problems that involve group comparisons are often more interesting, students of any level require develop strategies to compare data sets, motivates the need for and use of data graphical representations and get summaries (center and dispersion) of the data. Comparing groups is an activity that meets the components needed to support research on IIR: Make judgments, claims, or predictions. Draw on, utilize, and integrate prior knowledge (formal and informal) to the extent that this knowledge is available and articulate evidence-based arguments for judgments, claims, and predictions based on evidence (Zieffler et al. 2008).

The Context is another basic element in IIR it provides the uncertainty or doubt that evokes the need for inquiry and triggers suggestions from which an inquiry initiates, is central to giving meaning to an inference and provides an opportunity to evaluate the meaning of an inference (Makar, Bakker & Ben-Zvi, 2011). In this research an adaptation of the risk context proposed by Kahneman and Tversky (1984) on accepting a gambling game is used. This type of context engages students to solve the problem and encourages them to justify their answers (Orta & Sanchez, 2014). Kahneman and Tversky point out that "the paradigmatic example of decision under risk is the acceptability of a gamble that yields monetary outcomes with specified probabilities" (p. 341). Consider the following problem: The gains of realizations of n times the game A and m the game B are:

Game A: $x_1, x_2 \dots ; x_n$

Game B: $y_1, y_2 \dots ; y_m$

Which of the two games would you choose to play in?

The solution is reached by following a flow diagram: 1) Compare $\bar{x}$ and $\bar{y}$, 2) if $\bar{x} \neq \bar{y}$ then chose the Game whose mean is the greatest; 3) if $\bar{x} = \bar{y}$ then there are two options: 3a) Choose any game, 3b) Analyze the dispersion of data in each game and choose according to risk preferences. These preferences can be defined as generalizations of the attitudes to reject or seek the risk identified by psychologists:

> The preference for a sure outcome over a gamble that has higher or equal expectation is called risk averse, and the rejection of a sure thing in favor of a gamble of lower or equal expectation is called risk seeking (Kahneman & Tversky, 1984, p. 341).

It is worth noting that in a game, the dispersion of winnings (including losses) can be considered a measure of risk. Let's say that a preference is motivated by risk aversion when an option whose data have less dispersion over another whose data have greater dispersion is preferred. The decision is motivated by risk seeking when the option whose data have greater dispersion is chosen.

Methodology

The participants in this research were 63 kinder garden pre-service teachers, from a public school in

Mexico City. The problem shown in Figure 1 was used to know the future teachers' ideas and it was solved before initiating the course of statistical information processing (SEP, 2012) in a time of 60 minutes. The pre-service teachers´ arguments were analyzed starting from key words: *losses* and *winnings*. Subsequently the solution strategies were differentiated *win more* (by comparing the sum of the winnings of each game or the ratio between winnings and losses of each game), *lose less* (by comparing the sum of the losses of each game), and *win the same in both games* (by comparing the sum of all quantities of each game). Finally, three analysis categories of comparison were stablished: *comparison of informal mean*, *comparison of the ratio between winnings and losses*, and *comparison between losses or winnings*.

Analysis and results discussion

The analysis of the responses of the pre-service teachers began differentiating them according to the chosen game and, later, they were categorized based on the strategy of comparison with which the future teachers argued their election. Thirty-six pre-service teachers chose game 1, 21 game 2, and six answered any. In the following section there appear examples of the strategies of comparison used by the teachers in training.

> In a fair, the attendees are invited to participate in one of two games, but not in both. In order to know which game to play, John observes, takes note and sorts the results of 10 people playing each game. The losses (-) or cash prizes (+) obtained by the 20 people are shown in the following lists:
>
> Game 1: 15 -21 -4 50 -2 11 13 -25 16 -4
>
> Game 2: 120 -120 60 -24 -21 133 -81 96 -132 18
>
> a) If you had the possibility of playing only one of the two games, which one would you choose? Why?

Figure 1. Problem.

Comparison of informal mean

Six pre-service teachers justified their answer arguing the winnings or initial investment were the same (49), which resulted from adding all the numbers in each game (comparison of informal mean). Figure 2 shows an example in which a future professor replied that she would choose either of the two games.

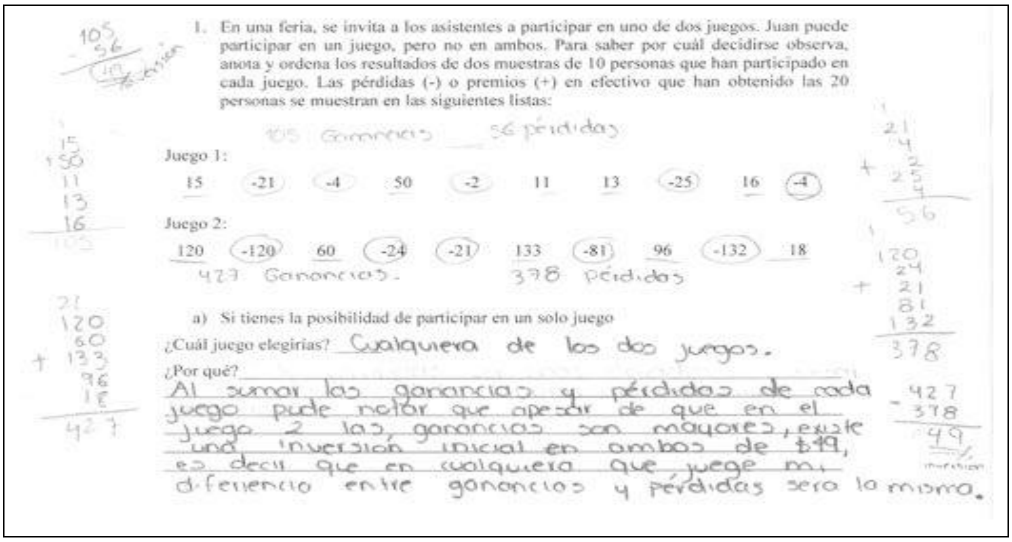

Figure 2. Comparison of informal mean.

It is possible to observe that the pre-service teacher added the winnings and the losses of every game, later she calculated the difference between they and got 49. The justification for her choice was: *Adding up the winnings and losses of each game I noticed that despite the fact that in the game 2, the winnings are greatest, there is an initial investment in both of $49, that is to say that any game who I play my difference between winnings and losses will be the same*. In these cases the future teachers perhaps not considered the risk involved.

Comparison of the ratio between winnings and losses

The answers chosen in game 1 are divided in two types: in the first one, the choice argument is based on the comparison of the ratio between the winnings and the losses of the games (25 answers), choosing game 1 since one could win almost twice as much one could lose as opposed to the losses in game 2 $\left(\frac{105}{56} > \frac{427}{378}\right)$. An instance of the justifications of this type was like the following one: *Because of the data reflects that in this game there are more likely to going out winner since the number winners almost duplicates that of losers and although it was less quantity the gained that the game 2, in the 1 it is safer to win even little in game 2 I would not play because although earn larger amounts of similarly lost much* (see Figure 3). In this example it is observed, on the one hand the use of the ratio between the winnings and the losses and, on the other the risk aversion since part of the argument says that *it is safer to win even little*.

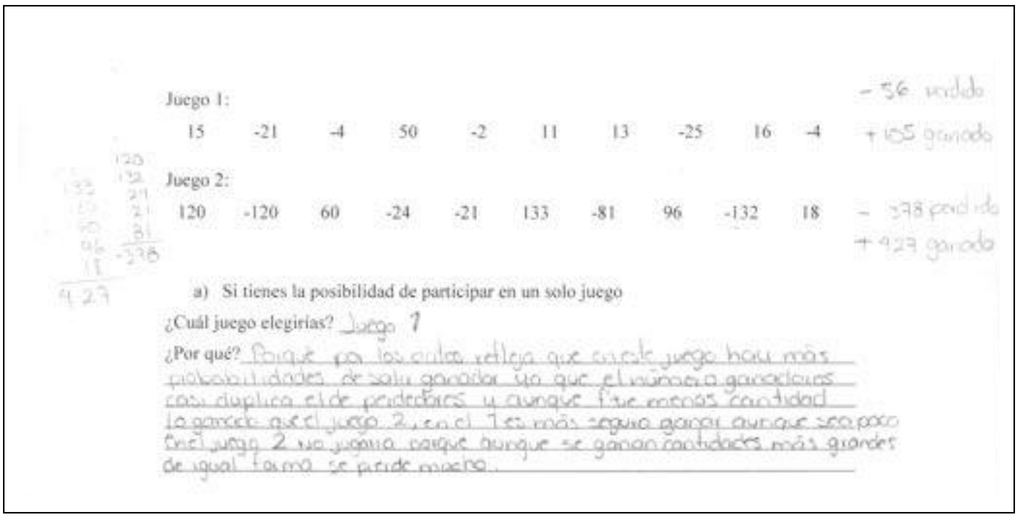

Figure 3. Comparison of the ratios.

Comparison between losses or winnings

In the second type, out of the answers chosen in game 1, the arguments were based in the comparison of the sum of losses in games 1 and 2 (6 answers).

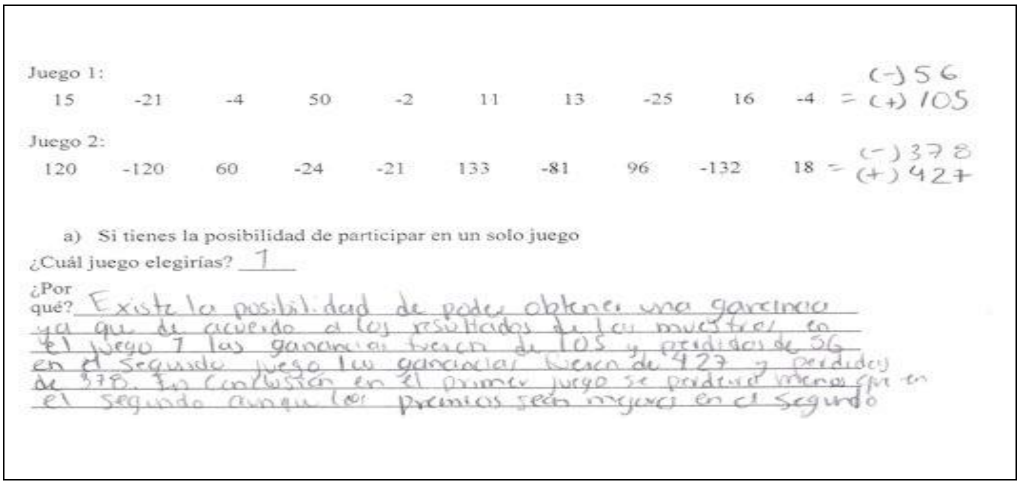

Figure 4. Comparison of the sum of losses.

The losses in game 1 were fewer (56 < 378), an instance of this strategy is the next one : *There is the possibility to obtain winnings according to the results of the samples in the game 1 winnings were 105 and the losses 56 in the second game winnings were 427 and the losses 378. In conclusion in the first game it will get lost less than in the second one although the winnings are better in the second one* (see Figure 4). In this response is perceived risk aversion, since what is claimed is losing less although in other game the winnings are better.

The answers in which game 2 was picked (20 answers) were based on the comparison of the sum of winnings (427 > 105). Figure 5 shows an example in which the sum of the winnings of each game was compared. The chosen game was number 2 and the justification was: *Invested more and apparently lost more but in equivalence to the 1 win more*. In this example in addition to the use of the maximum winnings to decide between a game and another, the risk seeking is observed since at the end of the justification one comments *I win more*.

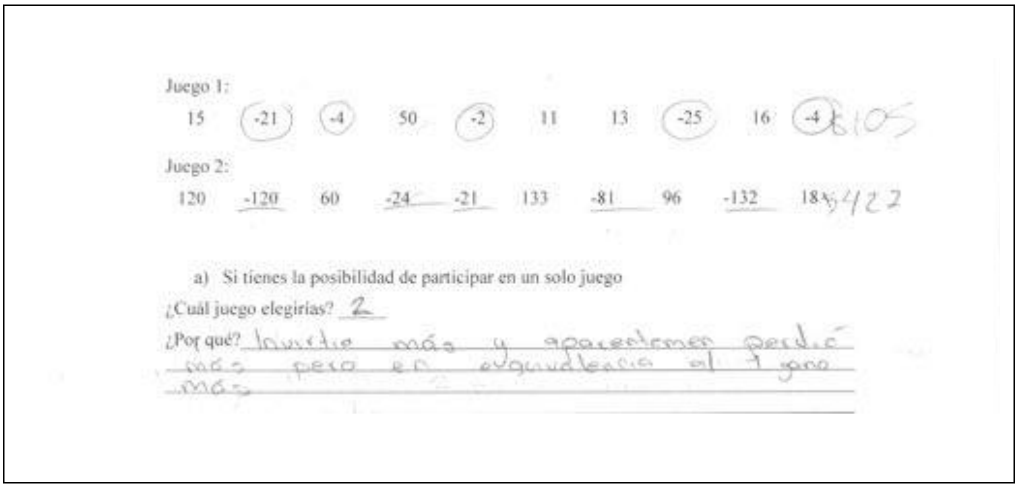

Figure 5. Comparison of the sum of winnings.

In regard to the arguments of the future teachers the initial strategy was to add the winnings and losses of each of the games and on the basis of these results substantiate their answer. We see from these results that the pre-service teachers do not use formal statistical knowledge as measures of central tendency and dispersion to justify their choice; they depend on day-to-day/informal knowledge: additions, subtractions, and ratios. The absence or little use of statistical concepts, such as the arithmetic mean has been reported in other studies of comparisons of data sets (Gal et al. 1989, Watson & Moritz, 1999). Only some pre-service teachers used an *informal arithmetic mean*. In the answers in which game 1 was chosen, we can sense that the answer revolved around an aversion to risk or a refusal to participate in game 2 since the latter had the highest losses. In the answers in which game 2 was chosen, we see that the answer is given by a tendency towards risk since there are more winnings. The future teachers’ solving strategies may be classified as follows: answers based on the comparison between the sum of winnings or losses; answers that choose any game comparing the sum of winnings and losses (informal arithmetic mean); and answers that compare the ratio between winnings and losses in both games. Although the pre-service teachers realize judgments and articulate them on the basis of the evidence that they have, they do not use statistical formal knowledge like measures of center and dispersion. The context proposed considerably lowers the excessive use of prejudices and beliefs when making inferences compared with García-Rios (2013) in which high school students based their answers on their beliefs and in their personal knowledge about the context and not in the data of the problem. It is necessary to stress the reflection upon the use and meaning of statistical concepts during pre-service teachers training because if they have solid and structured statistical tools, they will have a better performance as individuals and in the classroom.

REFERENCES

Ben-Zvi, D. (2006). Scaffolding students’ informal inference and argumentation. In A. Rossman & B. Chance (Eds.), *Working cooperatively in statistics education: Proceedings of the Seventh International Conference on Teaching Statistics*, Salvador, Brazil. Voorburg, The Netherlands: International Statistical Institute. Retrieved from https://www.stat.auckland.ac.nz/~iase/publications/17/2D1_BENZ.pdf

Castro Sotos, A. E., Vanhoof, S., Van den Noortgate, W., & Onghena, P. (2007). Students' misconceptions of statistical inference: A review of the empirical evidence from research on statistics education. Educational Research Review, 2(2), 98-113. doi:10.1016/j.edurev.2007.04.001

Estrada, A. (2007). Evaluación del conocimiento estadístico en la formación del profesorado. Uno. Revista de Didáctica de las Matemáticas, 45, 78-97. Retrieved from http://web.udl.cat/usuaris/z4084849/docs/uno45.pdf

Gal, I., Rothschild, K. & Wagner, D.A. (1989). Which group is better? The development of statistical reasoning in elementary school children. Paper presented at the meeting of the Society for Research in Child Development, Kansas City, MO.

Garcia-Rios, N. (2013). Inferencias estadísticas informales en estudiantes mexicanos. In J. M. Contreras, G. R. Cañadas, M. M. Gea & P. Arteaga (Eds.), Actas de las Jornadas Virtuales en Didáctica de la Estadística, Probabilidad y Combinatoria (pp. 343-357). Granada, Departamento de Didáctica de la Matemática de la Universidad de Granada. Retrieved from http://www.jvdiesproyco.es/documentos/ACTAS/2%20Comunicacion%2035.pdf

Garfield, J. & Ben-Zvi, D. (2008). Developing Students’ Statistical Reasoning: Connecting Research and teaching. New York: Springer.

Kahneman, D. & Tversky, A. (1984). Choices, values, and frames. American Psychologist 39 (4), 341-350. Retrieved from http://web.missouri.edu/~segerti/capstone/choicesvalues.pdf

Makar, K., Bakker, A., & Ben-Zvi, D. (2011). The reasoning behind informal statistical inference. Mathematical Thinking and Learning, 13, 152–173. doi:10.1080/10986065.2011.538301

Orta, J. A. & Sánchez, E. A. (2014). Interpreting variation of data in risk-context by middle school students. In K. Makar, B. de Sousa, & R. Gould (Eds.), Sustainability in statistics education: Proceedings of the Ninth International Conference on Teaching Statistics, Flagstaff, Arizona, USA. Retrieved from http://iase-web.org/icots/9/proceedings/pdfs/ICOTS9_7F3_ORTA.pdf

Pfannkuch, M. (2006). Informal inferential reasoning. In A. Rossman & B. Chance (Eds.), Working cooperatively in statistics education: Proceedings of the Seventh International Conference on Teaching Statistics, Salvador, Brazil. Voorburg, The Netherlands: International Statistical Institute. Retrieved from https://www.stat.auckland.ac.nz/~iase/publications/17/6A2_PFAN.pdf

SEP. (2012). Procesamiento de Información Estadística. México: Secretaría de Educación Pública. Retrieved from http://www.dgespe.sep.gob.mx/public/rc/programas/lepree/procesamiento_de_informacion_estadistica_lepree.pdf

Watson, J. M. & Moritz, J. B. (1999). The beginning of statistical inference: comparing two data sets. Educational Studies in Mathematics, 37 (2), 145-168. doi:10.1023/A:1003594832397

Zieffler, A., Garfield, J. delMas, R. & Reading, C. (2008). A framework to support research on informal inferential reasoning. Statistics Education Research Journal. 7(2), 40-58. Retrieved from http://iase-web.org/documents/SERJ/SERJ7%282%29_Zieffler.pdf

A study about the knowledge required from teachers to teach probability notions in early school years

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Abstract : The goal of this article is to present a study about Brazilian teachers' knowledge on probability for teaching in the elementary school early years. This research was developed within the scope of a continued education course of the Education Observatory - a project on research and development by UNIAN/CAPES. Data presented here refer to the first phase of the research classified as *Diagnostic*. The theoretical basis to analyze content retention by the teachers is Tall and Vinner's notion of conceptual image. Regarding the knowledge that teachers should master, we chose the categories established by Ball, Thames and Phelps such as knowledge of core/specific content, knowledge of content and of students, knowledge of content and of teaching. The answers given by the teachers in the diagnostic tool revealed inconsistent conceptions about probability and its teaching. This finding was used as a starting point for the development process throughout the project's second phase.
Key words: Teaching Probability; Early Education Teacher Development; Mathematical Knowledge for Teaching.

Résumé : L'objectif de ce travail est de présenter une étude sur les connaissances des professeurs brésiliens pour enseigner la probabilité dans les premières années de l'école primaire. Cette investigation a été développée au sein d'un cours de formation continue de l'Observatoire de l'Éducation – un projet de formation et de recherche de l'UNIAN/CAPES. Les données discutées ici se rapportent à la première phase de la recherche appelée *Diagnostique*. À propos de la base théorique, en ce qui concerne la compréhension d'un contenu par les professeurs, on a utilisé la notion d'image conceptuelle, selon Tall et Vinner. Relativement aux connaissances qui doivent être maîtrisées par les professeurs, on a considéré les catégories établies par Ball, Thames et Phelps, telles que: connaissance du contenu commun/spécialisé, connaissance du contenu et de l'étudiant et connaissance du contenu et de l'enseignement. Les réponses des professeurs à l'instrument diagnostique ont révélé des conceptions inconsistantes à propos de la probabilité et de son enseignement, constituant ainsi un point de départ pour le processus de formation, au long de la deuxième phase.

Mots-clés: Enseignement de Probabilité; Formation de Professeurs dans les deux premières années; Connaissance de Mathématiques pour l'Enseignement

Introduction

This presentation refers to a research whose goal was to look into the knowledge required from teachers to teach probability in the early school years of elementary school. This research was developed within the scope of a continued education course of the Education Observatory - a project on research and development by UNIAN/CAPES - that involved 27 teachers at the public school network of the State of São Paulo, Brazil. The participants had a teaching degree in mathematics and attended the course voluntarily.
Data initially discussed in this presentation refer to the phase called Diagnostic, which consisted of

questionnaires and interviews used to identify the knowledge teachers had about probability and their conceptions regarding its teaching.

The second phase named *Development* - which will not be discussed here - was conducted according to the principles of the Design Experiments Methodology. This development process had as its assumption that a sequence of activities - initially exploring the notion of randomness followed by the notion of sample space, and then, by quantification of probabilities - enhances teacher knowledge improvement and/or reconstruction in relation to probability.

As for the theoretical basis regarding content retention by the teachers, we used the notion of conceptual image by Tall and Vinner. These authors consider the conceptual image notion as the cognitive structure that develops within a person's mind through rich experiences and studies about a particular mathematical concept. Such image involves impressions, visual representations, examples, applications, and verbal descriptions about the properties and processes of a given concept. In relation to the knowledge that teachers should master, we chose the categories established by Ball, Thames and Phelps such as knowledge of core/specific content, knowledge of content and of students, knowledge of content and of teaching. The authors focused specifically on the manner by which the teachers need to know a certain content in order to teach it, and also "whatever else teachers need to know about mathematics and how and where teachers could use such knowledge in practical terms" (Ball et al., 2008, p.4), on top of the pedagogical knowledge of content and knowledge of syllabus. Hence, the focus of studies developed by Ball et al. (2008) is on the teaching job, that is, about what teachers do when they teach mathematics and about their perceptions, understanding and mathematical thinking required for such job.

The answers given by the teachers in the diagnostic tool revealed inconsistent conceptions about probability and its teaching. This finding was used as a starting point for the development process throughout the project's second phase. In this process, definitions of probability from the geometrical and frequency viewpoints, and also its classical definition, were used for the study of probability and reflections about its teaching.

Teachers Knowledge

We used a questionnaire with 13 questions, assuming that the conceptual image would be composed by, for instance, identification of random phenomena; understanding of the different probability definitions and their respective limitations; meaning and quantification of sample spaces; probability quantification; relations between variables in double-entry tables; connections with different contents; different strategies for approaches; difficulties inherent to the process of construction of this specific knowledge.

Below, we present our analysis of some data that allowed for the outlining of the conceptual image that constituted, at that moment, the knowledge repertoire about the meaning of probability, sample space and probability quantification.

In relation to the question "how would you define probability? (Use your own words)", 20 teachers wrote a definition that can be associated to the classic definition, as evidenced in the excerpt below:

> Probability is written with two numbers; the first shows the total number of possible outcomes, and the second, the number of outcomes we expect. (Teacher 16)
>
> The probability of a result in a game of chance is a fraction: the numerator is the number of cases we want to get, and the denominator is the total. I say the probability of getting an even number when I roll a dice is of three chances in six, so I write 3/6. (Teacher 21)
>
> The probability is the chance we have to win a game. When I toss a coin I can have either a head or a tail, the probability of getting a head is 1 in 2, and the same for a tail, 1 in 2. (Teacher 9)
>
> Probability is the number of possibilities that you have to win and the result is a fraction. I remember we can write probability in percentage, for instance, the chances of a pregnant woman having a girl is 50%. (Teacher 11)

It is worth underscoring that many teachers in the group do not seem to understand that the probability of one event is a number; instead, they thought it was a code consisting of two digits: one that informs the quantity of desirable cases and one that informs the total quantity of possible outcomes. This conception draws forth some inconsistent conceptions, not only relative to probability, but also relative to rational number representations and meanings of fraction.
Instead of answering the question, some of the teachers chose to make comments about their own learning process about the notion of probability in high school, which they also confused with Combinations.

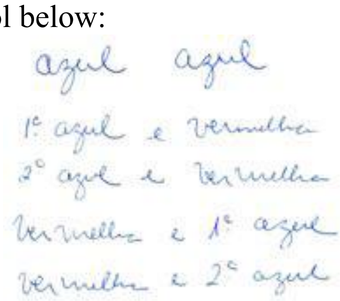
> I learned a little about probability, almost nothing, so I can't quite define it. (Teacher 3)
>
> For me to study probability I had to learn the Combinations and Permutations formulas in High School. (Teacher 18)
>
> Actually, I don't remember having learned probability when I was a student. I studied some probability in the Parâmetros when I studied the meanings of fraction. (teacher 27)

As for the question "Do you know more than one definition of probability?", all the teachers replied they did not; in fact, some of them were a little surprised by the question: "Is there more than one definition? I didn't know..." (Teacher 17). It is worth noting that these answers guided the follow-up discussion about the definition of probability and the approaches presented for the different school levels, mainly about the curricular guidelines regarding the need to work with this subject right from the early years.

About sample space and calculating probability

In order to look into the knowledge that teachers had regarding sample space, we proposed several questions and, more specifically, the following one: "One box has three balls; two blue ones and one red. If you pick two balls randomly, one at a time, which is the higher probability: getting two blue balls or one blue and one red?"
For this question, 21 teachers replied that the higher probability was to get two blue ones, because there were more blue balls. Three teachers stated that the probability was the same, but did not explain why, and other three teachers replied that the probability of getting one blue and one red ball was higher. Two of the teachers who answered correctly described the sample space of the event. In fact, by registering the sample space, the teachers could have noticed that there are twice as many combinations of blue-red than blue-blue; four ways to get blue-red (B1-R; B2-R; R-B1; R-B2) and only two ways to get blue-blue (B1-B2; B2-B1). If the teachers considered that two balls were picked up at the same time, the answer to the problem wouldn't have changed regardless of the different sample space (B-B; B1-R; B2-R).

We present this teacher's protocol below: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Protocol (Teacher 17)

In this protocol we can see that the teacher was able to differentiate two blue balls - the first and the second - when he described the possible blue-red combinations; however, he did not make such distinction for the blue-blue pair. Hence, the teacher only presented five elements of the sample space instead of six.

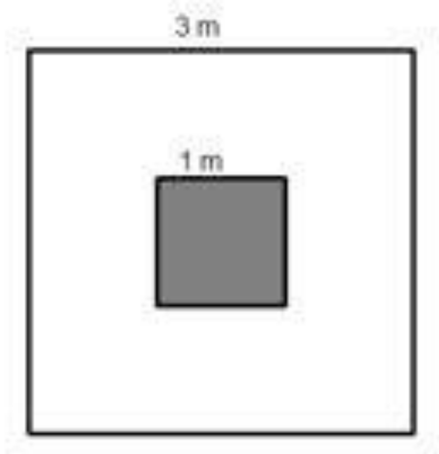
The answers to this problem reinforce that the sample space plays a key role in the understanding and calculation of event probability, even for very simple problems. According to Nunes and Bryant (2012), people frequently underestimate the sample space when solving probability situations.

We interviewed the three teachers who solved the problem correctly. One of them said that he replied to the question using simple intuition and could not justify his answer. Two other teachers stated that the reason for their correct answer was the fact that they wrote down all possible outcomes and they would not be able to solve the problem without that description. One of the teachers had a curious reply, to say the least:

> To me, it was evident that the higher probability was of two blue ones. I was going to answer like that, but then I thought, gee, if they are asking this it's because the probability is the same, or, conversely, getting one red and one blue is the higher probability. Then, I decided to write down the possibilities for the outcomes. Then I saw I was wrong. (Teacher 17)

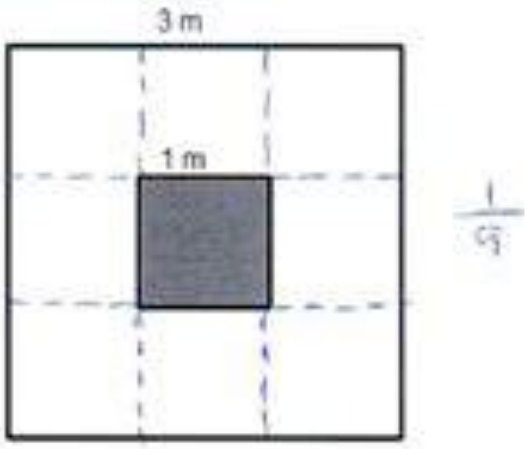
For our analysis of the knowledge that teachers had about calculating probabilities, we proposed, among other activities, the following problem:

- This picture represents a dartboard. The bull's eye is formed by two squares whose sides measure, respectively, 3 meters and 1 meter.



A player throws one dart and hits the bull's eye. What is the probability that she hit the bull's eye in the smaller square?

As for the third problem, which involved the geometrical definition of probability, the number of correct answers was not high: only four teachers chose the correct answer: $\frac{1}{9}$ which is the ratio between the areas of the smaller and bigger squares. The other teachers either did not solve or did not answer that the probability was of $\frac{1}{3}$, because they only considered the measurements of the sides of the square. It is interesting to note that two of the teachers that had the correct answer did not directly calculate the areas of the squares as the other teachers did, but they checked how many times the smaller square fitted in the bigger one, as shown in one of these teacher's protocol:



Protocol (teacher 17)

It is possible that these teachers, in this situation, did not explicitly consider the sample space as continuous, but rather by its "discreet" feature: the "sample space" was obtained through counting the nine same-size little squares inside the bigger square. The group's answers revealed inconsistent conceptions about sample space and, hence, about probability calculations. It was also possible to identify that aspects such as acknowledging the need to discuss the notion of randomness and the importance of organizing and describing the sample space using different representations were not part of the participants' conceptual image regarding the teaching of probability.

From the teachers' conceptual image of probability and its teaching, and from research results, we conceived and developed a continued education process aiming to enhance the knowledge base of these teachers about the teaching of this subject, as proposed by Ball, Thames and Phelps (2008). Reflections about the situations proposed during the Development Process, enhanced this knowledge base not only regarding the teaching of probability, but also about counting problems. This development process also enabled the implementation of innovations concerning probability in the 5th year of elementary school by the majority of the teachers participating in our research.

On the teaching and learning process of probability

Specialized content knowledge is considered to be the kind of knowledge that allows teachers, among other specificities, to identify the mistakes and the causes of such mistakes in materials produced by students. (Ball *et al.*, 2008). In this regard, we underscore that the teacher should not only master the notions and procedures of what he is to teach, but also develop other contents that can support teaching. According to Pietropaolo (2005), the teacher must have a *supplementary stack* of knowledge so that he can perform his role adequately.

However, by analyzing the answers given by the subjects of our research to the questions about concepts related to probability, we can assert that the great majority of teachers are not ready - prior to the formative process - to teach this matter in the early years. In other words, teachers have little knowledge about the specialized content required to teach probability.

For instance, we saw that many teachers did not consider the probability of one event as a number, but rather as an index composed by two numbers. We also identified that the great difficulty felt by the teachers concerning the theme stemmed from the description and quantification of the sample space. The multiplication principle was not part of the knowledge base of some of the teachers, either.

Notwithstanding this fact, in one of our diagnostic tool questionnaires we proposed questions about the teaching of probability, such as strategies they used in their lessons. Below, we present some of the questions raised about the teaching and learning of probability.

- The state of São Paulo's curriculum suggests that the notion of probability should be started at the final years of Elementary School. Do you agree with this proposition? Justify your answer.
- Have you taught, or do you teach probability in your math classes at the early years? If so, which strategies have you used?
- Have your students presented, or still present, any difficulties in learning the concepts related to probability? Describe your experience regarding the teaching of concepts related to this theme.

In relation to the first question, we had total unanimity among the teachers: although they believe probability is an interesting theme - mainly for children - they did not praise the state educational department's initiative to propose this subject: all of them replied that there was not enough time to develop the theme, due to the extension of the content proposed in the current syllabus. Teacher 19's answer gives a general idea regarding the group's position.

> I think probability is interesting, so much so that I enrolled in the course. I also think this subject can motivate the children. But we don't have time to teach this subject. We give priority to literacy and there is little time left for math. Then we have to teach numbers, the operations, the problems, a little about fraction, the metric system, money, tables and graphs. If I can hardly handle this, how can I teach probability if I have difficulty in it myself? And there is more: probability is never in the Saresp (*similar to STAs in the US*). I talk a little about probability when I teach fractions. (Teacher 19)

As for the second question, 16 teachers reported discussions with their students only in the following situations while teaching fractions: the probability to get a certain number when rolling a dice or a certain face when tossing a coin.

> I know that probability is in Data Treatment. In a course I attended at the Education Directory about the meanings of fractions, the teacher gave examples of probability: rolling dices, tossing coins, playing cards and picking balls from a box. In probability there is fraction, I just didn't quite understand if the meaning was part-whole or ratio. (teacher 7)

Other teachers reported, however, that they never approached the theme of probability in their classes. It is also important to highlight that when some of the teachers stated that they did teach probability in their classes, they were, in fact, referring to Combinations.

> I use something concrete when I'm teaching probability. In the problem of combining skirts and tops, I cut out the shapes and show the possible combinations. I think that kids see the combinations and learn. (Teacher 16)
>
> When I teach probability I use drawings to show all the combinations. I give them a problem with a table containing the prices, for instance, of three sandwiches and two beverages. I ask them to calculate the price of each combination. This makes it easier for the students to understand. (Teacher 20)

The answers given by our research subjects to the third question added little to what they had stated previously in the two first questions: the ones who really do some work with problems of probability claim that their students have little difficulty in the theme. This was an expected answer, since they rarely proposed situations involving probability and, when they did do it, it was through repetitive situations. The ones who pointed out difficulties actually meant the difficulties presented by children on counting. None of teachers mentioned the term "randomness" in their answers. As for the sample space, they only made indirect references to it.
Hence, and taking into account Ball*et al* (2008) ideas about the content pedagogical knowledge, these teachers would not have the required knowledge to teach probability in the early years of elementary school.

Diagnostics: a summary

By analyzing the results of this data collection as proposed by Tall and Vinner (1981), we consider that the conceptual image constructed by the majority of the participants of our study regarding the

teaching of probability in the early years was chiefly constituted by a field of problems for the application of ratio as one of the meanings of fractions. In other words, the probability of one event would always translate into a ratio between two whole positive numbers.
Besides this, the teachers' conceptual images did not present other views about probability, namely the algebraic and frequency definitions, a fact that restrained the scope of the proposed problems.
Hence, the study of probability would offer to these teachers few connections with other mathematical content and would be a less rich context to develop important cognitive skills.
The notion of sample space - a concept whose discussion might improve the understanding of probability calculation - was not part of the knowledge repertoire of specific contents accumulated by the teachers, suggesting that there are significant gaps in the pedagogical knowledge required to present such content to students. Some of the teachers barely mastered the multiplication principle.
Another point worth noting is the non-utilization of systematized procedures by the teachers, such as the tree diagram for naming and counting groupings in a sample space. Many researchers, such as Borba (2013), noticed that the use of tree diagrams allows for better understanding of combination problems.
In short, taking into account the categories proposed by Ball*et al* (2008), we conclude that our research subjects still did not have the required knowledge to teach notions concerning probability in the early years of school.
These results underscore the need to promote discussions about the relevance of notions concerning the theme of probability in the formative and/or continued development courses, and to discuss the difficulties felt by students when they start to construct this knowledge as well as the importance of its study at the different school levels.

REFERENCES

Ball, D. L., Thames, M. H., & Phelps, G. Content knowledge for teaching: what makes it special? *Journal of Teacher Education, 59(5), 389-407, 2008.*
Batanero, C. Didáctica de la Probabilidad. Universidad de Granada. Departamento de Didáctica de la Matemática. 2001. Available at: <http://www.ugr.es/~batanero>. Last accessed: July 22, 2013.
Batanero, C.. La comprensión de la probabilidad en los niños. ¿Qué podemos aprender de la investigación? En J. A. Fernandes, P. F. Correia, M. H. Martinho, & F. Viseu, (Eds..). *Atas do III Encontro de Probabilidades e Estatística na Escola. Braga: Centro de Investigação em Educação. Universidade do Minho, 2013.*
Gal, I. Towards "Probability Literacy" for all citizens: Building Blocks and Instructional Dilemmas. In Jones, G. A. (Ed), Exploring probability in school: Challenges for teaching and learning. p. 39-63. 2005.
Pietropaolo, R. C., *(Re) Significar a demonstração nos currículos da Educação Básica e da formação de professores de Matemática*. Tese (Doutorado em Educação Matemática). São Paulo: PUC/SP, 2005.
Tall, D. & Vinner, S. Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12, 151-169, 1981.*

Pedagogical use of tablets in mathematics teachers continued education

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Abstract: This article aims to identify conceptions shared by a group of five mathematics teachers and the way tablets were appropriated as a pedagogical tool in the context of teaching polynomial functions of the first degree. This group of teachers works at high schools at the São Paulo's public school network and participates in the Education Observatory Project. This qualitative research is in its final phase of analysis and is developed in the context of a continued education course involving the use of tablets and educational software. Studies of this nature are needed due to the fact that digital mobile technology is increasingly more accessible for students, and also because teachers need support in their process of appropriation and understanding of the cognitive potentialities of teaching mathematics using mobile technology. The analysis showed that the process of appropriation is gradual and, hence, continued education should create situations to enable teachers not only to learn how to operate the technological resources, but also to experience the possibilities of exploring mathematics with their students, attributing both personal and professional meanings to technology. This requires having experiences, dialogs, experimentation, sharing and also individual and collective reflections in a process of appropriation and knowledge construction and reconstruction.

Key words: Digital Technologies - Mathematics Education - appropriation - practice reconstruction

Résumé: Cet article vise à identifier les conceptions d'un groupe de cinq enseignants de mathématiques et de l'utilisation pédagogique crédit de sorte que la tablette dans le contexte de la fonction polynomiale de l'enseignement du 1er degré. Ce groupe d'enseignants agit à l'école secondaire de l'état de São Paulo et sont les participants au projet du Programme Observatoire de l'enseignement. Cette recherche qualitative, dans la phase d'analyse finale, se développe dans le cadre d'un cours de formation continue impliquant l'utilisation de comprimés et de logiciels éducatifs. La nécessité de ce genre d'étude est justifiée par le fait que les technologies numériques mobiles sont de plus en plus disponibles dans les mains des élèves et les enseignants ont besoin de soutien dans le processus d'appropriation et la compréhension du potentiel cognitif de l'enseignement des mathématiques avec la technologie mobile. L'analyse evidenciouque le processus d'appropriation est situations de devecriar éducation progressive et ensuite poursuivies où, en plus d'apprendre à exploiter les ressources technologiques, l'enseignant peut éprouver les possibilités d'explorer les mathématiques avec des étudiants, donner un sens personnel et professionnel. Cela nécessite des expériences, des dialogues, des essais, des actions et des réflexions individuelles et collectives dans un processus d'appropriation et de la construction et de la reconstruction de la connaissance.

Mots clés: Technologie Digital - Mathématiques Education - l'appropriation - reconstruction pratique

Introduction

Today, people in general and students in particular use digital information and communication technology (DICT) with great skills and seek to be constantly connected, communicating and accessing information from many regions in the world. This generation of learners - known as digital natives -is different from that of a few years ago. Children and young adults study and listen to music, communicate using the internet and do their homework, all at the same time. This kind of behavior is attached to the fast-paced way in which information reaches the individual and how it can be produced and made accessible through digital technology.

How do schools deal with this reality lived by their students? Are teachers prepared to teach this new paradigm of the digital culture society?

As far as using digital mobile technology at schools, the Brazilian government has encouraged its use since 2007 by means of various programs, among which is UCA Project (One-computer per Student, in Portuguese)[6] whose goal is the creation and socialization of different ways to use digital technology at public schools in Brazil, promoting the pedagogical use of DICT. Hence, many schools started to consider this new reality in which technology is in their students' hands. More recently, the ministry of education [7]invested in the purchase of tablets, which were initially delivered to high school teachers at public schools to be used as a pedagogical resource, enabling mobility and accessibility to digital content via Wi-Fi.

Although the experiences using digital mobile technology in the context of elementary education are recent, there are a few researches that derived mainly from the UCA Project, which analyzed the effects of having laptops in the students' hands and their educational use in the classroom. According to Almeida and Prado (2009), in the classroom routine the use of mobile computers can encourage new ways for students to relate with information, to learn and to teach, generating “[...] *changes in the relationships of all existing elements in this space and also in the way they act, which will push for changes in the school context*” (p.5).

The study by Mendes & Almeida (2011) conducted within the scope of the UCA Project in a public school in the north of Brazil, found that teachers pointed out the need to change the classroom organization, ranging from the physical settings to the way the class should be conducted. Mainly, it was necessary to review the syllabus and the didactic planning. They considered that the class became more dynamic, requiring, however, that teachers develop strategies so that the students would keep their focus on the subject content and, at the same time, could acknowledge the searches and discoveries made by the students from their interaction with the laptop.

In fact, before the appearance of mobile technology, schools used to have, at best, computer labs and the use of computers was dependent on previously arranged appointments and on the availability of access, as there was usually a single lab to service a great number of students and teachers in the same school. We are aware that schools reality changes with the insertion of mobile technology, and it presents new educational challenges together with amenities offered by the mobility and connectivity of devices such as tablets, iPods and smartphones. Especially because they are available for students and teachers, which characterizes, according to Eivazian (2012), as “[...] *a new paradigm of the use of technology in education*” (p.15).

Hence, there is a need to broaden and deepen studies about the use of digital mobile technology in school spaces, and consider the possible impacts in teachers practice. This digital mobile technology easily triggers curiosity and stimulates students' creativity, allowing for the collaboration among them for new discoveries, the agility to search for information and communication, while enabling new ways of interaction and learnings.

Particularly, touchscreen digital technology can bring new possibilities for the process of teaching and learning. In this aspect, the studies conducted by Arzarello et al (2013) underscore that “*the manipulation on tablet is different from that with mouse clicking, this kind of research investigates a new aspect of students’ behavior’s when using dynamic geometry software*” (p. 59). In term of

[6]For more information: <http://www.fnde.gov.br/programas/programa-nacional-de-tecnologia-educacional-proinfo/proinfo-projeto-um-computador-por-aluno-uca> Accessed on March 5, 2015.
[7]For more information: <http://portal.mec.gov.br/index.php?option=com_content&id=17479> Accessed on March 5, 2015.

researches conducted in Brazil, specifically in mathematics education using such devices, we highlight those by Bairral (2013), who points out the touchscreen potentiality for learning dynamic geometry, conform your words [...] *we assume that the handling of this type of environment should be seen as a cognitive tool that potentializes the learners' skills to explore, conjecture and construct different ways to justify them*. (p.8).

If this technology can potentialize cognitive processes, it is an indication of the need to understand how students and teachers use the resources in the school context, with focus on the syllabus content. Concomitantly to this demand, we also become instigated to explore and reflect upon teachers' preparation to make the best out of the potentialities present in digital technology for the processes of teaching and learning.

Acknowledging such needs and urgency, because this type of technology evolves very rapidly and its immediate access by children and young adults is almost instant after their appearance, our focus of study and research is on the continued education of mathematics teachers involving the pedagogical use of digital mobile technology.

Research Development

The goal of this study was to identify conceptions shared by a group of five mathematics teachers and the way tablet were appropriated as a pedagogical tool in the context of teaching polynomial functions of the first degree. This group of high school teachers works at the São Paulo's public school network and participates in the Education Observatory Project.[8]

This qualitative research - in its final phase of analysis - is being developed in the context of a continued education course involving the use of tablet and educational software. Data collection was made using the following: one questionnaire, semi-structured interviews, field logbooks and protocols of activities developed by the participating teachers during the course.

The questionnaire included questions related to the teachers' profiles and to the level of familiarization and knowledge of tablet regarding both personal and professional uses. The interviews were conducted after the course meetings, with the aim of listening to the teachers' accounts about the experiences lived in the course. Besides these, records were made of classroom observations and activities developed by the teachers using educational software for the exploration of different notations for the representation of functions, and, in particular, of graphs of polynomial functions of the first degree.

Data analysis about the process of technology appropriation by teachers has been developed using the theoretical basis provided by Sandholtz, Ringstaff & Dwyer (1997); Almeida & Valente (2011). These authors, while supervising the experiences regarding the implementation of technology in schools, identified that such process occurs gradually and starts by the adoption of technology and the mastering of operating for others who approach their pedagogical practice without using technology. For the latter, starting from a reflective process and the sharing among peers with the mediation of developers, makes the process of technology appropriation to evolve, instead of being used solely as an operating tool for creative and innovative practices (PRADO & LOBO DA COSTA, 2013).

According to Rabardel (1995), in the process of appropriation of digital technology, the human being have to develop an instrumentalization process and an instrumentation process. For this reason, the continued education in the context of pedagogical use of technology has to prioritize the construction - from the teachers - of new knowledge and the reconstruction of other types of

[8]The Education Observatory Project - funded by the Brazilian agencies CAPES/Inep - is developed through the partnership between post-graduate programs of universities and elementary public schools aiming at the development of researches and continued education actions to enhance education.

knowledge so as to integrate them to enhance instrumented activities.

Hence, based on these principles, the course being analyzed here sought to promote dialog and the integration among the participating teachers during the exploration of tablet resources in a contextualized manner, using mathematics content in the school syllabus. In this sense, we chose to focus on the use of Grapher and Geogebra no software on tablet to explore the content of polynomial function of the first degree.

Description and Analysis

The preliminary analyses identified the teachers' reaction when they interacted with the tablet, initially using the software Grapher during the continued education course. Grapher is a free software, easy to use and it allows for the creation of graphs of functions, changes in the background in the Cartesian plane, changes in the color of grids, zooming in and out images by tapping and opening with two fingers. This makes it possible to change scales, zoom effects and observe the particularities of these functions.

Figure 1 shows a Grapher screen shot representing the graph of function $f(x) = 2^x$ and two moments of exploration of the graph with touchscreen resources, demonstrating the possibility of zooming in and out to observe the variations in scale.

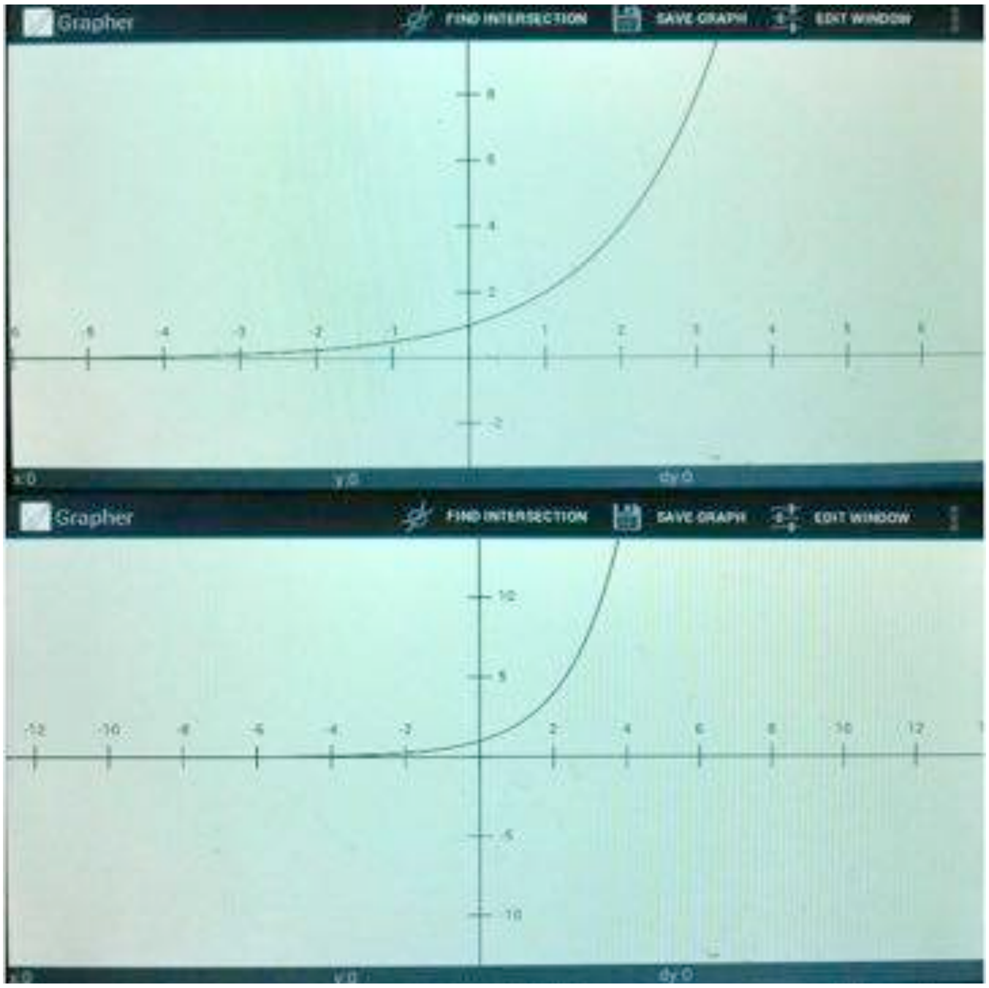

Figure 1: Grapher screen shot
Source: Author's file

Another example of use is related to the possibility of tapping the touchscreen to observe an intersection point of graphs of various functions designed on a single Cartesian plane on the tablet.

The teachers explored Grapher in the tablet by performing activities that involved functions. Some of them showed more difficulty in interacting with the touchscreen due to lack of familiarity, but even so, all of them reported the desire to learn how to use digital mobile technology, in particular tablet, in their classroom practice.

This group of teachers showed their awareness of the new social and educational reality, i.e., that the use of DICT is part of people's lives and they have to know their pedagogical potentialities to integrate them to the teaching of math.

Such acknowledgment is essential to start the process of appropriation of DICT because, as we mentioned before, digital mobile technology is in the students' hands, which echoes the teachers' concerns about how to prepare their lessons, i.e., how to teach using educational software and other resources available on tablet.

During the development, the teachers used tablet to teach polynomial functions of the first degree and discussed with their colleagues the pedagogical possibilities that they were able to identify, as shown in the testimonials below, in the situation where they were developing the creation of the graph f(x) = x-1, using Grapher and Geogebra.

-In the tablet it is easier because I can <u>show</u> to the student the linear coefficient in the graph and by zooming in and out the student can notice the changes in values, and that the line remains the same regardless of values of y (Teacher E’s log). *-It's easier and faster - just a tap* **-** *<u>to confirm</u> for the student, the same graph with different values. The student likes the looks, it's easier to notice and analyze the graphs and compare functions.* (Teacher B’s log) *-By using the software* [Grapher] *it's easier for the student <u>to identify</u> the points...* (Teacher B’s log)

Table 1: testimony of teachers

According to the teachers' testimonials it can be observed that even in an initial interaction with the software resources on the tablet, they pointed out a few possibilities, mainly in terms of making the viewing easier.

We noticed that the group of teachers adopted the same attitude usually taken in the classroom, in the sense that they present the graph to the students rather than give the task to the students so they can manipulate the software resources in the mathematical context, allowing for hypotheses formation and the elaboration of conjectures about the concepts at stake.

However, early in the process of appropriation it is usual for teachers to transfer all of what they habitually do when teaching a class using the chalkboard for a similar class, for instance, projecting slides, showing a graph on a screen. This means they produce "a clean version" of the same class using digital technology. In the intermediate phases of the process of appropriation, one example of what happens is when the teacher uses one of the resources of digital technology in isolation, as a complement to other teaching resources, rather than using the technology potentialities integrated with the goals of a certain activity.

The process of pedagogical appropriation of digital technology was also made clear through the reports of the majority of the teachers in this group, who made exceptions to the idea by underscoring that first it is necessary to teach using the chalk-and-board, and then explore the tablet as an aid to view and analyze graphs.

This means that the pedagogical appropriation of the use of digital technologies so as to integrate their resources to syllabus content requires teachers to go through a process of knowledge construction and reconstruction. In this sense, one of the teachers in the group who is already familiar with tablet and uses software to teach mathematics, expressed her understanding about the pedagogical use of DIC

-[...] *with technology, students explore and integrate with mathematical concepts and this requires another type of lesson...but we were not prepared for this*" (Teacher A's log)

Table 2: testimony of teacher

In fact, teachers have to be prepared to reconstruct their pedagogical practice to integrate digital technology, because the center of the educational process in this context is neither the teacher nor the student: it is in student's interactive process with the technology, between students, and between students and teacher, who conducts the pedagogical mediation.

Some considerations

The preliminary analyses of this research point out relevant aspects to be considered in continued education courses for mathematics teachers regarding the use of digital technology in teaching and learning processes.

In development processes, it is necessary to create situations where teachers learn not only how to operate the technological resources, but also how to attribute personal and professional meanings for their use. This requires having experiences, dialogs, experimentation, sharing and also individual and collective reflections in the group of professionals participating in a process of appropriation and knowledge construction and reconstruction, leading them to go through an initial process of instrumentalization that will evolve into instrumentation.

From the professional development viewpoint, continued education has to develop strategies that wake the willingness in teachers to learn throughout life and that, in this process, each person can also teach other persons ways to overcome challenges to develop their intellectual autonomy. This is a necessary quest so that in digital culture society we are able not only to be consumers of information but also producers who express and share knowledge in the technological network into a learning network.

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REFERENCES

Almeida, M.E.B. & Prado, M.E.B.B. (2009). Formação de Educadores para o uso dos computadores portáteis: indicadores de mudança na prática e no currículo. *Anais da VI Conferência Internacional de Tecnologias de Informação e Comunicação na Educação*. Braga, Portugal.

Almeida, M.E.B. & Valente, J. A. (2011). *Tecnologias e Currículo: trajetórias convergentes ou divergentes?* São Paulo: Paulus.

Arzarello F., Bairral M., Danè C. & Yasuyuki I. (2013). Ways of manipulation touchscreen in one geometrical dynamic software. In: E. Faggiano & A. Montone (Eds.). *Proceedings of the 11th International Conference on Technology in Mathematics Teaching ICTMT11*, Department of Mathematics. University of Bari, Italy. (pp. 59-64). Available for download at: <http://www.dm.uniba.it/ictmt11/download/ICTMT11_Proceedings.pdf>. Last accessed: February 20, 2015.

Bairral, M. A. (2013). Do clique ao touchscreen: Novas formas de interação e de aprendizado matemático. *36a Reunião Nacional da Anped*. Sistema Nacional de Educação e Participação Popular: Desafios para as Políticas Educacionais. Goiânia: Anped/UFG (pp. 1-18).

Eivazian, A.M.B. (2012). O Computador Móvel e a Prática de Professores que Ensinam Matemática em uma Escola do Projeto UCA. *Dissertação de Mestrado em Educação Matemática*. Universidade Bandeirante de São Paulo, Brasil. Available for download at: < http://www.matematicaepraticadocente.net.br/>. Last accessed: February 20, 2015.

Mendes, M. & Almeida, M.E.B. (2011). Utilização do laptop educacional em sala de aula. In: Almeida, M.E.B. & Prado, M.E.B.B. (orgs). *O Computador Portátil na Escola* - Mudanças e desafios nos processos de ensino e aprendizagem. São Paulo: Avercamp. (pp. 49-59).

Prado, M.E.B.B. & Lobo da Costa, N.M. (2013). O processo de apropriação das TIC e a reconstrução de novas práticas no ensino de matemática. *Actas VII Congreso Iberoamericano de Educación Matemática – CIBEM*. Montevidéu, Uruguai. (pp. 6973-6980). Available for download at: <http://www.cibem7.semur.edu.uy/7/actas/pdfs/588.pdf>. Last accessed: February 20, 2015.

Prado, M.E.B.B., Lobo da Costa, N.M. & Padilha, W.R. (2015). Tecnologias Digitais Móveis reconstrução de práticas no ensino de matemática. In: Lobo da Costa, N.M., Pietropaolo, R.C., Prado, M.E.B.B. & Campos, T.M.M. (orgs.) *Formação de professores de Matemática e suas Práticas: experiências e reflexões no contexto do Observatório da Educação*. In press.

Sandholtz, J.H., Ringstaff, C. & Dwyer, D.C. (1997). *Ensinando com Tecnologia*: *criando salas de aula centradas no aluno*. Porto Alegre: Ed. Artes Médicas.

Rabardel, P. (1995). *Les hommes et les technologies une approche cognitive des instruments contemporains*. Paris: Armand Colin.

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Investigating future primary teachers' grasping of situations related to unequal partition word problems

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Abstract : This contribution focuses on future primary teachers' grasping of situations related to unequal partition word problems. In the first part of the text we introduce an educational tool called Concept Cartoons, and investigate how it can be helpful in the process of identifying various aspects of the process of grasping of a situation. Our findings show that a suitably composed Concept Cartoon can be used to indicate understanding as well as misunderstanding and misconceptions. The second part of the text deals with various graphical representations of word problems, and their applicability in the process of solving unequal partition word problems. We show why schemes in the form of branched chains are not appropriate for representing the structure of this kind of word problems.

Résumé : Notre contribution est focalisée à la description des approches des enseignants du primaire en saisissant les situations basées sur les problèmes de type Parties-Tout. Dans la première partie, nous présentons un outil pédagogique appelé «Concept Cartoons» et nous le décrivons en tant qu'un outil d'aide pour identifier de divers phénomènes qui se manifestent en saisissant une situation. Nous constatons qu'un Concept Cartoon bien conçu peut être utilisé en tant qu'un indicateur de la compréhension, de la mauvaise compréhension et des erreurs. Dans la deuxième partie, nous proposons de diverses représentations graphiques (visualisations) des problèmes mathématiques et les possibilités de leur utilisation dans le processus de la résolution des problèmes de type Parties-Tout. Nous montrons les raisons pourquoi les schémas sous forme des „chaînes ramifiées" ne sont pas convenables pour la visualisation (représentation) de la structure de ce genre de problèmes.

Introduction

In the study presented here we focus on future primary school teachers, and consider the question of how to identify whether a future teacher grasps a situation related to a word problem with understanding (cf. Polya, 2004). Particularly we deal with unequal partition word problems. For our investigations we innovatively use an educational tool called Concept Cartoons.

Following up our research on problem posing presented at previous CIEAEMs (Tichá, 2009; Tichá & Hošpesová, 2013), we also consider the issue how using schemes for visual representation of the problem structure could help to grasp the situation related to the problem. We build on our recent experience showing that this approach might be helpful (Tichá, 2014; Tichá & Hošpesová, 2015).

Concept Cartoons

In mathematics education we make use of various schemes and visualizations. They help us to create a model of a problem or a record of its solution process. An educational tool called Concept Cartoons (CCs) can be also used for such purposes.

This tool was developed more than 20 years ago (Keogh & Naylor, 1993). Its original goal was to support teaching and learning in science classroom by generating discussion, stimulating investigation, and promoting learners' involvement and motivation (Naylor & Keogh, 2012). In later years the tool also expanded to other school subjects, including mathematics. Several years ago, the team of authors introduced a set of 130 CCs designed for classroom use in elementary school mathematics (Dabell, Keogh & Naylor, 2008).

Each Concept Cartoon (CC) is a picture presenting a situation well known to children, and a group of 5 children in a bubble-dialogue. The (mathematical) problem arises from the pictured situation, and sometimes is closely specified by the beginning of the text in the top left bubble – usually by the if-part of a conditional sentence. Texts in the other bubbles (and also the end of the text in the top left bubble) present alternative viewpoints on the situation and alternative solutions of the problem. One speech bubble is blank, with just "?" inside, to give a clear impression that there may be more alternative ideas that are not yet included in the dialog. See Fig. 1.

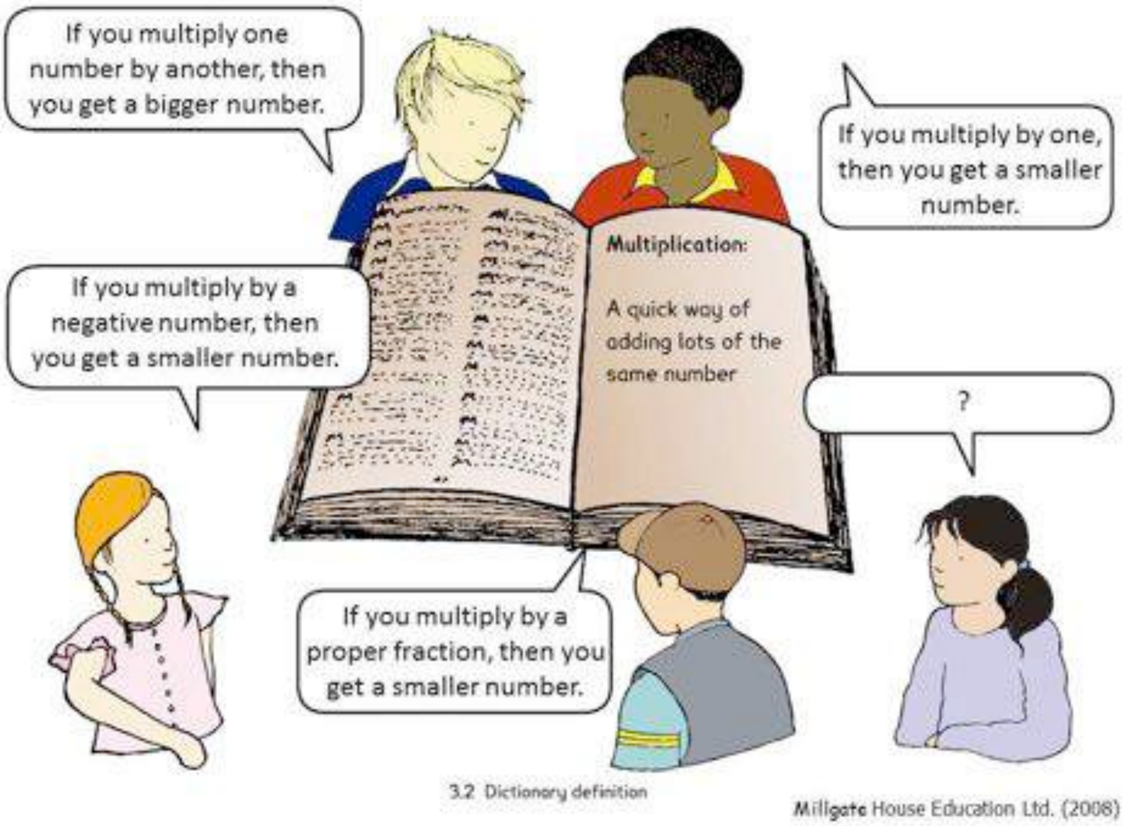

Figure 1: Concept Cartoon; taken from (Dabell, Keogh & Naylor, 2008), slightly modified.

The authors of the CC based the alternatives in bubbles on real classroom events or on common conceptions and misconceptions; they might also prepare some alternatives intentionally as authentic-looking unusual conceptions or misconceptions.

The situation pictured in the CC may be more or less open (with various ways of grasping, various ways of solving the problem based on the situation, or with multiple correct solutions to the problem – as in Fig. 1) or closed (with only one correct solution to the problem based on the situation – as in Fig. 2).

From the perspective of future primary school teachers' educators we feel the strength of CCs not only in teaching and learning, but also in diagnosing various types of teachers' mathematics knowledge: e.g. recently we have presented a study (Samková & Hošpesová, 2015) confirming that suitably chosen CCs allow to distinguish between subject matter knowledge and pedagogical content knowledge in the sense of Shulman (1986), and also between procedural and conceptual knowledge in the sense of Baroody, Feil and Johnson (2007). For that study we prepared a set of CCs, each of them presenting a closed situation leading to a calculation problem with one solution. We had two types of bubbles in these CCs: bubbles containing various procedures and their results, and bubbles containing just results. This combination of types of bubbles allowed us to investigate various aspect of teachers' knowledge (for details see Samková & Hošpesová, 2015).

Reported study (participants, methodology)

For the study reported here we chose a CC showing a closed situation leading to a calculation problem with one solution.

Since we did not plan to employ the blank bubble in this research, we replaced the "?" in the bottom left bubble by another alternative viewpoint, and offered there an intentionally prepared misconception. See Fig. 2.

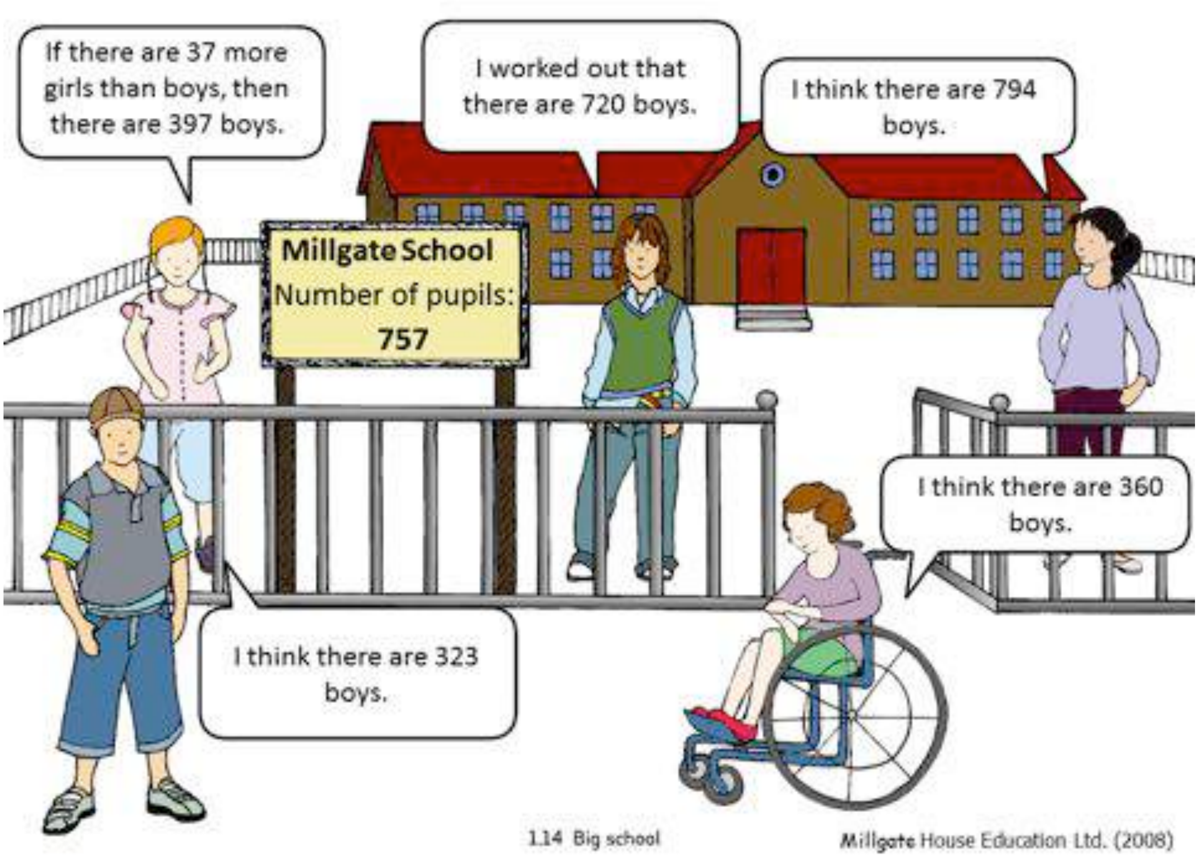

Figure 2: Concept Cartoon; taken from (Dabell, Keogh & Naylor, 2008), slightly modified.

Participants of our ongoing study are university master degree students – future primary school teachers. We collected data from them in two separate stages:

- In the first stage of the study we gave each participant a worksheet with the CC from Fig. 2. We asked them to decide which statements in bubbles are right, and to justify their decision. The participants worked individually, they wrote their conclusions on the worksheet. We put emphasis on the need to justify the decisions, in order to stimulate and deepen students' argumentation ability.
- In the second stage of the study we assigned the participants a word problem similar to the problem from the first stage, as a part of a standard written exam on arithmetic. We asked them not to use algebraic equations in the solution process, and to record the solution in detail.

The data from both stages of the study were processed qualitatively; we focused on aspects related to whether a participant grasps a situation with understanding. To be more precise, we monitored phenomena by which we characterized the process of *grasping of a situation* in our previous research (e.g. Tichá & Hošpesová, 2010), and phenomena which comply with the Polya's requirements for successful problem solution process (Polya, 2004).

By *grasping of a situation* we mean a process consisting of

- seeking and discovering the key phenomena of the situation and the relationships between them;
- insight into the subject of study;
- formulation of questions;
- searching for answers;
- interpretation of answers;
- evaluation of answers;
- identification of new questions and problems;
- continuing in a new process of searching using experience gained in previous activities

(Koman & Tichá, 1998).

Unequal partition problems

The problem outlined in the CC from Fig. 2 can be rephrased as a word problem:

> *There are 757 pupils in the Millgate School. Girls are 37 more than boys. How many boys are in the Millgate School?*

This word problem belongs to so-called unequal partition problems, i.e., problems of partition of a given quantity with the relationship between the parts expressed as a comparison of quantities (MacGregor & Stacey, 1998). In our case, both compared quantities are unknown.

In the second stage of the study we let students solve the following unequal partition word problem:

> *Tom and Carl have 68 marbles altogether. Carl has 14 marbles more than Tom. How many marbles has Tom?*

The situation with marbles is typical for unequal partition problems in our educational environment (cf. Novotná, 1997). These problems are usually solved either algebraically (i.e. using algebraic equations) or arithmetically (i.e. without equations, just by a sequence of arithmetic operations); graphical approach to the solution is not so common.

There are two prevailing arithmetic solution methods, based on two different representations of the situation: *sum-of-parts*, and *division-into-parts* (MacGregor & Stacey, 1998).

The case of *sum-of-parts* representation consists in searching one of the parts by taking away the extra quantity from the sum, and halving the reminder. In particular, solving the two word problems above results in calculating 757 – 37 = 720, 720 : 2 = 360 for the number of boys, and in calculating 68 – 14 = 54, 54 : 2 = 27 for the number of Tom's marbles. It is clearly seen from this representation that the unequal partition problem has a solution if the reminder is even, that means if the extra quantity and the sum have the same parity (both are even, or both are odd).

The case of *division-into-parts* representation consists in dividing the sum into two equal shares, and then adjusting these amounts by adding or subtracting half of the extra quantity. In particular, solving the two word problems above results in calculating 757 : 2 = 378.5, 37 : 2 = 18.5, 378.5 – 18.5 = 360 for the number of boys, and in calculating 68 : 2 = 34, 14 : 2 = 7, 34 – 7 = 27 for the number of Tom's marbles. We can see that if the solver can work with natural numbers only, then this method is not applicable for tasks with the extra quantity or the sum being odd.

Samples of actual findings

First stage of the study

i) Answers indicating understanding

Among the responses from the first stage of the study we revealed four different types of correct strategies:

- The most frequent one consisted in verifying (checking) of all offered alternatives. Such responses do not allow us to ascertain whether their authors know how to solve the problem and argue the solution procedure, but at least we can state that they grasped the situation successfully. Among these responses we revealed two different methods with diverse quality of the grasping process:
    - an analogue to guess-and-check method consisting of verifying every single offered alternative by calculating the number of girls and the number of all pupils for the given number of boys, and comparing such a number of all pupils with 757, e.g. by $397 + 37 = 434$, $397 + 434 = 831 \neq 757$ in case of the top left bubble;
    - a method using comparisons or estimates to reject immediately 794 and 720 for being too big.
- Less frequent was an arithmetic strategy consisting of a *sum-of-parts* method, i.e. of calculations $757 - 37 = 720$, $720 : 2 = 360$.
- Significantly fewer participants used another arithmetic strategy, a *division-into-parts* method, i.e. calculations $757 : 2 = 378.5$; $37 : 2 = 18.5$; $378.5 - 18.5 = 360$.
- Quite rarely appeared an algebraic strategy using an equation $x + (x + 37) = 757$, where x denotes the number of boys.

ii) Answers indicating misconceptions or misunderstanding

In the first stage of the study we revealed three different types of incorrect answers:

(a) Statement 720 is right, because $720 + 37 = 757$.

(b) No statement is right, because $757 : 2 - 37 = 341.5$ does not appear in bubbles.

(c) Statement 323 is right, because $(323 + 37) + 397 = 757$.

All three types indicate unsuccessful attempts to grasp the situation. Key phenomena of the situation and relationships between them were not discovered properly, the results were not verified with respect to the task, and on top of that – the author of the second answer was not even surprised by a decimal number as a result for the number of persons. We find interesting the fact that all authors of incorrect answers tried to justify somehow their answers.

Besides, the (c) misconception is a nice illustration of "take all numbers from the task, and do something with them" strategy – numbers 37 and 397 come from the top left bubble, and 757 comes from the information plate in the centre of the picture. There has to be noted that 323 is the intentionally prepared misconception we added into the blank bubble instead the question mark. Our previous experience had indicated that such a misconception might occur, and this suspicion was confirmed.

Second stage of the study

In the second stage of the study, the respondents mostly solved the word problem by the sum-of-parts method, i.e. $68 - 14 = 54$, $54 : 2 = 27$.

Among the incorrect solutions, only misconceptions analogical to (a) and (b) appeared.

The student, who made the (c) misconception in the first stage, solved the task correctly by the sum-of-parts method in the second stage. That attracted our attention, and we interviewed this student subsequently to realize that she had just learned the sum-of-parts method by rote, without understanding. In this particular case, CCs helped us to reveal a weakness in understanding which could not be revealed in the standard written exam.

Graphical representation, visualization

To our surprise, none of the participants provided a graphical solution, nor offered a visualization of the problem – despite the fact that they had already met with schemes of problem structure in math courses.

We see the issue of graphical solutions and visualisations as important, since we believe that understanding deepens through enriching the repertoire of representations (cf. Janvier, 1987). We consider an iconic representation provided by visualization as very valuable, and as a non-skippable component of the process of grasping.

In our conception, schemes of a problem structure are linear or branched chains in the sense of Kittler and Kuřina (1994). Samples from their primary school textbook you can see in Fig. 3. We use these schemes as the means of visualization, as a graphical representation either of the problem structure or of the problem solving procedure. Moreover, we use them as a diagnostic tool in future primary school teacher training (Tichá & Hošpesová, 2015).

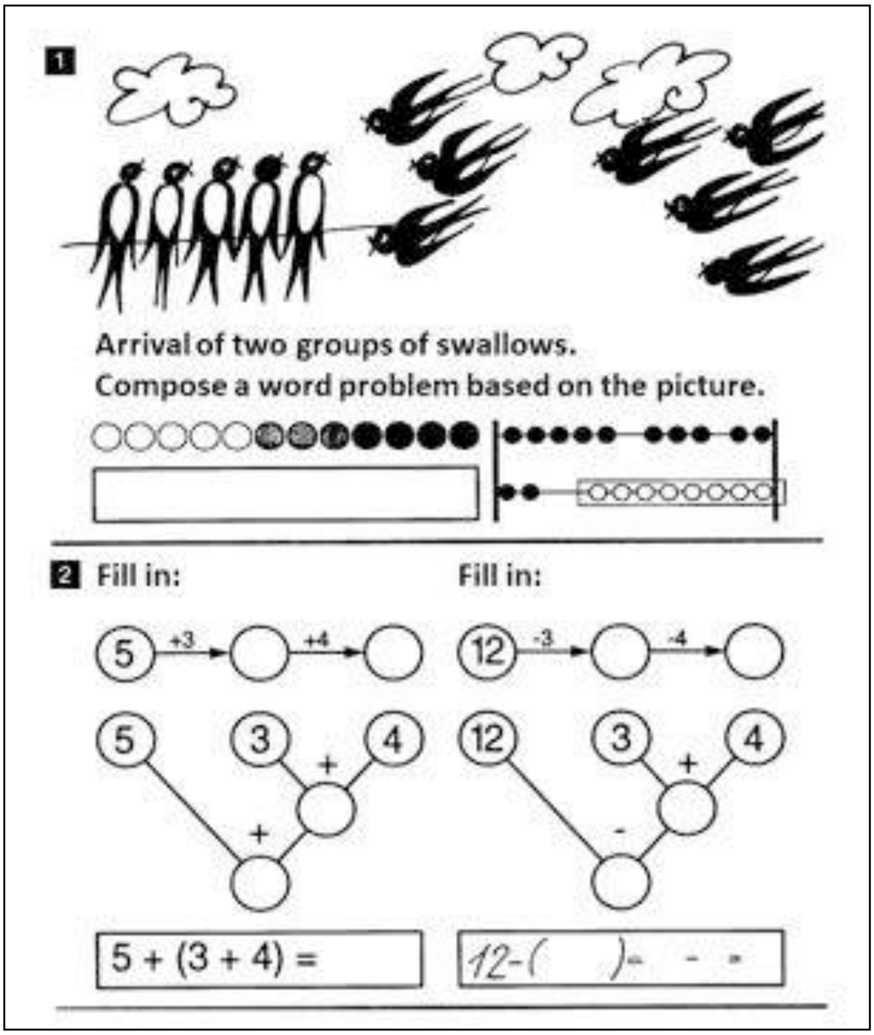

Figure 3: Linear and branched chains as schemes of a problem structure; taken from (Kittler &

Kuřina, 1994, p. 70), translated.

Similar schemes were introduced also by Nesher and Hershkovitz for representing the problem structure: "Using a scheme, in our view, constitutes a mapping between semantic relations underlying a given text and its mathematical structure. The scheme serves as generalized habit of action in a given situation." (Nesher & Hershkovitz, 1994, p. 1)

For some types of word problems such schemes may help to grasp the situation, e.g. for two-step word problems such as:

> *There are 15 green and 17 blue matchbox cars on a big shelf, and 9 red matchbox cars on a small shelf. How many matchbox cars are there?*

For these word problems we can depict all key phenomena of the situation and relations between them to create two separate sub-schemes (Fig. 4 left). These sub-schemes have one common element and their composition produce a compound scheme suggesting us how to solve the problem (Fig. 4 right). For comparison see hierarchical scheme in (Nesher & Hershkovitz, 1994, p. 8).

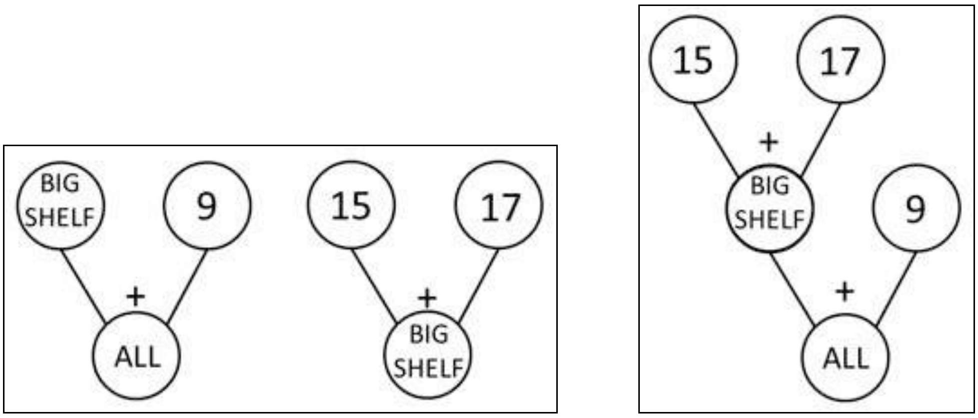

Figure 4: Schemes of the two-step problem with matchbox cars. Two separate sub-schemes (left), a compound scheme (right).

In the case of unequal partition problems the issue is more complicated. We may also create the sub-schemes (Fig. 5 left), but they have two common elements, and the compound scheme does not uncover the solution (in any rearrangement – see Fig. 5 middle, right). Concluded, in the case of unequal partition problems such schemes are not appropriate.

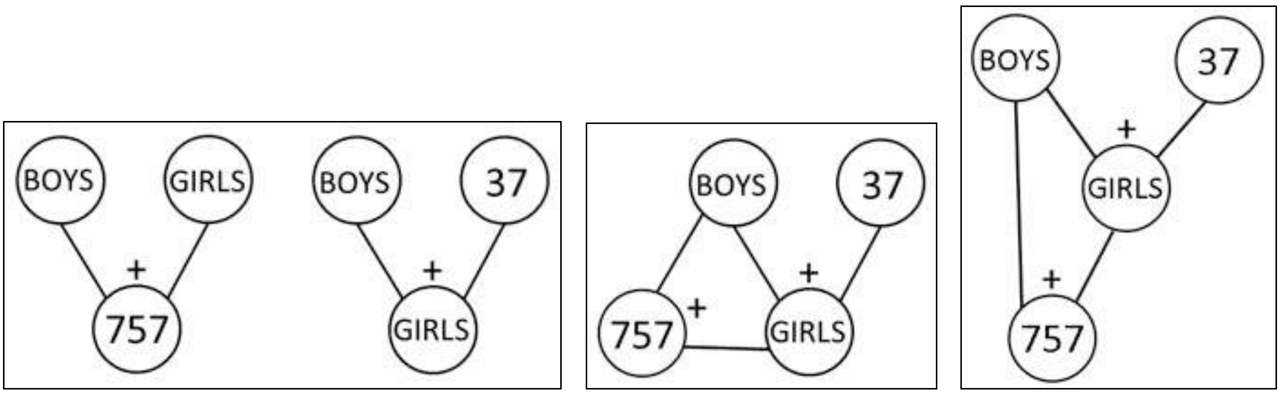

Figure 5: Schemes of the unequal partition problem outlined in Fig. 2. Two separate sub-schemes (left), a compound scheme (middle), a rearranged compound scheme (right).

For unequal partition problems we have to use a different kind of graphical representation. A suitable visualization can be obtained e.g. by a *segment model* (Novotná, 1997). This model serves just for getting an idea of the situation, thus the ratios of lengths of the segments are not supposed to correspond to the ratios of the numbers (Fig. 6 left). But our classroom experience show that students prefer to replace segments by rectangles, in order to be able to inscribe numbers inside, i.e. they use a *bar diagram* (Fig. 6 right).

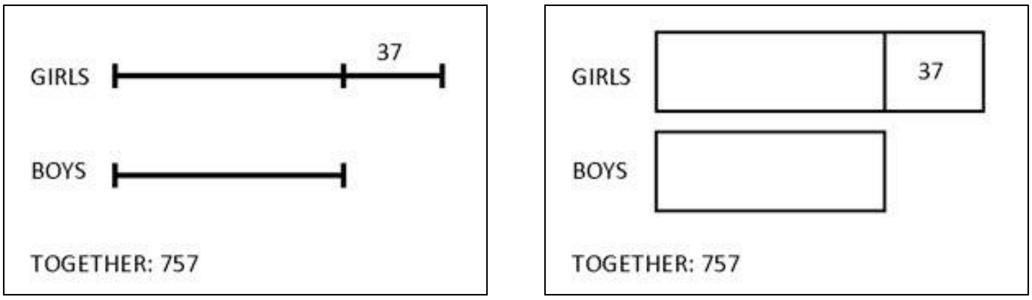

Figure 6: A segment model of the unequal partition problem outlined in Fig. 2 (left), a bar diagram of the same problem (right).

How to continue

As this text is a report of an ongoing study, we plan to continue in the research. We shall realize interviews with the participants of the study to reveal closer reasons for misconceptions that appeared in their responses, as well as reasons why none of them got use of a graphical representation.

As for CCs, preliminary findings of our study suggest that CCs could be used as a diagnostic tool for investigating future primary teachers' grasping of a situation. For the future we consider an interesting the issue of how particular features of CCs could help to reveal particular parts of the process of grasping.

We also plan to systematize CCs from the perspective of mathematics content. We aspire to create a set of CCs that would match Czech educational traditions, and cover regularly all important aspect of primary school mathematics.

Finally, this study shows how advantageous is the possibility to use a CC created by somebody else as a base for mediating our own thoughts and views. We may take such a CC, and adapt the content of its bubbles according to our intentions and previous experience. Our example with bottom left bubble in Fig. 2 illustrates how suitably chosen content of a newly added bubble can help to reveal a misconception that would stay unnoticed not only in the standard written exam, but probably also when working with the original version of the CC.

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REFERENCES

Baroody, A. J., Feil, Y., & Johnson, A. R. (2007). An alternative reconceptualization of procedural and conceptual knowledge. *Journal for Research in Mathematics Education,* 38, 115-131.

Dabell, J., Keogh, B., & Naylor, S. (2008). *Concept Cartoons in Mathematics Education.* Sandbach: Millgate House Education.

Janvier, C. (Ed.). (1987). *Problems of representation in the teaching and learning of mathematics.* Michigan: Lawrence Erlbaum Associates.

Keogh, B., & Naylor, S. (1993). Learning in science: another way in. *Primary Science Review,* 26, 22-23.

Kittler, J., & Kuřina, F. (1994). *Matematika pro 2. ročník základní školy. Pracovní sešit 1.* [In Czech: *Mathematics for the 2nd grade of primary school level. Workbook 1.*] Praha: MÚ AV ČR & ALBRA.

Koman, M., & Tichá, M. (1998). On travelling together and sharing expenses. *Teaching Mathematics and Its Applications,* 17(3), 117-122.

MacGregor, M., & Stacey, K. (1998). Cognitive models underlying algebraic and non-algebraic solutions to unequal partition problems. *Mathematics Education Research Journal*, 10, 46-60.

Naylor, S., & Keogh, B. (2012). *Concept Cartoons: what have we learnt?* Paper presented at the Fibonacci European Conference, Leicester. Retrieved from conceptcartoons.com/research.html

Nesher, P., & Hershkovitz, S. (1994). The role of schemes in two-step problems: analysis and research findings. *Educational Studies in Mathematics,* 26, 1-23.

Novotná, J. (1997). Using geometrical models and interviews as diagnostic tools to determine students' misunderstandings in mathematics. In *Proceedings of SEMT, Prague, Czech Republic, 1997,* 61-67.

Polya, G. (2004). *How to solve it. A new aspect of mathematical method.* Princeton and Oxford: Princeton University Press.

Samková, L., & Hošpesová, A. (2015). *Using Concept Cartoons to investigate future teachers' knowledge.* Paper presented at the Ninth Congress of the European Society for Research in Mathematics Education, CERME 9, Prague, Czech Republic; February 4-8, 2015.

Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher,* 15, 4-14.

Tichá, M. (2009). Problem posing in teacher training and as a research object in didactics: Two complementary perspectives. In *Proceedings CIEAEM 61 – Montréal, Canada, 2009, Vol. 2,* 445-449.

Tichá, M. (2014). On the way to discovering structure of word problems. In M. Pytlak (Ed.), *Proceedings of the Communication in the mathematics classroom, Poznan, Poland,* 43-45.

Tichá, M., & Hošpesová, A. (2010). Problem posing and development of pedagogical content knowledge in pre-service teacher training. In *Proceedings of CERME 6 – Lyon, France, 2010,* 1941-1950.

Tichá, M., & Hošpesová, A. (2013). Problem posing as a motivational tool in improving primary school teacher knowledge. In *Proceedings CIEAEM 65 – Turin, Italy, 2013, Vol. 1,* 307-316.

Tichá, M., & Hošpesová, A. (2015). *Word problems of a given structure in the perspective of teacher training.* Paper presented at the Ninth Congress of the European Society for Research in Mathematics Education, CERME 9, Prague, Czech Republic; February 4-8, 2015.

Sociocultural contexts as difficult resources to be incorporated by prospective mathematics teachers

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Abstract: The paper explains a collaborative research in which we built a cycle of training named "learning to teach citizenship through mathematics". During such transversal training cycle, it was observed the difficulty that future teachers have to introduce contextualized activities during their own mathematical practices as part of their didactical analysis.
Keywords: contextualized activities, didactical analysis, and teacher training

Résumé : Le document explique une recherche collaborative dans laquelle nous avons construit un cycle de formation intitulé "apprendre à enseigner la citoyenneté à travers les mathématiques". Durant ce cycle de formation transversale, on a observé la difficulté que les futurs enseignants doivent mettre en place des activités contextualisés au cours de leurs propres pratiques mathématiques dans le cadre de leur analyse didactique.
Mots clé: activités en contexte, analyse didactique, formation des professeurs

Introduction and theoretical background

This paper draws on our institutional training experience to reveal difficulties and successful elements regarding mathematics teacher training, in the frame of our Masters' degree on Teacher Training for Middle and High School Teachers, which is compulsory in order to be a Math Teacher in secondary schools in Catalonia (Spain). The degree involves a total of 600 hours of class. In our research, we start from the analysis of productions and reflections of the future teachers, as a fundamental process for teach them how to "redesign professional tasks". This experience is part of a research involving professors from six Latin American universities. The aim of this research is looking for the recognition of difficulties to design proposals enabling high school future mathematics teachers develop critical thinking and citizenship competencies through mathematics.

According to De Lange (1996), there are basically four reasons to integrate contextualized problems in the curriculum: a) facilitating the learning of mathematics, b) developing the mathematical skills of citizens, c) developing competencies and general attitudes associated with problem solving and d) allowing to see the usefulness of mathematics to solve both situations from other areas and everyday life situations. To contextualize and decontextualize as a set of processes, enables us to interpret mathematics as a tool of knowledge in order to establish a natural relationship with basic activities of human beings.

On the one hand, such a contextualized perspective promotes abilities in the use of techniques and mathematical models that explain situations of everyday life, and on the other hand, it also promotes the ability to evaluate its role in situations that exceed the needs of the private life of the individuals. All this will allow people to develop a perception of the nature incorporating the "mathematical" knowledge-basis, thus helping to make visible the mathematics (Niss, 1995). Drawing on such perspective, the contextualized look that has been described above suggested to us the consideration of five axes of analysis: (a) a look at the social aspects about the mathematical work; (b) an analysis of classically psychological problems such as the analysis of meanings, interactions, and processes of construction, which usually are present in the common part of the curricula of the training program; (c) an emphasis on the modeling perspective; (d) a global feature about experiencing mathematical practices and (e) an approach of teaching reflection based on the need of a set of different material, personal and theoretical mediators as didactic analysis tools, and evaluation tools.

The training cycle and the research associated

To focus such holistic perspective, it is clear that we must follow a path that will lead the future teachers of mathematics to: (1) knowing contextualized interesting experiences, (2) to identify what are their strong and weak points, (3) to have theoretical tools in order to analyze this type of practice, (4) to know how to organize and design contextualized practice, recognizing the type of mathematical knowledge involved, and the processes that are magnified, (5) to recognize the richness of processes that are involved in examples which also draw upon everyday life, from a modeling perspective, (6) to intertwine elements to integrate such practices in the construction of sequences in future teachers' school planning, and (7) to have elements to improve these practices, once implemented. From a realistic perspective, (8) training for citizenship and mathematical communication is also added as transversal issue.
The inter-agency proposal of training has been organized into three large blocks of tasks which will be described now in terms of their contribution to citizenship through mathematics education. In fact a classic psycho-socio-pedagogical block shows videos, texts and social problems affecting mathematics education. A mathematical perspective about modeling has been discussed from the Anthropological perspective and Project work.
Some historical problems (as the elasticity problem when Hooke did trials to obtain the proportionality between the force and distance changing) had been introduced emphasizing how they contributed to some changes in the history of humanity and in some of them to assume a sense of belonging to a particular community.
The second block of mathematical topics is focused on how it's possible to develop in the classroom processes through an interdisciplinary perspective, focusing on elements such as problems of housing, and in general, addressing issues of social and natural contexts in a class of mathematics. A specific project called "Tàndem" is presented observing how it works on the basis of social consumption, nutrition, and habitability problems.
A third block deals with different aspects related to mathematics education training. It deals with the knowledge of the evolution of theoretical approaches about teaching mathematics, as the realistic mathematics education. During this part, we discuss the idea that resources allow to set mathematically significant situations, where some of them are contextualized practices. Innovative experiences were shared as the role of teachers associations, the repository of resources of CREAMAT, and its contribution to teacher professional development.
A set of 11 professional tasks have been organized. Table 1 shows three examples of activities, in which some different contexts influence the relationship between Math and learning to improve Citizenship through mathematical activities when we implemented the professional tasks above cited.

Table 1. Some example of tasks in the unit

Main idea of the professional task	**Math content involved**	**Scope in the formation of learning to train in citizenship through mathematics**
Reflecting on citizenship transversal competencies and contextualized mathematics	Extra-mathematical contexts Mathematization	Competencies associated to different contexts
Integrating contextualized practices. Identifying the idea of connection. Rating the interdisciplinarity and math enculturation	Extra-mathematical: context natural world and cultural contexts. Connectivity analysis. It affects multiple representations. To interpret	Recognition of that from cultural elements to do mathematics, allows a possible empowerment of students, since models generate or interpret models developed by

	phenomena is recognized as research tasks of long-term projects	others. By insisting on aspects of the Ethnomathematics, we interpret the same mathematics as problem solving and modeling activity
Observations about the use of citizenship in the hot analysis of a mathematics school practice	Description and analysis of an implementation in the school.	Identification of citizenship characteristics, in their school proposal
Self-delayed analysis of school practice.	Epistemic analysis in which it's described the need of more tasks in the classroom experience	Observing the use of improvement criteria coming from the theoretical perspectives introduced for didactical analysis including citizenship ideas

About the research developmental process.

During the three year research process, we implemented and reconstructed the tasks in the unit. The global task planning and redesign in relation to didactical analysis, was described in detail in another article (Giménez, Vanegas and Font, 2013). The Project was built with the following traits: (a) Reinforcing cognitive and epistemic values through didactical analysis lenses in order to see the need for professional development as lifelong learning process. (b) Identifying the role of promoting Social Transversal competencies as citizenship, critical perspective, creativity, learning to learn through Math practices, interpreted as a modelling culturally developed human activity. (c) Analyzing social –cultural variables as family involvement, historical development. Introducing Reflective Collaborative Enquiry attitude, when doing and analyzing math practices, by using suitability criteria (OSA). (d)Improving self-confidence by doing Final Master's work as a first delayed self-reflective process. Our main aim was to identify epistemic difficulties for professional development changes. We search for immediate impact by analyzing final work.

We assigned a set of characteristics according to the use of indicators of citizenship and we compute the indicators according to different levels. In the table, we see the average of future teachers in each level, in order to see that we observe better results after years.

	Level 1	Level 2	Level 3
2012 (n=25)	52 %	29 %	19 %
2013 (n=28)	51 %	31 %	18 %
2014 (n= 49)	44 %	32 %	24 %
2015 (n= 47)	46 %	28 %	26 %

Table 1. Assigned levels for future teachers according the level of citizenship

In general, the future teachers had difficulties to relate didactical analysis to epistemic mathematical ideas. For instance, Future teacher 5 says *"When I did the didactic unit I didn't contextualize* *enough the exercises. Now, I think it's important to use activities proposed in the article: 'Algebra for all Junior High School students'.* In these kinds of sentences, we expected to talk more about the specific iterative algebraic approach as an explicit content in the article explained by the student. It's an example of the initial difficulties to accept the role of an epistemic and cognitive analysis. They talk about problem solving needs, not introduced in his practice, coming from Polya perspective. And some of them also explain the need to articulate the role of letters and unknowns recognizing the richness of processes in their practice.

It was observed that the students focus more on the dialogue than the mathematics involved. Future teacher 12 says, "*short challenges appear, with follow up questions in order to engage students in brief conversations just to clarify responses*", and many others as Student 6, talks about "*the teacher remains vigilant in order to ensure that classmates did not distract students*".

The future teachers had a few autonomy to apply in the design and implementation many learned knowledge. This aspect was considered a difficult problem to solve during redesign process because of institutional framework for the proposal, which did not deal a selection of schools.

Researching upon the reflections that future teachers made after their own practice in what is called Final work of Master (TFM), we see short references to the intentional teacher curriculum, and their declaration of intent regarding to integrate contextualized practices. We also found that future teachers improved their general didactical analysis of tasks, out of the citizenship arguments. They also relate some mathematical comments to the difficulties they found when using intra-mathematical contexts instead of contexts coming from the society. As an example, future teacher 8, relates the ambiguity in front to a theoretical article. He said about "*the need of searching analogies found because of an incorrect use of contextual framework*". He read a text from Reed to reflect about the use of two important variables influencing the decisions of the teacher.

> "*The context understood as a set of traits perceived in a certain problematic of real world involving objects, and facts"… But, the laws, principles, relations among quantities, and equations, constitute the structure of a problem*". It is interesting that the future teacher explain some conclusions from this discussion: "*the need to describe the similarities and differences among structures and surfaces of the source problems and aiming problems, because it influences the decisions about the equations presented to solve the aiming problems. It is also important to identify that familiarity can help the transference processes, but it also could be an obstacle to see the similarities and structural differences among problems"*.

We also analyze how the future teachers faced the use of internal and external connections in their school practice, and also in their self-reflection after the school practice. Some future teachers relate their comments to their previous background and explain the need for including applications to other disciplines.

> *As an economist, I can say the use of systems of equations to find equilibrium points, as intersection of different conditions, interpreted by curves of offer and demand… and planning problems of dead points… programming problems… We also use algebra as a process to solve engineering problems… chemistry problems, understanding digital images…* (St 12)

Some future teachers told about intra-mathematical connections, when algebraic systems of equations are used as referent knowledge for optimization problems…We assume that some of these knowledge must be introduced and adapt according the age of the students.

In terms o**f representativeness,** some teachers, tell us that it's needed not only a look for meanings, but to see a historical, epistemological and curricular perspective. When analyzing the particular case of algebra, the future teacher 8, for instance, proposed a set of ideas about the Arabic way of solving problems to be introduced next time. In this case, he just offer a reflection about "*considering algebra as part of cultural legacy*".

Conclusions

The experience shows that it is possible to overcome some classical epistemic view about math as a finished product, at least in their intentions, and focus a lot about social issues, perhaps due to our epistemic pressure and self-reflection about the Program itself.
Prospective mathematics teachers recognize school math activity as being involved in school math "interesting contextualized practices". As useful for life as possible, but, it remains the belief that contextualization takes a lot of time. The future teachers identify and exemplify in their practices the ideas of connectiveness and representativeness of knowledge and the ideas of interdisciplinarity and enculturation. Nevertheless, they didn't achieve to interpret how to develop citizenship and the influence of the critical thinking in their own practices.
Only half of the future teachers mention explicitly "citizenship" in their TFM, although in most of the cases we perceived some examples trying to introduce contextual elements as social problems, or cooperative work, in their redesigned activities. Likewise, they considered that the context is important not only to motivate students, but also to generate cross-disciplinary skills through mathematics. Almost all future teachers recognize the value of the critical dialogue to build excellent mathematical meanings.
Thus, for example, almost all of them mention the role of dialogue as an instrument to develop critical thinking. In some cases where future teachers developed small research we conjecture that they consider math as a way to offer opportunities and powerful tools to interpret phenomena. For instance, in one of the proposals of redesigned-activity, we see a student who initially had difficulties addressing the cone from a contextualized perspective, proposing the use of the LORAN (Long Range Navigation) navigation system to address the problem of a ship that had lost its location when navigating between two cities (Castelldefels and Torredambarra).
This was an example to explain the tapered path. This example allowed that future teacher to address complex situations that are not usual in high school math classes. Another student referred to the same idea when he said: "Contextualization links knowledge to a need and not a scheme set by the index of a book. Learn the concepts when you'll need them brings motivation to learning and strengthens the competence "learning to learn", since students learn how to make connections between concepts and their uses, and also allows you to use the learning in different contexts. If we use social contexts we will help them to develop their social skills and citizenship."
One well-known constraint for Teacher education, is that new curricula (in many countries) introduces good educational guidelines (as the need for contextualization, connections, citizenship, the role of professional reflection, and so on) just as "beginning sentences" when introducing the programs for disciplines as general aims. Therefore, the need of inter-related explanations to the mathematical content, that must be interpreted as resources. In "many of Teacher Training programs", there is not enough time for consolidating personal changes. The reflection about social variables as communication, implication, family involvement, …) Therefore, the need of relating pre-service and in service teacher training, in which young teachers should be self-motivated being included in reflective enquiry teams with teacher-researchers.

Acknowledgement

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REFERENCES

CASADAVALL, M. (2010). L'educació matemàtica a través del treball en contextos no matemáticas. Memòria del treball de Llicència d'estudis retribuïda. Generalitat de Catalunya. Retrieved http://www.xtec.cat/sgfp/llicencies/200809/memories/1877m.pdf

DE LANGE, J. (1996). Using and applying mathematics in education, In: Bishop et al., International handbook of mathematics education, Dordrecht, 1996, Kluwer A.P., pp. 49–97.

FONT, V. (2007) Comprensión y contexto: una mirada desde la didáctica de las matemáticas. En: La gaceta de la Real Sociedad Matemática Española, Vol. 10.2 (2007), 427– 442.

NISS, M. (1995). Las matemáticas en la sociedad. UNO. Revista de didáctica de las matemáticas. 6, 45-58. Barcelona. Graò.

VANEGAS, Y. M. (2013). Competencias ciudadanas y desarrollo profesional en matemáticas. Unpublished phD Universidad de Barcelona, Barcelona, España.

VANEGAS, Y.; GIMÉNEZ, J.; FONT, V. & DÍEZ-PALOMAR, J. (2014). Improving reflective analysis of a secondary school mathematics teachers program, In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36, Vol. 5, pp. 321-328. Vancouver, Canada: PME.

Instrumentation didactique des futurs enseignants de mathématiques.
Exemple de la co-variation

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Abstract : This work falls within the theoretical framework of the double instrumental genesis (Rabardel 1995, Tapan 2006). We are interested in the construction of techno-educational tools by teachers. We offer an analysis of two initial training sequences on the concept of co-variation using a dynamic geometry software. We highlight the fact that the didactic genesis is not necessarily correlated with advanced technical instrumentation. We use a classical educational tool (a priori analysis -- Charnay, 2003) to engage students in a didactic instrumental genesis.

Résumé : Ce travail s'inscrit dans le cadre théorique de la double genèse instrumentale (Rabardel 1995, Tapan 2006). Nous nous intéressons à la construction d'instruments techno-didactiques par les enseignants. Nous proposons l'analyse de deux séquences de formation initiale portant sur la notion de co-variation avec un logiciel de géométrie dynamique. Nous mettons en évidence le fait que la genèse didactique n'est pas nécessairement corrélée à une instrumentation technique poussée. Nous proposons une utilisation de l'outil pédagogique classique constitué par les analyses a priori (Charnay 2003) pour engager les étudiants dans une genèse instrumentale didactique.

Introduction

Le travail présenté ici repose sur une volonté de favoriser l'intégration de la technologie dans l'enseignement des mathématiques. Ainsi que le remarquent Leclère et *al.* (2007) :

> « La plupart des politiques visant à équiper les établissements ou à former les enseignants n'ont pas abouti à un développement important des usages en classe. Les recommandations incitatives, notamment dans les programmes officiels, n'ont pas suffi non plus à dynamiser les pratiques de façon significative. »

De nombreux chercheurs ont mis en avant la difficulté à changer les pratiques comme frein à l'intégration des TICS. Qu'en est-il pour les futurs enseignants qui sont en train de développer leurs propres pratiques? Bien que baignant dans une culture numérique, ils ne semblent pas plus enclins à intégrer la technologie dans leurs pratiques quotidiennes que leur ainés. Nous défendons ici l'idée qu'une bonne intégration de la technologie ne repose pas tant sur des compétences techniques poussées que sur une conscience des possibilités didactiques offertes par la technologie. Nous cherchons donc à développer une réflexion techno-didactique chez les enseignants, aussi bien en formation initiale que continue. Nous appuyons nos réflexions sur deux séquences de formation technologique dispensées dans le cadre des cours MAT3225 (Didactique de la variable et de la fonction) et MAT4812 (explorations mathématiques à l'aide de l'informatique) aux étudiants du Bac en Enseignement des Mathématiques de l'UQAM. Ces séquences portent sur le concept de co-variation, l'accent étant mis sur l'articulation entre les différentes représentations d'une fonction. L'objectif est double: amener les futurs enseignants à appréhender aussi bien qualitativement que quantitativement une situation de co-variation, et les amener à une réflexion didactique sur les possibilités offertes par les logiciels de géométrie dynamique pour articuler ces représentations.

Cadre théorique

Nous nous inscrivons dans le cadre théorique de la genèse instrumentale (Guin & Trouche, 1998; Rabardel, 1995; Trouche, 2007) en reprenant essentiellement l'idée que l'artéfact informatique devient un instrument à travers le double processus de l'instrumentation (l'artéfact conditionne l'action du sujet) et de l'instrumentalisation (adaptation de l'artéfact à l'usage qui en est fait) (Hoyles et Noss, 2003). Ainsi que le souligne Trouche 2007, l'analyse des processus d'instrumentalisation et d'instrumentation est un outil aidant à suivre les processus de conceptualisation

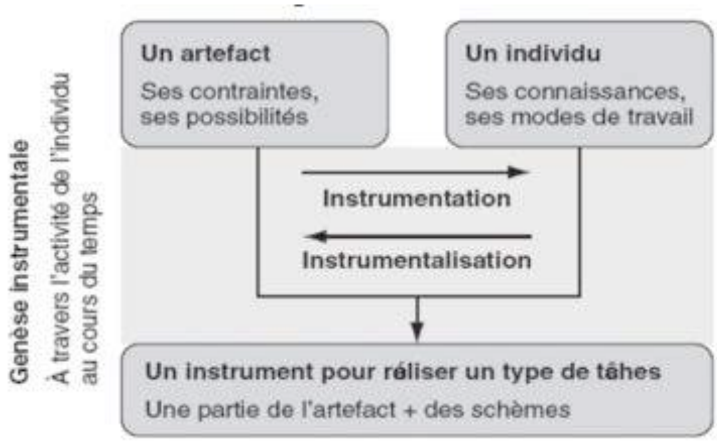


Figure 1. Genèse instrumentale (Trouche, 2007)

A travers ce double processus, c'est l'impact de l'instrument sur la conceptualisation mathématique et didactique qui nous intéresse. Nous reprenons l'idée d'une double genèse instrumentale proposée par Tapan (2006). Pour pouvoir intégrer de façon pertinente la technologie dans leur enseignement, les enseignants doivent non seulement construire des instruments leur permettant de résoudre eux-mêmes des tâches mathématiques, mais aussi et surtout des instruments didactiques leur permettant d'enseigner les mathématiques. Nous ferons donc la distinction entre le premier niveau d'instrumentation, que nous appelons instrumentation technique et le second que nous appelons instrumentation didactique. Dans le cas de futurs enseignants, on peut faire l'hypothèse que la genèse instrumentale didactique se fait en parallèle avec la genèse des connaissances didactiques et s'appuie sur des connaissances mathématiques dont certaines sont récemment acquises. C'est le cas pour la co-variation.

La partie psychologique de la genèse d'un instrument peut-être appréhendée par l'intermédiaire des schèmes mis en place. Nous reprenons ici la caractérisation de schème donnée par Vergnaud (1996): une organisation invariante de l'activité, incluant notamment des règles d'action, structurée par des invariants opératoires (connaissances implicites) qui se forgent au cours de l'activité et qui la pilotent. Ce sont les schèmes que le sujet constitue dans le cadre de la réalisation d'un type de tâches. Ces schèmes font partie intégrante de l'instrument construit par le sujet.

Notre objectif est de mieux comprendre le processus d'instrumentation didactique des futurs enseignants de mathématiques et de mettre au point des outils de formation permettant à la fois d'observer et d'accompagner leur genèse instrumentale.

Registres sémiotiques de la fonction

Les situations à l'étude ont été sélectionnées selon leur adéquation aux préconisations du MELS (Programme de formation de l'école québécoise -- Second cycle du secondaire en mathématiques -- MELS, p. 51)

> « Au cours de sa formation, l'élève […] développe son habileté à modéliser des situations…[…] Il améliore aussi sa capacité à évoquer une situation en faisant appel à plusieurs registres de représentation. Par exemple, les fonctions peuvent être représentées graphiquement ou sous forme de tableau ou de règle, et chacune de ces représentations est porteuse d'un point de vue qui lui est propre, complémentaire ou équivalent aux autres. »

Le programme fait ici explicitement référence aux registres de représentations sémiotiques tels qu'ils ont été définis par Duval (1993). De nombreuses recherches (Carlson 1998; Hitt 1998; Monk 1992; De Cotret 1985) soulignent les difficultés éprouvées par les étudiants avec le concept de fonctions et plus particulièrement avec l'articulation entre les différents registres de représentation. Nous avons sélectionné des situations qui s'appuient sur la géométrie dynamique pour permettre une conception plus globale de la co-variation, basée sur des allers et retours entre différents registres sémiotiques.

Duval (1993) définit trois critères constitutifs d'un registre sémiotique : l'existence de représentations identifiables, l'existence d'activités de traitement, c'est-à-dire de transformations d'une représentation à une autre au sein du même registre et enfin l'existence de mécanismes de conversion vers d'autres registres. Se basant sur ces critères, Passaro (2007) considère deux registres de représentation pour la fonction, que nous reprenons ici : le registre graphique et la situation. Nous nous intéressons ici à des situations de co-variation, c'est-à-dire une description verbale ou figurale d'une situation, accompagnée de l'identification explicite des grandeurs en co-variation. Nous considérons également un troisième registre : le registre symbolique, c'est-à-dire la représentation d'une fonction par une équation de type y=f(x).

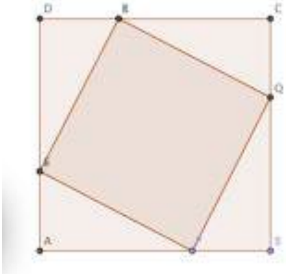
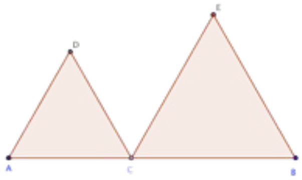
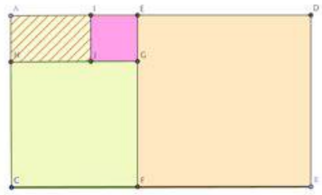
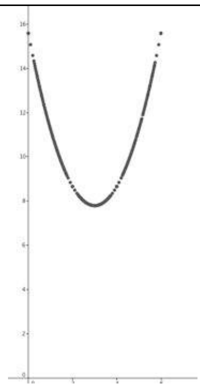
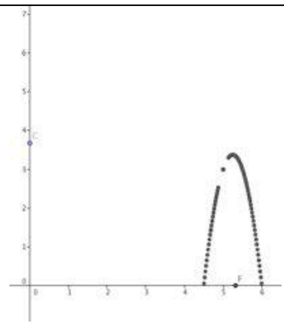
Les fonctions du déplacement

La géométrie dynamique repose sur l'utilisation du déplacement, c'est-à-dire la possibilité de déplacer les éléments d'une figure avec la souris. Les fonctions du déplacement ont été étudiées depuis longtemps. Restrepo (2008) propose une très bonne synthèse des différentes utilisations du déplacement (déplacement erratique, limité, guidé, discret, continu…) Elle montre également que l'utilisation de déplacement n'est immédiate ni pour les élèves ni pour les enseignants. Une utilisation efficace et pertinente du déplacement repose donc sur une instrumentation efficace. Ainsi que le souligne Soury-Lavergne (2011), « un savoir didactique sur la géométrie dynamique est le fait que le déplacement peut avoir plusieurs fonctions. ». Les situations que nous avons choisies reposent sur deux fonctions du déplacement :

- le déplacement pour explorer, qu'il soit continu (exhiber une continuité de configurations intermédiaires entre deux configurations données) ou discret (exploration de quelques configurations particulières),
- le déplacement pour visualiser la trajectoire un point, reposant sur l'utilisation de l'activation de la trace d'un point modélisant la situation fonctionnelle.

Situations exploitées

Le tableau ci-dessous récapitule les situations mises en jeu ainsi que leurs différentes représentations. Les trois situations sont construites sur un même schéma : on part d'un énoncé écrit décrivant la situation, accompagné éventuellement d'un dessin statique illustrant cet énoncé (situation 3). Les étudiants commencent par construire la figure modélisant la situation dans le logiciel. La construction de la figure est facilitée par les outils de construction disponibles dans le logiciel, de sorte que l'obtention de la figure repose principalement sur une compréhension des relations entre les différentes grandeurs géométriques. A partir de là, on s'attend à ce que les étudiants utilisent le déplacement des points mobiles pour explorer la situation. Ils ont ainsi accès à une exploration qualitative de la situation, et notamment la recherche des valeurs limites. Le graphe peut ensuite être obtenu depuis la figure, par activation de trace d'un point représentant la co-variation, et piloté en déplaçant le point mobile sur la figure. Le graphe peut également être obtenu à partir de l'équation de la fonction. Les étudiants peuvent utiliser la figure dynamique pour donner du sens au graphique obtenu. La juxtaposition et la comparaison de toutes les représentations permettent de faire des liens entre elles, de leur donner du sens et de comprendre la nature de la relation en jeu (il s'agit de relations quadratiques)

	Situation 1	Situation 2	Situation 3[9]
figural			
verbal	Le carré PQRS est inscrit dans le carré ABCD de côté 4 cm. P est un point mobile sur [AB]	Les triangles ADC et CEB sont équilatéraux. Le point C est mobile sur [AB]	Le rectangle CBDA représente un bâtiment de largeur 9 cm. Les pièces colorées sont des carrés. La pièce hachurée est rectangulaire.
Co-Variation	Aire de PQRS en fonction de AP	Somme des aires des triangles en fonction de AC	Aire de la pièce rectangulaire en fonction de la largeur du bâtiment.
Graphique			
Symbolique	$f(x) = 2x^2 - 8x + 16$	On pose AB=a $h(x) = \frac{\sqrt{3}}{4}(2x^2 - 2ax + a^2)$	$h(x) = -6x^2 + 63x - 162$

[9] D'après : http://eduscol.education.fr/bd/urtic/maths/index.php?commande=aper&id=2769

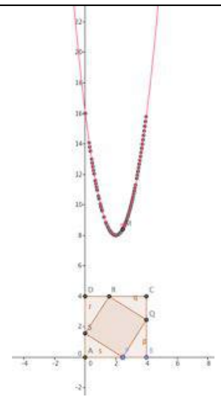
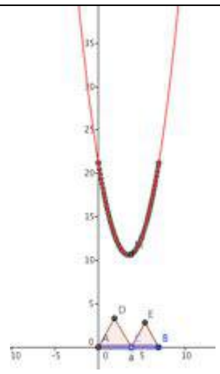
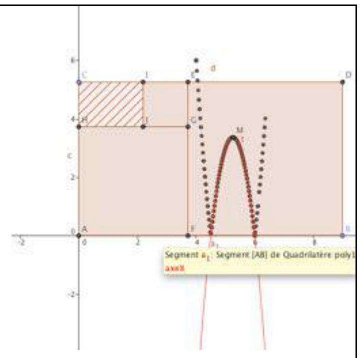
Juxtaposition et comparaison			

Tableau 1. Les situations utilisées

Séquence 1- cours MAT3225 : didactique de la variable et de la fonction

Ce cours est un cours de mathématique et de didactique. Il vise à la fois à renforcer les concepts mathématiques des étudiants et à les amener à un recul didactique dans une perspective d'enseignement. Le cours aborde les liens entre les différentes représentations d'une fonction. On s'intéresse ici à une séance destinée à sensibiliser les étudiants à l'apport possible des outils technologiques pour l'enseignement de la fonction. Lors de cette séquence, on présente en grand groupe la situation 1 ci-dessus. La figure est déjà construite, le formateur l'anime en avant. Les étudiants travaillent ensuite en dyades sur l'activité 2 avant un retour en grand groupe. Nous avons capté et analysé l'activité de deux dyades.

Déroulement de la séquence pour l'équipe 1

Cette équipe est formée de deux étudiantes qui ne maitrise pas tout à fait le logiciel GeoGebra mais n'ont aucune réticence à l'utiliser. Leur instrumentation est en cours et nous avons montré dans Tremblay & Venant (2015) que certains schèmes élémentaires comme « *distinction entre un point mobile et un point fixe* » ne sont pas encore tout à fait en place pour ces étudiantes. On voit par exemple dans l'extrait ci-dessous que le schème « *création d'un segment de longueur donné* » est découvert à l'occasion de cette activité :

> **Rosalie**: C'est juste pour savoir, AB, y'es fixe, c'est tout ?
> **Formateur**: Ouin, oui, c'est ça. Donc là on se donne un d'une longueur …
> **Rosalie**: Ah oui, je le vois là, ça c'est bon, regarde, regarde je suis en train de le voir... Regarde, tu fais - segment de longueur donnée.

Malgré cette instrumentation fragile, les deux étudiantes réussissent à construire un instrument intéressant pour explorer la situation. Construire la figure leur prend un certain temps (environ 4mn) mais à aucun moment elles ne perdent de vue l'enjeu mathématique de la tâche: explorer une situation de co-variation. Une fois la figure obtenue, elles prennent le temps de déplacer le point C et d'analyser les effets de ce déplacement. Il est à noter que malgré la fragilité de leurs schèmes, elles ont parfaitement instrumenté le déplacement pour explorer.

> **Florence**: Dans le fond! Mets le donc au centre, ils vont avoir la même aire.
> **Rosalie**: Ça c'est poly1, poly2, avec les couleurs, pis chaque fois ça donne, valeur de ce nombre là, c'est l'aire du polygone. Regarde tu vois, ils ont la même aire.
> **Florence**: Ok. Et là si j'en rapetisse un? [Elle prend la souris] c'est-tu des moitiés ?
> **Rosalie**: À 2.5
> **Florence**: Ok
> **Rosalie**: Tu passes à 2.71 à 24.
> **Florence**: Tantôt on avait 20 quelque d'aire, là on a...

C'est bien la nature de la co-variation qu'elles cherchent à élucider. Cependant leur utilisation du déplacement est discrète. Elles ne cherchent pas à se faire une vision globale du phénomène mais visent rapidement des positions caractéristiques du point C (au milieu, au quart du segment AB) et entrent directement dans une approche quantitative. L'instrument qu'elles construisent relève donc plus de l'instrument de mesure perfectionné que de la représentation dynamique. Cette observation les amène cependant assez rapidement à dégager la grandeur la plus pertinente à étudier : la somme des aires des deux triangles.

Florence: On est rendu quasiment à 27 là. Plus que 27 même.
Rosalie: De quoi là? Ça c'est 24 ça.
Florence: Si tu calcules l'aire totale des deux
Rosalie: Ah d'accord.

Elles utilisent pour établir une table de valeur en effectuant des mesures à bonds constants. Cette approche quantitative aboutit et elles en viennent à découvrir la nature quadratique de la relation.

Florence : Ah! That's it! (Elles se tapent dans les mains) C'est une parabole!

Elles cherchent ensuite à vérifier leur conjecture en traçant le graphique. Elles connaissent l'existence du schème « *activer la trace d'un point modélisant les grandeurs en relation* ». Elles sont même conscientes de certains invariants opératoires mais ne sont pas capables de les mettre en œuvre. Leur problème est essentiellement de créer les variables correspondant aux grandeurs en co-variation :

Rosalie : De quoi le sommet? Ah ouais... mais il faudrait faire le sommet avec une trace active.
Rosalie : on pourrait créer un point de coordonnées distance AC, mais ça c'est h, de h et i.

Avec l'aide du formateur, elles finissent par obtenir le graphique. Elles commencent alors une exploration plus qualitative, cherchant à faire le lien entre les calculs qu'elles ont effectués, le tableau de valeurs obtenu, le déplacement du point de coordonnées dynamiques sur le graphique et la position du point C.

Florence : Pourquoi qu'il part là ? Pourquoi qu'il est dans les airs ?
Rosalie : Ben, un, il ne va jamais y avoir une aire de zéro, tu comprends?
Florence : Ok donc là il est à 5, et la plus petite aire qu'on peut avoir c'est quoi ?
Rosalie : C'est à 5.
Florence : Qu'est-ce qu'on avait dit ? C'est à 21,5. C'est la plus petite somme qu'on va avoir.
Florence : Ah OK je comprends. Et puis le maximum c'est quoi ? C'est à 43,3 ?
Rosalie : on peut se rendre à AC=0 pis à AC=10. Il faudrait
Florence : j'comprends, j'comprends tout.

Elles sont interrompues à ce stade de leur exploration par le lancement de la mise en commun.

Bilan de l'activité pour l'équipe 1

L'instrument d'exploration construit n'est pas celui attendu, de ce fait l'exploration qualitative n'a pas réellement lieu. Les étudiantes restent dans une approche très quantitative. Du début à la fin, elles restent accrochées aux valeurs des aires affichées. On peut cependant supposer que si l'activité avait duré un peu plus longtemps, elles seraient entrées dans une exploration plus globale de la co-variation. Le fait que leur instrumentation technique n'est pas solide n'est pas tellement préjudiciable ici dans la mesure où le formateur travaille en permanence avec elle. On peut se demander jusqu'où elles auraient été si elles avaient été livrées à elles-mêmes. Si l'activité échoue à développer chez ces étudiantes une intuition globale d'une variation quadratique, elle leur permet cependant de donner du sens à la notion de graphique, avec des allers et retours systématiques entre la figure dynamique et le graphique, pour chaque position du point courant sur le graphique. On ne les voit à aucun moment prendre un recul didactique sur ce qu'elles sont en train de vivre, ni sur le rôle joué par la technologie dans cette exploration. Cependant, l'état de réflexion dans lequel elles se trouvent à la fin de l'activité les rend réceptives à la réflexion didactique proposée par le formateur durant la période de mise en commun.

Déroulement de l'activité pour l'équipe 2

L'équipe 2 est constituée de deux étudiants très à l'aise avec le logiciel, et la technologie en général, et fiers de l'être. Ils possèdent parfaitement les schèmes élémentaires. Ainsi construire la figure ne leur prend que 2 minutes, et encore parce qu'ils ont construits des triangles isocèles plutôt qu'équilatéraux. En revanche, ils sont concentrés uniquement sur les aspects techniques de la tâche et perdent facilement de vue les enjeux mathématiques et didactiques. Ainsi, après avoir construit la figure, ils sont obligés de revenir à l'énoncé pour se rappeler l'enjeu mathématique :

> **Olivier** : Maintenant qu'est-ce qu'on fait ? [Lisant la consigne au tableau.] On s'intéresse aux effets sur l'aire des triangles
> **Franck** : Ben fait afficher l'aire du polygone1, polygone2, polygone3, polygone 4. Fais juste t'assurer que 1 et 2 soient les triangles isocèles pis 3 et 4 les équilatéraux là.
> **Franck** : après ça met un curseur ben avec la somme des deux pis check qu'elle reste constante.

On voit que la réponse de Franck est purement technique. Son activité est complètement pilotée par les schèmes instrumentaux. Il ne cherche pas à anticiper quelles sont les grandeurs pertinentes ni la nature de leur relation. Il applique sans réfléchir le schème: « *afficher l'aire d'un polygone* ». En fait, il a une idée préconçue de la situation (la somme des aires des triangles est constante), et ne cherche pas à la mettre en question.

Les deux étudiants évacuent les enjeux mathématiques et didactiques pour se concentrer sur la réalisation d'un fichier GeoGebra le plus aboutit possible techniquement. Ainsi Olivier passe beaucoup de temps à peaufiner l'affichage des aires des triangles à l'aide de textes dynamiques au lieu de se contenter de l'affichage des variables correspondantes dans l'onglet algèbre du logiciel. Franck quant à lui cherche à tracer le plus de polygones possibles dans le même fichier. Il n'envisage pas cela comme une variable didactique mais plutôt comme un défi technologique.

> **Franck**: Ouais c'est ça, là ça va être mieux. Fait qu'on va avoir des triangles isocèles et des triangles équilatéraux. T'effaceras... ou tu mettras une couleur différente pour les autres triangles.
> **Olivier**: ouais c'est ça que j'comptais faire.
> **Franck**: Fait que comme ça on va être deux fois meilleurs que les autres puisqu'on aura deux triangles. Deux sets de triangles.
> **Nicolas**: Il y en a qui vont...
> **Franck**: Ouais et tu feras des hexagones après qui ont la même base. On va voir si avec des hexagones on va se faire battre.

C'est Olivier qui finit par ramener la tâche sur l'obtention de graphiques mais sans qu'aucune exploration de la figure dynamique ait eu lieu.

> **Franck** : pis après ça tu vas faire les hexagones.
> **Olivier** : tu veux pas qu'on voit comme… les courbes ?
> **Franck** : ben c'est juste pour qu'on… soit meilleurs qu'les autres (rires).
> **Olivier** : j'préfèrerais faire… aller voir la courbe à quoi qu'elle ressemble
> **Franck** : Ok d'abord.

Comme ils maitrisent parfaitement la création de points de coordonnées dynamiques, ils vont multiplier les traces actives. Leur but est de visualiser toutes les combinaisons possibles entre les différentes grandeurs présentes dans la figure (aire des divers polygones en fonction de la distance AC, aires d'un triangle isocèle en fonction de celle d'un triangle équilatéral…)
Ils obtiennent finalement trois courbes sur lesquelles ils ne prennent pas le temps de s'interroger, obsédés par l'idée d'en obtenir de nouvelles avec d'autres polygones. La mise en commun commence alors qu'ils peaufinent leur fichier par un jeu de couleur sur les courbes. Ils ne questionnent par le fait que plusieurs des points dynamiques génèrent la même courbe (avec 7 points dynamiques, ils obtiennent 3 courbes dont une droite). Ils ne s'interrogent pas non plus sur la nature des relations traduites par les courbes et ne reviennent jamais vers la figure dynamique.

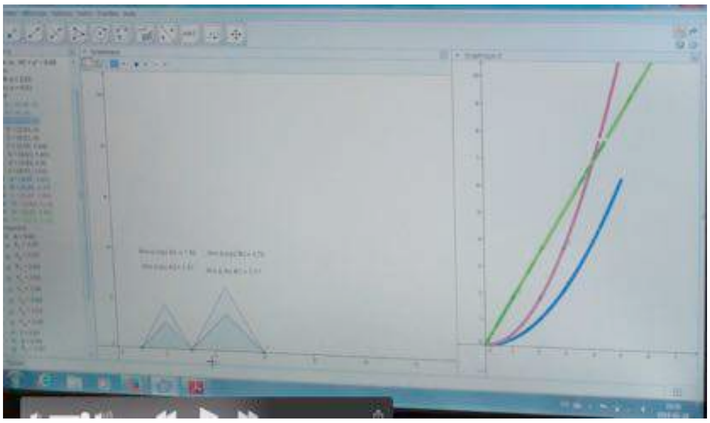


Figure 2. Courbes obtenues par l'équipe 2

Bilan pour l'équipe 2

Cette équipe n'a pas non plus construit l'instrument d'exploration et d'articulation des représentations attendu. Les enjeux technologiques prennent ici le pas sur les enjeux mathématiques. Les étudiants perdent de la nature mathématique de la tâche (explorer une situation fonctionnelle, comprendre la nature d'une relation de co-variation) pour entrer dans une sorte de défi technique. Il est difficile ici de conclure quant à leur conceptualisation de la co-variation, mais on note que le recul didactique attendu n'a pas lieu. A l'issue de l'activité les étudiants n'ont pas perçu les subtilités offertes par le logiciel en termes d'articulation des représentations, bien qu'aucun obstacle instrumental ne se soit interposé entre eux et la tâche. De plus, comme pour eux l'enjeu de la tâche est la réalisation technique du fichier GeoGebra, il n'est même pas sûr que la mise en commun, mettant l'accès sur les aspects didactiques de la situation leur ait été profitable.

Bilan pour la séquence 1

On peut dire que l'activité n'a pas atteint son potentiel ni pour une équipe ni pour l'autre. Les étudiants ne semblent pas mûrs pour prendre du recul didactique sur cette activité. Peut-être parce que la notion de co-variation est encore trop nouvelle pour eux et qu'ils adoptent plus volontiers une position d'étudiants que d'enseignants relativement à cette notion. On voit même avec l'équipe 1 que le graphique d'une fonction est un concept qui n'a pas encore pris tout son sens. A ce stade de leur formation, l'activité intervient donc plus comme une occasion d'approfondir leur propre connaissance du concept de co-variation. On voit, par le contraste entre les équipes 1 et 2 que le niveau d'instrumentation technique, et le fait d'être ou non en autonomie complète, jouent un rôle important sur le type d'activités cognitives qui vont avoir lieu. Les formateurs interviennent plus naturellement auprès des étudiants qu'ils savent moins à l'aise avec la technologie. Or, ce n'est pas parce que des étudiants sont autonomes du point de vue technique qu'ils développent un contrôle mathématico-didactique sur la tâche qu'ils réalisent. On voit qu'une très bonne instrumentation n'est pas forcément garante de la construction d'un instrument d'exploration mathématique et didactique. Le rôle joué par les formateurs est donc crucial. Ils sont responsables de la bonne orchestration des instrumentations didactiques (Trouche, 2007).

On voit donc que bien qu'étant des futurs enseignants, les étudiants ne se lancent pas naturellement dans une prise de recul didactique sur les activités qu'ils vivent. Dans cette séquence, c'est le moment de la mise en commun qui a suivi l'activité qui a été choisi pour les inciter à ce recul. Cependant nous n'avons aucun moyen de mesurer les effets concrets de cette façon de procéder. C'est pourquoi nous avons mis au point une deuxième séquence mettant en oeuvre un outil d'analyse (analyse a priori : Charnay (2003)) destiné à favoriser et recueillir les schèmes didactico-instrumentaux.

Séquence 2 : cours MAT4812- Explorations mathématiques à l'aide de

l’informatique

C’est un des derniers cours de la formation des futurs enseignants. A ce stade, les étudiants sont considérés comme des experts mathématiques et on cherche à provoquer chez eux une réflexion didactique. La séquence se déroule en deux temps :

Phase 1: situation 2

Le travail est collectif et se fait en grand groupe, en salle machine. Les étudiants manipulent en même temps que le formateur. La construction de la figure et la réflexion sont collectives et partagées. Les étudiants doivent ensuite proposer une analyse a priori de l’activité. Certains des étudiants ont déjà exploré la situation lors de la séquence 1 et d’autres non. L’accent est mis sur les aspects didactiques.

À la lecture des analyses a priori fournies durant la phase 1, on peut distinguer trois catégories d’étudiants :

La première catégorie regroupe les étudiants qui montrent par leur analyse de la situation qu’ils ont compris que l’enjeu est de travailler la notion de situation fonctionnelle et d’articuler différentes représentations d’une fonction. Ces étudiants maitrisent également les schèmes didactico-instrumentaux en jeu: construction d’un graphique dynamique par activation de la trace, utilisation d’un tableur pour générer le graphique, gestion des points mobiles, articulation entre les différentes représentations.

La deuxième catégorie regroupe les étudiants qui ont bien compris les enjeux didactiques autour de la co-variation mais maitrisent moins bien les schèmes instrumentaux en jeu. Comme on peut le voir dans l’extrait ci-dessous, l’utilisation de l’outil trace est envisagée de façon très technique, ce qui laisse supposer un manque de recul:

> « Ils [les élèves] peuvent définir deux variables, soit x = distance entre AC et y = somme des aires des triangles afin de pouvoir créer deux droites qui leur sont associées. Par la suite, en utilisant le point d’intersection de ces droites et la trace de ce point, les élèves peuvent bouger le schéma afin de voir la production de la trace. »

L’importance est donnée à la procédure à suivre plus qu’à l’instrument didactique construit. Ces étudiants concentrent leur attention sur le rôle que l’on peut faire jouer au tableur. Ils préconisent l’obtention du graphique à partir d’une table de valeur. Les deux façons envisagées d’obtenir le graphique (trace ou tableur) ne sont pas commentées dans une perspective didactique.
Ces étudiants ne perçoivent pas la possibilité d’articulation entre les différentes représentations offertes pas la technologie. Pour eux, le rôle de la technologie est essentiellement d’offrir des possibilités de visualisation et d’automatisation. Ils ont cependant perçu que le déplacement peut remplir plusieurs fonctions didactiques et sont capables de nommer celle qui est mise en jeu :

> « Les élèves peuvent utiliser le déplacement afin de bien visualiser la situation. Le déplacement leur permet d’explorer une infinité de cas possibles afin d’émettre une hypothèse. »

La troisième catégorie regroupe des étudiants qui maitrisent mal les enjeux didactiques de la situation: ils la considèrent comme ancrée dans le domaine de la géométrie, centrée sur les polygones réguliers et les calculs d’aire, tout en lui reconnaissant une composante algébrique :

> « Possibilité d’analyse algébrique (connaitre l’aire du deuxième triangle selon la base du premier triangle ainsi que la longueur du segment AB). »

Dans ce cas, les enjeux technologiques tournent autour de la construction de figures dans GeoGebra et l’exploration de différents cas possibles. La situation ayant été étudiée en classe, ces étudiants se rappellent qu’on a construit des graphiques mais on sent que le lien entre le graphique obtenu et la situation de départ est artificiel pour eux. Les stratégies proposées sont très procédurales. Elles décrivent les actions à effectuer dans le logiciel sans vraiment de vision globale.

Phase 2 : Situation 3

Les étudiants travaillent individuellement. Ils sont invités à explorer la situation selon trois modes : papier/crayon, tableur et GeoGebra. Ils doivent ensuite proposer une activité mettant en jeu cette situation avec des élèves du secondaire, en milieu technologique. Ils sont laissés libres dans le choix du logiciel à utiliser avec les élèves. Ils doivent fournir une analyse a priori de l’activité proposée.

On constate que les étudiants des catégories 1 et 2 reconnaissent dans la situation 3 la possibilité d’appréhender la notion de co-variation et de situation fonctionnelle. La différence dans leurs propositions respectives se situe au niveau de l’instrumentation de la trace. Les étudiants de la catégorie 1 analyse son utilisation :

> « L’élève doit se rendre compte qu’il n’y a en effet qu’une seule valeur maximale pour l’aire de la pièce (connaissances sur la quadratique). »

Ils anticipent également les difficultés des élèves:

> « Difficultés: Comment et sur quoi afficher la trace? ».

Les étudiants de la catégorie2 sont plus évasifs quant à l’obtention du graphique :

> « Il [l’élève] peut construire le graphique (avec la techno ou en papier-crayon) ».

Cela donne à penser que la technologie ne fait qu’automatiser les méthodes utilisées en papier crayon. Cela se confirme avec le choix presque systématique du tableur pour travailler la situation avec les élèves. Ces étudiants s’attardent donc davantage que les autres sur les enjeux didactiques de l’utilisation d’un tableur :

> « Les élèves passeraient progressivement vers une méthode plus algébrique (pour pouvoir répondre à la question) à partir des réponses obtenues par essais-erreurs grâce aux méthodes intuitives arithmétiques réalisées dans le tableur ».

Ce sont cependant ces étudiants de la catégorie 2 qui sont le plus sensibles à l’articulation entre les différents registres de représentation de la fonction :

> « But de l’activité: Amener les élèves à explorer une relation fonctionnelle et à la traduire dans différents modes de représentations (graphique, équation, tableau, etc.). Pour cela, ils devront dégager les variables dépendantes et indépendantes.»

Les autres étudiants situent cette activité dans le domaine de la géométrie pour un travail sur l’aire des rectangles. Les grandeurs en situation de co-variation sont identifiées mais le lien avec le concept de fonction n’est pas fait :

> « Les élèves doivent trouver de quelle manière la largeur donnée au bâtiment influencera les dimensions des autres pièces. ».

La résolution proposée est numérique, avec éventuellement un lien vers l’algèbre pour exprimer les relations entre les différentes grandeurs :

> « On veut faire travailler l’élève avec une inconnue, une variable et différentes expressions algébriques dans lesquelles on utilise l’inconnue de départ. »

Les différentes représentations sont envisagées sous l’angle de l’exploration de différents cas de figure. La mesure (ou affichage des valeurs) est omniprésente dans l’exploration envisagée.

> « En déplaçant les points non-fixes, il observe toutes les valeurs possibles des dimensions du rectangle brun et les écrits sur une feuille. Avec l’outil aire, on affiche l’aire de la région brune pour identifier les dimensions maximisant l’aire de la région brune. »

Le lien avec la notion de fonction n’est pas fait. L’utilisation de l’outil Trace n’est pas envisagée.

Bilan de la séquence 2 :

Les étudiants sont entrés dans une analyse didactique des situations travaillées, dans la mesure de leur propre compréhension des concepts à l'étude. L'instrumentation didactique qui est réalisée durant ces séquences est assez peu dépendante du niveau d'instrumentation technique. Lors de la phase 1 cela s'explique par le fait que l'activité est réalisée collectivement, et que donc les instrumentations des uns et des autres (y compris celle du formateur) se mettent au service d'une exploration collective. Si un étudiant est moins à l'aise pour certaines manipulations dans le logiciel, il se trouve toujours quelqu'un pour l'aider à s'approprier les schèmes sous-jacents. Dans la phase 2, on constate que les étudiants ne se tournent pas nécessairement vers le logiciel avec lequel ils sont le plus à l'aise, mais vers celui pour lequel ils maitrisent mieux les enjeux didactiques. Ainsi durant le cours, les étudiants ont été très sensibles au rôle intermédiaire entre arithmétique et algèbre joué par le tableur. Et on constate en effet que bien que les étudiants soient globalement plus à l'aise avec GeoGebra, ils proposent majoritairement de faire travailler les élèves dans le tableur. Ceci est à nuancer avec le fait que beaucoup d'entre eux ont vu dans cette situation l'occasion d'un travail sur les équations plus que sur les fonctions.

Conclusion

Nous voyons dans ce travail une avancée vers la mise en place d'une instrumentation didactique des futurs enseignants. Il apparaît qu'il ne suffit pas de faire travailler les étudiants en milieu technologique pour provoquer cette genèse. Le rôle du formateur est primordial et nous comptons approfondir cette question dans le cadre théorique de l'orchestration instrumentale proposé par (Trouche, 2003). Le rôle du formateur est très subtil car il ne doit être ni trop intrusif, ce qui empêcherait les étudiants de mettre en place leurs propres schèmes, ni trop extérieur car les genèses doivent être guidées. Le point important, qui nous conforte dans nos hypothèses de travail, c'est que la genèse didactique n'est pas nécessairement corrélée à une instrumentation technique poussée. Les observations de la séquence 1 nous confortent dans l'idée qu'il est plus important de sensibiliser les futurs enseignants aux choix didactiques plus ou moins conscients, sous-jacents à l'utilisation d'un logiciel pour une tâche mathématique, que de vouloir en faire des experts techniques. Une expertise technique sans conscience didactique ne mènera pas à une intégration efficace de la technologie dans l'enseignement.

La séquence 2 constitue un premier pas vers cette prise de conscience. Cependant, on voit que l'instrumentation didactique nécessite un recul didactique sur les notions travaillées. Les étudiants les moins sensibles aux apports possibles de la technologie sont ceux qui ont aussi du mal à cerner les enjeux pédagogiques et didactiques des situations proposées. Le cas de la co-variation est un peu particulier car de nombreuses recherches ont montré qu'il s'agit d'un concept difficile à appréhender de façon globale, même pour des étudiants en mathématiques. Nous proposons donc de poursuivre nos investigations sur des concepts pour lesquels nous serions plus à même de distinguer la connaissance didactique sur le concept de la connaissance didactique sur la technologie.

REFERENCES

Carlson, M. (1998). Physical enactment : a powerful representational tool for understanding the nature of covarying relationships ? In *Representations and mathematics visualization* (p. 63-77).

Charnay, R. (2003). L'analyse a priori, un outil pour l'enseignant! *RMT: potentialités pour la classe et la formation*, 199-208.

De cotret, S. R. (1985). *« Etude historique de la notion de fonction: analyse epistemologique et expérimentation didactique »*. Université du Québec à Montréal.

Duval, R. (1993). Registres de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de Didactique et de Sciences Cognitives*, *5*, 37-65.

Guin, D., & Trouche, L. (1998). Complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning*, *3*(3), 195-227. doi:10.1023/A:1009892720043

Hitt, F. (1998). Systèmes sémiotiques de représentation liés au concept de fonction. *Annales de didactique et de sciences cognitives,*.

Leclère, P., Simonnot, B., & Barcenilla, Javier, Dinet, J. (2007). Les freins à l’intégration des tice en classe. *TICE Méditerranée*, 1-11.

MELS. (s. d.). Programme de formation de l’école québécoise. Consulté 17 janvier 2015, à l’adresse https://www7.mels.gouv.qc.ca/dc/liste%5Forthographique/

MELS. (2000). Programme de formation de l’école québecoise - deuxième cycle du secondaire en mathématiques.

Monk, S. (1992). The concept of function aspects of epistemology and pedagogye. In G. H. Dubinsky (Éd.), *MAA notes and reports series* (p. 175-194).

Passaro, V. (2007). *Covariation Entre Deux Grandeurs Révélé Par Une Analyse Des Représentations Spontanées D ’ Élèves Du Premier Cycle Par Valériane*.

Rabardel, P. (1995). Les hommes et les technologies; approche cognitive des instruments contemporains. Consulté à l’adresse https://hal.archives-ouvertes.fr/hal-01017462/document

Restrepo, A. M. (2008). *Genèse instrumentale du déplacement en géométrie dynamique chez des élèves de 6ème*. Université Joseph-Fourier, Grenoble.

Soury-Lavergne, S. (2011). De l’intérêt des construction molles en géométrie dynamique. *Mathématice*.

Tapan, M. S. (2006). *Différents types de savoirs mis en oeuvre dans la formation initiale d’enseigants de mathématiques à l'intégration de technolgoies de géométrie dynamique.* Université Joseph Fourier- Grenoble 1.

Tremblay, K.-P., & Venant, F. (2015). La technologie au carrefour des cadres théoriques : une double analyse autour de la notion de co-variation. Montréal: GDM.

Trouche, L. (2003). Construction et conduite des instruments dans les apprentissages mathématiques : nécessité des orchestrations. Consulté à l’adresse https://telearn.archives-ouvertes.fr/hal-00190091/

Trouche, L. (2007). Chapitre 1. Environnements informatisés d’apprentissage : quelle assistance didactique pour la construction des instruments mathématiques ? *Perspectives en éducation et formation*. De Boeck Supérieur. doi:10.3917/dbu.flori.2007.01.0019

Vergnaud, G. (1996). Au fond de l‘action, la conceptualisation. In J.-M. Barbier (Éd.), *Savoirs théoriques et savoirs d‘action* (Pédagogie). PUF.

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L'orientation des enseignants de mathématiques et sciences sur les modèles constructivistes et transmissivistes d'enseignement.

Les résultats de la recherche Prisma sur les enseignants valdôtains des niveaux primaire et secondaire

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Résumé: L'article analyse la nature des processus d'enseignement et d'apprentissage développés par les enseignants valdotaines en vérifiant la présence des orientations transmissivistes ou constructivistes et en essayant d'enquêter sur les différences dans les approches des enseignants en mathématiques et disciplines scientifiques. Les données utilisées proviennent de la recherche de PRISMA, conduite par le Département de la surintendance des écoles de la Région Autonome Vallée d'Aoste et l'Université de la Vallée d'Aoste par une enquête administrée à l'ensemble des enseignants des écoles primaires et secondaires du premier degré de la région. Les énoncés permettant de mesurer l'orientation constructiviste ou trasmissiviste des enseignants ont été développés à partir de l'échelle d'attitude proposée par la recherche TALIS (*Teaching and Learning International Survey*, OCDE 2008). En utilisant la technique de l'analyse factorielle, à partir des réponses sur ces points, ils ont été extraits deux facteurs qui ont peut résumer dans le concepts des constructivisme et transmissivisme. Les notes factoriels ont ensuite été utilisés pour vérifier l'existence de différentes approches entre les enseignants de mathématiques par rapport à ceux des autres disciplines dans les différents niveaux d'enseignement.

Abstract: The article analyzes the nature of teaching and learning processes developed by teachers Aosta Valley by checking the presence of transmissivistes or constructivist orientations and trying to investigate the differences in the approaches of teachers in mathematics and scientific disciplines. The data used are from the PRISMA research, conducted by the Department of superintendent of schools of the Autonomous Region Aosta Valley and Aosta Valley University by a survey administered to all primary school teachers and secondary first degree in the area. The analysis measures the transmissivity or constructivist orientationthat teachers may have developed from the scale attitude offered by TALIS (Teaching and Learning International Survey, OECD 2008). Using factor analysis, based on the answers to these points, two factors were extracted, related to the concepts of constructivism and transmissivisme. The factorial scores were then used to verify the existence of different approaches between teachers of mathematics in relation to other

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disciplines in the different levels of education.

Introduction

Cet article présente des résultats concernant les modèles pédagogiques et didactiques des enseignants qui émergent de la recherche PRISMA (Projet de Recherche sur les Enseignements et Apprentissages Scientifiques et Mathématiques), réalisée dans la Vallée d'Aoste grâce à une enquête[11] qui a impliqué la population régionale des enseignants des écoles primaire et secondaire. En particulier, les concepts d'enseignement – apprentissage des enseignants sont analysées en utilisant les catégories analytiques du « transmissivisme » et du « constructivisme », sur lesquels on a comparé les orientations des enseignants de mathématiques et de la science avec ceux d'autres disciplines.

1. La recherche PRISMA. Objectifs et méthodologie

Conçue dans le but de promouvoir le développement des mesures et actions visant à l'amélioration de l'enseignement et de l'apprentissage des disciplines scientifiques et mathématiques dans les écoles de la région, la recherche PRISMA a été activée en collaboration entre le Département de la surintendance des écoles de la Région Autonome Vallée d'Aoste et l'Université de la Vallée d'Aoste.[12] Conjointement à la conception et à l'exécution de l'enquête sur l'ensemble du corps enseignant des écoles primaire et secondaire, des dispositions ont été prises pour la collecte systématique d'informations relatives aux établissements scolaires, dans le but de disposer d'un environnement adéquat dans lequel le corps enseignant est appelé à travailler.
La recherche a eu un caractère interdisciplinaire impliquant soit des pédagogues soit des sociologues. Parmi les premiers ont été se sont engagés soit les spécialistes afférents dans les domaines de la pédagogie générale soit de la didactique des mathématiques et des sciences, tandis que pour la sociologie ont été touchés les domaines de la sociologie de l'éducation et des politiques éducatives, ainsi que la sociologie de s sciences et des professions.
Les instruments qui ont été utilisés pour la collecte sont:

1. Un questionnaire distribué aux enseignants pour l'auto remplissage, en version complète (73 questions) pour les enseignants des disciplines mathématiques et scientifiques, focus spécifique de l'enquête et dans la version réduite et moins exigeant (36 questions) pour les enseignants des autres disciplines à l'égard desquels on voulait faire une comparaison seulement pour des questions transversales. Le questionnaire est constitué soit de certaines questions didactiques sur les spécificités valdôtaines, soit, surtout, de questions proviennent d'enquêtes par sondage les plus populaires nationales et internationales (VOSTS, NSTQ, VOSE, TIMSS, TALIS, etc.)[13], pour permettre la comparaison des résultats;
2. Des tableaux pour la collecte des données de contexte relatives soit à chaque établissement scolaire soit au système scolaire régional dans son ensemble.

Le questionnaire destiné aux enseignants a touché les domaines suivants:

- Biographie personnelle et professionnelle;

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[11] La soi-disante phase de *field* de la recherche PRISMA, c'est-à-dire la campagne de collecte des données, a été conduite au cours de la première moitié de l'année scolaire 2010-11.

[12] La recherche PRISMA dans son ensemble est coordonné par Piero Aguettaz e Chiara Allera Longo du Bureau soutien à l'autonomie scolaire – Département de la surintendance des écoles de la Région Autonome Vallée d'Aoste et par Fabrizio Bertolino du Département de Sciences humaines et sociales - Université de la Vallée d'Aoste.

[13] VOST - *Views on Science- Technology-Society, 1989;* NSTQ - *Nature of Science and Technology*, 2001; TIMSS - *Trends in International Mathematics and Science Study, 2003*; VOSE - *Views on science and education*, 2006; TALIS - Teaching and Learning International Survey, 2008.

- Rapport de l'enseignant avec sa connaissance personnelle;
- Rapport de l'enseignant avec sa profession;
- Les images de la science et des mathématiques ainsi que leur rôle dans la société;
- Attitudes à l'égard des processus d'enseignement et d'apprentissage;
- Organisation des processus d'enseignement et d'apprentissage en mathématiques et sciences;
- Les représentations du rôle social de l'école.

En ce qui concerne le contexte de travail des enseignants on a procédé à une collecte systématique d'informations liées aux domaines cités ci-dessous:

- Organisation des établissements scolaires;
- Présence et typologie de laboratoires;
- Projets et initiatives de formation des enseignants dans les domaines mathématiques-scientifiques-technologiques;
- Projets et initiatives adressés aux élèves dans les domaines mathématiques-scientifiques-technologiques;
- Activités et actions entreprises dans le contexte local, en se référant soit aux familles des élèves soit à la communauté sociale dans son ensemble.

En termes quantitatifs, les enseignants impliqués dans l'enquête ont été plus de 1300, parmi lesquels plus de 55% était constitué d'enseignants d'école primaire (96,4% de ceux-ci sont employés dans les structures publiques et le reste, 3,6%, dans celles privées) et le reste 45% par des professeurs d'école secondaire (répartis avec la même proportion des collègues de l'école primaire, 96,4% contre 3,6%, parmi les institutions publiques et celles privées).

Le taux de réponse obtenu de l'enquête a été plutôt élevé, en particulier dans les écoles primaires où il a dépassé 80%, en se situant complessivement au-delà des 70%. Sûrement ce résultat est liée à la participation active de 14 enseignants (8 primaires et 6 écoles secondaires) qui, comme «amis de la recherche» ont, dans une première phase testé l'instrument d'enquête, puis supervisé la distribution et la collecte des questionnaires à leur institution.
Le tableau 1 montre dans les détailles la composition de la population impliquée par l'enquête et les taux de réponse relatifs.

	Total enseignants			Enseignants des matières mathématiques-scientifiques			Enseignants d'autres disciplines		
	Population	Questionnaires remplis	Taux de réponse	Population	Questionnaires remplis	Taux de réponse	Population	Questionnaires remplis	Taux de réponse
École primaire (a)	749	596	79,6%	315	238	75,6%	434	358	82,5%
École secondaire du premier degré (b)	611	400	65,5%	102	86	84,3%	509	316	62,1%
Total	1360	996	73,2%	417	324	77,7%	943	674	71,5%

Tableau 1. Population intéressée par l'enquête PRISMA et taux de réponse[14].

[14] Mention: (a) Élèves de 6 à 11 ans si le parcours scolaire est régulier; (b) Élèves de 11 à 14 ans si le parcours scolaire

2. L'enquête sur les concepts d'enseignement et d'apprentissage des enseignants

Dans la recherche Prisma, l'enquête sur les concepts d'enseignement et d'apprentissage des enseignants a utilisé, entre autres outils théoriques, le binôme constructivisme contre transmissivisme. Ce binôme est considéré dans la littérature une catégorisation efficace des orientations de fond alternatives en matière de processus d'enseignement et d'apprentissage (De Sanctis, 2010), et a produit des solutions de mesures qui ont été adoptées par d'importantes recherches internationales. Prisma a choisi à ce propos de prendre comme référence l'outil sur l'échelle d'attitude développée par la recherche TALIS (*Teaching and Learning International Survey*) 2008, conduite par l'OCDE[15].

Selon les définitions diffusées dans la littérature, le concept traditionnel de type transmissif direct est basé sur la conviction que la connaissance peut être transmise efficacement en mettant en place un rapport hiérarchique avec les élèves et caractérisé par une gestion autoritaire et ferme de la classe et par la production des stimulations adéquates qui orientent clairement le processus d'apprentissage.

L'approche constructiviste considère au contraire la connaissance comme le résultat d'une construction active de l'étudiant, il adopte un concept systématique concentré sur la structuration du contexte dans lequel est réalisée l'activité d'apprentissage et préfère la sollicitation à diverses formes de collaboration (Calvani, 1998).

La recherche TALIS étudie ces deux approches diverses par une échelle de type Likert à deux dimensions, en fonction de laquelle à chaque enseignant répondant on attribue un score qui le positionne le long d'un continuum dont aux pôles se situent les modèles 'purs' constructiviste et transmissiviste.

L'échelle Likert développée par TALIS est constituée de deux groupes de quatre énoncés chacun, c'est à dire de descripteurs d'opérationnalisations retenus efficaces des deux constructions théoriques. Les énoncés expriment donc les traits de l'approche transmissive ou constructiviste et ont des positions successives alternées dans la batterie des questions.

Les enseignants braves/efficaces montrent la méthode correcte pour résoudre les problèmes
Le rôle de l'enseignant est celui de faciliter les processus de recherche réalisés directement par les étudiants
L'enseignement devrait être construit autour de problématiques pour lesquelles les réponses sont claires et correctes, et les concepts faciles à comprendre
Les étudiants apprennent mieux quand ils doivent trouver tout seul les solutions aux problèmes
Dans l'enseignement il faut fournir autant de connaissances possible
Les étudiants devraient trouver les solutions aux problèmes seuls avant que les enseignants leur

est régulier.

[15] TALIS (*Teaching and Learning International Survey*) est un recherche internationale sur les conditions d'enseignement et apprentissage développée sondant enseignants et dirigeants scolaires des écoles secondaire du premier degré publiques et privées. Pour plus d'informations on peut consulter la section dédiée a TALIS sur le website de l'OCDE: www.oecd.org/edu/school/talis.htmhttp://www.oecd.org/edu/school/talis.htm.

montrent comment faire pour les résoudre
Les enseignants ne devraient pas laisser que les étudiants développent des explications de manière autonomes, qui pourraient être fausses, mais plutôt donner des explications directes
C'est plus important apprendre à penser et à raisonner qu'apprendre des contenus spécifiques disciplinaires

Tableau 2. Question PRISMA sur les concepts d'enseignement des enseignants[16] (« Pour chacune des affirmations suivantes relatives à l'enseignement/apprentissage en général, indiquez combien vous êtes d'accord. S'exprimer sur toutes les affirmations »).

La recherche PRISMA a emprunté substantiellement la formulation originale[17] des énoncés, en intervenant cependant avec une atténuation relative de l'importance du climat de la classe (qui doit être « calme » comme exigence généralement nécessaire pour l'apprentissage efficace en référence au quatrième énoncé transmissif original TALIS) et davantage l'accent sur la directivité de l'enseignant (voir le quatrième énoncé transmissif PRISMA[18]). Le tableau 2 montre la formulation des énoncés adoptés en PRISMA.
En ce qui concerne les solutions de mesure adoptées, dans Talis les réponses fermées sur l'échelle des attitudes de Likert à quatre modes de réponse ont été disposées selon un motif symétrique à double polarité de désaccord-accord, à partir de lequel nous avons calculé les scores ipsatives[19]. Dans la recherche PRISMA nous avons adopté la solution à quatre mode de réponse et avec une seule polarité (de «pas du tout d'accord» à «tout à fait d'accord») qui peut être observé dans le tableau 2, et on n'a pas pris ex ante l'alternative entre les concepts transmissif et constructiviste, mais on a laissé l'émerger éventuellement à partir des données adoptant la technique de l'analyse factorielle oblique[20], c'est à dire faite sans l'hypothèse ni d'une relation entre les deux constructions opposition/alternativité, qui prend statistiquement la forme de corrélation négative (comme on dirait que l'équipe de recherche TALIS a plutôt fait), ni indépendance ou, avec la langue de l'analyse factorielle, l'orthogonalité des deux facteurs extraits. Ceci afin d'éviter précisément que l'alternativité ou l'indépendance des deux constructions pourrait subrepticement dériver par le

[16] Les énoncés impairs représentent expressions d'une attitude transmissive, tandis que les énoncés pairs représentent une attitude constructiviste.

[17] La formulation des énoncés adoptée par TALIS 2008 est représentée dans sa traduction en langue italienne in De Sanctis (2010). Pour la formulation originale en langue anglaise, veuillez consulter la publication OCDE « Creating Effective Teaching and Learning Environments: First Results from TALIS ». (www.oecd.org/education/school/43023606.pdf). Autres documents, en particulier le manuel technique de la recherche 2008 (OCDE 2009a), sont disponibles sur les website de l'OCDE (www.oecd.org/edu/school/talis.htm). Le lien au manuel technique mentionné est le suivant: www.oecd.org/education/school/44978960.pdf.

[18] Il s'agit de l'énoncé G, indiqué dans le tableau 2.

[19] L'échelle Likert TALIS est graduée dans la façon suivante: pas de tout d'accord, désaccord, d'accord, tout à fait d'accord. Les scores ipsatifs constituent une solution pour standardiser les réponses individuelles au but de réduire les effets distorsifs. En particulier, l'adoption des scores ipsatives dans la recherche TALIS en ce qui concerne l'analyse des attitudes a été dictée par la nécessité d'affronter les problèmes liés à l'application de l'analyse factorielle à un cadre cross-culturel qui présentent plusieurs différences dans les moyennes des indicateurs dans les pays considérés (OCSE 2009a, De Sanctis 2010). Les scores ipsatives sont calculées en décomptant le score moyenne obtenue par les huit énoncés soit à le score moyenne calculée sur les quatre énoncés qui constituent l'indice de transmissivisme soit sur le score moyenne calculée sur les quatre énoncés qui constituent l'indice de constructivisme. Puisque que la recherche PRISMA insiste au contraire sur un contexte subnational, les susdit problèmes ne se manifestent pas dans l'analyse factorielle.

[20] On a évidement réalisé des analyses factorielles soit avec rotations orthogonales soit avec rotations obliques, mais ces dernières présentent à notre avis un intérêt théorique plus élevé en non imposant l'alternative ex ante entre les deux constructs-facteurs de transmissivisme et constructivisme, mais plutôt en la testant empiriquement.
Les configurations de l'analyse factorielle reportée dans cet article sont les suivants: extraction nombre fixe de facteurs=2; maximum de vraisemblance, rotation oblique Oblimin avec normalisation de Kaiser.

réglage de la technique d'analyse.
En effet les deux facteurs extraits ont présenté les corrélations attendues avec les deux groupes des énoncés (voir le tableau 3 qui montre la matrice structure de l'analyse factorielle, ou les corrélations entre les facteurs et les énoncés); ils sont donc identifiés comme «Constructivisme» et «Transmissivisme». Ils sont également corrélés les uns aux autres d'une manière négative, comme implicitement supposé par la recherche TALIS, mais seulement faiblement, comme on peut le voir dans le tableau 4.

	Facteur	
	1 (Constructivisme)	2 (Transmissivisme)
B1.A Les enseignants braves/efficaces montrent la méthode correcte pour résoudre les problèmes	-,262	,621
B1.B Le rôle de l'enseignant est celui de faciliter les processus de recherche réalisés directement par les étudiants	,212	-,003
B1.C L'enseignement devrait être construit autour de problématiques pour lesquelles les réponses sont claires et correctes, et les concepts faciles à comprendre	-,057	,432
B1.D Les étudiants apprennent mieux quand ils doivent trouver tout seul les solutions aux problèmes	,671	-,159
B1.E Dans l'enseignement il faut fournir autant de connaissances possible	-,025	,453
B1.F Les étudiants devraient trouver les solutions aux problèmes seuls avant que les enseignants leur montrent comment faire pour les résoudre	,624	-,226
B1.G Les enseignants ne devraient pas laisser que les étudiants développent des explications de manière autonomes, qui pourraient être fausses, mais plutôt donner des explications directes	-,184	,369
B1.H C'est plus important apprendre à penser et à raisonner qu'apprendre des contenus spécifiques disciplinaires	,302	-,119

Tableau 3. Matrice de structure de l'analyse factorielle.

Facteur	1 (Constructivisme)	2 (Transmissivisme)
1 (Constructivisme)	1,000	-,230
2 (Transmissivisme)	-,230	1,000

Tableau 4. Matrice de corrélation des facteurs de l'analyse factorielle.

Sur la base des leurs scores factorielles, nous avons ensuite effectué des analyses de la variance univariée: le modèle adopté a pris comme variables dépendantes les variables cardinales liées aux scores factorielles «Constructivisme» et «Transmissivisme» et comme variables indépendantes la variable dichotomique de la matière scolaire, c'est à dire mathématiques et/ou sciences versus les autres matières. Les analyses de la variance ont été réalisées soit sur tous les enseignants répondants, soit séparant les enseignants des écoles primaires de ceux de l'école secondaire du premier degré.

3. Les résultats de l'analyse: l'orientation constructiviste des enseignants de mathématiques et/ou sciences

L'analyse de la variance des scores attribués aux répondants sur le deux facteurs «Constructivisme» et «Transmissivisme» – les résultats sont présentés dans les tableaux 5 et 6 – montre que les différences entre les enseignants de mathématiques et/ou science et ceux d'autres matières sont toutes significatives au test F (p <0,05). Le test de signification statistique dans ce cas est de peu d'importance car il a été choisi de faire correspondre l'échantillon de l'enquête à la population de référence et il a été donc établie pour chaque enseignant une probabilité d'inclusion dans l'échantillon égal à 1. Toutefois, bien que on a été établi de interviewer l'entière population d'enseignants, un peu moins du 30% d'eux n'a pas répondu au questionnaire, en posant des problèmes de représentativité des résultats obtenus. Mais si on suppose que la non-participation à l'enquête n'est pas influencée de façon significative par l'orientation constructiviste plutôt que transmissiviste de chaque enseignant, on peut assumer que les différences dans les scores factorielles sont significatives pour la totalité de la population d'enseignants

			N	Moyenne	Écart type	Erreur type	Intervalle de confiance 95% pour la moyenne	
							Limite inférieure	Limite supérieure
Tous les enseignants	Transmissivisme	Mathématiques/sciences	274	-,20868	,77053	,04655	-,30032	-,11704
		Autres disciplines	554	,10321	,72518	,03081	,04269	,16373
		Total	828	,00000	,75445	,02622	-,05146	,05146
	Constructivisme	Mathématiques/sciences	274	,22298	,71884	,04343	,13749	,30848
		Autres disciplines	554	-,11028	,80193	,03407	-,17721	-,04336
		Total	828	,00000	,79071	,02748	-,05394	,05394
Enseignants	Transmissivisme	Mathématiques/sciences	19	-,13727	,79218	,05658	-,2488	-,02568

école primaire			6				7	
		Autres disciplines	280	,09609	,78238	,04676	,00405	,18813
		Total	476	,00000	,79397	,03639	-,07151	,07151
	Constructivisme	Mathématiques/sciences	196	,11103	,67299	,04807	,01622	,20583
		Autres disciplines	280	-,07772	,79220	,04734	-,17091	,01548
		Total	476	,00000	,75047	,03440	-,06759	,06759
Enseignants école secondaire premier degré	Transmissivisme	Mathématiques/sciences	78	-,36721	,74724	,08461	-,53568	-,19873
		Autres disciplines	274	,10453	,72231	,04364	,01863	,19044
		Total	352	,00000	,75285	,04013	-,07892	,07892
	Constructivisme	Mathématiques/sciences	78	,29174	,84267	,09541	,10175	,48173
		Autres disciplines	274	-,08305	,80239	,04847	-,17848	,01238
		Total	352	,00000	,82513	,04398	-,08650	,08650

Tableau 5. Résultats de l'analyse de la variance (ANOVA) univariée avec disciplines d'enseignement comme variable dépendante (dichotomisée en 'mathématiques/sciences' versus 'autres disciplines') et le score factorielle de transmissivisme et constructivisme comme variable indépendante.

Les résultats présentés dans le tableau 5 montrent comment les enseignants de mathématiques et/ou sciences se situent sur des positions systématiquement plus constructivistes par rapport à leurs collègues d'autres disciplines, par ce qu'ils ont des scores moyens négatifs en relation au facteur « Transmissivisme » et positifs pour le facteur « Constructivisme », à la différence de leurs collègues d'autres disciplines lesquelles moyennes de score ont une tendance inverse. Cette tendance se manifesta, comme on peut l'observer dans le tableau 5, soit que nous considérons conjointement les niveaux d'écoles, soit que les enseignants de l'école primaire et ceux de la secondaire sont analysés séparément.

			Somme des carrés	df	Moyenne des carrés	F	Signifiance
Tous les enseignant	Transmissivis	Intra-group	17,833	1	17,833	32,52	,000

s	me	e				5	
		Inter-groupe	452,895	826	,548		
		Total	470,728	827			
	Constructivisme	Intra-groupe	20,361	1	20,361	33,861	,000
		Inter-groupe	496,697	826	,601		
		Total	517,058	827			
Enseigrants école primarie	Transmissivisme	Intra-groupe	6,279	1	6,279	10,152	,002
		Inter-groupe	293,156	474	,618		
		Total	299,435	475			
	Constructivisme	Intra-groupe	4,107	1	4,107	7,391	,007
		Inter-groupe	263,413	474	,556		
		Total	267,520	475			
Enseignants école secondaire premier degré	Transmissivisme	Intra-groupe	13,512	1	13,512	25,504	,000
		Inter-groupe	185,428	350	,530		
		Total	198,940	351			
	Constructivisme	Intra-groupe	8,529	1	8,529	12,953	,000
		Inter-groupe	230,445	350	,658		
		Total	238,973	351			

Tableau 6. Résultats de l'analyse de la variance (ANOVA) univariée avec disciplines d'enseignement comme

variable dépendante (dichotomisée en 'mathématiques/sciences' versus 'autres disciplines') e la score factorielle de transmissivisme et constructivisme comme variable indépendante

L'orientation plus constructiviste des enseignants de mathématiques et/ou sciences, et en particulier ceux de l'école secondaire, où les caractéristiques disciplinaires sont plus significatives et stables, trouvée en Vallée d'Aoste ressemble à un résultat intéressant de la recherche PRISMA, pour plus dans le seul pays de l'OCDE qui, parmi ceux étudiés dans l'enquête TALIS, a présenté une prévalence d'orientations transmissives parmi les enseignants de l'école secondaire de premier degré (De Sanctis, 2010).

REFERENCES BIBLIOGRAPHIQUES

Bramanti, D. (Ed.) (1998). *Progettazione formativa e valutazione*. Roma : Carocci.

Calvani, A. (1998). Costruttivismo, progettazione didattica e tecnologie. In D. Bramanti (Ed.), *Progettazione formativa e valutazione*. Roma : Carrocci.

De Sanctis, G. (2010). *TALIS. I docenti italiani tra bisogni di crescita professionale e resistenze*. Torino: Fondazione Giovanni Agnelli.

OCDE (2009a). *TALIS 2008. Technical Report*. Paris : OCDE. Repéré le 11 mars 2015 à www.oecd.org/education/school/44978960.pdf

OCDE (2009b). *Creating Effective Teaching and Learning Environments: First Results from TALIS*. Paris: OCDE. Repéré le 11 mars 2015 à www.oecd.org/dataoecd/17/51/43023606.pdf

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A Pedagogical Coaching Design Focused on The Pedagogy of Questioning in Teaching Mathematics

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Abstract: This study presents the impact of a pedagogical coaching designed for pre-service mathematics teachers, focusing on the pedagogy of questioning in teaching. The aim was to promote their understanding of why it is crucial to be knowledgeable about questions and questioning strategies, and to support them in developing skills to plan and pose quality and timely questions that encourage student thinking, effective classroom interaction and further learning. Using action research we examined the effectiveness of the design conducted within the context of their teaching practicum.

Résumé : Cette étude présente l'impact d'un encadrement pédagogique conçu pour les enseignants de mathématiques pré-service, en mettant l'accent sur la pédagogie du questionnement dans l'enseignement. L'objectif était de promouvoir leur compréhension des raisons pour lesquelles il est crucial d'être bien informés sur les questions et les stratégies d'interrogation, et de les aider à développer les compétences nécessaires pour planifier et poser des questions de qualité et en temps opportun qui encouragent la réflexion des élèves, l'interaction en classe efficace et plus d'apprentissage. Par la recherche-action, nous avons examiné l'efficacité de la conception menée dans le cadre de leur enseignement stage.

Key words: mathematics, pedagogical coaching, practicum, pre-service, questioning

Introduction

The teaching practicum plays a central role in teacher training programs. It fosters the pedagogical content knowledge (PCK) of pre-service teachers and helps in analyzing the effect of different theories through actual classroom observations preparing lessons, teaching and reflecting upon them. To this effect the college pedagogic supervisor plays an important role to facilitate the process and to support the development of student teachers, in order to maximize their professional gain from these practices. This role of the supervisor is too ambiguous and sometimes it can become complex in relation to school variables where the practicum is taking place, as well as in relation to the total number of practicum days that pre-service teachers are obliged to conduct throughout their training program. This number ranges between 25 and 90 days, depending on the type of the teacher education program in which each student teacher is enrolled at the college.
The supervisor's effort to deal with the various pedagogical issues and mathematics knowledge for teaching, within the intensive individual and group conference sessions within the practicum raises a feeling of concern, inefficiency and discomfort, and the need for seeking new ideas and practices that may enhance her professional contribution to the pre service teachers. Accordingly, we plan a pilot study, focusing on a central pedagogical issue in relation to mathematics teaching and learning, and work on it with enough depth through a number of practicum days.

The pedagogy of questioning in the mathematics classroom

One major aspect of any successful teaching and learning process is the interaction between the teacher and the students. It is argued that in order to have good classroom interaction, teachers should pose questions. Cotton (1998) found that in K-12 education, teachers' methods of questioning were the second most used teaching skill after lecturing. According to Schuster & Anderson (2005), good questions can set the stage for meaningful classroom discussion and learning, yet the power of questioning lies in answering. They contend that teachers not only need to ask good questions to obtain good answers; they also must ask good questions to promote the thinking required to provide good answers. Furthermore, research shows that the average wait time teachers allow students to generate response is one second or less (Rowe 1974), within which no one can expect students to understand a question, process it, and formulate a response. Cazden (2001) found that waiting at least three seconds helps students give longer, more elaborate, and better responses and with more evidence of learning; it encourages more questioning, and increases student-to-student and student-to-teacher interactions and engagement. Although teacher guides provide direction and questions to ask, the teacher must devise good questions that will enable students to learn. However, research shows that teachers receive little training on how to ask, what to ask, and when to ask questions, and how long they should wait after they pose a question. According to Martino and Maher (1994), developing effective questioning skills may take years to develop, for it requires an in-depth knowledge of both mathematics and children's learning of mathematics. To this effect, the role of the pedagogic supervisors has become significant. In this study we designed a pedagogical coaching process to promote pre-service teachers' understanding of the pedagogy of questioning. We refer to the supervisors' role a coach rather than a supervisor, to highlight that there isn't any evaluation of performance (for grades) at any stage of the process with regard to this study as it seems to be in supervision. This approach encourages pre service teachers to collaborate and work better with the supervisor as a coach.

The research question

What, if any, impact does pedagogical coaching focused on the pedagogy of questioning, have on the professional preparation of pre-service teachers? How effective was the design within the context of a teaching practicum?

Participants

13 pre-service teachers, who participated in a teaching practicum one day a week at two secondary schools in 2013-2014. All held relevant academic degrees and have work experience in other fields and enrolled in two different teacher education programs to prepare for secondary school mathematics teaching. 8 of them enrolled for a one year teaching certification program, and the other 5 were in their second and last year of their study, to earn a master degree of teaching (M Teach) secondary school mathematics.

Research methodology

The research questions were approached through action research, which is a disciplined process of inquiry conducted by and intended for those taking the action (Sagor, 2000). Action research facilitates evaluation and reflection in order to implement changes needed in practice – both for an individual and within an institution. Martino and Maher (1994) stated that "The art of questioning may take years to develop for it requires an in-depth knowledge of both mathematics and children's learning of mathematics.

Data sources

Documents and protocols from the pedagogical coaching sessions conducted by the first author (underlined), who has been the pedagogical supervisor of the practicum, along with the professional guidance of the second author who is her post - doctoral mentor and reflective reports by the participating students.
The pedagogical coaching design consisted of the following four processes:

1. Pedagogic workshops: whole-group discussions and exercises after reading relevant literature (mostly in Hebrew)[i] and watching videotaped lessons.
2. Pre-lesson conversations between the individual student teacher and the pedagogical supervisor, to assist lesson planning that include effective questions and questioning strategies.
3. Focused classroom observations that include recording the lesson and use rubrics to analyze questioning practices and their impact on student learning.

Post-lesson pedagogical conferences and conversations based on analyzing data from the classroom observations and the student teachers' reflective reports.

Research findings

The findings here are based on data analyses collected from participants in at one of the two practicum schools. Various attempts made by the supervisor to implement the plan in the other school, where all of the participants were in the one year long teaching certification program showed unsatisfactory outcomes from the beginning. Their mathematics knowledge for teaching was not strong enough, and the focused coaching on questioning ended at its early stage.

Professional gains of the pre-service teachers from the focused coaching

The focused coaching promoted pre-service teachers' understandings and perspectives regarding the different types of questions and the use of questioning in instruction. They used the pre-lesson conversations effectively to plan quality questions and basic follow-up and probing questions based on anticipating students' conceptions and potential responses to mathematical tasks. They learned about the importance of "wait time" in enhancing student thinking and participation. In their few instructional practices, the student teachers used the planned questions to engage students in mathematical thinking and in facilitating productive classroom mathematical discourse at higher cognitive levels. In one of their practicum lessons two pre-service teachers were encouraged to let their 7th grade students determine, what sign would the sum of any two integers with different signs have, after practicing addition using arrows on the number line. They realized that even the mathematically weak students are able to learn effectively and participate actively, when given opportunities to do so. They highlighted this in their conclusion using the quotation: *"When you teach a child something you take away forever his chance of discovering it for himself."* (Piaget)
They continued to pay attention and focus on questioning processes in classrooms and showed interest in further studying the topic. For example, two participants conducted a study on questioning by recording and analyzing their mentor's lessons in a 9th grade mathematics class. They summarized their experience and achievements:

> *Until now, we did not pay enough attention to the issue of asking questions such as what questions to ask and when to ask them. We learned about the importance and the amount of questions in class and how to respond to students in order to promote their thinking.*

They also developed relevant knowledge and skills to characterize questions with respect to different cognitive levels and analyzed lessons from this viewpoint:

> *This study made us become aware of the impact of the rate structure for the questions that appear in it, about the types of questions that exist, which of them should be used and how much. We feel that we have been exposed to a new topic and learned about ways of promoting learning and class discussion.*

Effectiveness of the design as a pedagogical supervision strategy

The coaching design was found to be a productive pedagogical supervision strategy. Knowing the purpose and the emphasis of a specific supervision session based on consistent and relevant readings enhanced the pre-service teachers' active participation and interest. When writing lesson plans, the pre-lesson conversations were found effective in assisting the student teachers think on how they can engage students in active learning during instruction. The classroom observations done with the help of well-defined guides and rubrics, made it easier to hold topic focused post lesson conversations and reflections, minimizing the tension that both the teacher and the pedagogical supervisor usually experience regarding such conversations. Furthermore, it allows collection and systematic documentation of data about what has been accomplished and also indicates how it affects the professional development of each pre-service teacher. However, it was difficult to assess whether there were significant changes in the instructional skills of the pre-service teachers because of the small number of lessons that they were allowed to teach in their classes.

Discussion and implication

Overall, the focused coaching design helped the pre-service teachers develop a better understanding of the role that questions and questioning strategies can play to achieve effective student thinking and learning. They learned relevant skills needed to characterize and analyze different types of questions, how to effectively use questioning, and the importance of wait time as a powerful teaching tool in mathematics classrooms. The study also highlighted the difficulties that hindered us in assessing possible changes regarding pre-service teachers' instructional practices; within the practicum context the pre-service teachers had few opportunities to teach whole-class lessons, thus we were not able to collect enough data to draw adequate conclusions. Furthermore, to develop effective questioning skills teachers need to have a deep knowledge of mathematics for teaching. Hence, we assume that a focused coaching design to improve instruction may be more effective for teachers, who teach lessons on a regular basis.

REFERENCES

Cazden, C. B. 2001. *Classroom discourse: The language of teaching and learning* (2nd.ed). Portsmouth, NH: Heinemann.

Cotton, K. 1998. *Classroom questioning*. North West Regional Educational Laboratory. Retrieved from:
http://www.learner.org/workshops/socialstudies/pdf/session6/6.ClassroomQuestioning.pdfCotton (1998)

Martino, A. M., & Maher, C. A. (1994). *Teacher questioning to stimulate justification and generalization in mathematics*. New Orleans, LA: Paper presented at the Annual Meeting of the American Educational Research Association.

Rowe, M.B. (1974). Wait-time and rewards as instructional variables, their influence on language, logic and fate control: Parts I and II. *Journal of Research in Science Teaching, 11*, 81-84 and 291-308.

Sagor, R. (2000). *Guiding school improvement with action research*. Alexandria, VA: Association of Supervision and Curriculum Development.

Schuster, L. & Anderson, N. C. (2005). *Good Questions for Math Teaching: Why Ask Them and What to Ask, Grades 5-8*. Math Solutions Publications, Sausalito, CA.

בוזו שוורץ, מ'. (2008); גלטר, ג'. (1996); הרפז, י'. (ינואר 2000); וולף, ד'. (1998); שחר, נ'. (2006);
Vacc.N. N (1993)

WORKING GROUP 3A / GROUP DE TRAVAIL 3A

Classroom practices and learning spaces (K-8) /
Pratiques en classe et autres espaces d’apprentissage (K-8)

Working Group 3A / Group de Travail 3A

Classroom practices and learning spaces (K-8) / Pratiques en classe et autres espaces d'apprentissage (K-8)

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The theme of the WG#3a was concerned with classroom practices and learning spaces at the elementary level (K-8). Twelve papers were presented, divided into three main themes:

Theme 1. Experimentations of teaching and analyses / Expérimentations et analyse de l'enseignement

1. Adaptation de l'enseignement des mathématiques en contexte de collaboration et de coenseignement (C.Côté & D.Gauthier)
2. Dialogues as an instrument in mathematical reasoning (S.Lekaus & G.-A.Askevold)
3. Networking of theories as resource for classroom activities analysis: the emergence of multimodal semiotic chains (C.Sabena & A.Maffia)
4. One task, five stories: comparing teaching sequences in lower secondary schools (F. Morselli & M.Testera)
5. Les méthaphores - quels enjeux pour l'enseignement des nombres relatifs? (A.Raad Nawal)

Theme 2. Students / Élèves

6. An artefact for deductive activities: a teaching experiment with primary school children (Umberto Dello Iacono, Laura Lombardi)
7. Calcul mental et stratégies: un regard en termes de potentialities (J.Proulx)
8. The gesture/diagram interplay in grappling with word problems about natural numbers (F.Ferrara & M.Seren Rosso)

Theme 3. Teachers / Enseignants

9. Réflexion sur les obstacles culturels en enseignement des mathématiques (N.Bednarz & J.Proulx)
10. The role of the teacher in fostering an aware approach to problem-solving activities: the case of geometric problems that could be solved through the construction of equations (A.Cusi)
11. Investigating the intertwinement between the affective and cognitive dimensions of teachers: a possible way for surfacing the reasons of their decisions (M.De Simone)

The first two days were spent on the first theme (Experimentations of teaching and analyses / Expérimentations et analyse de l'enseignement, through presentations of the papers by the authors as well as a number of plenary discussions, as well as small group interactions on the issues followed by recapitulating points of interest that emerged in these small groups. Following the same structure, the third day was spent on the second theme (Teachers / Enseignants) and the fourth day on the third theme (Students / Élèves).

As well, on the fourth day, discussions were engaged in order to make a synthesis of salient points that were engaged with during the WG activities. We offer below a short outline of these ideas:

- *Mathematical culture* : The notion of mathematical culture was a recurring theme right from the start of the WG, with presentations focusing implicitly or explicitly on this issue. The WG participants thus engaged in discussions about how to create or immerse students in a mathematical culture. Through this discussion, specific elements related to mathematical culture were highlighted, like the place given to justification and argumentation, to gestures, to metaphors, to signs, to writting, to modelling, etc., in how mathematics is done in classrooms. This raised also the issue of students' and teachers' relation with mathematics (how it is perceived, understood, appreciated, etc.), leading to specific ways of doing and of being in mathematics.

- *Teaching practice* : Concerning teaching practice, through discussing the relevant role of what teachers actually do within the mathematics classroom, we have stressed the importance of taking account of teachers' own culture, their students' culture, their own relations to mathematics (and other issues), their students' relations, their expectations, and so on, toward mathematics. In other words, it made surface the crucial issue of the unavoidable intertwinement between the cognitive and the affective dimensions in the mathematics teachers' practices. Often, the rationale for teachers' decisions are visible through their affective involvement. Then, from a methodological point of view, importance can be given to the analysis of the non-verbal communication of the teacher (body language, prosody, and so on). From a pedagogical point of view, consequences of this were also highlighted. For example, the importance of becoming aware of this deep intertwinement was stressed, but not only for the researchers' community but also within the teachers' community.

- *Middle ground between theory and practice*: As a final point, the group was able to engage in discussions that respected a good balance between practical and theoretical issues. The entire group did not focus on drawing specific prescriptions for practice or recommendations, nor did the discussions focused on abstract or decontextualized theoretical discussions. This enabled participants to work on issues at a concrete level, as well as grounding them in solid conceptualisations, walking a fine line between each.

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Calcul mental et stratégies : un regard en termes de potentialités

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Abstract : This paper suggests an analysis of solutions, taken from research studies on mental mathematics (with other themes than numbers), in relation to their mathematical potential, and in not in relation to their adequacy or truthfulness. This analysis aims to illustrate the mathematical richness of strategies put forth in a mental mathematics context.

Résumé : Ce texte propose une analyse de solutions tirées d'une recherche sur le calcul mental (avec d'autres thèmes que les nombres) en fonction de leur potentiel mathématique, et non en fonction de leur caractère adéquat ou véridique. Cette analyse a pour but d'illustrer la richesse mathématique des stratégies mises de l'avant en contexte de calcul mental.

Introduction

Les recherches sur le calcul mental montrent de nombreuses retombées positives de cette pratique chez les élèves sur leurs habiletés de résolution de problèmes, leur sens du nombre, leurs habiletés papier-crayon avec les algorithmes standards et leurs habiletés d'estimation (Boule, 2008; Butlen et Pézard, 1992; Heirdsfield et Cooper, 2004; Schoen et Zweng, 1986). Pour Butlen et Pézard (1992), le travail du calcul mental permet aussi de faire avancer des compréhensions mathématiques que le calcul écrit n'arrive pas à faire car trop cadré, avec l'apprentissage de techniques et d'algorithmes suffisamment performants qui ne font pas sentir le besoin de penser plus loin qu'eux. D'autres, tels Murphy (2004), Poirier (1990) ou Threlfall (2002), insistent sur la nature vivante, sur-le-champ et spécifique des procédures de calcul mental, où les stratégies sont souvent développées et taillées sur mesure pour les problèmes à résoudre, ouvrant sur des façons alternatives d'appréhender les problèmes.

Ces travaux sont presque uniquement axés sur les nombres, ce qui fait émerger un intérêt pour le travail d'autres thèmes mathématiques par le calcul mental (e.g. algèbre, trigonométrie, fonctions, statistique, géométrie) et l'étude des procédures déployées dans ces contextes. Ceci oriente mon programme de recherche, qui a pour but d'étudier la nature de l'activité mathématique en contexte de calcul mental sur d'autres thèmes que les nombres. Je présente ici quelques exemples de ce travail pour comprendre de quoi il s'agît et pour illustrer le potentiel du calcul mental pour l'avancement des compréhensions mathématiques.

Le calcul mental sur d'autres thèmes que les nombres

Malgré que les travaux sur le calcul mental soient sur les nombres, les définitions existant dans la littérature s'adaptent bien à un travail sur d'autres thèmes mathématiques. En s'inspirant de la définition de Hazekamp (1986), qui résume ce qui est généralement entendu par calcul mental, le calcul mental sur d'autres thèmes que les nombres peut se définir comme la détermination de réponses à une question mathématique à l'aide d'une résolution mentale, sans papier-crayon ni aide matérielle.

Méthodologie et analyse des stratégies

Pour étudier l'activité mathématique déployée en calcul mental sur d'autres thèmes que les nombres, diverses expérimentations sous forme d'études de cas ont été réalisées à l'intérieur desquelles plusieurs participants (groupes d'enseignants ou d'élèves) doivent résoudre des tâches de calcul mental. L'organisation des séances suit la forme suivante: (1) une tâche est offerte au groupe oralement ou par écrit au tableau; (2) les participants ont 15-20 secondes pour résoudre individuellement la tâche; (3) au signal, les participants écrivent leurs réponses; (4) les stratégies sont partagées et discutées en plénière; (5) une autre tâche est offerte.

Les données proviennent des stratégies expliquées oralement par les participants (et prises en note par les assistants de recherche). L'analyse est centrée, suivant les travaux de Douady (1994), sur la fonctionnalité des stratégies développées dans ces contextes de calcul mental, amenant à entrer sur le potentiel de celles-ci pour faire avancer les compréhensions mathématiques en jeu. Plutôt que de regarder les stratégies en fixant sur leurs aspects (in-)adéquat ou (non-)réussis, celles-ci sont analysées en fonction de ce qu'elles permettent de faire mathématiquement, de où elles peuvent mener.

Pour donner un aperçu de ces stratégies et illustrer le potentiel mathématique de ce type de travail, trois exemples sont présentés : sur les opérations sur les fonctions, sur les systèmes d'équations et sur les équations algébriques. Ces exemples sont choisis pour leur caractère représentatif (voir Proulx, 2013a, 2013b, 2014 pour des détails supplémentaires sur les stratégies déployées).

Exemple #1 : équations algébriques

Lors d'une expérimentation sur la résolution d'équations algébriques, des futurs enseignants de mathématiques du secondaire devaient résoudre des équations de la forme Ax+B=C, Ax+B=Cx+D, Ax/B=C/D, Ax^2+Bx+C=0).

La Figure 1 illustre une des stratégies déployées pour résoudre l'équation 5x+6+4x+3=–1+9x. Le participant explique qu'il n'y a pas de solutions, car on peut rapidement voir 9x d'un côté comme de l'autre de l'égalité, et qu'on voit aussi, sans les additionner, que les nombres restants ne donnent pas une réponse équivalente. Ceci permet de réaliser qu'il n'y a aucun x qui peut faire en sorte que des nombres différents deviennent égaux.

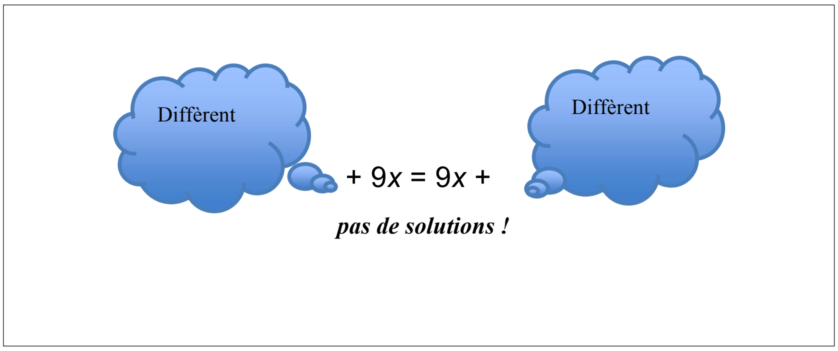

Figure 1. Exemple de stratégie de lecture globale de l'équation

On retrouve dans cette stratégie une idée de lecture globale de l'équation, permettant d'attester rapidement de la réponse, mettant de l'avant l'importance d'analyser l'équation à résoudre avant de se lancer dans sa résolution algébrique. De plus, elle permet de faire face à bon nombre de tâches complexes mais qui ont une structure analysable de façon globale. Par exemple, pour $x+x^2=2x^2+5-x^2$, on peut « voir » ce qui se répète de chaque côté de l'égalité, soit en x et en x^2, et ne pas les considérer dans la résolution, amenant à dire que $x=5$. Avec cette stratégie, le « bruit » provoqué par l'exposant 2 dans x^2, tout comme la présence d'une fraction dans l'exemple connu $x+\frac{x}{4}=6+\frac{x}{4}$ de Bednarz et Janvier (1992), est évité au profit d'une reconnaissance des répétitions non importantes pour la résolution. Cette stratégie, axée sur la recherche d'une valeur de x qui permet de rendre vraie l'équation, amène à traiter la résolution d'équations de façon globale par une analyse préalable de l'équation, et non par une entrée « tête baissée » sur sa résolution mécanique.

Exemple #2 : opérations sur fonctions

Lors d'une expérimentation en 5e année du secondaire, des élèves (15-16 ans) devaient résoudre graphiquement des tâches d'opérations sur fonctions. Deux (ou trois) fonctions représentées dans le plan cartésien sont projetées au tableau et les élèves doivent opérer sur ces fonctions et ensuite dessiner leur réponse sur une feuille dotée d'un graphique cartésien (avec la droite $y=x$ dessinée comme repère). La Figure 2 illustre une tâche où il faut additionner les fonctions f et g. Pour la résoudre, certains ont porté attention aux point suivants : (1) où f coupe l'axe des x (abscisse à l'origine), résultant à la valeur d'image en g (puisque l'image en f à additionner est 0); (2) où f et g se croisent et donnent chacune la même valeur d'image, résultant à une image de double valeur que celle à l'intersection; (3) où f et g croisent l'axe des y (ordonnées à l'origine), résultant d'un processus similaire au cas (2); (4) où g coupe l'axe des x comme dans le cas (1).

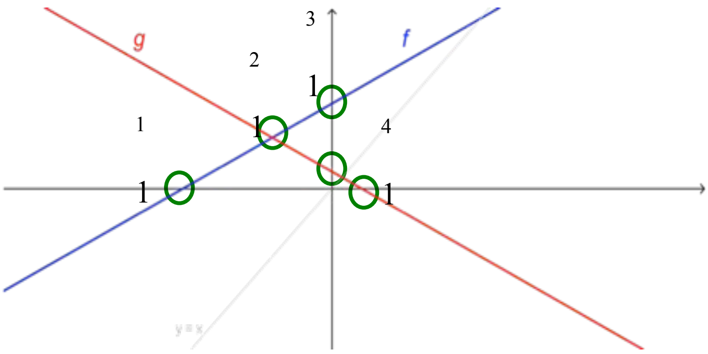

Figure 2. Les points mis de l'avant dans la résolution de la tâche d'addition des deux fonctions

Cette stratégie, axée sur des points « remarquables » qui permettent de déterminer par où passe la droite/fonction résultante, possède de plus une entrée pour travailler la multiplication de fonctions. En effet, s'attarder aux points 1 et 4 permet d'évaluer l'allure générale des images pour un x plus petit que celui au point 1 : avec des valeurs d'images négatives de f multipliées avec des images positives de g donnant des valeurs négatives dans leur multiplication, la même chose se produisant pour l'allure générale des images pour un x plus grand que celui en 4. Pour les valeurs en x situées entre celles pour le point 1 et pour le point 4, la multiplication des images donne une valeur positive, faisant entrevoir la quadratique (degré 2) créée par la multiplication des deux fonctions. Cette entrée par les points « remarquables » ouvre sur une sensibilité aux éléments graphiques significatifs pour les fonctions en jeu, ce qui peut de plus être lié à l'étude des points d'inflexions ou zéros (on obtient ici le tableau d'analyse [+ | – | +]).

Exemple #3 : systèmes d'équations

Dans le cadre d'une autre expérimentation, des enseignants du secondaire devaient résoudre diverses tâches sur les systèmes d'équations, données algébriquement au tableau, et ensuite dessiner leur réponse sur un plan cartésien (doté aussi de la droite repère $y=x$). Dans le cas de la résolution du système d'équations "$y=x$ et $y={}^{-}x+2$, la Figure 2 montre la réponse donnée par un des participants, soit la droite $x=1$.

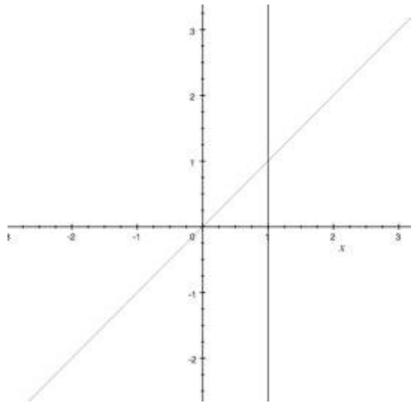

Figure 3. La droite $x=1$ donnée comme réponse pour résoudre le système "$y=x$ et $y={}^{-}x+2$"

Le participant a dessiné une ligne verticale, soit $x=1$, expliquant qu'il n'avait pas eu assez de temps pour trouver la valeur de y, mais que la solution devait être sur cette droite, car $x=1$ donne la même valeur pour les deux équations. À noter que la substitution de $x=1$ dans les équations donne la valeur en y (les équations étant sous la forme $y=\mathrm{m}x+\mathrm{b}$). Toutefois, dans ses manipulations algébriques pour trouver la valeur de x, l'emphase est sur la recherche d'un x commun qui donne la même réponse ($x=?$ et ${}^{-}x+2=?$) et non sur la recherche de la valeur de y, même si *c'est* la même valeur; chaque démarche étant vue séparément.

Malgré qu'incomplète, cette stratégie met de l'avant plusieurs possibilités mathématiques. La droite $x=1$ permet de représenter toutes les possibilités pour y pour résoudre le système, malgré qu'une seule valeur permettra de satisfaire les deux équations simultanément. De plus, cette droite $x=1$ représente la famille des solutions au système d'équations paramétrique "$y=\mathrm{x}+k$ et $y=-x+(2+k)$", entrant sur une étude des paramètres avec $k=0$ comme paramètre de ce système de solution (1,1). D'un autre côté, cet oubli de porter attention au y fait émerger une évidence intéressante que la valeur obtenue pour $x=1$ pour les deux équations est la valeur du y et donc que de faire ceci est aussi de travailler sur le y puisque la valeur en y doit satisfaire aux deux droites du système d'équations. De plus, le croisement de $x=1$ et de la droite repère $y=x$ est exactement où le point solution se situe, permettant d'insister sur le fait que la solution *fait partie* de chacune des deux droites du systèmes d'équations; des évidences souvent enfouies dans le travail algébrique dont cette stratégie permet de mettre de l'avant.

Retour sur l'analyse et remarques finales

L'analyse offerte, malgré que concise, a pour but d'illustrer le potentiel mathématique des stratégies mises de l'avant dans ces contextes de calcul mental sur d'autres thèmes que les nombres : lecture globale de l'équation et signification de l'égalité pour la résolution d'équations algébriques; multiplication de fonctions, tableau de représentation et puissance des points « remarquables » pour les opérations sur les fonctions; famille de solutions, possibilités de réponses, travail des paramètres et caractéristiques du point d'intersection pour la résolution du système d'équations linéaires. Ces cas servent à illustrer l'intérêt du travail de ces thèmes mathématiques par le calcul mental, provoqué par la recherche de stratégies économiques axées sur des propriétés particulières de la tâche (souvent différentes du travail usuel papier-crayon). Ces stratégies ouvrent en retour vers des façons riches de penser le travail sur ces thèmes mathématiques. En plus de ce qu'elles permettent de faire directement avec les tâches en question, il ressort de ces stratégies un potentiel intéressant, divers possibles, pour faire avancer vers de nouvelles compréhensions et activités mathématiques.

Références

Bednarz, N., & Janvier, B. (1992). L’enseignement de l’algèbre au secondaire : une caractérisation du scénario actuel et des problèmes qu’il pose aux élèves. *Actes du colloque international « Didactique des mathématiques et formation des enseignants »* (pp. 21-40), École normale supérieure de Marrakech.

Boule, F. (2008). *Le calcul mental au quotidien. Cycles 2 et 3*. Dijon, France: SCÉREN, CRDP de Bourgogne.

Butlen, D., & Pezard, M. (1992). Calcul mental et résolution de problèmes multiplicatifs, une experimentation du CP en CM2. *Recherches en didactique des mathématiques*, *12*, 319-68.

Douady, R. (1994). Ingénierie didactique et évolution du rapport au savoir. *Repères IREM n° 15,* 25p.

Hazekamp, D.W. (1986) Components of mental multiplying. In H.L. Schoen & M.J. Zweng (Eds.), *Estimation and mental computation. 1986 NCTM Yearbook* (pp. 116-126). Reston, VA: NCTM.

Heirdsfield, A.M., & Cooper, T.J. (2004). Factors affecting the process of proficient mental addition and subtraction: case studies of flexible and inflexible computers. *Journal of Mathematical Behavior*, *23*, 443-463.

Murphy, C. (2004). How do children come to use a taught mental calculation strategy? *Educational Studies in Mathematics*, *56*, 3-18.

Poirier, L. (1990). Évolution du role et de l’importance du calcul mental dans les programmes d’études québécois. *Bulletin de l’Association mathématique du Québec, 2*, 5-10.

Proulx, J. (2013a). Mental mathematics, emergence of strategies, and the enactivist theory of cognition. *Educational Studies in Mathematics*, *84*(3), 309-328.

Proulx, J. (2013b). Le calcul mental au-delà des nombres : conceptualisations et illustrations avec la résolution d’équations algébriques. *Annales de didactique et de sciences cognitives*, *18*, 61-90.

Proulx, J. (2014). Mental mathematics and operations on functions. *Proceedings of joint PME-38 and PME-NA 36* (vol. 5, pp. 17-24).Vancouver, Canada: PME.

Schoen, H.L., & Zweng, M.J. (Eds.) (1986). *Estimation and mental computation. 1986 NCTM Yearbook*. Reston, VA : NCTM.

Threlfall, J. (2002). Flexible mental calculations. *Educational Studies in Mathematics, 50*, 29-47.

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Réflexions sur les obstacles culturels en enseignement des mathématiques

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Abstract: This paper suggests an analysis of mathematics teachers' understandings in terms of cultural obstacles, one of Brousseau's epistemological obstacles dimension. This analysis focuses on the solving of tasks about Cartesian graphs, and its comparison with other groups of professionals making use of graphs (bankers, economists). Far from aiming to outline conceptual difficulties in teachers, this analysis underlines mathematical understandings strongly anchored in the school system context, whereas other groups of professional have their understandings anchored in their own work context.

Résumé : Ce texte propose une analyse de compréhensions développées par des enseignants de mathématiques en termes d'obstacles culturels, une des dimensions mise en évidence par Brousseau (1989) dans son travail sur les obstacles épistémologiques. Nous nous sommes intéressés à la manière dont des enseignants abordent différentes tâches relatives aux graphiques cartésiens comparativement à la manière dont d'autres groupes de professionnels (banquiers, économistes) abordent leur travail sur les graphiques. De cette analyse émerge non pas des erreurs commises au plan conceptuel, mais bien une certaine compréhension mathématique des graphiques fortement ancrée dans le système scolaire, alors que, comparativement, d'autres groupes de professionnels développent de toutes autres compréhensions, elles-aussi ancrées dans leur contexte de travail.

Introduction – la notion d'obstacle

Dans les années 1970-1980, la recherche en didactique des mathématiques a été marquée par un effort de compréhension du processus d'élaboration des connaissances chez l'apprenant et de la manière de traiter ces connaissances pour les faire évoluer (Bednarz & Garnier, 1989), certaines d'entre elles pouvant s'avérer extrêmement résistantes au changement. C'est dans l'analyse de ce dernier scénario que la notion d'obstacle épistémologique est apparue, conduisant à accepter ce que Brousseau appelle des sauts de complexité dans la construction des connaissances par opposition à la recherche de progressions continues (Brousseau, 1976, 1989, 1998, 2010[1]). Derrière la notion d'obstacle épistémologique (Bachelard, 1938), il y a en effet l'idée que les connaissances ne peuvent être construites que par des progrès relativement discontinus : « En fait, on connait contre une connaissance antérieure en détruisant des connaissances mal faites, en surmontant ce qui dans l'esprit même fait obstacle ». L'apprentissage entraine des ruptures cognitives et poursuivre dans cette voie va conduire les chercheurs à réexaminer l'interprétation des erreurs, jusqu'ici attribuées à un disfonctionnement ou une absence de connaissance (voir Bélanger, 1990-91). La reconnaissance des obstacles va dès lors devenir une des clés essentielles pour pouvoir concevoir un apprentissage des mathématiques contribuant à une évolution des connaissances.

En dégageant les caractéristiques communes à tous les exemples que donne Bachelard, Brousseau met en évidence des éléments permettant d'identifier de tels obstacles épistémologiques : « (1) un obstacle épistémologique est une connaissance ou une conception, pas une difficulté ou un manque de connaissances ; (2) cette connaissance produit des réponses adaptées dans un certain contexte, fréquemment rencontré (sinon elle n'aurait pas existé) ; (3) hors de ce contexte, elle engendre des

[1] Dans cet article récent, Brousseau commente et replace dans leur contexte historique et scientifique les textes déjà anciens portant sur la notion d'obstacle (Brousseau, 1976, 1989, 1998).

erreurs, des « réponses évidentes » mais fausses ; (4) cette connaissance résiste aux contradictions auxquelles elle est confrontée et à l'établissement d'une meilleure connaissance (ce remplacement est difficile à cause de la persistance des avantages que cette connaissance avait procuré ; autrement dit il ne suffit pas de posséder une connaissance meilleure pour que celle qui faisait obstacle disparaisse, il faut l'identifier, la renier explicitement et incorporer sa négation aux connaissances nouvelles) ; (5) Néanmoins, malgré la prise de conscience de son inexactitude, elle tend à réapparaitre de façon intempestive et opiniâtre » (Brousseau, 2010). D'autres types d'obstacles seront également repérés et une classification en sera proposée selon leur origine (Brousseau, 1989) :

- Épistémologique : inscrit dans l'histoire des connaissances et dans la culture, et dont le rejet a dû être intégré explicitement ;
- Didactique : conséquence de certains choix didactiques et d'une certaine manière d'approcher l'enseignement ;
- Ontogénétique : typique d'une étape du développement, apparaissant naturellement au cours du développement du sujet ;
- Culturel : connaissances d'origine diverse véhiculées par la culture et toujours présentes, une sorte de cécité de la culture.

Ce dernier type d'obstacle est, de tous les types, celui qui a été le moins développé en didactique des mathématiques. C'est de ce type d'obstacle que traite ce texte, dans une intention de creuser cette idée d'obstacle culturel, en puisant pour cela à un exemple tiré d'un travail effectué avec des enseignants de mathématiques du secondaire. Cet exemple nous permet d'illustrer comment une certaine « culture mathématique » (reliée ici à un ordre d'enseignement donné, soit le secondaire) peut façonner ses propres obstacles[2].

Repères méthodologiques

Nos données sont tirées d'un projet de recherche-formation (Bednarz & Proulx, 2010) mené avec des groupes d'enseignants du secondaire provenant d'écoles de la grande région de Montréal (deux groupes provenant d'écoles différentes, 8 enseignants par groupe, intervenant à différents niveaux scolaires (12 à 17 ans)). La recherche s'est étalée sur un an et demi, à raison de 15 sessions d'une journée par mois. Durant les sessions, les activités étaient organisées autour de tâches mathématiques touchant à différents contenus mathématiques (fractions, fonctions, algèbre, périmètre-aire, volume, trigonométrie). De plus, ces tâches étaient articulées à la pratique des enseignants, c'est-à-dire ancrées dans des situations d'apprentissage et d'enseignement des mathématiques en contexte de pratique : réponse ou question d'élèves, vignettes de classe, problèmes de manuels scolaires, matériels didactique, questions d'examen, etc. Les enseignants étaient invités à s'engager dans ces tâches mathématiques, d'abord en petits groupes puis en grand groupe, dans le but d'explorer et discuter les idées mathématiques sous-jacentes.

Dans ce qui suit, nous analysons un exemple portant sur le travail des graphiques. Une interprétation spécifique du graphique émerge de l'analyse, conceptualisée par nous en tant

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[2] La culture est ici définie au sens de Hall (1959) et renvoie non seulement aux convictions partagées de ceux qui s'y rassemblent, et à la partie explicite de cette culture relevant des dimensions institutionnelles (programmes, évaluations, par exemple), mais aussi et surtout à sa partie informelle et implicite qui relève pour une grande part des connaissances, des savoir-faire et des manières de faire partagées par ceux qui la constituent et s'y reconnaissent (voir aussi Artigue, 2004).

qu'obstacle culturel, et qui met de l'avant la façon dont une certaine culture scolaire peut façonner des concepts mathématiques.

L'interprétation graphique : la vision fonctionnelle comme obstacle culturel

L'extrait suivant est tiré d'une discussion dans un des groupes d'enseignants au sujet de l'exploration de différents graphiques. Alors qu'ils devaient anticiper ce que leurs élèves pourraient répondre à ces tâches, une discussion a émergé concernant la pertinence de telles tâches pour travailler l'interprétation graphique (la Figure 1 illustre deux de ces tâches graphiques, tirées du Shell Centre for Mathematical Education, 1985).

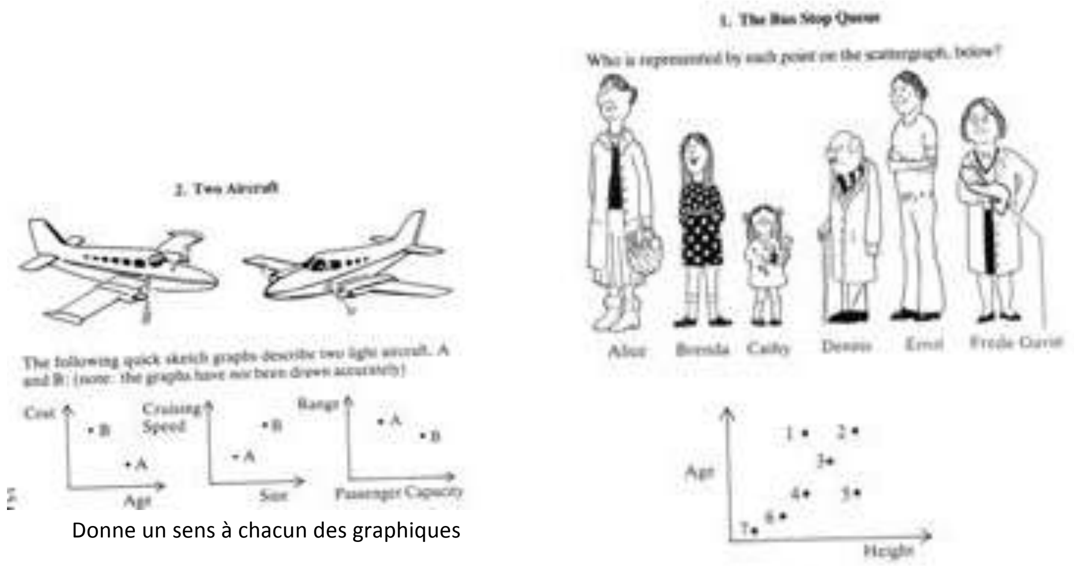

Figure 1. Exemple de deux tâches sur les graphiques distribuées aux enseignants

> Le débat sur la valeur de ces tâches sur les graphiques est démarré par Marie, une enseignante de 8e année: « Quelle est la pertinence de ce type de graphique s'il n'y a aucune relation entre les valeurs pour les deux axes? C'est contraire à ce que nous tentons de développer chez nos élèves, soit l'idée de variable dépendante et indépendante et de la relation entre les deux. ».
> Marie par la suite apparait ambivalente sur la valeur de ce type de graphique : « D'un côté, je trouve ces graphiques intéressants parce qu'ils forcent une interprétation des axes par l'élève, ce qui représente une de leurs difficultés que j'ai observée chez eux lorsqu'ils travaillent avec les graphiques. Ils ne lisent tout simplement pas les axes. Mais d'un autre côté, je ne suis pas très confortable avec l'idée de présenter des graphiques où il n'existe pas de relation entre les valeurs impliquées. ».
> D'autres enseignants seront du même avis, dont Clara, enseignante de 8e et 10e années : « Quelle est la pertinence de ces graphiques s'il n'y a pas de relation entre les variables? Un graphique est intéressant pour voir la corrélation entre les deux variables (e.g. en statistiques, pour l'étude du comportement d'une série de données et sa tendance, sa corrélation) ou si ceci correspond à une relation fonctionnelle. S'il n'y a pas une et une seule valeur qui correspond à une autre valeur spécifique de la variable indépendante, nous n'avons pas une fonction et ce n'est pas intéressant puisque nous ne pouvons pas parler de fonction réciproque ou inverse, par exemple. ».
> À travers la discussion, tous les enseignants étaient d'accord avec le fait que dans le cas des avions, il n'y avait aucune pertinence à utiliser un graphique pour représenter la situation, soulignant plutôt qu'une table de valeurs serait plus adéquate.

Les réactions des enseignants (sur un ton, il faut le dire, de surprise et exprimant un peu de frustration face à des tâches où les données n'étaient pas corrélées) mettent en évidence une compréhension spécifique et partagée des graphiques: des manières de travailler et voir les graphiques qui se constituent dans leurs pratiques, lorsqu'ils les enseignent et qui deviennent, en quelque sorte, *ce que sont* les graphiques pour eux. Les enseignants voient les graphiques, les conçoivent, comme représentant des relations entre deux variables; l'idée de la relation entre ces deux variables, de la variable indépendante à la variable dépendante, étant centrale et orientant vers une certaine « lecture » du graphique. Le sens donné aux graphiques se voit en quelque sorte transformé par l'activité professionnelle des enseignants : le graphique est ici pensé en termes d'une corrélation éventuelle, d'une anticipation d'une certaine variation, de la recherche d'une règle reliant les variables, ou encore d'un travail sur les paramètres[3], sur la recherche des fonctions inverses ou réciproques (autant de circonstances, d'usages, qui donnent sens à cette manière de voir partagée et qui sont indexées à cette conceptualisation).

Cette vision du graphique n'est pas inadéquate (elle fonctionne effectivement dans certaines circonstances, celles énoncées entre autres ci-haut). Toutefois, elle contraste, par exemple, avec celle qui ressort d'une étude ethnographique menée auprès d'un groupe de banquiers par Noss et Hoyles (1996). Pour les banquiers, les graphiques sont vus comme une manière de rendre compte de données, d'en fournir une image, comme une façon rapide et simple d'illustrer des données. Les banquiers, contrairement aux enseignants, s'attardent en ce sens à la forme du graphique selon les caractéristiques de sa représentation (nombre d'éléments représentés, disponibilité de la couleur, public visé, fréquence des variables, échelle des axes, etc.) et non à la relation entre une certaine quantité et une autre.

Cette lecture « fonctionnelle, de la variable indépendante à la variable dépendante » du graphique va aussi être confrontée à d'autres interprétations possibles, interrogeant sa pertinence. Ceci devient évident lorsqu'on compare cette interprétation avec la façon avec laquelle les économistes « lisent » les graphiques, tel que le montre Cognard (1996). Ceux-ci considèrent en premier lieu la valeur en ordonnée et ensuite celle en abscisse dans les graphiques d'offre et de demande, ce qui pourrait amener nos enseignants du secondaire à affirmer que les économistes lisent les graphiques « à l'envers »![4]

Ces exemples constrastés d'interprétation du graphique, des enseignants aux banquiers en passant par les économistes, illustrent les ruptures épistémologiques possibles auxquels pourraient être confrontés les étudiants du secondaire dans le passage d'un contexte à l'autre (notamment des mathématiques à économie) comme le souligne Noss (2002): "As mathematical knowledge is embedded in new settings and activities, it undergoes an epistemological and cognitive transformation" (p. 55).

Le contraste entre ces trois groupes de professionnels distincts fait ressortir des interprétations du graphique différentes et montre bien l'ancrage culturel de cette connaissance. La lecture fonctionnelle du « graphique » par les enseignants est une connaissance partagée par ces derniers, intégrée inconsciemment dans leurs pratiques (mais qui la constituent en retour), et relève en ce sens de ce que Hall (1959) nomme le plan informel de la culture (ici une culture mathématique qui se constitue au quotidien de l'action des enseignants du secondaire).

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[3] Il y a ici un autre cas d'obstacle culturel souvent rencontré dans le travail scolaire au Québec au niveau des paramètres, et qui n'a été qu'effleuré durant cette recherche. Les paramètres, parce que majoritairement, sinon uniquement, étudiés lors du travail sur les fonctions dans le curriculum scolaire québécois, en viennent à être conçus comme rattachés à l'étude des fonctions et à rien d'autre, les enseignants ayant de la difficulté à sortir du contexte de fonction pour étudier des questions relatives aux paramètres. On est ici aussi en présence d'un obstacle culturel relié à une culture mathématique scolaire (celle du secondaire) qui forge et transforme une notion par son usage scolaire.

[4] Ce qui est exactement ce que les étudiants que nous avons dans nos cours à la formation des maîtres au secondaire disent des économistes !

Conclusion

Avec ce cas d'interprétation de ce qu'est, peut être et représente un graphique, nous sommes bien en présence d'un obstacle *culturel,* au sens où cette compréhension, qui constitue une connaissance construite par les enseignants dans leur fréquentation professionnelle des mathématiques, est partagée par eux, au sein d'une certaine culture mathématique. On retrouve bien par ailleurs les caractéristiques d'un *obstacle* énoncées par Brousseau précédemment:

(1) celle d'une connaissance (une manière de voir les graphiques, de donner du sens, avec des circonstances associées) ;
(2) une connaissance qui fonctionne dans certains contextes familiers rencontrés par les enseignants (on travaille en effet le graphique sous forme fonctionnelle aux différents niveaux scolaires), mais
(3) qui peut engendrer, en dehors de ces contextes familiers, des réponses « évidentes » pour l'enseignant mais fausses : l'interprétation fonctionnelle, de la variable dépendante à la variable indépendante, est inadéquate dans d'autres situations, comme nous l'avons vu pour les banquiers et économistes, de sorte que si l'enseignant est confronté à une façon différente de raisonner venant de ses élèves (données séparées non corrélées, lecture en ordonnée avant, co-variation), il risque fort de dire que l'interprétation de l'élève est fausse et de ne pas l'accepter ; il peut être également amené à juger certaines questions non pertinentes et à ne pas les retenir, l'interprétation des graphiques en tant que représentation fonctionnelle étant vue comme allant de soi, et ce, même si les fonctions ne sont pas l'objet d'étude du graphique en cause;
(4) le remplacement de ces connaissances, comme on peut le voir dans ce qui précède, par des connaissances mathématiques plus appropriées, s'avère difficile à cause de la persistance des avantages que les enseignants y voient (les contextes d'école en mathématiques sont en effet tous « traitables » en vision fonctionnelle) ou encore à cause d'une certaine idée partagée de ce que sont les graphiques en mathématiques dans laquelle ils se reconnaissent ; les fonctions ayant une connotation en mathématiques plus « noble » que le travail sur des données séparées.
Nous sommes proches en ce sens de ce que Bachelard nommait obstacle épistémologique (inscrit dans ce cas dans une certaine culture de fonctionnement).

RÉFÉRENCES

Bednarz, N., & Janvier, B. (1992). L'enseignement de l'algèbre au secondaire : une caractérisation du scénario actuel et des problèmes qu'il pose aux élèves. *Actes du colloque international « Didactique des mathématiques et formation des enseignants »* (pp. 21-40), École normale supérieure de Marrakech.

Boule, F. (2008). *Le calcul mental au quotidien. Cycles 2 et 3*. Dijon, France: SCÉREN, CRDP de Bourgogne.

Butlen, D., & Pezard, M. (1992). Calcul mental et résolution de problèmes multiplicatifs, une experimentation du CP en CM2. *Recherches en didactique des mathématiques*, *12*, 319-68.

Douady, R. (1994). Ingénierie didactique et évolution du rapport au savoir. *Repères IREM n° 15,* 25p.

Hazekamp, D.W. (1986) Components of mental multiplying. In H.L. Schoen & M.J. Zweng (Eds.), *Estimation and mental computation. 1986 NCTM Yearbook* (pp. 116-126). Reston, VA: NCTM.

Heirdsfield, A.M., & Cooper, T.J. (2004). Factors affecting the process of proficient mental addition and subtraction: case studies of flexible and inflexible computers. *Journal of Mathematical Behavior*, *23*, 443-463.

Murphy, C. (2004). How do children come to use a taught mental calculation strategy? *Educational Studies in Mathematics*, *56*, 3-18.

Poirier, L. (1990). Évolution du role et de l'importance du calcul mental dans les programmes d'études québécois. *Bulletin de l'Association mathématique du Québec, 2*, 5-10.

Proulx, J. (2013a). Mental mathematics, emergence of strategies, and the enactivist theory of cognition. *Educational Studies in Mathematics*, *84*(3), 309-328.

Proulx, J. (2013b). Le calcul mental au-delà des nombres : conceptualisations et illustrations avec la résolution d'équations algébriques. *Annales de didactique et de sciences cognitives*, *18*, 61-90.

Proulx, J. (2014). Mental mathematics and operations on functions. *Proceedings of joint PME-38 and PME-NA 36* (vol. 5, pp. 17-24).Vancouver, Canada: PME.

Schoen, H.L., & Zweng, M.J. (Eds.) (1986). *Estimation and mental computation. 1986 NCTM Yearbook*. Reston, VA : NCTM.

Threlfall, J. (2002). Flexible mental calculations. *Educational Studies in Mathematics, 50*, 29-47.

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Adaptation de l'enseignement des mathématiques en contexte de collaboration et de coenseignement

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Abstract: The adaptation of teaching students with learning difficulties involves difficult pedagogical choices and the need to move towards more inclusive practices. In this perspective, the collaboration between teachers and resource teachers is an indispensable means, which is occasionally reflected in the choice of combining their professional expertise by practicing team teaching. Current research status of some adaptation actions that could be made in the context of team teaching and learning of mathematics. This doctoral research has identified and analyzed adaptation actions taking into account the operating procedures in collaboration between the teacher and the resource teacher supports the same groups-class. More specifically, the study refers to aid and interventions deployed by three teacher-education teacher dyads, likely to raise problems solving strategies with students of 2nd and 3rd cycles of primary school, during a sequence of team teaching.

Résumé: L'adaptation de l'enseignement aux élèves ayant des difficultés d'apprentissage implique des choix pédagogiques difficiles et le besoin de tendre vers des pratiques plus inclusives. Dans cette perspective, la collaboration entre les enseignants et les orthopédagogues constitue un moyen souvent incontournable, lequel se traduit occasionnellement dans le choix de combiner leurs expertises professionnelles en pratiquant le coenseignement. Les recherches actuelles font peu état des gestes d'adaptation qui peuvent être apportés en contexte de coenseignement et d'apprentissage des mathématiques. Cette recherche doctorale a permis de dégager et d'analyser des gestes d'adaptation en tenant compte des modalités de fonctionnement en collaboration entre l'enseignant et l'orthopédagogue prenant en charge le même groupe-classe. Plus spécifiquement, l'étude fait état des aides et des interventions déployées par trois dyades enseignant-orthopédagogue, susceptibles de mobiliser des stratégies de résolution de problèmes auprès d'élèves des 2^e^ et 3^e^ cycles du primaire, au cours d'une séquence de coenseignement.

État de la situation de l'intégration scolaire

Depuis plusieurs années, les enseignants de classes ordinaires manifestent le besoin d'être soutenus pour réaliser des adaptations liées à leur enseignement et répondre aux besoins de tous les élèves. Il devient ainsi possible de constater des carences dans la planification et la réalisation d'interventions destinées aux élèves en difficulté (Conseil supérieur de l'éducation, 1996). Par ailleurs, la présence en classe ordinaire d'élèves ayant des difficultés d'apprentissage est sans cesse croissante. En 1999, au Québec, les données ministérielles indiquaient que les élèves ayant un handicap, des difficultés d'adaptation ou d'apprentissage (EHDAA) formaient 12,42% de l'effectif scolaire. De ce pourcentage, la majorité présentait des difficultés d'apprentissage. Des données plus récentes dévoilent un accroissement de ce pourcentage, lequel est passé de 13,5 % en 2002-2003 à 16,0% en 2005-2006 (Gaudreau, Legault, Brodeur, Hurteau, Dunberry, Séguin et Legendre, 2008), puis à 18,4 % en 2009-2010, révélant ainsi une augmentation de 20 % en 7 ans (MÉLS, 2010). Ces élèves deviennent particulièrement touchés par le phénomène du décrochage scolaire et plusieurs d'entre eux quittent l'école sans avoir obtenu leur diplôme d'études secondaires (MÉLS, 2007).

Les difficultés en lecture et en mathématiques demeurent les plus préoccupantes. Environ 60 % des élèves qui ont des difficultés d'apprentissage en lecture auraient aussi des difficultés en mathématiques (Gersten et Chard, 1999). Le chevauchement de ces difficultés est attribuable à de faibles habiletés sur le plan du langage et de la littératie, mais est aussi lié à un déficit touchant les compétences mathématiques, notamment les processus stratégiques, les connaissances conceptuelles et la mémoire (Geary, 2005). Si certains élèves demandent uniquement des accommodations sur les plans physique ou matériel, d'autres ont besoin d'adaptations beaucoup plus substantielles sur les plans pédagogique et didactique (Goupil, 2007). Ces adaptations doivent également être prises en compte dans l'évaluation des apprentissages et être consignées à l'intérieur d'un plan d'intervention (MEQ, 2003).

Pour faire face à cette situation, les mesures d'aide sont généralement apportées dans le cadre d'un service orthopédagogique obligeant des rencontres individuelles ou en petits groupes d'élèves, se tenant à l'extérieur de la classe ordinaire d'appartenance. Or, l'une des principales critiques à l'égard de cette pratique est l'absence d'harmonisation des interventions et le manque de coordination entre les intervenants (St-Laurent, Giasson, Simard, Dionne, Royer, 1995). La collaboration entre enseignants et orthopédagogues semble ainsi constituer un moyen qui mériterait d'être davantage investigué, plus précisément en lien avec la conception et la réalisation conjointe de situations d'enseignement-apprentissage en salle de classe. Dans ce contexte, il s'est avéré indispensable de s'interroger sur les aides et interventions déployées communément par ces intervenants, et ce, plus particulièrement en mathématiques, en contexte de résolution de problèmes. Le présent article tente sommairement d'apporter un éclairage sur cette problématique, lequel repose sur un questionnement quant aux modalités de collaboration pouvant être établies entre les enseignants et les orthopédagogues dans la conception et la réalisation conjointes de situations d'enseignement-apprentissage prévoyant des adaptations pour des élèves ayant des difficultés d'apprentissage en mathématiques, puis, quant aux aides et interventions mises en place par ces partenaires au sein de la classe ordinaire.

Les objectifs poursuivis sont de décrire les modes de fonctionnement relatifs à la pratique du coenseignement en contexte d'implantation de ce type de collaboration, puis d'identifier et d'analyser les aides et interventions pouvant ainsi être apportées au sein de la classe.

Cadre conceptuel

La différenciation pédagogique et les adaptations de l'enseignement

Il existe une conception d'ordre didactique à l'effet que toute adaptation ait comme point de départ certains aménagements de situations d'enseignement-apprentissage aux besoins particuliers des élèves (MÉLS, 2006). Le fil conducteur de ces aménagements est celui de la différenciation pédagogique associée à l'intervention auprès des élèves en difficulté (Fuchs et Fuchs, 2007). Dans une visée inclusive, une adaptation destinée à des élèves en difficulté est aussi utilisée pour l'ensemble des élèves de la classe et devient ainsi profitable à tous (Nootens et Debeurme, 2010).

Les chercheurs Nootens et Debeurme (2010) dont la posture se veut inclusive distinguent deux sortes d'adaptations, les générales et les spécifiques. Les adaptations générales sont associées à des adaptations de routine (Fuchs, Fuchs et Bishop, 1992; Switlick, 1997) dont peuvent bénéficier à la fois les élèves en difficulté et les autres élèves de la classe. Ces auteures réfèrent à ce que le MÉLS (2006) nomme « flexibilité pédagogique », laquelle s'instaure selon l'émergence des besoins diversifiés pour l'ensemble des élèves (Nootens et Debeurme, 2010). De plus, ces adaptations, qualifiées de générales, servent de structures souples grâce auxquelles les interventions spécifiques peuvent avoir lieu. Les adaptations spécifiques renvoient ainsi au soutien particulier s'adressant à l'élève en difficulté. Elles concernent principalement le contenu notionnel et la tâche qui lui est reliée dans le but d'offrir un soutien aux difficultés à apprendre, à comprendre et à maîtriser le contenu scolaire (Fuchs et *al.,* 1995). Ces adaptations peuvent également se traduire sous la forme d'accommodements ou d'ajustements (Nootens et Debeurme, 2010).

L'adaptation de l'enseignement des mathématiques, ou son amélioration, constitue le fil conducteur des recherches menées dans le champ de la didactique des mathématiques depuis ses premiers développements, principalement sur la base de la théorie des situations didactiques (Brousseau, 1998) et de l'étude de ces situations par le biais de méthodes s'inspirant de l'ingénierie didactique (Artigue, 1988). Les positions adoptées intègrent les points de vue piagétien et vygostskien, soit suivant le postulat que le savoir construit par l'élève est le résultat de l'ensemble des gestes d'adaptation qui ont été posés pour lui, tout comme le résultat de son propre engagement et de sa capacité à réaliser la tâche qui lui est soumise en contexte de situations d'enseignement-apprentissage (Brun, 1994). La conception et la réalisation des situations d'enseignement-apprentissage s'effectuent en somme dans un système dynamique et ouvert (Rogalski, 2008). Certaines dimensions du travail de l'enseignant sont ainsi soumises à des contraintes, notamment dans des classes regroupant des élèves ayant des difficultés d'apprentissage (Perrin-Glorian, 1993). Les contraintes de temps de même que la prise en compte de certains obstacles pédagogiques émanant de l'hétérogénéité des élèves en sont des exemples (Mercier, 2002; Sarrazy, 2002). Il importe donc d'avoir une vision critique de ces dimensions contraignantes et de disposer de moyens pour les reconnaître. Si une certaine stabilité dans les pratiques des enseignants peut être observable (Pariès, Robert et Rogalski, 2008), plusieurs contraintes font en sorte que ces derniers effectuent malgré tout des choix qui diffèrent de ce qu'ils avaient anticipé au départ pour sa classe (Roditi, 2008).

La collaboration professionnelle et le coenseignement

Le milieu de l'éducation souscrit au développement de la collaboration entre les praticiens de l'enseignement et au besoin d'instaurer une culture de la collaboration dans la formation initiale en enseignement (Bourassa, Fournier, Goyer et Veilleux, 2013). La collaboration est la mise en relation des pratiques soutenant le développement professionnel des enseignants (Garcia et Marcel, 2011) et l'adoption de nouveaux dispositifs collectifs permettant une plus grande réussite des élèves (Hargreaves et Dawe, 1990). Elle s'instaure grâce à la dimension dialogique et au partage des savoirs entre les interlocuteurs, notamment dans la réalisation de travaux conjoints (Little, 1990; Martineau et Simard, 2011).

Les écrits récents sur la collaboration entre les enseignants et les orthopédagogues soulignent qu'il s'agit d'une pratique émergente (Tardif, 2007), mais aussi d'un élément clé dans le virage vers l'inclusion scolaire (Rousseau et Bélanger, 2004). Le coenseignement constitue une forme particulière de collaboration entre enseignants et orthopédagogues qui a attiré l'attention des chercheurs depuis quelques années. Cette pratique donne l'opportunité aux coenseignants de partager leurs connaissances et leur matériel, de briser leur sentiment d'isolement professionnel et de combiner leurs expertises afin d'améliorer leur enseignement auprès des différents apprenants, notamment pour les élèves à risque, ayant un handicap ou des besoins particuliers ((Trépanier et Paré, 2010), Volonino et Zigmond, 2007).

Les auteurs Gately et Gately (2001) ont défini le coenseignement comme une pratique selon laquelle l'enseignant du secteur général et celui du secteur de l'adaptation scolaire réunis, partagent la responsabilité de l'enseignement à l'ensemble des élèves de la classe, incluant les élèves ayant des besoins spéciaux. Il s'agit d'un moyen de répondre à la nécessité d'adapter les approches pédagogiques aux besoins de chaque apprenant et de satisfaire ainsi aux visées d'un enseignement inclusif (Murawski, 2003; Villa, Thousand et Nevin, 2008). La pratique du coenseignement est ainsi bien documentée. Pour en favoriser l'implantation, Trépanier et Paré (2010) en proposent différents modèles :

a) Le coenseignement en soutien : un enseignant intervient auprès du groupe-classe pendant que l’autre intervient auprès des élèves ciblés (*préenseignement*, *réenseignement* ou enrichissement).
b) Deux enseignants, deux groupes hétérogènes, un même contenu : l’enseignement porte sur un même contenu et les coenseignants interviennent sur des contenus similaires ou complémentaires. Les objectifs sont planifiés conjointement de même que les critères d’évaluation. Au terme de l’activité, un retour avec l’ensemble des élèves permet de faire le point sur les apprentissages réalisés.
c) Deux enseignants, deux sous-groupes homogènes, un enseignement complémentaire : l’orthopédagogue reprend le contenu enseigné avec un sous-groupe d’élèves en difficulté présentant des caractéristiques communes. L’enseignant poursuit son enseignement auprès de l’autre sous-groupe. Ce modèle permet d’approfondir un contenu avec certains élèves en utilisant des stratégies différentes pour un contenu déjà enseigné par l’un des coenseignants.
d) Des sous-groupes multiples, un contenu variable : ce modèle correspond à celui de l’enseignement en ateliers (*station teaching*), où les coenseignants interviennent en rotation ou dans une aire désignée, en divisant le contenu d’enseignement.
e) L’enseignement en équipe : ce modèle correspond à celui de l’enseignement de type « *team-teaching* », où les coenseignants enseignent en équipe. L’orthopédagogue peut se centrer davantage sur les stratégies d’apprentissage ou d’organisation à privilégier pour la réalisation de l’activité.

Il est important de préciser qu’à la lumière des études effectuées sur le coenseignement, plusieurs facteurs critiques ont été identifiés. Cette pratique implique une planification commune et des retours réflexifs sur le travail effectué conjointement ainsi que sur les ajustements à apporter aux besoins diversifiés des élèves (Murawski, 2003). De plus, le coenseignement exige le partage des responsabilités et un haut degré d’implication des acteurs (Friend et Cook, 2003), une part d’enthousiasme et de complicité (Zigmond et Magiera, 2001), une familiarité avec le curriculum et une philosophie commune sur l’enseignement et l’apprentissage (Isherwood et Barger-Anderson, 2007).

La résolution de problèmes et ses caractéristiques

Cette étude s’intéresse aux pratiques d’adaptation de l’enseignement réalisées en lien avec la résolution de problèmes mathématiques. Les recherches conduites sur la résolution de problèmes ont permis de comprendre que les difficultés des élèves se situent essentiellement dans la compréhension et l’interprétation des énoncés de même que dans la mise en relation du résultat de cette compréhension avec les procédures de résolution (Stern et Lehrndorfer, 1992; Thevenot, Barrouillet et Fayol, 2004). À la lumière de ces recherches, Geary (2005) propose plusieurs axes d’interventions susceptibles de contribuer à l’adaptation de l’enseignement en contexte de résolution de problèmes : 1) Miser sur la compréhension des énoncés en les rendant plus explicites et en faisant référence à l’univers des élèves; 2) Aménager les énoncés en fonction de la chronologie des évènements et en plaçant la question en tête d’énoncé; 3) Entraîner les élèves aux procédures de résolution de problèmes en valorisant l’élaboration de représentations graphiques; 4) Favoriser l’apprentissage des schémas des problèmes en lien avec les procédures utilisées à partir d’exemples de problèmes résolus.

De plus, selon Baroody et Coslick (1998), différentes composantes cognitives, métacognitives et affectives influent sur la résolution de problèmes. Les composantes cognitives concernent la capacité à utiliser les connaissances acquises dans un nouveau contexte et l’habileté à comprendre un problème, à l’analyser et à le résoudre sans devoir faire appel uniquement à la mémoire, aux procédures et aux règles. Les composantes métacognitives font appel à la capacité de réfléchir à son propre processus cognitif, à reconnaître qu'une solution est vraisemblable ou non, à évaluer sa démarche tout au long du processus. Les composantes affectives sont reliées à une réaction émotionnelle positive découlant de la confiance dans sa capacité à résoudre un problème, au fait de percevoir les mathématiques comme une matière intéressante, à la capacité à persévérer et à prendre des risques, à la conviction que les erreurs fournissent une occasion d'approfondir et d'améliorer sa compréhension. Ces diverses composantes cognitives, métacognitives et affectives présentes en contexte de résolution de problèmes sont prises en considération dans l'analyse des données issues de cette étude.

Méthodologie

La posture de la recherche et la cueillette de données

Pour la réalisation de cette recherche, la méthodologie prend appui sur le modèle de l'ingénierie didactique (Artigue, 1988) et sur celui de l'ergonomie cognitive (Rogalski, 2008). Cette approche a permis d'accéder au déploiement des pratiques conjointes des enseignants et des orthopédagogues dans leurs manifestations auprès de leurs élèves. Il s'agissait plus précisément de pouvoir aborder ces pratiques selon la manière dont elles se structurent au préalable, puis s'organisent et se vivent en situation de classe, *in situ*, sous l'impulsion des interactions avec les élèves.

Divers outils de collecte ont été utilisés, principalement pour la prise de données avant, pendant et après la réalisation conjointe des situations d'enseignement apprentissage: entretiens semi-dirigés, observations filmées en classe et tenue d'un journal de recherche. Les données ont été recueillies auprès de trois dyades formées d'une enseignante et d'une orthopédagogue et dans la cadre de l'instauration d'une démarche de collaboration entre ces acteurs donnant lieu à la réalisation de situations d'enseignement-apprentissage dans les classes des trois enseignantes concernées (2 classes de 4^{e} année et 1 classe de 5^{e} année). Pour chacune des dyades, les analyses ont été effectuées sur l'ensemble des séances observées portant sur des activités de résolution problèmes, soit en moyenne quatre périodes d'enseignement des mathématiques par dyade. Les aides et interventions mises en œuvre en vue de mobiliser des stratégies de résolution de problèmes ont été recensées et analysées, et ce, en tenant compte des dimensions didactiques liées à la dynamique et à l'évolution des situations. Les différents regards posés par les partenaires sur la conception et la réalisation de ces situations ont également été pris en compte.

Les analyses

Les analyses ont permis de mettre en évidence trois profils de collaboration distincts, ayant des retombées en lien avec la conception et la réalisation des situations d'enseignement-apprentissage. Différents constats ont pu être émis, lesquels montrent surtout une collaboration fragile découlant d'une coplanification centrée sur la tâche et non sur l'ajustement des situations d'enseignement-apprentissage aux besoins et caractéristiques des élèves. En outre, l'enjeu a surtout été de doter les élèves d'une procédure particulière de résolution de problèmes, celle-ci ayant été justifiée en favorisant le principe d'un découpage séquentiel du processus de résolution de problèmes : 1) l'identification de la question; 2) l'identification des données permettant de répondre à la question; 3) le traitement des données retenues; 4) l'émission de la réponse en lien avec la question. Outre l'appropriation de cette procédure par les élèves, les aides et interventions déployées ont montré des besoins plus spécifiques pour lesquels les enseignantes et les orthopédagogues ont dû réagir selon leur propre expertise, à partir de leur propre référentiel opératif (Marcel, Dupriez et Périsset Bagnoud, 2007) et en fonction de leur savoir pédagogique respectif (Shulman, 1986b)). Ces aides et interventions se sont avérées la plupart du temps discordantes, non planifiées, non coordonnés et issues d'une conception de l'adaptation de l'enseignement ne faisant pas consensus.

Cependant, à la lumière des interactions avec certains élèves, quelques aides et interventions sont apparues pertinentes pour ce qui est des stratégies de résolution de problèmes mobilisées et mise en œuvre dans les classes : 1) guidage sur la compréhension des énoncés du problème pour les rendre plus explicites; 2) accentuation sur la recherche de liens entre la question et les indices fournis (relations) orientant la prise de données utiles; 3) aide à l’élaboration des représentations graphiques en lien avec les procédures utilisées; 4) verbalisation des solutions (venant des élèves) en vue d’un regard critique sur la démarche employée; 5) appui à l’emploi de stratégies de vérification visant à juger de la pertinence des réponses par un retour sur la question.

La dyade 1

L’analyse des résultats a montré que les aides apportées au sein de la dyade 1 sont demeurées centrées sur des stratégies de recherche de solutions à suivre pas-à-pas, sans pour autant tenir compte des stratégies des élèves et de leur compréhension des problèmes. Elles ont été le plus souvent prises en charge par l’enseignante, posées dans l’optique d’un guidage quant aux procédures exigées et axées sur le recours à une méthode de découpage séquentiel du processus de résolution de problèmes. En retour, les élèves ciblés et identifiées comme ayant des difficultés en mathématiques n’ont pas toujours été en mesure de développer les stratégies mathématiques nécessaires pour résoudre les problèmes. Les solutions ont été livrées. Notons que le besoin de répondre simultanément aux demandes des élèves ayant des difficultés et à celles de l’ensemble des élèves de la classe a pu induire ce comportement de la part de l’enseignante. Cette analyse rejoint les propos de Weiss et Lloyd (2003) qui soutiennent que l'attitude de l'enseignant et l’absence d’une planification conjointe sont des facteurs limitatifs amenant l’orthopédagogue à jouer un rôle d’assistant plutôt que celui d'un véritable coenseignant. Cette absence d’un travail conjoint pour la planification des situations d’enseignement-apprentissage diminue alors la possibilité d’effectuer des interventions spécialisées et individualisées (Weiss et Lloyd, 2003).

La dyade 2

Les résultats ont démontré une contribution importante de l’orthopédagogue dans le cadre de ce projet, tant au moment de la planification des situations d’enseignement-apprentissage qu’au moment de la réalisation de ces situations. Une posture visant la complémentarité des approches a été adoptée par cette dernière. Cette posture rejoint les paramètres correspondant à celui du coenseignement en équipe (*team teaching*) rapportés par Cook et Friend (1995). Dans ce modèle, les coenseignants partagent la responsabilité de l’enseignement à l’ensemble des élèves, et ce, à partir d’une même situation planifiée et conçue de façon à permettre une répartition équitable des responsabilités selon les expertises de chacun (Argüelles, Hughes et Schumm, 2000). Partant de ce principe, l’enseignant peut présenter la leçon et l’autre enseignant, en l’occurrence l’orthopédagogue, peut intervenir en donnant des exemples et en expliquant des stratégies significatives à retenir (Bauwens, Hourcade et Friend, 1989). L’une des particularités de ce modèle, qui caractérise le fonctionnement en collaboration adopté dans la dyade 2, est qu’il implique la confiance mutuelle et le respect de l’autre (Cook et Friend, 1995), ce qui autorise la prise de risques, les échanges ou les interactions rapides et fluides (Gately et Gately, 2001).

La dyade 3

Les résultats font preuve d’une contribution importante de l’orthopédagogue tout au long des 3 séances de coenseignement. Cependant, des difficultés de coordination avec l’enseignante ont eu des répercussions importantes sur le plan didactique. Nous avons en effet mis en évidence des divergences de conceptions sur le sens et l’utilité de l’apprentissage de la démarche de résolution de problèmes dans les premiers moments d’interaction avec les élèves. Nous avons aussi relevé une entrée difficile dans la tâche et un climat de tension et d’insécurité chez les élèves, jumelés à des difficultés d’arrimage entre les procédures mises en œuvre et celles recommandées par la commission scolaire, prises comme référence. Des moments de reprise et de stagnation par rapport à la tâche à accomplir ont aussi crée un climat de confusion chez les élèves. Une confiance limitée de la part de l'enseignante à l'endroit des propositions de l'orthopédagogue désirant se tourner du côté des stratégies des élèves a également été mise en évidence.

Conclusion

Dans un contexte de coenseignement impliquant la présence de l'orthopédagogue en classe, il convient de reconnaître que la contribution de ce dernier peut avoir une influence sur la dynamique des situations mises en place. Une multitude de gestes et de décisions mis en œuvre à la fois par l'enseignant et par l'orthopédagogue se doivent alors d'être *contextualisés* dans le fonctionnement plus général de la classe (Robert et Rogalski, 2002). Les conclusions qu'il est possible de tirer à la lumière des résultats et analyses des aides et interventions observées démontrent que celles-ci ont été effectuées de manière spontanée et réactive, sans avoir été concrètement planifiées préalablement. Au fil de nos observations et analyses, la plupart des aides et interventions nous ont semblé être pratiquées dans un esprit convenant à des adaptations potentiellement importantes en lien avec les stratégies de résolution de problèmes pouvant être mobilisées. Cependant, en reconstituant les différents scénarios afin de resituer ces comportements sous un angle didactique, nous avons constaté certains dysfonctionnements à l'issue desquels il y a lieu de mettre en doute la portée adaptative des aides et interventions déployées ainsi que la capacité des acteurs concernés à ajuster leurs pratiques respectives.

Les analyses ont montré que ce ne sont pas tant les enjeux conceptuels qui ont dominé dans les situations mises en place, ni même les stratégies cognitives ou métacognitives mobilisant des connaissances (Baroody et Coslick, 1998), mais plutôt la résolution de problèmes en tant que moyen ou démarche à respecter. Cette façon de faire se rapproche d'un enseignement explicite des stratégies de résolution de problèmes et est couramment employée en contexte scolaire. Elle comporte toutefois plusieurs risques, dont ceux rapportés récemment par Theis dans un ouvrage investiguant l'apprentissage par l'intermédiaire de la résolution de problèmes (Theis et Gagnon, 2013). Nos analyses montrent en effet plusieurs manifestations freinant la compréhension du problème et du même coup l'emploi de stratégies de planification et de recherche de solutions : une centration sur le repérage et la transcription des informations contenues dans le problème ainsi que des attentes valorisant le respect des étapes, la clarté et l'organisation des traces de la démarche plutôt que les composantes conceptuelles de la tâche. L'aspect technique et procédural ainsi qu'une vigilance accrue auprès des élèves en difficulté pour que ceux-ci empruntent des stratégies de recherche de solution déterminées l'ont emporté sur les enjeux didactiques. Les difficultés à situer l'état des connaissances antérieures des élèves de même qu'à prêter attention aux obstacles au déploiement des stratégies de résolution de problèmes pouvant être rencontrés ou se présentant en cours d'interventions afin d'ajuster les interventions conjointes sont les facteurs explicatifs que nous avons tenté de faire ressortir dans la discussion des résultats.

Les résultats ont amené également à conclure à certaines distances et incertitudes quant aux points de vue des partenaires sur l'adaptation de l'enseignement et de leurs pratiques habituelles nécessitant des ajustements à l'endroit des élèves estimés en difficulté. Plusieurs pistes d'explications ont été avancées, convergeant vers une formation et des expertises qui débouchent rarement sur une culture de collaboration. Par ailleurs, le besoin d'harmoniser les pratiques et de tendre à une complémentarité des approches demeure une préoccupation présente. C'est ainsi que nous avons pu observer une certaine volonté de la part des partenaires d'envisager d'autres possibilités de collaborer au terme de leur participation à la recherche, et ce, notamment dans la perspective d'une complicité à développer dans la planification des situations ayant potentiellement des répercussions positives sur les élèves.

Une position de nature prospective par rapport aux conclusions et aux limites de notre recherche a par ailleurs été mise de l'avant, et ce, de façon à reconnaître l'importance de poursuivre les efforts de recherche visant à soutenir la formation et le développement professionnel des enseignants et des orthopédagogues au regard du défi de l'inclusion et de l'adaptation de l'enseignement dans une perspective de partenariat et de collaboration invitant au pilotage de pratiques nouvelles.

BIBLIOGRAPHIE

Argüelles, M. E., Hughes, M. T. et Schumm, J. S. (2000). Co-teaching: A different approach to inclusion. *Principal*, *79*(4), 48-51.

Artigue, M. (1988). Ingénierie didactique. *Recherches en didactique des mathématiques, 9*(3), 281-308.

Baroody, A. J. et Coslick, R. T. (1998). *Fostering children's mathematical power: An investigative approach to K-8 Mathematics Instruction.* Mahwah, NJ: Erlbaum.

Bauwens, J., Hourcade, J. J. et Friend, M. (1989). Cooperative teaching: A model for general and special education integration. *Remedial and Special Education, 10(2)*, 17-22.

Bourassa, B., Fournier, G., Goyer, L. et Veilleux, A. D. (2013). Faire le point, comprendre et réviser sa pratique professionnelle : une recherche collaborative auprès d'intervenants d'un service de placement étudiant. Dans B. Bourassa, G. Fournier et L. Goyer (Dir.), *Construction de savoirs et de pratiques professionnelles : le double jeu de la recherche collaborative* (p.9-40). Québec : Presses de l'Université Laval.

Brousseau, G. (1998). *La théorie des situations didactiques.* Grenoble: La Pensée Sauvage.

Brun, J. (1994). Évolution des rapports entre la psychologie du développement cognitif et la didactique des mathématiques. Dans M. Artigue, R. Gras, C. Laborde et P. Tavignot (Dir.), *Vingt ans de didactique des mathématiques en France* (p.67-83). Grenoble: La Pensée Sauvage.

Cook, L. et Friend, M. (1995). Coteaching : Guidelines for effective practices. *Focus on Exceptional Children, 28*(3), 1-16.

Friend, M. et Cook, L. (2003). *Interactions: Collaboration skills for school professionnals.* New York: Longman.

Fuchs, L. et Fuchs, D. (2007). Implementing responsiveness to intervention. *Teaching exceptional Chidren, 39*(5), 14-20.

Fuchs, L. S., Fuchs, D. et Bishop, N. (1992). Instructional adaptation for students at risk. *Journal of Educational Research, 86*(2), 70-84.

Fuchs, L., Fuchs, D., Hamlett, C., Phillips, N. B. et Karns, K. (1995). General educators' specialized adaptation with learning disabilities. *Exceptional Children, 61*(5), 440-458.

Garcia, A. et Marcel, J.-F. (2011). Travail partagé et processus d'apprentissage professionnel chez des enseignants documentalistes de l'enseignement agricole public français. Dans L. Portelance, C. Borges et J. Pharand (Dir.), *La collaboration dans le milieu de l'éducation* (p.125-142). Québec : Presses de l'Université du Québec.

Gaudreau, L., Legault, F., Brodeur, M., Hurteau, M., Dunberry, A., Séguin, S. P. et Legendre, R. (2008). *Rapport d'évaluation de l'application de la Politique de l'adaptation scolaire*, déposé au ministère de l'éducation, du Loisir et du Sport du Québec, Direction de l'adaptation scolaire. Montréal : Université du Québec à Montréal.

Gately, S. et Gately, F. (2001). Understanding coteaching components. *Teaching Exceptional Children, 33*(4), 40-47.

Geary, D. C. (2005). Difficultés d'apprentissage en arithmétique: Le rôle de la mémoire travail et de connaissance conceptuelle. Dans M.-P. Noël (Dir.), *Approches neuropsychologique et de développement des difficultés d'apprentissage en mathématiques* (p.169-191). Belgium: Solal

Gersten, R., et Chard, D. (1999). Number sense: Rethinking arithmetic instruction for students with mathematical disabilities. *The Journal of Special Education*, *33*(1), 18-28.

Goupil, G. (2007). *Élèves en difficulté d'adaptation et d'apprentissage.* Boucherville : Gaëtan Morin.

Hargreaves, A. et Dawe, R. (1990). Paths of professional development: contrived collegiality, collaborative culture, and the case of peer coaching. *Teaching and Teacher Education, 6*(3), 227-241.

Isherwood, R. S. et Barger-Anderson, R. (2007). Factors affecting the adoption of co-teaching. Models in inclusive classroom: one school's journey from mainstreaming to inclusion. *Journal of Ethnographic & Qualitative Research*, *2,* 121-128.

Little, J. W. (1990). The persistence of privacy: Autonomy and initiative in teacher's professional relations. *Teacher College Record, 91*(4), 509-536.

Marcel, J.-F., Dupriez, V. et Périsset Bagnoud, D. (2007) Le métier d'enseignant : nouvelles pratiques. Dans J.-F. Marcel, V. Dupriez, D. Périsset Bagnoud et M. Tardif (Dir.), *Coordonner, collaborer, coopérer : de nouvelles pratiques enseignantes* (p.7-17). Bruxelles : De Boeck.

Martineau, S. et Simard, D. (2011). Regard sur le dialogue comme condition de la collaboration en stage ou en accompagnement mentoral. Dans C. Borges et J. Pharand (Dir.), *La collaboration dans le milieu de l'éducation. Dimensions pratiques et perspectives théoriques*. (p.13-25). Québec: Presses de l'Université du Québec.

Ministère de l'Éducation du Québec (2003). *Les difficultés d'apprentissage à l'école. Cadre de référence pour guider l'intervention*. Québec : Gouvernement du Québec.

Ministère de l'Éducation, du Loisir et du Sport, Gouvernement du Québec (2006). *L'évaluation des apprentissages au secondaire.* Québec : Ministère de l'Éducation.

Ministère de l'Éducation, du Loisir et du Sport, Gouvernement du Québec (2007). *L'organisation des services éducatifs aux élèves à risque et aux élèves handicapés ou en difficulté d'adaptation ou d'apprentissage.* Québec : Gouvernement du Québec.

Ministère de l'Éducation, du Loisir et du Sport (2010). *Rencontres des partenaires en éducation. Document d'appui à la réflexion. Rencontre sur l'intégration des élèves handicapés ou en difficulté d'adaptation ou d'apprentissage.* Québec : Gouvernement du Québec.

Mercier, A. (2002). La transposition des objets d'enseignement et la définition de l'espace didactique. *Revue française de pédagogie,* 141(1), 135-171.

Murawski, W.W. (2003). *Co-teaching in the inclusive classroom: Working together to help all of your students find success*. Medina, WA: Institute for Educational Development.

Nootens, P. et Debeurme, G. (2010). L'enseignement en contexte d'inclusion : proposition d'un modèle d'analyse des pratiques d'adaptation. *Nouveaux cahiers de la recherche en éducation, 13 (2),* 127-144.

Pariès, M., Robert, A. et Rogalski, J. (2008). Analyses de séances en classe et stabilité des pratiques d'enseignants de mathématiques expérimentés du second degré. *Educationnal studies of mathematics, 68,* 55-80.

Perrin-Glorian, M.-J. (1993). Questions didactiques soulevées à partir de l’enseignement des mathématiques dans les classes faibles. *Recherche en Didactique des Mathématiques, 33*, 20-25.

Robert, A. et Rogalski, J. (2002). Le système complexe et cohérent des pratiques des enseignants de mathématiques : une double approche. *Revue canadienne de l’enseignement des sciences, des mathématiques et des technologies*, *2*(4), 505-528.

Roditi, E. (2008). Des pratiques enseignantes à la fois contraintes et personnelles, et pourtant cohérentes. Dans F. Vandebrouck (Dir.), *La classe de mathématiques, activités des élèves et pratiques des enseignants* (p.73-94*)*. Toulouse : Octarès.

Rogalski, J. (2008). Le cadre général de la théorie de l’activité. Une perspective de psychologie ergonomique. Dans F. Vandebrouck (Dir.), *La classe de mathématiques: activités des élèves et pratiques des enseignants* (p.23-30). Toulouse : Octarès.

Rousseau, N. et S. Bélanger (2004). *La pédagogie de l'inclusion scolaire.* Québec, Presses de l'Université du Québec.

Sarrazy, B. (2002). Effects of variability of teaching on responsiveness to the didactic contract in arithmetic problem-solving among pupils of 9-10 years. *European Journal of Psychology of Education*, *17*(3), 321-341.

Shulman, L. S. (1986b). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Stern, E. et Lehrndorfer, A. (1992). The role of situational context in solving word problems. *Cognitive Development, 7*(2), 259-268.

Saint-Laurent, L., Giasson, J., Simard, C., Dionne, J. J., Royer, E. et coll. (1995). *Programme d’intervention auprès des élèves à risque. Une nouvelle option éducative.* Boucherville: Gaëtan Morin.

Switlick, D. M. (1997). Curriculum modification and adaptation. Dans A. Bacon (Dir.), *Teaching students in inclusive settings: From theory to practice* (p.225-251). Boston: Needham Heights.

Tardif, M. (2007). Pratiques, collaboration et professionnalisation des enseignants. Dans J.-F. Marcel, V. Dupriez, D. Périsset Bagnoud et M. Tardif (Dir.), *Coordonner, collaborer, coopérer* (p.171-179)*.* Bruxelles : De Boeck.

Theis, L. et Gagnon, N. (2013). *L'apprentissage à travers les situations-problèmes. Bases théoriques et réalisation pratique.* Québec : Presses de l'Université du Québec.

Thevenot, C., Barrouillet, P. et Fayol, M. (2004). Représentation mentale et procédures de résolution de problèmes arithmétiques : L’effet du placement de la question. *L’Année Psychologique, 104*, 683-699.

Trépanier, N. et Paré, M. (2010). *Des modèles de service pour favoriser l’intégration scolaire.* Québec: Presses de l’Université du Québec.

Villa, R. A., Thousand, J. S. et Nevin, A. I. (2008). *A guide to co-teaching.* Thousand Oaks, CA: Corwin Press.

Volonino, V. et Zigmond, N. (2007). Promoting research-based practices through inclusion? *Theory Into Practice, 46*(4), 291–300.

Weiss, M. D. et Lloyd, J. (2003). Conditions for co-teaching: Lessons from a case study. *Teacher Education and Special Education, 26,* 27-41.

Zigmond, N. et Magiera, K. (2001). Co-teaching. *Current Practice Alerts, 6,* 1-4.

The role of the teacher in fostering an aware approach to problem-solving activities: the case of geometric problems that could be solved through the construction of equations

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Abstract: In this paper I present the results of a study aimed at analysing the role played by the teacher in fostering an aware approach to the use of algebraic language as a tool to support problem solving. In particular, I will present the analysis of a class discussion focused on the resolution of a geometric problem that could be solved through the construction of a suitable equation. This analysis, performed referring to a theoretical construct I developed in previous studies, will enable to characterise how the teachers should try to behave when facing this kind of problem solving activities in their classes.

Résumé: Dans cet article, je présente les résultats d'une étude visant à analyser le rôle joué par l'enseignant dans la promotion d'une approche consciente à l'utilisation du langage algébrique comme un outil pour soutenir la résolution de problèmes. En particulier, je vais vous présenter l'analyse d'une discussion en classe axée sur la résolution d'un problème géométrique qui pourrait être résolu grâce à la construction d'une équation appropriée. Cette analyse, effectuée référant à un outil théorique Je développé dans les études précédentes, permettra de caractériser la façon dont les enseignants devraient essayer de se comporter face à ce genre d'activités de résolution de problème dans leurs classes.

Introduction

The aim of this paper is to introduce a theoretical tool that could support the analysis of the role played by the teacher in fostering an aware approach to the use of algebraic language as a tool for problem solving. Precisely, I chose to focus on problem solving activities that are typical of the Italian school tradition: geometric problems that could be solved through an algebraic approach, that is through the construction of suitable equations.
In performing this analysis, I refer to a theoretical construct that I developed in a previous research (Cusi & Malara 2009, 2013): the one of teacher as a "*model of aware and effective attitudes and behaviours*" (in the following $M_{AE}AB$). The theoretical construct will be presented in the next section, where I will also outline the frame within which this kind of activities could be analysed. In the subsequent sections, after a presentation of the research hypothesis, aims and methodology, I will present the analysis of some excerpts from a class discussion. The results of this analysis and its theoretical implications will be discussed in the last section.

Theoretical frame

The analysis of the teaching-learning processes connected to problem solving activities that involve a complex interplay between algebraic and geometric reasoning could be performed according to different perspectives: (1) the general factors connected to failure or success in problem solving processes, (2) the use of algebraic language as a thinking tool, (3) the dynamics involved in problem solving processes when the problem solver has to refer to visual representations, and (4) the role played by the teacher in fostering students' development of the fundamental competencies necessary to effectively carry out this kind of tasks. As I previously declared, the focus of my research is on the fourth perspective. However, analysing the role of the teacher requires to take also the other perspectives into account.

As regards the first perspective, Schoenfeld (2010) describes the process of decision making in problem solving as an iterative one: (a) an individual enters into a particular context with a specific body of resources, goals, and orientations; (b) he/she activates certain pieces of information and knowledge that become salient; (c) he/she establishes specific goals and makes decisions, consistent with these goals, regarding what directions to pursue and what resources to use (if the situation is not familiar, decision-making is made referring to the subjective expected values of available options); (d) implementation begins and monitoring (effective or not) takes place. Schoenfeld observes that routines aimed at particular goals have sub-routines, which have their own subgoals. When a subgoal/goal is achieved, the individual proceeds to another goal or subgoal. If things don’t seem to be going well, the individual decides to change goals or to identify other pathways to try to achieve them.

The identification of a suitable strategy to be adopted during the kind of activities I am analysing depends also on the student’s capability of effectively using algebraic language as a tool to construct reasoning (second perspective). Thanks to my research studies (Cusi, 2009) I was able to highlight that an effective use of algebraic language as a thinking tool requires the development of three main key-competencies: (a) being able to activate appropriate conceptual frames and to effectively coordinate different frames (Arzarello et al. 2001); (b) being able to apply appropriate anticipating thoughts (Boero, 2001); and (c) being able to coordinate algebraic and verbal registers on both translational and interpretative levels (Duval, 2006).

Since the context of the problem solving activities I chose to focus on is the geometric one, these activities also involve a continuous reference to visual representations. Duval (2002) has identified different fundamental dynamics that characterise the problem solving processes when the problem solver has to refer to visual representations (third perspective): (1) the problem solver draws an initial figure starting from the text of the problem; (2) different sub-figures can be identified (their identification does not necessary depend on the way the starting figure has been constructed); (3) in order to develop reasoning, the problem solver needs to highlight connections between statements (theorems, definitions, properties…) and the sub-figures that could be recognised.

The discussion I briefly carried out in relation to all these aspects enables to stress on the delicate role played by the teacher in developing this kind of activities in the class. In my previous research, after having identified the key-competencies that support the processes of reasoning’s construction through algebraic language, I focused on the role played by the teacher and I identified a set of roles which outline a profile of a teacher who is able to effectively guide his/her students to the development of the three key-competencies (Cusi and Malara 2009, 2013). The $M_{AE}AB$ theoretical construct, which is the result of this study, highlights the approach of a teacher who consciously behave constantly aiming at “making thinking visible” (Collins et al., 1989), in order to make his/her students focus not only on syntactical or interpretative aspects, but also on the effective strategies adopted during the activity and on the meta-reflections on the actions which are performed.

A first group of roles typical of a teacher who poses him/herself as a $M_{AE}AB$ are those performed when he/she tries to carry out the class activities posing him/herself not as a “mere expert” who proposes effective approaches, but as a learner who faces problems with the main aim of making the hidden thinking visible, highlighting the objectives, the meaning of the strategies and the interpretation of results. The roles belonging to this group are: (a) Investigating subject and constituent part of the class in the research work being activated; (b) Practical/ Strategic guide; (c) “Activator” of interpretative processes and anticipating thoughts (see Cusi & Malara 2013 for a more detailed analysis of these roles).

The second group of roles refers to the phases of the activities during which the teacher becomes also a point of reference for students to help them clarify salient aspects at different levels, with an explicit connection to the knowledge they have already developed. The roles belonging to this second group are: (d) Guide in fostering a harmonized balance between the syntactical and the semantic level; (e) Reflective guide; (f) “Activator” of reflective attitudes and meta-cognitive acts (see Cusi & Malara 2013 for a more detailed analysis of these roles).

Research hypothesis and aims

As I stated at the beginning, the main aim of this study is to analyse, referring to the $M_{AE}AB$ construct, how the teacher carries out, during class discussions, activities focused on geometric problems that could be solved through the construction of suitable equations. I chose these activities because they involve an interesting interplay between fundamental components at different levels: (1) the use of algebraic language as a thinking tool; (2) an effective reference to visual representations; (3) a good coordination between the verbal register, the symbolic register and the register of the geometric figure; (4) the activation of the processes of decision-making typical of problem solving activities (in particular, the identification of appropriate goals and an effective monitoring of the strategies that are carried out).
My hypothesis is that the $M_{AE}AB$ construct could represent a useful tool to analyse the role played by the teacher during this kind of activities, with the aim of highlighting the objectives of the teachers' interventions during class discussions and the effectiveness (or not) of these interventions. At the same time, I am aware that the $M_{AE}AB$ construct was conceived referring to a specific mathematical activity, therefore there is a need of refining it, identifying possible other roles that are typical of problem solving activities involving a reference to visual representations.
The analysis of the roles of the teacher in carrying out this kind of activities has therefore two main aims: (1) to identify the roles, highlighted by the $M_{AE}AB$ construct, which are mainly performed by the teacher during this kind of activities; (2) to refine these roles, with reference to these specific activities, and to introduce other possible roles, to better characterise how the teacher should behave when a problem requires to develop reasoning referring also to visual representations.

Research methodology

To pursue the two aims introduced in the previous section, I analysed the transcripts of the audio-recordings of five class discussions (on five different problems) conducted by one teacher who had already been involved in my previous study (Cusi & Malara, 2009). The teacher was selected because the analysis of her way of behaving in the class, during the previous experimentations, showed that her approach was effective in making cognitive and metacognitive processes visible and in fostering meta-level reflections.
I performed the analysis of the transcripts referring to the $M_{AE}AB$ construct to discuss the teacher's interventions. In particular, I focused on the interventions aimed at enabling the students: (a) to reflect on the choice of the most appropriate unknown; (b) to share their strategies and make them explicit; (c) to interpret the results in relation to the problem; (d) to identify appropriate goals and to learn how to monitor the strategies that are carried out. To exemplify how the analysis has been performed, in the following paragraph, I will present the analysis of some excerpts from one of the five class discussions. The discussion focuses on a problem that will be briefly presented in the next section.

The problem on which the analysed activity is focused

The discussion I am going to analyse refers to the following problem:

The height of an isosceles trapezoid is 4/15 of the sum of the bases. The smaller base is 4/7 of the larger base. The perimeter is 288cm. Find the area of the trapezoid and the lengths of its diagonals.

The resolution of this problem could be subdivided into six main key-moments. In the following table, these six key-moments are highlighted and commented (column 2), referring to a possible outline of the resolution process (column 1). Since the aim of this brief analysis is to highlight the dynamics that intervene when algebraic language is used as a tool to support problem solving, I will not analyse the last part of the resolution process (that is the phase during which the solution of the equation is used to determine the area of the trapezoid and the lengths of its diagonals).

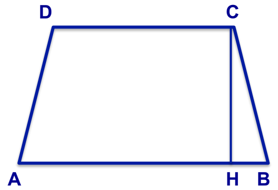
Possible sinthetic outline of the resolution process	**Key moments in the resolution process and reflections on the involved competencies**
 $CH = \frac{4}{15}(AB - CD)$ $CD = \frac{1}{7}AB$ $2p = 288cm$	(1) Construction of the reference figure, identification and representation of the hypothesis (the relation between the height of the trapezoid and the sum of the bases, the relation between the two bases, the length of the perimeter).
$AB = x$; $CD = \frac{1}{7}x$ $CH = \frac{4}{15}\left(x - \frac{1}{7}x\right) = \frac{4}{35}x$	(2) Identification of an appropriate unknown and reference to the hypothesis to represent the other unknown elements as expressions containing only the chosen unknown. This requires the activation of an appropriate anticipating thought to understand what is the most effective choice.
$2p = 288cm$ $\rightarrow AB - CD + 2CB =$ $= 288$ So, it is necessary to write CB as an expression of x.	(3) Identification of the relation (the length of the perimeter) to which it is useful to refer to construct the equation that enables to determine the unknown. This enables the identification of a goal that will guide the subsequent steps of the resolution process. The first step is, indeed, to identify the element (the length of CB) that should be written as an expression of x.
$CB^2 = CH^2 + HB^2$ because of the Pythagorean theorem	(4) Reference to theoretical statements (the Pythagorean theorem) to identify a further relation to write the length of CB as an expression of x. This requires the identification of a useful sub-figure (the one that is highlighted if we draw the height CH) and of the connections between this sub-figure and the theoretical statements to which we can refer to highlight further relations that involve the unknown elements. Moreover, it requires the activation of an anticipating thought that enables to identify a further sub-goal: to represent the length of CB as an expression of x, it is necessary to represent the length of HB as an expression of x.
$HB = \frac{1}{2}(AB - CD) =$ $= \frac{3}{14}x$ $CB^2 = \left(\frac{4}{35}x\right)^2 - \left(\frac{3}{14}x\right)^2$ $\rightarrow CB = \frac{17}{70}x$	(5) Identification of the relation that enables to write the length of HB as an expression of x. This requires the identification of a sub-figure and of the corresponding statement (since ABCD is an isosceles trapezoid, HB is the semi-difference of the two bases), useful to highlight the relation between HB, AB and CD. The construction of the correct symbolic expression that represents HB as a semi-difference of the two bases requires a good coordination between the symbolic register and the register of the geometric figure.
$x + \frac{1}{7}x + 2 \cdot \frac{17}{70}x = 288$ … $x = 140$ …	(6) Construction and subsequent resolution of the equation that enables to determine the unknown. The interpretation of the solution in relation to the specific data, which is a fundamental step toward the following resolution phases, requires a good coordination between the symbolic register and the register of the geometric figure.

Table 1. The key-moments in the resolution process

Analysis of some excerpts from a class discussion

In the following table, some excerpts of a class discussion about the problem are presented (column 1) and analysed according to the $M_{AE}AB$ construct (column 2). Because of space limitations, the excerpts were chosen to focus on the parts of the class discussion during which the teacher (T) guides her students to reflect on their strategies and construct a suitable equation.

Excerpts from the class discussion	**Interpretations of the roles played by the teacher with reference to the $M_{AE}AB$ construct**
Initially, T reads the text of the problem, constructs the reference figure with her students and asks them to think about possible strategies to solve the problem. The students work in small groups. Then T, after having written, guided by her students, the hypothesis of the problem, asks them to present the approaches they tried to adopt to solve it.	
(1) T: I wrote the data. D, An and Al, did you find a strategy? (2) D: We considered AB=x	
(3) T: (*Speaking with the other students*) Their idea was to consider AB=x, so they indicated AB as an unknown. Why? (*Speaking with the three students*)	T plays the role of *activator of metacognitive acts*: she asks to the three students to make the reasons, underlying their choice of the unknown, explicit.
(4) D: Because, in this way, we can find CD. (5) An: Because, if we know CD, then we can also partially find CH. (6) T: Partially? (7) An: No, we found it!	
(8) T: D is saying … Why did his group choose AB as an unknown? Because, if we consider x=AB, from this relation (*she is indicating the expression* $CD=\frac{4}{7}AB$) we can immediately find CD as an expression of x, without introducing another unknown. Then, if I have AB and CD, what can I find from the first relation? (*She is indicating the expression* $CH=\frac{4}{15}(AB-CD)$).	T plays the role of *reflective guide*: she recalls what D and An observed to make their cognitive and metacognitive processes really explicit. In this way, the other students could identify their way of reasoning (in choosing the unknown) as a model to which they could refer.
(9) S: CH (10) T: CH. Do you agree? (*Speaking with the rest of the class*). Did the other groups use the same strategy or did they choose different unknowns?	
In the following part of the discussion, D, An and Al introduce the equalities $CD=\frac{4}{7}x$ *and* $CH=\frac{4}{15}\left(x-\frac{4}{7}x\right)=\frac{4}{35}x$. *They declare that they were not able to go on. After that, L, P, A (another group) intervene, presenting the difficulties they faced in identifying a good relation to be used to construct the equation that could help in solving the problem. A suggests to refer to the relation AB+BC+CD+DA= 288 (that is the information about the perimeter). P suggests that CD could be substituted with the expression* $\frac{4}{7}x$.	
(50) T: P says "We have written AB=x, so we can substitute AB with x and CD with 4/7x". What problem do we still have?	T plays the role of *reflective guide* because she is trying to make P's strategy explicit. Moreover, she acts as a *practical/strategic guide* in trying to make the students focus on what she defines "a problem": BC and DA still have to be written as expressions of x.
G says that, with her group, she introduced two unknowns (AB=x and AD=s) with the aim of constructing a system with two equations. The first equation she proposes is the one derived from the perimeter (AB+BC+CD+DA=288). The second equation is derived from the relation AD=(288-AB-CD)/2.	

(55) T: G, your group referred to the same relation to construct two equations. Didn't you? In fact, the first is "the sum of the sides of the trapezoid is 288". The second is: "AD is he semi-difference between the perimeter and the sum of the bases". (56) G: Yes, we determined the side (AD) from the perimeter.	T plays the role of *activator of reflective attitudes*: she focuses on G's strategy to highlight the problem connected to her choice of using the same relation to construct two different equations.
(57): T: (*Speaking with the whole class*) If I construct a system of two equations and the two equations represent the same relation, what kind of system is this? Think about the calculations to be performed… (58) G: It (*the solution*) will be "for every x". (59) T: There are two equations that represent the same relation, written in different ways… What kind of system is this? *Whispering.* (60) G: It is an indeterminate system.	T plays the role of *guide in fostering a harmonized balance between the syntactical and the semantic level*: she makes the students reflect on the meaning of indeterminate system in relation to this specific problem.
T stresses that using the same relation two times entails the construction of a system with to equivalent equations, that is an indeterminate system. Some students ask for further clarifications.	
(68) T: We wrote AB+CD+2AD=288. We know that AB=x and CD=4/7x. If we were able to write CB as an expression of x, this (*she indicates the equality* AB+CD+2AD=288) would be an equation in x.	T poses herself as a *constituent part of the class in the research work being activated.* Moreover, she acts as *practical/strategic guide* in trying to make a possible sub-goal explicit: to write also CB as and expression of x.
(69) J: Our idea was to try to use the Pythagorean Theorem, that is to calculate … since we knew CH (*as an expression of x*) and also CB … (70) T: We do not know CB (*as an expression of x*). (71) J: From the other equation … (*J is trying to suggest to refer to the previous equation to write CB as an expression of x*).	T plays the role of *activator of metacognitive acts*: she is trying to help J in making his reasoning explicit.
(72) T: A new idea! CHB is a right triangle. We know that, if we have a right triangle, the Pythagorean Theorem is valid. Our aim is to find a strategy to write CB as an expression of x, to be able to write one equation with only one unknown. Are we able to write BC as an expression of x?	T plays different roles: - *reflective guide*, because her aim is to make J's approach be shared by the whole class; - *activator of anticipating thoughts*, because she makes the goal of this phase of the resolution process explicit; - *activator of reflective attitudes*, because she is sharing the reasons why the strategy proposed by J could be effective. To help the students understand J's strategy, she makes them focus on a sub-figure (CHB) to help them identify this sub-figure and its connection with the theoretical statement quoted by J (the Pythagorean Theorem).
In the following part of the discussion T is guided by the students in: (1) *writing HB and then CB as expressions of x; (2) simplifying the expression of CB, interpreting it as the length of a side of the trapezoid; (3) constructing a suitable equation and solving it; (4) interpreting the solution. In the end, she makes the class reflect again on the effectiveness of the adopted strategy.*	

Table 2. Analysis of some excerpts from a class discussion

Results

This analysis enabled me to show that the $M_{AE}AB$ construct could represent an effective tool to describe how the teacher acts when working with students on this kind of problem solving activities and to highlight the objectives of her interventions. Moreover, it enabled me to think about a

possible refinement of the characterisation of the roles highlighted by the $M_{AE}AB$ construct, with reference to these particular activities:
(a) *Investigating subject*, when the teacher asks students to give suggestions about how to go on with the activity, without judging what they say, and she intervenes with the aim of making them feel involved in the activity as a group;
(b) *Practical/ Strategic guide*, when she poses herself, in front of the problem, as an inquirer who aims at sharing the thinking processes and discussing the possible strategies to be activated;
(c1) *"Activator" of interpretative processes*, when she makes the students interpret the different algebraic expressions constructed during the discussion, with reference to the geometric figure, or when she makes them analyse the obtained results with reference to the problem situation;
(c2) *"Activator" of anticipating thoughts*, when she makes the possible goals or sub-goals explicit, in order to make them be shared by all the students, and recall these goals in order to make students monitor and control the activated strategies;
(d) *Guide in fostering a harmonized balance between the syntactical and the semantic level*, when she discusses possible problems arisen (for example, the construction of an indeterminate system), trying to highlight the interrelation between the processes that characterise the algebraic resolution of the problem and the corresponding meanings;
(e) *Reflective guide*, when, in front of a student who proposes an effective approach to the resolution of a problem, she asks him/her to make his/her thinking processes explicit, or she repeats what has been said by the student stressing on the reasons subtended to his/her approach, or she asks to other students to interpret what he/she said;
(f) *"Activator" of reflective attitudes and meta-cognitive acts*, when she poses meta-level questions aimed at making the students evaluate the effectiveness of a strategy and reflect on the effects of a choice that was made during the resolution process. In this way she promotes students' monitoring of their strategies.
Thanks to this analysis, I was also able to identify another role that could be introduced to analyse how the teacher should behave when a problem requires to develop reasoning referring also to visual representations: the role of "*supporter to the visualisation processes"*. The teacher plays this role when he/she explicitly refers to specific sub-figures to make the students focus on them and identify the connections between these sub-figures and suitable theoretical statements. In this way, he/she is evoking what Duval (1999) defines perceptual and discursive apprehensions. A further step in this research will be to analyse other problem solving activities where visualisation plays a central role, to better characterise the idea of "supporter to the visualisation processes", identifying possible sub-roles connected to the different apprehensions that could be evoked.

REFERENCES

Arzarello, F., Bazzini, L., and Chiappini, G. (2001). A model for analyzing algebraic thinking. In Sutherland R. & Al. (Eds.), *Perspectives on School Algebra* (pp. 61-81). Netherlands: Kluwer Publishers.

Boero, P. (2001). Transformation and Anticipation as Key Processes in Algebraic Problem Solving, in Sutherland, R. & Al. (Eds.), *Perspectives on School Algebra* (pp. 99-119). Netherlands: Kluwer Publishers.

Collins, A., Brown, J.S. and Newman, S.E. (1989). Cognitive Apprenticeship: Teaching the Crafts of Reading, Writing and Mathematics! In L.B. Resnick (Ed.), *Knowing, Learning, and Instruction: Essays in Honor of Robert Glaser* (pp. 453-494). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cusi, A. (2009). Interrelation between anticipating thought and interpretative aspects in the use of algebraic language for the construction of proofs. In V. Durand-Guerrier, S. Soury-Lavergne and F. Arzarello (Eds.), *Proc. of 6th CERME Conference* (pp. 469-478). INRP.

Lyon, France.

Cusi, A., and Malara, N.A. (2009). The role of the teacher in developing proof activities by means of algebraic language, In M. Tzekaki & Al. (Eds.), *Proc. PME 33*, (vol. 2, 361-368), Thessaloniki: University of Thessaloniki.

Cusi, A., Malara, N.A. (2013). A theoretical construct to analyze the teacher's role during introductory activities to algebraic modelling. In B. Ubuz et al. (eds.), *Proc. of Cerme 8* (pp. 3015-3024). Antalya (Turkey).

Duval, R. (1999). Representation, vision and visualisation: cognitive functions in mathematical thinking. Basic issues for learning.

Duval, R. (2002). Proof understanding in MATHEMATICS: What ways for STUDENTS? *Proc. of 2002 International Conference on Mathematics: Understanding Proving and Proving to Understand*, pp. 61-77.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103–131.

Schoenfeld, A. H. (2010). *How We Think: A Theory Of Goal-Oriented Decision Making And Its Educational Applications*. New York: Routledge.

Investigating the intertwinement between the affective and cognitive dimensions of teachers: a possible way for surfacing the reasons of their decisions

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Abstract: This paper is part of my Ph.D. thesis in which I attempted to bring together cognitive and affective dimensions of mathematics teachers' practice, often considered separately. In particular, during my Ph.D. research, I tried to construct a theoretical framework for the study, first reviewing Habermas' theory of rationality and its criticism. Then, I extended this framework into the emotional sphere drawing both on the neuropsychological research and on the research on affect in mathematics education. Lastly, I used the new framework to analyse the rationality and emotionality of two mathematics teachers using data from interviews and classroom video recordings.

Résumé*:* Cet article fait partie de ma thèse de doctorat dans laquelle j'ai essayé de lier les dimensions cognitives et affectives de la pratique des enseignants de mathématiques, souvent considérées séparément. En particulier, au cours de ma recherche de doctorat, j'ai essayé de construire un cadre théorique pour l'étude, en examinant d'abord la théorie de Habermas sur la rationalité et ses critiques. Ensuite, j'ai élargi ce cadre pour inclure la sphère émotionnelle, en faisant appel aux recherches neuropsychologiques et à la recherche sur l'affect dans l'enseignement des mathématiques. Enfin, j'ai utilisé le nouveau cadre pour analyser la rationalité et l'émotivité de deux enseignants de mathématiques en utilisant les données des entrevues et des vidéos faites en classe.

Introduction

The aim of this paper is to focus on the diversity of two teachers involved in my research. This diversity does not completely depend on their decisions. What radically changes is their affective involvement during the classroom process. Looking at the affective involvement, I can infer something about the reasons why they make those decisions and not others. These reasons are deeply related to teachers' expectations that are made visible through their affective engagement during their activities. In particular, I chose to plan the teaching experiments while they explained linear equations because they constitute a crucial mathematical topic in the Italian national curriculum at high school. Indeed, for example, it is one of the first mathematical content in which there is the delicate "shift" from the arithmetical world to the algebraic one. Hence, it is a mathematical topic that should require particular attention.

Literature review

A lot of research in different fields supports the grounded hypothesis of my work, namely the intertwinement between the affective and cognitive factors in the activity of a human being. In particular, I will present the research in neuropsychology and, then, the research in mathematics education. I choose to follow this order, because many of the studies developed in mathematics education are based on the neuroscientific results. In his wide research, Damasio has scientifically proved that emotions play a significant role in the decision-making of the subject. Damasio writes that "certain levels of emotion processing probably point us to the sector of the decision-making space where our reason can operate most efficiently." (Damasio, 1999, p. 42). Moreover, he defines, together with Immordino-Yang, emotion as "basic form of decision-making, a repertoire of know-how and actions that allows people to respond appropriately in different situations." (Immordino-Yang and Damasio, 2007, p. 7). In the last years, research in mathematics education has progressively perceived the existence of a mutual interaction between the affective sphere and cognition in mathematics learning (Zan, Brown, Evans & Hannula, 2006). The recent research in mathematics-related affect has considered different affective concepts such as values, identity, motivation, and norms. However, Zan and colleagues (Zan, Brown, Evans & Hannula 2006) have recognised the limited use of emotion in mathematics education research, even if it should be one the essential concept. Indeed, they pointed out "how repeated experience of emotion may be seen as the basis for more 'stable' attitudes and beliefs" (Zan, Brown, Evans & Hannula 2006, p. 116).

Theoretical framework

In order to analyse the intertwinement between the affective and cognitive factors in the decision-making processes of the teacher, I decided to focus on the notion of "emotional orientation", offered by Brown and Reid in 2006. Going deeply into this notion, drawing on Maturana (1988), Brown and Reid refer the notion of emotional orientation to the "criteria for an acceptance of an explanation by a members of a community" (p. 181). Furthermore, they considered emotions at the basis of these criteria. Hence, this concept seems to be a useful tool to study the entanglement between rational and emotional aspects in the decision-making process. Indeed, as the words suggest, the orientation of a subject, namely the decisions she makes, are "emotional", that is affected by emotions in a certain way. But there is still a methodological problem of how, practically, the intertwinement between emotions and cognition can be analysed. Hence, I sketchily present an adaptation of the theoretical framework of the emotional orientation in order to speak practically about these two sides of the same coin. I define the "emotional orientation" of a subject (e.g. a teacher) in terms of "the set of her expectations": the term "expectation" is connected to her "emotions of being right" when she uses specific criteria for accepting an explanation by a community (e.g. a class) rather than other ones (Ferrara & De Simone, 2014). The most difficulty encountered in studying emotions is their "visibility" and, then, their "certain" identification. In this context, when I speak of emotion of the teacher I will refer to her emotionality, namely the set of "behaviours that are observable and theoretically linked to the (hypothetical) underlying emotion" (Reber & Reber, 2001).

Entangled with the notion of emotional orientation, I will draw on the theory of rational behaviour, a philosophical speculation offered by Habermas in 1998. This philosophical lens has been re-elaborated and adjusted to mathematics education by a working group constituted by many researchers of our field (e.g. Boero and colleagues, 2010, 2014). Habermas speaks of the concept of discursive rationality in relation to a rational being involved in a discursive activity. For Habermas, a rational being is a human being "who can give account for her orientation towards validity claims." (Habermas, 1998, p. 310). The philosopher clearly explains that the discursive rationality has three different roots: the knowing, the acting and the speaking. Each of these roots constitutes a different component of rationality. The knowing is related to the epistemic rationality, the acting is related to the teleological rationality and the speaking is related to the communicative rationality. In particular, the epistemic rationality is connected to the justification of the knowledge at play; the teleological rationality is connected to the conscious actions a human being makes for accomplishing a goal; the communicative rationality is connected to the "speech oriented towards reaching understanding". Habermas explicitly stresses that these three different components of rationality are always present in the discursive activity of a rational being and they are deeply intertwined. In fact, a rational being acts in a certain way to achieve a goal, drawing on a specific knowledge, communicating in a precise manner.

Summarizing, I will investigate the intertwinement of rationality and emotions in the decision-making processes of the teacher drawing on the notion of emotional orientation, analysing the decisions the teacher takes through the interplay among the different components of rationality.

Methodology

The participants of my Ph.D. research were 3 teachers and their grade 9 classrooms, in a scientifically oriented secondary school in Piedmont. All of them were explaining the mathematical topic of linear equations. The teachers were selected assuming that rationality and emotions are proper of human beings and with the purpose of having different emotional orientations. Each teacher was first interviewed and asked about her personal beliefs on the topic of linear equations, on algebra in general and on how she uses the didactical materials. Each interview lasted roughly twenty minutes and was videotaped with one camera facing the interviewer and the subject. Then, the whole class activities conducted by the teacher and the students' working group activities were also videotaped. All voice and bodily movement during the interviews and the classroom activities were recorded. The videos were transcribed for data analysis.

The structure of analysis is constituted by two different steps. In the first step, from the a-priori interview, I identified some potential expectations of the teacher. I use the adjective "potential" because, at the level of the a-priori interview, I cannot be sure that these expectations are actually those that drive the action of the teacher in classroom. Hence, analysing the whole class activity coached by the teacher and looking at her "emotional indicators" (facial expressions, gestures, prosody and so on), expressions of her expectations, I am able to check if the potential expectations are actually proper of the teacher. The "emotional indicators" inform me about the emotionality of the teacher, being the visible markers of her affective involvement. Detecting the different expectations of the teacher, I sketch her emotional orientation as the set of the expectations she has for her teaching. In the second step of my analysis, I observe the decisions of the teachers through the three components of rationality (epistemic, teleological and communicative) and, simultaneously, looking at the emotional indicators, expressions of their expectations, I am able to say something about why teachers took those decisions and not others.

Data analysis

In this paper, I compare two different teachers involved in my research, focusing on their intertwinement between emotions and rationality in their decision-making processes within the classroom activity. In particular, I would like to stress that even if, at the "rational" level, they seem to make similar decisions, they are deeply different at the level of their affective involvement. This because they are moved by diverse reasons in making those decisions and not others.

For space constraints, I will just concentrate on one expectations of the two teachers: they both have the expectations regarding the previous knowledge of the students, but these expectations, by their very nature, are deeply different.

The case of Lorenza

I identified different expectations that constitute the emotional orientation of Lorenza, but for the limited space, I show just one of them. From the interview, I identified her *expectation about the validity of the previous knowledge* of the students that can be used for constructing the new one. With "previous knowledge", I mean what students have learnt both in the middle school and with her. In order to identify this expectation, I collected the moments of her interview from which this expectation could be detected. Lorenza explicitly declared: "*Usually, I begin to treat linear equations starting from their previous knowledge in order to see whether it is valid, or whether the students have misinterpreted the various procedures that they have been taught in the previous years. Anyway, I begin a new topic starting from the knowledge that the students already have*". This excerpt is just one passage of the interview in which she speaks of the previous knowledge she thinks her students have. Actually, looking at her classroom activity, this expectation is reflected in several moments of her practice. The following part is just one short example in which her expectation about the validity of the previous knowledge of students is actually visible.

1 T: Where does the concept of equivalence come from, eh? [*she frowns: Fig. 1*], we have already studied it, who remembers when we have spoken of equivalence, do you remember? [*tone of voice proper of a statement not of a question*] Do you remember [*Fig. 2*] the equivalence relation, never [*she shakes her head*], never [*nervously smiling: Fig. 3*], we have done the relations, do you remember? We have defined the equivalence relations, those of admitted	 *Fig. 1*  *Fig. 2*  *Fig. 3*
2 S1: those symmetric	
3 T: yes	
4 S1: reflexivity, symmetry, transitivity	

Lorenza has just explained the concept of equivalent equations, as those that admit the same set of solutions. The action of Lorenza of linking equivalent equations to equivalence relations is aimed to show that the relation among equivalent equations is an equivalence one. At the beginning, she decides not to introduce immediately the term "equivalence relation", rather she gives to students some clues about it (#1: "Where does the concept of equivalence come from?", "we have already studied it", "who remembers when we have spoken of equivalence?"). Unfortunately, none seems to remember what she is asking. Hence, Lorenza becomes more explicit, introducing the term "equivalence relation". This action comes along with a particular tone of voice proper of a statement and not of a question, even if she is asking if students remember what is an equivalence relation. Simultaneously, she says "equivalence relation" (#1) and she makes the gesture in Fig. 2, miming the past. Hence, the teleological emotionality of Lorenza involves both her actions to recall equivalence relations (rational key) and her hope and need that students are able to remember them (emotional key). This emotional key is revealed by the affirmative tone of voice of the question "Do you remember?" (#1) that could testify her expectation that students actually remember it; by her gesture of miming the past to give students a hint about when they have already spoken about it. As you noticed, from the first row of the discussion (#1), her speech is full of emotions, indeed, beyond the involvement of the whole class in the discourse, she makes many questions one after the other and she repeats many times the verb "to remember". Always in the #1, when none responses to her, she repeats for two times the adverb "never" smiling in a quite nervous way, as it is showed in Fig. 3. This emotional speech constitutes the communicative emotionality of the teacher. She cannot be plain in her discourse, then her expectation that students remember and re-elaborate the previous knowledge cannot be hidden. She expresses it through her somatic language and her prosody.

For space constraints, I can't show other analyses of Lorenza, in which she requires that students recall something previously done, but I would like to stress that she has very often the same attitude shown in the previous excerpt.

The case of Carla

The following passage comes from the a-priori interview of another teacher involved in my research, Carla. In this passage, she speaks about her conception on the role of the previous knowledge of students:

"*For me (sighing), the greatest problem I'm trying to solve-I realised and it's becoming dramatic in the last years-is the problem of the stability of knowledge, in that I feel that in many class- rooms, (speeding up) a part from the good ones, students don't remember what we did and, for me, this is serious. That is, for example, in grade 10, I would like to refer to something that I did before, on which I have even insisted, without having to repeat it entirely (...) the big problem to solve, in that I persist a lot, is being able to find a way for constructing a core (miming a base with her hand), a base of knowledge (miming a list with her open hand), of abilities that stay. For me, aside from time economy-'cause, maybe, it's a bit annoying having always to recall-it's really a matter that has to do with cognitive science, I don't know, I wouldn't know how to face it, but it's becoming a generalized problem, then (.) we should look for (...) the problem is looking for meaningful activities that allow (...) fixing things.*"

As you can see in the brief piece above, for Carla, classroom culture is very important for having knowledge stability. On this belief she develops her *expectation of constructing new knowledge from what has been already done in the classroom*. The expectation determines choices, for example, as I will show below, she introduces the "properties of linear equations" by calling back the "law of monotonies" for equalities that were explained at the start of the year (laws, related to the substitution property, according to which adding/subtracting the same number to, or multiplying/dividing by it, both sides of an equality does not change the equality). As you can easily notice, students will be able to construct themselves this mathematical content.

1 T: Instead [*highest pitch*] for the second "law of monotony" [*nodding, she stops in speaking and she bites her lips: Fig. 4*]?	 *Fig. 4*
2 S3: if we multiply or divide [*Carla nods, remaining with lips as in Fig. 4*]	
3 S5: by a number not equal to zero [*Carla nods, remaining with lips as in Fig. 4*]	
4 S7: both sides [*Carla nods, remaining with lips as in Fig. 4*]	
5 S3: we obtain an equivalent equation to the given one [*Carla nods, remaining with lips as in Fig. 4*]	
6 T: [*nodding*] do you all agree?	
7 Ss: yes	

In particular, in the excerpt just presented, the teacher is speaking of second monotony law (#1). When students construct that knowledge, she remains for all the time in the same attitude that is constituted by her closed fist, her nodding and tightening her lips (#2, #3, #4, #5: Fig. 4). Hence, there is an epistemic emotionality of Carla that is not only the knowledge of the "laws of monotony" for equalities (rational key), but also the fact that Carla hopes that students remember what they have already done with her for equalities. This way they can see the analogy with the case of equations, given that she is expecting that students consider equations as open statement as equalities (emotional key). This emotional key is shown, for example, by her emblematic attitude when one student after the other construct all together the principles of equivalence in #2, #3, #4, #5 (Fig. 4), by her pauses, her nodding etcetera. The teacher is engaged in the discursive activity with the class not only from a rational point of view, but also from an emotional point of view: she has hopes, expectations but also needs. Then in her speech are reflected both these sides because the teacher cannot be just pure rational in her discourse. For this reason, I speak of communicative emotionality of Carla. It is disclosed by her posture, her gestures, her stopping in speaking, the tightening of her lips and so on. These information allow to say something about why she decides to act in a certain way. For example, very often, she stops in speaking, she tightens her lips, she pauses, she makes the gesture that mimes the "bringing out" of the knowledge from her students, probably, because she hopes that students continue the discourse, having learnt with her to construct new knowledge, to see analogy, to give justification of what they do and to work on example before institutionalizing the knowledge.

Discussion

From the analysis of each teachers, it emerges that all of them have the expectations that students remember the previous mathematical knowledge. Then, all of them recall the totality of the previous mathematical contents in classroom, but with very different affective involvements, because they are driven by different reasons.

For example, in different passages of her lessons that I could not include in this paper, Lorenza uses rhetorical questions. Probably, she thinks that students know already the answer because they are drawing upon the valid and already accepted previous knowledge. In addition, she nervously smiles when they don't remember it. Moreover, when she asks questions about previous knowledge she is expecting an answer as it is easily understood from her facial expressions, but, simultaneously she uses an affirmative tone of voice. Probably, she would like that the previous facts are already valid for students such that they can naturally surface them. This discussion is in line with the expectation about the "classroom culture" I found from the a-priori interview.

On the contrary, Carla increases the tone of voice when she wants to recall previous facts, raises her eyebrows and mimes class culture in her "fist". In addition she makes the gesture of specifying, she pronounces, she pauses and she nods. The first difference from Lorenza is that Carla mimes the classroom culture having it in her fist (Fig. 4), while Lorenza mimes just the past (Fig. 2). This could be seen as a hint of the fact that Carla wants to hold in her fist the previous knowledge. Possibly, it is important for her not just to remind students that they have already seen it, rather she wants to employ it to construct the new one. Furthermore, if Lorenza never makes pauses in order to push students to make a step forward from the recalled knowledge (except for her atypical questions), it seems clear that Carla, instead, wants students to speak in order to develop previous facts. In fact she increases the tone of voice to draw the attention of students first on what they have recalled. Then, she raises her eyebrows and she pauses in order to expect a reaction from the class and she nods before they answer. Probably, she is quite sure that it is actually possible for students to construct new knowledge. This is also proved by the fact that she is disappointed when students don't react as she is expecting and, indeed, she uses an irritated tone of voice.

Conclusion

This comparison does not aim to "judge" in a certain manner the work of the teachers. It just serves to reflect upon their decisions, trying to reinforce the following thesis. There are different emotional aspects that characterize each teacher and their decisions are often made visible through them. Habermas speaks just of "rationality", but the whole activity of the subject is not only discursive, because there is an entire dimension of emotional aspects that intervenes, and that cannot be captured by Habermas' notion of 'rationality'. For this reason, I have also accounted for the emotionality of the subject. Moreover, the rationality and the emotionality are strictly intertwined, constituting an *unicum*. Hence, we propose to invent the term *ræmotionality* as a neologism to refer to the intertwinement between the rationality and the emotionality of the teacher: this way we can go beyond the Cartesian dualism body/mind towards a holistic view that could be described through the concept of ræmotionality. In a further development of the research, the concept of ræmotionality could also be considered as a theoretical lens through which analysing the activity of the teacher.

If I had considered just the speech of the teachers, without accounting for the audio and the video, I would have found not so many differences among the teachers. Inserting also the emotional elements that constitute a significant part in their teaching, as prosody, gestures, facial expressions and so on, I found three really different and complex ræmotionalities. Actually, this corresponds to the global impression that one has when looking at the video.

REFERENCES

Boero, P. & Planas N. (2014). Habermas' construct of rational behavior in mathematics education: new advances and research questions. In P. Liljedahl, C. Nicol, S. Oesterle, D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education*, Vol.1, 205-235.

Brown, L., & Reid D.A. (2006). Embodied cognition: Somatic markers, purposes and emotional orientations. *Educational Studies in Mathematics, 63*(2), 179-192.

Damasio, A. (1999). The Feeling of What Happens: Body and Emotion in the Making of Consciousness. San Diego, CA: Harcourt.

Ph.D. thesis: "Rationality in mathematics teaching: the emergence of emotions in decision-making", (2015), Dipartimento di Matematica "Giuseppe Peano", Torino.

Ferrara, F. & De Simone M. (2014). Using Habermas in the study of mathematics teaching: the need for a wider perspective. In P. Liljedahl, C. Nicol, S. Oesterle, D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education*, Vol.1, 223-228.

Habermas, J. (1998). *On the pragmatics of communication*. Cambridge, MA: MIT Press.

Hannula, M. (2012). Exploring new dimensions of mathematics-related affect: Embodied and social theories. *Research in Mathematics Education, 14*(2), 137-161.

Immordino-Yang, M.H., & Damasio, A. (2007). We feel, therefore we learn: The relevance of affective and social neuroscience to education. *Mind, Brain, and Education, 1*(1), 3-10.

Maturana, H.R. (1988). Reality: the search for objectivity or the quest for a compelling argument. *The Irish Journal of Psychology, 9*(1), 25-82.

Reber, A. S. (1995). *The Penguin dictionary of psychology*. Penguin Press.

Reid, D. (1999). Needing to explain: The mathematical emotional orientation. In Orit Zaslavsky (Ed.) *Proceedings of the Twentieth-third Annual Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 4 pp. 105-112) Haifa, Israel.

Zan, R., Brown, L., Evans, J., & Hannula, M. (2006). Affect in mathematics education: An introduction. *Educational Studies in Mathematics, 63*(2), 113-121.

An artefact for deductive activities: a teaching experiment with primary school children

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Abstract. In this paper we describe an experiment in which we want to explore the relationship between language and developmental processes of logical tools through "linguistic-manipulative" activities in primary school classrooms. Social and cultural developmental psychology and mathematical logic act as framework for the research. The teaching experiment regards the assertive aspects of the language and it is based on the construction of "system of axioms" and "deductive chains". A crucial point is the use of an artefact specifically designed and constructed for these activities. By a qualitative analysis of the activities we can observe how the children processed the activities and how the "transformation" of deduction processes in the manipulation of "linguistic objects", can support in the children the developing of abilities of mastering simple deductive processes.

Resumè: Dans cet article, nous décrivons une expérience dans laquelle nous voulons explorer le rapport entre le langage et les processus de développement de compétences logiques grâce à des activités de «manipulation de la langue» dans l'école primaire. La psychologie du développement social et culturel et la logique mathématique sont les bases théoriques de cette recherche qui concerne les aspects de la langue et qui est basé sur la construction de «systèmes d'axiomes» et «chaînes déductives». Un point crucial est l'utilisation d'un artefact spécifiquement conçus pour ces activités. L'analyse qualitative des activités nous permet d'observer comment les enfants les effectuent et comment la transformation du processus de déduction dans la manipulation des «objets linguistiques» peut soutenir le développement des capacités de maîtrise des processus déductifs simples.

1. Introduction and theoretical background

The aim of this research is to exploit the potential of formal deductive logic in the teaching-learning processes. In particular, in this work we want to explore the relationship between the manipulation of linguistic objects and the development of logical-deductive abilities, through a teaching experiment in a primary school. This study is a part of a wider research project, whose aim is analysing the relationship between language and its objectification, and the development of logical abilities in problem-solving situations in primary school (Coppola, Mollo, Pacelli, 2010).

The general idea underlying this study is that mathematical logic could be a useful tool in the educational field if it is seen not as a universal tool to give a foundation of mathematics, but as a "local" tool useful to represent steps of the deductive activity also in a not mathematical field. Therefore, the notions of "system of axioms" and "theorem" are interpreted, respectively, as available information (not complete, in general) and new information deduced from the available one (Coppola, Gerla, Pacelli, 2007). The activities of the experiment concern the assertive aspects of language and are based on the construction of a "system of axioms" and "deductive chains". A crucial point is the use of an artefact specifically designed and built for these activities.

Our research hypothesis is that the artefact, assisting the "transformation" of deduction processes in the manipulation of "linguistic objects", can support in the children the developing of abilities of mastering simple deductive processes, with problem-solving strategies. A further objective is to stimulate in the children a reflection about the deducibility as an activity related only with the available information, not involving own beliefs or preconceptions.

The theoretical background of the research is based on the one side on social and cultural developmental psychology (Vygotskij, 1934), on the other side on some aspects of mathematical logic. These theories are the lenses through which we look at the role of language. Language is seen as an "artefact" not only as a tool for communicating and structuring the world, but above all as an object to manipulate and to reflect on.

> "The idea of artefact is very general and encompasses several kinds of objects produced by human beings through the ages: sounds and gestures; utensils and implements; oral and written forms of natural language […]" (Bartolini Bussi, Mariotti, 2009).

More precisely, in the activities carried out by the children during the teaching experiment, the "artefact language" was constructed writing every assertion and every logical connective on magnetic cards, which could have been subsequently manipulated.

Many researches in cognitive psychology (Wason, 1966) highlighted limits in hypothetic-deductive reasoning abilities in adult people, in opposition to the piagetian ideas about the completeness of the development of logical thinking in the adult age. These limits are related to the difficulties in representing a deductive structure when the logic complexity necessary for a proof increases. Moreover, there are limits due to the interference between the information available in a text and the information coming from own knowledge, background and context. With regard to this issue, the role of "implicatures" is important to be considered (Ferrari, 2004).

> "The contexts in which everyday reasoning develops and those in which deductions, which (also) logic deals with, are built are completely different, with different acceptability and coherence criteria" (Dapueto, Ferrari, 1988).

2. Our artefact

The artefact used in the teaching experiment was designed by us and built by a carpenter. It consists of several cards, such as small whiteboards, on which you can write with coloured markers and you can clear with an eraser. The cards are magnetic, so they can be attached to a magnetic board. By composing the cards sequentially, we can obtain sentences of first order logic. This enables the children to "write" a system of axioms expressing the informative content of a text and to manage the axioms to build deductive chains successively. In such a way, a process of objectification of the language is carried out, and this is the basis of our research.

The whole experiment is in accordance with the idea that there are fragments of classical logic sufficiently simple to be introduced in primary school. In particular it is inspired to the logic programming paradigm in which the system of axioms is defined by ground atomic formulas (facts), universal atomic formulas (atomic rules) and implications (rules). By the artefact, it is easy to transform the structure of a sentence in a configuration of cards and transform the deductive process into a manipulation of cards.

Our artefact replaces the paper cards used in previous experiments (Coppola *et al*., 2010). Indeed the paper cards have to be necessarily prepared before the experiment by the researcher and this makes limited the intervention of the children. Moreover, the paper cards are not rewritable and, therefore, it is necessary to build different cards for each new text. In addition, the paper cards are not convenient to use because they cannot be fixed to the desk. Finally, in regard of the age of the children, the physical structure of magnetic cards (weight, shape, colour,...) plays a not secondary role in the learning process of the children.

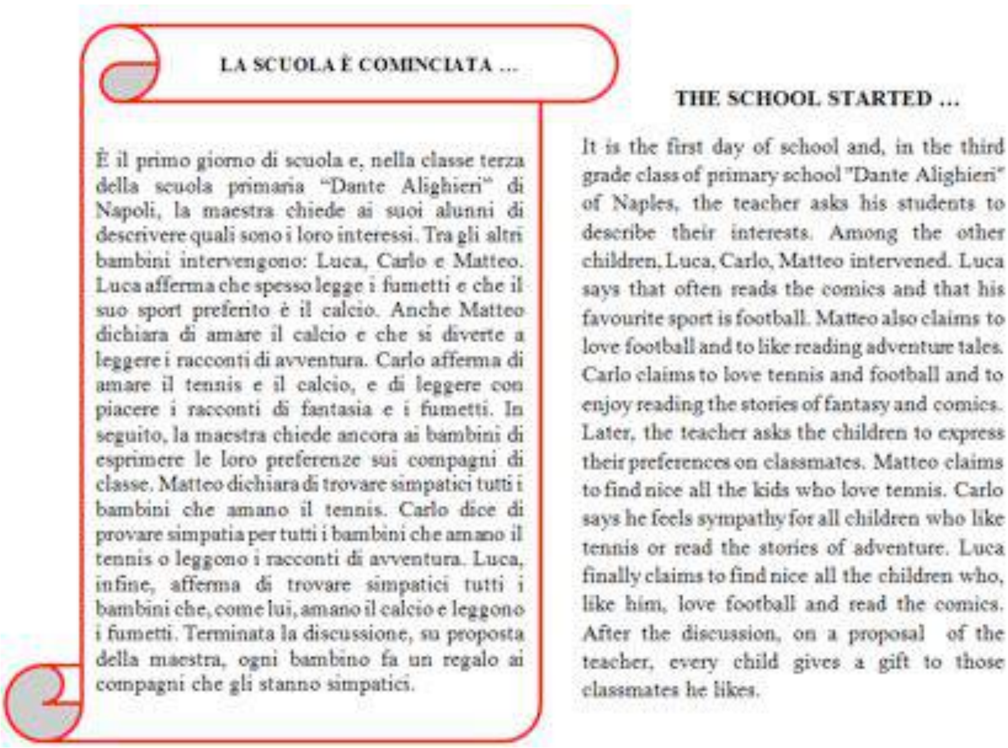
3. The teaching experiment

In this paper, we describe a teaching experiment carried out in a primary school (Istituto Comprensivo "G. Vespucci" in Forino, Avellino). The participants were 30 students of the two third grade classes of the school, joined together to carry out the specific activity. The teaching experiment was made up of four phases, all video-recorded:

- First phase: Reading and understanding a text;
- Second phase: Text analysis and extraction of knowledge;
- Third phase: Deduction;
- Fourth phase: New deduction and interviews.

3.1 Reading and understanding a text phase

The students did the activity individually. We gave the children a text concerning a real life situation (Figure 1).



LA SCUOLA È COMINCIATA ...

È il primo giorno di scuola e, nella classe terza della scuola primaria "Dante Alighieri" di Napoli, la maestra chiede ai suoi alunni di descrivere quali sono i loro interessi. Tra gli altri bambini intervengono: Luca, Carlo e Matteo. Luca afferma che spesso legge i fumetti e che il suo sport preferito è il calcio. Anche Matteo dichiara di amare il calcio e che si diverte a leggere i racconti di avventura. Carlo afferma di amare il tennis e il calcio, e di leggere con piacere i racconti di fantasia e i fumetti. In seguito, la maestra chiede ancora ai bambini di esprimere le loro preferenze sui compagni di classe. Matteo dichiara di trovare simpatici tutti i bambini che amano il tennis. Carlo dice di provare simpatia per tutti i bambini che amano il tennis o leggono i racconti di avventura. Luca, infine, afferma di trovare simpatici tutti i bambini che, come lui, amano il calcio e leggono i fumetti. Terminata la discussione, su proposta della maestra, ogni bambino fa un regalo ai compagni che gli stanno simpatici.

THE SCHOOL STARTED ...

It is the first day of school and, in the third grade class of primary school "Dante Alighieri" of Naples, the teacher asks his students to describe their interests. Among the other children, Luca, Carlo, Matteo intervened. Luca says that often reads the comics and that his favourite sport is football. Matteo also claims to love football and to like reading adventure tales. Carlo claims to love tennis and football and to enjoy reading the stories of fantasy and comics. Later, the teacher asks the children to express their preferences on classmates. Matteo claims to find nice all the kids who love tennis. Carlo says he feels sympathy for all children who like tennis or read the stories of adventure. Luca finally claims to find nice all the children who, like him, love football and read the comics. After the discussion, on a proposal of the teacher, every child gives a gift to those classmates he likes.

Figure 1. The text given to the children

After reading it, the children answered some comprehension questions and they explained their answer (Figure 2).

SCUOLA PRIMARIA "G. VESPUCCI"	SCUOLA PRIMARIA "G. VESPUCCI"
Nome ________ Data ________	Name ________ Date ________
Rispondi alle seguenti domande e spiega il motivo della tua risposta.	Answer to the following questions and explain your answer.
Dalla lettura del testo, puoi dire che:	*After reading the text, you can say that:*
1) **Matteo legge i racconti di avventura?** SI NO	1) **Does Matteo read adventure tales?** YES NO
2) **Luca trova simpatico Carlo?** SI NO	2) **Does Luca like Carlo?** YES NO
3) **Carlo ha 8 anni?** SI NO	3) **Is Carlo 8 years old?** YES NO
4) **Carlo trova simpatico Luca?** SI NO	4) **Does Carlo like Luca?** YES NO
5) **Matteo fa un regalo a Carlo?** SI NO	5) **Does Matteo give as a gift to Carlo?** YES NO
6) **Luca fa un regalo a Matteo?** SI NO	6) **Does Luca give as a gift to Matteo?** YES NO

Figure 2. Questionnaire

We collected their written productions and subsequently there was a collective discussion, led by us, to bring out comments on the activity.

3.2 Text analysis and extraction of knowledge phase

The children worked in a collective manner. Guided by us, they analysed the text, extracting key information, and writing them down on the whiteboard in everyday language, one by one. The information was divided into facts and rules and the children highlighted the "variable" words and the logical connectives (Figure 3).

Figure 3 – Facts and Rules in everyday language

Subsequently, we showed them our artefact to reproduce the facts and the rules written in everyday language on the whiteboard. The children, guided by us, wrote every part of each sentence on the cards, using

- different colours to differentiate constants and predicates,

- small cards to represent logical connectives and variables.

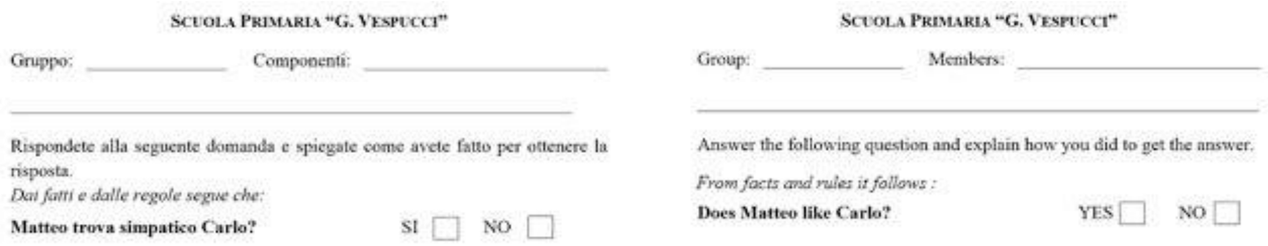
Then, they attacked the cards on the magnetic whiteboard, one by one and, in this way, they rebuilt the complete system of axioms (Figure 4).

Figure 4 – Facts and Rules with the artefact

In accordance with logic programming paradigm, the students wrote the rules placing the antecedent before the consequent. This made the visualization of the logical calculus more evident. In addition, the children represented the variables in a way to remember the kind of things or people they denoted. In fact, in the rules there were children that "change" and the majority of students chose to use the drawing of a child's face. In this way there was a difference from the conventions of the logicians (and more generally of mathematicians) that use letters as x, y, z, … referring to different objects. Thus, the language was objectified becoming a concretely manipulable tool.

3.3 Deduction phase

Students worked in cooperative groups, supported by us. We gave to each group the following material: a magnetic whiteboard, magnetic cards, markers of different colours and erasers. Each group had to evaluate whether a fact could be deduced or not through the previously built system of axioms, answering to questions similar to the ones of the questionnaire of the first phase (Figure 5).


SCUOLA PRIMARIA "G. VESPUCCI"

Gruppo: ____________ Componenti: ____________________

Rispondete alla seguente domanda e spiegate come avete fatto per ottenere la risposta.
Dai fatti e dalle regole segue che:
Matteo trova simpatico Carlo? SI ☐ NO ☐

SCUOLA PRIMARIA "G. VESPUCCI"

Group: ____________ Members: ____________________

Answer the following question and explain how you did to get the answer.
From facts and rules it follows :
Does Matteo like Carlo? YES ☐ NO ☐

Figure 5 – Example of question

In order to answer, we suggested to the children to build deduction chains using the magnetic cards. The algorithm, proposed by us to build the deduction chain, was similar to the strategy used by the Prolog language. This procedure is a backward or goal-oriented process: it starts from the goal and, going backwards, considers only the rules and facts useful to its achievement. The students translated this procedure in a sequence of configurations of cards applied on the whiteboard. After finding their own answer, each group showed it to the whole class, presenting and justifying the built deductive chain (Figure 6).

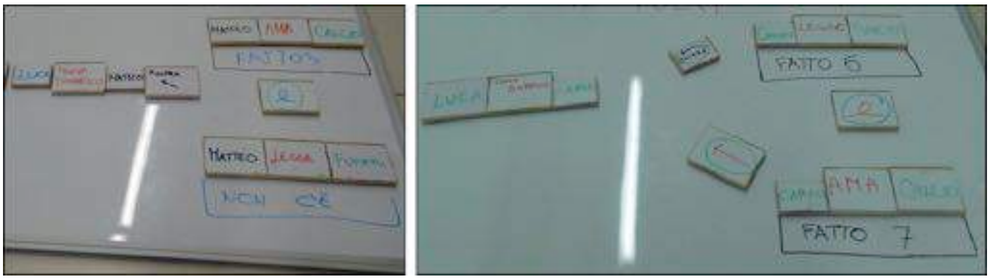


Figure 6 – Examples of deductive chains

3.4 New deduction and interviews phase

Students worked in pairs without our support. We showed a new text shorter than the first one. Each pair built the new system of axioms, using the artefact, as a sequence of configurations of cards applied on their whiteboard (Figure 7).

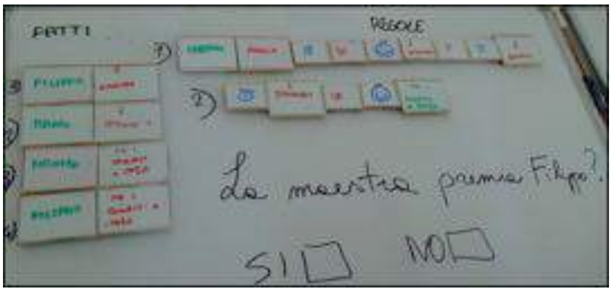


Figure 7 – The new system of axioms

Each pair played the activities autonomously, recalling the work done in the previous phase. Then, each pair had to evaluate whether a fact could be deduced or not through the built system of axioms, building deductive chains, like previous phase. (Figure 8).

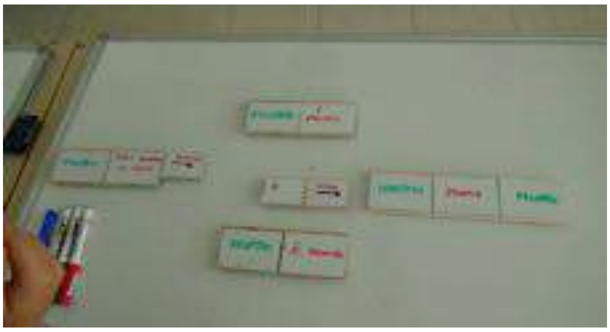


Figure 8 – Example of deductive chain

At the end of all the activities, some children were interviewed and we asked them to reconstruct their experience in order to encourage a reflection on the activity.

4. First Results

Here we present some of the results of a preliminary analysis of the experiment carried out. We analysed the data provided by the questionnaires and videos of reading and understanding a text phase, data provided by the videos and the written answers of deduction phase, data provided by the videos of new deduction and interviews phase.

4.1 Results of reading and understanding a text phase

The analysis showed that, during the first phase, most of the students gave correct answers only when the information was present explicitly in the text. Some children also managed to do simple deductions that required the use of a rule whose antecedent was devoid of logical connectives. However, they ran into difficulties when it was necessary to use more rules to answer the question or when in the antecedents there were the connectives "and" and "or" (most of the children showed not to know the difference between them). Referring to the question, "Does Luca give a present to Matteo?" in which they had to use the two rules: "A child gives a present to another child if he likes him" and "Luca likes another child if he loves football and reads comics", a student stated: "*I had difficulties in answering this question "Does Luca give a present to Matteo?" because I could not understand the answer*". Besides, as it emerges from the analysis of the answers to the questionnaires, it seems that pupils didn't follow only the information in the text, but they referred to their personal experiences. In particular, with respect to the third question: "Is Carlo eight years old?" all the students answered positively for the following reasons: "*Because if he is in the third class he must be 8 years*"; "*Because we are eight too and we are attending the third class*" (Figure 9).

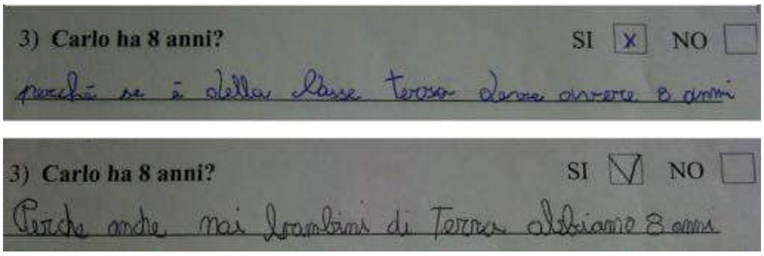


Figure 9 – Protocols 16 and 22[5]

As a matter of fact, the text was about children who attended the third class of a primary school (in Italy most of the students attending the third class of primary school is 8 years old). However, in the text there is not any rule that allows to deduce "logically" Carlo's age (this fact was not present in the text). Among the participants, there were also seven-year-old children. One of these answered to the same question: "*He is eight years old, because otherwise he would not attend the third class*" (Figure 10).

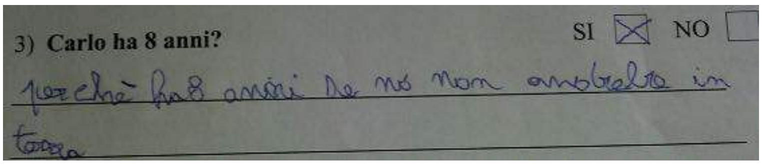


Figure 10 – Protocol 10

Then, during the discussion, he stated: "*I am an exception and if also Carlo was an exception, it would be written in the text*". From the explanations given by the children it emerges how they answered also referring to "moral rules", for example, "*Luca likes Carlo because everyone has to love each other*", "[…] *because being nice to each other is often paid back in the same coin*" (Figure 11). These rules not only were not present in the text but were not part of the assertive logic in the strict sense

[5] The protocols of the children are numbered from 1 to 30.

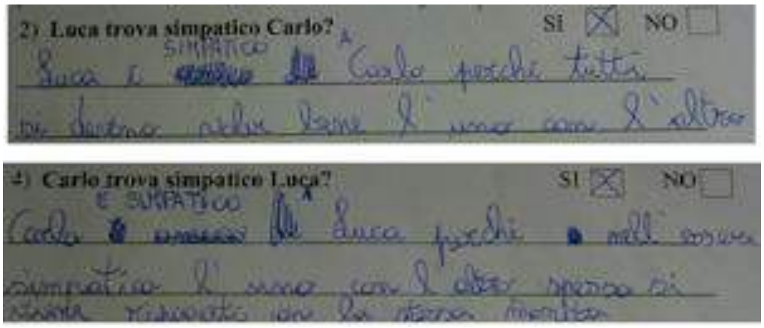


Figure 11 – Protocol 3

4.2 Results of deduction phase

By the analysis of the written answers and videos related to the third phase of deduction characterized by cards' manipulation, it seems to emerge that pupils, working together, were able to find answers to questions by the construction of simple deductive chains. Each group, "tracing forward" the built chain, managed to expose to others the deduction obtained arguing every single step. During the activity of creation of the deductive chains, they started a discussion about the role of connectives found in the rules. For example, a child looking at the configuration of the cards placed on the whiteboard, said: "*In the rule there is "or" and for this reason if they agree on one thing, they don't need the other; in that case there is the "and", so they must be checked both things*". Moreover, there were some interesting answer explanations written by some groups related to the question: "Does Luca like Matteo?". In this case, the answer was not deducible from the text because the rule "Luca likes another child if he loves football and reads comics" was not verified, Matteo loves football but he does not say to read comics. The children wrote their explanations in the following way: "*Luca doesn't like Matteo because he likes only children who love football and read comics, otherwise Matteo loves football but it is not written in the text that he reads comics*" (Protocol G1); "[…] w*e saw that Matteo does not read comics but loves football. However, Luca likes Matteo if he reads comics and loves football. Then we discovered that the answer is negative*" (Protocol G2); "*The answer is no because Matteo loves football but he doesn't read comics, therefore, Luca doesn't like Matteo because the connector "and" means both things*" (Protocol G3), (Figure 12).

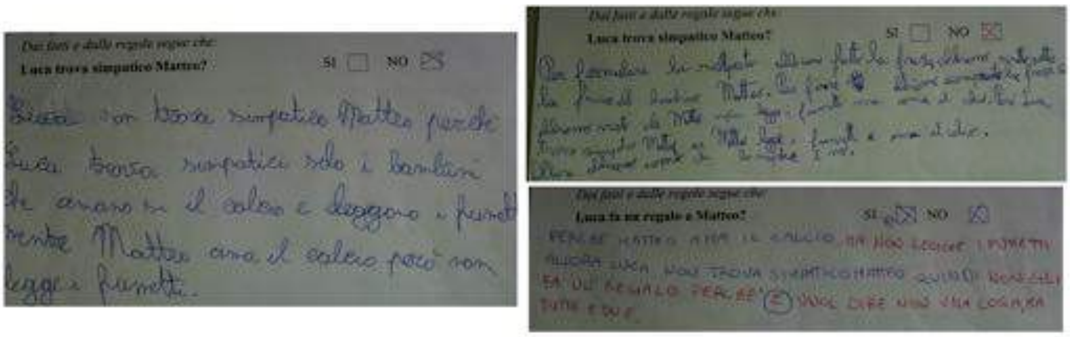


Figure 12 – Protocols G1, G2 and G3[6]

From these reasons, it seems to emerge that after the activity carried out through the use of the artefact, some children had developed a greater awareness about the "logical" use of the conjunction and disjunction. However at this stage it seems unclear the distinction between "not deducible" and "false". In addition, students showed they had understood the use of a variable within a rule. As a matter of fact some groups argued in this way: "*To formulate the answer we did the sentence, we wrote Matthew under the child's face. Finally, we broke down the sentence; we formulated a sentence to get at the answer and the child became Matteo* […]" (Figure 13).

[6] The protocols of groups are numbered from G1 to G4.

Figure 13 – Protocols G3

4.3 Results of new deduction and interviews phase

Analysing the videos related to the phase of the new deduction, we observed that each pair was able to create the system of axioms corresponding to the new text and to construct deductive chains autonomously, using the artefact. Even during this phase, students were asked the question: "Is Carlo eight years old?", and once again pupils, forgetting the system of axioms used until that moment, answered starting from their own experience. However, during a collective discussion where a reflection on the concepts of "falsehood" and "not deductibility" was stimulated, the children showed they had understood the difference. Here is an extract of an interview:

Interviewer: What did you used to answer the question: Is Carlo eight years old?
Child 3: I used the fact: Carlo attends the third class.
Interviewer: What rule did you use?
Child 3: I used the rule: every child who attends the third class is eight years old
Interviewer: Is there this rule in the text?
Child 3: No, it is a rule that I have added.
Interviewer: Now how do you respond to the same question?
Child 3: Carlo should not be eight years old because usually students in the third are 7 years, we cannot define it.

The individual interviews, carried out at the end of the experiment, showed that the children acquired full knowledge of the experience done. Here is an extract of an interview:

Interviewer: Describe the activities you did.
Child 11: […] *we wrote on pieces of wood the facts and rules that were in the text.*
Interviewer: How did you answered to the questions?
Child 11: We answered the questions using the rules that we needed, replacing the child's face with his name […] *moving the pieces* […] *and then we deduced the answer.*

After a few months, we went back to school and we gave the same students a new text. We asked them to read the text and answer some questions and to explain their answers, without our artefact. Analysing the answers of children, we observed that children referred to previous activities and made deductions from the facts and rules in the text, without using "moral rules" and personal experiences. In addition, some children were able to make deductions, using the logical connectives correctly. For example, a little girl wrote: "*I got the answer seeing the rules and Mario says to invite who play with the tablet or watching television series, and Martina watches television series*" (Figure 14).

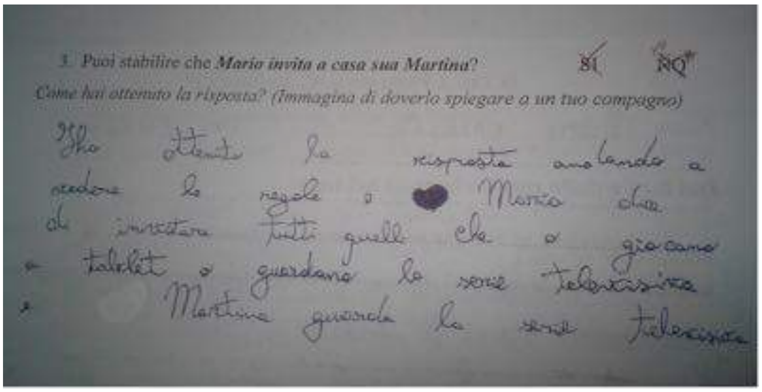


Figure 14 – Protocol 30

5. Conclusions

According to the researches regarding the use of artefacts in the educational field (Bartolini Bussi, Mariotti, 2009), the use of the specific artefact seems to support learning opportunities, promoted through collective discussion above a common "object", all participants could refer to and manipulate. Therefore, the language became a concretely manipulable object. Children could make deductions visualizing the "physical" nature of a proof. Indeed, activities of such a kind should not have the same results using only copybooks or blackboard. From the teaching experiment, it seems to emerge that logical-deductive activities are possible to carry out already with primary school children. We carried out an teaching experiment like this with secondary school students (10-11 years old) and we are analysing the data. We are planning an activity, like this, with mathematical text. In addition, we would like to implement a software to simulate this activity.

REFERENCES

Bartolini Bussi, M. G., Mariotti, M. A. (2009). Mediazione semiotica nella didattica della matematica: artefatti e segni nella tradizione di Vygotskij, *L'insegnamento della Matematica e delle Scienze integrate*, vol.32 A-B n.3, 270-294.

Coppola C., Gerla G., Pacelli T. (2007), Un progetto di ricerca: logica matematica per lo sviluppo di abilità dimostrative, *Periodico di Matematiche*, 1, 41-52.

Coppola C., Mollo M., Pacelli T. (2010). Deduzione come manipolazione linguistica: un'esperienza in una scuola primaria, *L'educazione matematica,* Anno XXXI, serie 1, vol. 2 n°3, 5 – 22.

Dapueto C., Ferrari P.L. (1988). Educazione logica ed educazione matematica nella scuola elementare, *L'insegnamento della Matematica*, vol.11.

Ferrari, P.L. (2004). *Matematica e linguaggio. Quadro teorico per la didattica*. Bologna, Pitagora Editrice.

Vygotskij, L.S. (1934). *Thought and language.* Moscow-Leningrad: Sozekgiz.

Wason P. C. (1966), *Reasoning*. In B. Foss (ed.), New Horizons in Psychology, Penguin, Harmondsworth, 135-151.

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The gesture/diagram interplay in grappling with word problems about natural numbers

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Abstract: In this paper, we pursue a participationist vision of learning through which we discuss the way that two grade 5 children use diagrams and gestures as new resources to think of a word problem that is concerned with natural numbers. The gesture/diagram interplay reveals that the diagrams mobilize the mathematical concepts, bringing forth new spaces and structures and alluding to new temporal dimensions between objects. In addition, the diagrams embed and anticipate gestures, which can be shared within the classroom through the collective discussion. Drawing on de Freitas and Sinclair (2012), this view offers a way of rethinking, in this particular situation, the relationship between immovable numbers and movable bodies.

Résumé: Dans cet article nous poursuivons une vision participationniste de l'apprentissage par lequel nous discutons de la façon dont deux élèves de 10-11 ans utilisent des diagrammes et des gestes comme nouvelles ressources pour aborder un problème concernant les nombres naturels. L'interaction entre geste et diagramme révèle que les diagrammes mobilisent les concepts mathématiques, ils amènent à la création de nouveaux espaces et structures et évoquent de nouvelles dimensions temporelles entre les objets. En outre, les diagrammes incorporent et anticipent les gestes qui peuvent être partagés en classe grâce à la discussion collective. En faisant appel à la pensée de de Freitas et Sinclair (2012), cette perspective offre une manière de repenser, dans cette situation spécifique, la relation entre les nombres, considérés comme immobiles, et les corps, considérés comme bougeant.

Background perspective

In this paper, we discuss an activity in which a class of grade 5 children has faced tasks concerning properties about natural numbers, for example about the sum/product of two consecutive odd/even numbers, the product of three consecutive odd numbers, etc. The activity is part of a longitudinal research study aimed to promote elementary forms of algebraic thinking at primary school. The literature has evidenced that embodied forms of algebraic thinking, which were already inquired in adolescents, can be made accessible to young students. As Radford (2014) puts it, "students can effectively think algebraically at a young age." (p. 258), even though we need to recognise "forms of algebraic thinking that are not necessarily based on alphanumeric symbolism." (p. 259). The tasks of our study were designed with the idea of mobilising algebraic concepts in the early years and preparing the children for more sophisticated learning later on. They were mainly focused on patterns in given sequences and structures in simple situations. In particular, here we have chosen to centre on a task about the sum of two consecutive odd numbers, which did not require the use of any specific technology, tool, material or representation. The task was given in written form, using natural language; de Freitas & Zolkower (2014) would refer to it as a *word problem*, in our case pertaining to mathematical events (that is, to the properties that it uses). As de Freitas and Zolkower suggest (no matter whether they precisely focus on problems about motion), there are some events behind them: "that which is depicted in the problem, that which is enacted in reading the problem, and that which is affected through the imperative to solve the problem." (p. 3). When we consider student engagement with word problems, say the authors, it is important to keep in mind all these events, since they entail various kinds of activities. For example, in their study it is revealed that diagrammatic and gestural semiotic resources play a crucial role in student and teacher engagement with word problems that pertain to motion, "operating as visual-haptic devices for exploring the problem from different angles." (p. 2).

Starting from this, we focus on the particular learning space in which the children work and on the resources through which they grapple with the word problem and develop mathematical thinking in this context. Thus, instead of looking at the resources as furnished by the task, by the teacher or by the tools, we take on a completely different perspective and investigate *how* the children themselves bring forth resources and the kind of resources that they create, as well as the way in which they operationalize these resources in their thinking strategies. To do so, we pursue a non-dualist vision of learning that distances us from any split between body and mind, between perceptual and conceptual (see e.g. Nemirovsky *et al*., 2013; de Freitas & Sinclair, 2014; Radford, 2014). Our position is *not* acquisitionist, in that it shares Sfard (2008)'s participationist commitment, which moves far away from the Piagetian view of thinking as manipulating mental schemes and instead considers thinking as a form of communication. This allows us to focus more on the learners' ways of communicating in the mathematics classroom, that is, on their ways of talking, moving, gesturing, doing.

Within this perspective, to analyse the way that the resources are taken up as the children engage with the given task, we mainly adopt the view about the gesture-diagram interplay that is offered by de Freitas and Sinclair (2012). De Freitas and Sinclair propose to rethink the relationship between gesture and diagram through their coupling, suggesting new ways of conceptualizing agency and embodiment in the mathematics classroom. They draw on the French philosopher of mathematics Gilles Châtelet, who sees gestures as "capturing devices" and diagrams as "physico-mathematical" entities. In doing so, the authors trouble some consequences of "the Aristotelian division between movable matter and immovable mathematics", and claim that "mathematics and matter are fused together, and [that] any attempt to deny the materiality of mathematics is a reflection of our desire for an ideal and unsullied mathematical world." (p. 137). They argue that the diagrammatic and the gestural participate in each other's provisional ontology, being both dynamic pivotal sources of mathematical meaning. If gesture gives rise to the very possibility of diagramming, so diagram gives rise to new possibilities for gesturing. Like gestures, diagrams have mobility and potentiality and can be thought of not as static pictures that represent meanings but "*as events*", not divorced from the mathematical event (p. 138, *italics in the original*). This means that diagrams are not external representations of existing knowledge, but "kinematic capturing devices, mechanisms for direct sampling that cut up space and allude to new dimensions and new structures." (p. 138). In the apparently static body of a diagram, virtual folds of future alteration are latent, ready to make new unexpected gestures and movements suddenly emerge. So, the diagram is a material surface that is not inert part of the mathematical event but a site of *agency* in the very encounter between learner and diagram. This means that we can rethink agency as dispersed and distributed in the classroom, across the relations between learner and her material surroundings.

In the situation we consider here, it is of great interest the way that these types of events are entwined with those of which de Freitas and Zolkower talk about word problems. In the next section, we present the methodology that we have used in our study and the task that is the focus of this paper.

Method and task

The data, which is we analyse in this study, is part of a longitudinal classroom intervention aimed to develop early algebraic thinking at primary school in grade 1-5 (age 6-10). The study took place in a primary classroom in Italy, where the first author took on the role of the guest teacher and taught lessons together with the regular classroom teacher, while the second author was an active observer who filmed with a mobile camera all the activities in which the children took part: from individual tasks and group works to collective discussions led by the researcher.

The specific activity, which is the focus of this paper, involved pairs of children in a task called "The strange sum". The task required the children to compare two statements, imagined as uttered by two grade 5 learners: Francesca and Martina. The statements are both concerned with what can be affirmed about the sum of two consecutive odd numbers. They were delivered to the children through a worksheet with written instructions on it and two call-outs used to present the situation. Francesca's call-out states: "The sum of two consecutive odd numbers is always multiple of four", while Martina's call-out says: "The sum of two consecutive odd numbers is an even number". It is asked who is right and why. The children can extensively justify their choices and reasoning in the last empty page at the end of the worksheet, entitled "Space of reasoning".

For this paper, we report the experience of two grade 5 students, Lara and Francesco, when they were first answering the task together and then discussing with the researcher, and when Francesco was explaining this answer during the class discussion. The data selected concerns the movie and the written productions of the two children, and the movie of the discussion. The next section examines this data in light of the vision on the diagram/gesture interplay and in terms of how the diagrammatic and the gestural are resources that the children create and use in their thinking strategies to solve the task.

The gesture/diagram interplay

Lara and Francesco

In the previous years, the children have worked on variables generalising pattern rules. Typically, they have considered familiar labels to denote remote term numbers, for example Pippo and Pluto (names of cartoon characters). Thus, it was quite usual to find in discourse terms called term Pippo, term Pluto, and so on, other than specific terms like terms 12 and 50. Since the beginning of their work, Francesco and Lara discuss the task by taking numerical examples to verify if the given statements hold or do not in these cases. The diagram that the children create on their worksheet is shown in Figure 1. They write that Francesca and Martina are both right, and the fact that the sum of two consecutive odd numbers is even "is right because all the consecutive odd numbers summed up lie in the multiplication table of 4 (what Francesca told) (top of Fig. 1)."

Figure 1. Diagram with numerical examples

We can observe that the diagram hides the explanation of the children's answer. In fact, a structure arises from and within the numerical operations, and embodied dynamic aspects are present. First of all, the close line that frames the results of the various sums outlines the property that these numbers have in common, that is, they are all even, as marked by the word "PARI" (*even* in Italian) written on side on the first row, and by the repeated letter "P." (the first letter of the word) next to the following numbers. The diagrammatic line outlines a new mobile dimension, which mobilizes the hand eventually used to recognise that sequence as a sequence of even numbers and to outline it, what the children first use to stress that Martina is right. The arrow pointing to the very bottom of the page, under the numerical results, puts forward a new dimension again, in which, instead, the same numbers are recognised and mobilized themselves as multiples of 4. This virtual dimension now connects these numbers to the truth of Francesca's statement: "The result of the numbers is equal to the multiplication table of 4".

The use of the multiplication table even points out the general nature of the reasoning. The presence of the two arrows is interesting. The first arrow joins the resulting numbers with the statement about their belonging to the multiplication table. The second arrow shifts attention from identifying the common property to a justification of it, which is given by the two statements on the right of the figure (that start with the word "because"). The arrows establish an order for thinking, so we indicate them as the first and the second. Indeed, particular cases are necessary to recognise the property before being able to give an explanation of its validity. And, still, the justification makes use of the specific examples of the first two sums of two consecutive odd numbers. The children write: "Because, if we take the initial number (1), add +0 then we "carry" the other number to the next calculation, the other number is 4 times bigger than the previous initial number. So, make 0 (the starting number) +4 (the number from the previous initial number) =4 (the detachment between the numbers in the multiplication table)". This explanation is really rich, even though not so simple to be grasped. Francesco and Lara have noticed a structure that is shared by each couple of subsequent sums, and attempt to reveal it with their words, using the numerical examples as an aid. Thus, taking the first two sums, 1+3 and 3+5, one can notice that 3 ("the other number", used the first time) is common to both sums, while the difference between 1 ("the initial number", "the previous initial number") and 5 ("the other number", used the second time) is 4 (initially wrongly expressed with "4 times bigger than the previous starting number" and then suitably recalled in terms of "+4" first and then of "=4"). We can observe certain elements in the explanation that give it a structural character. For example, the use of "the other number" is always referred to the second addend in a sum. The "initial number" is used to speak of the starting point of a couple of sums. With "+" the children are looking at what they have to do if they want to find "the detachment" between the two sums, and thus between their results, which are "numbers in the multiplication table" of 4.

The metaphorical action of "carrying" 3 from the first to the second sum is a way of mobilising the fact that two subsequent sums always involve an addend, which is present in both and does not count in the calculation of the detachment. Of course, this action is embodied in the task itself and in the sum of two consecutive odd numbers. In fact, taking two subsequent sums means counting twice the central odd number out of three consecutive odd numbers. Also, the difference between numbers in two subsequent sums is marked by the use of "+" ("+0" is the difference between 1 and itself, while "+4" is the difference between 1 and 5, that is, "the number from the previous initial number"). The last "=4" outlines the detachment (the result of "mak[ing] 0 (the starting number) +4"), which is the most relevant aspect to understand the property that all the results have in common. The extension to the other couples of subsequent sums is implicit in this recourse to generalised subjects ("the initial number", "the other number", "the number"), which abandons the specificity of being taking into account just the two initial sums 1+3 and 3+5 making the reasoning work for all the couples of subsequent sums. The mobility of the common addend, the number that can be thought of as movable from one sum to the next, is also part of the thinking strategy. The mathematics, the numbers, the children, the words, the close line, the signs, the arrows are all part of this movement as well as of the extension of the reasoning. From this point of view, they are all sites of agency, far from being external representations of aspects that are implicated and embedded in the situation.

The mobility said above appears in the bad copy of the diagram with numerical examples, which the children produced on one side of their worksheet (Fig. 2a). Two new kinds of arrows can be seen in this dirty diagram. One kind gives an arrow that points from upper left to bottom right (e.g. to relate 1 with 5, 3 with 7, 5 with 9, 9 with 13, 13 with 17), the other kind is an arrow that instead points from upper right to bottom left (e.g. to relate 3 with itself in the following line, 5 with itself in the following line, etc.). The relations between the numbers in couples of subsequent sums, which are the essential part of the children's way of talking about the sum of two consecutive odd numbers in their written explanation, are arrows in this diagram. Little signs also connect couples of subsequent results to express a relation between them. Then, the double arrow is used in the generalisation of the answer in which the children use *PLUTO* as a parameter: "You can also do it with a generic number (*PLUTO*)" (incipit of Fig. 2b).

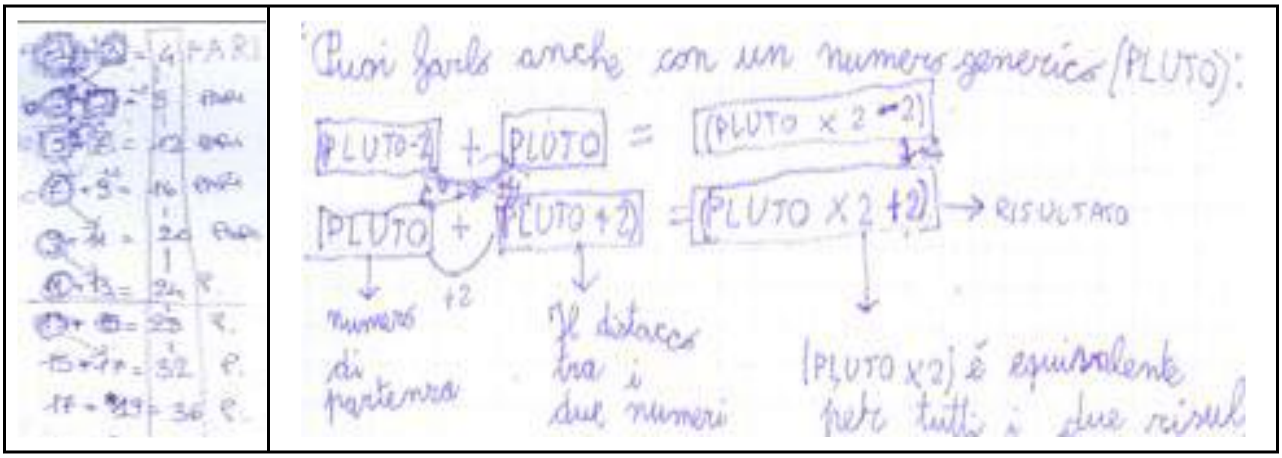

Figure 2. (a) Bad copy of diagram with numerical examples; (b) Reasoning using a generic *PLUTO*

The corresponding diagram shows that the thinking strategy of the children develops considering *PLUTO* as a starting odd number and the two subsequent couples: *PLUTO*−2, *PLUTO* and *PLUTO*, *PLUTO*+2. The sum of the coupled numbers gives rise respectively to: (*PLUTO*×2−2) and (*PLUTO*×2+2) (see Fig. 2b, top left), and (*PLUTO*×2) is "equal for both results" (Fig. 2b, bottom right). Interestingly, the second diagram is drawn in a distinct new page of the worksheet, as if distinct natures were recognized to the diagrams. Indeed, the first has a more arithmetic nature with respect to the second, which has an algebraic nature, even though the structure was already present in the first diagram.

In the algebraic diagram, the situation is even more intriguing from the cognitive point of view. The six numbers that generalise the situation for two consecutive couples of odd numbers and their resulting sums are joined through new arrows, two by two, in a mathematically coherent way, which captures what the children previously did with the numerical examples (see Figure 2a). Two pairs of arrows have distinct functions. The horizontal arrows relate two consecutive odd numbers in each row, pointing out the distance of 2 between them ("+2"). The slanted arrows brings one, directed to the left, the null distance between the two *PLUTO* ("+0"), and the other one, directed to the right, the distance of 4 between the first number, *PLUTO*−2, and the last number, *PLUTO*+2 ("+4"). In addition, the last vertical arrow between the two results underlines again a distance of 4 between them ("+4"), which is essential with respect to their being both multiples of 4. There also are three vertical arrows more on the bottom and one horizontal arrow on the right side. The three vertical arrows identify respectively (from left to right): the starting number, the difference between consecutive odd numbers, and the fact that *PLUTO*×2 is present in both results and, consequently, the gap is 4 (the children write under this arrow: "(*PLUTO*×2) is equal for all the two results, but one has the +2 and the other the −2, between the two sums there is a detachment of 4 numbers, so the two sums are part of the multiplication table of 4"). The horizontal arrow on the right side marks that in each row the result of the sum is present. Like for the instance of the numerical diagram, the arrows bring forth new structures between objects (the numbers in our case), which mobilize the potential joining gestures that Francesco and Lara used when they discovered the relationships just discussed. This interplay between gesture and diagram characterises the way in which the children reason, and can be also found in the work of other pairs of children working together, as well as in moments of comparison with the researcher, or in collective discussions.

Talking with the researcher

During pair work, the researcher was an active observer, and watched what the pairs were doing. When she came to Lara and Francesco, the two children had already dealt with the task and created the diagrams with numerical examples and the algebraic diagram of the previous section. The researcher asked them what they did and found. In this episode, the children discuss with her about why the sum of two consecutive odd numbers is always an even number (Martina's statement). They have just reasoned through few numerical examples on the fact that the result can be always seen as 2 times another number, which is the number between the starting two. In what follows, the researcher tries to shift attention to this kind of number to push towards generalisation.

Researcher: "What is twenty-four?" (*Points to the number in the numerical diagram*)

Children: "An even number"

Researcher: "And what is it equal to?"

Lara: "To twelve times two"

Francesco: "To twelve divided by two, that is, to twelve (together with Lara) times two"

Researcher: "Twelve times two. What is sixteen?"

Lara: "It is always the half (*Scrolls the vertical space between the two columns of odd numbers that are summed in the diagram; Fig. 3a*), the number that stays in the middle (*Mimes the number with her right hand jumping from one position on the desk to another; Figg. 3b, 3c*) between the odd numbers (*Mimes the two numbers with two fingers of her right hand open on the desk; Fig. 3d*), times two" (*Makes a short turning gesture above the desk to indicate the multiplication*)

Researcher: "If it is so, what would it become in the case of *PLUTO*, if the starting number is *PLUTO*, what happens?"

Francesco: "Here (*Points with his right hand index finger to the in-between space in the algebraic diagram*) it is *PLUTO* plus one, and it would be *PLUTO*, wait, it would be (*Keeps his right hand index finger in the previous position while gazing back at the numerical diagram*), hmm, *PLUTO* times two plus two (*Points with another right hand finger to the result in the algebraic diagram. Thinks, rises his gaze upward*), (two) times *PLUTO* plus one"

Researcher: "I didn't understand what would be, where this plus one goes to finish"

Francesco: "When here you look at in between (*Positions his right hand in the middle of the algebraic diagram, focusing on the line of the couple of odd numbers: PLUTO and PLUTO*+2*; Fig. 3e*) you go to *PLUTO* plus one. This number (*Shifts his right hand to point to* (*PLUTO*×2)+2*, while gazing at the* *numerical examples in the other diagram*) we have said that it is the result (*Shifts the right hand to the numerical diagram to indicate a numerical example, while looking back at the algebraic diagram; Fig. 3f*) of this (*Points with two of his left hand fingers to PLUTO and PLUTO+2 in the algebraic diagram, meaning the number in between; Fig. 3f*) times two. So it is *PLUTO* (*Goes back to the algebraic diagram with his right hand, even turning his whole torso towards that diagram*), at this point *PLUTO* (*Points with his right hand index finger to PLUTO*), *PLUTO* times two (*Jumps with the finger to PLUTO*×2 *in the result*). Yeah, right! *PLUTO* times two. *PLUTO* plus one, times two is *PLUTO* times two (*Points with the finger to PLUTO*×2 *in the result*), because there is *PLUTO* already, you multiply it by two, one times two is two (*Points to* +2 *in the result*), so plus two"

Researcher: "So, is that another way [to see (*PLUTO*×2)+2]?"

Children: "Yes" (*Nod*)

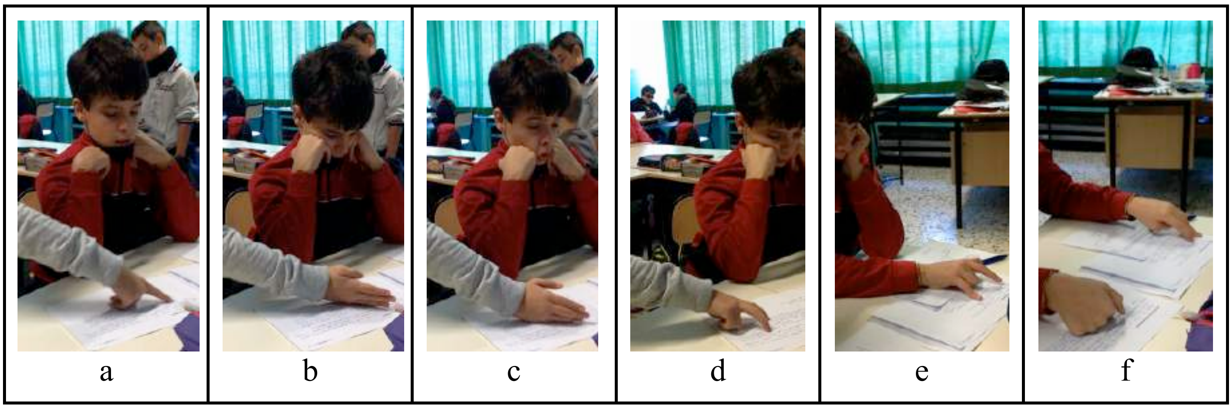

Figure 3. (a-d) Lara's scrolling the virtual middle numbers; (e, f) Francesco's pointing on the diagrams

The researcher has invited the children to draw attention to the structure of the resulting numbers and to their relation with the numbers in the sum, referring first to the numerical diagram. The children notice that the results are always the double of "the number that stays in the middle". We can see this in Lara's utterance, which first actualizes the virtual "half", also marking its positioning, through her scrolling gesture between couples of odd numbers, and then explains this positioning through the miming gesture of being in the middle. The gesture/diagram interplay is apparent from the way in which the numerical diagram gives rise to the two gestures that mobilise the general property of having as result 2 times the "half". The interplay becomes even more evident as soon as the researcher shifts attention to the parametric case of the couple (*PLUTO*, *PLUTO*+2), which is also the most challenging for the children due to its generalised character. In this case, both the algebraic diagram and the numerical diagram become part of this interplay. Francesco quite soon finds that, in the case of *PLUTO* as starting odd number, the half is given by "*PLUTO* plus one".

Much more time is necessary to explain to the researcher how this number relates to the result (*PLUTO*×2)+2. At this point, it is not only that the two diagrams capture gestures through which the mathematics is mobilised. Instead, the gestures (and eyes) also capture the mobility of diagrams, allowing Francesco to go back and forth between numerical examples and algebraic expressions, between the particular and the general, finding useful traces in the numerical to reason on the algebraic. So, Francesco's hand stops between *PLUTO* and *PLUTO*+2 exactly in the same way in which, before, Lara's hand has scrolled the numerical diagram, and has jumped and stopped on the desk to indicate the number in the middle. Francesco alternates his gaze with hand movement between the two diagrams before being with both hands on the desk, one on the numerical diagram and the other one on the algebraic diagram. It is through the comparison that Francesco comes to see the result (*PLUTO*×2)+2 as 2 times *PLUTO*+1. In fact, repeating the previous reasoning on the numerical, he is able to mobilise the fact that the number in the middle, *PLUTO*+1, when multiplied by 2, gives just the result (*PLUTO*×2)+2 (what is essential to understand why it is an even number, and thus to justify the correctness of Martina's statement).

The algebraic diagram gives rise now to the pointing gestures and gazes that match corresponding roles in couples of odd numbers. These gestures and gazes might have participated before in the generation of the diagram itself, and are embedded in it. For example, the little arch with +2, below the sign for the sum of the two odd numbers called *PLUTO* and *PLUTO*+2 (sign that is also present for the other couple (*PLUTO*−2, *PLUTO*), see Fig. 2b), actualizes the distance 2 between the two numbers, making room for the presence of the "half" as *PLUTO*+1 in Francesco's reasoning.

During the collective discussion

The entanglement between diagram and gesture, which we have observed in the case of the written work and in the interaction with the researcher, reappears during the collective discussion when Francesco shares his thinking strategy about Francesca's statement with the others. So, now the point is to justify why the sum of two consecutive odd numbers is a multiple of 4. Francesco comes to the IWB and writes on it the sums that, with Lara, they have considered in the algebraic diagram: (*PLUTO*−2)+(*PLUTO*), (*PLUTO*)+(*PLUTO*+2), and their results: respectively, ((*PLUTO*×2)−2) and ((*PLUTO*×2)+2). At this point, Francesco focuses first on the presence of (*PLUTO*×2) in both the results and claims that Francesca's statement is correct. To explain this, he begins gesturing on the IWB, recalling the arrows that he used in the numerical and algebraic diagrams, and expresses what follows:

Francesco: "We agree that the gap between these two numbers (*Jumps from* (*PLUTO*×2)-2 *to* (*PLUTO*×2)+2) is four. Thus..."

Researcher: "Why do we agree? Wait a moment, ask to the class (*Refers to the class*)"

Francesco: "How much there is between minus two and plus two? (*Points to* −2 *and* +2 *in the two results on the IWB; Fig. 4a*) Between minus two and plus two (*High pitch. Opens his hands vertically in the air in front, to mime a distance between two vertical positions, thinking of* −2 *as staying below and of* +2 *as staying above; Fig. 4b*)"

Children: "Four!"

Francesco: "Four numbers!"

Figure 4. (a) Pointing gestures to outline −2 and +2; (b) Miming the distance between the two results

Drawing attention to *PLUTO*×2, Francesco knows that the solution of the problem is hidden in the −2 and +2. He does not explicitly express this idea with words at the beginning, but we can clearly start perceiving it from his first jumping gesture that evoke the vertical arrow with the "+4" of the algebraic diagram, as well as the many "+4" in the numerical diagram, which he had created with Lara. The request of the researcher pushes him to make explicit to the whole class the reason of this distance 4 between the two results. Pointing to the numbers −2 and +2 in the results and opening his hands vertically in the air in front (Fig. 4), Francesco recalls in a different way two aspects that he has already pointed out: the presence of *PLUTO*×2 does not count at all for understanding "the gap" ("Between minus two and plus two" repeated twice), and the vertical arrow that indicates this gap in the algebraic diagram. Briefly speaking, the arrow signs that before brought forth new structures between numbers in the diagrams are now actualised and made present by Francesco's gestures, which also make tangible the mobility of the reasoning that was introduced through the diagrams. In this manner, the class can capture the gap between the two generic results, of which he is talking, without need for using algebra but just focusing on the numerical difference between the structures of the two numbers. This aspect is very relevant, because it also entails a new temporal dimension, through which looking at how the numbers are written, that is, at their structure, drawing attention to their similarities and differences (so, *PLUTO*×2 is a similarity while −2 and +2 are a difference, in this case). Once again, the gesture/diagram interplay is what comes to constitute the thinking process, and through it the embodied nature of the task is also revealed.

Conclusive remarks

The three episodes that we have analysed above have shown how the gesture/diagram interplay is an essential ingredient of the children's way of thinking about the given mathematical task. The children interlace the diagrammatic with the gestural and vice versa in finding strategies to reason on the correctness of the two statements of the task. The episodes have clearly highlighted what de Freitas and Sinclair (2012) claim, when saying that "[b]y adding a dotted line to the paper, a new dimension can be brought into being; an arrow might forge out new temporal relationships between objects. These excavations enable the virtual and the actual to become coupled anew" (p. 138). We have seen how the numerical and the algebraic diagrams, which Lara and Francesco have produced, have captured new spaces and structures, especially through the use of arrows, in many directions and with many purposes. Unexpected gestures, gazes and movements have emerged by the material surface of the diagrams as part of the mathematical event, actualising virtual folds of relationships between numbers. In the interplay of the diagrammatic and the gestural the encounter of the two children with the mathematics is unfolded, and doing mathematics becomes a dance of agencies that involves the interaction of learners with their material surrounding. The interplay offers mobility to the mathematics and inventiveness to the children in a way in which we can no longer think of diagrams as rigid representations of existing knowledge. Instead, they can be seen as embodied in a completely new manner. So, Lara and Francesco were able to create new resources in their learning space. The diagrams themselves are events, which become resources for the children as *capture technologies*, would say de Freitas and Sinclair. Indeed, they embed gestures that come to the fore when thinking strategies have to be shared within the classroom. In so being, they offer a rethinking of the relationship between immovable mathematics and movable matter and bodies, for which what constitutes learning essentially is engaging with the material world, through ways of talking, moving and feelings that are caused just by the engagement.

REFERENCES

de Freitas, E. & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics, 80*(1–2), 133–152.

de Freitas, E. & Sinclair, N. (2014). *Mathematics and the Body: Material Entanglements in the Classroom*. Cambridge: Cambridge University Press.

de Freitas, E. & Zolkower, B. (2014). Tense and aspect in word problems about motion: diagram, gesture, and the felt experience of time. *Mathematics Education Research Journal*. Online first. DOI: 10.1007/s13394-014-0139-2.

Radford, L. (2014). The progressive development of early embodied algebraic thinking. *Mathematics Education Research Journal, 26*(2), 257-277.

Sfard, A. (2008). *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*. Cambridge, UK: Cambridge University Press.

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Dialogues as an instrument in mathematical reasoning

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Abstract: This article reports on an ongoing project which focuses on mathematical argumentation and proving in form of written dialogues, inspired by the method of imaginary dialogues (Wille, 2011). Tasks are prepared in form of written dialogues between imaginary pupils discussing a mathematical problem, and pupils are invited to write their own dialogues continuing the mathematical discussion. In contrast to the original method that requires pupils to work individually, the pupils in the observed classrooms worked in small groups. In our study we analyze the types of arguments that are used in the imaginary conversations and other texts written by the pupils. We are also interested in identifying obstacles in the proving process. In this article we analyze excerpts of dialogues about an infinite sum of fractions written by pupils from two classrooms in Norway in grades 5 and 6.

Résumée: Cet article présente des premiers résultats d'un projet sur l'argumentation et les preuves en mathématique en forme de dialogues écrits, inspiré par la méthode des dialogues imaginaires (Wille, 2011). Des problèmes sont préparés en forme de dialogues écrits entre deux élèves imaginaires qui débattent une question en mathématique. Les élèves sont invités à continuer le débat en écrivant leurs propres dialogues. À la différence de la méthode originale où les élèves travaillent individuellement, les élèves dans les classes observées ont travaillé en groupe. Dans notre recherche nous analysons les types des arguments qui sont appliqués dans les conversations imaginaires et dans autres textes écrits par les élèves. Nous cherchons aussi à identifier des obstacles dans les processus de preuves. Dans cet article nous analysons des extraits de dialogues sur un problème de sommation infinie de fractions écrits par des élèves dans la cinquième et sixième année de leur scolarisation.

Background

The complex aspects of mathematical reasoning and proving have been widely studied, for example by de Villiers (1990), Hanna (2000), Harel and Sowder (1998). There is evidence of the fact that traditional approaches to teaching proof, based on the logical deductive presentation of proofs in the literature and mainly focusing on students' exposition to given proofs, have not been successful. An overview over research giving such evidence can be found in (Harel & Sowder, 1998). It seems to be difficult for students to move from empirical thinking, as done in everyday life, to logical-deductive thinking as required in understanding and producing mathematical proof. Reasoning outside mathematics is usually not deductive in nature. It is often based on examples which serve as basis for conjecturing and generalization with inductive or abductive methods or the use of analogies. Such forms of reasoning are also part of proving processes, but are usually not visible in the presentation of mathematical proof texts. One of the questions is therefore how to engage pupils in the multi-facet proving activities and how to support them in the transition from these activities to the production of proof texts including deductive reasoning. According to a study by Healy and Hoyles (2000), students preferred a narrative presentation of proof texts, for example due to its explanatory power. Pupils were also more successful in constructing correct proofs when presenting their arguments in words, often supported by examples or diagrams, rather than algebraically.

The method and research questions

This article is based on an analysis of mathematical texts written by pupils from two classrooms in Norway in grades 5 and 6. The tasks were presented to the pupils in form of short written dialogues of two imaginary pupils having a conversation about a mathematical problem. The pupils in the classroom were asked to continue writing the dialogues while they were investigating the mathematical problems. They were also encouraged to use drawings to support their thinking. This is inspired by the method of imaginary dialogues (Wille, 2011 & 2013). In contrast to the original method we did not require the pupils to work individually, but allowed them to work in small groups. The imaginary dialogues, being a fictive conversation between pupils who are talking to each other, not to the teacher, encourage the use of a narrative language that is close to the pupils' oral language. We believe that this will make it easier for the pupils to engage in mathematical reasoning, not being held back by possible problems caused by the obligation to use a more formal mathematical style. We also hope that this will reinforce the functions of explaining and exploring of proving (de Villiers, 1990), rather than only verifying.
In this article we were guided by two research questions:

1) Which types of argumentation can we find in the imaginary conversations?
2) Which types of obstacles emerge in the proving process?

We investigate these questions under the assumption that the pupils' understanding for proving will be reflected by those expressed by the imaginary pupils. In order to classify pupils' arguments we use the classification presented in (Reid & Knipping, 2010) which has four broad categories, *empirical*, *generic*, *symbolic* and *formal* arguments, each with a number of subcategories.

The task given to the pupils

The task analyzed in this article is the pupils' first encounter with mathematical reasoning in form of dialogues. The dialogue which was presented to the pupils (Figure 1) is a slight modification of a dialogue designed and used by Wille (2011). The task asks the pupils to find the limit of the geometric series $\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. In order to avoid improper fractions we started the summation with $\frac{1}{2}$ such that the limit of the considered series was equal to 1.
One of the difficulties with this task consists in the sum having infinitely many summands. The pupils cannot find the limit by applying only arithmetic methods. Some theoretical reasoning and detachment from the arithmetic processes is required. The limit of the series is an abstract object which, applying the theory of Sfard (1991) about the duality of mathematical objects and processes, requires of the learner to reach a state of reification. Therefore we expected that some of the pupils might think that the task could not be solved.
The text suggests both the use of drawings and the calculations of more summands. We therefore expected that pupils might use an inductive approach, producing more examples, in order to construct a hypothesis.

Two pupils are talking to each othe
Two pupils are talking to each other:

Elena:	Imagine that I add $\frac{1}{2}$ and $\frac{1}{4}$.
Janne:	OK. That is $\frac{3}{4}$.
Elena:	Now we add $\frac{1}{8}$ to the result and then $\frac{1}{16}$ to the next one and so forth.
Janne:	You mean something like this?
Janne is writing on a sheet of paper:	
	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$
Elena:	Exactly! That is getting quite big, or not?
Janne thinks...	
Janne:	I am not so sure. I believe it won’t be that big.
Elena:	Perhaps we can draw a picture or calculate the sums?
Janne:	That is a good idea. Maybe we’ll find out how large

Figure 1

First results and expected conclusions

In the following we will discuss the way the pupils use argumentation in their dialogues and discuss possible reasons for obstacles in the proving process when the pupils don’t come to a conclusion in their texts. Most of the dialogues collected in our study follow a typical pattern: in an initial part the protagonists establish a basis of knowledge considered helpful for further investigation followed by a section where the protagonists explore the given problem due to this basis; then the protagonists usually state a hypothesis and explore it; sometimes this is followed by a refined formulation of the hypothesis. Some protagonists come to a conclusion about the sum and establish new knowledge, while others get stuck in the exploration process and don’t know how to end it mathematically. These dialogues usually finish with some social phrases between the protagonists.

Janne:	Perhaps we must first add all the denominators and then the numerators?
Elena:	We can try it.
Janne tries to calculate on a sheet of paper.	
Janne:	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ makes $\frac{4}{30}$.
Elena:	But this cannot be right. If $\frac{1}{2}+\frac{1}{4}$ is $\frac{3}{4}$.
Janne:	Do you remember when we learned to compare fractions?
Elena:	Yes!
Janne:	So one half is as much as $\frac{2}{4}$!

Figure 2

In the excerpt shown in Figure 2 the pupils propose the following basis of knowledge: To add fractions one must add the denominators and the numerators. The protagonists soon discover that this cannot be right and reject this rule by a counterexample. They continue by establishing a new basis of knowledge (that 1/2 is the same as 2/4). By this we see that the pupils have some knowledge about methods in a mathematical argumentation process: the use of counterexamples and the mediation between different examples (rather than theories), which are functioning as generic prototypes, to investigate the correctness of the proposed rule.

Janne:	Fine, then we get: $\frac{7\cdot 2}{8\cdot 2}+\frac{1}{16}=\frac{14}{16}+\frac{1}{16}=\frac{15}{16}$
Elena:	OK. $\frac{15\cdot 2}{16\cdot 2}+\frac{1}{32}=\frac{30}{32}+\frac{1}{32}=\frac{31}{32}$
Janne:	This is a pattern. One takes the denominator times 2, and then the numerator is always one less than the denominator.
Elena:	Yes, you are right. Shall we continue?
Janne:	OK!
Elena:	How far do you think we get, and what do you think the last number will be? Large or small?
Janne:	$\frac{31\cdot 2}{32\cdot 2}+\frac{1}{64}=\frac{62}{64}+\frac{1}{64}=\frac{63}{64}$
Elena:	There will be no answer, this will continue to ∞ and we will not find the answer since this is a pattern.

Figure 3

In the pupils' approaches to the exploration of the problem, we find both purely *empirical* approaches, consisting in the calculation of more iteration steps, and argumentation based on pictorial *generic examples*. It could not be expected that the pupils at this stage of their mathematical education would provide a more formal argumentation.
The dialogue shown in Figure 3, written by three girls, does not go beyond the empirical approach. In this dialogue the protagonists discover a pattern in the calculations: after adding another fraction, the numerator is always one smaller than the denominator, but it seems that the protagonists cannot derive a conclusion from this finding. Instead they decide to continue the computational process of adding more fractions. It seems that the pupils have difficulties to reason abstractly about fractions and don't consider it to be possible to evaluate their quantity when numerator and denominator are not specific numbers. It seems further that the protagonists (and the pupils) have an understanding that the expected answer to a mathematical task is a "last" number reached by calculations, and that new insight or a conclusion derived deductively from a pattern is not a valid answer.

The number this will end with is big, but what is left is very small.
That means that the number gets big, but what is added gets smaller and smaller.
The quadrilateral/pizza doesn't get bigger. But the pieces on it get smaller and smaller:
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$
The number gets big, but we feel that this is wrong. There is 1 missing anyway.

So there is one missing anyway.

So the number gets almost a whole.
There is ONE piece missing before it gets a whole.

Figure 4

The text shown in Figure 4 was written by two boys. They produced a mathematical text consisting of drawings and written text that must be read together to understand the complete argumentation. They use several types of representation and semiotic systems to explain and refine their findings several times. It is not clear which number they are referring to that is getting big: this could refer both to the numbers in the denominators or to the limit. The boys seem to be capable of understanding and solving the problem, but the use of repetitions in the text suggests that they are striving to find an appropriate language to express their thoughts. Finally they make use of a generic visual argument for the fact that the limit is 1.

Summary

Most dialogues start with a section in which the protagonists establish a basis of knowledge. This seems to reflect a phase in the proving process in which the pupils activate knowledge considered useful to solve the task. There is evidence that the pupils have some knowledge about proving techniques like the use of counter examples or empirical methods to generate a hypothesis. The dialogues indicate that pupils who are changing between several types of representations of fractions, like drawings and calculations, are more likely to proceed from an empirical approach to a generic argument and are more likely to arrive at a conclusion. Obstacles in the proving process seem to be rooted in the lack of diversity of the chosen representations, the lack of ability to think about fractions in the abstract and a misconception about the nature of the task or the permitted responses, for example the expectation that the response must be a number resulting from calculation, not a conclusion derived by deductive reasoning.

REFERENCES

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, pp. 17-24.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational studies in mathematics 44*, pp. 5-23.

Harel, G., & Sowder, L. (1998). Students' Proof Schemes: Results from Exploratory Studies. *CBMS Issues in Mathematics Education* (Volume 7), pp. 234-283.

Healy, L., & Hoyles, C. (2000). A Study of Proof Conceptions in Algebra. *Journal for Research in Mathematics Education*, *31*(4), pp. 396-428.

Reid, D. A., & Knipping, C. (2010). *Proof in Mathematics Education.* Rotterdam: Sense Publishers.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics 22*, pp. 1-36.

Wille, A. (2011). Activation of inner mathematical discourses of students about fractions with the help of imaginary dialogues: A case study. *Proceedings of the 35th conference of the International Group for the Psychology of Mathematics Education*, (pp. 337-344).

Wille, A. (2013). Mathematik beim Schreiben denken - Auseinandersetzung mit Mathematik in Form von selbst erdachten Dialogen. I M. e. Rathgeb, *Mathematik im Prozess* (pp. 239-254). Wiesbaden: Springer Fachmedien.

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Networking of theories as resource for classroom activities analysis: the emergence of multimodal semiotic chains

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Abstract: This paper networks two different theoretical frameworks in order to describe teaching-learning processes in the mathematics classroom. Specifically, it considers the paradigm of multimodality (Arzarello et al., 2009), which addresses a wide variety of semiotic resources in the classroom context and analyses them with the Semiotic Bundle tool, and the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008), which focuses on the role that signs play in the transition from situated activities realized using artefacts to culturally shared mathematics. Basing on empirical analysis from a case study in middle school, the role of gestures performed by the teacher during a mathematical discussion is investigated, and the new construct of "multimodal semiotic chain" is introduced.

Résumé: Cette article mises en réseau deux cadres théoriques différents dans le but de décrire les processus d'enseignement-apprentissage dans la classe de mathématiques. Plus précisément, il considère le paradigme de la multimodalité (Arzarello et al., 2009), qui traite d'une grande variété de ressources sémiotiques dans le contexte de la classe et les analyse avec l'outil Sémiotique Bundle, et la théorie de la Médiation Sémiotique (Bartolini Bussi & Mariotti, 2008), qui met l'accent sur le rôle que jouent les signes dans la transition entre les activités situées réalisé en utilisant des artefacts aux mathématiques culturellement partagées. Basant sur une étude de cas en collège, le rôle des gestes à l'enseignant guidant une discussion mathématique sont étudiées, et le nouveau concept de «chaîne sémiotique multimodal» est présenté.

Introduction

As several researchers have remarked, many theoretical approaches have flourished in mathematics education, and in last years reflections on how different theoretical approaches can be used together in the same study has lead to a new branch of research, called 'networking of theories' (Bikner & Prediger, 2014). Within the networking theories approach, researchers seek to combine/compare/contrast/integrate different theories (usually two) in order to focus on a same research problem and to carry out data analysis in order both to have a better understanding of the data, and to investigate and reflect at meta-level on the theories used.

In our study we network two semiotic approaches: the former points to the role of multimodality in mathematics teaching and learning and has lead to the Semiotic Bundle tool of analysis (Arzarello et al., 2009), whereas the latter frames the role of artefacts and their exploitation by the teacher within the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008). As it will be discussed below, the two theories share some principles, and previous research has shown that their networking is a promising route in order to investigate the evolution of semiotic resources in the mathematics classroom processes (Maffei et al., 2009; Maffia & Mariotti, in press).

In this paper, we analyse a middle school mathematical discussion (as conceived by Bartolini Bussi, 1996) on the definition of altitude in triangles. Specifically, we will focus on the role of the teacher in managing the various semiotic resources during the collective construction of the definition. As a result of the data analysis —carried out through the double lens—we will show the emerging of a 'multimodal semiotic chain'.

Multimodality and the Semiotic Bundle tool of analysis

The perspective of multimodality in mathematics education has its roots in the psychological theories that emphasize the crucial role of the body in thinking and in knowledge development; in particular, it is related to the so-called embodied cognition perspective, but assigns also relevance to the social-cultural dimension, drawing on the work of Vygotskij and Vygotskian scholars.

The embodied cognition perspective is a stream in cognitive science that assigns the body a central role in shaping the mind. It has been brought to the fore in mathematics education by the provocative book *Where Mathematics Comes From* by Lakoff and Núñez (2000), and then applied by researchers in several studies (e.g., Nemirovsky 2003; Arzarello & Robutti 2008; Edwards 2009). This view has been supported by neuroscientific results[7], which Gallese and Lakoff (2005) interpret as indicating a *multimodal* character in the brain sensory-motor system These authors point out that "multimodal integration has been found in many different locations in the brain, and we believe that it is the norm. That is, sensory modalities like vision, touch, hearing, and so on are actually integrated with each other *and* with motor control and planning" (*ibid.*, p. 459).

Research in psychology and psycholinguistics have also stressed the role of gestures in communicating and in thinking. McNeill (1992) defines gestures as "the movements of the hands and arms that we see when people talk" (p. 1). This approach comes from the analysis of conversational settings and has been widely adopted in research studies in psychology, in which gestures are viewed as distinct but inherently linked with speech utterances. Research in a number of disciplines (such as psychology, cognitive linguistics, and anthropology) is increasingly showing the importance of gestures not only in communication, but also in cognition (e.g., Goldin-Meadow 2003; McNeill, 2005). Also in mathematics education, gestures have been paid attention by several researchers, interested in cognitive and communicative aspects of mathematics teaching/learning (Arzarello et al., 2009; Edwards, 2009; Nemirovsky, 2003; Radford, 2003; Roth, 2001).

In particular, Arzarello and colleagues (Arzarello, 2006; Arzarello et al., 2009; Sabena et al., 2012), stress that *mathematics teaching and learning have a multimodal character* and involve different perceptuo-sensory-motor activities, including speaking, acting with artefacts, and gesturing. They highlight that the processes of teaching and learning are shaped by resources of different kinds as words (written or spoken), extra-linguistic ways of expression (gestures, gazes, …), written representations (drawings, symbols,…) or tools (from paper and pencil to technological devices).

They can be framed as "signs" within a Vygotskian perspective (Vygotskij, 1931/1978), forming a sort of *semiotic bundle* through which not only communication, but also cognitive processes evolve. The semiotic bundle is a dynamic structure including signs that are produced by students and teachers during the mathematical lessons. Specifically, it consists in:

> a *system of signs* […] that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher. (Arzarello et al. 2009, p. 100)

Differently from other semiotic approaches, the semiotic bundle construct allows us to theoretically

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[7] These results concerns "mirror neurons" and "multimodal neurons": neurons firing when the subject performs an action, when he observes something, as well as when he imagines it (Gallese and Lakoff, 2005). Gallese and Lakoff use the notion of "multimodality" to describe a cognitive model in which there is not any central "brain engine" responsible for sense-making, controlling the different brain areas devoted to different sensorial modalities (which would occur if the brain behaved in a modular manner). Instead, there are multiple modalities that work together in an integrated way, overlapping with each other, such as vision, touch, and hearing, but also motor control and planning.

frame gestures and more generally all the bodily means of expression, as semiotic resources in learning processes. Furthermore, it allows at looking at their relationship with the traditionally studied semiotic systems. Such a notion appears adequate to analyse the multimodal character of learning, with respect to static and dynamic aspects, according to two kinds of analysis:

- *synchronic analysis:* focusing on the relationships between signs of a different nature (for example words and gestures) which are activated simultaneosly;
- *diachronic analysis:* focusing on the evolutions of signs as time goes by; for instance, a gesture may have a genetic function with respect to a certain diagram (Arzarello, 2006).

Through diachronic analysis, in gesture literature the phenomenon of 'catchment' has been identified (McNeill, 2005), to indicate the cases in which some gesture features are recurring in two or more (not necessarily consecutive) gestures. McNeill (*ibid.*) interprets catchments as indicating discourse cohesion, supported by the recurrence of consistent visuospatial imagery in the speaker's thinking. Using the semiotic bundle analysis, Arzarello and Sabena (2014) have pointed out the role of catchment with respect to logical aspects of mathematical arguments.

Synchronic and diachronic analyses have been also applied to focus on the semiotic resources activated by the *teacher* in classroom processes. In particular, 'semiotic games' have been described as those cases in which during teacher-students interaction, the teacher imitates a sign used by one or more students, and accompanies it with another kind of sign, in order to foster meaning evolution (Arzarello et al., 2009). The most typical case of semiotic game is when the teacher repeats a student's gesture, and accompanies it with proper mathematical words. Through a semiotic game, the teacher is not only referring to specific mathematical content, but he is also establishing, in an implicit way, that gestures are acceptable resources in the mathematics classroom.

Theory of Semiotic Mediation

The Theory of Semiotic Mediation (TSM, Bartolini Bussi & Mariotti, 2008) has been developed for modelling the relationship between an artefact that is used in the mathematics classroom to solve a specific task and the mathematics underpinning the artefact itself. The word 'artefact' has to be interpreted, according to Rabardel (1995), as any material or symbolic object designed for a particular goal (f.i. manipulative, software, symbols) and has not to be confused with 'instrument', which is the mixed entity constituted both of the artefact and the utilization schemes developed by a user. Also linguistic expressions can be considered as artefacts. Maffei and Mariotti (2011) define as *discursive artefacts* those sentences which have the potentialities to lead to the evolution of signs. This definition designate also what Sfard (2001) calls 'templates', referring to already known and used sentences in which a new word is inserted.

The main object of analysis of TSM is the evolution of signs along time, from contingent to culturally determined ones, according to a kind of diachronic analysis as defined in the previous section. This transition can be described as an example of passage from spontaneous concepts to scientific ones (Vygotskij, 1987).

Specifically, according to TSM, each artefact used in the mathematics classroom can produce signs that can be conceived as *mathematical signs*. But, when the student uses the artefact to accomplish a task, she can be unaware of the mathematics that is "inside" the object and then the signs that she produces are contingent to the solution of a particular task with that particular artefact: they are called *artefact signs*. The set of relationships between the artefact and the task together with those between the artefact and the related mathematics constitutes the *semiotic potential* of the artefact itself. As the signs produced through the activity with the artefact can be interpreted differently by the student and the mathematics expert,

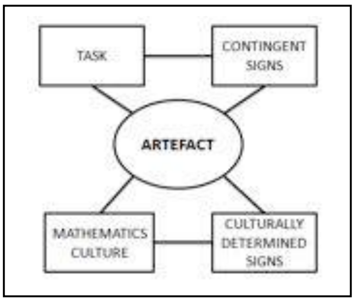

Fig.1. Artefact's polysemy

we can say that the artefact is polysemic (Fig.1).

The substantial difference between artefact signs and mathematical ones is in the context to which they refer: artefact signs refer to the activity realized with the artefact itself, they may be related to what Radford (2003) calls *contextual generalization* or to what Sfard (2008) defines as *routine-driven* use. Let's give an example: if a student uses the word 'circumference' referring to all the closed lines that are produced using a compass, she is using a sign that appears as a mathematical one but that can be far away from the culturally shared meaning that is the set of points with the same distance from the centre.

According to TSM a crucial educational aim is to promote "the evolution of signs expressing the relationship between the artefact and tasks into signs expressing the relationship between artefact and knowledge" (Bartolini Bussi & Mariotti, 2008, p.753). A central role is played by the teacher. As the teacher has the awareness of the semiotic potential of the artefact, she can guide a *mathematical discussion* with the aim of prompting such evolution of signs. A mathematical discussion orchestrated by the teacher is intended as a "polyphony of articulated voices on a mathematical object that is one of the motives for the teaching-learning activity" (Bartolini Bussi, 1996, p. 6).

During the process of semiotic mediation, the teacher uses many different signs, specifically to turn artefact signs into mathematical ones. All the signs that are used with this objective are defined as *pivot signs*. An example of pivot signs can be the hybridization of a word or sentence belonging to the artefact domain (as reference to parts of a manipulative, a command in a software, …) with other words coming from the mathematical culture. The set of artefact and mathematical signs, together with the pivot signs that are used to relate them, is called *semiotic chain*. Such a construct has been introduced in literature by Walkerdine (e.g. 1990) and applied to mathematics education by different authors. For example Presmeg (2006) defines this process of chaining as a sequence of abstractions that is created preserving the relationships to everyday practices of the students, and Hall (2000) stresses how the teacher can create chains to develop mathematical concepts drawing on everyday situation. The role of the teacher as an expert who works in the zone of proximal development (Vygotskij, 1931/1978) of her pupils is central in TSM because she is conceived as the only agent in the classroom who can guide the process of semiotic mediation through the construction of semiotic chains.

Research question and methodology

In this paper, we focus our attention on the students-teacher interaction during mathematical discussions, in order to answer the following research question:

How does the teacher use different semiotic resources (gestures in particular) in the construction of the semiotic chain?

Data come from videos of two lessons in grade 6 aimed to define the height of the triangle. The teacher and the students knew that there was a camera in the classroom for research purpose, which was no more specified. Information about the teacher's intentions or about students' productions were collected through an informal preliminary interview and a follow-up interview to the teacher.

Data analysis is carried out at a micro-analytical level, and combines tools from the two considered theories. Specifically, classroom discourses have been transcribed and coded looking at the evolution of signs both from the point of view of the context in which they are used (artefact, mathematical or pivot signs, according to the TSM) and the modality in which they are expressed (written symbols, words, gestures, using the Semiotic Bundle). Speech transcripts have been enriched with gestures images looking at the co-timing of words and gestures (or other semiotic resources). The resulting 'multimodal transcript' has then been analysed in detail, in order to seek for the elements that could provide the fabric of semiotic chains related to the meaning of height of a triangle. In the following section we summarize the classroom discussion with a selection of excerpts from the transcript, useful to show the results of the analysis (students' name have been changed, to preserve anonymity).

The classroom discussion

The discussion is aimed to introduce the definition of altitude in a triangle, drawing on the usage of the word '*altezza'* (altitude, height) in Italian colloquial language. Italian students usually are introduced to intuitive definitions of basic plane geometry concepts in primary school. Grade 6 is the first year of middle school, in which those concepts are re-thought in a more formal way.

Students are acquainted with *parallel* and *perpendicular* lines on the plane, and were also introduced to the definition of *distance* between point and line, as the segment line perpendicular to the line. Distance was also related to the idea of minimal path from a point to a line.

The emergence of a discursive artefact

The discussion begins with a brief brainstorming on the possible use of the term 'altezza' in ordinary language (see Fig. 2). The Italian word 'altezza' can be translated both as altitude and height. Figure 2 reports the different interpretations and associations that students relate to the words 'altezza' and 'alto' (high).

Among these proposed expressions, the teacher asks to select those that are more related to the geometrical context:

1 T: Now I ask you this: take your notebook and, from all these meanings that we have seen, from all these meanings, from all this expressions with the word 'high' that I wrote on the blackboard…can you write a list of those in which the word 'high' is used in a way that is similar to how we use it in geometry? […] Then you are going to tell me which one I have to underline and I delete the others.

Students write down their selection. Then they read their answers aloud, and the teacher underlines the chosen words at the blackboard (see Fig. 2):

Distance between feet and ground	High water
A person's tallness	Deep sea[2]
High temperature/pressure	High speed
High technology	High probability
Altitude: distance from sea level	High mark
High altitude	Triangle altitude
High density	High culture

Fig. 2. The classroom blackboard with the selected associations to the word 'altezza'.

2 Destiny: Distance between feet and ground.
3 T: Distance between feet and ground [*he underlines this expression at the blackboard*] Ok.
4 Sonia: Distance
5 T: Distance…?
6 Sonia: From sea level […]
7 T: Do we underline altitude too?
8 All together: Yes!
9 T: Distance between a point and sea level. So altitude [*Marco rises his hand*] Marika?
10 Marika: Deep sea.
11 T: Deep sea?
12 Marika: Because it is again a distance.
13 T: [*Fulvio rises his hand*] Fulvio?
14 Fulvio: High-water.
15 T: High-water. Why high-water?
16 Fulvio: It is the distance between the ground and how it is high…
17 Federico: Between the sea bed and the sea.

From this brief excerpt, we can notice that there is a kind of template sentence, which is repeated many times. It is "*the distance between ... and ...*". The teacher invites the students to identify it:

18 T: I am happy that you found so many many different ideas that can be used to express the idea of altitude. They all have something in common…

19 Federico: The distance. A person's tallness could be the distance between feet and head. Yes, the height (altezza)

A couple of months before, students were introduced to the definition of 'distance' between a point and a line, as the segment line that is perpendicular to the line and with one end-point in the point and the other one on the line. According to this definition, the altitude can be conceived as *the distance between a vertex and the opposite edge (the base)*. For this reason, the template "the distance between … and …" appears as a *discursive artefact* with the potential of leading to a mathematically consistent definition of altitude.

The appearance of gestures and their role in pivot signs

In order to reflect on the height of real objects, the teacher asks about the height of a mountain:

20 T: Can the height of a mountain be considered as the distance between something and something else? How can the height of a mountain be explained using the word 'distance'? [*The students are silent*] If it is true what some of you have said, that the word 'distance' has something to do with the word 'height'…

Fig. 3. Christian's gesture.

21 Christian: The height?

22 T: Using the word 'distance', how can you explain the idea of the height of a mountain?

23 Christian: The distance between… the top and…[*gesture pointing downwards, Fig. 3*]

24 T: Between the top [*pinching gesture as in Fig.4a*] Christian says…

25 Federico: And the base!

26 T: And the base [*gesture with open hand, palm upwards as in Fig.4b*]. So also in the height of a mountain, we can think at the highest point [*pinching gesture as in Fig. 4c*], like our height, the highest point on our head [*pinching gesture as in Fig. 4d*] and our base [*gesture with open hand, palm upwards as in Fig. 4e*] that are the feet. The height of the mountain [*pointing gesture as Fig 4f*] is the distance from the highest point of the mountain, with respect to the base of the mountain [*gesture as Fig 4g*], the ground [*gesture with open hand, palm downwards as Fig 4h*] where it rests. Now, if we think in this way at the height, the geometrical height, let's take a pencil and a ruler and I will let you apply this idea of height as distance, to the height of buildings. So, in a building, Alessandra, can you define the height using the word 'distance'?

Fig. 4a-h: The "highest point gesture" and the "base gesture".

27 Alessandra: Yes.

28 T: How? In a building, the height is…

29 Alessandra: The distance between the highest point and the ground.

30 T: The distance between the highest point and another point that is on the ground [*while talking he repeats both the types of gestures as shown in Figg. 4f and 4h*].

The teacher projects now the images of three real building and a fictitious one, one by one (Fig. 5), and asks the students to draw their altitudes. In such a task, students are expected to draw a vertical line that connects the highest point of the (image of a) building to the ground, that is *the distance between the highest point and the ground*. As the ground is represented as horizontal, the drawn altitude will be *perpendicular* to the ground itself.

Fig. 5. The buildings projected on the multimedia whiteboard.

Students draw the shape of the building on their notebooks and then represent the height through line segments. While they are drawing, a student asks if the height of a building is bigger if there are underground floors. A discussion about this topic takes place. Some students think that it would be better to consider the distance between the highest point and the underground while the teacher suggests that, in many cases, the street-level is considered as a reference.

It is interesting to notice that while saying 'highest point' the teacher and the pupils always point to a upper position, over the head or upon the head: we refer to such a gesture as the "*highest point gesture*". Furthermore, the words 'ground' and 'underground' are always co-timed with a horizontal movement of the open hand on a lower level: we call it the "*base gesture*". For example, while talking about the height of buildings, Marika says:

31 Marika: The distance between the highest point [*gesture in Fig.6b*] and… [*gesture in Fig.6e*]

The girl is completing her sentence with a gesture (Fig. 6b) instead of words. Figure 6 reports some other examples of gestures from the teacher and from another student as well.

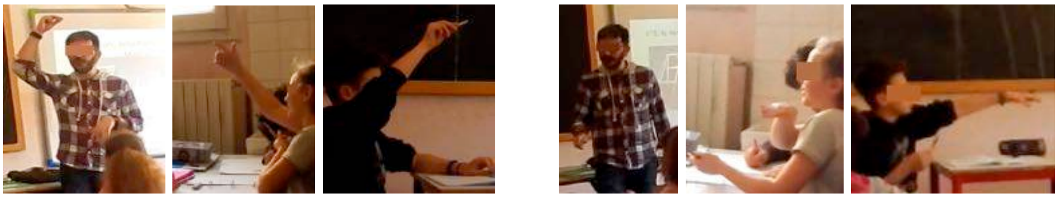

Fig. 6. Gesture catchment: Teacher and students repeating many times the "highest point gesture" and the "base gesture" during the discussion.

The repetition of similar gestures during the discussion is an example of *catchment*: this phenomenon is usually interpreted as providing structural cohesion to the co-occurring discussion (McNeill, 2005; Arzarello & Sabena, 2014). In our case, referring to TSM, such gestures can also be interpreted as carrying the germs of the expected *mathematical signs*. In fact, the highest point gesture is indicating an imaginary *point* while the base gesture consists of a linear movement of the open hand, as to represent a *line* or a *plane*. Teacher's hands anticipate the figural/graphical aspects of the geometrical terms that are crucial in the chosen definition of altitude. It seems here that the teacher, taking into account this didactical goal, is using this gestures as pivot signs to start a bridge between the expressions created to describe the heights of real objects, and the expected definition of altitude in a triangle. We can notice that the words 'point' and 'base', in this moment, refer to the to the "highest point" (line 26) and to the base of the body or of a mountain. They seems to be conceived as parts of real objects: in this sense they are still not mathematical signs but they have the potential to develop a semiotic chain. As the described gestures and these words appear often as co-timed, we can affirm that is the gesture-word couple to be used by the teacher as *pivot sign*.

Towards mathematical sign

At the end of the individual work, different solutions are shown at the blackboard. In particular it is noticed that sometimes the altitude can be external to the figure. Looking at students' productions, the teacher realizes that some pupils have drawn in a wrong way the altitude of the fictitious building (Fig.5d). The teacher draws Figure 5d on the blackboard and asks to Ester to draw the altitude. She draws a vertical line from the highest point downwards. Then the teacher asks:

32 T: How does it have to reach the ground? Rosi?

33 Rosi: Vertically.
34 T: Vertically! Why? How is the ground?
35 Many voices: Horizontal!
36 Destiny: It is flat.
37 T: It is flat, let's imagine it is flat and horizontal [*he does the "base gesture" with both his hands, Fig.7*] so in this case we have to reach it vertically.
38 Federico: Teacher, we have to reach it perpendicularly.
39 T: Wait a minute Federico.


Fig. 7. "Base gesture" with both hands

The teacher stops Federico. He prefers not to go directly toward the mathematical sign suggested by this student, because the lesson is going to finish in few minutes.

The discussion keeps going on in a second lesson. It starts from a new task, which consists in drawing the height of a boat while it is sailing a rough sea. Using the discursive artefact, the height of a boat is *the distance from the highest point and the boat's base* (that is the hull). The difference between this task and the building's one is in the position of the base, which is no more horizontal (as it was in the buildings in the previous lesson). Hence, the use of the discursive artefact within this task has the potential to trigger the discussion about the fact that the altitude is perpendicular to the base and not always vertical. Starting from different students' productions (Fig. 8), the teacher opens this discussion, which continues until pupils agree about the perpendicularity between altitude and base.

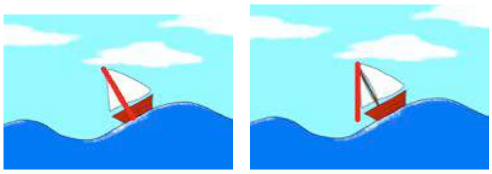
Fig. 8. Reproduction of the different students' productions discussed during the lesson.


Fig. 9a-b-c: Teacher's gestures during the second discussion.

40 T: So, this is one critical point, when we have to think about altitudes of objects without horizontal ground [*gesture in Fig. 9a*]. Because when the ground is horizontal there are no problems, you do the vertical line [*he draws an imaginary vertical line in the air, Fig. 9b*], you go downward [*he repeats the same gesture*] inside or outside the figure. But when the ground on which the figures lay is no more [*gesture in Fig. 9c*] horizontal, at that point we risk to make a wrong altitude. So, what can we write as a reminder to remember about this difficult point?
41 Marika: The perpendicular
42 T: So, I begin the sentence, then we see how we can conclude it. Let's keep in mind that the laying plane of an object can be… [*he begins to write it on the blackboard*]
43 Marika & Eleonora: atilt!
44 T: Atilt [*he keeps writing*]. In these cases, to draw the altitude of an object… Think about how you could complete this sentence to help your peers to avoid a future error. In these cases, to draw the altitude, we have to keep attention to…what? […]
45 Federico: The altitude has to be perpendicular [*he moves his hand vertically, Fig. 10a*] to the base [*"base gesture", Fig. 10b*]

The teacher insists on stressing the difference between vertical and perpendicular, but Federico interrupts him:

46 Federico: Teacher, but…what if […] there is a triangle which has the altitude outside [*he draws a vertical line in the air, as in Fig. 10a*] the laying base [*"base gesture" with one hand*] and then we have to draw the altitude in respect to the base [*"base gesture"*] of the triangle…to the laying base of the plane [*"base gesture" with two hands, Fig. 10c*] when the plane is waving. Where do we…

Fig. 10a-b-c: Federico's gestures speaking about the external altitude.

It is interesting to notice that, again, while formulating his question, Federico repeats the teacher's gestures co-timed with the same words (line 46): also gestures, besides words, are *shared*.

Then, the teacher introduces a new artefact that is already known by students: a Dynamic Geometry Software (DGS). He opens a file in which a acute-angled triangle, the line containing one of its base and the corresponding altitude (in red) are drawn. The teacher moves the vertex from which the altitude begins (point E in Fig. 11) and transforms the triangle with an obtuse angle (Fig. 11). He asks to pupils to describe this case and they say that the altitude is still perpendicular to the base but outside it.

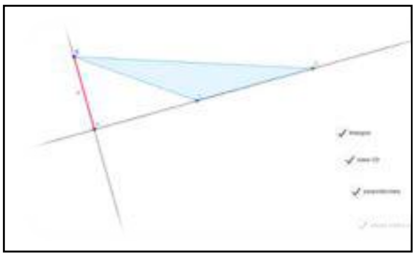

Fig. 11. The triangle in the DGS.

47 T: Pay attention! This is another very difficult point. It arrives perpendicular, and you say that it arrives perpendicular to the base. But, if I take away a thing from this drawing [*he makes invisible the line which the base belongs to*] Where does this altitude arrive to?

48 Federico: Perpendicular…to the extension of the altitude…ehm, of the base!

49 T: Ok. The altitude does not reach the base [*he points downward many times*] Why does it end outside the base in this case? Does it still arrive perpendicularly?

50 Mario: Of course!

51 T: To what?

52 Federico: To the base extension.

53 T: To the laying plane of the base [*he extends the gesture in Fig.9a opening his harms, Fig. 12*], so the extension of the base [*he repeats the gesture*]. So, in some case, the altitude starts from this famous highest point [*"highest point gesture"*] and it does not touch the base. Is it still perpendicular?

54 Some voices: Yes.

55 T: But…

56 Federico: It is external.

57 T: It is external and it does not touch the base [*"base gesture"*], it touches the extension [*he repeats the extension of the "base gesture"*] of the base. Let's write this.

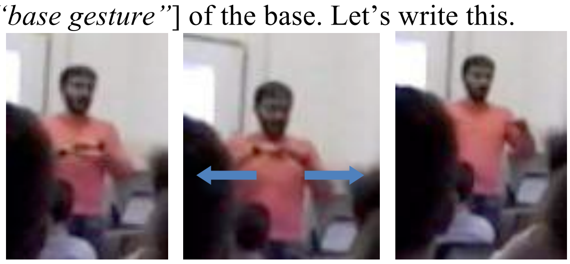

Fig. 12. Extension of the "base gesture" which correspond to the extension of the base.

As it can be noticed, the change in words (from 'base' to 'base extension') results in a gesture change (from gesture as in Fig. 9 to gesture as in Fig. 12), and in particular from a static to a dynamic gesture. The shared definition of altitude is evolving again. The rigorous mathematical definition will require another lesson, drawing on a task in which a the command "perpendicular" of the DGS will serve as an artefact to draw the *three* altitudes of a triangle.

Results and conclusion

At the beginning of the discussion, the teacher starts from the verbal expressions produced by the students to refer to the meaning of the word 'altezza' (altitude/height), and focuses their attention on the specific discursive artefact "*the distance between…and…*" (lines 18-30). The linguistic component of the semiotic bundle soon enriches with the gestural one. The first gesture is produced by Christian (line 24), in the attempt of completing the sentence asked by the teacher: the former part (referring to the top) is mentioned with the verbal resource, while the latter one (referring to the base) is expressed with the gesture of pointing downwards (Fig. 3).

Even if the teacher is not looking at the student, he is maybe influenced by the use of the gesture resource, because we can see that his following utterance (line 26) is accompanied by two kinds of gestures, and specifically:

- the "highest point gesture", i.e. a gesture pointing in a upper position, over his head or materially upon his head (Fig. 4a-c-d-f), when verbally referring to the "highest point" of a mountain or of a person's body, and
- the "base gesture", i.e. a gesture pointing to a lower position, below his foot (Fig. 4e-b-g).

These two gestures, and specifically the two *spatial locations* of the gestures are repeated in the entire discussion, showing a catchment that provides cohesion to the discourse, while it is slowly evolving towards the scientific meaning of height of a geometric figure. At the beginning the gestures mediate specific aspects of the objects at stake (being them a mountain, a human body or a triangle) by means of iconic features, later they will be associated to the written representation of geometric figures: it is through this iconic features kept in the catchment that the figural aspects of the triangle are called into the scene in continuity with the reference to objects from everyday life or the human body. Through the figural aspects, the geometric properties of the figure are then brought to the fore and discussed.

With a detailed diachronic analysis, we can see that, although the spatial locations of the gestures are kept during the entire discussion, the base gesture evolves: at first it is made with the palm upwards (Fig. 4e) and it is co-timed with the word "base" referred to the human body (line 26: "our base that are the feet"): we symbolize it with $\cup$. Then, it is made with the palm downwards (Fig. 4h), referred to the base of a mountain or buildings: we represent with $\cap$ this gesture. Later the gesture is made with two hands, with reference to the "flat and horizontal ground" (line 37, symbolically $\cap\cap$). After introducing the activity about the boat, the same gesture with two hands is repeated in a slanted inclination (Fig. 9c). Finally, the gesture with the two hands becomes a gesture made at the central part of the gesture space (just below the shoulders), with the two hands opening to stress the extension of the base of the triangle (Fig. 11, symbolized as $\overleftarrow{\cap}\overrightarrow{\cap}$). In the following table we sum up the diachronic analysis, correlating it with the synchronic analysis of words and gestures starting from the discursive artefact "*the distance between…and…*". Pointing gestures are represented with arrows according to the direction pointed at.

		Synchronically
Diachronically	24	Christian: The distance between **the top** and ↓
	26	T: [..] the distance from **the highest point of the mountain**, with respect to **the base of the mountain** $\cup$
	30	T: The distance between **the highest point** ↑and **another point that is on the ground** $\cap$
	31	Marika: The distance between **the highest point** ↑and $\cap$
	52-56	T: the altitude starts from **this famous highest point** ↑ […] It is external and it does not touch **the base** $\cap\cap$, it touches **the extension of the base** $\overleftarrow{\cap}\overrightarrow{\cap}$

Table 1. Schematic representation of the multimodal semiotic chain.

Through the detailed analysis with the Semiotic Bundle we are therefore able to describe the evolution of the signs used by the teacher to guide the discussion from the set of everyday meanings that the students associate to the word "height" (the starting point of the discussion) to the scientific concept (Vygotskij, 1987) that is chosen as goal for the lessons, namely height of a triangle as distance from a vertex to the opposite side, which is called base. This evolution forms a chain of multimodal signs that is schematically illustrated in Table 1. Analysing within a multimodal perspective the evolution of the discursive sign during the discussion, we have been able to identify a semiotic chain including different kinds of semiotic resources: we call it "*multimodal semiotic chain*".

Although we schematically illustrated it in a linear way (Tab. 1), we are not claiming that signs evolution occurs linearly in the classroom activities, nor in the meaning evolution for the students. An extensive analysis of these lessons and of the following ones would show that, on the opposite, the route for mathematical meaning productions in students is complex and not reducible to any linear path. However, the metaphor of the chain underlines that the various parts of the evolution are connected each other, and the analysis in multimodal sense may shed light on how they are connected. In our study, the initial artefact has a discursive nature, and the main semiotic resources are words and gestures (being the reference to written productions on the background). In this context, we showed that gestures may provide a glue to link the various signs within the semiotic chain. This result not only is coherent with previous results in gesture studies (see those on catchment in McNeill, 2005), but also specifies a specific function that gestures may have in the field of mathematics education.

Prompted by this result, many new questions arise. We point out three questions that we feel as most urgent. The first one is about the influence of the semiotic resources of the chain *on students*: this theme is only touched but not faced in this paper. The second one, of more theoretical nature, is about the nature of the semiotic chains when the starting point is a concrete artefact (as a manipulative or a software), as it is more typical in TSM's applications: can we still speak of multimodal semiotic chains? And what is the role of embodied resources therein? The third issue concerns the intentional use of multimodal semiotic chains by teachers. Future research is needed to answer these and possibly further emerging questions.

REFERENCES

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Latino americana de Investigación en Matemática Educativa*, Vol. Especial, 267-299.

Arzarello, F. & Robutti, O. (2008). Framing the embodied mind approach within a multimodal paradigm. In L. English (Ed.), *Handbook of international research in mathematics education,* 2nd revised edition. (pp. 716-745). Mahwah, NJ: Lawrence Erlbaum Associates.

Arzarello, F. & Sabena, C. (2014). Analytic-Structural Functions of Gestures in Mathematical Argumentation Processes. In L.D. Edwards, F. Ferrara & D. Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment* (pp. 75-103). Charlotte, NC (US): Information Age Publishing, Inc.

Arzarello, F., Paola, D. Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97-109.

Arzarello, F., Bazzini, L., Ferrara, F., Robutti, O., Sabena, C., & Villa, B. (2006). Will Penelope choose another bridegroom? Looking for an answer through signs. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proc. PME 30* (vol. 2, pp 73-80). Prague, CZ:

Charles University, PME.

Bartolini Bussi M.G. (1996). Mathematical discussion and perspective drawing in primary school. *Educational Studies in Mathematics*, 31, 11-41.

Bartolini Bussi, M.G., & Mariotti, M.A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English (Ed.), *Handbook of International Research in Mathematics Education (second edition).* Routledge.

Bikner-Ahsbahs A. & Prediger, S (Eds.), *Networking of Theories as a Research Practice in Mathematics Education*. ZDM-Series Advances in Mathematics Education. New York: Springer.

Edwards, L. (2009). Gestures and conceptual integration in mathematical talks. *Educational Studies in Mathematics*, *70*(2), 127–141.

Gallese, V., & Lakoff, G. (2005). The brain's concepts: the role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology*, *21*, 1–25.

Goldin-Meadow, S. (2003). *Hearing gestures: How our hands help us think.* Cambridge, MA: Harvard University Press.

Hall, M. (2000). *Bridging the gap between everyday and classroom mathematics: An investigation of two teachers' intentional use of semiotic chains*. Unpublished doctoral dissertation, The Florida State University.

Lakoff , G., & Nùñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago, IL: University of Chicago Press.

Maffia, A., & Mariotti M.A. (in press). Introduction to arithmetical expressions: a semiotic perspective, *Proc. CERME 9*, Prague, CZ: ERME.

Maffei, L., & Mariotti M.A. (2011)**.** The role of discursive artefacts in making the structure of an algebraic expression emerge. In M. Pytlak, E. Swoboda, & T. Rowland (Eds). *Proc. CERME 7* (pp.511-520). Rzeszòw, PL: ERME.

Maffei, L., Sabena, C. & Mariotti, M.A. (2009). Exploiting the feedback of the Aplusix CAS to mediate the equivalence between algebraic expressions. In M. Tzekaki, M. Kaldrimidou & C. Sakonidis (Eds.). *Proc.s PME 33* (vol. 4, pp. 65-72). Thessaloniki, GR: PME.

McNeill, D. (2005). *Gesture and thought*. Chicago, IL: University of Chicago Press.

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In N. A. Pateman, B. J. Dougherty, & J. T. Zillox (Eds.), *Proc. PME 27* (vol. 1, pp. 105–109). Honolulu, HI: PME.

Presmeg, N. (2006). Semiotics and the "connections" standard: signifi cance of semiotics for teachers of mathematics. *Educational Studies in Mathematics*, *61*, 163–182.

Rabardel, P. (1995). Les hommes et les technologies - Approche cognitive des instruments contemporains. Paris: A. Colin.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning*, *51*(1), 37–70.

Roth, W. M. (2001). Gestures: Their role in teaching and learning. *Review of Educational Research*, 71(3), 365-392.

Sabena, C., Robutti, O., Ferrara, F., & Arzarello, F. (2012). The development of a semiotic frame to analyse teaching and learning processes: examples in pre- and post-algebraic contexts. In L. Coulange, J. P. Drouhard, J. L. Dorier, & A. Robert (Eds.), *Recherches en Didactique des Mathématiques, Numéro spécial hors-série, Enseignement de l'algèbre élémentaire: bilan et perspectives* (pp. 231–245). Grenoble: La Pensée Sauvage.

Sfard, A. (2001). Learning mathematics as developing a discourse. In R. Speiser, C. Maher, & C. Walter (Eds.), *Proc. of PME-NA 2* (pp.23-44). Columbus, Ohio: PME-NA.

Sfard, A. (2008). *Thinking as communicating: human development, the growth of discourses and mathematizing*. Cambridge University Press.

Vygotskij, L. S. (1931/1978). *Mind in society. The development of higher psychological processes.* Edited by M. Cole, V. John-Steiner, S. Scribner, & E. Souberman. Cambridge, MA, and London: Harvard University Press.

Vygotskij, L. S. (1987). Thinking and speech. In R. W. Rieber, & A. C. Carton (Eds.), *The collected works of L. S. Vygotskij*. New York: Plenum Press.

Walkerdine, V. (1990). *The mastery of reason*. New York: Routledge.

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One task, five stories: comparing teaching sequences in lower secondary school

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Abstract: This contribution comes from the experience of the "Language and argumentation" project, carried out since 2008 by the Mathematics Department of the University of Genoa. The project is aimed at designing, experimenting and refining task sequences for a smooth and meaningful approach to argumentation and proof in lower secondary school. Within the project, teachers collaborate in designing task sequences and carry out them through cycles of experiment and refinement. In this contribution we compare the development of the same task sequence in five classes with five teachers, drawing some preliminary conclusions in terms of process understanding, task design and teacher professional development.

Résumé: Cette contribution se situe dans le contexte du projet "Langage et argumentation", mené depuis 2008 par le Département de Mathématiques de l'Université de Gênes. Le projet vise à concevoir, expérimenter et affiner séquences de tâches pour une approche significative à l'argumentation et à la preuve dans l'enseignement secondaire inférieur. Dans le projet, les enseignants collaborent à la conception des séquences de tâches et réalisent des cycles d'expérimentation et de raffinement. Dans cette contribution, nous comparons le développement de la même séquence de tâche dans cinq classes avec cinq enseignants, en proposant quelques conclusions préliminaires en termes d'analyse des processus, conception des tâches et développement professionnel des enseignants.

Introduction

This contribution is set within the "**Language and argumentation**" Project (lower secondary school strand), carried out by the Mathematics Department of the University of Genoa since 2008. The aim of the project is to design, experiment and refine task sequences for a smooth and meaningful approach to argumentation and proof in lower secondary school. A key principle of the Project is the strong collaboration between researchers and teachers in all the phases, from the design to the analysis of the task sequences (Morselli, 2013). The lower secondary school team is currently made up of 6 members: the author FM, a researcher in mathematics education, the author MT, an experiences teacher and teacher educator, two teachers with at least 10 years of teaching experience (EZ, with a university degree in mathematics, and EQ, with a university degree in chemistry), two teachers with less than 10 years of teaching experience (EP and OR, with a degree in biology).

A usual way of working is doing cycles of experimentations, where each teacher carries out her own teaching sequences and shares with the colleagues the results of the experiment. The author FM usually acts as an observer during the class sessions. In the experimental phase, the degree of variability left to each teacher is quite high, provided that his/her choices are discussed a priori or analysed a posteriori by the whole team. The discussion of the results in each class is a promising moment of reflection on students' processes and may lead to an improvement of the task sequence as well as to an occasion for teachers' professional development. We may say that the "Language and argumentation" project had also the final aim of fostering the professional growth of a new generation of teachers-researchers. Indeed, during the project the teachers did not only implement in their classes the innovative task sequences, rather they were involved in theoretical reflection and a posteriori analysis, thus becoming a community of inquiry (Jaworski, 2006).

In this paper we focus on the potential of **sequence comparison**. A task sequence was carried out by five teachers and a cross analysis of the results was performed by the team of teachers and researchers. We analyse the different sequences that took place and draw some conclusion in terms of task design, process understanding and professional development.

Theoretical background

Concerning the approach to proof in lower secondary school, two educational goals are to be attained: from one side, fostering the development of argumentative and linguistic competences (thus, seeing argumentation as strictly linked to proof, see Durand-Guerrier et al., 2012), from the other side, promoting the first encounter with mathematical proof. Problem solving activities, where students are asked to explain their solving process and justify their choice, are seen as important activities so as to foster the development of an argumentative attitude in the classroom. A smooth and meaningful approach to proof requires the students' progressive acquisition of basic content knowledge, but also the ability to manage (from a logical and linguistic point of view) the reasoning steps and their enchaining (modes of argumentation) and the ability to communicate the arguments in an understandable way. This is also in line with the idea that learning proof is approaching a form of rationality, as expressed by Morselli & Boero (2009), who proposed an adaptation of Habermas' construct of rationality to the special case of proving, showing that the discursive practice of proving may be seen as made up of three interrelated components:

> "- an epistemic aspect, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning […];
>
> - a teleological aspect, inherent in the problem solving character of proving, and the conscious choices to be made in order to obtain the aimed product;
>
> - a communicative aspect: the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture". (Morselli & Boero, 2009, p. 100)

The theoretical construct of rationality is a reference for the Project. Two types of argumentation are fostered in the task sequences: argumentation at content level, as a part of the solving and proving process, and argumentation at meta-level, as a means for fostering reflection on the practices of mathematical problem solving and proof related to the three components of rationality. Didactical methodologies such as group work and mathematical discussions (Bartolini Bussi, 1996) are widely used.

Research questions

As outlined by Morselli (2013), the cycles of design, experimentation, analysis and refinement characterize the teamwork. The team analysis may lead to a change in the task formulation or in the sequencing of the tasks. Furthermore, additional tasks, especially tasks fostering reflection, may be inserted. Each teacher may suggest modifications to the task sequence. Finally, any task sequence cannot be completely set up a priori, because it depends on the students' processes and products. Some tasks are "open" and must be set up during the experimentation. The teacher, with the cooperation of the team, must be able to evaluate *"on the spot"* the emergence of issues to be deepened, and to analyse students' products and promote students' own reflection on productions. These features raise two crucial issues: 1) providing the team with theoretical tools to analyse the processes; 2) finding a good balance between fixed goals and flexibility. To explore these issues, we performed a cross analysis of what happened in the five classes, with the five different teachers, starting from the same task.
The cross analysis is guided by the following questions: *A) Are there differences within and across classes in terms of students' individual argumentations? Is it possible to use rationality as an analytical tool and as a guide for planning the classroom discussion? B) Did the mathematical*

discussions develop in different ways? Which dimensions of rationality emerged during the discussions? Was there some occasion for argumentation at meta level?

Method

The task sequence we deal with was inspired by a formative assessment unit of the MARS project (http://map.mathshell.org/materials/lessons.php). The task sequence was adapted to the aim and principles of the "Language and argumentation" project. Namely, moments for mathematical discussion were inserted, as well as specific moments of reflection and comparison between individual processes and between products. The mathematical content at issue was ratio as a way of comparing quantities. The long-term goal of the task sequence was a first approach to proportional reasoning. The common starting point was the following first task:

> *Guglielmo loves organizing parties with his friends. When he and his friends get together, Guglielmo makes a fizzy orange drink by mixing orange juice with soda. On Friday, Guglielmo makes 7 liters of fizzy orange by mixing 3 liters of orange juice with 4 liters of soda. On Saturday, Guglielmo makes 9 liters of fizzy orange by mixing 4 liters of orange juice with 5 liters of soda.*
> *Does the fizzy orange on Saturday taste the same as Friday's fizzy orange, or different?*
> *If you think it tastes the same, explain how you can tell.*
> *If you think it tastes different, does it taste more or less orangey? Explain how you know.*

Totally, the sequence was implemented in three classes of grade 7 (grade 2 of lower secondary school, in the italian school system; here we will refer to them as classes 2A, 2B and 2C) and two classes of grade 8 (grade 3 of lower secondary school in the italian school system; here we will refer to them as classes 3A and 3C), in the period from march to may 2015. The class 2C of teacher OR (grade 7) had already approached ratio as a way of comparing quantities. The class 3C of teacher EZ (grade 8) had already studied proportions in a formal way. A priori, the team agreed that comparing the way of dealing with the same task having at disposal different knowledge and experience could be an interesting point of reflection.

Analysis

Class 2A (teacher MT)

17 of the 18 students answered that the frizzy orange on Saturday will taste the same as the fizzy orange on Friday. 4 students just wrote down that the taste is the same, without giving any explanation. Two students wrote down that the taste is the same without any justification, but highlighting that the only difference is in the total quantity of fizzy orange.

11 students also added some explanation for the fact that the taste does not change. Among them, one student just told that the taste does not change because *"The quantity of soda is always greater than the quantity of orange juice" (Gia)*. Two students proposed a sort of "additive" reasoning: "*You add one to both, and (the taste) does not change" (Der).* Five students talked about a "balance" between orange juice and soda: "*For me the taste is the same because Guglielmo added two more liters, balancing orange juice and soda" (Vic).* Among these answers, we highlight the use of expressions such as balance, proportion, equilibrium: *"The quantity of orange and that of soda are in proportion respect to the number of liters" (Gius).* Three students crafted their explanation on the fact that the difference between soda and orange juice is always 1 liter: *"Soda is always greater of 1liter than orange juice" (Ce).*

We may note that additive, balance and difference argument may be considered as equivalent. Argumentations are clear and well crafted, but they start from the assumption point that adding equal quantities does not change the taste. In terms of rationality, we may say the lack is in both epistemic (non acceptable premise) and teleological (what are the good ways to compare recipes?) dimensions.

One student, *Fr*, wrote down fractions (3/7, 4/9 on a line, 4/7, 5/9 on another line) and turned them into a representation with the same denominator (63); we argue his idea was to compare fractions, even he did not accompany the fractions by any explanation. He did not even answer on the sheet to the second question. His position was clarified during the subsequent class discussion. In terms of rationality, we may say his explanation lacks in the communicative dimension.

Class 2B (teacher EQ)

Four students answered that the fizzy orange on Friday tastes different from the fizzy orange on saturday, while 13 of students answered that the taste does not change. Among those who answered that the taste does not change, two students referred to a sort of balance between the ingredients (see *Sab* in the bulleted list selection), four students referred to the fact that 1 liter is added to both ingredients ("*For me the taste will not change 3+4=1+1=4+5 liters*", *Ser*), three students argued that the difference between ingredients is always 1 ("*The taste is the same because the difference between the two ingredients is the same as that of friday, there is one liter in plus added to soda and orange juice then the taste will not change*", *Ir*). Interestingly, scientific terms such as proportion and ratio were often used in an "everyday" meaning, e.g. *"It tastes the same because each ingredient is increased in proportion to the other; in this case, the ingredients increase their proportion of 1 liter"* (*Al*).

Among those who answered that the taste changes, two students gave an argumentation linked to a "realistic" interpretation of the text. Among them, *Tu*: "*For me it doesn't work because since I don't like mixing all those drinks and once I tried it and it was disgusting. But there are people who like mixing, for me they can do what they want but I think it is not healthy.*

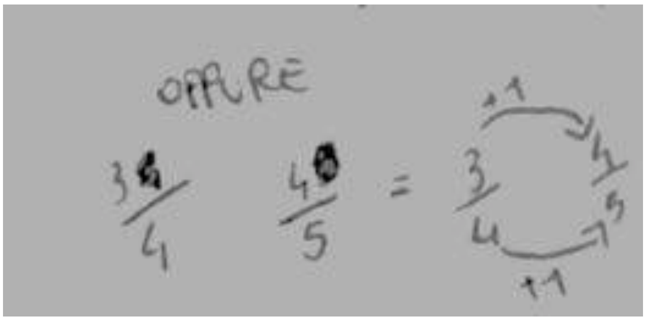
One student argued that the taste changes, but the argumentation was not complete, she just referred to the varied quantity of ingredients: "*That of Saturday will have a different taste than that of Friday because the number of liters changes. I can know it by seeing how much liters he puts in each day" (Sa).* One student proposed an argumentation based on a sort of transformational view of the problem:

> *Because if we take away the 1 liter difference between orange juice and soda from the drinks of Friday and Saturday, there will be equal quantity of the two ingredients. But if we add the liter of soda that we previously had taken away, the drinks acquire a different taste, because 1 liter of soda in 8 liters of frizzy orange mixes in a great quantity, being 8 liters a great quantity maybe it is not even possible to perceive it, while you perceive more 1 liter of soda in 4 liters of fizzy orange. (Az).*

The last argumentation is promising because it links together all the main issues that were used by students as warrants for argumenting the no-taste change (difference of 1 liter between ingredients; from Friday to Saturday you add 1 liter of each ingredient), but turns them into arguments for the change of taste, thanks to a transformational reasoning. *Az* was efficient in sharing her argumentation with peers during the subsequent discussion. The crucial point was to refer to previous experiences in science. Her explanation is efficient in terms of epistemic, teleological and communicative dimensions of rationality.

Class 2C (teacher OR)

The teacher OR had already presented ratio as a way for comparing increases, giving the example of age comparison: if we take two adults that have 5 ages of difference (say, 50 and 55 years old), and two kids that have 5 years of difference (say, 1 and 6), the age difference is the same but in the second case it is more "meaningful" because 5 years over 1 is much more than 5 years over 50. Faced to the fizzy orange problem, however, only three students over 17 answered that the taste is not the same. 5 students answered that the taste will not change because the two ingredients were increased of the same quantity (additive argument). Among them, one student referred to fractions, but he focused separately on numerator and denominator, in terms of addition:



6 students answered that the taste is the same because the difference between ingredients is always 1 ("*I think the taste is the same because the difference between soda and orange juice is the same*", *Lo*). One student spoke about ratio between quantities, but actually she meant difference ("*Because the quantity increases, but not in measures very different from each other, because the ratio between the two ingredients is one, in both cases. Only the final quantity changes*", *Sca*). One student combined the two arguments (addition and difference) ("*For me the taste is the taste because he just added 1 liter to both ingredients then the difference will be the same*", *Ma*).

Among those students who answered that the taste changes, one student gave an answer based only on "common sense" and a on a misunderstanding of the text (he compared the taste of the same mix on Friday and Saturday, arguing that after one day the fizzy taste is less evident): "*The taste is different because if you keep it more time you perceive more the taste of orange because it is thicker*" (*Ste*). Another student, *Lia*, argued that the taste will change and her argumentation, although not complete, seemed to refer to concentration ("*I think the taste will be different, it will have less orange taste in comparison to Friday because on Saturday he mixed 4 liters of orange juice and 5 liters of soda, then there will be more soda taste*"). One student, *Rio*, compared ratios in terms of fractions (note that he used the term "consequent" instead of "denominator"): "*No, because ratios are different ¾ and 4/5 that, having the same "consequent" it would be 20 and then 15/20 would be that of Friday and 16/20 that of Saturday, which tastes more orange*". The idea of comparing fractions is efficient (teleological rationality) and correct; the explanation could be improved in terms of communicative rationality, in order to make the classmates involved into the reasoning.

The analysis of this class is interesting because individual answers do not differ very much from results of the classes that were not taught ratios in a formal way. The subsequent discussion, however, was less problematic, since *Rio* recalled to the classmates the teacher's explanation and all the students agreed on the use of ratios as a way for dealing with the problem. One point of reflection, however, was the efficiency of the fizzy orange task as a way for giving meaning to ratio.

Class 3C (teacher EZ)

The teacher EZ had already presented ratios, fractions and proportions in the earlier school-year. 6 students answered that the taste does not change, proposing additive arguments ("*The taste will be the same, it will not change because he added the same quantity of liters to both ingredients, the quantity changes but not the taste*" (Leo). 2 students answered that the taste does not change, without any intelligible justification ("*The taste is the same because if you add the liters of orange juice and soda the result is equal to the liters of fizzy orange then the taste is the same*", *Ser*) or referring to a generic comparison between ingredients ("*The taste does not change because the quantities of soda are always greater than the quantities of orange juice*", *L1*).

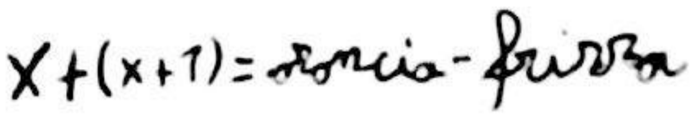
3 students answered that the taste does not change referring to the increase of quantity ("*The taste of the frizzy orange of Saturday is the same than that of Friday, because in order to make 9 liters of fizzy orange you need more soda and orange juice than for 7 liters, but the taste is the same*", *Sa*) or to an idea of proportion ("*For me the fizzy orange of Saturday has the same taste as that of Friday, because the proportion between orange juice and sprite is the same, he just did a little more than on Friday*", *Sca*).

8 students answered that the taste is not the same, but 6 of them used different arguments, not always correct and/or complete. The student *Mar*, for instance, just proposed an alternative recipe to obtain the same taste ("*You should do 5 liter of orange juice and 6 liters of soda*"). Two students relied on common sense; their answers may also be read in terms of misunderstanding of the text (e.g "*The taste is different because, after a while, the fizz goes away, but if you add a liter the taste does not change*", *Do*). Two students just referred to the varied quantity of ingredients (e.g "*On saturday he prepares more liters than on Friday*", *Ces*). Only two students reported some interesting argumentations. *Ric's* explanation, although not complete, may be referred to concentration ("*It has more taste because if on Saturday Guglielmo adds more liters and Friday he adds less, the taste gets more intense, because you add more orange juice in comparison to Friday*"). Finally, *L2* used fractions to compare ratios: "*I think the taste will be different because on Friday the ratio was ¾ while on Saturday it is 4/5. The taste will be more orangey, because ¾ is less than 4/5*". Her explanation is correct, even if it could be improved in terms of communicative dimension in order to involve her peers.

Class 3A (teacher EP)

The classroom, within the "Language and argumentation" project, had followed a two-years sequence with a special emphasis on detecting and representing algebraically numerical regularities. Faced to the fizzy orange task, 20 students answered that the taste does not change, while 6 students answered that the taste changes.

Among those who answered that the taste does not change, 4 students just mention that only quantity varies, without giving any explanation for the non-variation of taste ("*Because only the quantity of ingredients and the total quantity of fizzy orange varies*", *Nora*). 5 students referred to additive arguments ("*The fizzy orange has the same taste because the quantity of orange juice increases, but also the quantity of soda increases/decreases at the same rate. The only difference is in quantity*", *An*), 1 student to difference ("*It will have the same taste because in both the drinks there is more liter of soda than orange juice*", *En*). Moreover, 10 students used the term "proportion", actually referring to the fact that going from Friday to Saturday you add 1 to both ingredients and then you keep the same difference between the ingredients (e.g. "*The taste is the same because the numbers are in proportion*", *Ul*). We also mention one student (*Lor*) that, maybe influenced by the previous experiences within the project, wrote in symbols the additive relationship between the two ingredients:



In terms of rational dimensions, we see that *Lor's* attempt is not directed to the good goal (teleological dimension): representing the relationship between the two ingredients in one single recipe (say, that of Friday) is not the way of studying the relationship between recipes (of Friday and Saturday). The risk is to loose the final goal, and put a great effort in symbolic representation.

Among those 6 students who answered that the taste changes, two students gave no explanation, one referred to common sense, influenced by a misunderstanding of the text ("*If you open it on*

Friday, on Saturday it has a different taste, still of orange but less fizzy" (*Bar*), three students gave explanations only in terms of different quantities ("*For me it has more taste because the liters of orange are more than on Friday*", *No*).

Discussion and preliminary conclusions

The analysis of individual argumentative processes shows a low variability within and across classes in terms of solving strategies and explanation modes of representation. In each class, additive arguments were predominant. Surprisingly, the reference to additive arguments is relevant also for those classes who already had some formal introduction to ratios and even proportions. Another common point is the "everyday" use of scientific terms such as proportion and ratio. Moreover, in each class, except for class 3A, there was one correct individual argumentation that could be used for the subsequent classroom discussion, even if each of those argumentations is different in terms of dimensions of rationality.

The first round of cross comparison, performed with teachers during the experimentation, brought to the fore two emerging issues: the need of a shared meaning for scientific terms (proportion, ratio) and the importance of sharing with students the meta-knowledge concerning ratio and fractions as a way for comparing increases.

Classroom discussions were very different in terms of mathematical contents at issue and argumentation at content and meta level. A crucial role was played by each teacher's interventions and her choice to bring to the fore different dimensions of rationality. For instance, in teacher MT's classroom a key moment was the reflection on the need to compare different fractions in order to compare the "recipes". Fraction being the "way of representing" the recipe, fraction comparison was introduced according to the final aim of choosing the recipe. In this way, fraction comparison was linked to a teleological dimension of the solving process. Meta level discussion referred to the need of comparing fractions.

On the contrary, in teacher EQ's classroom the use of fractions came from a reflection on the use of specific terms (ratio, proportion) in "everyday language" way. Starting from the fact that such terms have a scientific meaning, the teacher moved the discussion on what could mean ratio in the case of the fizz-orange, and this led to representing the recipes in terms of fractions. In this case, the introduction of fractions was linked to an epistemic and communicational need. Afterwards, the need of comparing fractions emerged, but in this classroom it turned in terms of representing fraction as decimal numbers. Meta level discussion referred to the meaning of ratio and fraction.

Cross comparison performed after the experimentation lead to deeper reflection on the educational potentialities of the sequence. The task sequence led students to work on connected issues such as fractions, decimal numbers, ratio. The sequence also helped to make connections with science (concentration) and to work at meta-mathematical level on the use of mathematical constructs to model and solve everyday life problems.

An emerging issue is the importance of cross comparison, performed by the team of teachers, also professional development. This point was evidenced by the teachers' comments during the cross analysis of the task sequences.

REFERENCES

Arzarello, F. & Bartolini Bussi, M.G. (1998). Italian Trends of Research in Mathematics Education: A National Case Study in the International Perspective. In J. Kilpatrick & A. Sierpinska (Eds.), *Mathematics Education as a Research Domain: A Search for Identity*, pp. 243-262. Kluwer, Dordrecht.

Bartolini Bussi, M.G. (1996). Mathematical discussion and perspective drawing in elementary school. *Educational studies in mathematics 31*(1-2), 11-41.

Durand-Guerrier, V., Boero, P., Douek, N., Epp, S. S., Tanguay, D. (2012). Argumentation and proof in the Mathematics classroom. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in Mathematics Education, New ICMI Study Series 15* (pp. 349-367). Springer Science + Business media B.V.

Morselli, F. & Boero, P. (2011). Using Habermas' theory of rationality to gain insight into students' understanding of algebraic language. In Cai, J. & Knuth, E. (Eds.), *Early algebraization. A global dialogue from multiple perspectives* (pp. 453-481). Springer.

Morselli, F. (2013). The "Language and argumentation" project: researchers and teachers collaborating in task design. In Watson et alii (Eds), *Proceedings of ICMI Study 22 – Task design in mathematics education*, 487-496. Available online: http://hal.archives-ouvertes.fr/hal-00834054.

Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, 9, 187-211.

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Les metaphores
Quels enjeux pour l'enseignement des nombres relatifs?

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Résumé : Dans ce travail, nous avons choisi de nous placer résolument dans un cadre interne aux mathématiques afin de montrer la première fréquentation des nombres relatifs. Nous voulons, par ceci, mettre en relief la mise en texte de ce savoir, afin d'analyser des organisations mathématiques à enseigner autour de l'introduction des nombres relatifs et de leur somme. Ce qui nous permettra d'identifier la construction chez les élèves d'un rapport personnel idoine à ces nombres, ainsi qu'au calcul sur et avec.

Abstract: In this work, we decidedly chose to place ourselves in an internal frame of mathematics in order to show the first contact with relative numbers. Through this work, we intend to highlight the textualisation of this concept in order to analyze mathematical organizations to teach the introduction of relative numbers and their sum. This will allow us to identify the way students construct a personal connection suitable to these numbers as well as to calculations based on these numbers and using these numbers.

Problématique

Une des finalités premières de l'enseignement des mathématiques est, selon Brousseau (2000) le développement de la personnalité rationnelle de l'élève et l'apprentissage des comportements sociaux relatifs à l'établissement de la vérité. Les situations extra mathématiques apparaissent dans l'enseignement des mathématiques à tous les niveaux. Elles s'apparentent essentiellement à des mises en scènes (*des métaphores*) destinées à motiver le travail des techniques mathématiques enseignées qui constituent des réponses canoniques à des questions éventuellement non formulées. Alors que Ashton (1994) déclare que la caractéristique essentielle de la métaphore est qu'elle implique dans la recherche de sens.

Les contenus mathématiques enseignés sont toujours les mêmes au Liban, malgré la réforme des programmes en 1997[8], parce que la tradition l'emporte. Il est bien de les enseigner depuis toujours, mais le problème qui se pose est celui d'un mauvais traitement de ces objets mathématiques qui se répète. Pour ceci, nous retrouvons toujours des élèves qui agissent par *psittacisme*, ils répètent des termes ou des notions sans en comprendre le sens.
Ce qui contredit l'idée de Douady (2000), que les mathématiques sont un lieu où il est possible de mettre les élèves en situation d'avoir à faire des prévisions, de les tester, et d'obtenir des réponses pour lesquelles les démonstrations apportent la certitude.

Pour concrétiser les nombres relatifs et rendre leur enseignement « économique » les professeurs, en suivant les manuels, tentent d'automatiser l'activité demandée aux élèves en soutenant les routines enseignées par diverses métaphores comme par exemple : « si un négatif exprime une descente, lui additionner un nombre qui exprime une remontée et selon l'intensité de ce mouvement contraire, le résultat sera une descente ou une remontée ». Ces métaphores sont traditionnelles et elles se présentent comme une aide pédagogique. Les élèves libanais de la classe de EB6 (Sixième) confrontent, pour la première fois, ces nombres qui s'expriment par des dettes, des altitudes, des

[8] Programme libanais: décret-loi n° 10227, date 8 mai 1997.

profondeurs, soit des grandeurs pouvant être comptées comme augmentations ou diminutions, ascensions ou descentes, …, bref des grandeurs orientées, qui demandent une certaine jonglerie de l’esprit ce qui constitue une rupture importante avec les nombres manipulés jusque-là qui évaluaient les grandeurs dans leur seule intensité. La notation habituelle des nombres négatifs utilise le signe « – » qui est, pour eux, lié à une opération, « la soustraction », ce qui contribue à accroître les difficultés du calcul sur ces nombres.

Notre hypothèse est que l’outillage des métaphores, qui sont supposées aider l’élève à acquérir des nouveaux concepts, pourrait éventuellement rendre plus complexe l’interprétation du contenu présenté à partir du moment où la métaphore ajouterait des données d’informations qui ne peuvent pas être liées à la réalité à chaque moment. Les élèves appliquent alors un ensemble de recettes, des techniques n’ayant aucun fondement. A force de « faire des applications de leur leçon », ils réussissent certains calculs.

Dans le but de faire ressortir quelques métaphores liées à la pratique de l’enseignement des nombres relatifs, nous allons nous restreindre aux activités introductives des nombres relatifs et de leur somme. Cette étude prend appui sur le contenu de deux chapitres d’un manuel, de la classe de EB6 adopté par plusieurs écoles privées libanaises.

Le programme

Le programme (1997) étudié est celui en vigueur. Il fait état des notions à enseigner, qui ne sont pas le fruit d’un hasard mais d’une nécessité elle-même liée à certains types de tâches que les concepteurs souhaitent que les élèvent réussissent.
Nous chercherons en vain dans le programme des indications qui donne de l’intérêt de l’étude des « Nombres Relatifs » et plus spécifiquement des nombres négatifs. Ils sont au programme depuis toujours, mais ils n’ont jamais constitué un objet savant, c’est apparemment une « création didactique » (Mercier 2008). Nous ne retrouvons qu’une liste de tâches et de techniques sans fournir à l’enseignant des stratégies qui favorisent « le sens des nombres négatifs ».
Un nombre positif est noté par « +a » et un nombre négatif est noté « -a » et pour accomplir la tâche : « additionner « +a » et « -a », les enseignants font apprendre à leurs élèves la technique « supprimer les parenthèses, puis le signe + devant a et enfin le signe de l’addition. Ce qui donne (+a) + (-a) = a – a. Un recours à l’axe numérique est exigé par les rédacteurs du programme pour présenter les nombres relatifs, les comparer, les additionner et les soustraire. L’usage de la calculette est conseillé pour soustraire à ce niveau.

Le manuel scolaire

Le manuel scolaire n’est pas seulement un artefact culturel qui participe à l’organisation cognitive et sociale du savoir (Lebrun, 2007), mais aussi un *système social*, qui n’est pas autonome, l’environnement social pourra influencer son évolution d’une année à une autre, tout enseignement de bonne qualité passe par la qualité du (ou des) manuel(s) scolaire(s) utilisé(s) (Rocher, 2007). Chopin (1998), qui a étudié l’histoire des manuels du primaire, de toutes les disciplines, en France, considère que le manuel est un livre du maître, ou plutôt un guide qui impose à l’enseignant en plus du contenu scientifique, la méthode d’enseignement. Alors que, pour Assude et Margolinas (2005), le manuel scolaire est un outil pour analyser le curriculum et les processus de transposition didactique interne (Chevallard 1991). Son analyse permet de regarder les modes d’organisation des savoirs inférer et de mettre en évidence les contraintes institutionnelles qui pèsent sur l'enseignement de tout objet de savoir.

Ainsi, pour décrire quelques organisations mathématiques relatives au « Nombres relatifs » et à « Opération sur les nombres relatifs »[9], nous avons eu recours à l'étude du manuel : *Mathématiques, deuxième cycle, 6ème année, EB6, Collection Puissance, Al-Ahlia, Edition 2009*[10], en usage jusqu'à présent dans différentes écoles libanaises privées[11]. Nous cherchons à mettre en évidence la manière et les conséquences du choix des auteurs de la vie du savoir « Nombres Relatifs ».

1 - Activités introductives des nombres relatifs

Pour répondre à la première tâche « *Découvrir de nouveaux nombres-Notation »,* annoncée au début du chapitre « Nombres relatifs », les auteurs du manuel proposent deux activités où les nombres relatifs sont construits par modélisation. La « *plaque d'ascenseur* » et les « *thermomètres* » sont les métaphores qui font « exister » ces nombres.

A - La première activité modélise une « *plaque d'ascenseur* », qui n'est pas une nouveauté pour un élève de EB6, c'est vrai, elle est bien de son vécue. Les enfants de dix ans sont capables de prendre l'ascenseur seul. La plaque du livre est numérotée de *-3* à *2*, verticalement, en désignant par « RC » le niveau séparateur entre les trois étages indexés par un chiffre muni d'un signe « - » *(-3, -2, -1*) et les deux autres indexés par un chiffre sans signe (*1, 2*). Cette représentation verticale montre l'altitude de chaque étage par rapport à l'étage de référence « Rez-de-chaussée : RC ». Le « zéro » n'a pas de trace.
Pour répondre à la question « *Donne la signification des autres boutons* », l'élève n'a qu'à reprendre la phrase donnée dans le contenu de l'activité *« le bouton 1 correspond au premier étage, le bouton -1 correspond au premier sous-sol* » et faire le changement convenable des nombres. Mais dans cette plaque il n'y a pas une symétrie, le bouton 3 est absent, alors que -3 y est. Le *zéro* n'a pas de trace, l'élève supposera que le bouton RC est le point de départ pour s'éloigner de la rue principale tant vers le haut que vers le bas. Pouvons-nous dire que conformément aux commentaires du programme « *On admettra, pour une grandeur donnée, une valeur initiale à partir de laquelle la grandeur pourrait prendre des valeurs positives et négatives ».* Pourquoi le bouton indexé par deux lettres RC est la valeur initiale d'un comptage de *-3* à *2*, et comment « la grandeur » peut prendre des valeurs négatives ?
Signalons que dans d'autres ascenseurs, les plaques sont numérotées différemment. Certaines sont numérotées …, L2, L1, GF, 1, 2,.., pour d'autres les numéros des étages sont par deux sur chaque ligne.
Comment l'élève va-t-il faire le lien entre n'importe quelle plaque d'ascenseur et l'ordre des entiers relatifs ? Comment l'axe des nombres relatifs pourrait « exister » dans l'institution, à partir de la tâche associée « *plaque d'ascenseur* ». Pourquoi le bouton indexé par deux lettres « RC » est la valeur initiale d'un comptage vers *-3* ou vers *2*, et comment le « zéro » sera-t-il identifié ? Quelle signification est donnée pour « zéro » ?

B - La deuxième activité représente trois « *thermomètres* » identifié par A, B et C, et ils sont numérotés en degré Celsius, de -5 à 45, par unité de 5. Dans le thermomètre A, le mercure s'arrête à (-5), dans le thermomètre B, le mercure s'arrête à (5) et dans le thermomètre C à (-10). Pour les auteurs puisque « 5 est au-dessous de « zéro », alors il est noté (-5) et il est dit nombre négatif ». Il est demandé de l'élève de compléter le trou par « au-dessus » pour que 5 du thermomètre B soit

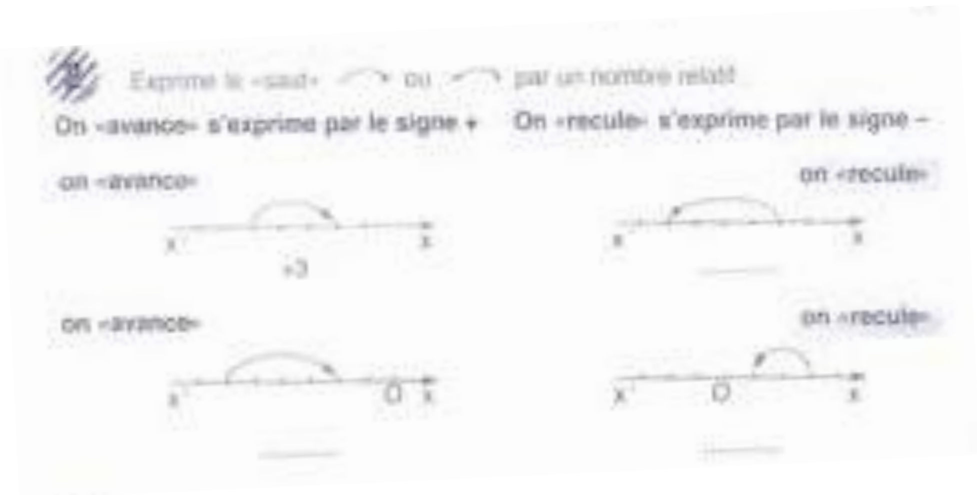
[9] Nous analyserons les activités liées à la somme des nombres relatifs.
[10] Cette édition est toujours la même à ce jour.
[11] Signalons, qu'au Liban, les écoles privées ont la liberté du choix des manuels, contrairement aux écoles publiques pour lesquelles le ministre de l'éducation impose les manuels.

noté (+5), et il est dit « nombre positif ». Les auteurs demandent de l'élève de compléter la phrase concernant la graduation du thermomètre C.
« *La température indiquée par le thermomètre C est ….de … notée …..*
… est un nombre…. »
Le « zéro » qu'elle est son sens ici ? Est-il un nombre ou un état ?

Pouvons-nous dire que ces deux activités répondent à l'un des axes de la réforme de 1997 du curriculum « *La méthode d'enseignement préconisée consiste à partir des situations réelles, vécues ou familière pour montrer qu'il n'y a pas de divorce entre les mathématiques et la vie quotidienne. Cette pratique des mathématiques amènera l'élève à l'intelligence des modèles conceptuels dont il comprendra l'efficacité grâce au transfert des apprentissages réussis* ». Les auteurs du manuel, en suivant les exigences du curriculum, n'ont pas fait un lien entre les entiers naturels existants dans les programmes de toutes les classes antécédentes, depuis la maternelle, et les entiers relatifs. Quel transfert des apprentissages fera l'élève des plaques des ascenseurs et des thermomètres ? Quelle signification donnera-t-il aux nombres négatifs qui se trouvent en coupure avec ses connaissances antérieures ? Les nombres négatifs sont tombés sur sa tête par un travail métaphorique laissant le « zéro », tantôt absent (le cas de l'ascenseur) tantôt présent (le cas du thermomètre), une valeur à admettre dans la liste des nombres proposés pour induire les relatifs qui sont laissés sans aucune spécification sur leur identité et leur fonctionnalité. Ils représentent tantôt un état : « il fait -5°C, je suis à l'étage -2», tantôt une variation : « la température a baissé de 5°C, l'ascenseur est descendu de 3 étages ».

2 - Activités introductives de l'addition et de la soustraction des nombres relatifs

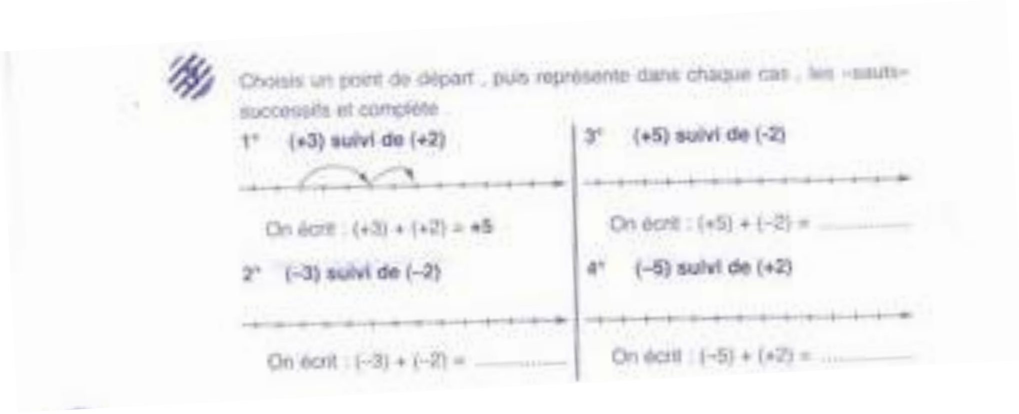
Pour répondre à la tâche « *Addition des nombres relatifs* » et « *Soustraction des nombres relatifs* » les auteurs ont adopté la technique des « sauts », comme le montre les trois documents suivants :



Extrait du manuel p 116

Les auteurs du manuel font une correspondance entre un nombre et deux ostensifs graphique et gestuel. Le nombre positif est un avancement et le nombre négatif est un recul.
Dans la deuxième ligne pour « on avance….», l'élève est sensé remplir le vide par +4 et par -2 pour « on recule … ». Le point « O » quel est son rôle sur ces axes ? Pourquoi il n'a pas de place sur les deux axes en première ligne? Où est l'emplacement du « zéro » sur ces axes ? Tout ceci constitue un usage non familier de la droite graduée.
Dans cette activité, les signes des nombres relatifs sont en lien serré avec le mouvement et non pas

avec leur position par rapport à « zéro » comme dans le cas des thermomètres.



Extrait du manuel p 116

La somme correspond au bilan de deux déplacements successifs. L'enchaînement des deux déplacements indiqué par « suivi de », pour les auteurs, correspond au (+) de l'addition, donc à un *opérateur*. La mise en avant de ce jeu est frappante dans l'algorithme d'addition de deux nombres relatifs. Sa valeur mathématique se pose.

La technique des sauts successives est une activité motivante pour l'élève, il va trouver du plaisir pour sauter, peut-être avec son crayon sur l'axe tracé sur le livre, en suivant les flèches, peut-être que l'enseignant traduira ce jeu en action réelle en classe. Mais, quelle connaissance mathématique est à acquérir ? Ceci explique certaines conceptions erronées des élèves : reculer 3 fois de 5 sauts les place à -15 (-3 × 5), pour eux l'acte de « reculer 3 fois » est « -3 » comme dans l'activité ci-dessus. Ils utilisent, par un geste mathématique métaphorique, une procédure d'addition réitérée.

Comment l'enseignant pourra-t-il convaincre ces élèves ? Par quelle métaphore « monter/descendre » ou « avancer/reculer » ? Faire comprendre aux élèves l'addition, par la mise en scène d'un déplacement crée un énorme obstacle au calcul avec les nombres relatifs. Vergnaud (1988) a montré à propos des problèmes additifs qu'il est difficile pour un enfant de se représenter une situation où deux transformations sont composées pour en former une troisième, et de calculer le bilan, alors que l'on ne connaît pas la valeur de l'état initial.

Extrait du manuel p 118

La tâche « *Soustraction des nombres relatifs* » a aussi pour technique « les sauts », mais cette fois cette technique est fondée sur une ostension déguisée. Elle est introduite par un modèle concret pour rechercher la transformation additive équivalente.
La technique de la soustraction de +9 et de + 5 est claire, et facile à appliquer. Par ostension l'élève retient les étapes et les applique pour calculer cette fameuse soustraction de deux nombres positifs.
Regardons ensemble le contrat institutionnel pour soustraire deux nombres de signes opposés (l'exemple de : *Complète*). L'élève n'est plus devant le *modèle soustractif*, il est face au *modèle additif*, alors que la tâche question d'étude parle de soustraction des nombres relatifs.
Essayons maintenant de suivre les deux points de cette technique pas à pas pour calculer +9 + opp (+5), si un élève pense qu'il doit refaire la même technique, puisque la question « *Complète* » vient juste après la technique à deux étapes de la soustraction de deux nombres positifs.
En premier point il est demandé de « représenter les deux nombres par **deux « sauts » de même départ** »[12]. L'élève doit sauter vers l'avant de 9, donc + 9, puis du même point de départ, il doit sauter l'opp (+5) qui est -5, donc il recule de 5 du même point de départ.
En deuxième point, l'élève doit « exprimer à l'aide d'un nombre relatif, le saut allant de **l'extrémité du second saut à celle du premier** », ce nombre serait +14, qui est différent de + 4. Ici, l'élève ne voit plus une « soustraction » la représentation sur l'axe de cette opération le place devant une flèche groupant les deux autres. Cet ostensif graphique est le représentant d'une « somme ». Or la réponse +4 est le résultat d'une différence. Comment alors, pour cet élève l'addition +9 + opp (+5) pourra être l'équivalente de la soustraction +9 – (+5) ???
Si l'élève applique la technique de l'addition pour +9 + opp (+5), il va sauter en avant 9, puis reculer de 5 (deux sauts successifs) pour avoir +4 qui est le *reste* de +9 – 5.
Quelle aide ont donné les auteurs du manuel à l'élève pour qu'il comprenne la soustraction des nombres négatifs. Ils ont été dans le souci de lui donner une image métaphorique mentale à la soustraction, en utilisant un modèle soit disant concret, *in situ* l'élève est placé devant la difficulté de donner du sens à la transformation de la soustraction en addition et de la valider.

L'expérimentation

Pour décrire le rapport personnel des élèves de la classe de EB6 au savoir « Nombres relatifs » et leur bénéfice de l'usage des métaphores dans l'apprentissage au fil du temps, nous avons construit deux exercices couvrant la somme et la différence de deux nombres relatifs. La passation de ces exercices, fut sur deux moments. Le premier moment : en EB6 (28 élèves), à la fin de l'apprentissage de l'addition et de la soustraction des nombres relatifs. Le deuxième moment : au début de l'année scolaire suivante en EB7 (Septième, 25 élèves des 28 de la classe de EB6).
Les élèves étaient à leur table. Ils étaient prévenus que ce test n'étaient pas noté, mais tout simplement une expérience pour évaluer leurs connaissances. Le temps disponible était de 30 minutes. Chaque élève disposait d'une feuille, sur laquelle il devait répondre, et dont une partie est consacré pour le brouillon, qui nous permettra de relever les hésitations de l'élève et même parfois sa façon de réfléchir. Nous avons interdit l'usage de la calculatrice.

Exercice I :
(+4) + (-9) = - 5 Pourquoi ?
(-3) + (-9) = - 12 Pourquoi ?

Nous avons posé « Pourquoi » dans l'espoir que l'élève retrace comment peut-il arriver à la réponse ? Et ceci en expliquant la procédure qui a permis de produire cette réponse. Serait-elle celle du « saut » ?

[12] Nous avons repris la même typographie.

Exercice II :

(+4) + (+8) = *(+7) + (-2) =* *(-3) + (+8) =*
(-11) + (+2) = *(+4) - (+8) =* *(-7) – (-2) =*
(-3) – (+8) = *(-11) – (+2) =*

Dans cet exercice, l'élève est amené à calculer. Nous avons choisi les mêmes nombres relatifs pour les deux opérations, dans un but d'éliminer la difficulté de la grandeur du nombre (Abou Raad, 2003).
L'analyse des copies des élèves pour l'exercice I, nous a montré que 20 des 28 élèves de la classe de EB6 ont justifié par la métaphore du saut pour le premier calcul « *J'avance de 4, puis je recule de 9 et j'obtiens la réponse* », et 10 ont justifié le deuxième calcul ainsi « *C'est une addition de deux nombres de signe commun. On prend le signe commun et on fait l'addition* ».
En revanche, arrivés en EB7, aucun de ces élèves n'a justifié le premier calcul et 8 des 25 élèves ont justifié le deuxième calcul par « *C'est une addition de deux nombres de mêmes signes, alors on place ce signe après l'égale et on additionne* ».
Ces résultats nous laissent dire que les acquis de la classe de EB6 sont perdus en EB5. Les élèves en EB6 ont su appliquer l'apprentissage de la métaphore « saut » qui est sensée faciliter la compréhension de la règle des signes et contribuée à une mémorisation plus efficace. Mais il s'avère qu'ils se sont trouvés incapables de l'outiller plus tard.
Nous nous engageons maintenant dans l'analyse de l'exercice II. Nous présentons dans un tableau les effectifs de réussite (R), de l'échec (E) et de non fait (NF) pour chaque calcul.

	EB6			**EB7**		
	R	**E**	**NF**	**R**	**E**	**NF**
(+4) + (+8)	**26**	**2**	**0**	**25**	**0**	**0**
(+7) + (-2)	**23**	**5**	**0**	**20**	**5**	**0**
(-3) + (+8)	**20**	**8**	**0**	**21**	**4**	**0**
(-11) + (+2)	**19**	**5**	**4**	**19**	**4**	**2**
(+4) - (+8)	**18**	**10**	**0**	**10**	**10**	**5**
(-7) – (-2)	**22**	**5**	**1**	**10**	**10**	**5**
(-3) – (+8)	**16**	**11**	**1**	**8**	**9**	**8**
(-11) – (+2)	**17**	**10**	**1**	**8**	**9**	**8**

Les effectives de réussite de ces 8 expressions sont plus élevés du côté des élèves en EB6. L'écart de réussite, des élèves en EB7, entre les expressions opérant sur la somme de deux nombres relatifs et les expressions opérant sur la différence de deux nombres relatifs est large. L'imitation mécanique des procédures est perdue avec le temps. Les élèves n'ont pas le contrôle théorique de la réussite de leur action. Ils ont retenu les étapes de calcul par ostension afin de les appliquer.
De ce fait, nous pouvons dire que la référence aux modèles concrets métaphoriques s'est révélée être un obstacle à la compréhension d'opérer avec des nombres relatifs et en particulier avec des nombres négatifs. Le signe « - » n'a pas une identité, puisque sa fonctionnalité est accordée à des métaphores (l'axe orienté, températures, altitudes, chronologie, plaque d'ascenseur, …).

Conclusion

L'analyse de ces activités présentées aux élèves pour introduire les nombres relatifs et pour y opérer, nous laisse dire que les élèves ne vont apprendre que des « savoir-faire » fragiles et de peu de portée. Les nombres relatifs présentés par des métaphores n'ont aucune signification mathématique. Ils sont présentés comme une *réalité évidente pour l'élève,* malgré les différents rôles que leur donnent les auteurs du manuel (les nombre relatifs indiquent tantôt des déplacements, tantôt des positions), qui sont dans le souci de donner à l'élève une image des nombres relatifs, en utilisant un modèle soit disant concret. I*n situ* l'élève est placé devant la difficulté de donner du sens aux nombres relatifs et en particulier aux nombres négatifs. Ces nombres paraissent comme des « choses » à prendre sans se demander pourquoi elles sont là et quelle est leur sens. Ce qui nous laisse dire comme Chevallard (2000) que l'enseignement des mathématiques est une visite d'un musée ou encore l'enseignement de réponses toutes faites véhiculées par la tradition, alors même que les questions motivant les réponses enseignées ont été perdues. Et comme le dit Brousseau (1987) enseigner un objet de savoir afin que les élèves le « conceptualisent » c'est enseigner. Donc la *conceptualisation* de ce savoir n'est pas à discuter puisque l'enseignant n'a rien à enseigner.
L'analyse des copies des élèves nous laisse dire que, ces élèves, par routinisation appliquent un ensemble de recettes des techniques n'ayant aucune théorie. A force de faire des applications, ils réussissent les calculs instantanément. Mais avec le temps, ces recettes s'oublient. Cette irruption d'usages métaphoriques supposée aider à contrôler des manipulations formelles, heurte le bon sens qui cherche à maintenir les significations antérieurement attribuées aux nombres quand il faudrait chercher à conserver les propriétés permettant les pratiques du calcul automatisé. L'élève est réduit à un *robot* qui applique des techniques identiquement à l'exemple résolu devant ses yeux. Il n'est plus « *l'élève-artisan laborieux* » de Chevallard (1999).

Nous souhaitons que, dans le futur le plus proche avec une nouvelle réforme, l'enseignement des mathématiques puisse offrir un modèle de construction du savoir dans le temps afin que les élèves puissent garder en vue le sens concret des calculs effectués.

BIBLIOGRAPHIE

Abou Raad N. (2003). *Difficultés rencontrées par les élèves de la classe de EB8 à la fin de l'apprentissage de la factorisation*. DEA, Université Libanaise, Faculté de Pédagogie.

Assude T & Margolinas C (2005). Aperçu sur les rôles des manuels dans la recherche en didactique des mathématiques. In E. Bruillard (Ed.), *Manuels scolaires, regards croisés*. (pp. 231-241). Caen : CRDP Basse-Normandie.

Ashton E. (1994). Metaphor in context: An examination of the significance of metaphor for reflection and communication. *Educational Studies*, 20(3), 357-366.

Brousseau G. et Brousseau N. (1987). *Rationnels et décimaux dans la scolarité obligatoire*, Publication de l'I.R.E.M. de Bordeaux.

Brousseau G. (2000). Que peut-on enseigner en mathématiques à l'école primaire et pourquoi ? Repères IREM, n° 38, pp.7-10.

Chevallard Y. (1991). *La transposition didactique. Du savoir savant au savoir enseigné*. Grenoble. La Pensée Sauvage.

Chevallard Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique, *Recherches en didactique des mathématiques*, 19 (2), pp.221 − 266.

Chevallard Y. (2000), *Enseignement insensé, enseignement raisonné et créativité sociale.* www.aix-mrs.iufm.fr.

Chopin A. (1998). Une perspective historique. *Les cahiers pédagogiques*, 369, 9-11.

Douady R. (2000). Que peut-on enseigner en mathématiques à l'école primaire et pourquoi ? Repères IREM, n° 38, pp.7-10.

Lebrun M. (2007). *Le manuel scolaire d'ici et d'ailleurs, d'hier à demain*. Presses de l'université du Québec.

Mercier A. (2008), Pour une lecture anthropologique du programme didactique, *In Education & didactique*, v. 2, n°1, pp.7 – 40.

Rocher G. (2007). Le manuel scolaire et les mutations sociales. In M. Lebrun (ed.) *Le manuel scolaire d'ici et d'ailleurs, d'hier à demain*. Québec. Presses de l'université du Québec.

Vergnaud, G. (1988). *Question de représentation et de formulation dans la résolution des problèmes mathématiques.* Annales de didactique des sciences cognitives, Strasbourg.

Le manuel scolaire : Mathé*matiques, Cycle Moyen,* 6ème année*, EB6* Collection Puissance Al-Ahlia, Edition 2009.

Programme libanais: décret-loi n° 10227, date 8 mai 1997.

WORKING GROUP 3B / GROUP DE TRAVAIL 3B

Classroom practices and learning spaces (from grade 9) /
Pratiques en classe et autres espaces d’apprentissage (à partir du college)

Working Group 3B / Group de Travail 3B

Classroom practices and learning spaces (from grade 9) / Pratiques en classe et autres espaces d'apprentissage (à partir du college)

Peter Appelbaum, Monica Panero

Working Group 3B was focused on secondary and higher education. The work began with a review and reformulation of questions from the conference discussion paper, which led to the following initial topics for discussion:

1. How ICT are useful and how can we have evidences that they are useful?
2. Multiple and different kinds of representations and the relationship with visualization.
3. Role of discussion in a mathematics teaching/learning environment.

The discussion around the utility of ICT has been enriched by research studies in different contexts and with different kinds of technology. Therefore, first of all, we realized the importance of specifying for whom or what they can be useful: For the teacher? For the student? For the teaching-learning processes?

We explored the potentialities of a teaching approach blending an e-learning platform, online tasks with a well-argued solution, online feedback and discussion meetings on the online work (Albano & Pierri). We investigated the use of the WebQuests method to lead students in their researches on the web and the use of online collaborative documents to process the collected information within an extra-curricular course on fractals (Alfieri). The analysis of an experience of graphing motion with WiiGraph for fostering students' reasoning on spatio-temporal mathematical relationships (Ferrara & Ferrari) contributed with interesting issues on the students' engagement and learning through "playing" mathematical instruments. Three studies on the use of GeoGebra combined with paper and pencil have been presented (López & Guzmán; Serpe & Frassia; Guzmán & Zambramo) and led us to discuss other questions: Is there no new questions introduced because of ICT? Are the pedagogical, didactical and research concerns the same as paper and pencil or the same as physical objects manipulation?

The debates around technology, and in particular the potentiality offered in terms of visualization, raised another important issue: multiple and different representations and their relationship with visualization. We recognized that in mathematics we do not work directly on the object, but we actually work on one (or more) of its possible representations. Does the representation co-construct the mathematical objects *with* the mathematical learner/thinker? Different contributions in the group allowed us to investigate this question in-depth. Olsher & Hershkowitz proposed to analyze the process of solving a "visual-pattern-problem" starting from the algebraic expression used to solve it. They analyzed how the visual strategies that are behind the algebraic components of the given expression are detected and reconstructed. Discussion on representation and visualization led us to ask a constellation of other questions we have to cope with as mathematicians, as teachers, as teacher educators, and as researchers: Is one of our goals in mathematics education (at all levels) to help students as they make transitions among different representations? Do mathematicians generalize from collections of concrete representations to make conjectures and then to prove these conjectures about the "concrete" specific cases? If different students use different representations, do they have different understandings? We wondered whether visualization is a process or a product, and how the links among mathematical experiences with representations of the mathematical object(s)-representations in turn affect the experiences. We finally decided to consider two possibilities: (a) visualization as a complex set of tools with which one manages representations; and, (b) visualization as a form of representation at the same level as language, i.e., concrete, contextual, abstract symbolism. Managing these different points of view can be related to the teacher's objective in terms of how the teacher wants students to visualize, and leads to a non-linear combination of interactions and relationships rather than direct empirical causations.

This debate fostered further discussion about the utility of technology in mathematics education. When we work on/with/through a mathematical object with technology, we have to ask what kind of representations the software is producing. Different issues have been addressed:

- Although we usually assume the student/mathematician is creating a representation, might it be more useful to see the representation as enunciating simultaneously the student/mathematician and the mathematics?
- To say that the technology represents a concept or relationship that we do not have access to is nonsense, because how would we represent something that we did not beforehand have access to?

Discussion around representation led us to assume the importance of the transition between mathematics formulas and procedures and common language. Orozco Vaca and Zazueta proposed a teaching strategy based on writing as a metacognitive tool, by providing students with a grid of questions for leading their problem solving activity. Thus, a strategy can be, for instance, that the students describe what they do in the exercise, while they are doing it. For the teacher, this is extremely important; but it is also often difficult to understand the meaning of what a student is saying. There can be many meanings when we talk, and we ourselves are often not even certain about what we are intending to mean until later, when we have looked back upon what we have said. Moreover, we want students to understand what the other students are saying, so they can discuss with each other; once we consider the complexity of co-constructed meanings in a constantly shifting and mediated environment, we need also to reconsider what we are even studying in the first place.

This issue led us to explore the role of discussion in a mathematics teaching/learning environment: What about students as a resource for their peers and for the teacher? But in a class there are several students; this multiplicity introduces complexities and problems. Kotarinou et al. presented a remarkable example of a year-long, interdisciplinary, didactical intervention giving to students the possibility of expanding the boundaries of official school mathematics discourse while actively involved in (re)negotiating their own learning processes.

Regarding the interaction between a teacher and students, we focused also on the specific role of the teacher in guiding the construction of knowledge and practices in the classroom. We particularly analyzed how the semiotic resources, and especially the gestures, can contribute to the mathematical discourse (Panero) as well as the physical discourse (Salinas & Guzmán) around the theme of variations. Do the gestures help at a theoretical level, but also at a practical one in the construction of techniques? And, to come back on the second point of discussion, are the gestures part of the visualization processes, or are they representations?

On one point we all agreed: the mathematical object emerges from all of the semiotic activity, and all the used semiotic resources and representations (mediated by the technology or not) contribute to disclose it.

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From rote procedures to meaningful ones: a blended semiotic approach

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Abstract: This paper proposed a blended learning approach for supporting undergraduate students to overcome rote learning practices and for helping them to reflect on the rules and relations of the mathematics they are doing. To this aim a learning activity has been set up: it consists in time-restricted online tasks (to solve problems and write argumentation to justify what done), online individual feedbacks from teacher/tutor, weekly face-to-face meeting to discuss the online work. Among the added values of such blended setting we highlight the chance of achieving practices focusing on crucial topic such as argumentation, which are not allowed in traditional lectures based on one-to-hundred/s communication. With respect to written argumentation, here we have investigated students' protocols, according to the frame of the functional linguistics.

Résumé: Cet article propose une approche d'apprentissage mixte dont l'objectif est de soutenir les étudiants de premier cycle dans les pratiques de l'apprentissage par cœur ou rote learning et les aider à réfléchir sur les règles, les relations et les domaines mathématiques abordés. Dans ce but, une activité d'apprentissage a été mis en place, qui comprend: des tâches en ligne en un temps limité (résoudre des problèmes et écrire l'argumentation expliquant les solutions), des évaluations individuelles en ligne par l'enseignant / tuteur, des rencontres hebdomadaires en face-à-face pour discuter du travail effectué en ligne. Parmi les valeurs ajoutées de cette configuration mixte, il est mis en évidence la possibilité de parvenir à une pratique centrée sur des aspects cruciaux tels que l'argumentation, pas fournie dans les classes traditionnelles basée sur une communication un à cent. En ce qui concerne l'argumentation écrite, cet article étudie les protocoles des étudiants, selon le cadre de la linguistique fonctionnelle.

Introduction

In this paper, we discuss a learning activity in blended setting implemented as support in an undergraduate linear algebra course for freshmen students in engineering. Experience shows that students in such faculty where mathematics is, with no doubt, a "service" domain, sometimes confuse such "service" domain with "procedural" mathematics vs "conceptual" mathematics (Hiebert&Lefevre, 1986). This means that many students do not care for understanding, but just for "operating" in some ways by rote learning, without being aware of what they are doing and why they do that way. Moreover, Italian setting of undergraduate courses, which consists in plenary lectures to one hundreds of students or more, just allows the teacher to state theoretical results (i.e. definitions, theorems and proofs) and to show some examples of solution techniques of typical problems. In this academic setting, deeper and actual conceptual understanding, required to pass the exam, is committed to the personal responsibility of each student. E-learning platforms allow to support and engage students out lecture time by meaningful online activities, designed and implemented according to the outcomes of research in mathematics education (Albano&Ferrari, 2008, 2013, Albano, 2011). The activity we refer to here consists in online time-restricted tasks and post face-to-face discussions among students and teacher/tutor. The online tasks require the students to solve some of previous final exam problems and to give suitable argumentations to support their solving procedures. Individual online feedbacks are provided on the students' products and a face-to-face meeting are followed to discuss on that.

The benefits of writing-to-learn are widely recognized (Morgan, 1998). In particular, we assume that forcing written argumentations through our online task can raise the student's reflections on the rules and relations of the mathematics he/she is doing. This can promote a shift from procedural to conceptual knowledge, fostering understanding how and why the procedures work and connecting procedures with their conceptual underpinnings (theorems, definitions and their characterizations, properties of mathematics objects, and so on). Moreover, written language can be used as a semiotic mean of objectification that is the student's intentional use of language in a social process of meaning production in order to achieve awareness of what he/she is doing (Radford, 2002).

An argumentation is first a piece of written (verbal, figural, symbolic) text and is strongly affected by both the student's competence in language and in the specific mathematical contents of the task (Ferrari, 2004). Here we do not adopt a specific theory of argumentation, but regard argumentation as a piece of text, written for more or less explicit purposes. We will investigate students' arguments in the frame of Halliday's Systemic Functional Linguistics (Halliday, 1985; O'Halloran, 2005). The importance of focusing on the linguistics functions is due to two opposite aims of the communication in mathematics education: on one hand, the need for effective representation of mathematical concepts and procedures and, on the other hand, the need for effective communication with other people. The context affects communication: the context of *situation*, which is space, time, the participants as individuals, the context of *culture*, concerning beliefs and knowing related to the participants and to the topics of the communication. Circumstances of communication can suggest the individuals to use different registers (intended as a linguistic variety): *colloquial* registers, mainly used in spoken everyday communication, and *literate* registers, mainly used in written communication among educated people. More details about such themes in mathematics education can be found in Ferrari (2015).

Methodology

Universities are aware that, due to the increasing use of the Internet by people, introduce the technology in education may help reduce the gap between students' out-of-school and educational experience. Sometimes, the use of technology in universities has been limited to considering an online platform as a repository of resources (course notes, exam results, technical information...).
The elearning modalities may offer the possibility to plan the personalization of individual paths as well as activities involving collaboration among students but also between student and teacher. In the context of Linear Algebra course, we created and followed up a blended course, which was piloted with almost 70 first year engineering students in the 2014/2015 academic year.
Linear Algebra course is composed by 60 hours, (5 per week), split into 3-hours of theoretical lecture and 2-hours of exercises. Examination consists in two parts: first, a written is required to be passed in order to access to an oral examination (discussion), in which the student is required to master definitions and theorems (including understanding of the proofs). The written examination, to be carried out over three hours, is composed by five or six exercises related to the topics illustrated during the course asking to provide for each of them appropriated explanations. The final mark depends on both the written and oral examination.
Often, students in faculties such as engineering where mathematics is, with no doubt, a "service" domain, confuse such "service" domain with "operational" mathematics. This means that they do not care of understanding, but just of "operating" in some ways, without being aware of what they are doing and why they do that way. Our purpose is to try to overcome these modes of procedure and refer students to justify what they do in the resolution of the exercises.
Generally, they deliver a set of counts without any justification, leaving the reader the task of understanding what they are doing, why and how those operations are linked to the question of the exercise/problem.
The students were enrolled in the blended course from the beginning, part of the first face-to-face lesson was dedicated to explaining to them how to use the platform, and what activities were available on it. The learning resources that they can delivery through the platform include theoretical materials for deepening topics from a theoretical point of view (including videos reproducing explanations of theorems and their proofs) and practical point of view (worked-out

exercises and problems), and self-assessment quizzes (close-ended questions). They also had access to a private email service where they could exchange comments and questions about the activities among students or with the teacher.

Besides these, we have exploited the open-ended questions (tasks – "compito a casa"), suitably temporized, in order to set the online tasks previously described.

The students have been split into two groups, which have worked on tasks of the same type during two different periods of the term. We have assigned the participants of the two groups distributing them homogeneously according to the entry test mark.

Each three days a week, we chose a problem among the ones of the previous written examinations and related to the topic just seen at the lectures. It has been assigned to the students as web tasks (using the platform) by using the functionality "compito a casa". The temporization of the activity induces students to carry out the assigned task in a fixed set time and at the same time, give them the possibility to work wherever they want and at any moment of the day. An example of the assigned web tasks t is the following:

> Solve the question 2) c) of the exam problems of 14 January 2014, and justify little by little the procedures you apply in the solution of the problem as you were explaining to a friend that cannot do this exercise and you wanted to make him clear how and why it is solved in the way you're doing.

We have chosen previous written examinations to motivate them, as students' interest is closely related to passing the written examination.

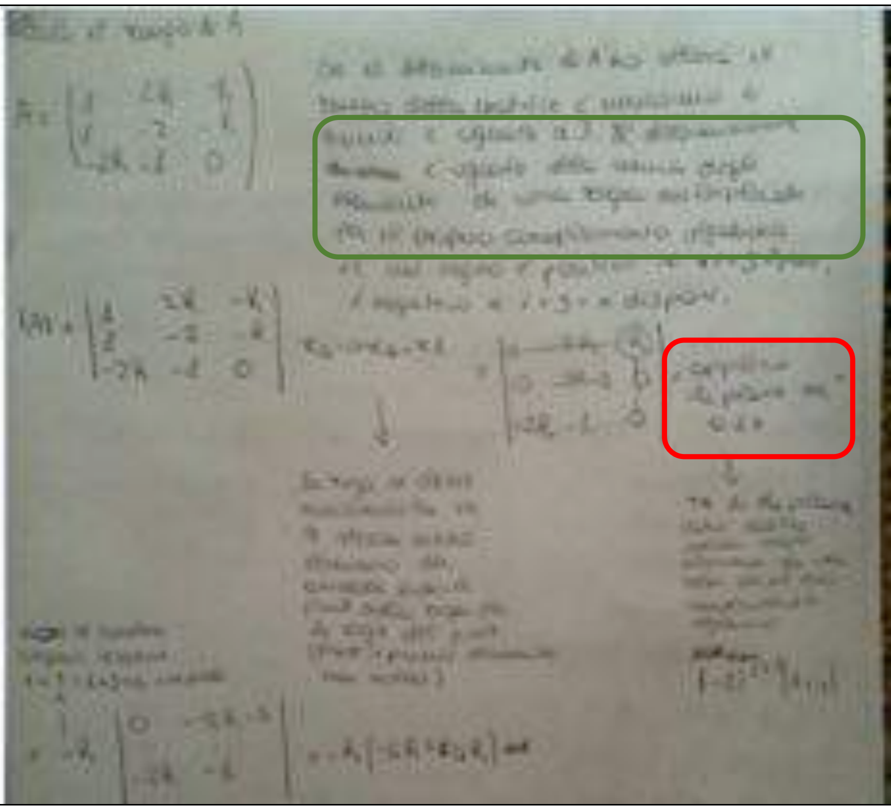
The students were required to make them in the three days. After that, we checked the students' products and sent them feedbacks with the evidence of the errors together with some sketch of explanations of the errors. We don't say them exactly which errors they made but we provide them with our feedback on their product consisting in some specific questions to provoke the student' reflections on his/her own errors.

After they received our feedback, we invite them to come in our room during "reception time" (at least 2 hours per week). In this "informal session" the single student can express his/her own difficulties and/or we can address them specifically, and in individually learning path. This latter depends on the interactions between us and the students and it is guided by the difficulties encountered during interaction which are made clear also by means of punctual questions we pose to the students depending on what we observe or we supposed to be the trouble at stake.

Sample of blended interactions

In the following we will show samples of the interaction online between the students and the teacher and its relation with the face-to-face one

Let us see a protocol containing the computation of the determinant of a matrix, together with the student's explanations (through the application of Laplace theorem).



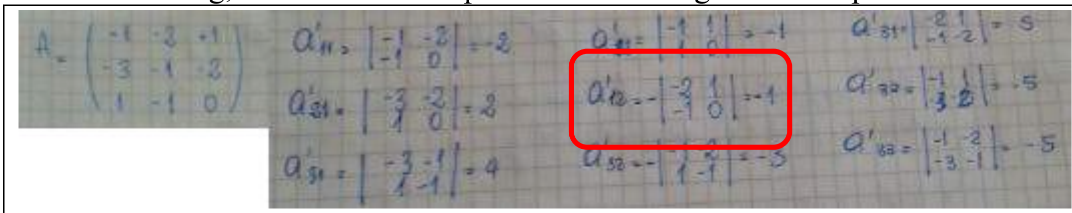
Teacher's comment: Is the Laplace theorem applied to an element, for example $a_{1,3}$? In the Laplace development that you consider, are you using a row?

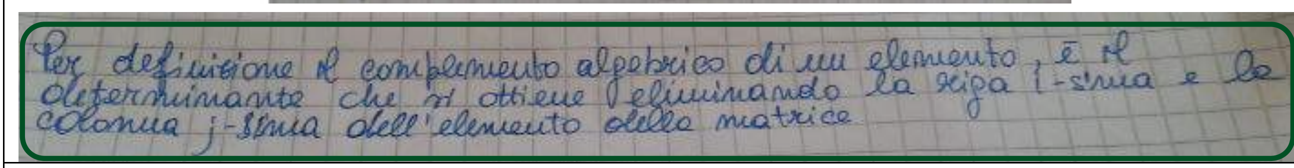
Table 1.The Laplace's theorem application.

If you look exclusively to the computations, it seems that everything is fine. But looking at the writings (red box), we have "*I apply the Laplace theorem to the element* $a_{1,3}$". At the same time, the writings in the green box refer to the correct statement of the Laplace theorem with respect to a row. Face-to-face discussion has made evident that this student, as others, has learnt by rote to use elementary operations on the rows so that to have a column with only one non-zero element. Then he/she knew that applying Laplace theorem to that column consists in considering the only product of that element and its algebraic complement. It came out that he/she was not able to apply Laplace theorem to a column with more than one non-zero element, he/she always considers just one element in the chosen column: he/she did not understand Laplace theorem indeed! Analogously, he/she learnt by rote what he/she wrote in the green box, without being aware of its procedural meaning.

This is an example of the usefulness of both written and spoken texts to detect and fix some fundamental bugs in the students' knowledge.

In the following, we show the computations of the algebraic complements of elements in a matrix.





Teacher's comment: Is the written definition compliant with the calculation made?

Table 2. Computations of the algebraic complements

The student's definition (green box) misses "multiply by $(-1)^{i+j}$", but he/she computes correctly (red box). When the student came to the face-to-face meeting, he/she could not yet see the difference between the computation performed and the definition s/he wrote. Then the teacher proposed to the student to read the his/her definition while s/he (the teacher) would act as a performer. As the student expresses words, the teacher converted them into operations. It was at this point that the student realized the incongruence between what he/she has in mind (corresponding to what he/she did correctly) and what he/she wrote. Thus, he/she grasped the difference between the object "algebraic complement" and the process of its computation and the link between them.
Let us consider the request of the inverse image of a vector, given a homomorphism,. Here some students' writings and teacher's comments:

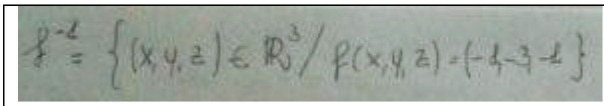
 $f^{-1} = \{(x,y,z) \in \mathbb{R}^3 / f(x,y,z) = (-1,-3,-1)\}$	*Teacher's comment*: Which is the difference between f^1 and f^1 (-1,-3,-1)?

Table 3. Computations of the algebraic complements

The teacher question would the student focus on his/her use of the symbol f^1 he/she used referred to a set. During the face-to-face discussion, the teacher can investigate if in such use the student implies f^1(-1,-3,-1) or if it is not clear to him/her the difference between the inverse function and the inverse image of a vector. Some students have troubles in distinguishing conceptually a function from its value in a point. The same occurs in case of inverse function and inverse image of a point. Thus, they learn some procedures, even correct, without being aware of what they are doing, as well shown by the following protocol.

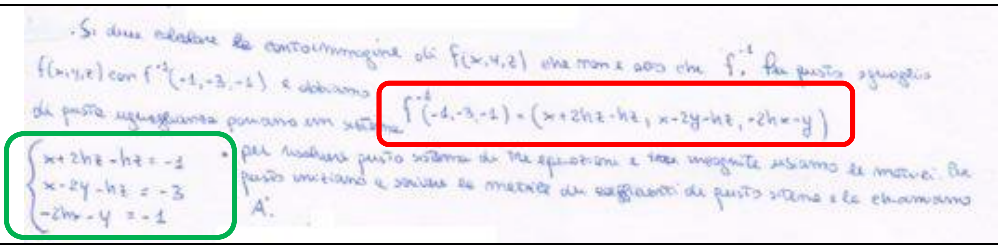
 $f^{-1}(-1,-3,-1) = (x+2hz-hz,\ x-2y-hz,\ -2hx-y)$ $\begin{cases} x+2hz-hz=-1 \\ x-2y-hz=-3 \\ -2hx-y=-1 \end{cases}$ A.
Teacher's comment: Which is the difference between "inverse image of *f(x,y,z)*" and f^1? What do you mean as f^1 (-1,-3,-1)? Is it an element of the domain or of the codomain of the function? And *f(x,y,z)*? What means the equality in the red box? How do you obtain the linear system from the equality you wrote (formula)? What happens to "f^1"?

Table 4. Inverse function and inverse image

The student says by words and in symbols (red box) that in order to compute the inverse image of (-1,-3,-1), it is needed to equal f^1 (-1,-3,-1) and *f(x,y,z)*. Then, he/she says that this equality is made concrete by a linear system (green box) corresponding to (-1,-3,-1)=*f(x,y,z)*. The student does not care of the symbol f^1 previously written in the red box! From oral discussion, it came out that such symbol was void of meaning for him/her!
A further critical issue in general concerns the relation between the posed question and the outcomes of the performed process.
Let us consider, for instance, the following conclusion given by a student:

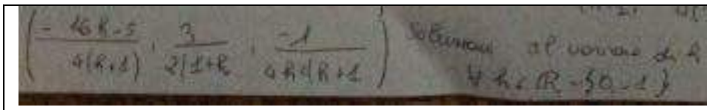
 $\left(\frac{-16k-5}{4(k+1)}, \frac{3}{2(1+k)}, \frac{-1}{4k(k+1)}\right)$ Soluzione al variare di k $\forall k \in \mathbb{R}-\{0,-1\}$	*Teacher's comment*: Which is the connection between the founded solution and the exercise's request? What happens for h=0,-1?

Table 5. Relation between the posed question and the computations outcomes

Some students make some computations but, at the end of the process, they are not able to come back to the question. In this case, the student writes "the solution" of the system, but there is no more reference to the required inverse image. Moreover, the question concerns all the real values of the parameter h, but the student misses the cases where the system has not a single unique solution.

Written argumentations' analysis

In this section, we want to analyse the written argumentations of the students, under the lens of multisemioticity and multivariety and their use in the context of the task.
First, we can say that the students use both symbolic and verbal language to perform the task, as well as they use both colloquial and literate registers. We have focused our attention to their intertwined use with respect to the specific requirement of "explaining what they do".
From the analysis of the protocols, we can outline mainly three type of verbal argumentations:

- Theoretical recalls, such as definitions, with incoherent operational procedures;
- Theoretical recalls, such as characterizations or properties, which give reasons of the performed procedures;
- Description of an algorithm, such as a verbal sequence of steps for solving the problem and performing the exercise.

We report some protocols, which seem to be meaningful and representative, and how we analyzed them, using the above described categories.

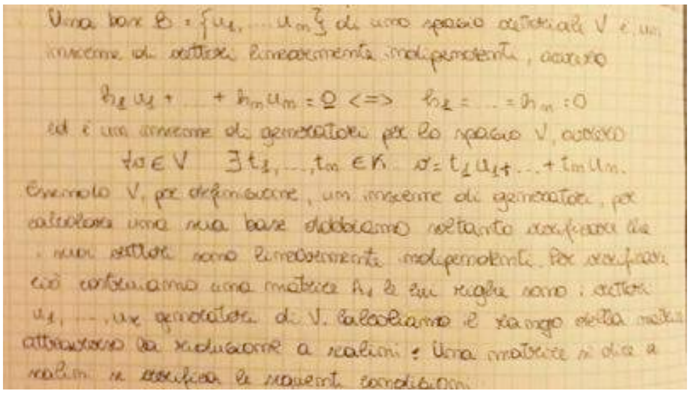

Una base B = {u1, ... un} di uno spazio vettoriale V è un insieme di vettori linearmente indipendenti, ovvero
h1u1 + ... + hnun = 0 <=> h1 = ... = hn = 0
ed è un insieme di generatori per lo spazio V, ovvero
∀v∈V ∃t1, ..., tn ∈K v = t1u1 + ... + tnun.
Essendo V, per definizione, un insieme di generatori, per calcolare una sua base dobbiamo soltanto verificare che i suoi vettori sono linearmente indipendenti. Per verificare ciò costruiamo una matrice Av le cui righe sono i vettori u1, ..., un generatori di V. Calcoliamo il rango della matrice attraverso la riduzione a scalini: Una matrice si dice a scalini se verifica le seguenti condizioni

Fig. 1: Protocol PE1.D17

PE1.D17 A basis B={u_1,…u_n} of a vector space V is a set of linearly independent vectors, that is $h_1u_1+\ldots+h_nu_n=0 \Leftrightarrow h_1=\ldots=h_n=0$ and it is a set of generators for the space V, that is $\forall v\in V$ $\exists t_1,..t_n \in K: v=t_1u_1+\ldots+t_nu_n$. Being V by definition, a set of generators, in order to calculate its basis we must only verify that its vectors are linearly independent. To verify this, we build a matrix A_v whose rows are the vectors $u_1,\ldots u_n$ generators of V. We calculate the rank of matrix by row-echelon reducing.

In Fig. 1, we can find theoretical recalls that are not linked to operational procedures: the student refers to the definition of linear independence but does not use it for performing the exercise; he/she uses the characterization in terms of rank of a matrix, and in particular computes such rank by means of the row-echelon reduction.

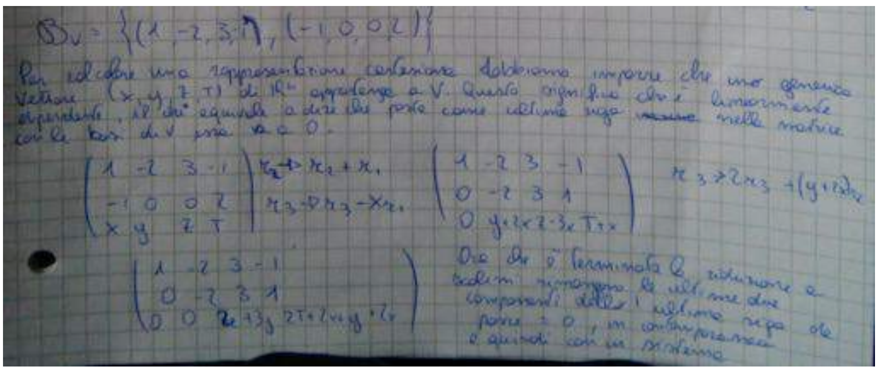

Bv = {(1,-2,3,-1), (-1,0,0,2)}
Per calcolare una rappresentazione cartesiana dobbiamo imporre che un generico vettore (x,y,z,t) di R⁴ appartenga a V.

Fig. 2: Protocol PE1.D06

PE1.D06 In order to compute a Cartesian representation we must impose that a generic vector (x,y,z,t) di R^4 belongs to V. This means that it is linear dependent, which is equivalent to say that posed as last row in the matrix with the bases of V it will become zero. […] Now that the row-echelon reduction is completed the last two components of the last row remain to be posed to 0, contemporarily and then with a system.

In Fig. 2 we can see how the student justifies how he/she constructs a certain matrix and why he/she reduce it in row-echelon form and pose to zero the last row in order to obtain a homogeneous linear system. Each step is suitably linked to the theoretical underpinnings, such as the belonging to a vector space, the linear dependence, its characterization and so on.

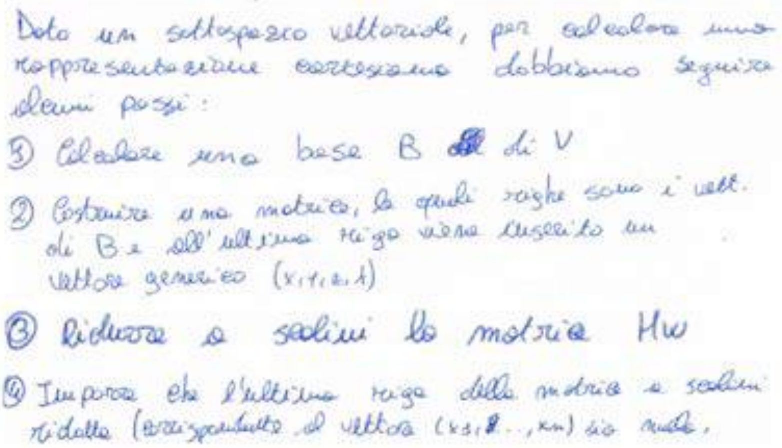

Dato un sottospazio vettoriale, per calcolare una rappresentazione cartesiana dobbiamo seguire alcuni passi:
1) Calcolare una base B di V
2) Costruire una matrice, le quali righe sono i vett. di B e all'ultima riga viene inserito un vettore generico (x,y,z,t)
3) Ridurre a scalini la matrice Mw
4) Imporre che l'ultima riga della matrice a scalini ridotta (corrispondente al vettore (x1,..,xn)) sia nulla.

Fig. 3: Protocol PE1.D18

PE1.D18 Given a vector subspace, for calculating a Cartesian representation we must follow some steps: 1) To calculate a basis B of V; 2) To build a matrix whose rows are the vectors of B and in the last row we put the generic vector (x,y,z,t); 3) To row-echelon reduce the matrix M_w; 4) to impose that the last row of reduced matrix (corresponding to vectors (x_1,..x_n)) is zero.

In Fig. 3 the student simply gives an ordered verbal list of procedures to be performed in order to calculate a Cartesian representation. No comments are given to justify why to do in such a way in order to solve the posed question.

Sometimes, the analysis of the verbal part has made evident some difficulties, such as confusion among various known results (Fig. 4) or actual gaps in knowing (Fig. 5). Let us see in more details.

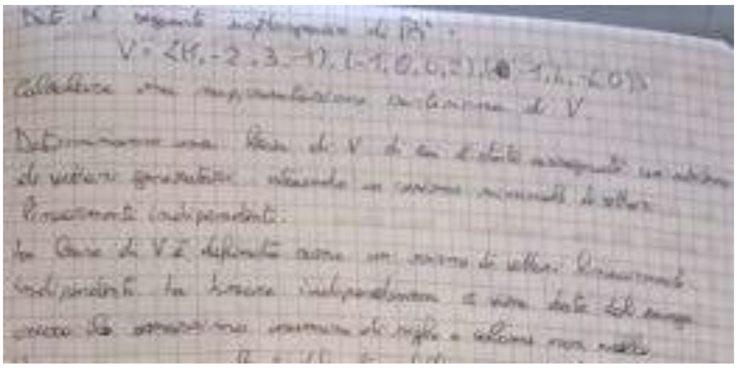


Fig. 4: Protocol PE1.D19

PE1.D19 We determine a basis of V known a set of vectors generators extracting a minimal set of linearly independent vectors. The basis V is defined as a set of linear independent vectors. The linear independence is provided by the rank that is the maximum number of non-zero rows or columns.

In Fig. 4 we can see that the student seems to recall two facts: on one hand, he/she links the linear independence with the rank of a matrix; on the other hand, he/she gives as characterization of the rank the one referred to a row-echelon matrix, not to a generic matrix.

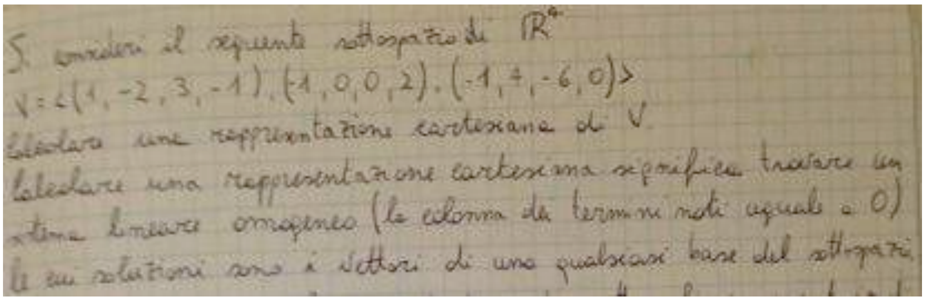


Fig. 5: Protocol PE1.D22

PE1.D22 To calculate a Cartesian representation means to find a homogeneous linear system (the column of the known terms is equal to zero) whose solutions are the vectors of any basis of the subspace.

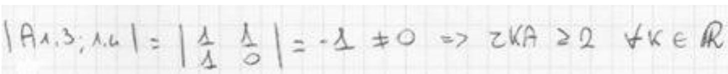
Fig. 5 shows that the student considers as solutions of the Cartesian representation of a vector space only vectors belonging to a basis.

Verbal argumentations accompany operational procedures, which are added with "argumentation" in symbolic language. Comparing verbal and symbolic statements, we can distinguish various cases:

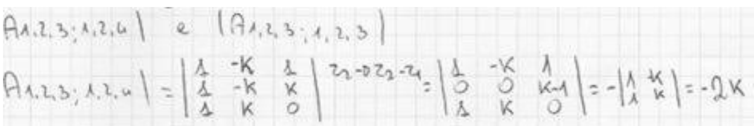
- Discrepancy between what the student verbally affirms to do and what he/she actually says in symbols (Fig. 6);
- Confirmation in symbolic language of errors made evident by verbal argumentation (Fig. 7);
- Correct use of symbols and symbolic argumentation, but accompanying by short verbal argumentation, which can highlight some misconceptions to be verified in face-to-face meetings (Fig. 8).

Fig. 6: Protocol PE1.B20

PE1.B20 We build a minor of order 2:



After having found a non-zero minor, we build some 3-minors that including rows and columns of the non-zero 2-minor



As we can see, the student declares to do something but he/she does something else, in fact, the latter 3-minor considered does not satisfy the condition concerning the inclusion stated before.

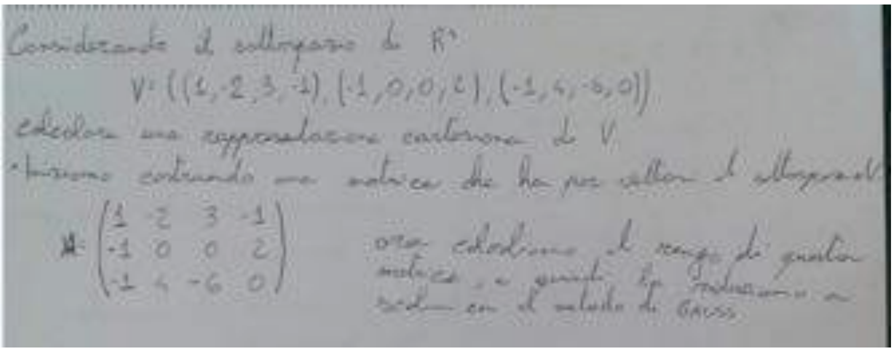
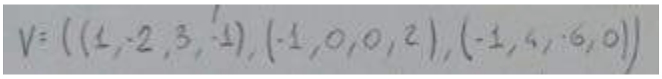
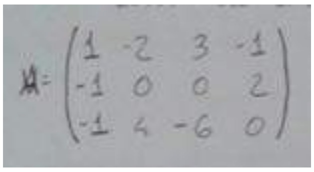


Fig. 7: Protocol PE1.D23

PE1.D23 To calculate a Cartesian representation of the subspace V of R^4



We start by building a matrix whose vectors are the subspace V



Now we calculate the rank of this matrix and so we reduce in row-echelon form through the Gauss method.

In the case of Fig. 7, we first observe that the student rewrites the given question and, in doing this, substitutes the original symbols < > by brackets. Looking just at the symbols, we can doubt on the awareness of semantic of different kind of brackets, but looking at the verbal argumentation, we can state that he/she actually thinks that only the vectors listed constitute the vector space.

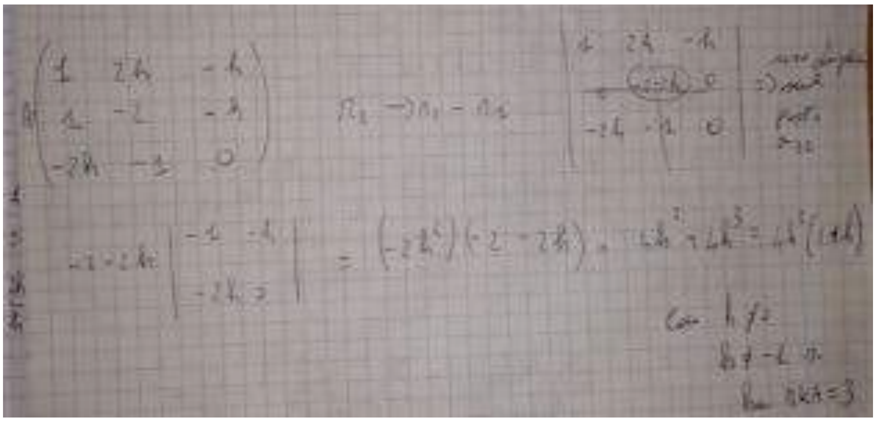


Fig. 8: Protocol PE1.C26

Fig. 8 shows a student' work where symbolic language is mainly used, sometimes referring to the underlying process (see the two cross lines in the computation of the determinant). The calculations made are correct and so the symbolic language seems, but the student writes "I use Laplace on the place a_{22}". This suggests the teacher to further investigation on the correct understanding of the Laplace theorem in face-to-face meeting (see the above discussion concerning Table 1).

Some pieces of the performed tasks are equipped with only symbolic argumentations. These can be correctly used and referred to operational explanations, such as in Fig. 9.

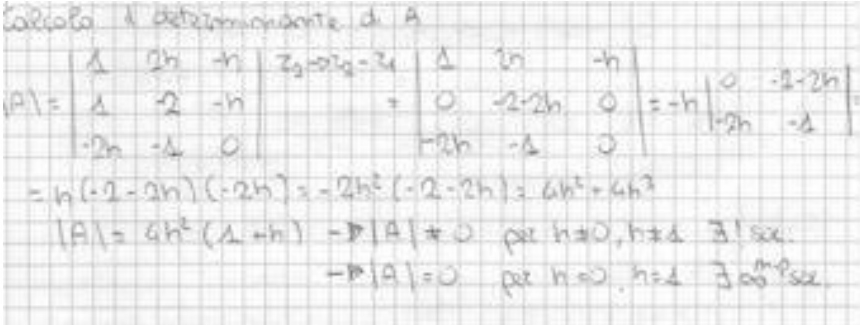


Fig. 9: PE1.C20

Some cases show a correct use without complete awareness of the semantic underpinning some sequence of symbols and interrelations. This is the case of students who are not able to use what they have found symbolically, even if correctly discussed (Fig. 10).

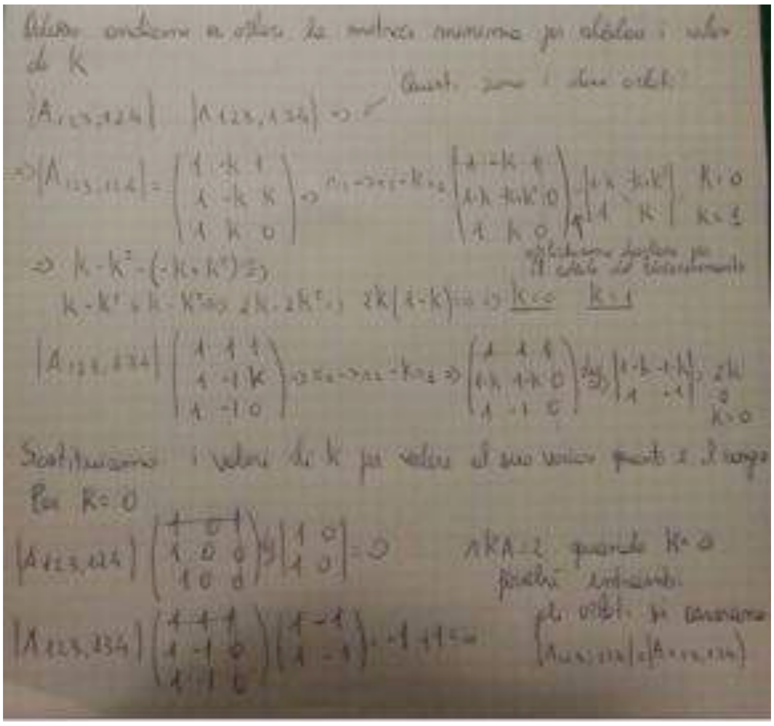
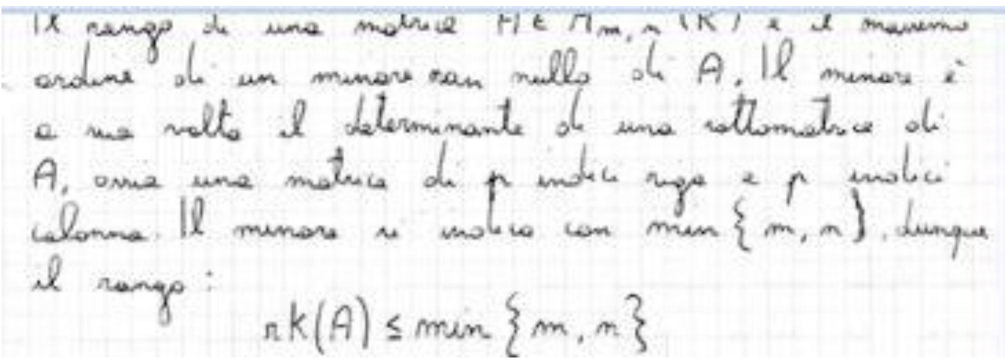


Fig. 10: Protocol PE1.B13

Some other cases seems to use correct use of symbolic language, such as in Fig. 11 if you look at the last row. Actually, verbal description highlights an erroneous meaning, different from the conventional one, of the symbol “min{*m*,*n*}”, which denotes “the minor”. According to the student, the last symbolic row has no sense!



Il rango di una matrice $M \in M_{m,n}(K)$ è il massimo ordine di un minore non nullo di A. Il minore è a sua volta il determinante di una sottomatrice di A, ossia una matrice di p indici riga e p indici colonna. Il minore si indica con min{m, n}, dunque il rango:

$rk(A) \leq \min\{m, n\}$

Fig. 11: Protocol PE1.C35

PE1.C35 The rank of a matrix A in $M_{m,n}(K)$ is the maximum order of a non-zero minor of A. The minor is in turn the determinant of a submatrix of A, that is a matrix of p row indices and p column indices. The minor will be denoted by $\min\{m,n\}$, then the rank:

$$rk(A) \leq \min\{m,n\}$$

Finally, we note that, in some other cases, the symbolic language is used in a colloquial register: from the strict grammatical view of point, we have incorrect writings, but if we look at them under the Radford objectification, we can found that they refer to and reproduce a process in mind. In Fig. 12, the student writes the matrix and rewrites the first two columns, then traces some lines with different colors (probably indicating the difference between the sign of the product to take into account), such lines have also a versus (probably indicating the order the student uses for the product).

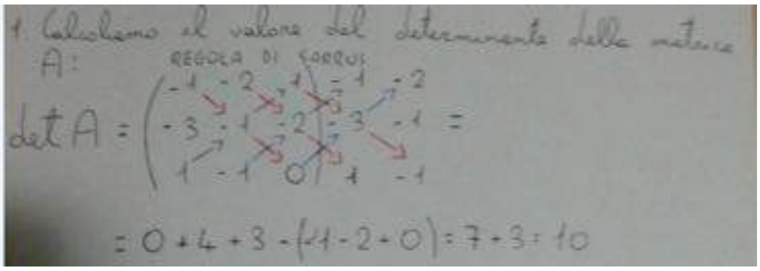


Fig. 12: Protocol PE1.A16

Conclusions

In this paper, we proposed a blended learning approach for trying to overcome the operational modes that the students adopt related to mathematics for helping them to achieve some awareness what they are doing and why they do that way.

E-learning environments have a great potential in the context of higher education. They provided opportunities to use multiple representation systems, to create activities of construction and treatment of semiotic representation, to develop several new forms of communication (regardless of time and distance) and to foster self and peer assessment processes. We exploited the potential of e-learning to pursue an important educational goals: the devolution of awareness into the learner in his/her interaction with mathematics. To this aim we used a specific open-ended questions of the learning platform (tasks – "compito a casa"), that requires the students to solve samples of exam problems, with the additional request of written argumentation to justify what they do. The added value to use this functionality is its temporization that have a double purpose: on one hand it induces students to carry out the assigned task in a fixed set time, keeping pace with the face-to-face lectures, and, at the same time, give them the possibility to work wherever they want and at any moment of the day. Experience shows that nowadays freshmen students need to be supported in individual extra-lectures work and such kind of time-restricted online activities can be a useful tool to this aim. The blended setting, online tasks and teacher/tutor's feedback on the students' products, and post face-to-face deeper discussions during extra weekly sessions has allowed to make realistic a template of learning activity which is not realizable for large group of students in traditional setting. Moreover, it allows keeping the focus on a crucial topic for meaningful understanding, such as argumentation, which is not foreseen in communication one-to-hundred/s, that is the case of the traditional courses.

In brief, from the analysis of the protocols we have found the use of verbal and symbolic languages, which can be correct or incorrect. When both on hand, referred to a single process, they can be coherent or not each other. Moreover, both of them can have been used in a colloquial or literate register. What we have considered important for learning is not the register used, but the coherence of the text showing a certain level of understanding. From the analysis, it seems us that in some students, who seems to use literate registers, simply rewrite definitions or theorems from books, but the theoretical references they do are not appropriate to justify the process of solution made. We can argue that maybe the aim of their communication here is not to answer to the task requirement (that is to justify what they do) but to make the teacher convinced that they "know" the theory underpinning, where "to know" is interpreted as reproduction of books.

REFERENCES

Albano, G. (2011). Mathematics education: teaching and learning opportunities in blended learning. In A. Juan, A. Huertas, S. Trenholm and C. Steegmann (eds): *Teaching Mathematics Online: Emergent Technologies and Methodologies* – pp. 60-89.

Albano, G., Ferrari, P.L. (2008). Integrating technology and research in mathematics education: the case of e-learning. In Garcia Peñalvo (ed.): *Advances in E-Learning: Experiences and Methodologies* (132-148).

Albano, G., Ferrari, P.L. (2013). Linguistic competence and mathematics learning: the tools of e-learning. *Journal of e-Learning and Knowledge Society (Je-LKS) (eISSN 1971-8829)*, Vol. 9, No. 2, pp. 27-41.

Hiebert, J., Lefevre, P. (1986). *Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis* (pp. 1–27). Hillsdale, NJ: Erlbaum.

Ferrari, P.L. (2004). Mathematical Language and Advanced Mathematics Learning. In Johnsen Høines, M. & Berit Fuglestad, A. (Eds.), *Proceedings of the PME 28*, Vol. 2 (pp.383-390), Bergen (Norway), Bergen University College Press.

Ferrari, P.L. (2015). Language and argumentation in the solution of problems with graphs. In K. Krainer & N. Vondrová (Eds.), *Proceedings of CERME9*.

Halliday, M.A.K. (1985). *An introduction to functional grammar,* London: Arnold

O'Halloran, K.L. (2005). *Mathematical Discourse. Language, Symbolism and Visual Images*. London: Continuum.

Morgan, C. (1998). *Writing Mathematically. The Discourse of Investigation*. London: Falmer Press.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, 22 (2), 14–23.

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L-system Fractals: an educational approach by new technologies

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Abstract: Living in a technological era it is essential that teachers and pupils have regular access to technologies. They support and advance mathematical sense making, reasoning, problem solving and communication. L-system fractals is an interesting topic to show the use of new technologies in an educational approach.

Résumé: Vivre dans une ère technologique, il est essentiel que les enseignants et élèves ont régulièrement accès aux technologies. Ils favorisent et promouvoir la création d'un sens mathématique du terme, raisonnement, résolution de problèmes et de la communication . Fractales de L-System est un sujet intéressant pour illustrer l'utilisation des nouvelles technologies dans l'approche pédagogique.

Introduction

Digital technologies impact more and more all aspects of the personal, social and professional people's life, pupils and teachers included. New technologies are spreading at an incredible speed. Social networks numbers exponentially increase and these tools become more and more influential; they deeply impact our communication, as well as our relationships with authorities and institutions. All these changes impact learning processes and teacher's work at school and outside, the relationships between teachers and pupils. They create new challenges and responsibilities. Artigue M. (2013) suggests interesting and shared questions:

- What is the advantage in mathematics education of the incredible amount of accessible information and resources?
- What is the advantage of the change of social communication and Internet facilities for creating and supporting communities of mathematics teachers and pupils?
- How to use the new affordances to give pupils more autonomy, to develope mathematical topics?
- How to develop a mathematics education more open on the outside world and still faithful to the epistemology of the discipline?
- How to use these new opportunities to better address the specific needs our pupils may have?

The geometrical topic about L-system fractals is an example of an educational activity in which the use of new technologies is essential firstly to manipulate geometric objects, to create new conjecture and demonstrate new hypotheses; secondly it is necessary to communicate outside the classroom and to share experiences.

L-System Fractals.

Fractals are a "good" geometrical topic for the application of new technologies, because after the study of affinity transformations by pencil and sheet, it is impossible to plot a fractal imagine or make conjecture to think new fractals, without using a software.

What a fractal is

Benoit Mandelbrot is the father of fractal theory. In the past fractals were regarded as mathematical monsters, because of their unusual properties. In 1975, Mandelbrot called them fractals, from the Latin word *fractus*, meaning *fraction*. Fractal Geometry is around us, into the clouds, rivers, nature (Mandelbrot B.B. 1998):

"Euclidean geometry is unable to describe the complexity of nature, (because) by observing the nature, we are able to see that clouds are not spheres, mountains are not cones, coastlines are not circles but they are complex geometrically objects."

Fractals are geometrical figures, that are characterized by unlimited repetition of the same shapes of a more lowered sequence. Fractal's proprieties are: self-similarity, scaling laws and not integer dimension. There are different types of fractal: IFS (iterated function systems) and L-system.

What an L-systems fractal is

L-systems fractals have been conceived as a mathematical theory of plants development. After the incorporation of geometric features, plant models, expressed using L-systems, become detailed enough to allow the use of computer graphics for realistic visualization of plant structures and their development processes. Aristid Lindenmayer (1925-1989) was an Hungarian biologist who developed a formal languages called Lindenmayer Systems or L-systems to generate fractals. They were introduced as a theoretical framework to study the development of a simple multicellular organisms and subsequently applied to investigate higher plants and plant's organs.
The central concept of L-systems is rewriting. In general, rewriting is a technique to define complex objects by successively replacing parts of a simple initial object using a set of rewriting rules or productions.
In 1968, Aristid Lindenmayer, introduced a type of string-rewriting mechanism, subsequently termed L-systems.
In L-systems we can define a string as an ordered triplet $\mathbf{G = (V, \omega, P)}$ in which:
1. $\mathbf{V}$ is a finite set of symbols called alphabet;
2. $\boldsymbol{\omega} \in \mathbf{V}^+$ is a non-empty word called axiom ($\mathbf{V}^+$ is the set of all non-empty words over V);.
3. $\mathbf{P}$ is a finite set of production**:** $\mathbf{P \subset V \times V^*}$, $\mathbf{V^*}$ is the set of all words over V.
P defines how the variables can be replaced with combinations of constants and other variables. A production (a, $\boldsymbol{\omega}$) $\in$ P is written as a $\rightarrow \boldsymbol{\omega}$. The letter a and the word $\boldsymbol{\omega}$ are called the predecessor and the successor of this production, respectively. It is assumed that for any letter a $\in$ V, there is at least one word $\boldsymbol{\omega} \in V^*$ such that a $\rightarrow \boldsymbol{\omega}$. If no production is explicitly specified for a given predecessor a$\in$ V , the identity production a $\rightarrow$ a is assumed to belong to the set of productions P (Lindenmayer, Prusinkiewicz,1990).
The formal language is not enough for building imagines or for creating geometric patterns. At this point it is essential working by multimedia tools, choosing a software to convert some numerical and symbolic codes in imagines.

L-system Fractals: what technology?

One of the geometric systems that computer graphics use for the L-system's generation is called Turtle Geometry. The basic idea of turtle interpretation is given below.
A state of the turtle is defined as a triplet (x,y,α) where the Cartesian coordinates (x, y) represent the turtle's *position*, and the angle α, called the *heading*, is the direction in which the turtle is facing.
Given the step size *d* and the angle increment *δ,* the turtle can respond to commands represented by the following symbols:

1. **F** (it moves forward a step of length *d* the state changes to (x' = x + dcosα, y'= y+dsinα, α). A line segment between points (x, y) and (x', y') is drawn;
2. **f** (it moves forward a step of length *d* without drawing a line);
3. + (it turns left by angle δ the state changes to (x,y,α+δ));
4. – (it turn right by angle δ, the state changes to (x,y,α-δ)).

Given a string ν, the initial state of the turtle (x_0, y_0, α_0) and fixed parameters *d* and *δ*, the *turtle interpretation* of ν is the figure drawn by the turtle in response to the string ν. Specifically, this method can be applied to interpret strings which are generated by L-systems (Prusinkiewicz P.,1999).
For example, in the following three approximations of the Koch snowflake. These figures are obtained by interpreting strings generated by the following L-system:
1.axiom **ω: f++f++f** start angle: 90°, turn angle 60° (it corresponds to the initiator or starter figure of the fractal) (Fig.1a).
2. production **p: f=f-f++f-f** (it corresponds to the generator of the fractal) (Fig.1b).
3. Koch snowflake at sixth generation (Fig.1c).

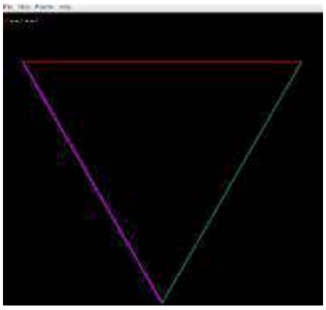
Fig. 1a


Fig. 1b

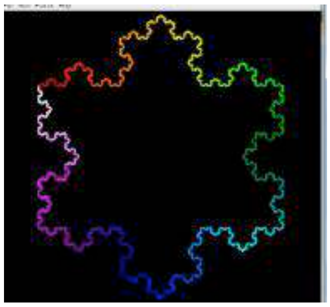
Fig. 1c

The figures has been made by Fractal Grower. It is a Java software for Growing Lindenmayer Substitution Fractals (L-systems) created by Joel Castellanos, Department of Computer Science, University of New Mexico (http://www.cs.unm.edu/~joel/PaperFoldingFractal/paper.html). We use it just for educational purpose.
The software is easy to use and above all, its versatility fosters the exploration of fractals and their properties. For example it is possible to change the start figure but the final fractal does not change. As you can see in the below figures (Fig. 2a, 2b, 2c) the start figure is an hexagon, not a triangle, the final fractal is that shown in Fig. 1c:

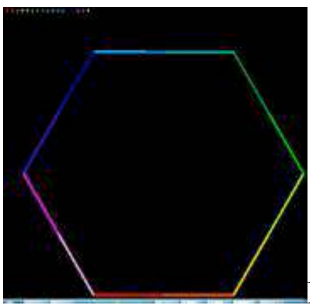
Fig. 2a

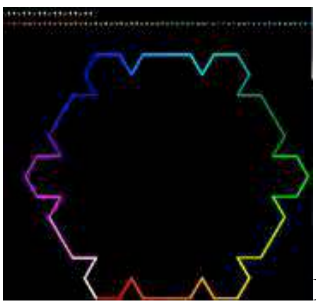
Fig. 2b

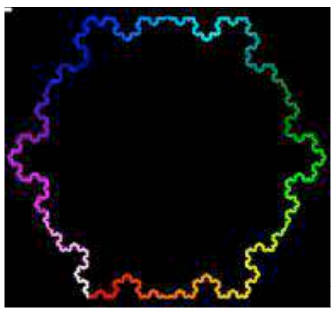
Fig. 2c

Fractal Grower, or any other similar software, allows to manipulate geometric objects graphically, for example translating, turning or reducing them, it helps to understand how the geometric transformations work or why the numerical codes change in dynamic figures, so that pupils can increase their curiosity, their imagination, their geometrical knowledge.

Below there are some pictures and codes of L-system fractals generated by Fractal Grower (Fig. 3, Fig. 4):

Fig. 3. A fractal

Axiom **: f-f-f-f ;**Production: **f=ff-f-f-f-f-f+f** at the sixth generation

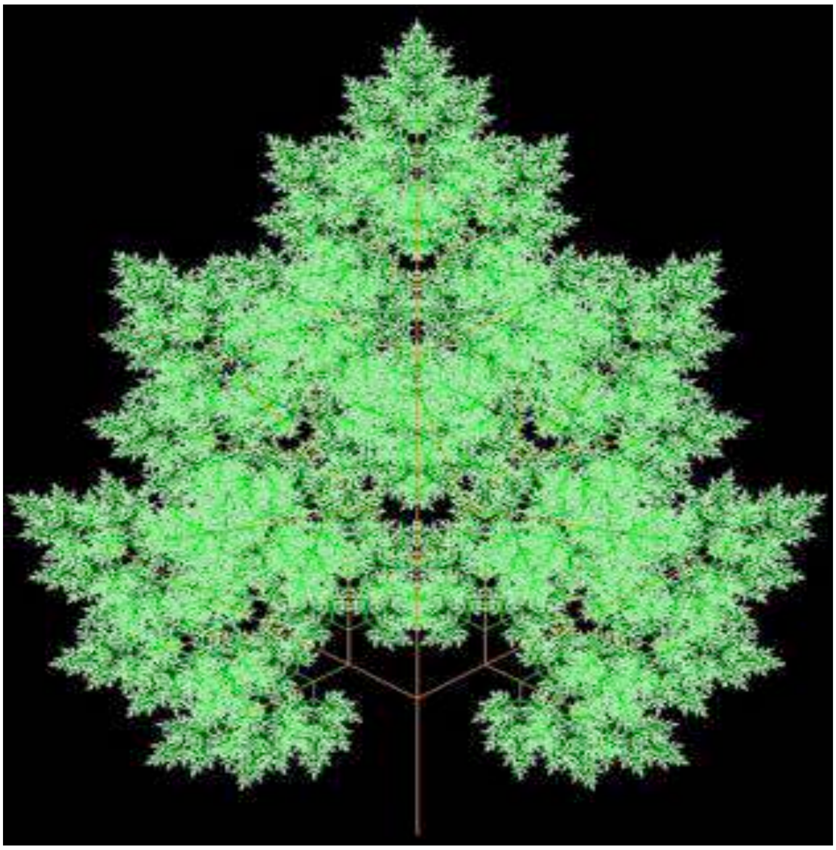

Fig. 4. A fractal tree

Axiom **ω:af;** Production: **f= a![---f][+++f]![--f][++f]!f** at sixth generation.

L-system Fractals: what educational approach?

In our school a team of teachers (me included) have been participating since eight years to the project Mathematics & Reality, a national project managed by Prof. Primo Brandi and Prof. Anna Salvadori (University of Perugia). The project deals with making proposals to develop unsuspected and educational relationships between mathematics and the real world.
Mathematics & Reality (M&R) is an innovative project of educational mathematics which central point is a dynamic interaction between the real world and the mathematical world, which is achieved by the language of mathematical models. The mathematical model of a real-world event is a process of rationalization and abstraction that allows to analyze the problem, to describe it as an object and create a simulation using the universal symbolic language of mathematics. Since 2006 we have been proposing many different mathematical laboratories attended by around 1,000 pupils. The M&R project is at the moment one of the most important extra-time activity of the school.
In M&R project, Fractal Geometry is the topic of my educational activity. It has three different phases: *Fractal Class, Fractal Webquests, Fractal Cloud learning.*

First step: Fractals Class

Fractal Class is an extra-time course about fractals. It takes 16 hours (one two hour-lesson per week). Around 50 pupils per year on average attended this course (totally 300 pupils since 2006). During the first lesson goals and contents are proposed to the pupils.
Goals:

- modelling the world around us using affinity transformations;
- building the most famous fractals (Sierpinski’s Triangle, Koch’s snowflake, etc.);
- plotting fractals by software tools;
- making conjectures and simulations by using free software;
- mixing traditional teaching and new technologies.

Contents:

- Geometric transformations and matrices: composition of geometric transformations; inverse of a geometric transformation, affinity transformations: rotation, contraction, translations.
- Iterated Function System Fractal by genetic code: evolution of an iterative process of figures, attractors and fixed figures.

During this phase, the following educational approaches are applied:
• interactive lesson: between the teacher and pupils. The teacher proposes the open questions, and the class must answer or fill the missing parts of a theory by searching on Internet.
• Cooperative learning: it allows horizontal communication improving the cooperation among pupils, encouraging their participation and their involvement without inhibition, promoting their skills.

Second step: Fractals WebQuests

Lessons and school courses are not the only source of information for pupils. Living in the digital era means to exploit a flux of information, this is an advantage for making student more autonomous in his knowledge process. Before the Fractal Course ended, the teacher proposes some research topics about Fractal Geometry to whole class. It is a suggestion for a more deep study about fractal geometry after the end of the course and, at the same time, it is a new step of the activity, to which a few pupils joined spontaneously. The main aim of this additional activity in M&R project is aimed to the participation at M&R National Congress, held every year at the University of Perugia. During the conference pupils make a presentation about their research work and compete for the Best Presentation in Mathematics. Along these years experienced with pupils many works have been produced such as:

1) "To make a tree… it takes an L-fractal-system" by Jessica Rubino, Chiara Punturiero, Ersilia Paparazzo.
2) "Fractal snowfall in Catanzaro" by Lucia Mazza, Stefano Caglioti, Sara Virgilio, Emmanuele Benedetto.

Both works have been presented by pupils at National Congress of Maths & Reality in Perugia in 2012 and 2014. The first topic was also presented in Gotheborg in Euromath (European Mathematical Congress for Students) in 2013. Both works represent an example of mixing affinity transformations, matrices and L-system fractals by using different free digital tools.
In this phase the webquests process is applied. This educational approach has the main goal to discover additional information on a particular topic and to provide some product using the gathered information. This teaching methodology was created by Bernie Dodge at University of San Diego (USA) in 1995 and today is recognized at international level. Dogde (2001) defined it as:

an inquiry-oriented activity in which most or all of the information used by learners is drawn from the Web. WebQuests are designed to use learners' time well, to focus on using information rather than looking for it, and to support learners' thinking at the levels of analysis, synthesis and evaluation (p. 6).

Internet is a chaotic space, conveying a myriad of information, endless realities, news, experiences; the cyberspace lives on exchanges, enriching overall product with multiple contribution. This chaotic knowledge must be deciphered, selected, structured, otherwise your search could seem sterile and superficial. Webquests permits to avoid getting lost into the network and to economize the time. To achieve that efficiency and clarity of purpose, WebQuests should contain at least the following parts (Dogde B. 2001):

1. An *introduction* that sets the stage and provides some background information;
2. A *task* that is doable and interesting;
3. A *set of information* sources needed to complete the task;
4. A *description* of the process the learners should go through in accomplishing the task;
5. A *conclusion* that brings closure to the quest.

A few lessons are needed to gather data and elaborate the structure of the research topic. During this step topic is assigned, the structure of the work is decided and the free software is chosen together by pupils and teacher. In the framework of both works there are four parts:

1. fractals in general (what a fractal is, the most important properties about fractals);
2. mathematical contents (affinity transformations and matrices);
3. L-system fractals (theoretical definitions about L-system fractals and their codes);
4. Logo (or Turtle)-language and free software for plotting imagines of fractals. Among them we used Fractal Grower, how it said before.

Thank to this activity, pupils acquire new skills. They learn to look for information on the web, to select the most relevant parts and to apply the most suitable among them. It is therefore a working strategy strongly characterized by cooperative work and problem-solving.

Third step: Fractals Cloud-Learning

After fractal course and fractal webquests, teacher works with groups of pupils to improve research and create multimedia presentations for M&R congress. When the course ended, pupils and teacher can not interact one each other at school, so technology offered new opportunities in improving teaching and learning. They enable individuals to customize the working/learning environment

using a range of tools to meet personal interests and needs. This is the reason why it has been explored the educational potential of 'cloud computing'. Web 2.0 tools give the choice to interact and cooperate each other in a social media dialogue as creators of user-generated content in a virtual community. Examples of Web 2.0 technologies include social networking sites, blogs, wikis, video-sharing sites. They also have the potential to promote sharing, openness, transparency and collective knowledge construction. During our experience we use Google Drive (as a storage of files) and Google Docs (as virtual learning environment). Inside this space pupils and teacher can exchange and share ideas and information and they can work at the same time – despite not being in the same classroom – in order to achieve the final version of the multimedia presentation. The advantages of using Google Docs are:
1. multiple people can work at the same time on the same document and everyone can see people's changes as they make them, and every change is saved automatically;
2. everyone can collaborate in real time over chat. If more than one person has the document open, just click to open a group chat. Instant feedback is possible without leaving the document.
In this new learning environment teacher acts more systematically as advisor, guide and supervisor, as well as provider of the frameworks for the learning process of pupils. They have greater responsibility for their own learning in this learning environment, as they look for, find, synthesize and share their knowledge with others. During the cloud-learning phase, pupils and teacher work together and study the theoretical content of L-system fractals, they analyze and simplify theory to include it in the presentation.

Results

The most important goal of this work is to show new opportunities generated by ICT in learning and teaching geometric contents. L-system fractals demonstrate to be a good topic to get this achievement. In fact, Web2.0 tools are fundamental and essential in order to:
1. improve theoretical contents about L-system fractals theory;
2. create new virtual learning and teaching spaces, in which the student is left alone to choose the materials, to organize autonomously theoretical and technological topics.
Pupils are able to make new conjectures and create new fractals initially studying the appropriate geometric transformations and then checking the results with the help of the recommended software.
Results obtained after the activity are:
1. enhancement of mathematics skills (geometric transformations, iterative processes and functions, matrices and determinants);
2. enhancement of the mathematical competence in the sense of the definition given by PISA4:

Mathematical skill is the ability of an individual to identify and understand the role that mathematics plays in the real world, to operate based assessments and to use mathematics and confront it in ways that meet the needs of the life of that individual as citizen exercising a constructive role, committed and based on the reflection.

The educational activity applied in this project fosters the development of the personality and the attitudes of the pupils, supports them during their educational and emotional growing.
Also the teacher has advantages testing this activity:
- discovering new educational virtual environments where to share and cooperate with pupils and colleagues;
- exploring new methods to renew teaching and learning of mathematical topics.
New challenges have yet to be experienced about the educational approach, submitted in this paper, among them: to extend it to several mathematical topics and, even more difficult, to engage the whole class.

REFERENCES

Artigue, M. (2013). Teaching Mathematics in the Digital Era: Challenges and Perspectives. In Y. Baldwin (Ed.), Proceedings of HTEM, July 14th-19th, 2013. Brasil, 1-20

Barnsley M.(1993). Fractal Everywhere, Morgan Kaufmann Pub.

Brandi P., Salvatori A. (2004).Modelli matematici elementari, Bruno Mondadori.

Dogde B.(2001). Five Rules for Writing a Great WebQuest. Learning & Leading with Technology. 28 (8), 6-9.

Drijvers P. (2012). Digital Technology in Mathematics Education: Why it works (or doesn't). ICME-12 2012.

Gowers T. (2004). Matematica: un'introduzione. Einaudi.

Jaworski, B. (2005). Learning Communities In Mathematics: Creating an Inquiry community between teachers and didacticians. Papers of the British Society for Research into Learning Mathematics, Research in Mathematics Education, 7, 101-120.

Katz R. (2009). The Tower and the Cloud: Higher Education in the Age of Cloud Computing, Educause.edu

Lindenmayer A., Prusinkiewicz P. (1990). The Algorithmic beauty of plants. Electronic Version. Springer-Verlag. New-York.

Mandelbrot B.B. (1998) The fractal geometry of nature W.H. Freeman and Comp. New York.

Prusinkiewicz P. (1999). A look at the visual modeling of plants using L-systems. Agronomie, 19(3-4), 211-224.

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Being Collaborative, Being Rivals: Playing wiigraph in the Mathematics Classroom

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Abstract: This paper describes an activity that involved a 9th grade class in graphing motion with WiiGraph, an interactive software application that uses the two Nintendo Wii remote controls to graphically display the position of users. Potential interactions with WiiGraph allow the students to experience spatio-temporal mathematical relationships through tasks in which they can be collaborating or competing with each other, offering kinaesthetic learning challenges. We analyse the ways in which these challenges affect the doing/doer of mathematics, as well as the mathematics itself, giving rise to new mathematical subjectivities. To do so, we pursue a participationist vision of learning with tool, which joins the view about playing mathematical instruments by Nemirovsky et al. (2013) and that about distributed agency by Rotman (2008) and de Freitas and Sinclair (2014), for which the human learner always reinscribes herself into a mathematics that is changed by its encounter with the technology in use.

Résumé: Cet article décrit une activité proposée à des élèves de 14-15 ans. Elle est basée sur l'analyse des représentations graphiques des mouvements obtenus avec WiiGraph, un logiciel interactif qui utilise les deux télécommandes Nintendo Wii pour afficher graphiquement la position des utilisateurs. Les interactions potentielles avec WiiGraph permettent aux étudiants de vivre des relations mathématiques spatio-temporelles à travers des tâches qui les engagent dans des défis kinesthésiques pour l'apprentissage où ils peuvent collaborer ou être compétition. Nous analysons comment la composante kinesthésique affecte l'activité mathématique ainsi que le sujet impliqué dans cette activité provoquant de nouvelles subjectivités mathématiques. Pour mener cette analyse, nous choisissons une vision participationniste de l'apprentissage avec les instruments, qui joint l'idée de «jouer les instruments mathématiques» proposée par Némirovsky *et al*. (2013) et celle de «l'action distribuée» par Rotman (2008) et de Freitas et Sinclair (2014), pour lesquels l'apprenant se réinscrit toujours dans des mathématiques qui sont modifiées par sa rencontre avec la technologie utilisée.

Introduction

In their work about playing mathematical instruments, Nemirovsky and colleagues (2013) assume a *non dialectic* approach to mathematical tool use that questions any dualism between perceptual and conceptual, "between bodily, tool-mediated expression and mental structures or schemes" (p. 376), which would entail an acquisitionist *rather than* a participationist view of learning. While, in an acquisitionist perspective, learning is conceptualised in terms of mechanisms that students are expected to acquire, pursuing a view of the second kind entails recognising the students' ways of talking, moving and feeling as communicating and learning within the classroom. Nemirovsky *et al.* propose that mathematical thinking and learning inheres in transformations in lived bodily experience. Lived experience has to be intended as a temporal flow of perceptuomotor activity, that is, perceptuomotor activity "is always permeated by expectations, recollections, fantasies, moods, and so on" (p. 378). These aspects are part of the ways in which students speak, do, and feel as well as of their changes over time. The authors are interested in analysing this flow in the context of using what they call a *mathematical instrument* (which will be discussed in the next section). In the study, they point out that dynamic geometry environments (DGEs) and motion detectors are both families of mathematical instruments that have received much attention from research. For example, they argue, dragging in a DGE can support important integration of motoric and perceptual aspects, being the first given by dragging geometric elements in a diagram by hand and the second by visual feedback on the whole diagram. Motion detectors instead involve similar aspects in how they allow students to explore the modelling of their own body movement by means of real-time graphs.

Thinking about tool use in teaching and learning mathematics, we should be concerned not only with the students' ways of communicating, but also on how the mathematics taught and learned changes in interacting with the tool. As Rotman (2003) claims, "the effect of the computer on mathematics (pure and applied) will be correspondingly far-reaching and radical; that the computer will ultimately reconfigure the mathematical matrix from which it sprang and will do this not only by affecting changes in content and method over a wide mathematical terrain, but more importantly by altering the practice and overall nature, and perhaps the very conception we have of mathematics." (p. 1675). In his more recent book *Becoming besides ourselves*, Rotman (2008) adds that mathematics is permanently altered by its encounters with new technologies, thus the future of mathematics will see a reduction of an alphanumeric hegemony and, especially, a move toward visual and dynamic mathematical expressions. Drawing on the image of the post-alphabetical world of Rotman, Sinclair (2014) argues that this will imply a new kind of sensory politics at play, since digital technologies have been "steadfast in their acts of dissensus" in relation to the dominant regime of sense-making. The dynamic geometry triangle, for example, carved out "a new dimension for time, thus changing what was taken to be common sense about how you saw, drew and talked about triangles." (p. 175).

What is intriguing in this discourse is the kind of *mathematical subjects* that students are becoming in learning mathematics with technology, or, briefly speaking, the fact that "in these environments driven by the hand or body, the human is constantly reinscribing herself into the idealized, abstract mathematics." (Sinclair, 2014, p. 168). It is through this 're-inscription' that the students' encounters with the mathematics in terms of their lived-in experience become sites of *agency* in the classroom, and that agency is constituted across the learning situation, being no longer attributed to the individual learner as the *only* centre of mathematical activity, but to her engagement with the material surrounding. In this perspective, we pursue a participationist vision of learning with tool aligning with this re-inscription of the self or, following Rotman, with the increasingly break down self, who becomes plural, distributed and *besides* herself, in a word: *post*human. We do this in the specific context of graphing *motion*, which we find challenging for the particularly important position taken on by motion: concerning the way that mathematics struggles to reckon with time, in terms of the learners' kinaesthetic engagement, and for its being so constitutive of self.

Agency and mathematical instrument

The vision we adopt in this paper troubles the traditional conceptualist idea that materials and tools have confined properties of their own. Instead, it has an *interactionist* nature for which materials and tools are not inert but are constantly interacting with each other and with the human body. In so being, it is far from the Cartesian reading of the human body, according to which the human body is exceptional in its freedom and will, and the human mind animates the body while other bodies lack any such *agency*. We follow here De Freitas and Sinclair (2013, 2014), who propose an alternative theoretical view of the body, drawing on materialist theories that consider freedom and agency as "dispersed across human and non-human agents" (de Freitas & Sinclair, 2014, p. 39), without centring "human will or intention in the orchestrating of experience" or conceiving "the human body as the principal administrator of its own participation" (*ibid*., p. 19). Instead, they unbind the body from its skin for shedding light on the ontologies of both body and mathematics, and breaking "with binaries that set organic against inorganic, and animate against inanimate, so that matter might be re-animated more generally and seen in terms of potentiality and emergent generative power" (de Freitas & Sinclair, 2013, pp. 459-460). In so doing, they rethink the boundaries of the body in the mathematics classroom, so that boundaries are constantly re-created and *assemblages* emerge as the *body* and as the unit of analysis in the learning experience of students.

This distances us from discourses of identity, causality and determinism. It also opens room for a discourse of subjectivity that, rather than locating knowing in the individual body, attempts to adequately address the collective social body. In line with posthumanist theories of subjectivity, subjects "are constituted as assemblages of dispersed social networks" (de Freitas & Sinclair, 2014,

p. 33), and the human body has to be conceived in terms of distributed networks where the material and the social are fused. Attention is shifted to distributing agency across a network of interactions, the properties of which are constantly changing (Rotman, 2008). This is an even more fascinating position when referred to interactions with tools. For example, Rotman talks about "becoming beside ourselves" to capture the new acentred sense of subjectivity that is emerging this century, in part, claim de Freitas and Sinclair (2014), "because of new digital technologies that herald and hail a network 'I' which thinks of itself as permeated by other collectives and assemblages" (p. 36). This vision problematizes the idea that any one part of the assemblage is the source of action, intention or will. Plural agency entails the formation of new assemblages and new folds upon the working surface, where digital tools are like everything else in being materialities that do not have determinate boundaries. Instead, they operate within the relations of the ever-changing assemblage participating in the network 'I'. Thus, distributing agency allows theorizing the role of tools in learning mathematics in ways that the traditional cognitivist could not. Rotman would outline how mathematical activity co-involves the concepts, the learner and the material world, including tools, and that this co-involvement means that "mathematical activity does not just produce more mathematics (or more learning), but also produces a new person in a new material world" (de Freitas & Sinclair, 2014, p. 109). In this sense, subjects constantly reinscribe themselves into the mathematics.

Focusing on learning mathematics with tool, we propose to join the vision of distributed agency with the notion of *mathematical instrument* introduced by Nemirovsky *et al*. (2013). Nemirovsky and his colleagues propose an alternative perspective for tool use, which talks about mathematical knowing as constituted by (not dialectically related to) embodied tool use, and attempts to avoid a deterministic view in line with the Cartesian reading of the human body in interaction with tools. In the particular context of a science museum exhibit, these researchers study emergent fluency with a mathematical instrument as a way to access certain kinds of mathematics. In so doing, they "locate tool fluency as well as mathematical thinking and learning in the process of perceptuomotor integration when learners engage with others and physical artifacts" (p. 378). In this view, a mathematical instrument is defined "as a material and semiotic device together with a set of embodied practices that enable the user to produce, transform, or elaborate on expressive forms (e.g., graphs, equations, diagrams, or mathematical talk) that are acknowledged within the culture of mathematics." (p. 376). The term instrument is used, instead of tool for example, because it intentionally connotes the culture of music, where one cannot speak of "a violinist's expertise as something divorced from the quick movements of her fingers over the strings and the trained dance of her eyes across a musical score" (p. 377). In a similar way, one cannot talk about mathematical expertise divorcing it from the "skillful motoric and perceptual engagement with the tools of the discipline." (p. 377). So, the metaphor of *playing mathematical instruments* is used to conceptualize mathematical instruments analogously to musical instruments. Nemirovsky and colleagues claim: "[f]luent use of a mathematical instrument allows for culturally recognizable creation in mathematical domains, just as musical instruments enable practitioners to produce distinct kinds of music that members of musical communities acknowledge." (p. 373). Then, emergent tool fluency is described in terms of perceptuomotor integration, a phenomenon that occurs when the perceptual and motoric aspects of using a mathematical instrument are intertwined. The relevant point is that, for these researchers, perceptuomotor integration is constitutive of mathematics learning processes and is common to using a wide variety of tools, not only those considered in their study.

WiiGraph as a resource in the classroom

In the context of graphing motion, the paper presents an activity in which we used the Nintendo Wii as a resource for the learning of mathematics. The choice is not casual, since the Wii offers with its devices new game experiences to its users, with lots of possibilities in terms of kinaesthetic and proprioceptive engagement. The main devices are the Wii Remotes (Wiimotes), the controllers for playing games with the Wii, and the Balance Board, a platform through which it is possible to play some specific games related to balance situations. In our study, we introduced in the mathematics classroom WiiGraph, an interactive software application, which works with the aid of two controllers. WiiGraph has been developed and manifactured at CRMSE, the Center for Research in Mathematics and Science Education, of San Diego State University, by R. Nemirovsky and colleagues. It allows the users to see their positions (as distances from the Wii sensor bar) displayed on a graphical area. In particular, according to various options, it is possible to work with two coloured lines of position versus time on a single Cartesian plane, or adding to these lines on the plane an operation of the two positions, again as a function of time, among the four operations of sum, difference, product and division. In this case, two students can be asked to move in front of the sensor bar and to explore what kind of line is the new one, or to *collaborate* for moving in order to get a given graph line as a result of a certain operation.

WiiGraph also provides many opportunities for other interesting experiences. We give some examples. On the one hand, the two students can be required to move for matching a target maze. In this situation, the learning experience embeds the challenge between the two for being as precise as possible with creating the target graph. The students are *rivals* and they 'play the game' to *compete* in getting the best score. On the other hand, the learners can be challenged by the teacher to produce a Versus graph of the two positions, which essentially plots an ordered pair of the distances of each learner over time. For instance, tasks can involve the creation of plane figures, like a rectangle, a circle, and so on. In this situation, the students need to be *collaborative* for moving to obtain the given figure. It is also possible having the students to collaborate for challenging the machine to match a target operation line or versus graph.

By engaging the students in grappling with these challenges via competitive or collaborative, embodied kinaesthetic interactions, these technologies can be valued resources to gain meaningful insights into temporospatial mathematical relationships, and to approach the concept of function in the context of modelling motion through kind of gaming experiences. Below, we propose some of the potential interactions with WiiGraph that we designed to foster new dynamic learning spaces. Moreover, we will discuss how, through these interactions, the mobile and the kinaesthetic change the doing/doer of mathematics, as well as the mathematics itself, also bringing about new challenges for teachers.

Interactions with WiiGraph

WiiGraph works with two Wiimotes connected via Bluetooth to the computer where the software is running and via infrared technology to the sensor bar of the Wii. The software allows displaying the location of the two controllers over time on a single Cartesian graph area, while users move holding them in front of the sensor bar, in a large space for interaction devoted to embodied explorations. A display area is needed to project the computer screen so that the interactions can be shared within the classroom context. In order for a graph session to start, each Wiimote has to be directed steady at the sensor bar, in which case a coloured diffuse circle appears on the graph area. With the circle visible for both controls, WiiGraph produces two real time coloured graph lines corresponding to the two controls' movements (the graphs of the two functions $a(t)$ and $b(t)$, being a and b the distances of the controllers from the sensor). Several graph types, composite operations, targets and challenges can be set. The two that are the main focus of this paper are the "*Make your own Maze!*" target and the $a+b$ operation that can be activated in the case of the *Line* Graph type. The first option allows for the creation of a target maze, with a certain complexity (given by number of inflection points, thickness, tension and layout of the line), which becomes the challenge target for the users. Figure 1a shows an example of a session of this type, also giving the final score of the two "players". As a resource for learning, this first option is captivating to design tasks that make pairs of students to be rivals, each player against the other, challenging each other to get as close as possible to given graphs. The second option permits to have a third coloured graph on the screen, which results from the sum of a and b over time, that is the graph of the function $(a+b)(t)$. As a resource for learning, this option is intriguing if we think of tasks that require couples of players to obtain specific lines as sum graphs (this can be realised through plain tasks or using a "*Make your own Maze!*" target for the sum). In these second kinds of tasks, the students need to be collaborative to reach their goal. Obviously, the same can be replicated with a different operation among the four that are available in the software.

WiiGraph also offers another interesting modality for designing collaborative tasks, that is, the *Versus* Graph type. This type plots an ordered pair of the distances of each user over time (with the production of the ordered pair that remains implicit). Practically, in this case, graphs are obtained by pairing the two users' functions of position versus time, one for the horizontal axis and the other one for the vertical axis. So, the spatial graph corresponding to two parametric functions (where the parameter is time) is constructed. As a resource in learning mathematics, this option may favour tasks that require the students to create shapes or plane figures, like for example rectangles, rhombuses, and circles (see a trial for the circle in Fig. 1b), thus entailing the need for an interactive coordination between collaborating users. The case of the circle is of very concern whether we think of approaching it as limit of regular polygons with a progressively increasing number of sides, and in relation to the study of the sinusoidal functions as characteristic components of circular motion.

We also believe that the modalities of being collaborative/being competitive deserve lots of attention from the pedagogical point of view. They offer great potential for new learning spaces, where, beyond the mobile and the kinaesthetic, the influence/interference between students who are "playing" the tools *together* are a compelling resource in the teaching situation, which might engage learners in completely new manners and at differently deep levels, use their strengths and curiosities to fuel sophisticated mathematics, as well as stimulate discussions about the values of collaboration and competition.

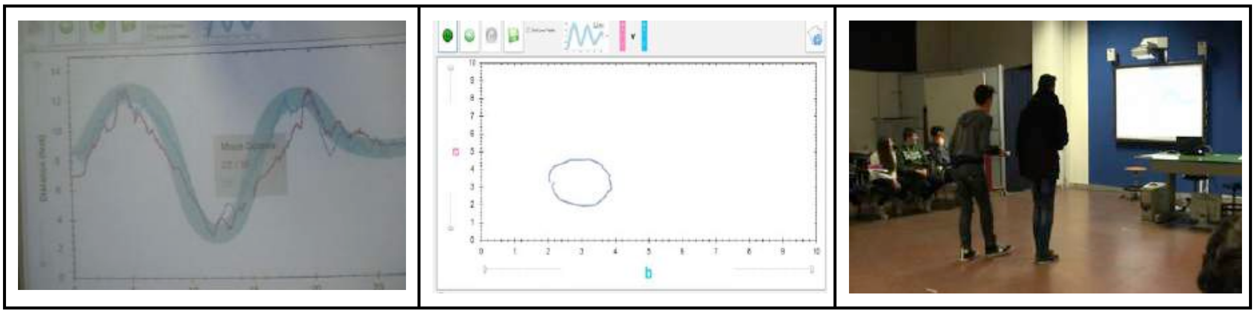

Figure 1. (a) "Make your own Maze!" and (b) Versus Graph sessions; (c) Students playing

Method

Our study has involved a 9th grade group of 30 students and their regular teacher in mathematical investigations through the use of WiiGraph. The study lasted a total of 9 two-hour weekly meetings, which were preceded with an individual questionnaire about believes on function and graph. The activities were carried out in a lab room used as a laboratory space for mathematics lessons. We set the room projecting the computer screen on an IWB, with the students sat around the interaction space, apart from the two holding the controllers. This way, all of them could watch the on-line acquisition of WiiGraph and the real-time creation of the graphical lines. All the experiences and tasks were designed in collaboration with the class teacher along the entire course of the study. In addition, during the meetings the students worked in various manners. They took part in some individual tasks, group works and collective discussions led by the researchers (the two authors), who were always present in the classroom. All these phases were also filmed through the use of one or two mobile cameras. Data for the analyses comes from the movies and from the written productions of the students. In the next section, we present examples from the classroom and we briefly discuss them using the theoretical commitments that we have elaborated above.

Discussion of classroom examples

Episode 1. Being rivals or *Make your own Maze!*

The first example concerns a task with a competitive nature. The task asked the students, divided into groups of three, to choose one among them for a challenge. The challenge consists of competing with a class-mate from another group to match a target graph, using the "*Make your own Maze!*" modality of WiiGraph. This modality furnishes as a target a graphical line with a given shape and thickness, which can be set up through the options. Different target graphs were given to different couples of competing groups. For example, Figure 1a shows the light blue tick graph that was the target for the first two players: Emanuele and Oliver. We can see that this line had two humps. The other two lines in the Figure were obtained through the movement of the two students, and captures the functions $a(t)$ and $b(t)$.

Before moving in front of the sensor, each player received suggestions from his group-mates about the suitable movement to perform. Then, the students moved (Fig. 1c) and Emanuele won. He got a score of 28 out of 35 (blue line) against 22 out of 35 (pink line), obtained by Oliver.

In their being rivals, Emanuele and Oliver are affected by their moving close to each other, as well as by the feedback that the software gives them in terms of the real-time lines appearing on the screen. This is apparent in how the two students move during the challenge, when their tension towards winning is clearly manifested by their posture and voice engagement (for example, their laughing). At the end, Oliver is unsatisfied about his result, which he associates to having made a "mistake", as he affirms when the researcher asks the students if they have reflections about their experience:

Oliver: I made a mistake on the second [hump] (*Points towards the hump on the graph, from his chair*) to go up, because I had to move faster. I didn't realise that

Researcher: Where, do you say? (Invites him to better explain at the IWB)

Oliver: Here I have really gone out (*Without touching the board but being quite close in front, mimes with his open right hand the increasing pink line piece, which is out of the target graph: Figg. 2a and 2b. The gesture occurs in a plastic way with the head and body softly tended towards the IWB so as to go along with the slope of the line*), because if I would have moved faster (*His right index finger runs along the corresponding piece of the target graph exactly staying within its thickness: Fig. 2c. Turns to look at the researcher, laughing*) I would had been more in the graph (*Repeats the previous gesture moving his finger faster than before*)

Figure 2. (a) Oliver's gesture and posture for the line's slope; (b) Following the target line

The brief episode shows how Oliver bridges the 'distance' between his "mistaken" pink line and the corresponding target graph in the graph area. His words are relevant in the discourse, especially in their being entangled with other components, like his ways of moving and feeling. At first, Oliver uses the verb "to go out" to speak about the pink line piece that goes out of the target thickness, as witnessed by his miming gesture of the line. He is comparing the two graphs but, at the same time, the subject "I" associated with the motion verb fuses the line and the motion experience (see the use of the past tense and of "really"). In doing so, it is constituted as a plural "I", which is distributed between the space and time of the motion challenge, the space-time graphs and the feeling of being mistaken, accentuated by Oliver's plastic and soft bodily movement so close to the IWB although without touching it (Figg. 2a and 2b). In a second moment, Oliver justifies ("because") his having "gone out" through the logical "if" that suddenly brings into discourse a new dimension: that is, a new virtual pink line related to a new "faster" movement. This time, Oliver first runs his finger along the suitable piece of graph, making present the virtual line within the thickness of the target graph, while he explicitly speaks of his motion experience through the use of the verb "to move". Then, the changed speed with which he repeats the running gesture actualises the higher speed of the new imaginary movement, while also actualising again the new pink line and its slope. The new line means to be "more in the graph". On the one hand, this last verbal expression seems to underline the imaginative presence of the student "inside" the graph, merging the line and the motion experience again, in contrast with the present situation (past in movement) in which Oliver is outside of the graph. On the other hand, it also seems to actualise the mobility of the pink line in the situation: the line assumes different slopes depending on the movement.

Interpretation of Episode 1

We can interpret the episode saying that Oliver reinscribes himself into the temporospatial nature of his challenge with Emanuele. He goes back to the spatial and temporal dimensions of the motion experience, by thinking of the new speed ("faster"). But he does this in the present spatio-temporal dimension of the graph ("on the second", "out", "in the graph"), by actualizing in word and gesture the virtual line, which only comes to exist in relation to the current line. Perceptual and motoric aspects of the challenge are both recovered through this actualization. Oliver is 'moving' back and forth, between the graph space and the interaction space, making the slope of the line and the speed of the motion experience change together, one depending on the other. The assemblage of meaning develops through his ways of doing, gesturing, talking and feeling (like his last laugh), which also mobilize the mathematics at play. In his being part of this changing assemblage Oliver is learning with tool, starting to gain some fluency with WiiGraph used in the *Make your own Maze!* modality.

Another aspect that we see as relevant in the episode concerns the fact that the lived-in challenge with Emanuele is present in Oliver's tension towards correcting himself and finding a way of getting a better graph, which would imply a better score. The particular experience of being rivals, allowed by the software in this case, is not neutral with respect to the ways in which Oliver communicates with the class. For example, his use of the adverb "really" seems to underline the force and affect of what exactly occurred and to justify his losing of the challenge. Briefly speaking, in this moment, it is likely that Oliver is not simply thinking of him as being mistaken because he was not able to be "more in the graph", just in terms of a line that goes out from the target thickness. Rather, for him it is much more a question of not being able to be "more in the graph" *with respect to* his rival.

Episode 2. From competing to collaborating

The other couples of competing groups were given the same challenge, with new target graphs, and the result of each challenge was discussed collectively. After this phase, the competing groups were asked to join in order to face a written task together. The task was essentially made of two requests. The first request was formulated as follows:

> "You have tried to move, competing with each other, in order to be as faithful as possible to a given graph on the screen. You have also obtained a measure of your accuracy (precision with

respect to the complexity of the graph). Write down your reactions and sensations just after the challenge (distinguishing the voices of the two groups)."

A second request asked the students to give advice to an imaginary friend, who has to compete (but never used WiiGraph), and to explain the strategies to reach a score as good as possible.

Emanuele's group and Oliver's group produced a protocol in which they drew their target graph, and two answers for the first request, which are very similar between each other (see Fig. 3). Oliver and his group-mates wrote: "Our sensations are: that of changing speed in order to create a different slope to create different kinds of "humps". The second difficulty lies in finding the starting point such that it is possible to have a "secure" point from which the graph is to be arisen." (Fig. 3, left). Emanuele and his group-mates, instead, wrote: "Our difficulties were little. I did not have so many difficulties apart from the change of speed in the curves. Our main difficulty, which was making us to lose points, has been finding the correct starting point" (Fig. 3, right).

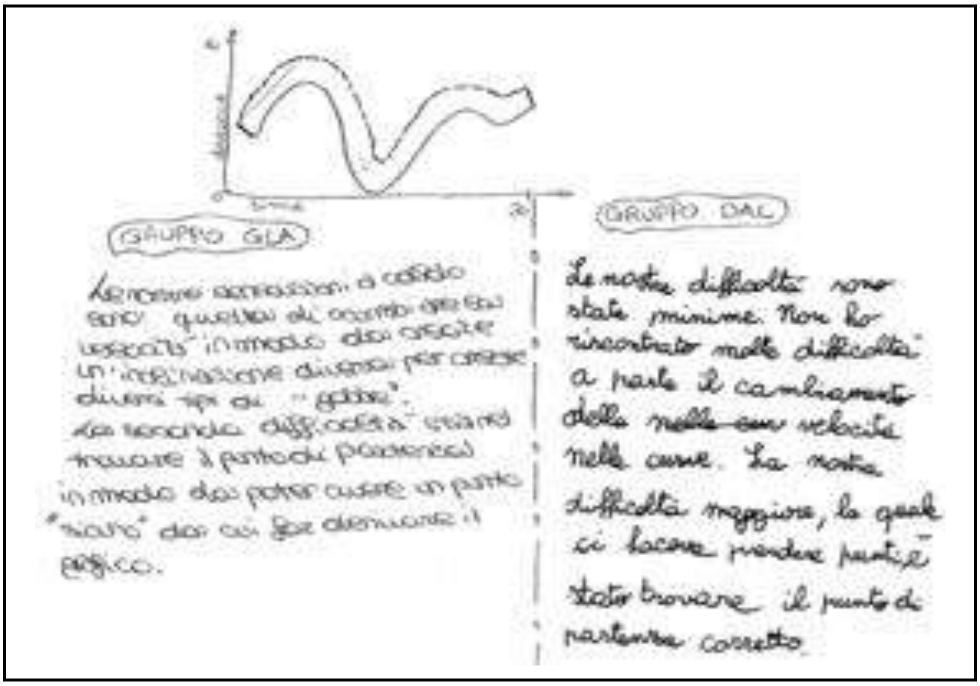

Figure 3. Emanuele's group (right) and Oliver's group's (left) difficulties to match the target graph

As a second point, the answer to the request about advices for the imaginary friend, and strategies to adopt, claims:

> "The relevant thing is staying inside the blue thickness and holding the controller towards the sensor, and remembering that the closer you are to the sensor, the closer you are to zero on the graph. A thing to look at is that the steeper the piece is, the bigger speed will have to be, in order to remain inside the thickness. One has to move in a direction perpendicular with respect to the sensor." (Fig. 4)

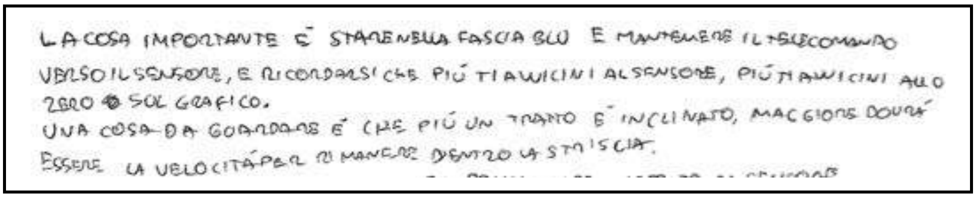
LA COSA IMPORTANTE È STARE NELLA FASCIA BLU E MANTENERE IL TELECOMANDO VERSO IL SENSORE, E RICORDARSI CHE PIÙ TI AVVICINI AL SENSORE, PIÙ TI AVVICINI ALLO ZERO SUL GRAFICO.
UNA COSA DA GUARDARE È CHE PIÙ UN TRATTO È INCLINATO, MAGGIORE DOVRÀ ESSERE LA VELOCITÀ PER RIMANERE DENTRO LA STRISCIA.

Figure 4. The answer to the second request of the written task

After dealing with the second request, the group could use WiiGraph to check its suggestions. The six students came in front of the sensor to try a new challenge, in line with their strategies. The striking thing is that, at this point, Emanuele and Oliver changed the way of playing (and being) in the challenge, by performing their movements without competing but collaborating with each other. They decided together the starting point and the changes in speed, which were the difficulties that both expressed in the first part of the written task. This was so unexpected that the researcher wondered whether the two students were not supposed to be in a competition. Oliver responded with a resolute "No, no, in a collaboration", immediately before starting. Thus, the students moved in coordination with and next to each other, so much that they proceeded, silent and focused, with the same pace and their facing arms touching each other, keeping the controllers very close (Fig. 5).

Figure 5. Emanuele and Oliver collaborating with each other

Interpretation of Episode 2

We can interpret this second episode pointing out the way that fluency with the tool is progressively growing for the students, even in a phase in which they are solving a written task. Perceptual and motoric features of the previous competing experience come evidently out in the answers given by the students. Starting from the initial request, the two groups share the critical aspects entailed in matching a target graph, that is, the starting point of movement and the speed at which to move. The question of speed is significant concerning perceptuomotor integration in the use of tools. While, for Emanuele and his group-mates, different speeds mean different slopes, which, in turn, mean "different kinds of "humps"", Oliver and his group talk about change of "speed in the curves". In both cases, the idea of speed comes with the graphical shapes assembling the visual and kinaesthetic experiences involved in the creation of a graph with WiiGraph.

This fluency is present even in the answer to the second request, where the steepness of the curve is captured in terms of speed, and being closer to zero *is* being closer to the sensor. But fluency is becoming functional, on the one side, to the original competitive task, for which one has the goal of remaining "inside the thickness" as much as possible (or, in other words, of winning the game). On the other side, it is clear that the new task is making way for the students' becoming collaborative, through focus on advices to be given to somebody else (the imaginary friend), who never took part in similar experiences with the tools, and on consequent strategies of motion. The task is becoming challenging for the students in a new manner: they want to play the game and win. The nature of the task implies a shift in the students' interaction with the software, from their being rivals to their becoming players against the computer, thus collaborative with each other. This is shown in a powerful way through the need for checking the correctness of the written suggestions, for which attention is no longer on the two players who are competing but on their capacity of saying a third person *how to play*. The assembling of meaning here grows through the ways in which the students talk about accuracy in the situation (through difficulties, suggestions, strategies), and move, again animating the curves and the mathematics at play. In being part of the changing assemblage, the six students are learning and gaining more and more fluency with the particular use of the software.

Episode 3. Being collaborative or *a+b* in Line

The third episode is about a task born with collaborative nature. The students first encountered the Line modality for the sum $a+b$, moving in front of the sensor without knowing the chosen options. Through the kinaesthetic experience, they discovered that a third real-time line was appearing on the screen, and was produced by adding the values of *a* and *b* over time (see Fig. 6). After this, the researcher led a collective discussion about how to obtain through the new modality a horizontal line, and about whether there exists a unique way or not. Then, in the following week, the students joined their groups and faced a written task on the sum. In particular, the students had to imagine to be in a situation in which a class-mate is absent and were asked how they would describe the activity of the previous week, and how they would explain the functioning of *a*+*b*.

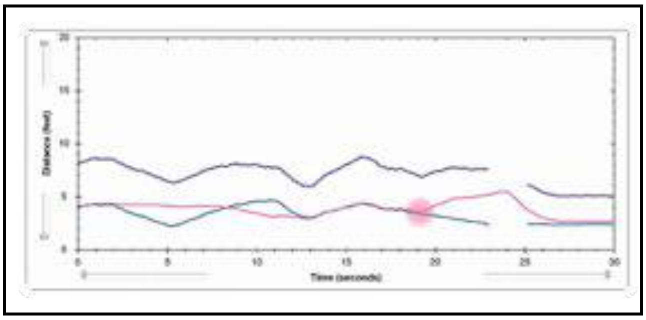

Figure 6. Line modality for the sum *a*+*b*

Alessandro and Luisa worked with Massimiliano, who was really absent the week before, when the students encountered the sum for the first time. This episode focuses on an initial brief moment of group work, in which Alessandro and Luisa explain to Massimiliano the meaning of the sum. The three students are sitting around a table, each on one side. The short dialogue develops as follows:

Luisa: Then, there are two people, that their graph, that is, each of them performs a movement (*With the pen in her right hand mimes some humps in the air in front, looking at Massimiliano*), which is on the graph (*Gazes and points with the pen to the graph area of WiiGraph*) and, that is, the graph (*Mimes again the humps in the air with her right hand, the pen in the left hand*) is the sum of these movements of two people (*Looks at Alessandro, smiling*), and so

Alessandro: It is as if there was, then, that is, it is as if, say, there were three people, that is, there are two people (*Turns and looks at the interaction space where the people should be*), who perform two movements (*Mimes the two people moving with his two open right hand fingers little moving back and forth in the air, gazing to the interaction space: Fig. 7a*), and it is as, that is. If they stay, one at 1 [feet] and one at 2 [feet] (*Looks back at the interaction space*), it is as if there really was a third person, who moves (*Turns towards the researcher, mimes a quick movement in front of him, with his right hand moving a little forward in front of his torso: Fig. 7b*) at 3 [feet] (*Turns again towards the interaction space*). It is a sum, that is, that is get as *a* plus *b* equal to *c* (*Looks at Massimiliano*)

Massimiliano: Ah, yeah (*Nods*)

Luisa: As if there was a *c* (*Looks at Massimiliano*)

Alessandro: Right

Luisa: That is, there the movement is that of *c* (*Repeats the previous hump gesture in the air with her right hand: Fig. 7c*)

Figure 7. Luisa and Alessandro explaining *a*+*b*

Interpretation of Episode 3

We can interpret this episode looking at the two ways in which Luisa and Alessandro explain to Massimiliano how the graph of *a*+*b* works. Luisa mainly talks of what happens on the graph area, thinking of the movements of two people as being "on the graph". Reference to the two people is interesting in terms of her fluency with the tool, since her lived-in perceptual experience of the two graphs of *a* and *b* is embodied with the motoric aspects of the experience for which two people have to move at the same time ("each of them"). The two people are the starting point. The sum depends on them, since "the graph" is thought of in terms of "these movements of two people". But Luisa is not precise in her explanation as she witnesses by gazing and smiling to Alessandro so as to looking for help. Alessandro modifies her original idea introducing a "third person", and adding much more precision to the explanation. In doing so, he specifically talks about an imaginary situation ("as if", repeated many times), which would involve "three people" and would take place in the interaction space (as shown by the insistently act of turning towards it). There, the third person would "really" move, and move together (in collaboration with) the two people who are performing the "two movements" that still are the starting point. Agency is plural and distributed among all the actors in the situation, which are, beyond the students, the imaginary moving people, the tools, the graphs, the sum of functions. The relation between the two original people and this third person (that marks the need for collaboration) is clearly expressed by means of the numerical examples of 1, 2 and 3 feet. Then, it is transformed into the algebraic expression *a*+*b*=*c*, which implies that a variable of position ("As if there was a *c*") could be also attached to the movement of the added person ("there the movement is that of *c*"). The beautiful example shows that the lived-in experience of the students with WiiGraph is crucial in the assembling of meaning for the sum function. The ways of moving, gesturing, gazing, talking and feeling of Luisa and Alessandro reveal their perceptuomotor engagement and fluency with the software. In playing the instrument, the new graph of the sum is mobilized and learned, even adding a form of imaginary collaboration among movers that will presuppose the presence of a new mover. All these aspects can be also found in the written explanation that the group produced. In fact, the students wrote: "The two people move in front of the sensor in the same way in which they moved the other times but, on the graph a third movement is represented, which is the sum of the first two". They also added that "the first thing is that it is necessary to collaborate", and the introduction of *c* is fundamental in this respect as shown in Fig. 8.

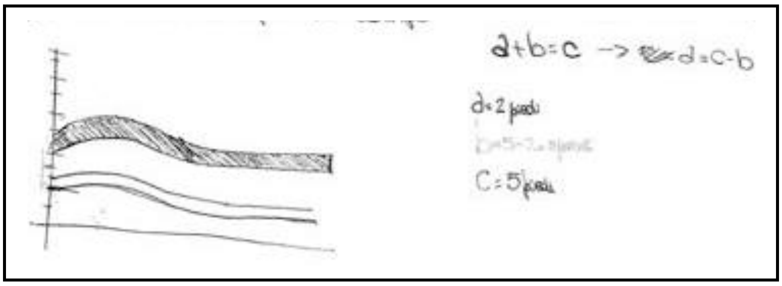

Figure 8. Written protocol for a+b

Conclusive remarks

In this paper, we have pursued a non-dualist participationist vision of learning mathematics with tool. This vision moves from drawing on Rotman's idea that the self is always breaking down and reinscribing, becoming plural, distributed and posthuman, through her engagement with the material surrounding, from which we cannot separate learning. We have considered the particular context of graphing motion, by the aid of a software application called WiiGraph, which allows students to reason on a graphical approach to function using in the same time two controllers of the Nintendo Wii. The aspect of playing (with) the controllers (in the usual experience, devices to play games) was fused with the aspect of playing WiiGraph and the Wiimotes as mathematical instruments in the classroom. Students were never alone in playing with the software. The game experiences available in WiiGraph occasioned lots of possibilities, with respect to the learners' kinaesthetic and proprioceptive engagement, as well as in terms of competitive/collaborative engagement between learners. From the didactic point of view, they became a matter of interest and motivation towards the mathematics. From the pedagogical point of view, they gave rise to issues about task design, becoming a very resource for learning, implying new unexpected and unexplored teaching and learning spaces. The discussion of some classroom episodes has shown that the students gain fluency with the devices and with the software, and develop mathematical understandings, through their lived-in experience in which perceptual and motoric aspects are continuously entangled. All the encounters of the students with the mathematics are made of recollections, expectations, fantasies and moods related to their embodied and kinaesthetic interactions with the tools. The only site of agency is not the human body of the students. Instead, agency is distributed and constituted across the learning situation. The body that is always becoming is that of the students and of their relations with all the human and non-human agents that are involved in the situation. In this perspective, we can talk about learning in terms of this moving assemblage. The mathematics itself is transformed and changing in the assembling of meaning, along various dimensions, like the digital and the material.

REFERENCES

de Freitas, E. & Sinclair, N. (2013). New materialist ontologies in mathematics education: The body in/of mathematics. *Educational Studies in Mathematics, 83*(3), 453-470.

de Freitas, E. & Sinclair, N. (2014). *Mathematics and the Body: Material Entanglements in the Classroom*. Cambridge: Cambridge University Press.

Nemirovsky, R., Kelton, M.L. & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education, 44*(2), 372-415.

Rotman, B. (2003). Will the digital computer transform classical mathematics? *Philosophical Transactions of the Royal Society A, 361*(1809), 1675-1690.

Rotman, B. (2008). *Becoming Besides Ourselves: The Alphabet, Ghosts, and Distributed Human Beings*. Durham: Duke University Press.

Sinclair, N. (2014). Generations of research on new technologies in mathematics education. *Teaching Mathematics and Its Applications, 33*(3), 166-178.

Vector subspaces generated by vectors of $\mathbb{R}^n$: Role of technology

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Abstract: In this work we report how the use of technology (e.g., Geogebra) promotes –in students– learning the concept of vector subspace generated by vectors of $\mathbb{R}^n$, $n > 3$. Two cognitive theories support this research: *shifts of attention* (Mason, 2008) and *representations* (Duval, 1999, 2003, 2006). Eleven college students (between 19 and 26 years old) from an instituto tecnológico located in Mexico City participated in the research. Data was collected by doing group interviews (groups of two or three members); students worked on previously designed Activities[1] using Geogebra and pencil-and-paper. Our results show that implementing the use of technology and pencil-and-paper, the students correctly determined vector subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$, and partially of $\mathbb{R}^n$, $n > 3$.

Résumé: Dans ce papier nous reportons comment l'utilisation de la technologie (e.g., Geogebra) favorise – chez les étudiants– l'apprentissage du concept celui de Sous-espace vectoriel engendré par des vecteurs de $\mathbb{R}^n$, $n>3$. Deux théories de type cognitive appuient cette recherche: celle appelé comme *Change d'attention* (Mason, 2008) et celle des *Représentations* (Duval, 1999, 2003, 2006). Dans cette étude ont participé 11 étudiants (d'environ 19 à 26 années), du niveaux supérieur d'un Institute Technologique de la ville Mexico, au Mexique. La collecte de données a été faite par des entrevues en équipes de deux ou trois étudiants. Dans les entrevues, les étudiants ont réalisé des Activités, désignées préalablement, en utilisant Geogebra et ainsi que papier/Crayon. Nos résultats nous suggèrent que, avec l'usage de la technologie et ainsi que de papier/crayon, les étudiants ont pu déterminer correctement les sous-espaces vectoriels de $\mathbb{R}^2$ et de $\mathbb{R}^3$, et aussi partialement de $\mathbb{R}^n$, $n > 3$.

Background and research question

Research in mathematics education –about teaching and learning of linear algebra concepts– report learning difficulties of this area in conventional teaching environment (pencil-and-paper). With respect to this problematic, several authors such as Dorier, Robert, Robinet and Rogalski (2000, 2011), Sierpinska (2000), among others, have detected that such difficulties are related with the *obstacle of formalism*, which is closely linked with the learning of concepts of vector space or related, e.g., set generated by a vector of $\mathbb{R}^n$, linear combination of vectors of $\mathbb{R}^n$, set generated by vectors of $\mathbb{R}^n$, $n\geq 2$, etc.

With the purpose of overcoming the obstacle of formalism several researchers (e.g., Gol & Sinclair, 2010; Stewart & Thomas, 2010, among others) have implemented the use of technology to reduce the learning difficulties of linear algebra concepts. Research reports can be found where they use CAS (*Computer Algebra Systems; e.g., Matlab, Mathcad, Maple, Derive, etc.*) and DGS (*Dynamic Geometry Software; e.g., Sketchpad; Cabri Geometry II* –farther on was *Cabri*–, and more recently *Geogebra, etc.*).

About the use of CAS –with respect to the teaching of linear algebra concepts– Harel (1997) reports the outcomes from using Matlab as a didactic medium, in the study of basis of space columns of echelon form matrices reduced line by line using CAS. According to Harel, the results obtained by using that CAS were satisfactory, which induces to believe that CAS potentially enhance the learning of abstract linear algebra concepts. Nonetheless, in the research of Pruncut (2008), when using CAS Maple, as a didactic medium, the outcomes did not turned as expected with respect of

[1] The Activity or Activities (upper case) in this paper, refers to the activities that the students did with the purpose of leading them to understand linear algebra concepts.

the students thinking about linear algebra concepts. Related research show that the use of CAS as didactic mediums does not enhance the learning of the abstract concepts.

Gol (2012), being aware of the works of Pruncut, search for alternative software to use so that students would develop geometric intuitions that would allow them the learning of the eigenvalues and eigenvectors concepts. The author acknowledged in the research that such goal could not be achieved by using CAS due to the static limitations of this kind of software while working with mathematical objects, and decided to go for the sketchpad (DGS) as an alternative so that the students would understand in a reflective way such concepts.

From the research of Gol as well as others where the DGS was use as didactic mediums it can be inferred that the DGS software play an important role in the learning of concepts, in particular of linear algebra. For example, Sierpinska, Dreyfus and Hillel (1999) showed that with the use of Cabri, the students learned mathematic concepts of linear algebra; it was emphasized that it could not been accomplished in a pencil-and-paper environment (p. 217). Furthermore, Sierpinska (2000) used such software with the students with the purpose of avoiding the obstacle of formalism. In that research, the author states that students were capable of expressing themselves in ways such as: "the vectors were the same or not", "parallel vectors", "vectors that have the same direction", etc. (p. 210), while the students that worked in conventional environment of pencil-and-paper were not capable of "visualize" vector properties like the ones mentioned above.

Recent research (e.g., Soto & Romero, 2011; Uicab & Oktaç, 2006, among others) show that the use of dynamic software, with the emphasis of overcoming the obstacle of formalism in the learning of concepts related with vector spaces, leads to satisfactory but not definitive results. Researchers that have implemented the use of DGS as a mediator in the learning of abstract linear algebra concepts have reported how it is achieve –by the students– the learning of the concept of linear transformation of vectors (Soto & Romero, 2011; Uicab & Oktaç, 2006); eigenvalues and eigenvectors (Gol & Sinclair, 2010); linear dependency of vectors (Aranda & Callejo, 2010); linear combination of vectors, linear dependency or independency of vectors and linear transformations (Andreoli, Beltrametti, & Rodríguez, 2009); vector space basis, set generated by vectors and linear dependency and independency of vectors (Stewart & Thomas, 2010). Furthermore, Aydin (2014) studied the main tendencies of the role of technology with respect to the learning of abstract concepts in linear algebra.

The results of these researchers put in evidence that the learning of concepts of linear algebra is promoted by the use of technology, despite, there still are some difficulties to fully achieve it. In this paper, it is aimed to respond the question: How does the use of the DGS (e.g., Geogebra) influences in the learning of the concept of vector subspace of $\mathbb{R}^n$, $n>3$?

Conceptual framework

This research is based on the cognitive theories of *shifts of attention* (Mason, 2008) and *representations* (Duval, 1999, 2003, 2006). The theory of *shifts of attention* is based on three basic concepts: *attention*, by the means that the observation –by the student– takes place, which allows to *sustain*, *discern*, *relate*, *perceive*, and *reason* the topic being studied and without it, would not be possible to make sense what the student learned; *being aware of…*, consist in verifying if the student makes sense of what he/she aims to know, and precise the knowledge/senses of what the student already knows, and the *attitude* as a means of willingness of the student to want to learn.

The theory of *representations* (Duval, 1999, 2003, 2006) is based in two concepts: *semiosis,* as apprehension or production of semiotic representations (as a means that the individual uses to externally expose his/her mental representations –images about an object– achieved by the use of signs: natural language, algebraic formulas, geometric figures, among other), and *noesis*, cognitive doings such as the learning of an object.

The data was analyzed considering the areas where the two theories coincide, which are: (1) *object association*, that considers that fact that the learning of concepts (in mathematics) can be shown in several representations; (2) *cognitive position*, the change of awareness from *implicit* to *explicit* (Mason, 2008) that occurs when the student can *see* (Duval, 2003) and willingly manages the topics being studied as a whole, and (3) *the student makes sense of what she or he aims to know*, it is recognized when students master the concepts *attention*, *being aware of…*, and *attitude* (Mason, 2008), and can support –with relative ease– the transition from one type of semiotic representation to another (Duval, 2003) of the mathematic topic being studied.

Methodology

Participants. In this research, eleven college students participated (between 19 and 26 years old) from an instituto tecnológico located in Mexico City. The students took a linear algebra course where technology was not used to teach the class. They were chosen by their teacher according to their willingness (*attitude*) to collaborate and previous academic performance in their linear algebra course (pencil-and-paper course environment). This research was conducted by assembling four groups of two or three members. Data was collected by doing group interviews, done directly by one of the authors of this paper, on predesigned activities with the aim to recollect evidence of the work of the students. None of the students had previous experience using Geogebra to solve linear algebra activities, hence some training was provided before starting to gather data.

Design and implementation of the tools for the gathering of data. As part of the employed methodology in this research the following was designed: a *Pre-test*, three Activities, and a *Post-test*. The students approach individuality the *Pre-test*, and the *Post-test*, but not the three Activities which were conducted in the groups. Previous to the training for the teaching of the software commands related to current discussed subject, the students took a *Pre-test* with the aim to know the software mastering level of the students with respect to vector subspace generated by vectors of $\mathbb{R}^n$, $n>3$.

After taking the *Pre-test*, the students commenced the Activities; then after finishing the Activities the *Post-test* was conducted. The thematic content of the three Activities are: Activity I: Set generated by a vector of $\mathbb{R}^2$ and $\mathbb{R}^3$, Activity II: Linear combinations of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, and activity III: Set generated by vectors of $\mathbb{R}^2$ and $\mathbb{R}^3$. The data collection took place in a classroom and each collecting section lasted approximately three hours.

The activities. The Activities were designed by the authors of this paper and implemented by one of them, which will be called $\mathbf{I_{nv}}$. An interview was conducted alongside the Activities and was videotaped. During the interview, the analysis and thoughtful thinking about the claims of the students were favored by via in depth discussions aimed at learning concepts. It was permitted to the students to ask questions with respect to understating the questions being asked. The previous work of the students was discussed and analyzed at the beginning and at the end of each section.

Data analysis and discussion of results

Due to length limitations, only the data collected by a single group (Team 4) composed by two students ($\mathbf{E_1}$ and $\mathbf{E_2}$) would be presented in this paper.

Activity I: Set generated by a vector of $\mathbb{R}^2$ and $\mathbb{R}^3$. The initial discussion in this Activity was about the concept of scaling a vector. The students did not have difficulties in accurately and efficiently interpreting the new vector generated by scaling a vector; for convenience, in this paper, the new vector will be called the *scalar product vector*. From a list of vectors of $\mathbb{R}^2$ and $\mathbb{R}^3$ that students had sketched in pencil-and-paper, and in Geogebra, $\mathbf{I_{nv}}$ asked: How do you define a vector from the list of vectors, is it a scalar product vector? $\mathbf{E_1}$ Answered:

1 $\mathbf{E_1}$: Because any scalar that multiplies a vector is [*resulting vector*] contained in the same straight line [*where the initial vector is located*].

2 **I$_{nv}$**: In the expression $-2\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}-2\\-4\end{pmatrix}$, the scalar -2 that multiplies $\begin{pmatrix}1\\2\end{pmatrix}$ Is "any" scalar?

3 **E$_1$**: No, it is just some specific scalar [*right after*, ***E$_2$*** *added to the answer of his team mate*].

4 **E$_2$**: Because there is only a scalar [*multiplying the vector*], that if it multiplies the vector would result in a [*single*] scalar product vector (*cognitive position*, Mason, 2008; Duval, 2003).

Following, the students write in their own words how they understand the relation of the scalar product vector (see Figure 1).

If) Take into account the proposed activities in Ia) to Ie), How do you define a vector that is scalar product vector of another?

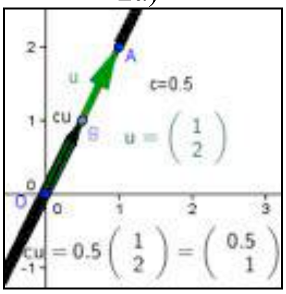
Porque hay un escalar y solo uno que si multiplica al primer vector nos resulta otro vector.

Because there is only one scaler that if multiplies the first [initial] vectors results in another vector.

Figure 1. Definition of a scalar product vector.

With the aim so that students make sense of the amount of the scalar product vectors, **I$_{nv}$** requested to the students to trace in Geogebra the scalar product vectors $\begin{pmatrix}1\\2\end{pmatrix}$ (see Figure 2a), and **I$_{nv}$** asked: Are all scalar product vectors [*vectors generated by scaling the initial vector u*] of u shown on the computer screen? **E$_1$** answers:

2a)



2b)

Ij) Think about what you see on the computer screen and answer in a justified manner the question, have you been able to construct all the scalar product vectors of $\begin{pmatrix}1\\2\end{pmatrix}$?

No se logran ver, pero asumimos que si porque hay una infinidad de escalares que vienen de $-\infty$ a $+\infty$ en dirección del vector $\begin{pmatrix}1\\2\end{pmatrix}$.

One cannot see them but we believe so, because there are an infinite number of scalars that come from $-\infty$ to $+\infty$ in the direction of the vector $\begin{pmatrix}1\\2\end{pmatrix}$.

Figure 2. Visualization in Geogebra of the scalar product vectors of the vector *u* of $\mathbb{R}^2$.

5 **E$_1$**: Yeah, though there is a limit marked [*pointing to the computer screen*] we know the straight line goes to infinity.

6 **I$_{nv}$**: You are stating that there is an infinite number of scalar product vectors *u*, but on the screen all those scalar product vectors of $u=\begin{pmatrix}1\\2\end{pmatrix}$ can be seem? (See Figure 2b)

7 **E$_2$**: Well not, but we assume [*that the scalar* c *shapes up the product scalar vectors of* $u=\begin{pmatrix}1\\2\end{pmatrix}$] and that the vector cu goes to the origin location to the minus infinity and to the plus infinity.

8 **I$_{nv}$**: What geometric location describes point B [of Figure 2a)]?

9 **E$_2$**: Is it an entire straight line?

10 **I$_{nv}$**: It is not a segment of a straight line?

11 **E$_1$**: Because there is an infinite number of scalars that generate a straight line [*then makes a correction*]; no, because there is an infinity number of vectors [*scalar product vectors of u*] that generate a straight line [*the student makes sense of he/she wants to know* Mason, 2008;

Duval, 2003].

I1) According to what you see on the computer screen explain, in the following rectangle, the claim: "The vector u = $\begin{pmatrix}1\\2\end{pmatrix}$ along with their scalar product vectors (the vectors 'cu'), generate a straight line".

Si generan una linea recta, porque hay una infinidad de escalares que si multiplican al vector $\begin{pmatrix}1\\2\end{pmatrix}$ generan una recta infinita

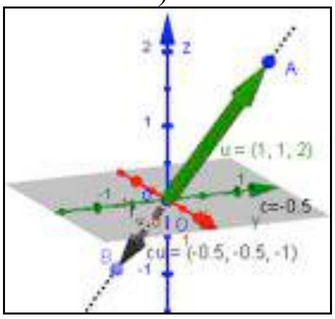
Yes, they do generate a straight line, because there are infinite scalars that can multiply the vector $\begin{pmatrix}1\\2\end{pmatrix}$ to generate an infinite straight line.

Figure 3. Response of the students to question asked by $\mathbf{I_{nv}}$ in line 8.

In Figure 3, the students explain the responses to the line 8.

The work of the students in Geogebra allowed them to visualized (*cognitive Position*, Mason, 2008; Duval, 2003) the concept of scaling a vector in $\mathbb{R}^3$ (see figures 4a) and 4b).

4a)



4b)

Io) According to what you see on the computer screen explain, in the following rectangle, the claim: "The vector 'u' along with their multiples (the vectors 'cu') generate a straight line".

Si genera una linea recta, porque el vector u = $\begin{pmatrix}1\\1\\2\end{pmatrix}$ junto con sus multiplos genera la recta que los contiene

It does generate a straight line, because the vector u = $\begin{pmatrix}1\\1\\2\end{pmatrix}$ along with their scalar product vectors generate a straight line that contains them.

Figure 4. Visual representation in Geogebra of the vectors generating by scaling an initial vector u of $\mathbb{R}^3$.

With the work done by the students up to the Task Io)[2] it allow $\mathbf{I_{nv}}$ to ask how do you define the concept of a set generated by a vector. The silence of the students reflected the trouble they had in answering the question. Hence, $\mathbf{I_{nv}}$ asked they question in a different way: Think about the work you (the students) have done up to the Task Io) then answer, if there is a relation between "the get generated by a vector u" and "the set of product vectors (vectors generated by the scaling of the initial vector u)"? $\mathbf{E_1}$ gave the following opinion:

12 $\mathbf{E_1}$: We could say that the set generated [*by a vector u*] is the straight line that contains the initial vector along with the product scaled vectors making a straight line [*following* $\boldsymbol{E_2}$ *added to the answer of his classmate.*]

13 $\mathbf{E_2}$: Both of the set of vectors are the same [*strongly stating*]: "any set generated by a vector is a set of the scalar product vectors of the initial vector", and "any set of the scalar product vectors of the initial vectors is a set generated by an initial vector", [*therefore*] the set of vectors are the same [*the student* $\mathbf{E_2}$ *makes sense of what he wants to know*, Mason, 2008; Duval, 2003].

14 $\mathbf{I_{nv}}$: According to what you have seen in the computer screen with respect to the Activities done up to this point. How do you define a set generated by a vector of $\mathbb{R}^2$ and $\mathbb{R}^3$?

15 $\mathbf{E_2}$: According to what we have seen in the files […] we could observed that a vector along with its scalar product vectors generate a straight line that contains them in $\mathbb{R}^2$ and $\mathbb{R}^3$,

[2] The Task or Tasks (upper case) in this paper, refers to the specific work that the students did for the activities. For example, "Task Ia)" refers to task "a" of Activity I.

which is the same as the set generated by a vector: that is, the scalar product vectors of a vector would now be named a set generated by a vector: that is, the scalar product vectors would now be called a set generated by a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$, which generate a straight line. [***E***]

Activity I end it by requesting the students to determine the set generated by two vectors of the vector spaces of $\mathbb{R}^2$, $\mathbb{R}^4$, $\mathbb{R}^5$ and $\mathbb{R}^n$. They gave their answers (Column D from Task Iy) from Figure 5) in terms of the number of zero rows from the echelon formed matrix from these vectors (see Column C from Figure 5), since this matrix representation would allow them to determine whether the vectors of Column A are or are not scalar product vectors (see Column F). In Figure 5, can be observed how the students accurately responded for Row 5 (position 5D), but in rows 3, 4 and 6, did not know what to answer (positions 3D, 4D, and 6D). The aim of $\mathbf{I_{nv}}$ was that the students to accurately answer, for the Activity III, Activities such as the ones planned in positions 3D, 4D, and 6D from Task Iy) (see Figure 5); for this, it was necessary to study for Activity II the concept of linear combination of vectors, Activity that is discussed below.

	A	B	C	D	E	F
1	*Set of vectors*	*Matrix associated with the vectors*	*Echelon form matrix*	*Set generated by the vectors of Column A*	*Does it generate a straight line?*	*Justification of the answer in Column E*
2	$\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}-2\\-4\end{pmatrix}\right\}$	$\begin{pmatrix}1 & -2\\2 & -4\end{pmatrix}$	$\begin{pmatrix}1 & -2\\0 & 0\end{pmatrix}$	[illegible]	Yes	The vectors are scalar product vector of each other
3	$\left\{\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}2\\-3\end{pmatrix}\right\}$	[illegible]	[illegible]	[illegible]	No	los vectores son no multiplos entre ellos The vectors are not scalar product vectors of each other
4	$\{(-5,3,2,1),(3,4,5,2)\}$	[illegible]	[illegible]		No	los vectores son no multiplos entre ellos The vectors are not scalar product vectors of each other
5	$\{(2,-3,1,1,0),(4,-6,2,2,0)\}$	[illegible]	[illegible]	[illegible]	Yes	los vectores son multiplos entre ellos The vectors are scalar product vectors of each other
6	Vectores con *n* componentes $\{(1,2,\ldots,5),(1,1,\ldots,-5)\}$	$\begin{pmatrix}1 & 2 & \ldots & 5\\1 & 1 & \ldots & -5\end{pmatrix}$	[illegible]		No	los vectores son no multiplos entre ellos The vectors are not scalar product vectors of each other

Figure 5. Task Iy), set generated by two vector in the vector spaces of $\mathbb{R}^2$, $\mathbb{R}^4$, $\mathbb{R}^5$ and $\mathbb{R}^n$.

Activity II: Linear combination of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. Activity two starts by discussion when are two vectors equal to each other. From a list that the students had sketched in pencil-and-paper, and in Geogebra, $\mathbf{I_n}$ asked: When are two vectors equal to each other? $\mathbf{E_1}$ and $\mathbf{E_2}$ answered:

16 $\mathbf{E_1}$: It can be when [*the vectors*] have the same magnitude and direction [*does not senses the meaning of a vector*].

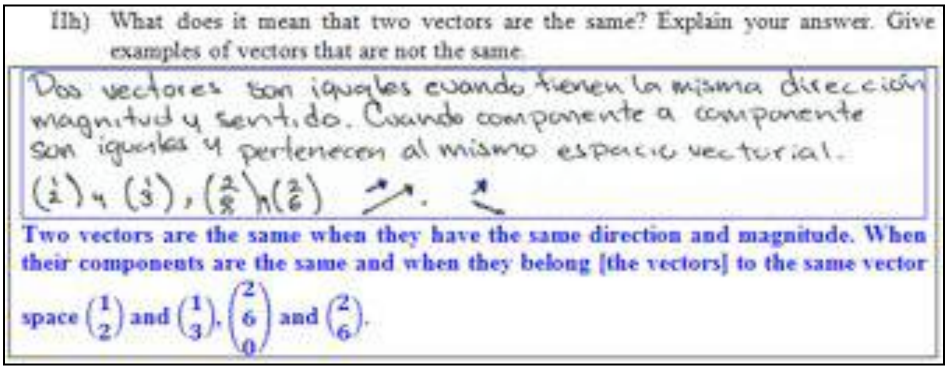
17 $\mathbf{E_2}$: I remembered that two vectors were [*are*] the same when component to component are the same [*does not mention if the vectors are in the same vector space*].

18 $\mathbf{I_{nv}}$: What it says (in line 17) (talking to $\mathbf{E_2}$), Are you contradicting $\mathbf{E_1}$? What you are stating (in the line 16) contradicts (asking $\mathbf{E_1}$) $\mathbf{E_2}$?

19 $\mathbf{E_1}$: It does not contradict, because $\mathbf{E_2}$ is talking about an initial and final component that is the

same as magnitude [***E_1*** *does not answered adequately, but tries to associates objects in two representations*, Mason, 2008, Duval 2003. ***E_2*** *tries to complement the answer*].

20: **E_2**: Because, if given a vector [*in $\mathbb{R}^2$ or in $\mathbb{R}^3$*] that if displaced to another place in its own space [*vector space*] it conserves its magnitude… [***E_2*** *pauses for a second but does not continues*].



IIh) What does it mean that two vectors are the same? Explain your answer. Give examples of vectors that are not the same

Dos vectores son iguales cuando tienen la misma dirección magnitud y sentido. Cuando componente a componente son iguales y pertenecen al mismo espacio vectorial.
$\binom{1}{2}$ y $\binom{1}{3}$, $\begin{pmatrix}2\\6\\0\end{pmatrix}$ y $\binom{2}{6}$

Two vectors are the same when they have the same direction and magnitude. When their components are the same and when they belong [the vectors] to the same vector space $\binom{1}{2}$ and $\binom{1}{3}$, $\begin{pmatrix}2\\6\\0\end{pmatrix}$ and $\binom{2}{6}$.

Figure 6. Definition, algebraic and geometric, of the concept of vector equality.

During a section of the interview (lines 16 to 20), the students were not able to adequately define the concept of vector equality; however, they improved their response when they wrote the answer (algebraic and geometric, o*bject association*, Mason 2008; Duval, 2003, see Figure 6).

Following, **I_{nv}** conducted the discussion with the aim that students would learn the property of enclosed vector addition that are not scalar product vectors of $\mathbb{R}^2$. **I_{nv}** asked, given that the vectors $\binom{1}{-3}$ and $\binom{3}{2}$ are in $\mathbb{R}^2$, where is the sum of the scalar product $\frac{1}{2}\binom{1}{-3} + (-1)\binom{3}{2}$? **$E_1$** responded the question:

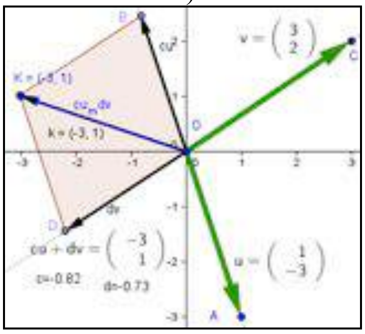
21: **E_1**: In $\mathbb{R}^2$, because the vectors $\binom{1}{-3}$ and $\binom{3}{2}$ are located in $\mathbb{R}^2$, besides it doesn't make sense that the sum of the scalar product vectors gives, for example, a vector of $\mathbb{R}^3$. [*the students makes sense of what he wants to know*, Mason, 2008; Duval, 2003.]

22 **I_{nv}**: In how many ways can you choose the scalars *c* and *d* when summing the scalar product vectors $c\binom{1}{-3} + d\binom{3}{2}$, and in which vector space is located?

23 **E_2**: In an infinite number of ways and is located in the $\mathbb{R}^2$ vector space.

24 **I_{nv}**: Is there a vector that cannot be written as a sum of the scalar product vectors $\binom{1}{-3}$ and $\binom{3}{2}$?

7a)



7b)

IIq) Observe in IIp) that $k = \begin{pmatrix} a \\ b \end{pmatrix}$, a,b∈ ℝ can be any vector in the plane. Given the vectors $u = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $v = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ do scalar product vectors exist of those vectors, such that their sum is vector "k"? If your answer is yes, calculate these constants "c" and "d". In the following rectangle make the necessary calculations (just like in IIn) and, write your answer in the suspensive points.

$cu + dv = k$

$c\begin{pmatrix} 1 \\ -3 \end{pmatrix} + d\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ $\begin{pmatrix} c + 3d \\ -3c + 2d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ $\left(\begin{array}{cc|c} 1 & 3 & a \\ -3 & 2 & b \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & \frac{2}{11}a - \frac{3}{11}b \\ 0 & 1 & \frac{3}{11}a + \frac{1}{11}b \end{array}\right)$

$\begin{pmatrix} c \\ -3c \end{pmatrix} + \begin{pmatrix} 3d \\ 2d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ $\begin{cases} c + 3d = a \\ -3c + 2d = b \end{cases}$

$c = \frac{2}{11}a - \frac{3}{11}b$

$d = \frac{3}{11}a + \frac{1}{11}b$

7c)

IIr) The values "c" and "d" that you calculated in IIb) o IIq) satisfy the relation cu + dv = k, Does a unique pair of constants "c" and "d" exist that satisfy that relation? Justify your answer in an algebraic way (like in IIq) and in a geometric way (like in IIp).

Forma algebraica. Solo existe un valor para c y d que multiplicados por los vectores $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ y $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ nos resulte el vector $\begin{pmatrix} a \\ b \end{pmatrix}$

Forma geometrica. Solo existe un ajuste de los multiplos de los vectores $u = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ y $v = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ que nos de el vector $k = \begin{pmatrix} a \\ b \end{pmatrix}$.

Algebraic way. Only a single value exists for c and d that multiplied by the vectors $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ results in the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

Geometric way. Only a single adjustment can be made so that the scalar product vectors of $u = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $v = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ results in the vector $k = \begin{pmatrix} a \\ b \end{pmatrix}$.

Figure 7. The students justified the existence of a unique pair of *c* and *d* that satisfies $c\begin{pmatrix} 1 \\ -3 \end{pmatrix} + d\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$.

The students simultaneously answered "No", to the questions asked by $\mathbf{I_{nv}}$ in the line 24, but did not justified their answer. Following, $\mathbf{I_{nv}}$ asked the student to calculate *c* and *d* from the equation $c\begin{pmatrix} 1 \\ -3 \end{pmatrix} + d\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ by geometric means using Geogebra, if $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ (see Figure 7a) and by an algebraic way using pencil-and-paper with the purpose that student would think about the unique existence of the values of *c* and *d,* despite what the resulting vector would be (K in the Figure 7a), which they reason it during the Tasks IIq) and IIr), respectively, from the Figure 7b) and 7c).

With the aim that students would write the definition of linear combination of vectors, the following discussion section took place.

25 $\mathbf{I_{nv}}$: In what vector space is the result of the sum of the scalar product vectors $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$?

26 $\mathbf{E_1}$: In $\mathbb{R}^3$ (and added), every time a sum of the scalar product vectors is done the result has to be [*is*] in the same vector space that we are referring to [*the same vector space that the scalar product vectors are in*, o*bject association*, Mason, 2008; Duval, 2003].

27 $\mathbf{I_{nv}}$: With the experience you have gained when using Geogebra or the traditional pencil-and-paper way, what do you think is the outcome gained due to de adding of the scalar product vectors?

28 $\mathbf{E_2}$: That we could make adjustments [*–In Geogebra– of each scalar product vector that is included in the addition*] in such a way that it outcomes a resulting vector, but this new vector belongs to the same space from which was generated [*same space as the individual vectors that were summed*], it has to belong to the same space [o*bject association*, Mason,

2008; Duval, 2003, ***E₁** adds*].

29 **E₁**: The resulting vector belongs to the same space as the vectors that are being summed [***E₂** interrupts*].

30 **E₂**: Due to the nature of the two vectors in a vector space, the sum of them gives a vector that is in the same space [*vector space*].

The work in the previous section of the interview (lines 25 to 30) permitted the students to recollect sufficient supporting arguments to allowed them to accurately interpret that the sum of the scalar product vectors of any number of vectors, of a certain vector space, is another vector of the same vector space [*the students makes sense of what he wants to know*, Mason, 2008; Duval, 2003, see Figure 8].

IIy) Take into account the result in IIt) –or more general in IIu)– How could you argue that the *sum of the scalar product vectors* of any number of vectors, of a certain *vector space*, is another vector of the same *space*?

Por la naturaleza de dos vectores en un cierto espacio vectorial, la suma de multiplos entre ellos genera un vector dentro del mismo espacio.	**Due to the nature of the two vectors in a certain vector space, the sum of the scalar product vectors generate a vector inside the same [vector] space.**

Figure 8. Interpretation by the students of the result of summing the scalar product vectors.

Following, the $\mathbf{I_{nv}}$ directed the interview with the idea that students would accurately define the concept of linear combination of vectors, and asked, according to the Task IIdd) (see Figure 9), What is a linear combination of vectors? **E₁** answered:

31 **E₁**: When the sum of the scalar product vectors result in a [*new*] vector of the same vector space.

32 $\mathbf{I_{nv}}$: What is the relation between a sum of the scalar product vectors and the concept of linear combination of those vectors?

33 **E₂**: The linear combination of vectors … (**E₁** interrupts).

34 **E₁**: Is the sum of the scalar product vectors.

35 $\mathbf{I_{nv}}$: According to the experience gained by doing the Activities in Geogebra and by using pencil-and-paper, define what is a linear combination of vectors?

36 **E₂**: Is a resulting vector in that same space (**E₁** interrupts).

37 **E₁**: Is the resulting vector of the sum of the scalar product vectors, which is located within the same space.

38 $\mathbf{I_{nv}}$: $\mathbf{I_{nv}}$ [*Directs towards **E₁** and asked*] What **E₂** said (line 36) contradicts what you mentioned (line 37)? (**E₂** answers).

39 **E₂**: No, because in here it also says that the sum of the scalar product vectors is what we have been working on, which would outcome in a resulting vector […] but now it would not be called sum of the scalar product vectors, instead it would now be called linear combination of those vectors [*being summed*] [*the student makes sense of what he/she wants to know*, Mason, 2008; Duval, 2003].

After the above discussion took place (lines 31 to 39) and with the experience gained by working in an environment using Geogebra and pencil-and-paper, $\mathbf{I_{nv}}$ asked the students to define the concept of linear combination of vectors. The response of the students is showed in Figure 9.

Activity II ended with Task IIdd) shown in Figure 9. With the this Activity along with Activity I allowed the students to gain more skills to be able to address Activities that would direct them to the concept of set generated by vectors of different vector spaces which were discussed in Activity III.

IIdd) Without caring which scalar product vector is taken from any vector, nor the quantity of vectors in the *vector spaces* $\mathbb{R}^2$ or $\mathbb{R}^3$, the sum of those scalar product vectors always gives another vector of the same *vector space*. This property consisting of the sum of scalar product vectors is called *linear combination of those vectors*. According to the activities done with Geogebra and with pencil-and-paper in this activity write a definition of this concept.

A partir de geogebra y las actividades a papel y lápiz podemos definir la combinación lineal como el vector resultante de la suma de multiplos entre dos vectores	**Starting with Geogebra and the activities with pencil-and-paper, one can define the linear combination as the resulting vector of the sum of the scalar product vectors.**

Figure 9. Definition by the students of the concept of linear combination of vectors.

Activity III: Set generated by vectors of $\mathbb{R}^2$ and $\mathbb{R}^3$. From a list of vectors of $\mathbb{R}^2$ and $\mathbb{R}^3$ that the students wrote in Activity III, the students sketched in Geogebra the sum of the scalar product vectors from that list. Following **I_{nv}** asked:

40 **I_{nv}**: What set of vectors generate two scalar product vectors $\mathbb{R}^2$?

41 **E_1**: The straight line that contains the vectors [*cognitive position*, Mason, 2008; Duval, 2003].

42 **I_{nv}**: What are the characteristics of an echelon form matrix composed of two vectors either column or row of $\mathbb{R}^2$ that are scalar product vectors?

43 **E_2**: The echelon form matrix has a row of zeros.

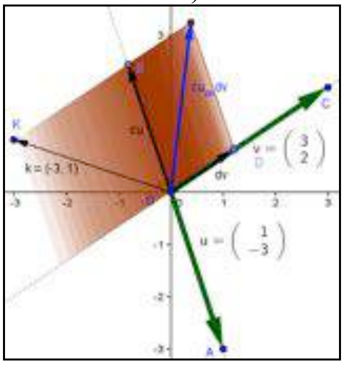
The purpose of the **I_{nv}** in asking those question was so that the students would relate the characteristics of the echelon form matrix of a set of *n* vectors of $\mathbb{R}^n$, $n \geq 2$ to the space generated by the set of vectors. Following, it was studied the vectors that are not scaled product vectors $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ aiming that the students would determine to which set of vectors generate these vectors. For these reason, it was asked to the student to work on a file in Geogebra where the sketched vectors where present (see Figure 7a). Then, according to what the students where observing on the computer screen, **I_{nv}** asked:

44 **I_{nv}**: If the vector is located in the interior of the parallelogram (OBKD in Figure 7a), Can a linear combination of the vectors $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ that is that vector [vector $cu_m dv$ in Figure 7a] be found?

45 **E_2**: Can the scalar product vectors be adjusted so that the parallelogram would be the vector k, even if the vector is found in the limit of the parallelogram and even if is out of the parallelogram.

Figure 10a) shows the end result of adjusting a couple of times the scalar product vectors of $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ in such a way that if adjusting them an "infinite" number of times the union of the parallelogram would "cover" the plane *xy*; in Figure 10b) the students justify the reason why these vectors generate the plane *xy* [*cognitive position*, Mason, 2008; Duval, 2003].

10a)



10b)

IIIj) In the file AIII 2D 2 activate Automatic Animation of the points B y D. From the point of view of the work done on Geogebra, how do you argue the statement: "The union of the parallelogram that form the scalar product vectors of the vectors $u=\begin{pmatrix}1\\-3\end{pmatrix}$ and $v=\begin{pmatrix}3\\2\end{pmatrix}$ is the plane xy". Deactivate the animation in the points B and D.

Es el plano xy porque los vectores u y v no son multiplos entre si y los paralelogramos que forman sus multiplos cubre todo el plano xy.

It is the plane xy because the vectors u and v are not multiples of each other and the parallelograms that their multiples form cover the entire xy plane.

Figure 10. The students interpreted the set generated by the vectors $\begin{pmatrix}1\\-3\end{pmatrix}$ and $\begin{pmatrix}3\\2\end{pmatrix}$.

By doing Task IIIq) shown in Figure 11 the students were able to determine (Column F, Task IIIq) the set generated by two vectors of $\mathbb{R}^2$ in terms of the echelon form matrix (Column D, Task IIIq). Under such conditions, it allowed the students to classify the vector subspaces of $\mathbb{R}^2$ (see Figure 12).

	A	B	C	D	E	F
1	*Vectors*		*Matrix*	*Echelon form matrix*	*Linear combination* cu + dv	*Set generated by the vectors "u" and "v"*
2	"u"	"v"				
3	Example $\begin{pmatrix}1\\2\end{pmatrix}$	Example $\begin{pmatrix}-2\\-4\end{pmatrix}$	Example $\begin{pmatrix}1&-2\\2&-4\end{pmatrix}$	$\begin{pmatrix}1&-2\\0&0\end{pmatrix}$	Example $c\begin{pmatrix}1\\2\end{pmatrix}+d\begin{pmatrix}-2\\-4\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$: $x=c-2d$, $y=2(c-2d)$, $y=2x$, $-2x+y=0$	Example Straight line in $\mathbb{R}^2$: $y=2x$, $\{\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\2\end{pmatrix}\}$
4	Example $\begin{pmatrix}1\\-3\end{pmatrix}$	Example $\begin{pmatrix}3\\2\end{pmatrix}$	Example $\begin{pmatrix}1&3\\-3&2\end{pmatrix}$	$\begin{pmatrix}1&3\\0&11\end{pmatrix}$	Example $c\begin{pmatrix}1\\-3\end{pmatrix}+d\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$: $x=c+3d$, $y=2d-3c$	Example $\mathbb{R}^2$
5	$\begin{pmatrix}4\\1\end{pmatrix}$	$\begin{pmatrix}1\\\frac{1}{4}\end{pmatrix}$	$\begin{pmatrix}4&1\\1&\frac{1}{4}\end{pmatrix}$	$\begin{pmatrix}4&1\\0&0\end{pmatrix}$	$c\begin{pmatrix}4\\1\end{pmatrix}+d\begin{pmatrix}1\\\frac{1}{4}\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$ $x=4c+d$ $y=c+\frac{1}{4}d$ $-x+4y=0$	$\{\begin{pmatrix}x\\y\end{pmatrix}=\lambda\begin{pmatrix}4\\1\end{pmatrix}\}$
6	$\begin{pmatrix}-2\\3\end{pmatrix}$	$\begin{pmatrix}1\\4\end{pmatrix}$	$\begin{pmatrix}-2&1\\3&4\end{pmatrix}$	$\begin{pmatrix}-2&1\\0&11\end{pmatrix}$	$c\begin{pmatrix}-2\\3\end{pmatrix}+d\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$ $x=-2c+1d$ $y=3c+4d$	$\mathbb{R}^2$
7	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$c\begin{pmatrix}1\\0\end{pmatrix}+d\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$ $x=c$ $y=0$	$\{\begin{pmatrix}x\\y\end{pmatrix}=\lambda\begin{pmatrix}1\\0\end{pmatrix}\}$ $y=0$
8	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$c\begin{pmatrix}0\\0\end{pmatrix}+d\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$ $x=0$ $y=0$	$\{\begin{pmatrix}x\\y\end{pmatrix}=\lambda\begin{pmatrix}0\\0\end{pmatrix}\}$ $=$ $\{\begin{pmatrix}0\\0\end{pmatrix}\}$

Figure 11. Task IIIq), vector subsets of vectors of $\mathbb{R}^2$ defined as a function of the echelon form matrix.

12a)

IIIt) Take into account the results in Column F of the table of vectors IIIq), and in the following rectangle make a list of all the other possible *sets generated by* and by pair of vectors "u" and "v" of $\mathbb{R}^2$ (that are different from *vector cero*).

* Conjunto de vector cero $\{\begin{pmatrix}0\\0\end{pmatrix}\}$
* $\mathbb{R}^2$
* Toda recta de $\mathbb{R}^2$ que pasa por el origen

***Set of zero vector $\{\begin{pmatrix}0\\0\end{pmatrix}\}$**
***$\mathbb{R}^2$**
*** Any straight line of $\mathbb{R}^2$ that crosses the origin**

12b)

IIIpp) Take into account the results of Column F of the former table of vectors, and in the following rectangle make a list of *all* the possible *sets generated by* any three vectors "u", "v", and "w" of $\mathbb{R}^3$.

· Conjunto que solo contiene al vector cero.
· Cualquier recta de $\mathbb{R}^3$ que pasa por el origen.
· Cualquier plano de $\mathbb{R}^3$ que pasa por el origen
· $\mathbb{R}^3$.

***Set that only contains the zero vector**
***Any straight line of $\mathbb{R}^3$ that crosses the origin**
***Any plane of $\mathbb{R}^3$ that crosses the origin**
***$\mathbb{R}^3$**

Figure 12. List of vector spaces of $\mathbb{R}^2$ and $\mathbb{R}^3$.

From the results generated in Task IIIq) (see Figure 11) the students wrote a list of vector subspaces of $\mathbb{R}^2$ (see Figure 12a) and similar Activities of set of vectors of $\mathbb{R}^3$ leading the students to write a list of the subspaces of $\mathbb{R}^3$(see Figure 12b). However when $\mathbf{I_{nv}}$ asked the students about the subspaces of $\mathbb{R}^1$, $\mathbf{E_2}$, answer: $\mathbb{R}^1$and the set that contains the zero vector (the trivial subspaces of $\mathbb{R}^1$). Despite the lack of graphic representations of vectors of $\mathbb{R}^n$, $n \geq 4$, the students *associated* (Mason, 2008; Duval, 2003) the set generated (subspace of $\mathbb{R}^1$) with the number of zero rows of the echelon form matrix initially composed of vectors either row or column of set of *n* vectors of $\mathbb{R}^n$. This way, $\mathbf{I_{nv}}$ requested from the students to explained: Which are the characteristics of the echelon form matrix of four vectors of $\mathbb{R}^1$? And what are their corresponding vector subspaces of those vectors. They answered the following:

46 **E$_1$ y E$_2$**: [*The echelon form matrix*] does not have rows of zeros and it generates $\mathbb{R}^1$; it has a row of zeros, generates a *tri-dimensional vector hyper-space* [*that passes through the origin*]; it was two rows of zeros and it generates a *hyper-plane* [*that passes through the origin*]; if it has three rows of zeros and it generates a straight line [*that passes through the origin*] and if all its rows are zeros and generates a zero [*zero vector*].

Figure 13a) shows how the students determined the subspaces generated by four vectors of $\mathbb{R}^1$ starting from the number of zero rows of the echelon form matrix of these vectors, while in Figure 13b) it is requested to the students to determine a set of vectors that would generate a *tri-dimensional hyper-space* of $\mathbb{R}^n$.

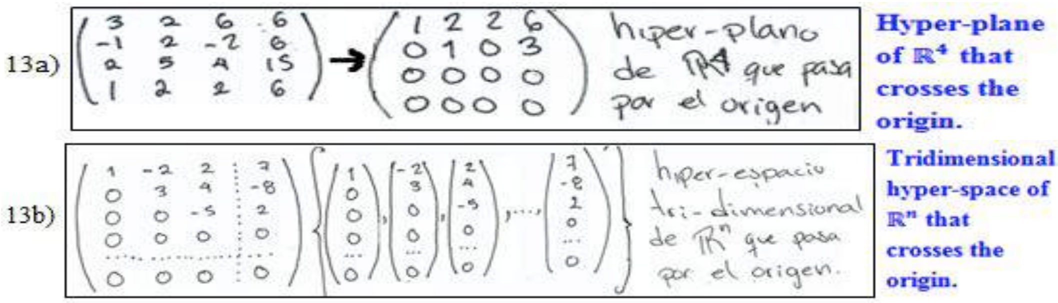


Figure 13. Subspace generated by four vectors of $\mathbb{R}^1$ and a set of vectors of $\mathbb{R}^n$ that generates a subspace of this vector space.

Conclusions

The available data collected from the Activities completed by the Team 4 along with the data analysis allowed us to infer a partial answer to the question: How does the use of the DGS (e.g., Geogebra) influences in the learning of the concept of vector subspace of $\mathbb{R}^n$, $n > 3$? The data analysis indicates that when students implemented the use of technology, they were able to improved learning [to relearn in a deeper thinking way] the concept of vector subspace generated by vectors of $\mathbb{R}^n$. The improvement was reasonable for $n = 2$ and $n = 3$, but if $n \geq 4$ then the improvement in their learning was only partially achieved.

By implementing the use of the echelon form matrices allowed the students to determine the vector subspaces generated by vectors of $\mathbb{R}^n$, $n \geq 4$. Via this way, we could observed that technology partially promotes the learning of the concept of subspaces generated by vectors $\mathbb{R}^n$ and that the pencil-and-paper environment, extends the understanding of the concepts of subspaces of $\mathbb{R}^2$ and of $\mathbb{R}^3$. However, there still are difficulties in the understanding of this concept when students try to extrapolate the concept for sets of vectors of $\mathbb{R}^n$, $n \geq 4$.

REFERENCES

Aranda, C., & Callejo, M.L. (2010). Construcción del concepto de dependencia lineal en un contexto de geometría dinámica: un estudio de casos. *Revista latinoamericana de investigación en matemática educativa, 13(2)*, pp. 129-158.

Andreoli, D.I, Beltrametti, M.A., & Rodríguez, V.C.J. (2009). Construir un puente al álgebra lineal en el entorno Cabri. *Revista Premisa, 11(43),* pp. 21-32.

Aydin, S. (2014). The role of technology in the teaching linear algebra. *Studies in Modern Society, 5(1)*, 117-128. ISSN 2029-431X

Dorier, J.L., Robert, A., Robinet, J., & Rogalski, M. (2011). On a research programme concerning the teaching and learning of linear algebra in the first-year of a French science university. *International Journal of Mathematical Education in Schiece & Technology, 2000, 3(1)*, pp. 27-35. DOI: 10.1080/002073900287354

Dorier, J.L., Robert, A., Robinet, J., & Rogalski, M. (2000). *The obstacle of formalism in linear algebra*. En Dorier, J-L. (Ed.), On the Teaching of Linear Algebra, pp. 85-124.

Duval, R. (1999). *Semiosis y pensamiento humano: registros semióticos y aprendizajes intelectuales* [Sémiosis et Pensée Humaine. Registres sémiotiques et apprentissages intellectuels, Peter Lang S. A. Editions scientifiques européennes, 1995]. Universidad del Valle, Instituto de Educación y Pedagogía, Grupo de Educación Matemática (Ed.), Cap. 1 and Cap. 4.

Duval, R. (2003). «Voir» en mathématiques. En Filloy, E. (coordinador), Matemática Educativa*: Aspectos de la investigación actual*, Fondo de Cultura Económica, pp. 41-76.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, pp. 103–131.

Gol, T.S. (2012). *Using Dynamic geometry to explore linear algebra concepts: the emergence of mobile, visual thinking*. *Doctoral Thesis*. Simon Fraser University, Canada.

Gol, T.S., & Sinclair, N. (2010). Shifts of attention in DGE to learn eigen theory. In Pinto, M.F. & Kawasaki, T.F. (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 33-40.

Harel, G. (1997). The linear algebra curriculum study group recommendations: Moving beyond concept definition. In Carlson D., Johnson, C, Lay, D., Porter, D., Watkins, A, \& Watkins, W. (Eds.). *Resources for Teaching Linear Algebra*. MAA Notes, Vol. 42, 107-126

Mason, J. (2008). Being mathematical with and in front of learners: attention, awareness and attitude as sources of differences between teacher educators, teachers and learners. In T. Wood & B. Jaworski (Eds.), *The International handbook of mathematics teachers education:* Vol. 4. Rotterdam, the Netherlands: Sense Publisher, pp. 31-56.

Pruncut, A. (2008). *A study of students' theoretical thinking in a technology-assisted environment*. *Master's Thesis-Concordia University*: Montreal.

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J. L. Dorier (Ed.), *On the teaching of linear algebra in question* (pp. 209-246). Netherlands: Kluwer Academic Publisher. ISBN: 978-0-7923-6539-6

Sierpinska, A., Dreyfus, T., & Hillel, J. (1999). Evaluation of a teaching design in linear algebra: the case of linear transformation. *Recherches en Didactique des Mathématiques, 19(1)*, pp. 7-10.

Soto, J.L., & Romero, F.C. (2011). El concepto de transformación lineal: una aproximación basada en la conversión gráfico-algebraica, con apoyo de GeoGebra. In F. Hitt & C. Cortés

(Eds.), *Formation à la recherche en didactique des mathématiques* (pp. 38-49). Canada: Loze-Dion. ISBN: 978-2-9235-6554-5

Stewart, S., & Thomas, M.O. (2010). Student learning of basis, span and linear independence in linear algebra. *International Journal of Mathematical Education in Science and Technology*, 41(2), 173-188. DOI: 10.1080/00207390903399620

Uicab, R., & Octaç, A. (2006). Transformaciones lineales en un ambiente de geometría dinámica. *Revista Latinoamericana de Investigación en Matemática Educativa, 9(3)*, pp. 459-490.

Expanding contexts for teaching Upper Secondary school mathematics

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Abstract: This paper describes how the reading of a literary work—The Sand Reckoner, concerning the work and life of Archimedes—created the opportunity of an expanding learning space. Tools and resources of different knowledge domains (funds of knowledge) came together to transform traditional classroom practices where students were actively involved in (re)negotiating their own learning processes as well as their conceptions of mathematical discourse. The research analyzes a year-long, interdisciplinary, didactical intervention based on a teaching experiment methodology; 10th grade students in a public school in Athens brought together different funds of knowledge and Discourses that coalesced to both destabilize and expand the boundaries of official school Mathematics Discourse.

Résumé: Le présent article décrit comment la lecture d' une œuvre littéraire -Le Reckoner du Sable, concernant les travaux et la vie d'Archimède- a créé la possibilité d' un espace d'apprentissage en expansion. Des outils et des ressources des différentes domaines de connaissance (fonds de connaissance) se sont réunis pour transformer les pratiques pédagogiques traditionnelles où les étudiants ont été activement impliqués à la (re)négociation de leur propre processus d' apprentissage ainsi que leurs conceptions de Discourse mathématique. La recherche analyse une intervention didactique interdisciplinaire annuelle, basée sur la méthodologie de l' expériment pédagogique. Les élèves d'une classe de seconde, d' un Lycée public d'Athènes, ont rassemblés des différents fonds de connaissance et des Discourses dont la synthèse a déstabilisé et repoussé les limites de la Discourse Mathématique de l'école officielle .

Introduction

Moje and his colleagues (2004, p. 41) argue that the active integration of multiple funds of knowledge and Discourse is important to support youth in learning how to navigate the texts and literate practices that are necessary for 'survival' in secondary schools. In what comes to be called a 'hybrid' or 'third' space, these seemingly oppositional categories can actually work together to generate new knowledge, new Discourses, and new forms of literacy (Moje, Ciechanowski, Kramer, Ellis, Carrillo & Collazo, 2004). A number of studies have examined the integration of the literacy practices and texts of different domains of knowledge, funds of knowing, and Discourse in Science learning in school (Barton & Tan, 2009; Barton & Tan, 2008; Basu & Barton, 2007; Moje 2004; Moje 2001); others have examined ways of constructing a 'third space' for improving mathematics teaching/learning (Razfar, 2012; Flessner, 2009; Thornton, 2006).

In our paper we claim that introducing the reading of literary works, performing arts and other similar practices in teaching Geometry has the potential to create a hybrid (third) space, with new tools and new Discourse—a blend of standard and non-standard mathematics Discourse—where a richer repertoire of students' participation possibilities is enabled.

The study

Our aim in this article is to discuss how a group of adolescent students engaged in the cross-curriculum project, "In the footsteps of Archimedes," through the reading of the literary work, "*The Sand Reckoner*," which concerned the work and life of Archimedes. Students were encouraged to communicate mathematics through performing arts (using mind and body) and through a variety of practices connected to reading literature. These classroom experiences challenged students' stereotypic images of what constitutes Mathematics knowledge and Mathematics learning (as outcome as well as procedure) through the realization of the relative connections among the disciplines.

In this project we explored the following main research questions:
How can the reading of a literary work in the Mathematics classroom create a (hybrid) third space in which students can renegotiate the dichotomy of the Discourses of Science and Humanities? How might students' experience of the expanded mathematical space transform their conceptualizations of mathematics and motivate their participation?

Data selection

The use of ethnographic research techniques (i.e. participant observation and interviewing) helped us to gather empirical evidence concerning students' experiences; the majority of students' activities were also videotaped and analysed. Semi-structured interviews aimed to explore how students themselves perceived and processed their experience of participating in the project through mind and body. Our data collection was completed with a questionnaire given to the students at the end of the project implementation.

The Project in practice

The project was carried out for one whole school year in a State Lyceum in an inner city school in Athens, Greece, with one class of 10th grade students (12 girls and 12 boys), the participation of 2 teachers, and the teacher librarian. Throughout the project, 28 different thematic teaching interventions and activities were carried out, most of which lasted 2-4 teaching periods. Nearly every week, teaching sessions with different themes were held according to the references that were in the chapter of the book that the students regularly used and that they had read prior to the class activity. In this way, the teaching of different mathematical topics was been presented as the elaboration of meanings constructed during the reading of the book. A mixing of different tools coming from different contexts were used, challenging dichotomies such as body-spirit, formal-informal learning, listening-doing, etc.

The whole investigation was integrated at the end of the school year in a performance addressed to students' parents—an application of their learning further motivated their participation.
In the next paragraph some of the teaching issues concerning Mathematics are presented, followed by the practices and tools that were exploited. In almost all sessions, different worksheets were used.

- Measurement in Antiquity: The value of our decimal system.

The system of naming large numbers from the work, 'The Sand Reckoner,' by Archimedes: revision of the properties of powers. (Dramatisation)

- The Unsolved Geometric Problems of Antiquity (an introduction to cubic routes). Neusis construction: Proposition 8 from the Book of 'Lemmas' by Archimedes (demonstration of instruments for the solution of the geometrical problems, reading relevant extracts from the *"Parrot's Theorem"* by Denis Guedj.
- Resolving mathematical problems with the use of physics principles**.**

Finding the area of a parabolic section and the volume of a sphere by Archimedes (documentary film about Archimedes' Palimpsest)

- Patterns in mathematics (activity *"Balancing mobiles"* from the book *''Mathematics from History, The Greeks''* of M. Brading)
- The relationship of Mathematics and Physics with music (Power Point presentation, 'Rhythm and numbers', music by cello)
- Graphs of functions, the slope and the rate of change of a function (*"Fortune line"*: a graph to show the fluctuations in the intensity of feelings of one of the novels' heroes)
- Mathematics in our life: 'Maths in society','Women mathematicians'

'Maths and Nature','Maths and War' 'The wonderful world of fractals'
(Thematic exhibition, 'Radio broadcasts')

- The moral and political responsibility of the modern scientist.

('role-playing roundtable discussion')

- Applied and pure Mathematics (expressive reading of extracts from the second 'Socratic dialogue' by the Hungarian mathematician A. Renyi)

Discussing the results

Students' experiences

Both from the responses we got in the semi-structured interviews with the pupils, and from the observation during the whole process, it seemed that the pupils were motivated within these expanded contexts, and that they became creative and cooperative through their engagement in new teaching practices. Students referred positively to the whole project and highlighted that the whole process was beneficial not only at a cognitive but at an affective level as well. The following quotes are indicative:

- *'The activities widened my knowledge',*
- '*It made me learn how to do a research'*,
- '*It gave us the opportunity of expressing our creativity and the special skills each student has',*
- '*The round table discussion motivated me a lot. 'It was fun'*
- *'The spirit of cooperation was strengthened'*

Because our sample was small for quantitative analysis, we are attempting to present some data just to enhance our qualitative, anecdotal evidence: Eighteen from the twenty-three students would like to repeat similar activities in the following year, and thirteen consider these activities could be a regular part of the teaching process. Pupils considered that they gained from the whole program: the skill of cooperating 69,5%, knowledge 54%, abilities 43,5%, self-confidence 50%, critical ability 37,5%, creativity 37,5%. They also found teachers cooperation effective (14) and interesting (17).
To the question, ''which activities did you like more and why?'' twenty of the twenty-three students chose the ones with Drama-in-Education techniques. The students defined these activities as being original, interesting, and different. Their main reason for identifying these activities was the possibility of working together (7 answers) while at the same time expressing their creativity. Moreover, students were surprised to have the chance to use both mind and body in doing mathematics.

The experiential character of learning: Student interviews revealed that Drama-in-Education techniques were the tool that mediated mathematical knowledge by offering the experiential dimension, motivating students to become active learners who owned their own learning (November, 2012). *'You think the same way as the hero of the book...' 'I was anticipating the experiential activities with greater interest than the other ones'.*

The role of creating the third space: Moje et al. (2004) speak about hybrid space where different funds of knowledge and Discourses coalesce to destabilize and expand the boundaries of official school Discourse. In our classroom intervention this view was affirmed, as demonstrated in the following student quotes:

'I was impressed the way all disciplines were integrated',
'A whole world opened up that is waiting to be explored',
'We managed to make connections between Mathematics and civilization and to learn that everything around us is connected to Mathematics'.

The nature of mathematical knowledge: In the responses we got in the interviews with the pupils, the impact on their perceptions of mathematics is emphasized rather than the cognitive level: '*It* *made me realize that Mathematics is a living entity', 'It is not only what we learn at school but something more beautiful and attainable'.*

This further reveals the differences between student and teacher perspectives on the learning activity; whereas the teacher is focusing on the cognitive outcomes, the students are experiencing the third space in terms of aesthetics and emotions (Appelbaum & Scott Allen 2008).

In conclusion: The use of a literary book in school mathematics teaching, as well as the fusion of varied mediation tools and disciplinary discourses, may be understood in terms of the contextual, affective, and attitudinal approaches to a curriculum where mathematics may be humanized. These experiences also enable opportunities for incorporating unconventional and informal practices that generate a third space where the different Discourses establish a dialogue; Students in this third space have the opportunity to renegotiate both mathematics concepts and their own, personal perception of what constitutes mathematics knowledge. Most importantly, students in the third space can understand this perception as *their own* perception of mathematics, one they could carry with them into future studies.

REFERENCES

Appelbaum, P. & Scott Allen D. (2008). *Embracing Mathematics: On Becoming a Teacher and Changing with Mathematics*. New York: Poutledge.

Basu, J., Barton, Calabrese, A. (2007). Developing a sustained Interest in Science among Urban Minority Yuth. *Journal of research in Science Teaching, 44*(3), 466-489.

Barton, Calabrese, A. C. & Tan, E. (2009). Funds of knowledge and Discourses and Hybrid Space. *Journal of research in Science Teaching, 46*(1), 50-73.

Barton, Calabrese A. & Tan, E. (2008). Creating Hybrid Spaces for Engaging School Science Among Urban Middle School Girls. *American Educational Research Journal* Vol. 45, No. 1, pp. 68 –103.

Flessner, Ryan (2009) Working toward a third space in the teaching of elementary mathematics. *Educational Action Research*, *17*(3), 425-446.

Moje, E. B., Ciechanowski, K. M., Kramer, K., Ellis, L., Carrillo, R., & Collazo, T. (2004). Working toward third space in content area literacy: An examination of everyday funds of knowledge and discourse. *Reading Research Quarterly, 39*, 38–72.

Moje, E.B., Collazo,T., Carrillo, R., & Marx, R.W. (2001). "Maestro, what is 'quality'?": Language, literacy, and discourse inproject-based science. *Journal of Research in Science Teaching*, *38*, 469–496.

November, A. (2012). *Who owns the Learning? Preparing students for success in the digital age*. Blooomington, IN: Solution Tree Press.

Razfar, Aria (2012). Vamos a Jugar Counters! Learning, Mathematics Through Funds of Knowledge, Play, and the Third Space , Bilingual Research Journal: *The Journal of the National Association for Bilingual Education, 35*(1), 53-75.

Thornton, S. (2006). Moving into Third Space — High School Students' Funds of Knowledge in the Mathematics Classroom. In Peter Grootenboer, Robyn Zevenbergen and Mohan Chinnappan (Eds.) *29th annual conference of MERGA,* (pp.504-511). Camberra Adelaide.

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Objectification of the concept of variation about the quadrature problem

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Abstract: in this article we report the results obtained when implementing Activities related with the concept of variation about the quadrature problem with high-school students (grade 11) in Mexico City using paper-and-pencil and technological environments (e.g., GeoGebra). This is a qualitative research and is supported by the Theory of Objectification (Radford, 2006, 2008, 2014). The data collection was done by video-recording the students' work while solving the Activities, worksheets and software generated files with GeoGebra. Our results show that the use of paper-and-pencil and technology promote the objectification of the concept of variation among students.

Résume: dans ce papier nous reportons les résultats issus lors d'implémenter, avec des étudiants d'école secondaire 5 (degré 11) dans la ville de Mexico, au Mexique, des Activités liées avec le concept de variation sur le problème des quadratures, dans les environnements papier/crayon et ainsi que technologique (e.g., Geogebra). Cette recherche est de type qualitatif et elle est appuyée sur la Théorie de l'Objectivation (Radford, 2006, 2008, 2014). La collecte des données nous l'avons faite par vidéo-enregistrement lorsque les étudiants ont été en train de résoudre les Activités, feuilles de travail et ainsi que des fichiers génères par le software Geogebra. Nos résultats nous suggèrent que l'utilisation de papier/crayon et ainsi que celle de la technologie favorisent l'objectivation du concept de variation chez les étudiants.

Background and research problem

Educational mathematics research on the concept of variation (e.g., Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Vasco, 2006; among others) shows that the teaching of topics involving variation and change is commonly done in a traditional way through algorithms (using only paper-and-pencil) and lacks visual and geometric arguments. In this article we seek to answer the question: How does solving tasks [quadrature problem] designed in paper-and-pencil and technological environments influence the process of objectification of the concept of variation in the students?

Theoretical framework

This study is supported on the Theory of Objectification (Radford, 2006, 2008, 2014) founded on the concepts of *labour*, knowledge, knowing, and learning. This theory considers learning as a social process that goes back and forth between knowledge and the self. Through labour the subject transforms knowing into objects of conscience to give room to learning; however, such transformation (knowledge mediation) does not occur in an isolated way; it demands a labour together with the other, and during this labour not only is knowledge transformed but also the subject who acquires it, the one who learns (Radford, 2014, p. 138). One of the aims of the Theory of Objectification is to make evident how the subject learns the cultural knowledge through social interaction and semiotic means of objectification (Radford, 2003, 2005) as signs (written, verbal or gestural) and artifacts, fundamental sources of meaning production (Radford, 2006).

Indeed, the objects, the tools, the linguistic resources, and the signs which the subjects use intentionally in the processes of signification to carry out their actions and reach their goals constitute the so called semiotic means of objectification (Radford, 2003, p. 41). Among these, the gesture stands out, particularly the one made with the hands which may move in time and space with the purpose of conveying ideas. This goal is not always achieved with either written or spoken language.

Method

This is a qualitative research. The participants were 12 high-school students (ages 16 and 17) from different classrooms (grade 11) coursing Analytic Geometry in a Mexico City school. The students were chosen by their teacher taking in account their good performance in mathematics. They were divided in six teams of two members each and were video-recorded while they solved the Activities. All the Activities were solved in paper-and-pencil and technological (GeoGebra) environments. The data collection was done through video-recording, work sheets and software generated files. Due to limitations of the workspace, in this article we only report the work of one team solving two of the Activities.

Data analysis and result discussion

The discourse made by the students during the development of the Activities implemented is a way of social interaction, a social practice; a reflection influenced by artifacts, either material or cognitive, such as gestures, language and objects, among others (Radford, 2006). It is a joint labour in which both the knowledge and the subject that acquires such knowledge are transformed. Therefore, in order to carry out the data collection, this article considers the relationship between the different ways in which the students clarify and convey ideas with language and artifacts while they discuss and reflect together (Radford, 2014), as well as the students' written answers in each Activity.

Here, Activities A1 and A2 are described; we show the excerpts of the discussion and the reflection held by the students E1 and E2 from Team 1 when solving the activities (A1 in a paper-and-pencil environment, and A2 in a technological one with GeoGebra), and the data obtained from their work is analyzed.

Description of Activity A1 (paper-and-pencil environment)

In Figure 1, the straight lines l_1, l_3, and l_5 are parallel to the straight lines l_2, l_4 , and l_6, respectively, while H is the midpoint of $\overline{EG}$.

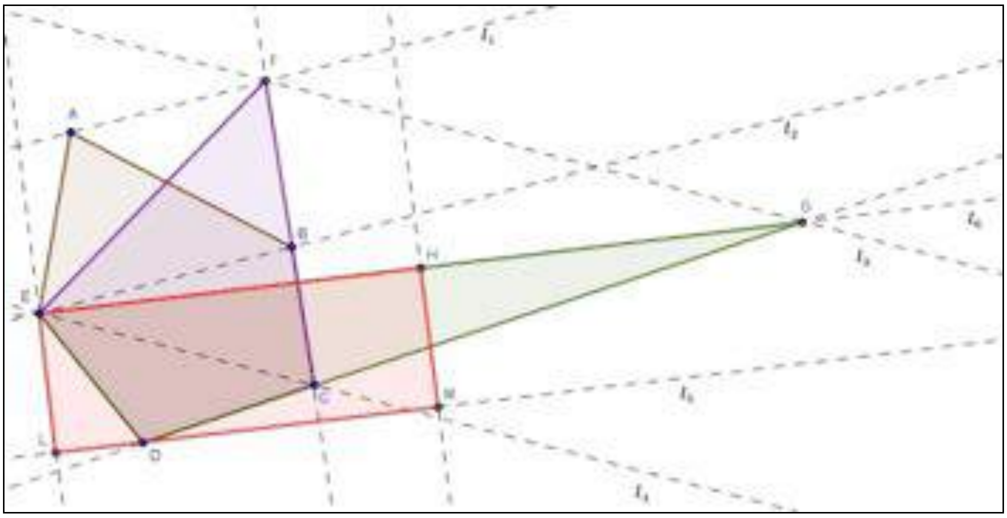


Figure 1. Polygons of equal areas but different perimeters.

(a) Use paper-and-pencil to prove that the area of:

(i) ⬟*ABCDE* [3] and the area of ◊*CDEF* [4],

[3] ⬟*ABCDE* refers to any pentagon whose vertices are the points A, B, C, D, and E.
[4] ◊*CDEF* refers to any quadrangle whose vertices are the points C, D, E, and F.

(ii) ◊*CDEF* and the area of △*DEG* [5],

(iii) △*DEG* and the area of ▯*EHML* [6] are equal.

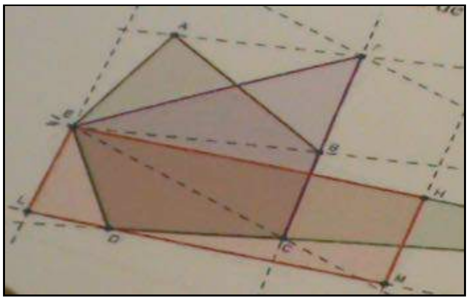
(b) Provide convincing arguments to demonstrate that the areas of all these polygons are equal.

Development and analysis of Activity A1

(a) (i) Equality of the areas of ⬟*ABCDE* and ◊*CDEF*

E1: Uhm, these [⬟*ABCDE and* ◊*CDEF; Figure 2a*] have areas in common [*points with his finger at* ◊*BCDE; Figure 2b*] and these [*points with his finger at* △*ABE and* △*FBE*] have the same height.

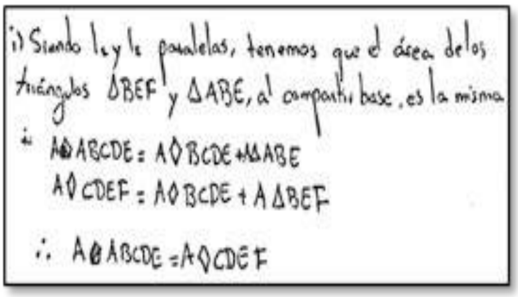
E2: Yes. [*E1 immediately starts writing the proof shown in Figure 3a*].



(2a)

(2b)

Figure 2. In (2b), E1 uses a gesture to explain to E2 why the areas of ⬟*ABCDE* and ◊*CDEF* shown in (2a) are equal.



i) Siendo l_1 y l_2 paralelas, tenemos que el área de los triángulos ΔBEF y ΔABE, al compartir base, es la misma
∴ A⬟ABCDE = A◊BCDE + AΔABE
A◊CDEF = A◊BCDE + AΔBEF
∴ A⬟ABCDE = A◊CDEF

(3a)

i) Being l_1 and l_2 parallel, we have that the areas of triangles △*BEF* and △A*BE* is the same as they share base.
If $A($⬟$ABCDE) = A(◊BCDE) + A(\triangle ABE)$
$A(◊CDEF) = A(◊BCDE) + A(\triangle BEF)$
∴ $A($⬟$ABCDE) = A(◊CDEF)$

(3b)

Figure 3. (3a) Written proof by E1 and E2 that the areas of ⬟*ABCDE* and ◊*CDEF* are equal. (3b) English translation of answer given by E1 and E2 in section (a) (i) of A1.

(a) (ii) Equality of the areas of ◊*CDEF* and △*DEG*

E1: This area [*points with his finger at* △*CDE*] is common to both [*points with his finger at* ◊*CDEF and* △*DEG*] and these two [*points with his finger at* l_3 *and* l_4] are parallel. [*See Figure 4a*]

E2: Oh, yes. That's right.

[5] △*DEG* refers to any triangle whose vertices are the points D, E, and G.
[6] ▯*EHML* refers to any rectangle whose vertices are the points E, H, M y L.

E1: Then these two triangles [*points with his finger at* $\triangle CEF$ *and* $\triangle CEG$] are equal.

E2: Yes, this point [*he refers to the intersection point of the side* $\overline{CF}$ *of* $\lozenge CDEF$ *and the side* $\overline{EG}$ *of* $\triangle DEG$]…

E1: Mmh…

Next, E2 refers to the common region between $\lozenge CDEF$ and $\triangle DEG$. If P is the intersection point of the side $\overline{CF}$ of $\lozenge CDEF$ and the side $\overline{EG}$ of $\triangle DEG$; then, the region common to both polygons is $\lozenge CDEP$.

E2: Yes well, when we cut the common [*he refers to* $\lozenge CDEP$], well, not to this [*he refers again to* $\lozenge CDEP$], but to this [*he refers to* $\triangle CDE$ *and points at it with his finger*]. [*See Figure 4b*]

E1: Yes, this little triangle [*points with his finger at* $\triangle CDE$*; Figure 4b*], now the others [*he refers to* $\triangle CEF$ *and* $\triangle CEG$] have… Ok… [*He means that the areas are equal*].


(4a)

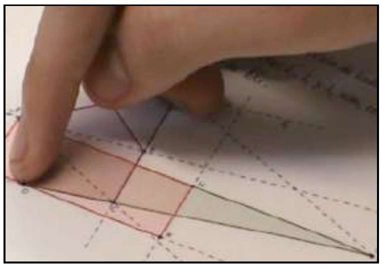
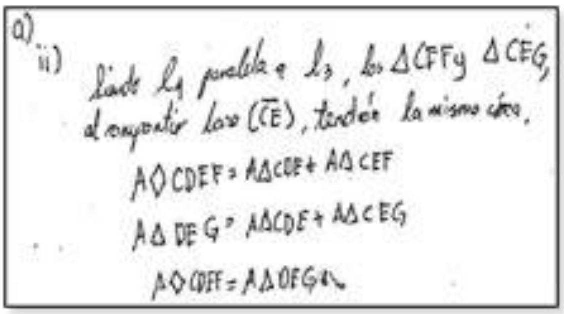

(4b)

Figure 4. E1 and E2 use gestures; in (4a) to identify the parallels l_3 and l_4 related with $\lozenge CDEF$ and $\triangle DEG$, and in (4b) to identify the region common to these polygons.

Now, the students write down their answer on paper (Figure 5a).



(5a)

ii) Being l_4 parallel to l_3, $\triangle CEF$ and $\triangle CEG$ will have the same area as they share base ($\overline{CE}$),

$A(\lozenge CDEF) = A(\triangle CDE) + A(\triangle CEF)$

$A(\triangle DEG) = A(\triangle CDE) + A(\triangle CEG)$

$A(\lozenge CDEF) = A(\triangle DEG)$

(5b)

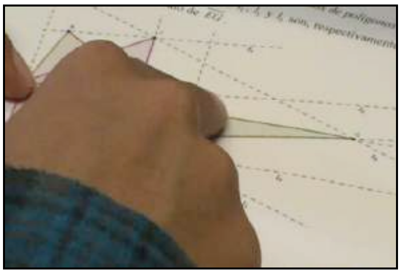
Figure 5. (5a) Written proof by E1 y E2 that the areas of $\lozenge CDEF$ and $\triangle DEG$ are equal. (5b) English translation of the answer given by E1 and E2 in section (a) (ii) of A1.

(a) (iii) Equality of the areas of $\triangle DEG$ and $\square EHML$

E1: Now, for $\triangle DEG$ and $\square EHML$ (Figure 6a). This, as they told us, is the midpoint [*he refers to H, but at the same time points with his fingers to the points E and H; Figure 6b*] well yes, it's done. [*He means it is enough with that; it is proven.*]

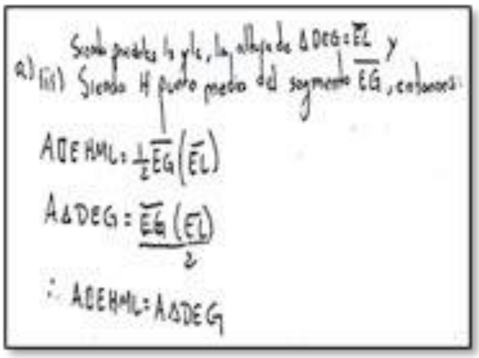
E2: Oh, yes, it is the same. Because this and this are parallel [*points with his finger at* l_6 *and* l_5*. Right away, E1 starts writing down the proof shown in Figure 7a*]…

(6a)



(6b)

Figure 6. E1 uses gestures to explain to E2 the equality of the areas of $\triangle DEG$ and $\square EHML$.



(7a)

iii) Being l_5 and l_6 parallel, the height of $\triangle DEG = \overline{EL}$ and being H midpoint of segment $\overline{EG}$, then:

$$A(\square EHML) = \frac{1}{2}\overline{EG}(\overline{EL})$$

$$A(\triangle DEG) = \frac{\overline{EG}(\overline{EL})}{2}$$

$$\therefore A(\square EHML) = A(\triangle DEG)$$

(7b)

Figure 7. (7a) Written proof by E1 and E2 that the areas of $\triangle DEG$ and $\square EHML$ are equal. (7b) English translation of the answer given by E1 and E2 in section (a) (iii) of A1.

(b) Provide convincing arguments to proof that the areas of all these polygons are equal.

E2: Well, is one is equal to the other, and the other is equal to the other, well that's it… [*He means that each and every area involved is equal to the others*].

E1: Ok

Figure 8a shows the written answer the students provided for this section.

(8a)

b) If

$A(⬟ABCDE) = A(\diamond CDEF)$,

$A(\diamond CDEF) = A(\triangle DEG)$ ($A(\square EHML)$) no

and $A(\triangle DEG) = A(\square EHML)$

then:

$A(⬟ABCDE) = A(\diamond CDEF) = A(\triangle DEG) = A(\square EHML)$

(8b)

Figure 8. (8a) Written answer in section (b) of A1 provided by E1 and E2. (8b) English translation of the answer given by E1 and E2 in section (b) of A1.

During the development of this Activity, the students talk using colloquial language which is inaccurate in occasions while, in their written answer, they combine symbolic and colloquial language in an adequate manner. In the three sections of (a) in this Activity, E1 and E2 use a sign (gesture) to provide clarity to their explanation and justification as to why the areas of the polygons in question are equal, as well as to distinguish the other polygons involved in their speech. They discuss and reflect (social interaction) adding previous knowledge related with the area of a triangle and with the properties of parallel straight lines (as a result of their knowledge mediation), which they turn into action from the start (as shown in the dialogues and in Figures 3 to 7). The moment E1 and E2 point with their finger at different points, straight lines, and regions in Figure 1 to convey their ideas and justify their answers gives sense to the equality of the areas, first in (a) (i) and (a) (ii) with an addition (see Figures 3 and 5), and then in (a) (iii) by pointing out that the segment $\overline{EH}$ is half of $\overline{EG}$ (see Figure 7).

Description of Activity A2 (technological environment)

The students open the GeoGebra file in which Figure 1 of Activity A1 has been reproduced and they manipulate the dynamic construction. In this Activity the students are required the following:

(a) Explain in a clear manner how it is possible to find a square of equal area to this polygon [quadrature of the polygon] from any polygon given [regardless of the number of sides].

(b) Argument why the equality of the areas of the polygons is maintained every time one of their sides is eliminated, and why their perimeter varies.

(c)

Development and analysis of Activity A2

(a) Explain in a clear manner how it is possible to find a square of equal area to this polygon [quadrature of the polygon] from any polygon given [regardless of the number of sides].

E2: Of course, they don't change their proportions [*he means the equality of the areas of the polygons while E1 manipulates the construction*].

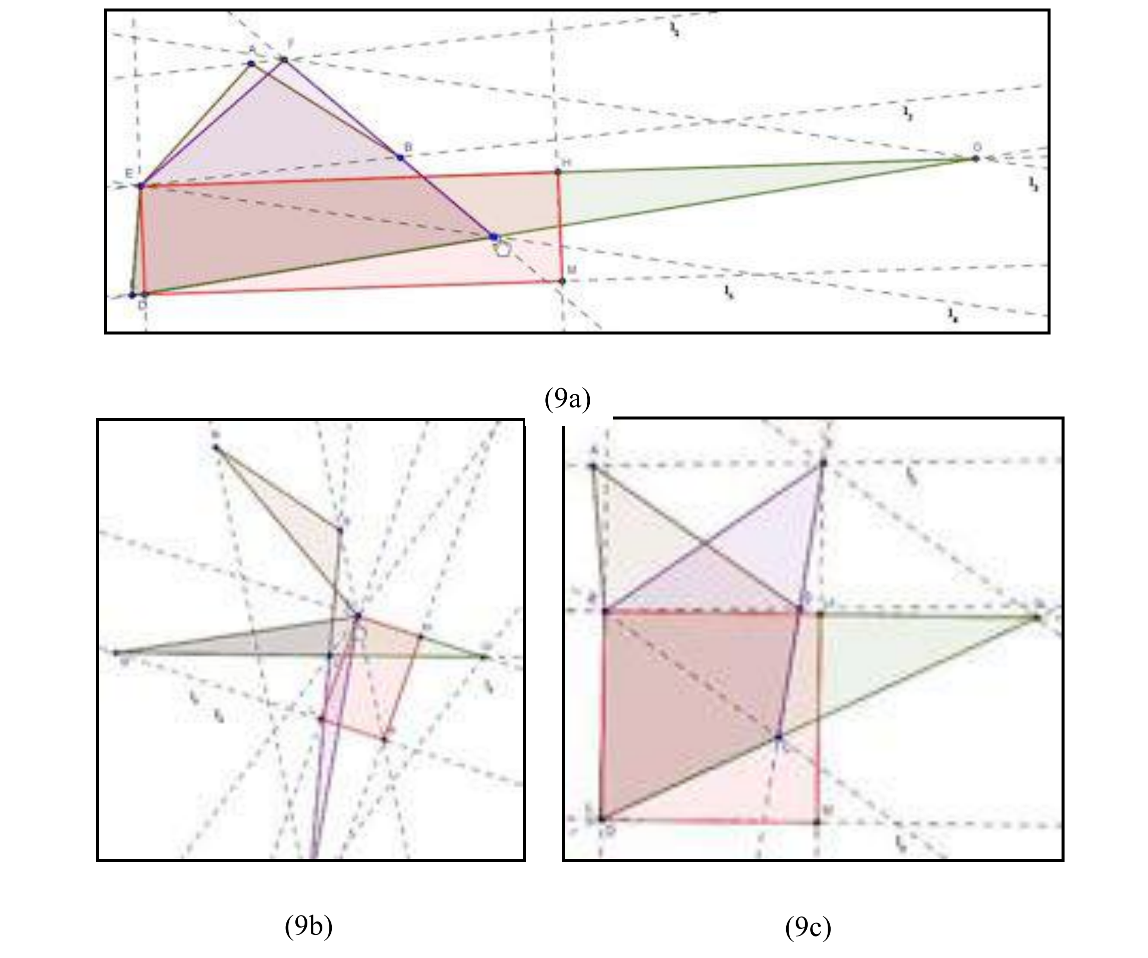
E1: No, because everything moves on the parallel lines [*see Figure 9*]. Mmh, how is it possible to find a square of equal area from a polygon?

E2: From any, right? [*He means any polygon*].

E1: Well yes, It is true for any, that is regardless…

E2: As long as their proportionality is maintained [*he means the areas of the polygons*].

E1: Ok, we may get a square from any polygon, but how do you explain that?



(9a)

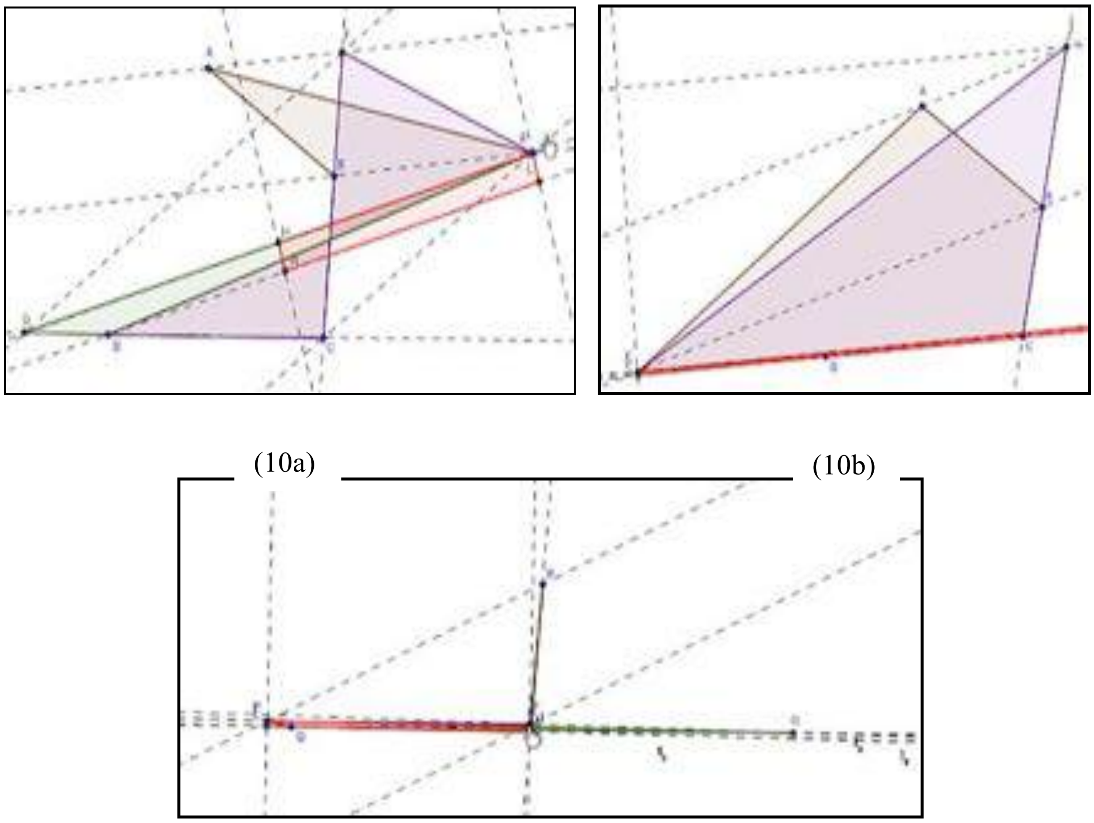
(9b) (9c)

Figure 9. (9a) Approximation from the pentagon to the quadrangle. (9b) Intersection of the sides of a same polygon. (9c) Approximation from the given rectangle to a square.

E2: That means, we could say that it may be found with the analytical process of equalizing the areas and finding the variables…

E1: Yes, but [...] is right it is true for any polygon [...] Oh, ok, then it is possible… [*To construct a square of equal area to that of the given polygon while approximating the given rectangle to a square; Figure 9c*]. It is the same, isn't it? That as long as it does not go through any of its sides [*as long as the sides of a polygon do not cut each other*] the pull [*of any vertex of* ⬟*ABCDE*]…

E2: Yes, as long as it [*any vertex of the polygon*] does not extend infinitely and touch the parallels [*of any vertex of the given polygon*] its proportionality is maintained [*he means the equality of the areas of all the polygons*]. [*See Figures 9 and 10*]



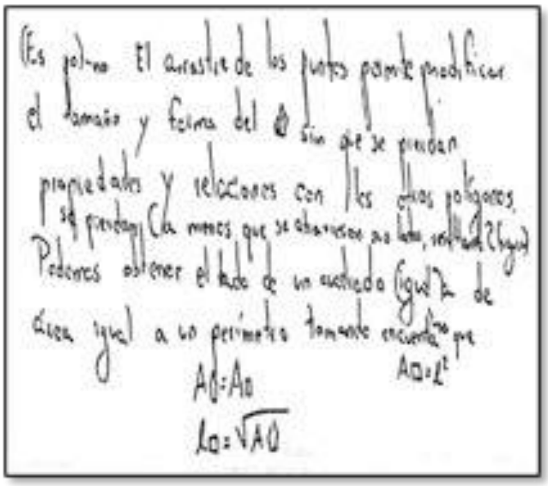
(10a) (10b)

(10c)

Figure 10. (10a) Intersection of the sides of the polygons. (10b) Some overlapped parallels. (10c) Polygons degenerated into straight lines.

E1: Their proportions are not lost unless their sides [*of a same polygon*] are crossed. [*See Figures 9 and 10*]

Next, the students start writing down their answer on paper (see Figure 11). E1 and E2 explore the construction; pull each and every of the vertices of the pentagon ⬟*ABCDE* over all the work area; they observe at moments that some of the polygons are deformed until they are confused with either others (convex or otherwise) or a straight line or point while two parallels get as close as they want or if they separate or get close to the vertices of ⬟*ABCDE* as much as they want (Figure 10). They say that if certain rules are not followed, it is possible that the proportions of the polygons are lost and that their areas are not equal; however, they do not validate such conjecture, which is not entirely true since disconnected polygons maintain their areas. Nevertheless, in general terms, the students use GeoGebra as a semiotic means of objectification since with it they think, reflect and act to provide a response to this section (see Figure 11).


(Es posible) El arrastre de los puntos permite modificar el tamaño y forma del ⬟ sin que se pierdan propiedades y relaciones con los otros polígonos se pierdan (a menos que se atraviesen sus lados, creando 2 figuras) Podemos obtener el lado de un cuadrado igual de área igual a un perímetro tomando en cuenta que

$A_{\square} = \ell^2$

$A_{⬟} = A_{\square}$

$\ell_{\square} = \sqrt{A_{⬟}}$

(11a)

The pull of the points allows modifying the size and shape of the ⬟ without losing the properties and relationships with the other polygons are lost (unless their sides are crossed, creating 2 figures).
We can obtain the side of a square of equal area to a perimeter taking into account that

$A_{\square} = \ell^2$

$A_{⬟} = A_{\square}$

$\ell = \sqrt{A_{⬟}}$

(11b)

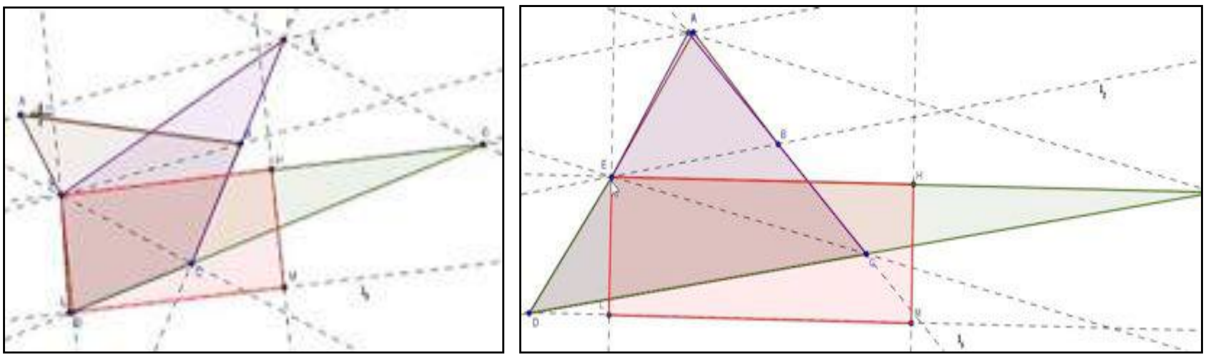
Figure 11. (11a) Written answer to section (a) of A2 provided by E1 and E2. (11b). English translation of the answer given by E1 and E2 in section (a) of A2.

(b) Provide argumentation as to why the equality of the areas of the polygons is maintained every time one of their sides is eliminated, and why their perimeters vary.

E1 movers the dynamic construction (see Figure 12) and continues the dialogue.

E1: Yes, these [*areas*] when certain properties are fulfilled…

E2: Because it doesn't matter the sides it [*the polygon given*] has, the properties of such figures [*he means the other polygons*], despite their sides, will match the proportionality between them [*he means the proportions between the sides of the polygons*]. And their perimeters change because they are ultimately based on the number of sides. [*See Figure 12b*]



(12a) (12b)

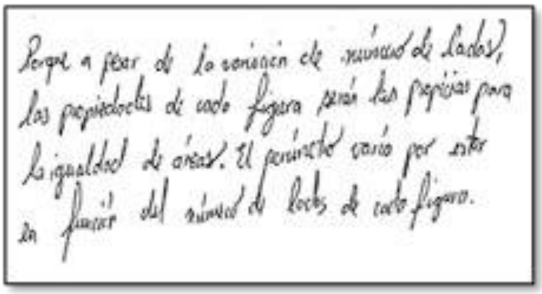
Figure 12. Polygons of variable perimeters and equal areas. (12b) The pentagon and the quadrangle are confused and approximated to a triangle maintaining their areas.

E1: Yes.

E2: Are the properties of each figure [*maintained*] no matter their sides? Well, not that they don't matter… Just with that, the properties of each figure…

E1: It's not all the properties [*the ones maintained*], it's just that the area doesn't change.

Next, the students start writing down their answer on paper (see Figure 13). They manipulate the dynamic construction and argument that the equality of the areas of all these polygons is maintained if certain properties of the polygons are fulfilled, like proportionality, regardless the variation in the number of sides. However, this explanation is ambiguous. Using the software, they clarify, explain, and justify that the perimeter of each polygon depends on the number of its sides, so the perimeter of each polygon varies and its area remains constant. These facts are observed in the dialogue between the students, in Figure 12, and in the written answer they provided (Figure 13); it is evident they resort to the software to (dynamically) strengthen what they had studied in Activity A1. Above all, they understand that the perimeter of every polygon varies even when the areas of all are equal.


Porque a pesar de la variación de número de lados, las propiedades de cada figura serán las propicias para la igualdad de áreas. El perímetro varía por estar en función del número de lados de cada figura.

Because despite the variation in the number of sides, the properties of each figure will be suitable for equality in the areas. The perimeter varies because it depends on the number of sides of each figure.

(13a) (13b)

Figure 13. (13a) Written answer provided by E1 and E2 in section (b) of A2. (13b) English translation of the answer provided by E1 and E2 in section (b) of A2.

Overall, in sections (a) and (b) of Activity A2, E1 y E2 discuss and reflect (social interaction) with the aid of software. They add both the knowledge used and those acquired in Activity A1 (as a result of the mediation of their knowing) which they set in play again while they manipulate the construction with the software (as a means of semiotic objectification) to give sense to the equality of areas in all the polygons involved, and to understand how to find a square of equal area to the given polygon (Figure 11a) and realize what varies and what remains constant in the given construction. In fact, the students use GeoGebra as a semiotic means of objectification since using it they think, reflect, and act to solve this Activity and provide an answer to what is asked in every section.

Conclusions

In the activities designed for this study, in the pencil-and-paper environment, the test is carried out from a known hypothesis through a series of logical and valid reasoning using a given figure (fixed) until reach a result that confirms it; likewise, there is the possibility of raising new hypotheses and make conjectures. GeoGebra allows us to reason, argue and create new hypotheses and conjectures differently than in the static environment formulated. The dynamic construction can be manipulated using the *drag* tool and slippers, to better understand the geometric behavior of the object of study and justify the statement given. However, you should not ignore the work with paper-and-pencil, because that accounts for the knowledge that students bring into play, and how they organize them in written replies when they meet each of the activities through the use of colloquial and formal language.

The gestures used in Activity A1 allowed the students to generate and clarify ideas to provide arguments for their proofs; while they elaborated conjectures with the software used in Activity A2, they understood why it is possible to find a square of equal area to that of a given polygon and they managed to objectify the concept of variation [through a process] when they say that the perimeter of every polygon varies because it depends on de sides and that the only thing that remains unchanged is the area.

The design of the Activities (first, with paper-and-pencil, and then with technology) promotes the learning of the concept of variation inherent in the quadrature problem because the dynamic nature of the software (GeoGebra) allowed the students to validate or reject the conjectures made around a drawing on paper, to find properties that are not easy to detect in a paper-and-pencil environment, and to formulate new conjectures as when they say that the areas of the polygons are maintained as long as the sides of the figures do not cross or overlap each other and they do not grow infinitely or get reduced to a point. Our results suggest that the use of paper-and-pencil and technology promote the objectification of the concept of variation in students.

REFERENCES

Carlson, M., Jacobs, S., Coe, E., Larsen, S. & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. Journal for Research in Mathematics Education, 33(5), 352-378.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: a semiotic-cultural approach to students' types of generalization. Mathematical Thinking and Learning, 5(1), 37-70.

Radford, L. (2005). Why do gestures matter? Gestures as semiotic means of objectification. En H. L. Chick & J. L. Vincent (Eds.), Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 143-145). University of Melbourne, Australia.

Radford, L. (2006). Elementos de una teoría dela objetivación. Revista Latinoamericana de Investigación en Matemática Educativa, número especial, 103-129.

Radford, L. (2008). The ethics of being and knowing: towards a cultural theory of learning. In Radford, L., Schubring, G. and Seeger, F. (eds.), Semiotics in mathematics education: epistemology, history, classroom, and culture, Rotterdam, NL, Sense Publishers, pp. 215–234.

Radford, L. (2014). De la teoría de la objetivación. Revista Latinoamericana de Etnomatemática, 7(2), 132-150.

Vasco, C. E. (2006). El pensamiento variacional, la modelación y las nuevas tecnologías. En C. Vasco (Ed.), Didáctica de las matemáticas: artículos selectos. (pp. 134-148). Bogotá: Universidad Pedagógica Nacional.

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Visual Strategy and Algebraic Expression: Two Sides of the Same Problem?

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Abstract: It has been advocated that the search for patterns and their organization in mathematical language is a central component of mathematical thinking. Hershkowitz, Arcavi and Bruckheimer (2001) investigated problem solving processes of a "visual-pattern-problem" which start with visual strategies for reorganizing the visual pattern and ended in a formal algebraic expression as a solution. In this study we had decided to use the same problem for investigating also a process in an opposite direction: starting from a given algebraic expression, analyzing it in a way that uncovers the visual strategies which are behind the algebraic components of the given expression.

Resumè : Il a été préconisé que la recherche de modèles et de leur organisation en langage mathématique est un élément central de la pensée mathématique. Hershkowitz, Arcavi et Bruckheimer (2001) ont étudié les processus de résolution de problèmes d'un "problème de modèle visuel" ("visual-pattern-problem"). Ceux-ci commencent par des stratégies visuelles afin de réorganiser le modèle visuel pour aboutir enfin à une expression algébrique formelle. Dans cette étude, nous avons décidé d'utiliser le même problème pour enquêter aussi un processus dans la direction opposée: partant d'une expression algébrique donnée, l'analyser d'une manière qui décèle les stratégies visuelles qui sont derrière les composants algébriques de l'expression donnée.

Background

Many questions about the different roles of visualization in mathematics have been addressed in the last few decades of mathematics education research (Arcavi, 2003). Researchers have studied the ways in which children develop ways to describe visual patterns gradually grasping the basic concepts of algebra. Among the ways are the use of computer software that enables to perform visual manipulations in building visual patterns (Healy & Hoyles, 1999), and also software that enables the constructing of a model by visualizing sets of patterns gradually replaced by numbers and variables (Mavrikis, Noss, Hoyles, & Geraniou, 2013).

Hershkowitz et al. (2001) used a rich visual task (Figure 1) in order to invoke as many visual solution strategies as possible, that were categorized according to the visual-counting strategies used by the different solvers.

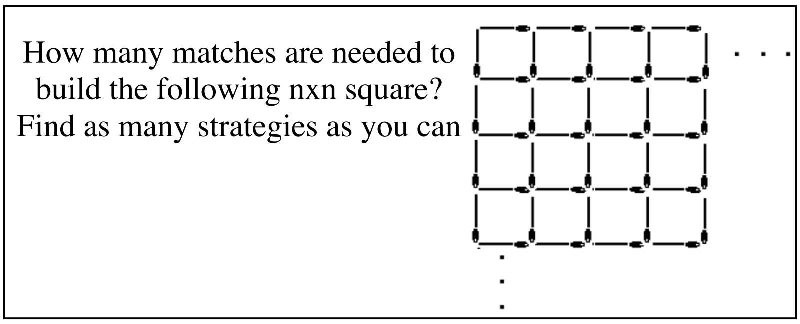

Figure 1: nxn square made of matches

This problem is representative of a whole class of "counting" situations in which the solutions' main steps can be described as follows: observation, recording and understanding of regularity, finding and applying a "visual-counting" strategy, generalizing and capturing the generalization in a symbolic-algebraic form (see Figure 2).

1. $4+3(n-1)\cdot 2+2(n-1)(n-1)$	3. $2\cdot 2(1+2+\cdots+n)$
2. $\frac{4n^2+4n}{2}$	4. $4\cdot\frac{n(n+1)}{2}$

Figure 2: Algebraic solution examples to the visual problem

The algebra expresses the visual-counting-strategies, and compresses them into expressions. This emphasizes that even though the problem is visual, without using algebra we would not have been able to write the solution in a compact way.

Methodology

Participants in the study were eight mathematics education graduate students from different programs in Israel. The participants took part in a group's activity that has two parts. In the first part participants were asked to individually find an algebraic expression that would represent the number of matches needed to build the nxn matches' square shown in Figure 1.

In the second part, participants were asked to go in the opposite direction; they were given six different algebraic expressions that represent solutions for the number of matches in the nxn square. The participants were then asked to choose an expression, to find a visual strategy for finding the number of matches in the nxn square, where the algebraic expression they choose is its solution. Then they had to explain to their peers the visual strategy they had found. Data sources included the individual written report, written by each student along his/her work's process, video documentation of the group activity, and field notes.

The analyses and interpretations of the students' reports were based to large extent on the findings of Hershkowitz et al. (2001). Next, we focused on categorizing the different characteristics of the visual strategies which are hidden behind the given algebraic expressions.

Results

Our initial analysis revealed three main characteristics in uncovering the visual pattern strategy which is the origin of the algebraic expression as a whole or its components. In the following we will illustrate these characteristics as they appeared.

The dual role of numbers in the algebraic expressions - a quantity and a visual-construction-unit

While trying to uncover the visual strategy which was the origin to algebraic expressions, we found out that the most popular component in the matches' square problem was "4". In some cases it was standing for a quantity. For example, when dealing with the expression: $4\cdot\frac{n(n+1)}{2}$, student C referred to the "4":

C: "Where does this four come from? I now need to decompose the, to decompose it to four parts. … The most natural way for me to get to four parts is to look at each triangle (One quarter of the square while divided to four parts by its diagonals).
In a different situation, student D was addressing the "4" as something else:
D: "… and I saw n four (nx4). OK?"

Instructor: "When you say n four what do you mean?"
D: "n four is n times square" [referring to a square construct made of four matches.]

Visual counting strategies behind arithmetic operations

Arithmetic operations within the algebraic expression might serve as hints for the visual counting strategies upon which the expression is constructed. For example, the division by 2 operator in the expression: $\frac{4n^2+4n}{2}$. This expression was the product of the visual counting strategy which was called by Hershkowitz et al. (2001), Shake and count. In this solution some of the visual constructs counted "shared" matches which are thus counted twice, so the solution process has to take into account fixing this double counting by division. When describing his way of constructing a visual solution suitable for this expression, student B referred to the operator:

"We have here four n squared, plus four n, divided by two. So first the division by two gives me a hint that something is counted here twice and at the end we divide".

Ambiguity in visual representations of algebraic expressions

One expected outcome was that each algebraic expression had several suggested visual solutions that could be described with it. This ambiguity manifested, for example, in the "3" included in the expression: 4+3(n-1)·2+2(n-1)(n-1). This expression was visualized commonly by what Hershkowitz et al. (2001) categorized as from one square on…As described visually in in figure 3, and explained by student A:

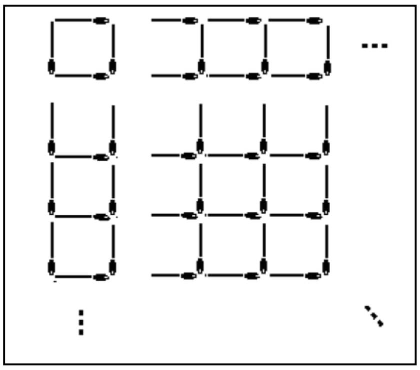

Figure 3: From one square on…

A: The first square [upper left] has four sides, four matches, and here we have n minus one [showing left column], and here we have n minus one [showing top row], and each one of these, we add three. …
[background] Wait. I did not understand. Which three matches?
A: This has four matches [upper left square]. Here we have n minus one squares. n minus one sides, like, we multiply by three. One, and two, and three [counting sides of U pattern marked in red in figure] one-two-three, one-two-three
Instructor: All right.
A: And also these [showing the turned U figures that make up the top row]. So we have four, plus two times n minus one times three.
For this case, although the U and the ɔ patterns are visually different, it took only one minor clarification for all of the students to accept that number "3" stands for them both, thus not explicitly giving any information about the visual solution apart from this part of the expression is constructed from three matches.

Expression form which is familiar as representing numerical counting strategy

Some expressions are familiar as the sum of numerical sequence. For example, the expression: $4 \cdot \frac{n(n+1)}{2}$. This expression was called by Hershkowitz et al. (2001) Starting with symbols. In this case, participants identified this expression as four times the sum of 1+2+3+…+n. This resulted in many instances of an arithmetic series that could be identified in the pattern in various ways. Two of which appear in Figure 4.

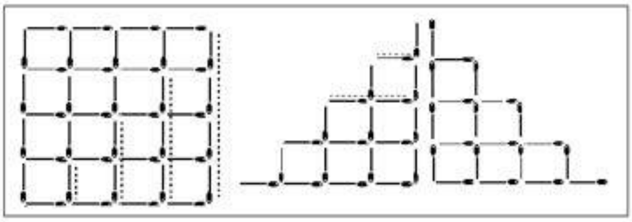

Figure 4: Two instances of 1+2+3+…+n (marked by dashed lines)
in the nxn square of matches

This exemplifies how visual reasoning can be guided, inspired and supported, a posteriori, by a symbolic expression known to be the solution. Therefore, the visualization process may not only involve the decomposition into units or the creation of auxiliary constructions, it may also be guided by a known symbolic result.

Concluding Remarks

Can a visual strategy be inspired by a symbolic expression? Our data show that yes, students might be familiar with the visual problem, and then may "see" the visual-counting strategy or even strategies in the certain algebraic expression given to them.

It is worth to note that this thinking direction is much more complicated than the opposite direction described at Hershkowitz et al. (2001) paper. At the previous paper the solver is going from the visual problem to look for appropriate visual solution's strategies, and at the end transform them into algebraic expressions. In this paper the solver has to go back and forth in tiny steps looking for hints that evolve from the components of the algebraic expressions.

REFERENCES

Arcavi, A. (2003). The role of visual representations in the learning of Mathematics. *Educational Studies in Mathematics, 52*, 215-241.

Healy, L., & Hoyles, C. (1999). Visual and symbolic reasoning in Mathematics: Making connections with computers? *Mathematical Thinking and Learning, 1:1*, 59-84.

Hershkowitz, R., Arcavi, A., & Bruckheimer, M. (2001). Reflections on the status and nature of visual reasoning - the case of the matches. *International Journal of Mathematical Education in Science and Technology, 32(2)*, 255-265.

Mavrikis, M., Noss, R., Hoyles, C., & Geraniou, E. (2013). Sowing the seeds of algebraic generalization: Designing epistemic affordances for an intelligent micro-world. *Journal of Computer Assisted Learning, 29(1)*, 68-84.

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Teaching the derivative in the secondary school

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Abstract : In this paper I want to focus the attention on a specific mathematical content of the secondary curriculum: the derivative concept. Drawing on the work done in my PhD thesis, I outline two moments of the didactic transposition (Chevallard, 1985) of this notion in the secondary school. More precisely, I focus on the introduction of the derivative through the problem of the tangent to a generic function within two textbooks and in the case of a teacher in her grade 13 classroom. The aim is identifying how the local dimension intervene in the work on the function that has to be differentiated.

Résumé : Dans ce papier je veux focaliser l'attention sur un contenu mathématique spécifique du curriculum secondaire : le concept de dérivée. En m'appuyant sur mon travail de thèse, je mets l'accent sur deux moments de la transposition didactique (Chevallard, 1985) de cette notion dans l'enseignement secondaire. Plus précisément, je me concentre sur l'introduction de la dérivée à travers le problème de la tangente à une fonction générique dans deux manuels et dans le cas d'une enseignante dans une classe de terminale scientifique (en Italie, élèves de 18-19 ans). L'objectif est d'identifier comment la dimension locale intervient dans le travail sur la fonction à dériver.

Introduction

In the secondary teaching, the derivative is one of the first and fundamental concepts of Calculus. It evokes and calls into question competences, notions and registers which are proper to the algebraic or the geometrical domain. In particular, it involves functions and their properties, limits and also geometrical objects such as the tangent line. At the same time, the derivative permits to solve several problems belonging to the Calculus domain, such as optimization problems, zeros approximation methods, primitives of functions, and many others. Therefore, the introduction of the derivative notion represents a crucial node for students, and also for teachers.

In Italy, this moment occurs at the last year of upper secondary school (grade 13). As I could verify working with some teachers during my PhD, the derivative notion is considered as a cornerstone in the mathematics curriculum of the last year of secondary school, with relevant applications to physics. Their experience shows teachers that learning the rules to differentiate a function is quite simple and automatic in terms of computation. Nonetheless, conceptualizing the derivative as a mathematical object, and in particular as a function, for then reemploying it as a tool (Douady, 1986), may not reveal so immediate. One of the aspects that make this process difficult is fostering, on the teachers' side, and grasping, on the students' side, a local dimension in the work on the function that has to be differentiated.

With a particular interest in the teaching practices with the derivative concept in secondary school, I articulated my study around the following research question: how does the local dimension intervene in the development of derivative-related practices? In particular, I focused on the role given to the local work on functions in the intended curriculum (mathematics to be taught) and in the implemented curriculum (taught mathematics) when such practices are introduced[7].

[7] The expressions *intended* and *implemented* curriculum were introduced by the Second International Mathematics Study (SIMS) in the 70s-80s, along with that of *attained* curriculum, which consists in "the mathematics that the student has learned and the attitudes that the student has acquired as a result of being taught the curriculum in school" (Mullis & Martin, 2007, p.11).

Theoretical framework

This study is grounded on the Anthropological Theory of the Didactic (ATD), which has been elaborated and disseminated by Chevallard during the last thirty years (Chevallard, 1985, 1992, 1999; Bosch & Gascón, 2006). My focus is on the didactic transposition of the derivative concept in the secondary school context. The didactic transposition is a process that

> "starts far away from school, in the choice of the bodies of knowledge that have to be transmitted. Then follows a clearly creative type of work – not a mere 'transference', adaptation or simplification –, namely a process of de-construction and rebuilding of the different elements of the knowledge, with the aim of making it 'teachable' while keeping its power and functional character." (Bosch & Gascón, 2006, p.53).

Within such a frame, I coordinate three theoretical lenses coming from three different theoretical approaches. In order to describe the intended and implemented teaching practices, I refer to the notion of *praxeology*, which is central in the ATD. A praxeology consists of a type of task, a technique to solve it, the justification that such a technique is efficient and the theoretical arguments that support this justification. For instance, among the praxeologies related to the derivative concept, it is particularly relevant the one that allows determining the equation of the tangent line to a generic function at a point. The peculiarity is that this praxeology has been already practised with conics in grade 9-11, with techniques and justifications whose validity is not extendable to a generic curve. From the teachers' point of view, reworking this praxeology permits to introduce the derivative as a fundamental tool to solve the type of task in the generic case. Thus, we distinguish two planes:

- the *mathematical praxeology* that has to be constructed around the type of task of determining the equation of the tangent ($T_{tangent}$ in the following);
- the *didactic praxeology* that consists in the teacher's organisation and management of the development of the mathematical praxeology, through different didactic moments.

I am referring here to the *model of the didactic moments* elaborated by Chevallard (1999) distinguishing different (but not ordered) steps in the construction and the practice of a praxeology that determine the teacher's didactic praxeology. More precisely, I am interested in

- the moment of the first (significant) meeting with the task;
- the technical moment, where a technique is developed or at least an embryonic form of it;
- the technological-theoretical moment, where the justifications for the techniques are formulated and grounded on a specific theory.

Working on $T_{tangent}$ entails introducing a local regard on the function that has to be differentiated. A key question is: how this local dimension is introduced?

In order to detect if and how a local work is done on the function, I use the lens of the *perspectives* (Rogalski, 2008; Maschietto 2002; Vandebrouck, 2011), that are different ways to regard a function while working on it. We can recognise that a certain perspective is adopted on a given function f if certain properties of f are exalted. One can be interested in a pointwise property of f that is valid at a specific point (e.g., $f(2) = 4$, $x=3$ is a zero of f). In this case, the enhanced perspective on the function is *pointwise*. Moreover, one can consider the function as a whole object or some global properties of it that are valid in a given interval (e.g., f is even, f is increasing in [0,1]). In this case, the enhanced perspective on the function is *global*. In addition, and this is the case of the differentiability property, one can concentrate on a local property of f that is valid in a neighbourhood of a given point (e.g., f is discontinuous in $x=2$, f has a maximum point in $x=1$). In this case, the enhanced perspective on the function is *local*, in the sense that it highlights property of the function that are valid on a family of neighbourhoods that contain the given point. It is not enough to know what value the function takes at that point, and it is not necessary to choose a particular interval: a local property is valid for an infinity of open intervals containing the point.

In order to recognise the perspective adopted on a function, it is certainly important to know what has been said or written about the function. Nevertheless, if we apply this lens for analysing teachers or students working on functional objects, we realise that it is the combination with other semiotic resources, different from the oral or written speech, that actually inform us of the adopted perspective. Therefore, I consider the semiotic bundle (Arzarello, 2006) as a third lens for analysing the semiotic resources activated by the teacher and the students during the work with functions, and their mutual relationships. The semiotic bundle is defined as a bundle of semiotic sets (speech, gestures, sketches, drawings, symbols, …), their internal relationships, and the coordination between two or more resources that are simultaneously active. The great variety of semiotic resources that can be activated by the teacher or the students while working on $T_{tangent}$ can exploit different registers of representation (algebraic, symbolic, graphical, etc.) on functions and reveal or hide a particular perspective on them. When different semiotic resources (e.g., speech and gestures) converge to underline the same perspective on a function, this unity may enhance such a perspective and foster its activation. However, it is also possible that two or more different semiotic resources simultaneously active highlight different perspectives on a function.

Through the presented theoretical framework, I approach my research question in the following terms: how the derivative-related praxeologies are constructed and how the local perspective intervene in this process?

Methodology

In this paper, as I said above, I focus on the transposition of the derivative in the intended curriculum and in the implemented one. I concentrate especially on scientific high schools, where we can suppose to find a more intensive local work on functions. As for the intended curriculum, I consider in particular the praxeology for determining the equation of the tangent in two of the most widespread textbooks in Piedmont (Turin region): Bergamini *et al.* (2013) and Sasso (2012). As for the implemented curriculum, I propose the case of a teacher dealing with the introduction of the derivative in her grade 13 classroom. It is one of the three case studies that I observed and analysed for my PhD thesis. The teacher used one of the analysed textbooks. I interviewed her before entering the classroom about her usual practices and plans for the lesson. I chose to present this particular case because something in the lesson led her to modify her usual practices. The interesting unexpected outcome is that her didactic transposition of the derivative did not turn out to be a transposition of the textbook one.

Textbooks analysis

As far as $T_{tangent}$ is concerned, in the analysed grade 13 textbooks, we can recognise the didactic transposition of one of the "scholarly" definitions of differentiable function (see for example Bramanti *et al.* 2000, p.171)

> DEF. Let f : $(a,b) \rightarrow \boldsymbol{R}$. We say that f has derivative at x_0 in (a,b) if
>
> $$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$
>
> exists and is finite. This limit is called first derivative of f f at x_0 and it is denoted with $f'(x_0)$.
>
> The straight line whose equation is $y = f(x_0) + f'(x_0)(x\text{-}x_0)$ is called tangent line to the graph of f at the point $(x_0, f(x_0))$.

Indeed, at the beginning of the chapter on the derivative within both Bergamini *et al.* (2013) and Sasso (2012), the following phases are developed.

First of all, the problem of the tangent to a generic function is introduced and the tangent line is seen as the limit of a dynamic secant line or a sequence of secant lines (see Fig. 1).

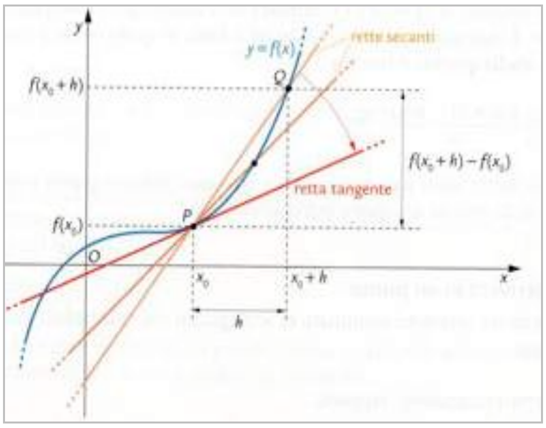

Fig. 1: The tangent as the limit of a sequence of secants (Sasso, 2012, p. 259).

Thus, firstly, the gradient of the secant line is found and then the limit as *h* goes to *0* is applied in order to obtain the gradient of the tangent line:

$$m \text{ secant: } \frac{f(x_0+h)-f(x_0)}{h} \xrightarrow{\lim\limits_{h\to 0}} m \text{ tangent: } \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

Finally, the derivative is defined as this limit if it exists and is finite.

Basing the analysis on the perspectives, we can notice that in a first phase pointwise and global perspectives on the function are activated with the work on the secant line. Then, the limit symbol is introduced and the local perspective suddenly intervenes.

Through the lens of the semiotic bundle (here we have *words+graph+symbols*), we can detect a potential, but rather implicit, activation of the local perspective. The local perspective on functions is potentially activable by a student who disposes of this material. Nevertheless, it is difficult without any mediation to correctly establish the relationships among the words that dynamically describe the static graph on the page (e.g., "Q is approaching P", "as Q gets closer and closer to P") and the introduced symbol: $\lim_{h\to 0}$.

Case studies analysis

Two of the observed teachers transpose the textbook transposition in their classroom. As it happens on the textbooks, they resort pointwise and global perspectives on the graph of a generic function, by using the secant line as an intermediary. Then, the introduction of a local dimension is delegated to the limit symbol, which is justified through terms of movement, such as "Q is approaching P" or "as Q gets closer and closer to P".

I propose the analysis of a different didactic transposition: the case of V. In the preliminary interview, to the question "*How do you introduce the derivative notion?*", V. answers that she usually starts with the tangent line definition. Indeed, in classroom she immediately poses the problem of defining the tangent to a generic function at a given point. In particular she asks to students: "*Which properties must a tangent line have?*". An open discussion arises and the students, as V. expected, recall all the operational definitions and techniques they used with conics. Within the model of the didactic moments, we can recognise that this first phase of the lesson is devoted by the teacher to a technological-theoretical discussion around the problem of the tangent in the case of a generic function. The main concerns indeed consist in defining the mathematical object they are working with and explaining why all the conics-related techniques are no longer successful. V.'s intention, as she declares before the lesson, is disarming the students of all their previously used techniques and introducing the derivative as a tool to solve the generic type of task $T_{tangent}$. Nevertheless, the discussion in classroom produces an unexpected but correct definition of tangent. Let us analyse the key moments of the discussion through the lenses of the perspectives and of the semiotic resources.

The first definition proposed by a student (S1) is: the tangent intersects the curve at a single point.

1	S1:	[the tangent] must intersect [the function] at a single point.
2	S2:	[…] But, if it is so, not all the points has a tangent line ... I'm imagining a sloped function (tilting his hand) then maybe the tangent line in that case could intersect the function in another point, right?
3	V:	[…] So, are you thinking of something like this? (She sketches the curve in Fig. 2)
4	S2:	Yes, there is the tangent line but it touches other points of the function.
5	V:	For example, if I search for the tangent line here? (She points at the maximum point on the curve, see Fig. 2) How do I imagine it?

The first definition (line 1) enhances a pointwise perspective on the function, exalted by the pointwise speech indicators "*at a single point*". Leaning on S2's global remark (line 2), V. proposes a graphical non-example (line 3, Fig. 2). Her global sketch of a whole section of the graph on an interval contextualises her pointwise pointing gesture on it ("*here*" in line 5, Fig. 2), in order to foster the students to look at the whole graph of the function, in a global perspective.

Another student (S3) proposes to localise this definition by adding "*in a suitable interval*", but the definition remains pointwise.

6	S3:	To avoid what S2 said, we can take a suitable interval (moving his two indexes up and down together, as in Fig. 3) where the tangent line satisfies our conditions.
7	V:	So, we limit the zone.
8	S3:	At that point, if I want a tangent line to a point in that interval, I can do it without any other intersection of the line in that interval.
9	V:	[…] So, we take a point, wherever we want, this one (x_0, y_0), we limit to a suitable neighbourhood (she sketches the situation at the white board, see Fig. 4) and what do we require there?
10	S4:	There, that the line intersects [the function] only at that point.
11	S5:	It is not enough.
12	V:	It is not enough, why? […] It could do so (she draws the situation in Fig. 5).

With his words (line 6), S3 makes a limiting gesture with his fingers (Fig. 3) that the teacher reproduces at the whiteboard as two vertical lines around the point (Fig. 4). Although the student's gesture (Fig. 3), strengthen by the teacher's sign on the graph (Fig. 4), has already the intention of enlarging the pointwise perspective of the first definition, the speech indicators at this stage are global for the student ("*a suitable interval*", line 6, "*in that interval*", line 8) and local only for the teacher ("*a suitable neighbourhood*", line 9). The proposed definition (line 10) amended by "*there*" that means "*in that interval*" falls again in the pointwise perspective. The local handholds are too weak to make the students enlarge their perspective. However, S5 recognises that this property is not enough (line 11) and, leaning on his intervention, V. makes a graphical counterexample that exalts the pointwise character of the given definition (Fig. 5).

Fig. 2: V.'s non-example to the first definition.

Fig. 3: S3's gesture for "a suitable interval".

Fig. 4: Teacher's sign | | reproducing the student's gesture.

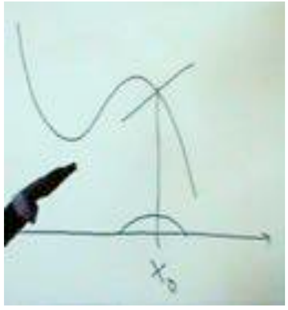

Fig. 5: V.'s counterexample to the first definition.

The third proposed definition is pointwise: the tangent is the perpendicular to the radius of the circle, to which one of the students locally adds "*the circle that best approximates the curve*". The teacher then recognises that it can be a correct approach, but technically too difficult for them.

Thus, another student (S6) further proposes: the tangent must all lie in the same region of plane.

13	S6:	It must all lie in the same half-plane, except for the point.
14	V:	What do you mean?
15	S6:	A function detects two half-planes.
16	V:	Yes. They aren't half-planes, but regions of plane.
17	S6:	Ok. And the straight line must always lie in the same region of plane.
18	V:	Yes. Always?
19	S6:	In the interval (measuring a short distance with his hands, as in Fig. 6)
20	V:	Locally. All we are saying is only local (she sketches two vertical lines on the white board) [...] Ok, S6. And if I draw a function like this (she sketches the curve in Fig. 7) and I ask you to find the tangent in this point (indicating the inflection point). Is there the tangent in that point or not?
21	S7:	It tends to coincide with the function.
22	S3:	It is like when we studied *sinx* that was asymptotically equivalent to $y = x$, isn't it?
23	V:	Did we have the tangent in that case?
24	S5:	There exists the tangent but the reasoning based on the regions of plane falls.

S6 formulates the global definition, speaking about half-planes, then corrected in regions of plane by V. (lines 13-17). Prompted by the teacher (line 18), he adds "*in the interval*" (line 19) making the same limiting gesture as before (see Fig. 3 and Fig. 6). V. again converts this gesture in two vertical lines on the whiteboard, but she strengthens the local feature of this shared sign, accompanying her sketch with the words ("*Locally. All we are saying is only local*", line 20). Then, she proposes a local graphical counterexample where the tangency point is an inflection point (Fig. 7). This graph not only fosters the students to reject the claimed property (line 24), but also evokes the case of $y = sinx$ and $y = x$ (line 21-22). In the previous months, V. has made the students work on remarkable limits like *sinx* over x going to 1 as x goes to 0, by speaking in terms of asymptotic equivalence and supporting graphically this property.

Fig. 6: Student’s gesture for “in the neighbourhood”.

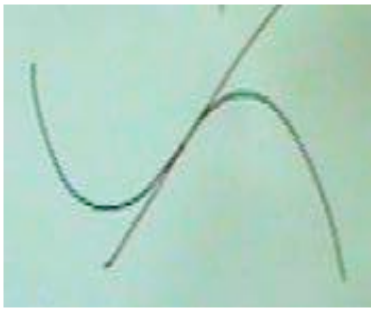

Fig. 7: V.’s counterexample to the student’s definition.

Thus, finally, a student proposes a correct local definition: the tangent is the straight line that best approximates the given curve in the neighbourhood of the point, and S6 writes an equality at the whiteboard (Fig. 8). We can interpret it as a previous technique (linked to the asymptotic equivalence property and the remarkable limits), re-employed at the level of justification in relation with the given definition.

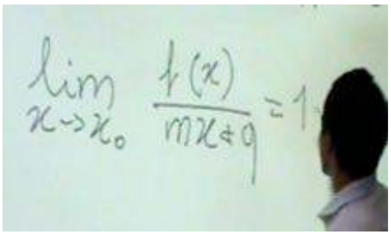

Fig. 8: Equality proposed by S6 to express that $f(x)$ and $mx+q$ are asymptotically equivalent.

V., confronted to this unexpected further development by the students, says: “*It is an approach that I have never tried before. Let’s try together now*”. The definition and the equality given by the students lead to the target technique, but through a technology that the teacher has not pre-prepared. V. works on S6’s justification and a graphical-symbolical reformulation of the type of task allows her to find the right technology from which the gradient of the tangent m can be deduced. In particular she applies a vertical translation to the x-axis of the vector $(0,f(x_0))$ which permits her to compare the infinitesimals CB and AB as x goes to x_0 (see Fig. 9). She accompanies this action on the graph by saying

25 V: Why it [the equality proposed by S6, Fig. 8] doesn’t give me the idea of asymptotic estimate? Because the asymptotic estimate is valid for infinitesimal quantities, which go to 0. Thus, here first of all I need an indeterminate form 0 over 0, the two quantities must go to zero, and then I compare the speed with which they go to zero.

By expressing CB and AB in symbols, she can finally write the equality in Fig. 10.

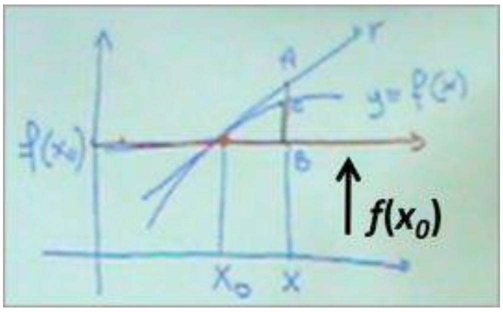

Fig. 9: V.’s graphical reformulation of $T_{tangent}$.

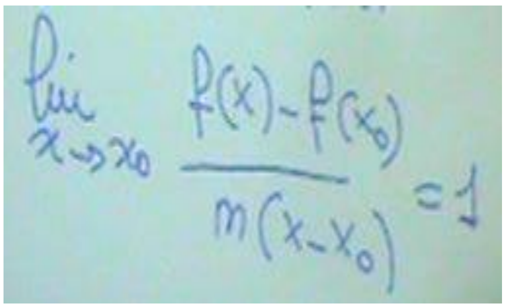

Fig. 10: The right technology to find m.

With the lens of the praxeology, V.’s local words, graph activity and symbols (Fig. 10) can be interpreted as the justification for the technique for finding m. Such a justification is supported, at the theoretical level, by the local asymptotic equivalence property. Within the model of the didactic moments, this second phase can be interpreted as the strict interrelation between the technological-theoretical moment and the moment of elaboration of a technique for determining the gradient m of the tangent and then for finding its equation. Starting from the theoretical definition of tangent and from the asymptotical equivalence property, the technology is formulated using the graph (Fig. 9), the speech ("*the asymptotic estimate is valid for infinitesimal quantities, which go to 0*", "*an indeterminate form 0 over 0, the two quantities must go to zero*", line 25) and the symbols (Fig. 10) in order to find the target technique for $T_{tangent}$:

$$tg\colon\ y - f(x_0) = m(x - x_0) \text{ where } m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

The complete mathematical praxeology for $T_{tangent}$ constructed by V. and her students is summed up in Table 1.

$T_{tangent}$	Determining the equation of the tangent to a function f at the point x_0.
Technique	$tg\colon\ y - f(x_0) = m(x - x_0)$ where $m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$
Technolo gy	Among all the straight lines passing through ($x_0, f(x_0)$), the tangent is the one that best approximates the function. The infinitesimal quantity $f(x)$ - $f(x_0)$ is asymptotically equivalent to the infinitesimal quantity $m(x\text{-}x_0)$ (see graph in Fig. 9). The condition $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{m(x - x_0)} = 1$ is then satisfied. Since mm is a constant we can bring it out of the limit sign, obtaining: $\frac{1}{m} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{m(x - x_0)} = 1 \iff m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$. $\frac{1}{m} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{m(x - x_0)} = 1 \iff m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$
Theory	- Definition of tangent as *the straight line that best approximates the curve in the neighbourhood of a point.* - Property of being asymptotically equivalent. - Analytic equation of a straight line. - Limit theory.

Table 1. Mathematical praxeology for $T_{tangent}$.

Discussion and conclusion

Analysing this classroom experience allows us to retrace the didactic moments of the development of a mathematical praxeology for determining the equation of the tangent. It comes out a didactic transposition of the derivative that is not transposed from the textbook transposition. The local perspective on the function that has to be differentiated is present in the speech (e.g., "*the asymptotic estimate is valid for infinitesimal quantities, which go to 0*", line 25). It is marked by the students' gesture (see Fig. 3 and 6) and by the teacher's sign at the whiteboard (see Fig. 4). This is a sort of *semiotic game* (Arzarello & Paola, 2007) where

> "the teacher uses one of the shared resources (gestures) to enter in a consonant communicative attitude with his students and another one (speech) to push them towards the scientific meaning of what they are considering" (Arzarello & Paola, 2007, p.23).

In the case of V., the relationship between the semiotic resources is more complex. Indeed, the teacher exploits one of the shared gestures, but without repeating it. In recalling it, she changes the semiotic resource, converting the gesture into the written sign "| |" and accompanying it with a meaningful mathematical speech, which prompts the students from a global perspective on "*the interval*" to a local perspective on "*the neighbourhood*". This semiotic game is an important strategy of the teacher for enhancing the local perspective. She starts from the way the students refer to an interval without specifying if their perspective is global or local on the function, and constructs on it the reasoning within a local neighbourhood.

A turning point for the work in the classroom is represented by the teacher's local counterexample of the tangent at the inflection point (Fig. 7). Through the graphical sign, V. unconsciously evokes in the students' mind the case of $y = sinx$ which is asymptotically equivalent to $y = x$ as x goes to 0. The recalling of this property and of the related praxeology leads the students to propose a completely local definition of the tangent enriching the theoretical base. In addition, one of the student proposes a hint of technology supported by such a theoretical base, introducing a possible symbolic formalisation. It is from this moment on that the teacher starts manipulating symbols. The local perspective, which has been gradually developed and strengthen by the definition of tangent, is thus transferred to symbols *lim* and $x \rightarrow x_0$ that are proposed by the students and borrowed by the local praxeology of the remarkable limits. The local dimension on the generic function f is conveyed by the reasoning in a neighbourhood, which is inherent in the idea of best linear approximation.

In the technological part of this praxeology, the justifying speech is centred on the definition of the tangent as the best linear approximation of the function in a neighbourhood of the point. There is no allusion to pointwise and global aspects of the function referring to a secant becoming tangent or to global increments that has to be reduced. The local perspective on the function permeates each part of the praxeology.

In conclusion, this didactic transposition of the derivative notion could represent a challenging but also powerful alternative to the traditional scheme, whose procedure appear sometimes obscure and artificial for students.

REFERENCES

Arzarello, F. (2006). Semiosis as a multimodal process. *RELIME. Revista latinoamericana de investigación en matemática educativa, 9*(1), 267-300.

Arzarello, F., & Paola, D. (2007). Semiotic games: the role of the teacher. In W. Jeong-Ho, L. Hee-Chan, P. Kyo-Sik & S. Dong-Yeop (Eds.), Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 17-24). Seoul: PME.

Bergamini, M., Trifone, A., & Barozzi, G. (2013). *Matematica.blu 2.0*, Vol.5, Libro digitale multimediale. Bologna: Zanichelli.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin*, 58, 51-63.

Bramanti, M., Pagani, C. D., & Salsa, S. (2000). *Matematica: calcolo infinitesimale e algebra lineare*. Zanichelli.

Chevallard, Y. (1985). *La transposition didactique*. Grenoble: La pensée sauvage.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en didactique des mathématiques, 12*(1), 73-112.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en didactique des mathématiques, 19*(2), 221-265.

Douady, R. (1986). Jeux de cadres et dialectique outil-objet. *Recherches en didactique des mathématiques, 7*(2), 5-31.

Maschietto, M. (2002). *L'enseignement de l'analyse au lycée : les débuts du jeu global/local* *dans l'environnement de calculatrices.* Doctorat dissertation, Université Paris 7 and Università di Torino.

Mullis, I. V., & Martin, M. O. (2007). TIMSS in perspective: Lessons learned from IEA’s four decades of international mathematics assessments. In T. Loveless (Ed.), *Lessons learned: What international assessments tell us about math achievement* (pp.9-36). Brookings Institution Press.

Rogalski, M. (2008). Les rapports entre local et global: mathématiques, rôle en physique élémentaire, questions didactiques. In L. Viennot (Ed.), *Didactique, épistémologie et histoire des sciences* (pp. 61-87).

Sasso, L. (2012). *Nuova Matematica a colori*. Edizione BLU per la riforma. Quinto anno. Vol.5. Novara: Petrini, De Agostini Scuola.

Vandebrouck, F. (2011). Perspectives et domaines de travail pour l'étude des fonctions. In *Annales de Didactiques et de Sciences Cognitives*, 16, 149-185.

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Using of the Cartesian plane and gestures as resources in teaching practice

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Abstract: in this article we report how a secondary school physics teacher in Mexico City used the Cartesian plane and gestures as resources to promote the understanding on the concept of reference frame among his students (between ages 16 and 18). The teacher's use of resources arose from a question posed to him by one of his students while solving a problem linked to the concept of acceleration. This is a qualitative research supported by the theory of use of resources as a means of reflection during teaching practice in the classroom. This theory is linked to the documentational genesis (Gueudet & Trouche, 2009; 2012), and was used in the analysis of our data. The data collection was done by video-recording of the class when the physics teacher worked with his students. Our results suggest that even this physics teacher has difficulties understanding the concepts related with the movement of objects.

Résumé: dans ce papier nous reportons comment un enseignant de physique d'une école secondaire 5 dans la ville Mexico, au Mexique a utilisé le plan cartésien et ainsi que des gestes comme ressources pour favoriser la compréhension de leur étudiants sur le concept de système de référence. L'utilisation de ressources par l'enseignant ont émergé à partir d'une question posée à lui par un de leur étudiants pendant la résolution d'un problème lequel est lié au concept d'accélération. Cette recherche est de type qualitatif, et elle est appuyée sur la théorie autour l'utilisation de ressources comme moyen de réflexion pendant la pratique de l'enseignant en salle de classe. Cette théorie est liée à celle de la genèse documentaire (Gueudet & Trouche, 2009, 2012), et elle a été utilisée dans l'analyse de nos données. La collecte de données a été faite par vidéo-enregistrement de la classe lorsque l'enseignent de physique travaillait avec leur étudiants en salle de classe. Nos résultats nous suggèrent que même cet enseignant de physique a des difficultés pour comprendre les concepts liés au mouvement des objets.

Introduction and research problem

Several works on mathematics practice agree in the need of paying close attention to the use of resources; in understanding what they are and the way in which they work as extensions of the teacher in school practice (Adler, 2000). This author points out that the resources used in teaching practice need to become a focus of attention (p. 221). Besides, she notes that the resources are not restricted to material objects. She classifies them in human (teachers, parents, and teacher's knowledge, among others), material (textbooks, calculators, and mathematical objects like the Cartesian plane, among others), and socio-cultural (language). Guzmán and Kieran (2013) state that the way in which resources support teachers or not in their efforts to solve problems in class, clearly has an impact in the students' experience in problem solving. For his part, Radford (2012) uses the term artifact to point out the relevance of researching their use and understand their influence in the teaching and learning processes.

Starting from the importance in the analysis of the teachers' use of resources in the classroom when solving problems, in this article we seek to answer the question: How are the non-physical resources [representations and gestures] used by the teacher so that his students give meaning to the reference frames?

Conceptual framework

Gueudet and Trouche (2009) propose a theoretical approach similar to the one proposed by Adler (2000) concerning resource conceptualization. For instance, they do not consider resources merely as those coming from material objects, but as all those that take part in the understanding and solving of a problem. In their proposal, these authors make the difference between resources and documents. So, documents are developed through which they call "documentational genesis." Documentational work is the core of the teachers' activity and of their professional development.

In the documentational genesis, the documents are created from a process in which the teachers build schemes of utilization of the resources for situations within a variety of contexts. The process is exemplified in the equation: document = resources + schemes of utilization (Gueudet & Trouche, 2009, p. 205). The schemes cover particular rules of action and are structured through the *uses* and the *operational invariants* during the activity. The *uses* correspond to the observable part of the scheme, which happens during the teacher's action. In contrast, the *operational invariants* correspond to the cognitive structure that guides the teacher's action. Then, the schemes are only observable through the actions the subject carries out when working with the resources (Gueudet & Trouche, 2009).

These authors use the term resource to emphasize the variety of artifacts that they consider and, at the same time, that an artifact (physical or psychological) is a cultural and social medium provided by human activity (e.g., computers and language); produced with specific purposes (e.g., problem solving, Gueudet & Trouche, 2009). During the subject's activity with the use of artifacts, there are two processes: instrumentalization and instrumentation; the first one occurs when the subject takes over the artifact and determines the way it is used. Instrumentation is the influence of the artifact over the subject's actions [activity] (Gueudet & Trouche, 2009).

So, in documentational genesis, one of the objectives is to conceptualize the teacher's activity goal-oriented, considered as social activity. Such consideration of the activity leads to paying attention to the social contexts of the different groups (Gueudet & Trouche, 2012) in which it is present. This way of conceptualizing the activity is linked to the interest of the authors in mediation and mediating artifacts. Hence, the theoretical proposal by Gueudet and Trouche (2009) is related to other research whose approach is semiotic mediation (Mariotti & Maracci, 2012; Radford, 2008; Arzarello, 2006, among others). Mariotti and Maracci (2012) consider an approximation over the teacher's role and describe how the use of artifacts when doing activities may be increased. They also analyze how the use of resources is related to their function as a tool of semiotic mediation. Radford (2008) takes the mediated characteristic of the thought in Vygotsky's sense to refer to the role of the artifacts during social practices. Vygotsky (quoted in Kozulin, 2000) considers that the tools and the psychological signs, like language, are used to control one's activity and to influence on others' activities; so, we think with and through cultural artifacts (Radford, 2008).

In one of his research works, Radford (2009) proposes that mathematical thought is not only mediated by written symbols but also by actions, gestures and other types of signals. Then, thought is produced as well through a sophisticated semiotic coordination of voice, body, gestures, symbols, and tools (Radford, 2009). In the same approach, Arzarello (2006); Arzarello, Paola Robutti and Sabena (2009) consider gestures as part of the resources activated in the classroom: speech, registers records, objects, etc. Hence, gestures are a resource (semiotic tools) used by both the students and the teachers in teaching and learning mathematics.

Method

This is a qualitative study. The research was carried out in a physics laboratory of a high school in Mexico City (grade 12) and the participants were: the physics teacher, who practices since over 30 years, and 11 students (ages 16 to 18). The students (grouped in teams of 3 and 4 members) carried out an experiment regarding a ball falling down an inclined plane to get distance-time Cartesian graphs and to interpret them in terms of the physics concept of reference frame. All the participants were video-recorded during the (five) sessions; they were given worksheets to guide the activities. After the five sessions, the videos of every session were analyzed besides gathering and analyzing the worksheets; no interviews with the participants were conducted. In order to fulfill the purpose of this research, we show excerpts of the moment (last session) when the teacher talks with Peter (*pseudonymous*). In this dialogue, they talk about the concept of reference frame, which arises from a question by another student (E1) related to the concept of acceleration. Data analysis is focused on the use of resources (gestures and concept of reference frame) by the teacher to answer to the student's question.

Analysis of the resources used by the teacher

We present some excerpts of the dialogue between the teacher [**Prof.**] and Peter [**P.**] after another student (**E1**) in Peter's team asked the teacher a question related to negative acceleration. We show moments during which the teacher uses the resources of the concept of reference frame and the use of gestures in his statements.

E1: Teacher, what would be an example of negative acceleration?

P.: It is that deceleration is to stop accelerating, that is, that a mobile stops little by little.

E1.: What would be an example of negative acceleration? [*Addressing the teacher*].

Prof.: It is only called deceleration until the moment you reach zero.

P.: Oh! Ok.

Prof.: Yes? You are decelerating until you reach zero. Which is related to negative acceleration. It's only that negative acceleration doesn't stop, it does not stop there. It goes on, right? For example, when you throw an object, it decelerates.

E1.: And it will have a negative acceleration if you pass zero.

Prof.: It always has a negative acceleration; it always has a negative acceleration, it's just that deceleration, you finish when it stops [*the object*]. And here it keeps on acting so that now [*he is interrupted by **E1** who says: "it is negative"*]. Then, the body goes up, but our acceleration, negative. Then, it stops [*when it reaches its highest point*], but because the acceleration continues, it goes down [*he means that the body changes direction with respect to the initial one*].

At first, a student (**E1**) asks the teacher a question related to what negative acceleration is. Here, the teacher retakes what Peter said: "It is that deceleration is to stop accelerating"; Peter tried to relate the negative acceleration [concept] with a known physical phenomenon; however, he [*the teacher*] is unclear in his explanation on the existing relation between deceleration and negative acceleration; regardless, he says: "Which is related to." The relation that the teacher establishes is as follows: when the teacher says "You are decelerating until you reach zero", he means the moment in which the mobile stops. When he says "It's only that negative acceleration doesn't stop, it does not stop there. It goes on…" he refers to negative acceleration (concept) obtained from a reference frame. That is, the teacher does not clarify how a concept such as negative acceleration may be used to analyze movement (deceleration). In his speech, the teacher implicitly argues that, from a reference frame, the direction of the object goes to where the acceleration has been determined to have negative sign. However, accelerations with positive sign (positive accelerations) may occur and the mobile would still decelerate (slow down), depending on the reference frame chosen by the observer; hence, the importance of choosing an adequate reference frame to analyze a certain movement of objects. In a textbook (Giancoli, 2006) used as support during the development of the courses in this school level, the author expresses the care that must be taken for not conceiving that deceleration necessarily means acceleration is negative. "When an object slows down, sometimes it is said that it is decelerating. But we must be careful: deceleration does not necessarily mean that the acceleration is negative." (p. 25). In reality, it means that the magnitude of the velocity decreases. Later, he provides an example and says: "when you throw an object, it decelerates". Here, the teacher does not clarify whether he is making reference to a vertical throw. We infer it is about that movement when he says: "Then, the body goes up […] Then, it stops, but because the acceleration continues, it goes down." The teacher's example corresponds now to a movement that has a behavior in two directions (when it goes up and when it goes down), but in which the orientation of the reference frame that shows the negative acceleration down is constant. He also fails to link the deceleration with the changes in velocity of the object. From that moment, the teacher's objective is that the students understand how the sign (positive or negative) of physics quantities (e.g., acceleration) are established; to do so, he will need to use the concept of reference frame. Until this point, we do not observe the teacher using any resource besides language. The dialogue with the students continues, and the teacher retakes the experiment[8] done by the students on an inclined plane:

Prof.: In the inclined plane [*Referring to the experiment done by the students*], where did you consider the positive goes to? [***P.** asks: "come again?"*], where did you consider the positive goes to in the inclined plane? [*The students get nervous*] Yes, you considered positive and negative, right? [***P.** anwers: "yes"*] Where was the positive going to?

P: Uhm… positive, well, what I managed to understand was that the positive was taken from, well from the part, how do you say? From the uppermost length, well the centimeters that were from the distance to the floor to the hundred and twenty centimeters that were on the practice. And from that point to the floor, well to where the rail [*referring to the inclined plane*] ended was, uhm, the plane.

The teacher uses the physics concept of reference frame as a resource (of the mathematical object type) to interpret the physics phenomenon. However, he uses it in an unclear way. The reference frame involves a direction (orientation) and a mathematical origin (which may coincide or not with the phenomenological origin). Here, the teacher refers only to the direction when he asks "where […] the positive goes to?", but Peter seems to refer to the (mathematical) origin of the reference frame, which he relates to the phenomenological origin, that is "from the part". This lack of understanding between the teacher and Peter may be due to the spontaneous way in which the teacher introduces the concept of reference frame. What is more, the moment when the teacher asks: "Where did you consider the positive goes to?", the other team members shy away from the conversation. Besides, the teacher provides no reason as to why he asks; at no moment he mentions "reference frame" either even when his objective is to use said resource so that Peter understands how the sign of acceleration is established [considered]. Attention should be paid when Peter says "what I managed to understand" since he refers to the way in which the experiment was conducted; while the teacher intends to carry out a conceptual analysis which guides the rest of the discussion with Peter. The discussion continues:

Prof.: Where did you consider the positive goes to? [*After Peter remains thinking, the teacher continues*] The orientations, positive, negative, are arbitrary, right? And if you take the positive goes up, that does not mean that the ball goes up. And if you take the positive goes down, that does not mean that the ball goes up. And if you take the positive goes diagonally, that does not mean that the ball goes up. The ball has its behavior and that's it [***P.** says: "yes"*], right? The other is the interpretation [***P.** says: "Oh, ok."*] Right? Where do you take the positive goes to?

P: Well, uhm, in the moment when the ball goes down.

Within the resource of reference frame, the teacher focuses his attention on the observer (when he says: "they are arbitrary"), who chooses the conventions to measure [interpret] the physics phenomenon (inclined plane), but he is not explicit about the student being the observer in the

[8] In the experiment, the students video-recorded the movement of an object (tennis ball) on an inclined plane. Later, they used the video and with the aid of software they obtained tables of values for the experiment; later, they sketched the graphs (with Excel) that represented the movement of the ball.

experiment. Thus, we see the *use* of the resource by the teacher (Gueudet & Trouche, 2009). For his part, now Peter focuses his attention on the phenomenon and starts linking it to the data collection. Nevertheless, he does not perceive the relation between the experiment (physics phenomenon) and the reference frame. It is here when the teacher uses another cultural-semiotic resource (gestures); i.e., a resource used intentionally by the teacher to achieve an objective (Gueudet & Trouche, 2009) and that at the same time, is used in a social process of meaning creation through the teacher's actions in a specific time and space (Arzarello, 2009; Radford, 2008) observed in the following excerpt.

Prof.: Where do you consider the positive goes to? Not the moment [*when*], but where to? [***P.*** *whispers to* ***E1****: "Help me!" see Figure 1*] I let this go [*He takes an object (mousepad) and placed it at a distance above the table; Figure 2a*] what is going to happen to it? [***P.*** *says. It's going to fall down*] [***Prof.*** *Lets the object go; see Figure 2b*] Where do you consider the positive goes to? [***P.*** *thinks*] Then, I help you. I consider the positive goes down. Then, this body [*referring to the object he is picking up at the same time and then lets fall down*], did its distance increased or decreased? [***P.*** *does not answer*].

Figure 1. Peter asks a fellow student for help.

From the excerpt above, we observe the social dimension of the moment when something more than a negotiation of meanings takes place. (Radford, 2008) concerning the resource the teacher is using. Peter feels nervous and uncertain when he says: "Help me!" (see Figure 1). After what Peter answered what he considered a reference frame when he said "what I managed to understand" (in the second excerpt), we found a moment when he could not answer to the teacher's question. Therefore, the interaction between Peter and the teacher plays a key role to mutually recognize the institutional meanings addressed. This social interaction affecting the individuals is decisive in Peter's learning process (Radford, 2008) and also in the way the teacher conveys knowledge.

Later on, the resource of gesture is used by the teacher to represent the physics phenomenon and he expects for Peter to understand the concept of reference frame; the cognitive part that guides the teacher's action (Gueudet & Trouche, 2009). In the teacher's gestures, there is only the representation of the physics phenomenon (Figures 2a and 2b), and with language he refers to the concept of reference frame when he says: "I consider the positive goes down." However, Peter seems to only observe the gesture without its meaning; what he observes is the phenomenon of the object falling down, the visible part of the resource (Adler, 2000). When the teacher says: "I consider the positive goes down", there is no element of either orientation or origin of the reference frame in his gesture. That is why when asking whether the distance increases or decreases, the teacher does not realize he needs an origin from which the distance of the object increases. In the following excerpt the teacher modifies his resource by reflecting.

Figure 2. Gestures as resources used by the teacher in two moments (2a, left), (2b, right).

Prof.: What is the distance? [*He picks up the object again; Figure 2a*] [***P.*** *says: Uhm… at X centimeters*] Zero, ok. Zero. What is the distance? [*He lowers the object a little with respect to the highest point at which he had held it; see Figure 3a*] Zero was here [*he points with his finger at where the object was at the beginning; see Figure 3b*].

P: Yes, uhm… we could say minus one or something like that.

Prof.: No, I cannot assign minus one. […] Positive goes down [*He places the object back where he had placed it at the beginning and lets it fall down*], I say, arbitrarily. What is the distance? [*He places de object below the initial point (zero for the teacher); see Figure 3c*].

Figure 3. Gestures as resources used by the teacher in three moments (3a, left), (3b, center) and (3c, right).

Reflecting upon the resource he is using, the teacher (Gueudet & Trouche, 2009) realizes that he needs an origin from which the distance can be measured, so he adds a new gesture (Figure 3b) that represents the origin of the reference frame and uses it in two moments; first, when he says "Zero, ok. Zero"; then, when he states "Zero was here." We infer the teacher's reflection on the movement of the object because of the development of the speech he used with the students when he tries to explain the meaning of the sign of acceleration. He had not previously used the gesture to represent the origin of the reference frame, but as the teacher's speech and action evolved, a new gesture (resource) appeared. After he lowers the object and asks: "What is the distance?" (see Figure 3a) the gesture appears to make reference to the origin of the reference frame (see Figure 3b). It is worth pointing out that the first moment also indicates that the origin of the reference frame (mathematical origin) coincides with the phenomenological origin (start of the movement), but the teacher neither says nor makes it explicit. The dialogue continues:

P: It goes down. [***Prof.*** *asks again: "what is the distance?" making reference to Figure 3c*] Oh!, I don't know.

Prof.: Well, calculate it more or less [*Someone from another team says: what's the distance between his finger and what he's holding?*].

P.: […] It's that I got nervous, let's see, let's say some ten centimeters. [***Prof.*** *lowers the object (he moves it closer to the table); see Figure 4a*] Like twenty-five. [***Prof.*** *lowers the object even more (he moves it closer to the table); see Figure 4b*] Like thirty-five, forty; let's say forty. [***Prof.*** *lifts the object and places it above his finger (origin); see Figure 4c*] Like at ten.

Prof.: Minus ten. [*At the same time,* ***P.*** *says: "oh, good point, minus ten"*].

Figure 4. Gestures as resources used by the teacher in three moments (4a, left), (4b, center) and (4c, right).

Peter does not understand why the teacher makes the gesture (Figure 3b) or what it references. He still refers to the movement that follows the object (physics phenomenon) when he says: "It goes down." It is only when another member of the team asks explicitly about the distance between the teacher's finger and the object that Peter manages to understand and keeps on answering as the teacher moves the object away from his finger (Figure 4b). Once the teacher determines the reference frame with its origin, he makes a new gesture (Figure 4c) so that Peter notices unsuccessfully the role orientation plays, represented here by the negative value [of the acceleration] that the object would have when placed above the origin of the reference frame. It is now that Peter focuses again only on the distance between the finger and the object (see Figure 4c) without relating it to the concept of reference frame. The excerpt of the dialogue that follows shows the closing of the discussion between the teacher and the students:

Prof.: Yes? Why?

P.: Because your considering that the positive goes down.

Prof.: And does that mean that this [*Referring to the object*] goes up? [***P.** says: "no"*]. No, then. Did the distance increase or decrease?

P.: No, none of them.

Prof.: How not? Let's see [*Raises the object to the starting point (origin) and lets it fall down*] It is not in the same place [***P.** says: "Good point"*]. Does the distance increase or decrease? [*He raises the object again and lets it fall*].

P.: It decreases.

Prof.: It increases! Didn't we say that positive goes down?

P.: Yes, it's true.

Peter answers correctly after the teacher used the convention (positive goes down). However, when the teacher asks him: "Did the distance increase or decrease?" Peter gets confused again. This [student's] confusion is caused by the teacher's question because it is ambiguous; the teacher is not clear as to what increases. The teacher expects an answer regarding the origin of his reference frame (finger). But if we take into account how he varies the distance with respect to the table, which is what the object comes closer to, then the distance decreases; in this sense, Peter's answer is right. The argument the teacher is not clear about says: "It increases! Didn't we say that positive goes down?" The orientation of the reference frame is independent from the distance that the object keeps with respect to the origin.

Conclusions

The gestures used as semiotic resources by the teacher complement another resource (reference frame) despite the fact that the physics concept of reference frame is a topic of study in itself. Here, the teacher used said concept as a resource so that the students understood the negative sign that the physics amount of acceleration may have. From Gueudet and Trouche's (2009) theoretical perspective, the concept of reference frame was observed as a resource from its *uses* and the continuous process of its development. It was observed that the resources are not isolated, but are part of a set of resources that the teacher uses with a purpose, and that he modifies from his reflection (Gueudet & Trouche, 2009). The predominant resources in the teacher (besides spoken language) were the concept of reference frame and gestures. The gestures contributed to create and to learn ideas (Arzarello, 2006). So, the way in which mathematical cognition is mediated as well by actions, gestures and other types of signs is shown; and that leads to pay attention to gestures as a source to observe the process of concept formation (Radford, 2009).

It was observed that in the teacher's gesture (Figure 3b) both the mathematical object (origin of the reference frame) represented with the index finger and the physics phenomenon are involved; in such a way that he places the object in movement within a reference frame by using a gesture. We must underline that regardless the use of resources generates new knowledge, the student's understanding of its physics meaning [of the concept of negative acceleration] turned out to be complicated and incomplete due to the semantic load of those resources. The results obtained in this research, part of which are reported in this article, allow us to state it is adequate to continue the research about the use of resources in teaching practice. Such research will help us to better understand the practice when teaching school physics concepts in classroom environments.

REFERENCES

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education, 3*(3), 205-224.

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Latinoamericana de Investigación en Matemática Educativa,* 9(4)*,* 267-299.

Arzarello, F., Paola, D., Robutti, O. & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70(2),* 97-109.

Giancoli, D. (2006). *Física. Principios con aplicaciones*. México: Pearson Educación.

Gueudet, G., & Trouche, L. (2009). Towards new documentation system for mathematics Teachers? *Educational Studies in Mathematics*, 71, 199-218.

Gueudet, G., & Trouche, L. (2012). Teachers´work with resources: Documentational geneses and professional geneses. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to ´lived´resources: Mathematics curriculum metarials and teacher development* (pp. 189-213). New York: Springer.

Guzmán, J. & Kieran, C. (2013). Becoming aware of mathematical gaps in new curricular materials: A resource-based analysis of teaching practice. *The Mathematics Enthusiast*, 10(1,2), pp. 163-190.

Kozulin, A. (2000). *Instrumentos psicológicos. La educación desde una perspectiva sociocultural.* España: Paidós.

Mariotti, M. A., & Maracci, M. (2012). Resources for the teacher from a semiotic mediation perspective. En G. Gueudet, B. Pepin & L. Trouche (eds.) *From text to ´lived´resources: Mathematics curriculum metarials and teacher development* (pp. 59-75). New York: Springer.

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring & F. Seeger (Eds.), *Semiotics in mathematics education: epistemology, history, classroom, and culture* (pp. 215-234). Rotterdam: Sense Publishers.

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, 70(3), 111-126.

Radford, L. (2012). On the cognitive, epistemic, and ontological roles of artifacts. In G. Gueudet, B. Pepin, & L. Trouche, (Eds.), *From text to ´lived´resources: Mathematics curriculum metarials and teacher development* (pp. 238-288). New York: Springer.

The deltoid as envelope of line in high school: A constructive approach in the classroom

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Abstract: This paper presents a design of educational module (EM) aimed on the one hand at retrieving to the teaching of the traditional syllabus some mathematical terms connected to geometrical constructions, and on the other to introduce, by virtue of the new issues posed by dynamic geometry software (DGS), different teaching perspectives from the traditional ones so that students can be encouraged to experience the true meaning of mathematical discovery. The focus is based on the tracing of the deltoid curve - triangular curve - derived by an envelope of lines. The peculiarity of EM lies in the methodological approach which privileges a practical strategy based on the combination of the usual tools like ruler and compasses with their modern versions offered by a DGS like GeoGebra. In the concluding paragraph we will report a coherent analysis of the method and the results obtained from the experimentation in class.

Résumé: Cet article présente une conception de module éducatif (ME), qui vise d'une part à la récupération à l'enseignement du programme d'enseignement traditionnel certains termes mathématiques liés à des constructions géométriques, et de l'autre d'introduire, en vertu des nouvelles questions posées par la géométrie dynamique logiciel (DGS), différentes perspectives d'enseignement de celles traditionnelles afin que les étudiants peuvent être encouragés à découvrir le vrai sens de la découverte mathématique. La mise au point est basée sur le tracé de la courbe deltoïde - courbe triangulaire - dérivée par une enveloppe de lignes. La particularité de ME réside dans l'approche méthodologique qui privilégie une stratégie pratique basée sur la combinaison des outils habituels comme règle et compas avec leurs versions modernes offerts par un DGS comme GeoGebra. Dans le paragraphe de conclusion, nous ferons rapport une analyse cohérente de la méthode et les résultats obtenus à partir de l'expérimentation en classe.

The beauty of shape is not, as people normally believe, that of living beings and of the paintings which represent them, but the rectilinear and circular beauty of figures, plane and solid, that can be obtained with compasses, ruler and square ruler. Because these are beautiful not, like the former, in a relative way, but in themselves and by their very nature.

(Plato, Φίληβος, 51c)

Theoretical Framework

The use of dynamic geometry software (DGS) continually opens up new teaching perspectives in the teaching-learning of geometry because it enhances the constructive aspect without detracting from deductive accuracy, from clarity of hypotheses and related consequences pertaining to the discipline (Hannafin et al, 2001; Laborde, 2001; Arzarello et al, 2002; Hohenwarter et al, 2008; Leikin et al, 2013). Thanks to DGS the graphic-constructive phase, both prior to the acquisition of some concepts and geometrical properties, and subsequently as verification and/or further study, is not only enjoyable, but also greatly helps teaching, as it offers both visualization and exemplification and/or exploration. In short the surveyor's traditional tools (ruler, square ruler, compasses), retrieved and simulated by DGS, on the one hand facilitate geometrical intuition, while on the other they raise and stimulate interest and the learner's imagination enabling speculation, which is sometimes immediately verifiable, thanks to the immediate computer *feedback* (Jones et al, 2000; Hollebrands, 2007; Hohenwarter et al, 2007; Ruthven et al, 2007; Baccaglini-Frank, & Mariotti, 2010). In this work we put forward an educational module (EM) with inter-disciplinary applications, which highlights the intrinsic relationship between geometry and drawing and has a dual objective: on the one hand to retrieve and reintroduce some mathematical terms connected to geometrical constructions in the teaching of the traditional syllabus, and on the other to show, by virtue of the new issues posed by dynamic geometry software (DGS), different teaching perspectives from the traditional ones so that students can be encouraged to experience the true meaning of mathematical discovery. In specific terms, the focus is based on the tracing of the deltoid curve - triangular curve - derived by an envelope of lines.

The envelope technique before the invention of DGS represented a real challenge for teaching, not only for reasons of timing in class, but also because it is difficult to manage, especially as far as visualization and graphic representation are concerned. Indeed, the definition of a curve, intended as a place of points which satisfy a given property, usually implies the geometric construction of one of its points with the classical tools like ruler and compasses (R and C) which today are effectively substituted by the corresponding tools offered by DGS. The connection between a drawing and a geometric object in everyday teaching practice is nearly always established through a process of approximation. This is based on the idea that with subsequent, better attempts the drawing can eventually achieve something close to the ideal figure.

Geometric constructions made with traditional tools (R and C) also fit this framework and are opposed to free-hand constructions in purely empirical terms of precision. In Italian higher secondary schools students come across constructions made with tools – when this happens – as part of the *Drawing and Art History* subject*,* something which in most cases reinforces the practical aspect of constructions and their separation from a geometric context.

The use of tools is then seen in practical rather than theoretical terms (Mariotti, 1995).

However, in this way a fundamental aspect is ignored and remains unknown to students: each tool contains some knowledge, which is useful for the solution of a particular class of problems. In this sense a geometric construction appears like a geometrical problem (Mariotti, 1996) whose solution can be worked out within a given theoretical framework. Indeed the essential didactic value of the Euclidean frame has always been the perception of its nature of a comprehensive frame which begins with the 'simple and evident' and progresses to the complex and 'non evident'.

Geometric construction, suitably contextualized in the teaching practice, helps the students to begin just this complex path which starts with the simple and evident (R and C) and moves on to the complex and 'non evident' in a tangible, critical and rigorous way.

The integrated tools offered by a DGS represent a valid aid along the way as they progress in the same way from what is predefined to what is made by the user. Therefore we can say that the ED favours a constructive approach based on a combination of traditional tools (R and C) and their modern versions offered by a DGS like GeoGebra. When integrating different tools in the classroom different dimensions have to be taken into account: the relation between the use of tool and learning, the role of the teacher in technology-rich mathematics education, and the characteristics of technological tools (Barzel et al., 2005).

Construction as graphic representation of a curve, as an envelope of lines, takes place then through a series of 'manipulative experiences' finalized at balancing the conceptual elements with the drawing of the figure (Fischbein, 1993) of the geometric object.

What is special about EM is its effective combination of the tools offered by a DGS like GeoGebra and the traditional ones (R and C) in so far as the latter instruments enable the learners to understand the logic underlying a DGS on the one hand, and on the other to appreciate the amazing abstraction process involved that since the classical age has made geometry not just a collection of empirical experiences but a rigorous theory as shown by Euclid in the '*Elements*'.

Design activities in classroom

> *A plane line can be considered as generated by the 'continuous' movement of a point or a straight line: in the first case, it is the place of all the positions of the moving point; in the second, it is the envelope of the positions of the mobile straight line.*[9]
>
> (Luigi Cremona, definitions related top lane lines)

Design activities in the classroom have a dual objective: raising students' curiosity and interest for geometry on the one hand, and on the other retrieving and consolidating geometric concepts and methods, already known to students in broad terms but which they have not fully mastered yet.
That is why we chose a practical approach based on 'manipulative experience' implemented through problem posing and problem solving in order to gain insight into Euclidean geometry. In particular, we set up two workshop activities which entailed the construction of a deltoid curve first using traditional tools (R and C), and then virtually by using the GeoGebra spreadsheet.

Activity One

The class is divided into groups, and each group gets the required material and tools (graph paper, ruler, compasses, protractor, pencils and rubbers); then the teacher sets the following task for the geometric construction:

- *Draw a circumference C with centre O and radius* $r = 2\ cm$*;*
- *Trace a diameter naming the two extremes A and B;*
- *Starting from the B point subdivide the circumference into n (n divider of 360) arches of equal width α, let B_i (i=1, …, n) be the second extremes of the arches (Example* $\alpha = 10°$*);*
- *Starting from the A point subdivide the circumference into n arches of equal width 2α, let A_i (i=1, …, n) be the second extremes of the arches;*
- *Trace the straight line passing through points A_i and B_i (i= 1, …, n).*

Materials and task interact motivating the students to come to terms with the new concept; the sheet of graph paper acts as a background which enhances the drawing and at the same time supports the students while they work with the required accuracy. In other words, making a construction with R and C to draw a deltoid curve means starting from other geometrical objects using only real tools; this requires not only good manual skills at drawing but also a first level of abstraction. The direct manual work is obviously centred on the role of the traditional tools in the construction of Euclidean geometry as a hypothetical-deductive science highlighting the process of idealization through which these real tools are turned into abstract ones, characterized by the properties expressed by the postulates. Once all the groups have finished the task the teacher (T) asks the students (S) some stimulus-questions. The examples below are to be intended as basic guidelines. Here is an excerpt from the protocol:

T: *the geometric construction obtained shows a new figure, let's try to understand better ……..*
S: *it's an unusual figure …. Because from the drawing of straight lines we get some border curved lines;*
S: *the construction obtained "generates a closed curve with three points";*
T: *have you come across any 'closed curves with three points" before?*
S: *I saw a similar construction in a museum where two "pictures" had been made with taught wires which replaced the drawn lines;*
S: *it makes me think of an equilateral triangle with the sides curved inward;*
T: *Which properties do you think verify the drawn straight lines?*

[9] Plücker,1839

S: *all the straight lines are tangent to the curve;*
T: *Are these properties casual or do they occur regularly and if so in which conditions?*
S: *so it is a curve obtained from all these tangent lines….*
T: *what happens if the measure of the radius varies from the circumference, does the figure remain the same or does it change?*

The methodology of guided discovery encourages the students to formulate the first conjectures and to explore further; after that the teacher introduces the concept of an envelope of curves, a term which contains simple but fascinating concepts. The set of traced straight lines forms a curve called *envelope* which can be assimilated to an arch of circumference; the ordered composition of these arches generates a curve with different shapes, open or closed, which is quite impressive.
The conversation triggered and guided by the teacher is very important because it avoids the construction of formal games and it educates to reasoning before formulating conjectures and hypotheses, stimulating creativity, intuition and the imagination. As a result, the majority of students continue to investigate, to tally up points, monitor, reflect and experiment trying to go deeply into the mathematics knowledge system; a small minority, instead, has reservations about the need to repeat the process of construction with different radius values, as it would be repetitive and therefore boring. To re-establish class motivation, the teacher can then suggest the repetition of the process of construction in the computer lab using the GeoGebra spreadsheet. To recap, the starting point of the activity is a declarative and static task of a problem posing nature which asks the students to carry out an imperative and functional piece of traditional work (tracing of the curve with R and C) finalized to a representation which has a lot of learning potential. Such a problem-posing activity in fact goes beyond the traditional logic of the repetitive execution of a drawing (same mechanical operations of subdivision) because at cognitive level it triggers the ability to understand and interpret knowledge (hermeneutics), to investigate (ability to discover and produce knowledge) and heuristic (ability to invent and create new knowledge).

Activity Two

In this second activity the students, again divided into groups, are asked to repeat the previous process of construction, but this time they should do so using the GeoGebra spreadsheet; therefore the "predefined objects" available on the tool bar will be used: point, medium point, circumference, rotation.
The steps of the solving algorithm are as follows:

1. Draw two points, A and B in the Euclidean plane;
2. Determine the medium point (C) of A and B points;
3. Trace the circumference with centre C passing by point B;
4. Construct point B', rotated of B by 10° angle with respect to C;
5. Construct point B_n', rotated di B by an angle $n \cdot 10°$ with respect to C ($n = 1, \ldots, 35$) (Figure 1);
6. Trace the envelope lines (Figure 2).

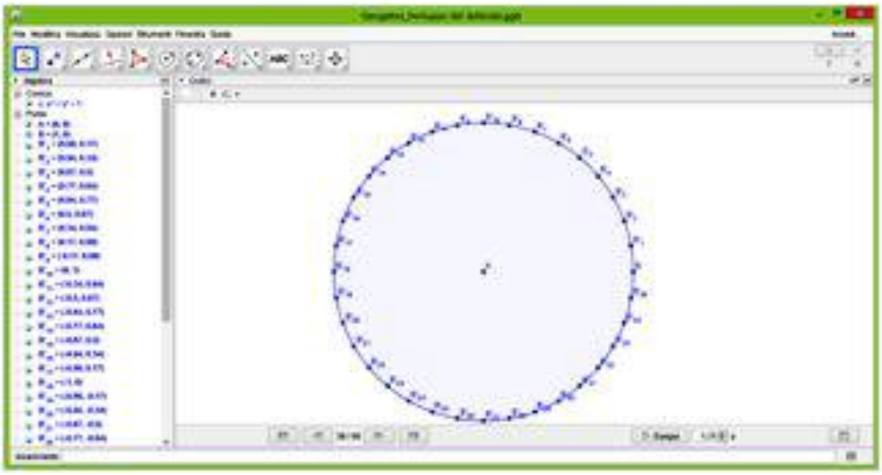

Figure 1. Subdivision of the circumference into arches with same width.

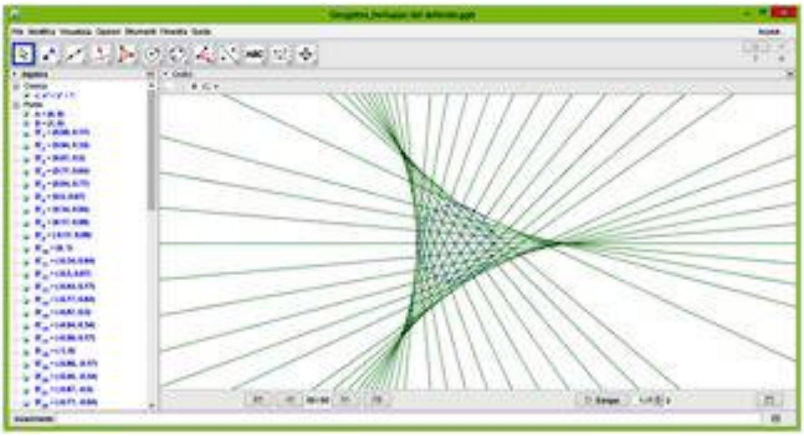

Figure 2. Output of the envelope of the deltoid curve.

The repetition of the construction algorithm helps the learner to understand and use the geometrical shapes in question; it provides the answer inherent to the question of the invariable: the drawing of the curve does not change for different values of the circumference radius. At this point the teacher tells the students that the triangular curve obtained is called deltoid due to its shape being similar to the Greek letter 'delta', and continues by saying that the discovery of the curve cannot be ascribed to a particular person due to its relation with another curve named cycloid (rolling curves) studied by Galileo and Mersenne as far back as 1599, and later conceived by the Danish Roemer in 1674 while studying the best shape for a gearwheel.
The first to actually seriously consider the deltoid curve was Euler in 1745 in relation to an optical problem, while later in 1856 Stainer studied the curve in such depth that it was nicknamed 'Steiner's ipocycloid'. In any case a more extended study of the curve within the context of its historical background could be the object of future lessons.
With regard to the present, the teacher begins a problem solving activity by asking the students to compare the geometric constructions made, noting analogies and possible differences.
A lively discussion takes place among the groups. The teacher keeps to herself.
At this point all the students agree that both constructions do not show any differences regarding the geometrical objects used; but they notice that "on the computer we can move points without having to start all over again". To sum up, from the discussion it becomes apparent that the repetition process should be streamlined; consequently the students are required to study a strategy for the resolution of the problem. Improving the previous process of construction becomes very important from a teaching point of view: the tools used previously are no longer sufficient.
This is when the GeoGebra slider tool can represent a valid aid. A numerical slider visible on the spreadsheet is thus created, finalized to the tracing of the straight lines so as to reduce the time

needed for the execution of the previous algorithm. This is a very tricky phase for the students because they have to take quite a big jump into abstraction: the difficulty lays not so much in the creation of the spider, but in identifying the object or objects to apply it to.
The numeric slider requires the allocation of a variable specifying the numeric interval (min and max) and the increase; the obstacle is represented by the identification of the geometrical object (there can be more than one!) which will be modified by the slider.
The basic geometrical objects of the construction are points A and B on which the slider will work; therefore the following points need to be defined:

- ✓ A' (rotated by A by an angle $n \cdot 10^\circ$ clockwise);
- ✓ B' (rotated by B by an angle $n \cdot 10^\circ$ anticlockwise);

from which follows the tracing of the straight line passing by points A' and B'.
The creation of the slider is fundamental because it represents the tool designated to further improve the construction process of the deltoid curve as an envelope of lines.
Below you can find the steps of the algorithm:

1. Draw two points, A and B in the Euclidean plane;
2. Determine the medium point (C) of A and B points;
3. Trace the circumference with centre C passing by point B;
4. Define a *slider* n (*n* whole number from 1 to 36);
   - Construct a point A', rotated of A of an angle $-20 \cdot n^\circ$;
   - Construct a point B', rotated of B of an angle $10 \cdot n^\circ$;
   - Trace the straight line passing by points A' and B' (Figure 3).

The envelope of the deltoid curve can be obtained moving the *slider* along (Figure 4).

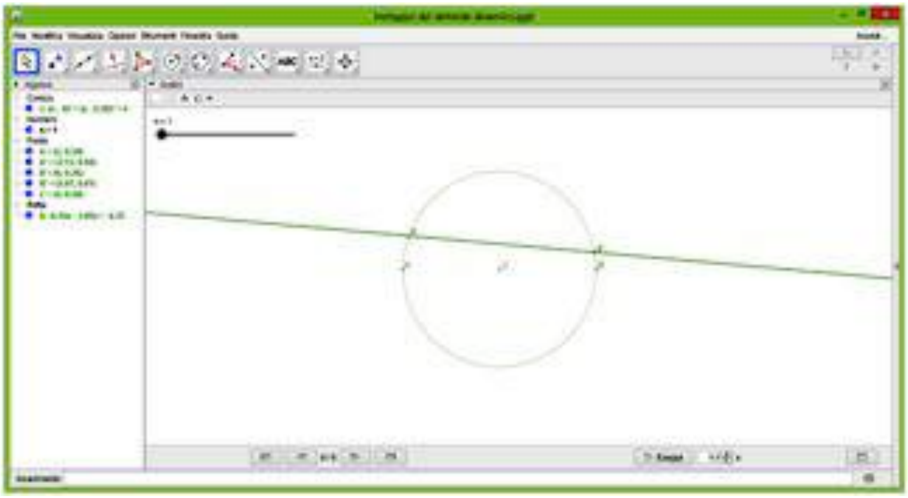

Figure 3. Partial output of the envelope of the deltoid curve.

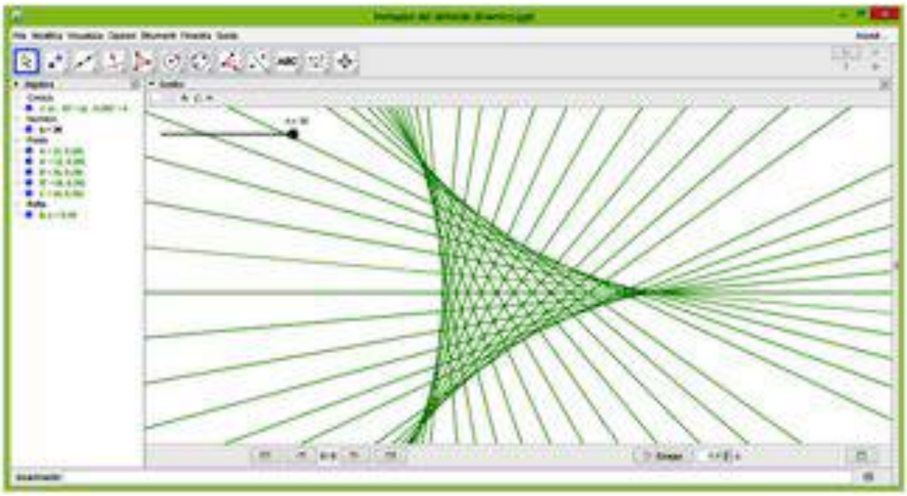

Figure 4. Output of the complete envelope of the deltoid curve.

This second activity carried out in the computer lab is very educational because it:

- implies the real comprehension of the algorithm of construction, through the simplification of the geometrical objects and their complexity;
- enables the learner to answer the question about the invariant of construction;
- adds value to the students' learning, because they get the opportunity to improve their skills and to become familiar with the formal and rigorous language of mathematics without being compelled to do so by a teacher.

To sum up, the second activity acts as support because it values intuition and at the same time enables the learners to make generalizations - impossible to attain with a static image - which lead to manage intuitive discovery within a rational framework, and ultimately to accept not only traditional deductive but also inductive processes of learning.

Concluding Remarks

The EM gives an opportunity for 'direct experience' at different levels with mathematical facts: students have really had the chance to work on geometrical objects constructively, exploring properties, formulating and testing conjectures, also thanks to the tools offered by GeoGebra software. Approaching geometric constructions once more and becoming familiar with the meaning of some terms, like the envelope, is extremely stimulating both for the students and the teacher. Using once again phrases like 'repeating the construction', 'the point subject to such a condition', etc. means to reflect on the progress made by teaching technologies in the last few decades.
In fact, the bond only to these instruments (R and C) is a kind of definition of 'geometric procedure' for the problem's resolution and, in some way, the idea of geometric construction anticipates the modern concept of algorithm that will be defined precisely many years later.
The geometric construction part from some data, which are the geometrical objects departure, and through a finite number of steps constituted by the elementary operations allows realizing a new object: the curve deltoid. The finiteness of the number of steps is an important aspect from the computational point of view. The choice to represent the deltoid curve with the envelope technique through the use of tools which combine tradition and modernity brings into the classroom alternative methods and procedures to the usual teaching approach, and introduces the students to the culture of Mathematics.
The methodology has been assisted by the practice of self-reflection enabling a coherent analysis of the characteristics of the two laboratory activities that, below, are shown in Figure 5.

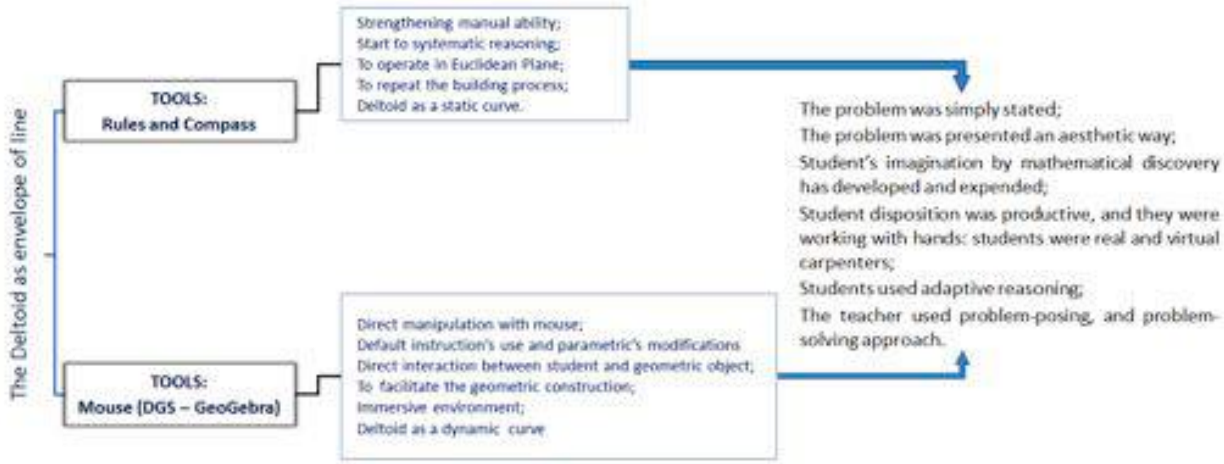

Figure 5. Activities descriptors.

This methodology values the very essence of geometric construction of a curve by starting from other given objects; at the same time the students have a chance to experiment new ways of working using the knowledge acquired, testing hypotheses along the way and reasoning both inductively and deductively while also broadening their cultural horizons.
Manipulative 'experience' has motivated the learners while at the same time the teacher becomes a facilitator, collaborator and guide. The close link between the tools used has transformed the operative phase into an active research procedure which in turn has led to the formulation of new concepts and the mastering of the procedure.
However, the EM is in no way limiting. The creative teacher can use the design as a springboard for new teaching initiatives which are instructive and engaging.
Geometry can be meaningful only if it expresses its relations with the space of experience [...] it is one of the best opportunities to mathematicise reality (Freudenthal, Mathematics as an Educational Task).

REFERENCES

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *Zentralblatt fur Didaktik der Mathematik*, 34(3), 66–72.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, *70*(2), 97-109.

Baccaglini-Frank, A., & Mariotti, M. A. (2010) Generating conjectures in dynamic geometry: The maintaining dragging model. *International Journal of Computers for Mathematical Learning*, *15*(3), 225–53.

Bazel, B.,Drijvers, P., Maschietto, M., & Trouche, L., (2005). Tools and technologies in mathematical didactics. In Bosch M. (Ed) *Proceedings of CERME 4* , 927-936, Spain.

Cremona L. (1862). Introduzione ad una teoria geometrica delle curve piane. Bologna: TiPI Gamberini e Parmeggiani, 23.

D'Amore, B. (1999). Elementi di didattica della Matematica. Bologna: Pitagora Editrice, 189-190.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in mathematics*, 24, 139-162.

Hannafin, R., Burruss, J., & Little, C. (2001). Learning with dynamic geometry programs: Perspectives of teachers and learners. *Journal of Educational Research*, *94*, 132-44.

Hohenwarter, M. & Preiner, J. (2007). Dynamic mathematics with GeoGebra. *The Journal of Online Mathematics and its Applications*, ID1448, vol. 7.

Hohenwarter, J., Hohenwarter, M., & Lavicza, Z. (2008). Introducing dynamic mathematics software to secondary school teachers: the case of GeoGebra. *Journal of Computers in Mathematics and Science Teaching*, 28(2), 135-146.

Hollebrands, K. (2007). The role of a dynamic software program for geometry in the strategies high school mathematics students employ. *Journal for Research in Mathematics Education*, *38*, 164-92.

Jones, K., Gutiérrez, A. and Mariotti, M. A. (2000) Proof in dynamic geometry environments. *Educational Studies in Mathematics*, 44, (1-3), 1-3. [Guest editorial for a PME special issue]

Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning*, 6(3), 283–317.

Leikin, R., & Grossman, D. (2013). Teachers modify geometry problems: from proof to investigation. *Educational Studies in Mathematics*, 82, 515-531.

Mariotti, M.A. (1995). Le rappresentazioni grafiche e l'apprendimento della geometria in D'Amore, B. (eds), *Insegnare ad apprendere la matematica in Aula: situazioni e prospettive*. Bologna: Pitagora, 47 -58.

Mariotti, M.A. (1996). Costruzioni in geometria. *L'insegnamento della Matematica e delle Scienze Integrate,* 19B, n.3, 261 - 288.

Plücker, J. (1839). Theorie, der algebraischen Curven. Bonn, 200.

Ruthven, K. et al., Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice. *Computers & Education* (2007), doi:10.1016/j.compedu.2007.05.013

Sitography

URL: mathematica.sns.it/media/volumi/460/CREMONA_curve_piane.pdf

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Writing as a Metacognitive Tool in Geometry Problem Solving

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Abstract: This work reports a teaching experiment, which explores the use of writing as a metacognitive tool in high school geometry problem solving. We develop a qualitative research study, to explore how explicit writing directives can help students to understand, organize and monitor the steps involved in the different phases of a cycle of activities for Geometry problem solving in the third year of secondary school.

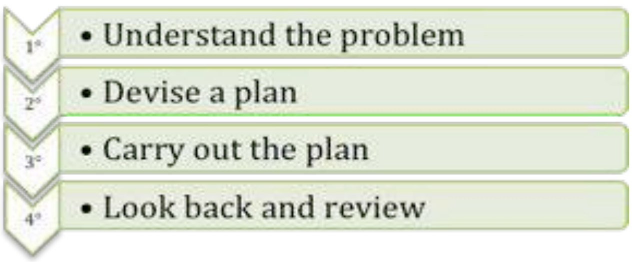
Résumé: Ce travail rapporte une expérience d'enseignement sur l'utilisation de l'écriture comme outil métacognitif dans la solution de problèmes de géométrie dans l'enseignement secondaire. Nous avons développé une étude de recherche qualitative, afin d'explorer comment les consignes d'écriture peuvent aider les élèves, de manière explicite, à comprendre, organiser et contrôler les étapes impliquées par les différentes phases d'un cycle d'activités pour la résolution d'un problème de Géométrie, pendant la troisième année du cycle secondaire[10].

Background and research problem

From 2006 to 2013, the Ministry of Public Education applied nationwide a guided multiple-choice evaluation in language, mathematics, and other subjects of the national curriculum in each school cycle of basic education. This test known ENLACE for its acronym in Spanish, helped to diagnose basic education in México, but an undesired side effect of this type of evaluation was that many teachers and pupils, in their efforts to improve the marks of their schools, overemphasized strategies to succeed in multiple choice tests, in detriment of the skills for dealing with open ended mathematical questions. In our work, with the purpose of improving students' problem-solving skills and foster reflective activity when working with open ended questions, we use explicit writing directives to help them expressing their understanding of geometry problems, and to organize, monitor and justify the steps for their solution.

Theoretical framework

In his classic book How to solve it, Polya (1957) provides an outline in four steps of the mathematical problem-solving process:



[10] Ndt. Équivaut à la classe de quatrième en France

Polya gives advice to educators on the four steps, and exemplifies them through a careful analysis of non-trivial mathematical problems accessible to high school or beginning undergraduate students. Additionally, he compiles a dictionary of basic heuristics like working backwards, exploring limit cases, making diagrams, solving simpler related problems, etc., in order to help the problem-solver to progress in difficult cases. Schoenfeld (1985a) delves further into Polya's ideas, studying the cognitive underpinnings and practical difficulties of mathematical problem solving, taking into account mastery of mathematical knowledge or resources, heuristics, executive or control issues and beliefs about mathematics and mathematical activity. In his own empirical research, Schoenfeld (1985b) encountered the major role that executive or control issues, have in the problem-solving processes. Control is concerned with the ways individuals use information at their disposal to take major decisions about what to do in a given problem. Control actions have global consequences in the evolution of solutions, as they determine which paths are taken or abandoned, and how resources are used. Schoenfeld studied the interplay of declarative knowledge, self- regulation and belief or intuition in these control processes, and described them as metacognitive phenomena (Schoenfeld1987, 1992).

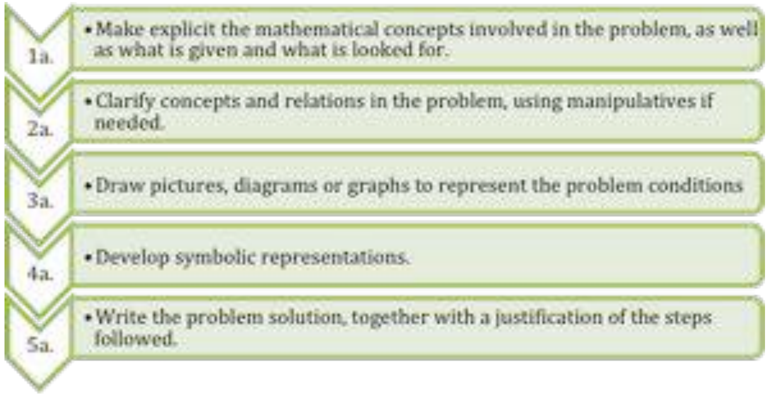
Metacognition and its entailments for learning and teaching have become important themes for mathematics education research in the last three decades (Schoenfeld 1985a, 1985b; Harman 1998; Collins et al. 2005). Metacognitive skills are needed in many school subjects, but according to Veenman (2012), they are honed mainly through four kinds of activities: reading text, problem-solving, discovery learning and writing.

Skillful reading and writing have great impact on problem solving activities. Hyde & Hyde (1991); Hyde (2006), have pointed out the importance, for students in basic education involved in mathematical problem solving, of explicitly trying to describe and represent mathematical concepts, questions, assumptions and solutions. In this way they identify and clarify previous knowledge engaged in the problem solving processes, and can better organize, monitor and reflect on their work, thus strengthening their thought. The philosophy is that language, mathematics and thought – in its cognitive and metacognitive dimensions-, are better fostered together.

A substantial part of Hyde's work comes from a deep analysis of the vast body of research on reading comprehension. Hyde (2006) identifies some of the most successful strategies in this area, and tries to understand how they work in order to contextualize and apply them in mathematical subjects, and in particular in mathematical problem solving. Hyde connects these contextualized strategies to the four phases of Polya's problem solving approach, and formulates a Braid Model of Problem Solving, in which a complex repertoire of questions intends to sort out many of the practical difficulties for implementing Polya´s four phases. In classroom use, these questions may be used as needed by the teacher combining them as well with different forms of collective participation. Our teaching experiment described below, draws heavily on Hyde's ideas, but emphasizes writing rather than reading comprehension, and uses a very simple orientation scheme because we want to give to the student accessible tools he can use without teacher´s assistance.

Methodology

In our teaching experiment, a five step plan for the problem solving process is followed:



The students are guided through these steps by the explicit directive of writing the answers to simple questions formulated in colloquial Mexican Spanish, whose rough English equivalent are: What are the givens? What do I need to find? Which mathematical idea am I using? Which drawings or diagrams can help me to find a solution? What steps will I follow to find a solution? How may I justify my solution? Are there other ways to find an answer?

A set of non-trivial geometrical problems was compiled from different sources, and reduced to 12 problems, after filtering and clarifying the statements working through them with the help of high school students and teacher trainees from Mexico City.

In the teaching experiment, the problems were worked out in 20 sessions of 45 minutes each, with a group of 10 highly motivated third year students from Secondary School 99 in Mexico City, working to prepare their admission exams for College. In the beginning four sessions work was done collectively. First and second session were devoted to the compilation of a glossary in which student made explicit and came to an agreement about the meaning of important concepts of high school geometry. Third and fourth sessions were used to explain the writing directives, and to work out and discuss collectively model examples of their application. The rest of the sessions the students answered individually worksheets for the 12 problems, having the guiding questions written on the blackboard as reminder. Data sources for analysis are the students’ written productions, and the field notes of the teacher-researcher.

Preliminary findings.

A detailed analysis of students´ productions is not yet complete, but we have noted that the writing directives for the 1st and 3rd steps of the problem solving plan were useful for all students. In most problems, 7 of the 10 students usually separately display givens, what is looked for, and one or several figures with the relations needed to solve the problem, while 3 of them regularly point out all these elements within a single figure. None of the 10 students made use of manipulatives to clarify some aspect of the problems. All students made an effort to give a clear, extended and complete justification of their answers, although 3 of them usually managed only to make an orderly account of the steps followed, with scarce justification or none at all. There is as well great variation in the quality of justifications in different problems in most students.

In the following, we will examine samples of student´s work in one of the 12 problems, whose statement is shown in Figure 1.

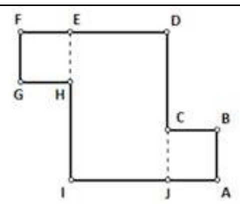
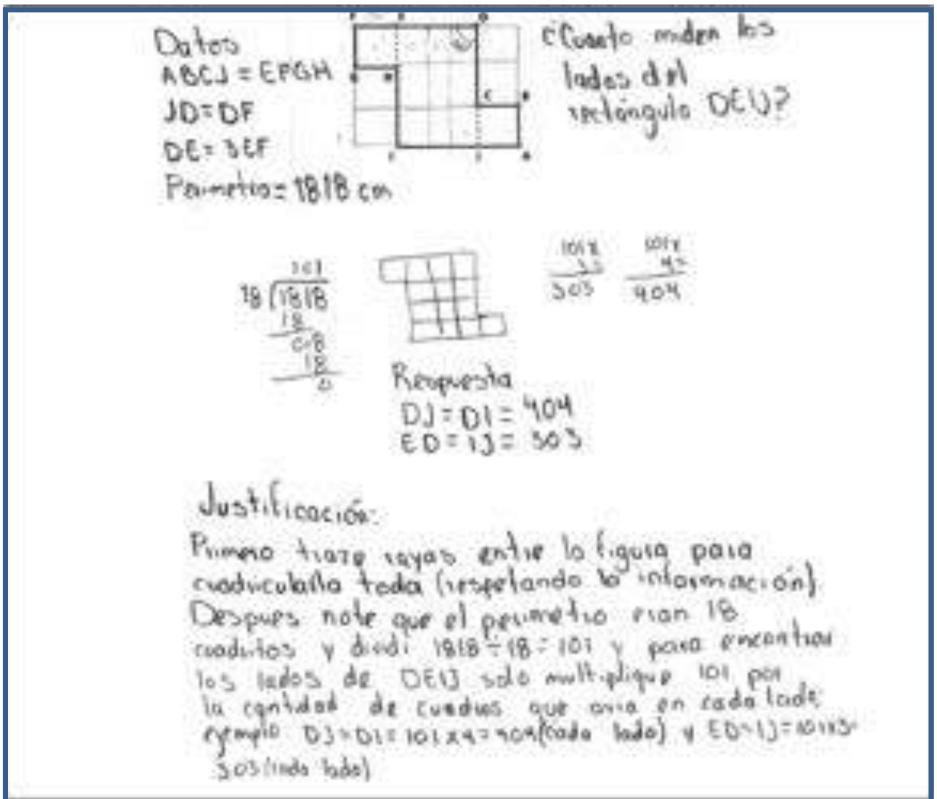
Problem. In the figure, ABCJ & EFGH are equal squares. JD = DF & DE = 3EF. The perimeter of the figure is 1818 cm. What do the sides of rectangle DEIJ measure? <u>Note:</u> The figure is out of scale.	

Figure 1. Problem statement

Figure 2 shows the answer given by student who correctly solves the problem using the simple intuitive strategy of decomposing the given figure on equal squares to recover the dimensions of rectangle DEIJ. He writes a brief, precise description of the procedure used, although makes some minor mistakes, i.e.: He writes boxes instead of box sides, but is referring to segments, as he unequivocally counts 18 in the perimeter. He also writes DJ=DI=404 instead of DJ=EI=404. It is worth noticing that he makes first an attempt to crosshatch the original figure in the worksheet, but corrects and then crosshatches his own drawing "complying with the given information" as he comments inside parenthesis in his writing.



Justification:

First I draw lines in the figure to crosshatch it all (complying with the given information) Then I noted that the perimeter was 18 boxes, and divided 1818÷18=101 and to find the sides of DEIJ only multiplied by the number of squares lying on each side, example DJ=DI=404(each side) ED=IJ=303(each side).

Figure 2. Sample worksheet with student's answer

We have seen many students using a similar crosshatching strategy with this problem. For instance Figure 3 shows work on the same problem by a student not participating in the teaching experiment. He directly crosshatches the given figure, without noticing the remark that the figure is out of scale and so, being unable to use the given conditions JD=DF and DE=3EF. He wrongly assumes that the length of the sides of the squares is given by the quotient between the perimeter and the number of squares, and proceeds to calculate the perimeter of DEIJ from that incorrect dimension.

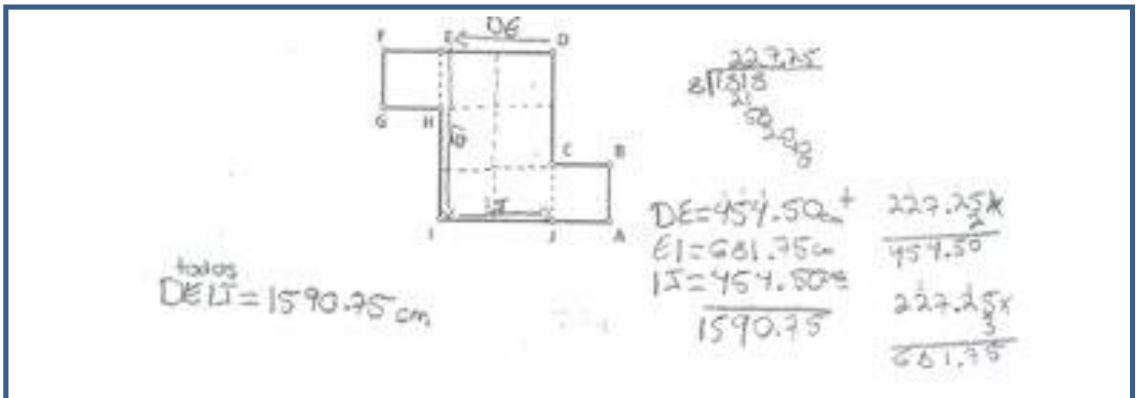
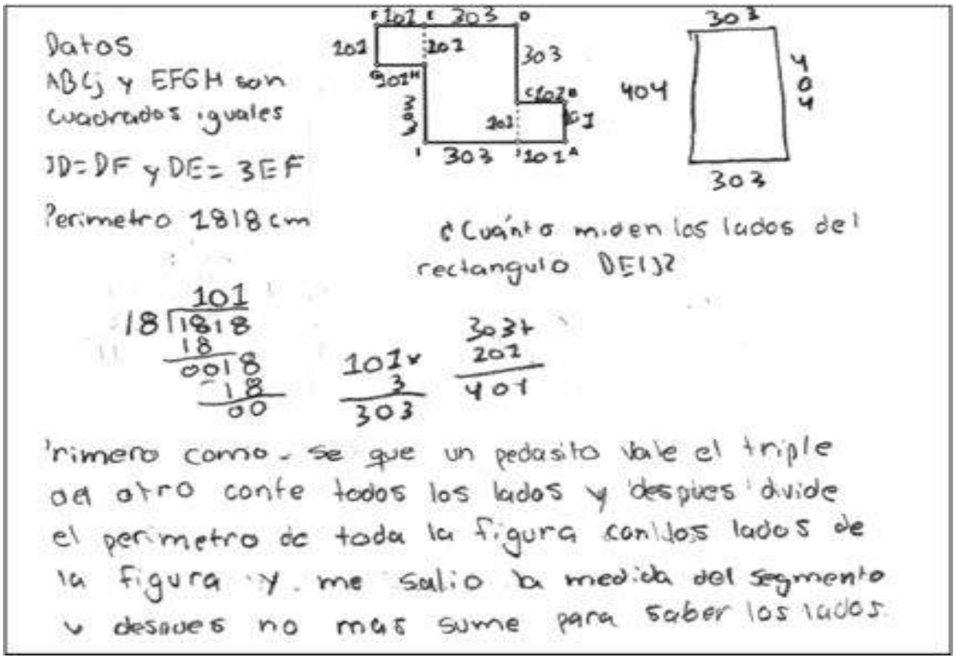


Figure 3. Worksheet of a student who did not take part in the workshop

From our point of view, identifying what is given and what is looked for in the problem formulation, and beginning to work explicitly writing these elements, makes a big difference for the students. Writing what is given provides an initial orientation which remains on sight, and functions as a control element that helps to correct mistakes and take into account relevant relationships and conditions.

Some students directly make use of the initial orientation provided by a complete, clear, and in orderly writing of what is given and what is looked for. Figure 4 shows the work of a student who draws the rectangle whose dimensions are required above the written statement of what is looked for, and uses the original figure as a reference whose sides he tags with the proper dimensions obtained by straightforward calculations using the given relationships. He grasps clearly what is given, and his justification globally describes the operations made, apparently considering them self-explaining. This brevity contrasts with the work of other students, like the example shown in Figure 5, who gives a detailed reconstruction of his train of thought, and writes in detail each relationship used, and each operation made. This student makes an effort as well to write in an orderly narrative sequence, clear visual disposition of mathematical expressions, and is as well one of the few students using punctuation marks.



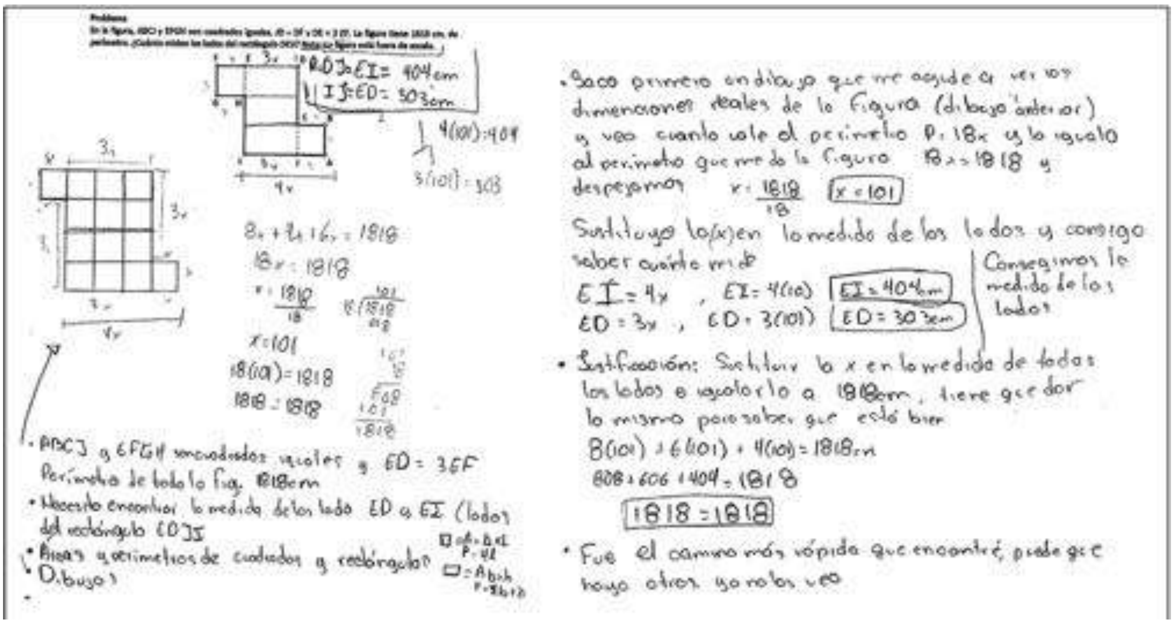
First since I know that a small piece value is three times the other I counted all the sides and then divided the perimeter of the whole figure by the sides of the figure and I got the measurement of the segment and then I just add up to know the sides.

Figure 4. Sample worksheet with student's answer

Well, first I know that the perimeter of the figure is 1818 cm and the figure has 8 sides, the base and the height are the same IA=DF, JD=IE. I said to myself that DE=3EF, but also that the figure is not in scale. So then I saw that on the side DF=4FE is the same as the side of IA=4JA, just like DC=3AB is the same as IH=3FG, so I added all of the little parts that I had and it added up to 18. 4 at FD, 4 at IA, 3 at DC, 3 at IH, 1 at CB, 1 at AB, 1 at FG, 1 at GH. And I divided 1818 ÷ 18 and got 101, and that's what FE measured, so I just multiplied 101 times 3 to obtain the base of ED and IJ, which was 303 cm and I multiplied 101 times 4 to get the height of the rectangle, and I got 404 cm.

Figure 5. Sample worksheet with student's answer

Besides working more directly with relationships some students mobilized other resources like using algebraic symbols. Figure 6 shows the work of one of such students, who also felt comfortable in mentioning the concepts used while giving a detailed account of the steps in his work.



ABCJ and EFGH are equal squares and ED=EF. Perimeter of all the figure. I need to find the measure of sides ED and EI (sides of the rectangle EDJI). Areas and perimeters of squares and rectangles. Drawings. First, I make a drawing that helps me to see the real dimensions of the figure (last drawing), and I see the value of the perimeter P=18x and I make this equal to the perimeter of the figure 18x=1818 and finding the unknown I get x=1818/18, x=101. Substituting x in the measure of the sides I get to know the measure of EI=4x, EI=404cm, ED=3x, ED=303cm (We got the measures of the sides). Justification: Substitute the x in the measure of all sides and equal to 1818, if have to add up the same, but you already know is all right. 8(101) + 6(101) + 4(101) = 1818cm. 808+606+404=1818. 1818=1818. It was the fastest way I found. Maybe there are others, I cannot see them.

Figure 6. Sample worksheet with student's answer

It is worth noticing that in his writing this student gives clues of developing self-regulation skills. To verify the correctness of the calculation of the unknown side x of the small square in the figure, substitution an addition of all parts should be made, "but you already know is all right", as a sound procedure has been followed. As the teaching experiment unfolded, students relied more and more on their own revision of written procedures rather than asking their classmates or teacher if solutions were right.

All 10 students finally managed to get correct answers for the 12 problems, but some of them only after revising several failed attempts. The writing directives were generally useful, but of course do not fully account for the success of the whole group of participants. An important factor was the sheer effort and quantity of work they invested in the workshop. They were studying for their admission exams for higher education and thus highly motivated. They were as well better prepared that average high school students not going to higher education. This may as well explain why none of the students feel the need to use concrete material to clarify the problems.

Conclusions

In the teaching experiment, the writing directives used to guide the problem solving process were generally useful for students. The 1st and 3rd steps of the problem solving plan helped students to clarify the problem statement. The simple action of identifying the data and relationships given in the problem formulation, as well as what is looked for, and begin to work with these explicitly written elements on sight, provides an initial orientation that helps students to take into account relevant relationships and conditions, avoid mistakes, and thus mobilize their knowledge in a proper direction. Particularly important as well, was the effort to produce clear and explicit justifications of the steps to get the solution. That effort helped to check their answers and to become aware of sources of error and facilitated the rectification of failed attempts of solution. These processes foster the development of self-regulation skills which manifested in the teaching experiment as students progressively relied more and more on their own revision of written procedures to assess the correction of their solutions.

REFERENCES

Collis, C., Israel S., Bauserman, K., Kinnuncan, K (2005) *Metacognition in Literacy Learning. Theory, Assessment, Instruction, and Professional Development*. London: LEA.

Hyde, A. & Hyde, P. (1991) *Mathwise: teaching mathematical thinking and problem solving*. Portsmouth, NH: Heinemmann.

Hyde, A. A. (2006). *Comprehending math adapting reading strategies to teach mathematics, K-6*. Portsmouth, NH: Heinemmann.

Hartman, J. (Ed.) (1998). Metacognition in teaching and learning: An Introduction. *Instructional Science* 26 (1-2), 1–3, ISSN 0020-4277 (Print) 1573-1952 (Online) DOI: 10.1023/A: 1003023628307.

Polya G. (1957). *How to solve it: a new aspect of mathematical method*. Garden City, NY: Doubleday.

Schoenfeld, A. H. (1985a). Metacognitive and epistemological issues in mathematical understanding. In E.A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 361-379). London: LEA.

Schoenfeld, A. H. (1985b). *Mathematical Problem Solving*. Orlando: Academic Press

Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A.H: Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 185-215). London: LEA.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D. Grows (Ed.), *Handbook for Research on Mathematics Teaching and Learning.* (pgs. 334-370). New York: MacMillan.

Veenman, M. (2012). *Metacognition in Science Education: Definitions, Constituents, and Their Intricate Relation with Cognition*. Metacognition in Science Education: Trends in Current Research. DOI: 10.1007/978-94-007-2132-6. ISBN 978-94-007-2131-9 USA: Springer.

WORKING GROUP 4 / GROUP DE TRAVAIL 4

Cultural, political, and social issues / Sujets culturels, politiques et sociales

Working Group 4 / Group de Travail 4

Cultural, political, and social issues / Sujets culturels, politiques et sociales

Pedro Palhares, Charoula Stathopoulou

In the first day, we have started with a presentation of some objectives for the group work. Besides that, two papers regarding Culture Constructions (both in History and for students ranging) were presented and discussed. The first, from Samuel Bello and Karin Jelinek, focused on the school selection process and monitoring of gifted students and analysing language games through Wittgenstein's concept. The second, from David Guillemette, brought us a discussion of the value history of mathematics in mathematics classrooms from a sociocultural point of view, after some conceptual elements of the theory of objectivation.

Second day was allocated for the papers regarding sociocultural factors in mathematics teaching. Three papers were in this slot. The first, from Vasiliki Chrysikou and Charoula Stathopoulou, was dedicated to the issue of teaching mathematics to students with severe intellectual disability. The focus of this paper was on the sociocultural factors that affect teaching and learning mathematics of three students, and the potential of home-school collaboration to promote students' active involvement during grocery shopping and money dealing. Second paper, from Filipe Sousa, Pedro Palhares and Maria Luísa Oliveras, tried to analyse the knowledge and the critical thinking level of students from two different cultural contexts (one in a fishing community the other in a more urban area) regarding the mathematical topic of symmetries, finding some slight differences between students of the two contexts in these aspects. Third paper, from Nina Bohlmann and Uwe Gellert, concerning students solving word problems, discussed the claim that standardized testing (re)produces the myth of mathematically illiterate students, but they argue the problem may rely on the standardized test and their designers and not on students' capability itself.

Third day was devoted to culture and language either as obstacles or resources. First paper was from Peter Appelbaum, Charoula Stathopoulou, Christos Govaris, and Eleni Gana and explored aspects of culture, its role as a resource or as an obstacle, discussing, through their experience in a project regarding the education of Roma Children, how it affects mathematics teaching in the classroom, considering that norms and practices in the classroom are mostly political rather than culturally embedded. The following two papers, second and third of the day were both presenting and discussing aspects of a European Comission funded project. The second paper, from Franco Favilli, attempted to describe a teaching unit, which aimed at overcoming the learning obstacle represented by the contrast between the simplicity of classroom language and the complexity of mathematics language. Third paper, from Hana Moraová, Jarmila Novotná and Andreas Ulovec, focused on the issue of coping with the increasing language diversity and presented some points regarding the implementation of a teaching unit, concluding that teachers, instead of detailed teaching units, what they really need is topics with different cultural origins that they can adapt to suit the needs of their particular group of students.

Fourth paper, from Lisa Boistrup and Eva Norén, discussed the issue of Swedish second language learners and their success in the national tests in mathematics in grade 5. They verified that some schools adapted the administration of the test to give second language students a better opportunity and other schools didn't and discussed it from an institutional perspective.

Fourth day was dedicated first to the issue of the complexity of mathematics teaching and learning through comparative studies and then to the preparation of the group report. First paper, from Benedetto di Paola, tried to understand the reasons why Confucian Heritage students have been performing better in PISA or TIMMS, by interviewing a Chinese teacher and exploring similarities and differences between East and West didactical approaches. Second paper, from Andreas Moutsios-Rentzos, discussed the role of perceived proximity in mathematics education as a crucial factor in the determination of the relevance of theoretical and empirical tools in mathematics teaching and learning research, by the consideration of a research project on proof.

For all the papers, we chose to limit the presentation to ten minutes, leaving 5 minutes for a reaction prepared in advance by another participant of the group, and ten minutes more for a generalized discussion. The group felt that this allocation of time and roles was extremely productive generating fruitful discussions and great involvement among the members of the group, as it was depicted in the last moment’s video recording.

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Culture is "Bricks, stones and tiles randomly thrown" (Λίθοι, πλίνθοι και κέραμοι ατάκτως ερριμμένα)

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Resumé: Nous explorons les aspects de la culture, son rôle comme une ressource et comme un obstacle, et comment tout cela informe ou désinforme l'enseignement des mathématiques dans la classe comme un espace où les normes et les pratiques sont surtout politiquement plutôt que culturellement intégré. Pragmatological matériau dérivé de notre expérience à travers le projet, «Education des enfants roms dans l'Epire, Iles Ioniennes, la Thessalie et la Grèce occidentale» illustre nos principaux points théoriques.

Abstract: We explore aspects of culture, its role as a resource and as an obstacle, and how all of this informs or misinforms mathematics teaching in the classroom as a space where norms and practices are mostly politically rather than culturally embedded. Pragmatological material derived from our experience through the project , «Education of Roma children in the Epirus, Ionian Islands, Thessaly and Western Greece» illustrates our key theoretical points.

Introduction

In our paper we are exploring aspects of culture, its role as a resource and as an obstacle, and how all of this informs or misinforms mathematics teaching in the classroom as a space where norms and practices are mostly politically rather than culturally embedded. Pragmatological material derived from our experience through a project regarding the Education of Roma Children[1] in Greece illustrates our key theoretical points.

How is "Culture" a resource, and how is "Culture" an obstacle, in mathematics teaching and learning?

Culture as a Resource

On the one hand, Alan Bishop's perspective from the 1980s offers mathematics educators a useful notion of "Mathematical Enculturation": Bishop's (1988a; 1988b) early work offered a broad, universal, set of intellectual practices that could be taken as "mathematical," independent of the particular social context: counting, measuring, locating, designing, playing, and explaining. The apparent universality of mathematics results, according to Bishop, from the universality of the adaptive, human goals that define these six types of activities, rather than the *a priori* nature of mathematical principles. Bishop assumed the cultural universality of the activities, but emphasized the diversity found in symbolic mathematical technologies produced by the activities within varying cultural contexts. If we begin thinking about mathematics education with this in mind, then mathematics education would have to be understood more as a subset of cultural practices within a

[1] Data for this paper is taken from the project, «Education of Roma children in the Epirus, Ionian Islands, Thessaly and Western Greece», 2010-2013 (E.U. Lifelong Learning, action code: 304263)

broader culture than as a tool independent of such a culture. That is, mathematics education would be one form of enculturation and acculturation within a culture. (See below, in the next paragraph.) Furthermore, if we take enculturation seriously, then we can ground pedagogical decisions in aspects of culture independent of the fundamental properties of mathematical thinking: Cultural variations in the ways that mathematics is practiced might be significant across teacher, student, family and community cultures represented in a school environment. Here culture provides a lens for comprehending what and how to attend to such variations, as well as a theory for what and how to attend to external to mathematics but impacting upon the experiences of teaching and learning mathematics that are taking place in an educational encounter. One consequence is the notion of a "Multicultural Classroom", a place where a variety of cultures come together and mix, creating hybrid identities, potential epistemological and linguistic conflicts, and a range of ways of valuing or devaluing mathematics as an academic subject. Whether dilemmas, paradoxes, conflicts, or miseducative, any of these confrontations and complexities would be an indicator of culture as a resource, through which better understanding and potentially powerful pedagogical interventions could emerge. (Appelbaum & Stathopoulou 2014)

The example of the experience of Roma students in the Greek educational system can be used to illustrate these ideas. For example, in the new National Curriculum (NC) and the Cross Thematic Curriculum Framework (CTCF) of the Greek Primary Education System, there is a shift from a the concealing of diversity in earlier policy documents toward a position that characterized the older national curriculum, to respecting cultural and linguistic diversity. A further examination of the national curriculum texts nevertheless points out issues related to a thorough network of limitations that continue to entrap multiculturalism as simultaneously important yet crystallized in stereotypical ways. In particular, the direction of the NC and the CTCF within this model of intercultural learning informed by cultural perspectives is founded on a static definition of culture and cultural differences, which works against its broader goals by promoting the reproduction of a stereotyped perception of the 'other(s)' (Govaris, 2015).

Roma pupils entering a Greek school can be understood as highlighting common cultural expectations of a 'normal' student through contrasts with the strengths and weaknesses that Roma youth bring with them into the learning experiences. The image of the "normal" student contains and expresses a set of expectations from teachers, and the educational system in general, especially regarding the knowledge capital and skills to bring a student in order to be able to participate successfully in teaching organized learning processes. The expectation of successful participation requires the ability of individual performance, which necessitates ongoing evaluation processes of the learning process of each student. In such a shape of thinking and student responses, a deterministic understanding among teachers, for example, about the strong influence of family background on students' school performance, in combination with the dominant perception that Roma students come from families with deficient cultural and linguistic capital, plays a catalytic importance for Roma students. Because teachers are encouraged to (only) think in cultural terms, they privilege the perception of different cultural backgrounds, inadvertently depriving Roma students the ability to be carriers of a 'normal' knowledge capital, skills and attitudes that a student needs to make a career without any problems at school, at least within the expectations that people have for these students. Teachers appreciate the Roma students' performance skills as lower than their non-Roma peers, therefore cultivating reduced expectations in terms of school success.

If we start from the premise that there are particular and unique ways of being mathematical, then school mathematics experiences are either experiences of enculturation, in which younger members of the culture are enabled to become more sophisticated, grown-up, members of that culture over time; or acculturation, in which there are power relations between those who are more or less sophisticated and experienced, and those whose identity is more closely affiliated with the dominant, school mathematical culture (and thus have advantages in these power relations); or both. In this sense, the term "learning" might be suitably replaced by enculturation, acculturation, or both. One consequence, or indicator, of this type of cultural circumstance, is that a subculture of the broader community is apparently used informally or formally as the measure of other cultures for the determination of what is accepted as legitimately mathematical or not, "more mathematical" rather than less, "good mathematics", and so on; it is this subculture that would be named a "dominant" culture.

In the example of Roma children in the Greek school system, the Roma culture has in particular ways been pre-determined by the expectations that teachers have for Roma children, because they try to be 'good', multiculturally-aware teachers. Students' culture of origin has already been assessed by the teachers. So the students are faced with specific assumptions (stereotypical or not) about "what" their culture "is", and how this culture affects their learning potential. Thus, "the" culture –as a resource or as an obstacle –is needed to be seen in relation to the image of teachers about culture, in relation with what teachers are expecting from different students (Roma), in other words, what teachers expect from the concept of culture itself. In the example of mathematics: in school it is widespread among teachers that the Roma culture is probably conducive to mathematics learning, as trade is at the heart of Roma life, etc. How this expectation determines their attitude to teaching is an empirical question. What is important is that teachers at the level of expectations are likely to perceive positive terms of cultural origin, to correlate positively with the objectives of the teaching of mathematics, searching to find points of contact with those who believe that students know.

At the same time, Roma students can be seen to exhibit the results of what is called stereotype threat: the students' achievement is adversely affected by their perception of themselves as representatives of all Roma, and thus of the potential of any Roma child to learn. In general, Roma youth in this study have incorporated in their self-image stereotypical representations of significant others (non-Roma) that affect their perceptions of their own potential. We could say that culture might be a resource for these learners; however, in this particular case it becomes more of an obstacle for these youth, who tend to create their own sense of self as a learner in general, and as a learner of mathematics in particular, within a particular school and cultural context.

Culture as an Obstacle

On the other hand, "Culture" as a conceptual tool carries with it a legacy of anthropological history, as well as political implications. It becomes confusing to tease out how mathematics in this broadly universal conception is and is not imbricated in a colonialist enterprise. The expectation of a universality at some level carries with it some elements of the Western, ideological framework of mathematics as neutral and distant from culture, so that an analysis on this macro level creates a continuity with that perspective, rendering local variations across cultures and subcultures less significant or seemingly irrelevant. This is because anthropology, as a European construct, created "other" non-European cultures as alien others, in its early development, and has been deconstructed to demonstrate that these early, simplistic tools, such as concepts of "culture", inscribe implicit forms of hierarchy and "epistemicide" –the erasure of any potential awareness or existence of alternatives (Parasekeva 2015). Every mathematical knowledge not historically- and culturally-embedded in Western mathematics is measured and defined in such an approach by the Western mathematics. Indeed, it has become central to the study of mathematics and mathematics education to view mathematics itself as the alien culture, into which learners must enter. In more politically nuance terms, mathematics is seen as a particular cultural construct within the Western, European colonialist enterprise (Barton, 1996, p.9; Appelbaum, 1995; Davis & Hersh, 1986).

With respect to the experience of the learner, culture thus can become an obstacle to educative experience. When speaking of early years of schooling, enculturation is not really apprenticeship within the child's own culture since a school curriculum has already decided before the arrival of the child what and how the child will learn. In this sense, no school experiences are possible to actually describe as "enculturation". We inherit the term "enculturation" from the title of Alan Bishop's (1988b) book; yet he himself later used the more appropriate concept "acculturation," given that every learner grapples with cultural conflicts (Bishop, 2002). A psychological approach that contrasts with the socio-cultural orientation discussed here would make an analogous distinction: enculturation can be understood as acquiring the characteristics of a subculture, in this case, mathematics, through being enmeshed within that culture, while acculturation would refer to "fitting in" to a cultural milieu by emulating the characteristics of those who are already members of that milieu (Kirshner, 2004). Again, the cultural perspective emphasizes how intercultural experiences are always bound up in unequal power relations that serve important roles in the experiences of those involved. We might say that school mathematics serves, through acculturation, important functions in social and cultural reproduction, contributing to the development of "reasonable" people who reason in particular ways, and who are also able to be governed by systems of power and established authorities (Cline-Cohen, 1982; Walkerdine, 1987; Appelbaum, 1995). On the other hand, an awareness of the special vocabulary of school mathematics, and the idiosyncratic ways of working as a student of mathematics that help learners succeed in such a context, offers useful ideas for supporting learners of mathematics who are not yet demonstrating mastery of the material. The particular kind of cultural approach discussed in this context, in this paragraph, distinguishes between the subject knowledge of a course in mathematics and the norms and expectations that teachers of mathematics might have for learners in the course. (Appelbaum & Stathopoulou 2014)

It is common in a typical Greek classroom, for example, that the teacher does not know what sorts of knowledge Roma children bring with them from everyday life into school. In this way, teachers cannot legitimate or exploit the funds of knowledge of Roma children through the school curriculum. A cultural analysis of school experience highlights in this respect the ways that school mathematics does not authorize Roma children's strengths and weaknesses, making them both invisible to those participating in the school curriculum and to the children themselves.

Sometimes described as "academic literacy," the norms and expectations for how one works and demonstrates learning in a school context have been shown to be teachable and assimilable when made explicit to the students, and when practiced as explicit ways of working (Appelbaum, 2008; Polya, 1945; Mason, et al., 1985; Brown & Walter, 2005; Cotton, 2010). To use Geertz's (1973) image of webs of signification, the culture of a mathematics classroom is the tangled interweaving of webs of meaning and interpretation brought to the classroom by teachers, learners, broader characteristics of the social milieu of the school and society, etc. The academic literacy expectations structure the potential interactions that occur in educational encounters, constraining and enabling activity, interpretations, expectations, fears and desires.

In our own recent research in the framework of the project on Roma children education we specifically examine the funds of knowledge that these children bring into the school; curriculum designed with these funds of knowledge in mind becomes far more effective in terms of student learning outcomes. Field work identifies funds of knowledge, and the classroom becomes a 'third space'/ hybrid space (Moje et al. 20040, where students are encouraged to speak about their knowledge of language and mathematics; the children make connections between everyday life experiences and school mathematics concepts, and inform the general understanding by all students, Roma and non-Roma, of mathematical concepts, through their everyday knowledge (Stathopoulou, et al. 2014).

We notice that, in a typical classroom context, Roma youth cultural differences (measured by comparison with the dominant group, by practical definition a problematic distinction) are considered by teachers as an obstacle—sometimes just their presence is an obstacle. The following quote from a teacher of primary school is characteristic: "My class is good; for good luck, I have no Roma children, so my work comes easier". Through our contribution, as part of the project «Education of Roma children…», we tried to respond to issues like this; we tried to transform such obstacles. Our main strategies were: a) research on the spot (on their community of origin) in order to access their funds of knowledge and b) to create spaces, hybrid spaces in the classroom, where discourses from the community meet typical classroom discourses, and the students' knowledge becomes accepted and in turn transformed in formal school knowledge, creating new academic opportunities for students. What emerged here, something stronger than cultural issues, are better characterized as *political* issues. Curricula, teacher training, and the broader educational parameters depict broader policies regarding education and students of minority backgrounds. (Stathopoulou, et al. 2014)

Culture as an Analytical Tool

Approaches to mathematics education and culture establish forms of reality and common sense through the application of distinctions, often without any clear attention to these distinctions. In this way, these approaches create implicit—sometimes explicit— assumptions about dichotomies such as in-school and out-of-school learning, formal and informal education, teaching and learning, mathematics and culture, student or teacher identity and mathematics, and so on. For example, if we carry out a project or teach a school mathematics lesson trying to make it more meaningful and relevant to some students in the classroom by noting that they are members of a non-mainstream subculture, we are reducing the uniqueness of each individual to a set of stereotypical assumptions from a generic caricature of this subculture. Each individual may or may not fit this set of assumptions. Indeed, most of the learners in this situation are members of multiple subcultures at the same time, and are in any given moment having experiences that resonate with cultural habits and dispositions from more than one of these subcultures. As researchers, mathematics educators wish to use categories based on cultural distinctions to analyze situations, because this seems like the only reasonable, common-sense way for us to make sense of the setting and the people in it. Yet, as soon as we use these distinctions, we are already aware of the variations within any given group that seem more extreme than differences between groups. And as soon as we try to take into account the variations within any given group, we are already aware of the ways in which these variations are inadequate to capture the variations within any one individual within that group. That is, borders between categories are permeable, so that, to keep this simple, say, a Catholic, Latina girl in a Chicago classroom may or may not be having an experience consistent with what her teacher might expect of a learner recently relocated from New Jersey with her Cuban-American, Jewish father, working in a small group with her Chicano best friend and a recent immigrant from Albania. In other words, each learner is determined to some extent by the cultural contexts that are part of their life; yet, as individuals, learners have a repertoire of behaviors and ways of making meaning out of experience that are specific to them.

Conflicts as Analytical Sites

It is increasingly challenging to exploit all resources available in the interests of mathematics learners, given the myriad of types of resources and locations of these resources; at the same time, the "resources" take on different meanings depending on one's cultural perspectives, one's understanding of others' cultural perspectives on these resources, and one's analysis of the conflicts that may or may not emerge. In this sense, again, culture is itself a resource for pedagogical theory. Conflicts exist in most discussions of education broadly conceived about the role of mathematics in the lives of children and adults — both in the present and in their futures, in terms of both individual and societal needs. These conflicts and associated confusions regarding the role(s) of mathematics are made more complex by the expectations for mathematics and mathematics learning, more or less culturally determined, that meet each other in educational encounters. Sometimes, mathematics is taken as a culture itself; funds of knowledge pedagogies offer ways that home cultures can be appropriated as resources; ethnomathematical critiques and approaches to teaching and learning become resources for changing practices to resonate with cultural expectations. These are not separate, analytic categories, but mutually informing strands of interwoven discourse..

Culture as a Term in Discourse about Mathematics Education

As a tentative conclusion, we suggest using the term "culture" to refer to aspects of cultural contexts, and more specifically, aspects of culture related to learning and knowledge, rather than to speak of "culture" in general. We do this to avoid the discontinuity that appears at school through dichotomies of formal and informal learning, distinguished by the role of a designer or evaluator of learning experiences not present in the learning context that is necessary for "formal" learning to take place. Culture, more broadly, is both "an historically transmitted pattern of meanings embodied in symbols, a system of inherited conceptions expressed in symbolic forms by means of which men communicate, perpetuate, and develop their knowledge about and their attitudes toward life" (Geertz 1973a, p. 89), and those "webs of significance" people themselves spin" (Geertz, 1973b, p. 5). Culture for mathematics education is a collection of bricks, stones and tiles randomly thrown, so that, after the fact we can see some mosaics and patterns and walls and buildings and surfaces and works of art that seemed to have been created from somewhere, but are, in the sense of Michael Polanyi (1974), a mere happenstance of our human qualities of perception: the tiles, stones and bricks come from the legacies of dominant cultures, colonialism, and local traditions; the magnificent works of art are created by the humans who pick up the pieces and place them in juxtaposition.

REFERENCES

Appelbaum, P. (1995). *Popular culture, educational discourse, and* mathematics. Albany, NY: State University of New York Press.

Appelbaum, P. (2008). *Embracing mathematics: On becoming a teacher and changing with mathematics*. New York: Routledge.

Appelbaum, P., Stathopoulou C, (2014). Critical Issues in Culture and Mathematics Learning, Chapter to be included in the *Handbook of International Research in Mathematics Education*, edited by Lyn English and David Kirshner. NY: Routledge/Taylor & Francis.

Barton, B. (1996). Ethnomathematics: Exploring cultural diversity in mathematics. Doctoral dissertation, The University of Auckland.

Bishop, A. (ed.). (1988a). Mathematics education and culture. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Bishop, A. (1988b). Mathematical enculturation: A cultural perspective on mathematics education. Boston: Kluwer Academic Publishers.

Bishop, A. (2002). Mathematical acculturation, cultural conflicts, and transition. In de Abreu, G. , Bishop, A., & Presmeg, N. (Eds.), *Transitions between contexts of mathematical practices*, pp. 193–212. Dordrecht: Kluwer.

Brown, S., & Walter, M. (2005). *The art of problem posing*. Mahwah ,NJ: Lawrence Erlbaum.

Cline-Cohen, P. (1982). *A calculating people: The spread of numeracy in early America*.

Berkeley, CA: University of California Press.

Cotton, T. (2010). *Understanding and teaching primary mathematics.* Edinburgh Gate: Pearson Education Limited.

Davis, P., & Hersh, R. (1986). *Descartes' dream: The world according to mathematics*. San Diego: Harcourt, Brace, Jovanovich.

Geertz, C. (1973a). Religion as a cultural system. In *The interpretation of cultures*, pp. 87-125. New York: Basic Books

Geertz, C. (1973b). Thick description: Toward an interpretive theory of culture. In *The interpretation of cultures,* pp. 3-30. New York: Basic Books

Govaris, Ch. (2015) Intercultural learning in CTCF / NC Elementary School: critical notifications In: Language Education-35 studies devoted to Napoleon Professor Mitsi (413-426) (edited by George Androulakis), Athens: Gutenberg.

Kirshner, D. (2004). Enculturation: The neglected learning metaphor in mathematics education. In D. McDougall & J. A. Ross (Eds.), *Proceedings of the twenty-sixth annual meeting of the International Group for the Psychology of Mathematics Education, North American Chapter*, Vol., 2, pp. 765-772). Toronto: OISE/UT.

Mason, J., Burton, L., & Stacey, K. (1985). *Thinking mathematically*. Reading, MA: Addison-Wesley.

Moje, E. B., Ciechanowski, K. M., Kramer, K., Ellis, L., Carrillo, R., & Collazo, T. (2004). Working toward third space in content area literacy: An examination of everyday funds of knowledge and discourse. Reading *Research Quarterly*, 39, 38–72.

Paraskeva, J. (2015). *Curriculum epistemicide: Towards an itinerant curriculum theory.* NY: Routledge.

Polya, G. (1945). *How to solve it: A new aspect of mathematical method*. Princeton, NJ: Princeton University Press.

Polyani, M. (1974). *Personal knowledge: Toward a post-critical philosophy.* Chicago: University of Chicago Press.

Stathopoulou, C., Govaris, C., Appelbaum, P., & Gana, L. (2014). Bringing students out- and in- school knowledge: teaching young Roma children mathematics and language through their everyday experience. In *Actes/Proceedings CIEAEM 66, Lyon, 21-25 Julliet/July 2014*, di Paola, B. & Fazio, C. (eds.), pp. 320-24. CIEAEM 66 Proceedings QRDM 24, Suppl. 1, WG4.pdf.

Walkerdine, V. (1987). *The mastery of reason: Cognitive development and the production of meaning*. New York: Routledge.

Lorsque les hautes compétences en mathématiques ne sont que des formes de vie

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Abstract : This is an issue that introduces a research that aimed to analyse the selection process and the school educational monitoring of students so-called "gifted", by understanding of ways in which "language games" and certain behaviours valued by school are in relation with their life forms. Methodologically, we use the Wittgenstein's concept of language games as well as "relations of power" and "games of truth" both based on Foucault to create the notion of "games of power-language" from which we make our analyses by highlighting that the subject so-called "Gifted student" is an effect, an invention of discursive practices. We also conclude that process of identification and selection of gifted students are based on strategies of comparison and ranking.

Résumé : Il s'agit d'un texte qui se réfère à une recherche qui a eu pour but d'analyser les processus scolaires de sélection et de renforcer le suivi éducatif des enfants qui sont appelés aujourd'hui au Brésil « porteurs de hautes compétences/surdoués », afin de comprendre les façons dont les jeux de langage et les conduites valorisés par ces processus sont en rapport avec leurs formes de vie. Méthodologiquement, nous utilisons les notions des jeux de langage chez Wittgenstein ainsi que des relations du pouvoir et des jeux de vérité chez Foucault pour créer le concept de « jeux de pouvoir-langage » à partir duquel nous faisons nos analyses en mettant en évidence que le sujet/porteur de hautes compétences n'est que l´effet de pratiques discursives et que les processus d'identification et de sélection sont tout d'abord des processus de comparaison et de classement.

Présentation

L'objet abordé ici se rapporte à la discussion sur « les questions culturelles, politiques et sociales » introduite lors du CIEAEM-67, ce qui concerne les hautes compétences pour l'apprentissage des mathématiques en mettant en question si elles sont effectivement liées au développement cognitif.

Nos idées sont basées sur une recherche de doctorat dont le but était d'analyser les processus scolaires de sélection et de renforcer le suivi éducatif des enfants qui sont appelés aujourd'hui au Brésil « porteurs de hautes compétences », afin de comprendre les façons dont les jeux de langage et les conduites valorisées par ces processus sont en rapport avec leurs formes de vie.

Méthodologiquement, il s'agit d'une recherche qui est appelée au Brésil « poststructuraliste » puisque cette perspective permet que l'on s'interroge sur les idées propres au structuralisme, en particulier sur la fonction du langage et l'essentialité, la rigidité et la fixité de leurs significations, ainsi que l'histoire des différentes façons d'être sujet[2].

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[2] Il faut souligner que grande partie des études sur le thème des hautes compétences l´exploitent sous une perspective cognitiviste de l´apprentissage et du développement. Cependant, ce travail vise à discuter les hautes compétences en mathématiques en se basant sur une théorie sociale qui les discute sous une perspective de relations de pouvoir et de

Ainsi, tenant compte des notions « wittgensteiniennes » des jeux de langage en rapport avec les formes de vie et les concepts « foucauldiens » de pratique discursive, de relations de pouvoir et de jeux de vérité, nous créons le concept de « jeux de pouvoir-langage ». Ce concept nous a permis d'analyser comment des enseignants de quelques écoles de la ville de Porto Alegre identifient, comparent et classifient leurs élèves en tant que sujets/porteurs de hautes compétences. Pour ce faire, nous accompagnons les activités de sélection[3] développées par les spécialistes de la SIR/AH[4] puis nous réalisons des entretiens avec non seulement trois élèves dits « porteurs de hautes compétences » en mathématiques mais aussi leurs enseignants.

Une fois les mouvements analytiques achevés, il nous a été possible de vérifier qu'il y a eu un déplacement et une réactualisation de la signification du terme « surdoué » par celui-ci appelé « de hautes compétences ». En même temps, on va confirmer que les sujets de hautes compétences sont produits par l'observation et la comparaison attentive et précise des performances des élèves mises en évidence ainsi et qui sont valorisés par les enseignants et par l'école. (Jelinek, 2013b; Jelinek et Bello 2014). Le développement des ces idées est présenté ci-dessous.

De surdoués vers porteurs de hautes compétences

Les hautes compétences seront ici considérées comme des pratiques discursives (Foucault, 1995) car par des règles et régularités qui leur sont propres, elles définissent et modélisent des actions, des conduites en produisant des subjectivités, des identités, c'est à dire des formes-sujets, par lesquelles on est visible, dicible, déchiffrable. Ainsi, la forme-sujet « porteur de hautes compétences » n'est pas seulement une variation du nom de l'anciennement dénommé surdoué, mais un ensemble de pratiques et comportements qui y sont liés. Il convient de noter ici que le langage est plus qu'un simple acte de la parole ou de l'écriture, il implique des façons de penser et d´agir en rapport avec nos formes de vie. Par la création de la notion de « jeux de langage », Wittgenstein (2008) apporte l'idée d'une normativité du langage par laquelle on constitue une réalité et ses significations dans certaines situations d'utilisation (Bello et Régnier, 2014). Le langage est le monde que nous habitons et que nous « pratiquons », à la fois instrument et construction (Paltrinieri, 2011).

De la même façon, les notions de relations de pouvoir-savoir et de jeux de vérité chez Foucault (2013, 2006, 2003) aident à comprendre comment on est devenu forme-sujet à partir des significations linguistiques crées. Il est important de remarquer que pour Foucault (2013, p. 146), le pouvoir ne doit pas être compris en tant que système oppressif, mais comme un ensemble de relations dont le but est d'agir ou de chercher à agir sur la conduite de l'autre. « (...) c'est lorsqu'il y a un rapport entre deux sujets libres et qu'il y a dans ce rapport un déséquilibre tel que l'un peut agir sur l'autre et que l'autre est 'agi', ou accepte l'être ». Selon Jelinek (2013a), pendant les années 1960s et 1970s les significations données aux *dits* surdoués étaient en rapport avec les domaines scientifiques (mathématiques et linguistiques) et artistiques (musique et arts visuels) et valorisaient les qualités telles que : la performance numérique, la mémoire, le raisonnement logique-mathématique, la vitesse de la pensée, toutes capables d'être mesurées par des tests de QI. Il y avait aussi une caractérisation liée à la vitesse de maturation physique et intellectuelle de l'individu. Au cours des années 1990s, les changements quant à la compréhension d'intelligence comme ceux proposés par Gardner (2001) en rapport avec l'environnement social et culturel des individus nous font croire en l'existence d'habilités et de performances différentes quand l'individu est *porteur*

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productions identitaires.

[3] Le processus de sélection développé par les spécialistes de la SIR/AH se compose d'une fiche d'auto-désignation devant *être remplie par les collègues des élèves de hautes compétences aussi que d'une fiche d'élementss d'observation des activités de classe* devant être remplie par leurs enseignants. Ces processus comptent également des entretiens.

[4] SIR/AH est l'abreviation de « Sala de Integração e Recursos para Alunos com Altas Habilidades/Superdotação » - Salon d'intégration et Ressources pour élèves de Hautes compétences/ surdoués.

d’au moins une de ces habilités. Ce changement de signification apporte au sens de porteur de hautes compétences plusieurs autres qualités qui devront être observées, telles que l’esprit de leadership, d’initiative et de collaboration.

Pour Jelinek (2013b), la compréhension contemporaine de « porteur de hautes compétences » n’est plus seulement liée au domaine des mathématiques, mais bien aux *comportements sociaux et économiques souhaitables* qui fonctionnent comme normes de classification et de comparaison des sujets scolaires. Les enfants dits de hautes compétences se gênent très facilement et font acte de distraction et de frustration à certains moments, ils résistent à la répétition par cœur, ils possèdent des pensées critiques et sont excessivement actifs, ce que l’on peut comprendre comme hyperactivité. Ce concept de « hautes compétences » devient un discours qui s’oppose à celui des difficultés de l’apprentissage.

Foucault (1995) a appelé discours les pratiques au milieu de relations de pouvoir, elles produisent des énoncés, établissent des vérités et des connaissances à un moment donné historique. Pour le philosophe, les pratiques discursives ne se réfèrent pas à l´activité d´un sujet en soi, mais à l´existence objective et matérielle de certaines règles par lesquelles ce sujet est constitué et/ou produit dans les relations sociales. La pratique discursive des hautes compétences permet non seulement que les identités se fixent, mais aussi que s´établissent des différences entre les individus. L´identité “porteur de hautes compétences” n´est pas seulement une production de sens, mais aussi une manière d´établir des appartenances à des formes-sujet déterminées.

Il existe ainsi, dans une pratique discursive des hautes compétences, un jeu de pouvoir-langage, c´est-à-dire un jeu linguistique qui la désigne, lui donne un sens et un jeu de pouvoir qui l´établit.

Le sens des hautes compétences à l’école

D’après le sens des jeux de langage ainsi que les formes-sujet, on peut s’interroger : de quelles manières les pratiques des porteurs de hautes compétences sont-elles en rapport avec leurs formes de vie?

Les chercheurs des hautes compétences sont unanimes en ce qui concerne le rapport entre le contexte dans lequel l’enfant vit et ses pratiques au moment des évaluations (Guenther, 2000, 2006; Silver, 2010; Sim-Sim, 2005; Winner, 1998), tant et si bien qu’un enfant sera porteur de hautes compétences en accord avec un groupe social et à un moment daté et situé. Autrement dit, un enfant possède de hautes compétences dans une forme de vie spécifique et non dans d’autres. Si l’intelligence est culturelle, alors les hautes compétences aussi.

Les pratiques d’identification des porteurs de hautes compétences dans lesquelles les enfants se distinguent vont mettre en évidence des situations, des objets, des savoirs, des conduites qui leur sont familières ou en rapport avec le quotidien. Autrement dit, les situations que les enfants doivent résoudre ou auxquelles ils doivent faire face ont *un air de famille avec leurs formes de vie*.

A notre avis, la fiche d’observation utilisée par les spécialistes à l’école pour identifier les enfants/porteurs de hautes compétences ne font que valoriser certaines conduites spécifiques qui sont significatives et importantes au point de vue de l’école.

Il est intéressant d´observer que parmi les 27 éléments dont se compose la fiche d´observation du professeur, 20 d´entre eux se basent sur des caractéristiques comportementales, les autres servent à comparer les individus de la classe, ce que nous pouvons identifier par l´expression *meilleurs en*:

FICHE D´ÉLÉMENTS POUR L´ OBSERVATION EN CLASSE

Indiquez pour chaque élément les deux élèves de votre classe, garçon ou fille, qui, à votre avis, presentent les s caracteristiques suivantes :

1)Les meilleurs de la classe dans les domaines du langage, de la communication et de l´expression :
2)Les meilleurs en mathématiques et sciences :
3)Les meilleurs dans les domaines de l´art et de l´éducation artistique :
4)Les meilleurs dans les activités extracurriculaires :
5)Les plus locaces, causeurs :
6)Les plus curieux, intéressés, questionneurs :
7)Ceux qui participent le plus et qui sont toujours présents, à l´intérieur et à l´extérieur de la salle de cours :
8)Les plus critiques envers les autres et eux-mêmes:
9)Ceux qui ont la meilleure mémoire, qui apprennent et retiennent facilement :
10) Les plus persistants, engagés, qui terminent ce qu´ils font :
11) Les plus indépendents, qui commencent leur travail et le font seuls :
12) Ceux qui s´ennuient le plus, les plus désintéressés, mais pas nécessairement en retard :
13) Les plus originaux et créatifs:
14) Les plus sensibles aux autres et les plus gentils avec les camarades de classe :
15) Ceux qui s´occupent du bien-être des autres ::
16) Les plus sûrs d´eux-mêmes :
17) Les plus actifs, perspicaces, observateurs :
18) Les plus capables de penser et de tirer des conclusions :
19) Les plus sympathiques et les plus aimés par leurs camarades de classe :
20) Les plus solitaires et ignorés ::
21) Les plus espiègles, drôles, chambardeurs :
22) Ceux que vous considérez les plus intelligents :
23) Ceux qui ont la meilleure performance en sports et exercices physiques :
24) Ceux qui se distinguent dans les activités manuelles et motrices :
25) Ceux qui donnent des réponses inattendues et pertinentes :
26) Ceux qui sont capables de diriger et de transmettre leur énergie pour encourager le groupe :
27) Y a-t-il dans votre classe un enfant avec d´autres talents spéciaux ? Lesquels ?

Table 1. Fiche d'observation utilisée par les spécialistes à l'école.

"Ceux qui ont une meilleure mémoire, qui apprennent et retiennent le plus facilement, qui sont les plus indépendents, qui commencent leur travail et le font seuls", ces éléments sont parmi ceux qui identifient un possible *porteur* de hautes compétences en mathématiques[5] et qui, bien que nous ayons affaire à un domaine dit exact, se réfèrent aussi à la conduite des individus.

Si nous regardons attentivement les éléments énumérés sur la fiche, nous voyons que les hautes compétences en mathématiques ne sont pas attribuées par leurs aspects cognitifs, mais se naturalisent par les aspects comportementaux. Et nous pouvons ajouter : que les vérités à la base de ces pratiques ne sont que des jeux constructeurs d´une objectivité pour une subjectivité – celui dit *porteur* de hautes compétences en mathématiques.

Ce que l´on cherche à élucider, c´est que ces conduites, observables dans le cadre des pratiques scolaires, sont des manifestations du respect des règles des *jeux de pouvoir-langage* qui constituent et sont constitués par ces pratiques. Des évidences de ceci ont aussi pu être identifiées à partir des entretiens réalisés dans les écoles tout au long du travail sur le terrain.

En réfléchissant sur certaines formulations des professeurs – comme par exemple *"il a toujours fait des observations intéressantes, qui nous ont surpris en classe, on ne savait pas où il allait chercher ça..."* – je me risque à dire que le discours est *explicite*, car il n´a ni significations occultes ni savoirs secrets, bien au contraire, il précise ce qui doit être accepté,– dans ce cas, que les hautes capacités sont en rapport avec les conduites exprimées par les autorités scolaires. Et c´est ceci qui doit être tenu pour acquis, c´est-à-dire comment les règlements qui supportent ces pratiques sont

[5] Pour faciliter la réflexion proposée ici, il faut se rappeler que pour être désigné *porteur* de hautes compétences en mathématiques, l´individu doit être signalé dans au moins trois des éléments suivants : 2,9,11,18 et 22.

comportementaux et servent de critères pour distinguer les états ou les comportements adéquats ou non.

On ne peut pas s´empêcher de remarquer que, même s´il s´agit de hautes compétences en mathématiques, les caractéristiques relatives à ces pratiques ne sont pas exploitées en profondeur par les professeurs et que les conduites sont explicites à l´extrème. Un exemple de ceci est la valorisation de ces élèves qui "donnent des réponses inattendues et pertinentes" dans le milieu scolaire, comme nous avons vu dans l´exemple donné. En considérant que la peespective de cette recherche a été les pratiques, il est possible de constater que le sujet des hautes compétences *devient* ce sujet *seulement* par sa conduite.

En ce sens, il ne vient pas comme un sujet essentialisé, a partir d´une condition innée, mais se produit comme sujet à partir de ses conduites qui fabriquent sa propre identification. Telles conduites agissent à partir de règles, elles aussi constitutives et constituantes des *jeux de pouvoir-langage*.

L´expression largement utilisée dans la fiche d´observation du professeur, est "meilleurs en" comme nous pouvons le voir à partir des éléments "les meilleurs de la classe dans les domaines du langage, de la communication et de l´expression", "les meilleurs en mathématiques et sciences", "les meilleurs dans les domaines de l´art et de l´éducation artistique", "les meilleurs dans les activités extracurriculaires", "ceux qui ont la meilleure mémoire", "ceux qui ont la meilleure performance en sports". Tels éléments associés aux caractéristiques énoncées par les professeurs, comme "c´est un élève distingué" ou "il était déja alphabétisé quand il est arrivé en seconde, ce qui est une rareté à l´école", ou encore, "il s´est toujours distingué", nous fait penser que la sélection des porteurs de hautes compétences se base fortement sur une norme comparative.

Bien que des caractéristiques d´individualisation soient valorisées et constituent en quelque sorte la plupart des questions utilisées pour définir et orienter un sujet porteur de hautes compétences, il ne suffit pas seulement de dire ceci ou cela. Il faut établir des moyens et des formes de comparaison, sous la forme de relations d´altérité, où interviennent des jeux de pouvoir-langage mêlés à des pratiques de sélection qui mettent en évidence des mesures et des postures d´ordres et de règles méritocratiques.

D'ailleurs, si auparavant un test de QI était employé pour comparer la performance d'un individu avec une « norme » de caractère théorique, aujourd'hui la fiche d'auto-désignation ne fait que comparer les individus les uns avec les autres.

Si cette opération de comparaison entre les sujet est une condition vitale pour que les jeux d´identification des conduites puissent fontionner, cette même fiche établira aussi une pratique de comparaison du sujet avec lui-même, puisqu´il devra réfléchir sur les différentes caractéristiques qui lui sont propres. Le sujet élève, dans ce cas, est amené à se reconnaître comme un sujet bon ou très bon, au moins dans une des catégories : mathématiques, arts, gymnastique, théâtre, sciences, créer des histoires, danse, leadership, lire, faire des recherches, sports, créativité, écrire, musique, amitié ou dans d´autres domaines que cite l´élève.

Le résultat de l'application de ces deux fiches est l'expression de la façon dont les jeux de pouvoir-langage saisissent des conduites des sujets scolaires en les transformant en conduites propres des *dits* « porteurs de hautes compétences ».

Par ailleurs, il est important de remarquer que d'autres formes de vie sont négligées ou soumises car les jeux de pouvoir langage les classent ; en fin de compte la valorisation donnée aux conduites des élèves considère des normes, des modèles et des pratiques qu'une société pense être les formes de vie scolaire adéquates et de hautes compétences. Pour conclure, nous pouvons dire que le sujet/porteur de hautes compétences est le sujet des jeux de pouvoir-langage. C'est la forme-sujet qui résulte de la soumission d'un individu aux règles du langage, du pouvoir, des vérités et des savoirs scolaires et par lesquelles il entre en mouvement et se rapporte à lui-même et aux autres, par lesquelles il construit une subjectivité pleine de désirs, d'intentions, de connaissances et de valeurs qui sont propres à sa forme de vie.

REFERENCES

Bello, S. E. L.; Régnier, J. C. (2014, Juillet). *Langage, subjectivité et pratique: qu'appelle-t-on réalité en éducation mathématique?* 66 CIEAEM, ENS – Lyon – France.

Foucault, M. (2013). L'origine de l'herméneutique de Soi. Paris: VRIN.

Foucault, M. (2006). *Ditos e Escritos V: ética, sexualidade, política*. MOTTA, M. (Eds.). (E. Monteiro & I. Barbosa, Trans.). Rio de Janeiro: Forense Universitária.

Foucault, M. (2003). *A verdade e as formas jurídicas*. Rio de Janeiro: Nau.

Foucault, M. (1995). *A arqueologia do saber*. (L. Neves, Trans.). Rio de Janeiro: Forense Universitária.

Gardner, H. (2001) *Inteligência: um conceito reformulado.* Rio de Janeiro: objetiva.

Guenther, Z. (2006). *Capacidade e Talento: um programa para a escola*. São Paulo: EPU.

Guenther, Z. (2000). *Desenvolver Capacidades e Talentos: um conceito de inclusão*. Petrópolis: Vozes.

Jelinek, K. (2013a). *A prática discursiva das Altas Habilidades em Matemática*. Bolema, Rio Claro (SP), v. 27, n. 45, p. 193-214.

Jelinek, K. (2013b) *A produção do sujeito de altas habilidades: os jogos de poder-linguagem nas práticas de seleção e enriquecimento educativo*. (Doctoral Thèse) Retrieved from: http://hdl.handle.net/10183/70606

Jelinek, K. R.; Bello, S. E. L. (2014). *Highly able mathematics learners from a pragmatic perspective of language*. RIPEM , 4 (2), 139-154. http://www.sbem.com.br/ojs/index.php/ripem/issue/view/12

Paltrinieri, L. (2011) Pratique et Langage chez Wittgenstein et Foucault. In: Gros, F. Davidson, A. *Foucault et Wittgenstein: de possibles rencontres* (41-77). Paris: KIME.

Silver, H. (2010). *Inteligências múltiplas e estilos de aprendizagem: para que todos possam aprender*. Porto: Ed. Porto.

Sim-Sim, I. (2005). *Necessidades educativas especiais: dificuldades da criança ou da escola*. Lisboa: Textos.

Winner, E. (1998). *Crianças superdotadas: mitos e realidades*. Tradução de Sandra Costa. Porto Alegre: Artmed.

Wittgenstein, L. (2008). *Investigações Filosóficas*. (M. Montagnoli, Trans.). Petrópolis: Vozes.

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Word Problems: Resources for the Classroom?

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Abstract: In recent discussions on students' practices for solving word problems, some researchers argue that the many students who fail on these problems do not understand the nature of the problem situation. Other researchers, on the contrary, point to the requirement of utilizing the appropriate degree of realistic considerations which is related to the difficult decision of how much and what part of the reality seems to be relevant according to the problem designer and the responses coding scheme. Apparently, the issue of the relevance of the problem situation is a main obstacle for generating the expected solution. In the specific context of standardised testing, the student's achievement of translating the word problem text correctly into a mathematical operation and of performing the operation correctly is often devalued by narrow expectations. Therefore, standardised testing produces a myth: the mathematically illiterate student. In this paper, we use classroom video data to substantiate the claim that the issue of the students' failure, or difficulty, of taking the problem situations into account 'correctly' might be more relevant for those researchers and assessment designers occupied with standardised testing than for students solving word problems in everyday mathematics classrooms.

Résumé: Au cours des dernières discussions sur les pratiques des élèves pour résoudre les problèmes de contexte, certains chercheurs soutiennent que les nombreux étudiants qui échouent sur ces problèmes ne comprennent pas la nature de la situation du problème. D'autres chercheurs, au contraire, indiquent l'obligation d'utiliser le degré approprié de considérations réalistes. Ce degré est lié à la décision difficile de combien et quelle partie de la réalité semble être pertinente selon le concepteur du problème et le système de codification des réponses. Apparemment, la question de la pertinence de la situation du problème est un obstacle principal pour générer la solution attendue. Dans le contexte spécifique des tests standardisés, le rendement de l'élève de traduire le problème de contexte correctement en opérations mathématiques et d'exécuter l'opération correctement est souvent dévalué par les attentes étroites. Par conséquent, les tests standardisés produit un mythe: l'élève sans culture mathématique. Dans cet article, nous utilisons les données vidéo pour étayer l'affirmation selon laquelle la difficulté des étudiants de prendre les situations des problèmes «correctement» en compte pourrait être plus pertinente pour les chercheurs et les concepteurs d'évaluation occupées avec des tests normalisés que pour les étudiants à résoudre des problèmes de contexte dans les classes de mathématiques de tous les jours.

Introduction

In recent discussions on students' practices for solving word problems, one mathematical task has been used paradigmatically to exemplify the intricacies and complexities of this kind of didactic material for assessing the students' mathematical knowledge (Cooper 1992, 2004, Gates and Vistro-Yu 2003, Gellert 2009, Murphy 1995, Palm 2009). The task has originally been used by the British Schools Examinations and Assessment Council (SEAC 1992) and it reads as follows: "This is the sign in a lift in an office block: This lift can carry up to 14 people. In the morning rush, 269 people want to get up in this lift. How many times must it go up?" This word problem is structurally similar, although less militaristic, to older assessment tasks used in the National Assessment of Educational Progress in the U.S.A., for instance, as discussed in Carpenter et al. (1983): "An army bus holds 36 soldiers. If 1128 soldiers are being bussed to their training site, how many buses are needed?" While some researchers argue that the many students who fail the assessments by providing answers like 19.21 to the lift item, or 31.33 to the bus item, or ignoring the remainder, do not understand the nature of the problem situation, others point to the requirement of utilizing the appropriate degree of realistic considerations, which Gates and Vistro-Yu (2003) describe as "the goldilocks principle of the reality behind mathematical problems – not too much, not too little – just enough" (p. 53). In this second perspective, for many students the complexity of the word problem is related to the difficult decision of how much and what part of the reality of queuing and transport seems to be relevant according to the problem designer and the responses coding scheme. Apparently, the issue of the relevance of the problem situation is a main obstacle for generating the expected solution.

In this paper, we use classroom video data to substantiate the claim that the issue of the students' failure, or difficulty, of taking the problem situations into account 'correctly' might be more relevant for those researchers and assessment designers occupied with standardised testing than for students solving word problems in everyday mathematics classrooms. We argue that recent decisions in many countries to react on the issue of the ignored problem context —which is an issue of standardised testing schemes— by including more "realistic type" word problems in the mathematics curriculum does a disservice to the teachers and learners in mathematics classrooms who intend to teach and learn mathematics, and not how to correctly respond to dysfunctional assessments of mathematical literacy. Rather than being empirical or theoretical, our paper is related to educational policy and the mathematics curriculum.

The Myth of the Mathematically Illiterate Student

In a teaching experiment, which we describe in Bohlmann, Straehler-Pohl and Gellert (2015), a class of sixth-graders and their mathematics teacher engage with the lift problem and with context variations of the same mathematical operation 269÷14. While the numerical facts are maintained, the context variations put the task in situations (road crossing, cable car), in which it is more or less probable that "real" people in "reality" would wait until it is their turn to cross the road, to enter the lift, or to take the cable car. The students solve the word problems and produce posters, which present their solution. The teacher then discusses with the students the varying degrees of fictitious behaviour that the tasks assume of the people involved. The students seem more and more confident to add what they consider as distortions of reality. At the end of the lesson, the teacher confronts the students' criticism of unreal problem situations with their mathematical solutions presented on posters. Note that Luke is a special student. It could be reconstructed from the video data that, on several occasions, the teacher positions him as the best student in class.

Teacher:	You have now mentioned quite a lot of things that do not fit to the task when you say, this is unrealistic. And yet all of you have made the poster. And found solutions. (3 sec.) Why didn't you consider all these things? (2 sec.) Tony?
Tony:	Because we never thought about it before?
Teacher:	Luke?
Luke:	Because we are supposed to solve the task and are not supposed to think about it.
Teacher:	Again, louder!
Luke:	We are supposed to solve the task like always, and we are not supposed to think about the actual situation, but simply find out the maths problem and solve it.

The teacher then reconfirms Luke's final statement. Apparently, successful participation in this mathematics class requires the students to act according to the strategy described by Luke: When confronted with a word problem, find out the expected mathematical operation and calculate the result. Don't distract yourself by thinking of what could happen if the problem situation were "real".

There is sufficient evidence to assume that people are able to deal with numbers correctly in everyday situations, independent of their proficiency of coping with school mathematical word problems (Lave 1988, Nunes, Schliemann and Carraher 1993, Rogoff and Lave 1984). Thus, as long as the students' preparation for everyday life matters is concerned, Luke's strategy does not seem to be too harmful. But what happens in forms of assessment in the classroom? Isn't this strategy inappropriate because a final answer of 19.21 is not fully meeting the teacher's expectations? It can be argued that the deviation of the answer "19.21" from the expected result, "20", is remarkable. However, in a context of everyday teaching, including the typical tests that teachers prepare for their students, "19.21" is rather close to what the teacher might expect: In order to reach "19.21", a student needs to correctly decide on the mathematical operation (division) that the problem situation requires —which is not trivial a task— and to correctly calculate 269÷14. In an everyday class context, "19.21" is probably credited with, say, three out of four points. Thus "19.21" is a quite good response to the word problem.

The situation turns out to be completely different under a standardised testing regime, where a solution "19.21" would normally be credited with zero points. In the specific context of standardised testing, the student's achievement of translating the word problem text correctly into a mathematical operation and of performing the division correctly is fully devalued. In this way, standardised testing produces a myth: students' incompetence to model "real-world" situations mathematically; or, in terms of the Programme for International Student Assessment (PISA), the mathematically illiterate student. As our classroom data illustrate, if asked to relate the textual representation of a problem situation to a "real-world" situation, the students refuse to uncritically carry out mathematical operations with the given numbers. They immediately recognise the crude simplification of problem situations generated by its textual representations in mathematics word problems.

In the aftermath of the first PISA and TIMSS results, mathematics curricula experienced conceptual shifts in many countries. In the case of Germany, "problem solving", "application of knowledge" and "mathematical modelling" have become the key concepts of the new mathematics curricula, thus promoting a re-orientation of the curriculum towards mathematics-in-the-real-world. Attempts are made to change the nature of the word problems from simply disguised mathematical problems to "more realistic", open tasks, taking the context outside the classroom and the students' experiences more seriously. Eventually, these claims perpetuate the myth of reference and the myth of participation (Dowling 1998) rather than enabling students to model "real-world" situations, and additionally still mask the hierarchy of forms of knowledge in the mathematics classroom.

The myth of students' incompetence to model "real-world" situations is one starting point of recent curriculum reforms. The reform pressures teachers and students to conceive of word problems as reifications of the usefulness of mathematics in the "real world". Teachers and students can detour this pressure by considering word problems a "genre which carries within its form echoes of related genres (for example, riddles, parables, puzzles and competitions of wit), none of which bears any simple, necessary relationship to a presumed practical 'reality'" (Gerofsky 2010, p. 62), and develop more sophisticated educational practices in the mathematics classroom that take students' meta-knowledge about word problems —as a resource for classroom interaction— into account.

REFERENCES

Bohlmann, N., Straehler-Pohl, H., & Gellert, U. (2015). Deconstructing the filtration of reality in word problems. *Quaderni di Ricerca in Didattica*, 24(Suppl. 1: Proceedings of CIEAEM 66), 142–145.

Carpenter, T.P., Lindquist, M.M., Matthews, W., & Silver, E.A. (1983). Results of the Third NAEP Mathematics Assessment: secondary school. *The Mathematics Teacher*, 76(9), 652–659.

Cooper, B. (1992). Testing National Curriculum mathematics: some critical comments on the treatment of 'real' contexts for mathematics. *The Curriculum Journal*, 3(3), 231–243.

Cooper, B. (2004). Dilemmas in designing problems in „realistic" school mathematics: a sociological overview and some research findings. In M. Olssen (Ed.), *Culture and learning: access and opportunity in the classroom* (pp. 183–202). Greenwich, CT: IAP.

Dowling, P. (1998). *The sociology of mathematics education: mathematical myths/pedagogic texts*. London: RoutledgeFalmer.

Gates, P., & Vistro-Yu, C. (2003). *Is* mathematics for all? In A.J. Bishop, M.A. Clements, C. Keitel, J. Kilpatrick & F.K.S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 31–73). Dordrecht: Kluwer.

Gellert, U. (2009). Zur Explizierung strukturierender Prinzipien mathematischer Unterrichtspraxis. *Journal für Mathematik-Didaktik*, 30(2), 121–146.

Gerofsky, S. (2010). The impossibility of 'real-life' word problems (according to Bakhtin, Lacan, Zizek and Baudrillard). *Discourse: Studies in the Cultural Politics of Education*, 31(1), 61–73.

Lave, J. (1988). *Cognition in practice: mind, mathematics, and culture in everyday life*. Cambridge: Cambridge University Press.

Murphy, P. (1995). Sources of inequity: understanding students' responses to assessment. *Assessment in Education*, 2(3), 249–270.

Nunes, T., Schliemann, A.D., & Carraher D.W. (1993). *Street mathematics and school mathematics*. Cambridge: Cambridge University Press.

Palm, T. (2009). Theory of authentic task situations. In L. Verschaffel, B. Greer, W. Van Dooren & S. Mukhopadhyay (Eds.), *Words and worlds: modelling verbal descriptions of situations* (pp. 3–19). Rotterdam: Sense.

Rogoff, B., & Lave, J. (Eds.) (1984). *Everyday cognition: its development in social contexts*. Cambridge, MA: Harvard University Press.

Schools Examinations and Assessment Council (SEAC) (1992). *Mathematics Tests 1992*, Key Stage 3. London: SEAC.

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A school for all? Political and social issues regarding second language learners in mathematics education

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Résumé: Pour étudier un aspect de l'équité en ce qui concerne l'apprentissage des mathématiques dans « une école pour tous », nous avons étudié comment les professeurs décrivent l'organisation de l'épreuve nationale de mathématiques. Les étudiants de cet étude sont en 5ème année scolaire (étudiants de onze à douze ans) et sont des élèves qui ont le suédois comme leur deuxième langue (Second Language Learners : SLL). Avec les données d'une enquête parmi les professeurs, aussi avec des profils de compétences pour les étudiants de 5ème année scolaire, nous avons effectué une analyse thématique. Les résultats indiquent qu'il y avait des écoles où les professeurs ont travaillé en conformité avec les instructions de l'épreuve, et, par conséquence, adaptés un organisation de l'épreuve qu'améliore les possibilités pour les étudiants SLL de montrer leur savoir en mathématiques. Ceci est cohérent avec l'intention exprimée dans les documents de règlement. Il y avait aussi des écoles où les professeurs décrivent plutôt des justifications de l'exclusion des étudiants SLL du test, qu'une adaptation de l'organisation du test selon les instructions. La, des mauvais résultats des étudiants SLL sont expliqués par problèmes de langue. Dans ces écoles, les étudiants SLL n'ont pas été invités à montrer leur savoir en mathématiques. Nous discutons ces résultats dans une perspective institutionnelle.

Abstract: To investigate one equity aspect regarding mathematics learning in "a school for all" we have investigated how teachers comment on their arrangements for Swedish second language learners (SLL) to succeed on the National Test in mathematics in grade 5 (students are 11–12 years old). With data from a teacher survey and competency profiles for students in grade 5 we have performed a thematic analysis. The findings indicate that there were schools where the teachers worked in line with the instructions of the test and, therefore, adapted the administration of the test to enable SLL students better opportunities to display knowing in mathematics. This is coherent with a view expressed in policy documents. There were also schools where the teachers did not write about how to adapt the test administration but rather justified the exclusion of SLL students from the test or explained SLL students' poor results due to language issues. In these schools the SLL students were not invited to display mathematics. We discuss these findings from an institutional perspective.

Introduction

This paper is relevant for Subtheme 4, Cultural, political, and social issues, and its purpose is to illuminate one aspect of equity issues in " a school for all", namely second language learners' equal access to a compulsory National Test (NT) in mathematics, which provides possibilities for students to show the mathematical knowing outlined in the national syllabus for mathematics.

Due to school reforms in Sweden (Skolöverstyrelsen, 1962) the compulsory education is an integrated school where everyone is welcome. "A school for all" has been commensurate with the view that the education should have a countervailing affect to help pupils who do not have "enough" prerequisites from home to gain equal access to education in terms of gender or social, cultural or economical factors. Practically "a school for all" means that all students are held together from grade 1 to grade 9, and hardly any ability grouping occurs. The education should be adapted to each student's individual potential. In the increasingly heterogeneous society, however, this type of school is challenged (Tallberg Broman, Rubinstein Reich & Hägerström, 2002).

One group of students that is in need of counteracting measures is Swedish second language learners (SLL) and this is clear in policy documents. One aspect here is that SLL pupils are entitled to instruction in Swedish as a second language and also get graded for that subject (Utbildningsdepartementet, 2011). They also have the right to study their mother tongue as a subject and to get instruction in their mother tongue at least two hours a week. The compensatory measures are also significant in relation to socio-economic status and parental education background.

To investigate one equity aspect regarding mathematics learning in “a school for all” we have investigated how teachers comment on their arrangements for Swedish second language learners to succeed on the National Test in mathematics in grade 5 (students are 11–12 years old).

Political issues and mathematics education

Although the political ambitions mentioned above are good, it turns out in practice that many students with less educated parents and many students with non-Swedish background do not do so well in school. Immigrant students learn to see themselves as an “inferior” kind of students. Based on the ideas that Swedish students are well-behaved and have a bright future, immigrant boys refer to themselves as “immigrant boys in the suburbs with poor grades” (León Rosales, 2010). They are affected by contextual factors including media that categorizes immigrant students with a deficient rationale. Teachers’ expectations and requirements to work with students individually, as well as local conditions, segregation, poverty and socio-economic status restrict student achievement (León Rosales, 2010). According to Klapp Lekholm (2008), 3-5% of the grades achieved in grade 9 in Swedish, English and mathematics are based on elements such as interest, motivation and parental involvement. Regarding mathematics teaching it is characterized by educational segregation, where teachers use different teaching methods according to their perception of student groups’ social and linguistic composition. This leads to lower expectations, which in turn leads to lower performance for children with lower socio-economic background or special immigrant groups (Hansson, 2011).

In recent years it has been shown that the socio-economic gap in mathematical performances has widened between students with high-and low-educated parents (Skolverket, 2013). Reports show that the gap is increasing and that multilingual pupils do worse than students with Swedish as their first language. Social selection to higher education remains and educational patterns are reproduced (SOU 2004: 47). Statistics show that there are differences between municipalities and schools in terms of pupils’ performance in National Tests[6] in grade 6 and 9. It may depend on school organization and how teaching is conducted, for example, different ways of working, teaching efforts and schools pupil composition. Parents with knowledge of the education system are enculturated in the ways of education in Sweden and can therefore enculture their children accordingly.

Immigrant students’ difficulties in school is seen as a result of aspects of the students’ background, not as the result of the teaching situation or environment. Mathematics teachers seem to treat students differently depending on whether they are boys or girls, and if they have Swedish or non-Swedish background (Moschkovich, 2007; Parszyk, 1999; Stathopoulou & Kalabasis, 2007).

Data collection and analysis

Swedish students in grade 5 and 9 have completed a National Test in mathematics from 1996 until 2010 and since then National Tests are given in grade 3, 6 and 9. The tests consist of different item formats such as short answer questions, questions which need more elaboration from the student and group tasks. The teachers assess the students’ performances drawing on assessment instructions included in the test material and they can also complete a competency profile for each student. The teachers are asked to answer a survey in order to give the test designers feedback. Here they can comment on test samples and on observations from the test situation in the classroom and the like. For this paper we have analysed 26 of the competency profiles for students from year 2005, and 16 from 2002. We have also examined 155 teachers’ surveys from 2005 as well as the teacher

[6] From 2010 mandatory national subject tests are held in grades 3, 6 and 9 of compulsory school to assess student progress.

instructions for the test.

A theoretical frame in this paper is the concept of institution (Douglas, 1986) and we view the teachers' answers as representative of the teacher perspective within the institution of school. In elaborating on the presence of institutions, it can be argued that mathematical assessments are situated in a context characterised by dominant (mathematics) education discourses, the use of artefacts developed over time, framings in terms of specific resources for learning, division of time, structures within and between schools, classification of students into schools and learning groups, established routines, and authoritative rules (Selander, 2008, drawing on Douglas, 1986).

We performed a thematic analysis in line with Braun and Clarke (2006). The process required a decision on whether to perform an inductive or a theoretical thematic analysis. We have performed a mainly inductive thematic analysis within the theoretical framing of viewing school as an institution where national tests are one part. In this paper we connect to the concept of institution in the discussion whereas the actual analysis was the interplay between the aim of the study and different phases of a thematic analysis, such as familiarizing with data, searching for themes, and defining and naming themes. Another decision, drawing on Braun and Clarke (2006), was that we adopted a semantic approach where "the themes are identified within the explicit or surface meanings of the data, and the analyst is not looking for anything beyond what a participant has said or what has been written" (p. 84).

Second language learning students opportunities to take the National Test

In the analysis, themes of teacher comments on second language learners in relation to the national test were construed. Three of these themes are outlined below. We also describe what is written in the teacher instruction in relation to the themes.

1. Second language learners did not get the opportunity to take the test:
One theme was teachers writing that some second language learners were excluded from taking the test. One justification could be that "Pupils attending the preparatory class did not take the test" or that "newly arrived pupils did not take the test". The reason for this was that the teachers saw language issues as impediments to the student taking the test. Since this was a test in mathematics and not Swedish, this would not have to be a reason for excluding students from taking the test. In the instructions for the test, there was advice for how to enable all students to take part in the test, for example that items could be explained or translated for students as long as the mathematics that was tested was not revealed. It was also clear from the instructions that students had the opportunity to display mathematical knowing in a variety of forms of expressions.

2. Some students did take the test but made low results due to language issues.
A second theme was that teachers wrote about how students' results were low due to language issues. Reasons that teachers mentioned were: "limited vocabulary"; "students' lack of comprehension"; and "did not understand Swedish". This theme can be related to the same part of the teacher instructions as described above and the heading for this information was Adaption of the test.

3. Second language learners get help in various ways when taking the test.
The third theme is more in line with the information in the teacher instructions on how to adapt the test for students in need. In this theme the teachers described how they and colleagues went about to adapt the test for second language students. It could be more frequent teacher-student interaction: "Someone reading for the student" or "Smaller group instruction". It could also be support from teachers with other competence than the regular mathematics teacher: "Mother tongue teacher in Arabic did translations" or "Swedish second language learning pupils did test with Swedish second language teachers". Some teachers described the adaptation as being about facilitating the language: "Minimizes texts – he/she is immigrant student", "SLL students received simplified words"; and

"SLL teacher explained words".

Concluding discussion

The findings indicate that there were schools where the teachers worked in line with the instructions of the test and, therefore, adapted the administration of the test to enable SLL students better opportunities to display knowing in mathematics. This is coherent with the view expressed in policy documents described in the introduction. There were also schools where the teachers did not write about how to adapt the test administration but rather justified the exclusion of the students from the test or explained SLL students' poor results due to language matters. In these schools the students were not invited to display mathematics knowing in the same way as the other students. Research has shown the items in National Tests to be more creative that teachers' own test (Boesen, 2006) so this could, in fact, be a true limitation for the SLL students in these schools.

In order to try to understand these findings we draw on institutional theory (Douglas, 1986). From this point of view the acts of teachers are seen as part of a broader institutional context with different frames and discourses. The teachers that tried to adapt the test taking for the SLL students acted according to the frames in the form of test instructions and this was also in agreement with the dominating discourse in policy documents. The other teachers acted according to other discourses which previous research has revealed (e.g. Moschkovich, 2007). Here the test taking is fair if all students are doing the test in the same way, and in such a discourse adaptions do not come into question (see Norén, 2011). Another institutional aspect refers to framings in terms of number of teachers and the possibilities to actually help students according to the guidelines in the teacher information. In schools where the National Test is made important not only for the teachers in the grade where the test is taken, but for all, there could be possibilities for allocating more teachers to the classes taking the test during the test period. In schools where this is not the case, the teachers may not have any opportunity to help SLL students in the same spirit as theme 3. In these processes, the heads of the schools are important.

REFERENCES

Boesen, J. (2006). *Assessing mathematical creativity: Comparing national and teacher-made tests, explaining differences and examining impact (*Doctoral dissertation). Umeå university, Umeå, Sweden

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*, 77-101.

Douglas, M. (1986). *How Institutions Think*. Syracuse, N.Y,: Syracuse university press.

Hansson, Å. (2011). *Ansvar för matematiklärande.* Diss. Göteborgs universitet.

Holland, D., Lachicotte Jr, W., Skinner, D. & Cain, C. (2003). Identity and Agency in Cultural Worlds. London, Cambridge: Harvard University Press.

Klapp Lekholm, A. (2008). *Grades and grade assignment: effect of students and school characteristics.* Diss. Göteborgs universitet.

León Rosales, R. (2010). *Vid framtidens hitersta gräns: Om maskulina elevpositioner i en multietnisk skola*. Botkyrka: Mångkulturellt Centrum.

Moschkovich, J. (2002). A Situated and Sociocultural Perspective on Bilingual Mathematics Learners. *Mathematical Thinking and Learning, Special Issue on "Diversity, Equity, and Matematical Learning"*, 189-212.

Norén, E. (2011). 30 Grade- eight students: Discourse switch and bilingual students solving text problems in mathematics. In M. Setati, T. Nkambule & L. Goosen (Eds.), *ICMI Study 21*

Conference: Mathematics Education and Language Diversity 16-20 September 2011, (pp. 292-300). Sao Paulo, Brazil.

Parszyk, I.-M. (1999). *En skola för andra: Minoritetselevers upplevelser av arbets- och livsvillkor i grundskolan*. [A school for others. Minority students' experianeces of work and life conditions in compulsory school] Stockholm: HLS förlag, Studies in Educational Science.

Selander, S. (2008). Designs for learning – A theoretical perspective. *Designs for Learning, 1* (1).

Skolöverstyrelsen (1962). Läroplan för grundskolan [Curriculum for compulsory school]. Stockholm: SÖ-förlaget, Skolöverstyrelsen.

Skolverket (2013*). PISA 2012 - 15-åringars kunskaper i matematik, läsförståelse och naturvetenskap: Rapport 398.* Stockholm: Skolverket.

SOU 2004: 47. Förtroendekommissionen. *Näringslivet och förtroendet. Betänkande.*
Stockholm: Fritze.

Stathopoulou, C., & Kalabasis, F. (2007). Language and Culture in Mathematics education: Reflections on Observing a Romany Class in a Greek School. *Educational Studies in Mathematics 64(*2), 231-238.

Tallberg Broman, I., Rubinstein Reich, L., & Hägerström, J. (2002). *Likvärdighet i en skola för alla. Historisk bakgrund och kritisk granskning.* Stockholm: Skolverket.

Utbildningsdepartementet [Ministry of Education]. (2011). Skolförorning [School ordinance]. SFS 2011:185.

Teaching Mathematics To Students With Severe Intellectual Disability: An Action Research

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Abstract: The present study is part of an educational action research, in which sociocultural factors that affect teaching and learning mathematics of three students with severe intellectual disability are explored. In this study, factors related to family and school environments are further presented. During the first cycle of the research, the teacher-researcher acquires a holistic view of the students' interests, needs and everyday experiences (funds of knowledge, etc.). As a result of the interaction with the students' mothers, the emphasis is placed on home-school collaboration and its potential to promote students' active involvement during grocery shopping and money dealings. The task-based approach used includes simulation, video modeling and community-based instruction. Students appeared eager to repeat grocery shopping and their mothers are motivated to allow students become active participants while making purchases.

Résumé: L'étude ci-présente fait partie d'une recherche-action éducative, où nous explorons les facteurs socioculturels qui influencent l'enseignement et l'apprentissage des mathématiques de trois étudiants ayant déficience intellectuelle. Dans cette étude, il est question d'une présentation plus poussée des facteurs liés à l'environnement familial et scolaire. Pendant la première étape de la recherche, l'enseignant-chercheur acquiert une vue holistique des intérêts des élèves, des besoins et des experiences quotidiennes (fonds de connaissances, etc.). En raison de l'interaction avec les étudiants de mères, l'accent est mis sur la collaboration entre la famille et l'école et sur son pouvoir d'inciter les étudiants à participer activement aux courses et aux transactions monétaires. L'approche appuyée sur les tâches inclut la simulation, la modélisation vidéo et l'instruction au sein de la communauté. Les étudiants ont apparu désireux de répéter les courses et leurs mères sont motivées pour permettre aux élèves de devenir des participants actifs tout en faisant des achats.

Introduction

During the last decades several researches have focused on teaching mathematics to students with intellectual disability. Researchers tend to use experimental or quasi-experimental designs (Browder & Grasso, 1999; Butler, Miller, & Lee, 2001). As Porter and Lacey have argued (2005: 49), this is based on a rather "controlled context" and does not reflect the complexity of real-world classroom settings.

On the other hand, research in mathematics learning has shifted from mastering a predetermined body of knowledge and procedures, to a more sociocultural approach (Goos, Galbraith, & Renshaw, 2004). On this basis, mathematics activities must be meaningful to the students and help them make sense of real world. This is vital for individuals with severe developmental disabilities, because many of them are not adequately prepared to live and work in their community (Xin, Grasso, Dipipi-Hoy, & Jitendra, 2005).

Research Setting

The research methodology is located within action research and, more particularly, falls into the "reflective practitioner" category (Schön, 1983). According to the participants in the National Invitational Seminar on Action Research held at Deakin University in May 1981 *"educational action research is a term used to describe a family of activities in curriculum development, professional development, school improvement programs, and systems planning and policy development, These activities have in common the identification of strategies of planned action which are implemented, and then systematically submitted to observation, reflection and change.*

Participants in the action being considered are integrally involved in all of these activities." (Carr & Kemmis, 2004: 164).

The first writer is the teacher-researcher, while the second writer and two more university teachers are the critical friends. The teacher-researcher teaches mathematics to the nineteen students (aged 16 to 36 years old) of a secondary special school located in a rural area of Greece. The participants of the research are three students with severe intellectual disability (Katerina-16 years old, Vangelis-24 years old, Anastasia-29 years old), who constitute one of the six classes of the school. In the research participate, also, their mothers, since these are the persons of the family more involved in students' education and caring.

While Vangelis is able to count up to thirty-nine and Katerina up to twenty-seven, Anastasia can count up to ten by rote. They can write and recognize numbers up to ten, but they can't compare numbers at the abstract level (i.e., without using objects or drawings) (Butler et al., 2001). They have difficulty maintaining one-to-one correspondence while counting a collection of up to ten objects, but they can recognize that the last number word used to count a collection has special significance because it represents the total number of objects. Finally, they are able to mentally determine sums up to three (i.e., 1+1, 2+1).

The project proceeds through a spiral of cycles (plan-act-observe-reflect). The main purpose of the research is to study the sociocultural factors that affect teaching and learning mathematics of students with intellectual disability within real-world settings. In this study, factors related to family and school environments are further presented. The data collection methods rely mainly on observations and interviews. The teacher-researcher keeps a self-reflexive research diary, which contains observations inside and outside of the classroom, thoughts, ideas, explanations and interpretations. Some lessons, the interviews of the students' mothers and the meetings at the end of each cycle with the critical friends, are tape-recorded and analysed by the teacher-researcher.

Findings from the First Cycle

The general idea that the teacher-researcher identified at the beginning of the research is that there is little connection between school mathematics and students' interests and everyday math experiences. At the first cycle, the teacher-researcher realized that math skills were presented "decontextualized" (Bishop, 1988: 180) and isolated from students' everyday activities. This is exactly the opposite of what professionals call "criterion of ultimate functioning" (Browder & Cooper-Duffy, 2003). According to this, teachers should consider whether skills are functional (i.e., usable in daily life) and age appropriate (i.e., relevant to students' chronological age rather than developmental age) for students with severe developmental disabilities.

Since this is the first year the teacher-researcher teaches in this particular school, it was necessary to acquire a holistic view of the sociocultural factors that affect students' teaching and learning of mathematics. Interviews with mothers showed that they are rather protective with their children and provide them with limited chances of math experiences in everyday life. Both girls have been present many times during grocery shopping, but none mother reported active involvement of their daughters. On the other hand, Vangelis' mother reported that he has gone to the grocery store alone for several times. During this, Vangelis is able to recall two or three items and an employee at the grocery store would find the items, keep the money and put the change in a plastic bag. All mothers expressed the desire their children acquire purchasing skills necessary for everyday independent living. At the same time, it seemed rather difficult to believe that their children could be able to achieve independence on such skills and tended to lower their expectations.

Observations in the natural environment of the school and the classroom, in combination with interviewing students and their mothers, provided the teacher-researcher with a holistic view of children's needs and transformed the attention from the mathematics classroom to home-school collaboration as an essential role in students' learning mathematics.

During the Second Cycle

In order to help students acquire math skills necessary for everyday life, the students, their mothers and the teacher-researcher should work together toward common goals. All mothers agreed that they wanted their children to learn to go grocery shopping and make purchases. As a starting point, the teacher-researcher evaluated students' ability to go grocery shopping in a simulated setting. In the math classroom there is a corner with empty grocery items including milk, biscuits, toothpaste, etc., and a desk used as a counter where students pay money. Another student had the role of the shopkeeper and each of the three students was asked to role play grocery shopping. At the end of this activity it was remarkable to find out that none of the students had taken money before visiting the store. Also, Anastasia and Vangelis forgot to take the purchased items after leaving the store. At this point, the main teaching objective is students be able to go grocery shopping. The task-based approach used includes simulation in a classroom setting, video modeling and community-based instruction.

The teacher-researcher used a realistic and life-like scenario by informing the students that "the school principal has assigned them to buy what is necessary for the school Carnival party". All students started brainstorming and agreed to visit a local supermarket a day before the party. Firstly students had to make their own grocery list by cutting the desired pictures of items from leaflets (an adaptation used because students have difficulty reading or writing) (see Fig. 1). This activity was generalized with their mothers and making family's grocery list.

Figure 1. *Students make their grocery list*

For the next lesson, the teacher-researcher made a videotape of a model going grocery shopping in community setting. Video modeling was used as an instructional technique. This videotape was also given to the students' mothers in order to watch it, to interact with their children and to motivate them to allow students take an active role during grocery shopping. As an application of the video modeling instruction, the teacher-researcher asked the students to role play the steps of grocery shopping in simulated classroom setting. The teacher-researcher was the cashier and each student role played the customer.

The next step was community-based instruction, where all three students took their shopping list and a prespecified amount of money (a five euro note) (an adaptation used because students have difficulty giving a specific amount of money) and went grocery shopping. At the grocery store each student fulfilled the shopping steps by having the necessary prompting from the teacher-researcher, according to their needs (see Fig. 2). The whole process was videotaped and these video recordings were used to make three different videotapes, one for each student's family.

Figure 2. *Students go grocery shopping in a community setting*

Findings and Discussion

Through the task-based approach students were able to make meaningful connections between their everyday experiences and mathematics activities being engaged in classroom. All students kept their motivation during the whole teaching process and asked the teacher-researcher to repeat the grocery shopping activity. On the other hand, mothers reported that it was the first time their children participated in such activities with the school and they are very pleased and thankful. After families watched the videotapes, mothers stated very proud of their children. The students became the “star” of the family for a few minutes and they described the moment with happiness.

This process managed to make a step towards social inclusion of the three students. Since practicing in the natural environment where skills will be used is essential, the teacher-researcher motivated mothers to engage their children during the grocery shopping process. Students could make their own grocery list and shop independently while mothers could do their grocery shopping. The girls’ mothers reported that after watching the videotape they engaged their daughters during grocery shopping for more than one times. Vangelis’ mother gave him the opportunity to find the location of two items in the grocery store, but did not engage him fully at the shopping procedure. The teacher-researcher explained that it is necessary for Vangelis to participate during the whole process and manage to fulfill independently all the steps. Both Vangelis and his mother were motivated and promised to try it next time.

Although the three students may lack prerequisite skills needed to make purchases, such as counting money or making change, it is important to give them the opportunity to fully participate in the grocery shopping procedure (Mechling, Gast, & Bartholde, 2003). The research is still in progress and next steps include teaching purchase skills to the students by using the next dollar strategy (Browder & Grasso, 1999). Therefore, intense home-school collaboration is important to ensure that the outcome will be relevant to the students’ family routines (Westling, 1996).

REFERENCES

Bishop, A. J. (1988). Mathematics education in its cultural context. *Educational Studies in Mathematics, 19*(2), 179-191.

Browder, D. M., & Cooper-Duffy, K. (2003). Evidence-based practices for students with severe disabilities and the requirement for accountability in “No child left behind”. *The Journal of Special Education, 37*(3), 157-163.

Browder, D. M., & Grasso, E. (1999). Teaching money skills to individuals with mental retardation: A literature review with practical applications. *Remedial and*

Special Education, 20(5), 297-308.

Butler, F. M., Miller, S. P., Lee, K., & Pierce, T. (2001). Teaching mathematics to students with mild-to-moderate mental retardation: A review of the literature. *Mental Retardation, 39*(1), 20-31.

Carr, W., & Kemmis, S. (2004). *Becoming critical: Education, knowledge and action research.* London: Taylor & Francis e-Library.

Goos, M., Galbraith, P., & Renshaw, P. (2004). Establishing a community of practice in a secondary mathematics classroom. In B. Allen & S. Johnston-Wilder (Eds.), *Mathematics education: Exploring the culture of learning* (pp. 91-116). London: RoutledgeFalmer.

Mechling, L. C., Gast, D. L., & Bartholde, S. (2003). Multimedia computer-based instruction to teach students with moderate intellectual disabilities to use a debit card to make purchases. *Exceptionality, 11*(4), 239-254.

Porter, J., & Lacey, P. (2005). *Researching learning difficulties: A guide for practitioners.* London: Paul Chapman.

Schön, D. A. (1983). *The reflective practitioner: How professionals think in action.* New York: Basic Books.

Xin, Y. P., Grasso, E., DiPipi-Hoy, C. M., & Jitendra, A. (2005). The effects of purchasing skill instruction for individuals with developmental disabilities: A meta-analysis. *Exceptional Children, 71*(4), 379–400.

Westling, D. L. (1996). What do parents of children with moderate and severe mental disabilities want? *Education and Training in Mental Retardation and Developmental Disabilities, 3*, 86-114.

Can we learn from "outside"? A dialogue with a Chinese teacher: the "two basics" as a meaningful approach to mathematics teaching

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Abstract: Since many years Confucian heritage students (Chinese ones in particular), acquire leading positions in numerous international scientific programmes and display excellent performance in international assessments as PISA or TIMMS (OECD, 2013). To understand the "reasons" of this excellence we tried to explore some aspects of the cultural background of teaching practices and classroom life in those countries. With this aim a Chinese teacher was interviewed; we asked him about principles, values and beliefs and their impact on teaching/learning Math in classroom. The paper discusses what emerged from this dialogue and in particular from the idea of the "Two Basics" mathematics teaching approach, typical for the Chinese educational context. Furthermore, this work tries to underline (in a implicit or explicit way) similarities and differences between East and West didactical approaches and to define a sort of integration of these in order to improve a better mathematics education for all students.

Résumé: Depuis de nombreuses années, les élèves de l'héritage confucéen (les chinois en particulier) atteignent des positions de leader dans de nombreux programmes scientifiques internationaux et obtiennent d'excellentes performances dans les évaluations internationales PISA ou TIMMS (OECD, 2013).
Pour comprendre les «raisons» de cette excellence, nous avons essayé d'explorer certaines caractéristiques du contexte culturel des pratiques d'enseignement et de la vie scolaire dans ces pays.
Avec cet objectif, nous avons interviewé un professeur chinois; nous lui avons demandé à propos des principes, des valeurs et des croyances et du leur impact sur l'enseignement /apprentissage des mathématiques en classe. L'article traite de ce qui a ressorti de ce dialogue et en particulier de l'idée des "Deux Principes", l'approche à l'enseignement des mathématiques typique du contexte éducatif chinois.
En outre, ce travail veut mettre en évidence (de manière implicite ou explicite) similitudes et différences entre les approches didactiques de l'Est et de l'Ouest, et veut définir une sorte d'intégration entre eux dans le but de parvenir à un meilleur enseignement des mathématiques pour tous les élèves.

Cultural aspects related to Chinese mathematics educational context: an overview of possible values and principles

A general overview of the cultural aspects that could be the linked to education in China is related to the social status of school and the value that School as institution has for teachers, students and families. In this area of remarkable economic development, with the connected social changes, school in fact is seen as an instrument of social promotion. Many expectations of families and students are pinned on the result in English and mathematics. So students study very seriously till late. In the Confucian view school has to be difficult. Teachers must assign "many" homework and require great commitments. The authority of teachers is not discussed and the confidence of students is granted.

According to Zhang et alii (2004), the didactic aspects related to this and the implication on the way of teaching/learning in classroom could be summarised in five points:

- *Rituals:* in Chinese, Korean and Japanese school, lessons follow a repetitive pattern that always includes welcoming the teacher, summary and review of past lessons, various other activities depending on the grade level, and a final greeting. This habitual structure is heartening for students and makes the lesson a separate space in their lives, where they can concentrate and where specific rules apply.
- *Progressivity:* a good part of the lesson is devoted to summary and review of past lessons. Then you can add some new contents. Knowledge is built with steady rhythm, step by step, with great consistency.
- *Rote learning strategies and significance:* memorization, repetition, repetitive exercises are fundamental in teaching practices in these countries, but it seems that students develop considerable metacognitive skills, which appears like a paradox . In *Chéng chú tōng Bian běnmò* (乘除 通 变 本末 "Full Explanation on multiplication and division", 1274) the Chinese mathematician *Yang Hui* (杨辉 1238-1298) wrote a list of important skills and knowledge for students, indicating timetables and ways to learn: 260 days in all. According to him, it takes a day to figure out a method (an algorithm or a procedure) and you have to work on it for two months to seize its meaning and applications without forgetting. But, for example, once you understand the method of addition (加法 jiāfǎ), studying subtraction (减 jiǎn) you can reverse all the exercises made by adding the differences to subtrahends. This gives the method its "origins" and reduces the time of study only 5 days. So understanding significance is the target.
- *Relational approach:* in Chinese or Japanese textbooks whenever you find a rule, you find nearby the reverse one; whenever you learn an operation, you learn the reverse one; whenever you see an identity, like in polynomial algebra, you find the same written in reverse form. So you learn together addition and subtraction, polynomial algebra and factorization, and so on… That means that you are not just learning some object and procedure, but their mathematical relations too.
- *Visualization:* schemas, pictures, drawings, blackboard sketch, magnets, stickers, page layouts, graphic arrangement in books, animations and screens, all are strongly used by eastern teachers to represent, organize, and explain mathematical properties and objects at school. Visualization tradition is still alive. Students remember images that mean concepts, not especially words.

In particular, about Mathematics, a cultivated person in China has to know many mathematics and a clever person can do mental calculations quickly and accurately. Mathematics as a subject has big space in school timetable.

If we look at the history of the Discipline, Chinese mathematics has been so important that one of foundational myths of the country is related to magic squares (幻方 *huàn fāng,* the myth is the *Luo river magic square* 洛書 *luò shū*). Mathematics is up to now one of the traditional *Six Arts* (六藝 *liù yì*) required in young lords education and played a strategic role in imperial bureaucrats curriculum and examinations (Nicosia, 2010).

Due to historical reasons, in many Eastern countries, including Japan, Korea, Singapore, and China (Mainland China, Taiwan, and Hong Kong), and even Russia, mathematics educators emphasize the importance of foundational training more than is usually seen in the West.

In all the Confucian heritages the didactical consequence of this approach is the principle of the "two basics", that is the most typical teaching framework in all these eastern countries (Zhang, 2004).

In China the "two basics" principle (basic knowledge and basic skills) in mathematics teaching is a broad and loose concept without a strict definition. According to Zhang (2004), its general meaning is that, between "solid foundation" and "individual development and creativity", although both are important, foundation is the more important. As a Chinese proverb declares: "although a tower is beautiful its groundwork is more important".

With the aim to better understand this important principle hidden in Chinese teaching framework, and its possible link to mathematical competences acquired by Chinese students (as evidenced in international competitions), we met Lai in Guangzhou last year and we interviewed him. With his help we tried to define a general framework of cultural background of teaching practices and classroom life in China. Furthermore, according to his point of view, we tried to define a possible link between these cultural situations, these social values and the excellent training that students showed in international assessments as PISA or TIMMS (OECD, 2013).

The "two basics": solid foundation and individual development. A dialogue with a Chinese teacher

Lai is a Chinese teacher; he lives/teaches Mathematics in a Upper Secondary School in Guangzhou (廣州) that it is the biggest city in the south of China in the region of Guangdong. He's teaching since 1978 and he is considering an expert Professor in his school. We met him last year in May, we spent 3 month with him observing his typical way of teaching and its impact in class. After each lesson we discussed with him exploring what happened in class and why.

This paper discusses what emerged from an informal dialogue with him about social and cultural aspects related to the Chinese school and their impact of the teaching/learning phase in mathematics. This dialogue was registered after the first Lai Math lesson at his Upper Secondary School.

In particular we report what we discussed and what Lai put in evidence about the "two basics" principle and its value for the Chinese teaching approaches and its characteristic.

Researcher: *What do you mean with Two basics?*
Lai: *Two basics means "Basic Knowledge" and "Basic Skills". According to the Chinese prospective, skills can be developed into knowledge and knowledge can induce skills. So … knowing and practicing. In fact, skills and knowledge overlap at some examples. You have to understand that there are two kinds of knowledge, one is direct and it is possible to acquire it through direct exploration and investigation. The second kind of knowledge is indirect knowledge. It came from imitation, repetition, memorization … We have two basics that are the slabs, these slabs connected together by ability. The possible success of problem solving in Math but also in the other subject depends on whether such slabs are connected.*
Researcher: *What do you mean with connection?*
Lai: *Connection of pieces means connection of knowledge. The quality of foundation depends on number of reinforcement and also the quality of the concrete activities … so we can say that … the slabs are joined together.*

Trying to better explain his idea, he had drawn the Model of learning cycles in the "two basics" as follow:

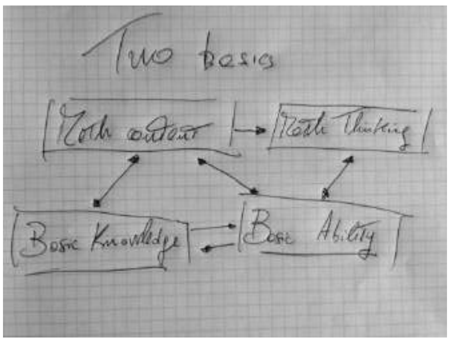

Fig.1 Model of learning cycles in the "two basics" by Lai

After that Lai explains the drawing:
Lai: *If we think to a mathematical content, we can summarise the process in this way. I hope to be clear for you. The first step is imitation, with teacher's guidance.*

Researcher: *Imitation? What do you mean?*
Lai: *Imitation means students need to observe and internalize, personalize. Ok?*
Researcher: *Yes but in which way?*
Lai: *it depends on teacher and students … The second step in fact is the intervention by teacher. In this phase we have criticism, correction of concepts, strong examples for the concept, and a brief summary of knowledge learned in a trunk and practice.*
Research: *But how to avoid having practice become mechanical repetition?*
Lai: *because … yes … only in this way we can have a real understanding by students. The two basics provide the following principles that are the base of a significant learning:*
1. *memory leads to recognition and become intuition,*
2. *good speed of operation provides grounds for efficient thinking,*
3. *using deduction and reasoning to sustain precise logical thinking,*
4. *rising of standard through variation of problems and learning process.*

Lai: The third step is then abstraction process by students, which is a kind of internalization and self monitoring. By internalization, students with the help of teachers can connect different knowledge, and deduce new knowledge.

Researcher: *And what about the internalization and self monitoring? How can facilitate it?*
Lai: it depends on the teacher. On the examples discussed with students, the time allocated to this activities … It is difficult, I tried to summarize. There are two levels of teaching. The first one is using daily lives context aimed to capture the interest of students. The second level is the learning process through abstraction. Teachers teach mathematics based on the process of "correspondence, variation, induction and deduction".

After a brief pause linked to the difficulties of the dialogue Lai argues:

Lai: *Correspondence means to map the concept of mathematical problems to another problem with variation*[7]*; it also involves relations of expression, structure, meaning … We will discuss more about it. It is complex. Tomorrow I will show you some practices in classroom. These practices refer to the "routine" daily practice commonly accepted by Chinese teachers.*

The model of learning cycles in the "two basics" applied to variation problems: an example form Primary School

To better discusses his idea of variation and trying to generalize this Chinese practice Lai shws a mathematics book (published in Japan) for Primary grade and in particular discusses the exercise N.1 (pages 62 and 63) reported below:

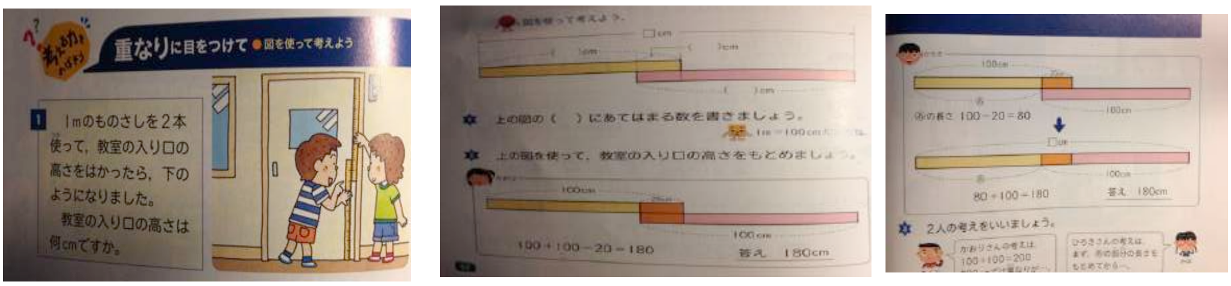

Fig.3 Use of variation problems in Primary textbook (pages 62 and 63)

According to Lai, the typical approach proposed by the text in this exercise is significant to "facilitate" the students strategy solution of the proposed problematic in order to work on their

[7] In this paper we don't discuss the typical Chinese variation approach to mathematical problem solving. Some interesting didactical considerations on this subject are in Bartolini Bussi, Sun, Ramploud, (2013); Spagnolo, Di Paola (2010); Di Paola, Ramploud (2013).

memorization procedure leads to recognition and intuition; to favour their efficient and logical thinking and to generalize through variation of problems and learning process.

Lai: *In general in a mathematics book it is easy to note how the use of schemes (as the ones proposed in this exercise) is totally linked to a didactics focused on the research of solving methods, rather than on the presentation of isolated contents. This happen in many western book. Do you agree?*
Researcher: *What do you mean?*
Lai: If you look for example to these pages (he refers to 1 pages 62 and 63) it is clear … when observing the associated schemes … the operation of addition and subtraction and the strong interdependence between them is clear, they are proposed in the same context, their use is evident for students. This approach is typical in Chinese curriculum. I think it is not typical in your country. The operation requested to students are linked to the idea of variation as I said before. All Chinese students are able to solve this problem, all are able to read it with different "lent" and to move inside the mathematics calculation requested by the interdependence between the operation of addition and subtraction together. The representation help also students to conceptualize this strongly relation and to generalize it.
…
Researcher: *And what about the model of* "two basics"? Why you shows me this exercise?
Lai: *Do you remember what I said on the" two basics" model? This is an example of connection of pieces of knowledge. I shows it because this is one of the mail point of the Chinese curriculum, an example of the philosophical idea of two basics to practical activities in classes trough the use of variation.*

...

Lai is right, if we look at a typical Italian textbook (we can generalize to the western textbook), the approach is different. The idea of variation is distant from the western culture, it is instead a key point of the curriculum of China and all the Confucian area for all the grades. It is possible to find some example just from the pre-school grades as showed below.

Fig.3 Use of variation problem in pre-school Chinese textbook

Conclusion

Can we learn from "outside"? Can we *contaminate* (Ramploud, Di Paola, 2013) our typical didactical approaches with something "different" in order to enrich it? We think yes.
The tables below tries to summarize what emerged from the dialogue with Lai and in particular discusses some possible key aspects stressed by Lai on the typology of mathematical activities that are typically propose in Chinese classroom and the teaching/learning time related to their implementation according to the two basics approach at School.

Mathematical activities	• Activities mainly individual or in a big group, • Learning by imitation: "small steps" teaching instead of free discover, • Great importance to mental calculation and memorization: "practice makes perfect", • Few importance to proofs, • Strong importance to the relationship between comprehension and manipulation, • Emphasis on the teaching of problem solution schemes.
Time	• Students are required to adapt to the pace of progress setted for most students by the teacher, • Teachers present the main mathematics contents as quickly as possible to avoid students spending too much time on winding paths.

Table 1. Typical didactical approaches in China

Is it possible to integrate in our western didactical practices some different activities and teaching proposals coming out from abroad (for example from China) in a sort of continuous dialogue from distances and cultures crossing? We think yes.
The different cultural background in teaching and learning hidden in the Easter context can, according to us, help us to better understand our own background and, on the other side, be pertinent, for example, to better understand the excellent performance in international assessments of Confucian area.
To study similarities and differences between our western typical teaching approaches and the Chinese one could represent for teacher and researchers only the starting point to adapt possible new "multicultural approaches" in own classes.

REFERENCES

Bartolini Bussi, M.G., Martignone, F. (2013). Cultural issues in the communication of research on Mathematics Education. *For the Learning of Mathematics*, Vol. 33, pp. 2-8.

Bartolini Bussi, M.G., Sun, X., Ramploud, A. (2013). A dialogue between cultures about task design for primary school. In C. Margolinas (ed.) *Proceedings of ICMI Study 22*, Task Design in Mathematics Education. (Vol. 1) 551-559. Oxford.

Cai, J., Knuth, D. (eds.) (2011). *Early Algebraization: A Global Dialogue from Multiple Perspectives*. Springer.

Di Paola, B. (in press). Teaching/learning mathematics in multicultural class: a new challenge for researchers, teachers and students. A case study in Italian classroom with Chinese immigrated students, *Proceedings of CTRAS Conference,* Long Beach, California, USA, 2015.

Nicosia, G.G. (2010). *Cinesi, Scuola e Matematica*, Bologna, Italia

Ramploud, A.; Di Paola, B. (2013). The Chinese perspective of variation to rethink the Italian approach to word-problems from a pre-algebraic point of view. In Fazio, C. (ed.), *Proceedings of CIEAEM 65*. Torino.

Spagnolo, F., Di Paola, B. (2010). *European and Chinese cognitive styles and their impact in teaching mathematics*. Berlín, Springer-Verlag.

Sun, X. (2011). An Insider’s Perspective: “Variation Problems” and Their Cultural Grounds in Chinese Curriculum Practice. *Journal of Mathematics Education*, Vol. 4, No. 1, pp. 101-114.

Zhang, Dianzhou and Dai, Zaiping. (2004). The “Two Basics” mathematics teaching approach and open ended problem solving in China, *Proceedings ICME 10*.

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Mastering Mathematics, Mainstream and Minority Languages

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Abstract: The multicultural nature of modern society constitutes one of the most significant changes to have influenced schools in many European countries, especially at primary and middle school level. The teacher is seldom aware of the need to rethink and if necessary modify his/her methodological and pedagogical approach. This attitude is even more evident in maths teachers who often consider their subject universal and culture-free.

Little has been done in Europe as far as maths teaching in multicultural contexts is concerned. The different languages and cultures present in the classroom make the teaching/learning process even more arduous than it already is, especially for pupils from minority cultures and/or with a migrant background or for Roma pupils.

A teaching unit, designed in a European Commission funded project, is described. Its aim is to provide teachers with a tool to help their pupils to overcome the learning obstacle represented by the contrast between the simplicity of classroom language and the complexity of mathematics language. Teachers have to bear in mind, however, that the language used in class is an element of further complexity for pupils from minority cultures with a different mother tongue.

Résumé: La nature multiculturelle de la société moderne constitue l'un des changements les plus importants à avoir influencé les écoles dans de nombreux pays européens, notamment au niveau de l'école primaire et du collège. L'enseignant est rarement conscients de la nécessité de repenser et si nécessaire modifier sa/son approche méthodologique et pédagogique. Cette attitude est encore plus évident dans professeurs de mathématiques qui considèrent souvent leur sujet universel et sans culture.

Peu a été fait en Europe dans la mesure où l'enseignement des mathématiques dans des contextes multiculturels est concerné. Les différentes langues et cultures présentes dans la salle de classe font le processus d'enseignement/apprentissage encore plus ardue qu'elle ne l'est déjà, en particulier pour les élèves issus de minorités culturelles et/ou issus de l'immigration ou pour les élèves roms.

Une unité d'enseignement, conçu dans un projet financé par la Commission européenne, est décrite. Son objectif est de fournir aux enseignants un outil pour aider leurs élèves à surmonter l'obstacle de l'apprentissage représenté par le contraste entre la simplicité de la langue en classe et la complexité du langage des mathématiques. Les enseignants doivent garder à l'esprit, cependant, que le langage utilisé en classe est un élément de complexité supplémentaire pour les élèves issus de minorités culturelles ayant une langue maternelle différente.

Rationale

Mathematics teachers feel the necessity for training and materials which reflect the needs of their classes in terms of linguistic and cultural differences. Their pupils from minority cultures and/or those with a migrant background encounter even more difficulties than their native classmates in acquiring fundamental mathematics skills.

The above mentioned needs have been identified in several research studies carried out as to multicultural and inclusive education ([1], [4]), the role of the foreign language in mathematics learning ([2], [5], [7]) and the educational approach and methodologies for mathematics education in multicultural classrooms ([3], [6]).

The *M³EaL project* [·] aims to identify teaching strategies for teachers and activities for pupils who

[·] This paper is part of the dissemination activities of the Project *Multiculturalism, Migration, Mathematics Education* allow both to approach the challenges and facing them satisfactorily. The methodological tools used, to be considered innovative compared with the standard routine of the mathematics classroom, are the following:

- Great attention to the language used in order to provide suitable compromise between the simplicity of classroom language and the complexity of mathematics language, bearing in mind, however, that the language used in class is an element of further complexity for pupils from minority cultures with a different mother tongue;
- Proposals for didactic units for the mathematics classroom which facilitate interdisciplinary extensions and which are inspired, above all, by practical problems and situations from everyday life and from different cultures.

These methodological tools should, in general, help to make all pupils more interested and motivated to learn mathematics; in particular, enable pupils with different cultures and languages to overcome some of the difficulties they encounter in maths due to these very differences: the teaching of mathematics by using aids to activate different thought processes and skills which otherwise risk remaining latent because of language shortcomings.

Moreover, the above-mentioned methodological tools facilitate the appreciation of the positive aspects of different cultures and create favourable conditions for intercultural dialogue in the classroom, thus creating an inclusive educational setting.

A further innovative aspect is the contribution from language specialists to the communication and intercultural issues of the teacher training activities.

A teaching unit from M³EaL project

The teaching unit aims to provide teachers with a tool to help their pupils to overcome the learning obstacle represented by the contrast between the simplicity of classroom language and the complexity of mathematics language, bearing in mind, however, that the language used in class is an element of further complexity for pupils from minority cultures with a different mother tongue.

The teaching unit has been designed by the *M³EaL project* coordinator Institution, Centro interdipartimentale per l'Aggiornamento, Formazione e Ricerca Educativa – C.A.F.R.E. of the University of Pisa (Italy), and already piloted in the project participating schools selected by CAFRE and two further project partners: Ecole Supérieure du Professorat et de l'Education – E.S.P.E. of the University of Paris-Est Créteil (France) and the University of Thessaly (Greece).

Its primary target group are mathematics teachers in primary and lower secondary schools in socio-culturally diverse areas, the secondary target group consequently consisting of students from cultural minorities and/or culturally deprived groups.

The educational aims of the teaching unit can be roughly divided into general and mathematical aims.

Among the *general aims* we can consider:

- The appreciation of the positive aspects of different cultures.
- The creation of favourable conditions for intercultural dialogue in the classroom, and, therefore, an inclusive educational setting.

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Project partners: University of Vienna, Austria – Charles University in Prague, Czech Republic – University of Paris-Est, Créteil, France – University of Thessaly, Greece – University of Pisa, Italy – University of Siena, Italy – University of Agder, Norway

Project website: http://m3eal.dm.unipi.it

- The development of awareness and critical attitudes towards the use of language and its interpretation.
- The awareness of how important it is to use specific and unambiguous language.
- The capacity to state the reason for the choices made and used during the activity.

Among the *mathematical aims* we can consider:

- The increase of the learners' capacity to understand and to elaborate the mathematical discourse.
- The improvement of the ability of reading and understanding mathematics textbooks and word problems.
- The improved usage of mathematical language.
- The reinforcement of the knowledge of mathematical glossary.
- The development of the ability to find a proper balance between the natural language and the mathematical language.

The teaching unit should lead teachers to reflect upon a number of aspects:

- Difficulty in using mathematical language correctly: uncertainties, doubts and mistakes shown in understanding the written texts express the need to favour, in teaching, the verbalisation process, which induces students not only to make their ideas explicit but also to try and make it in a clear and correct way to make them understood.
- The need to use the linguistic instrument appropriately, its use is a fundamental step towards the construction of knowledge, although it requires a considerable time for maturation.
- The need to develop activities like this one, because they offer information about pupils' knowledge, the conceptualisation level they have reached, possible gaps, and misconceptions. This information is fundamental to be able to intervene in the classroom with appropriate and well planned teaching actions.

The teaching unit consists of five main activities. If possible, all the activities should be carried out in small groups, each of which including a minority pupil at least.

- **Analysis of a textbook** (*Reading* and *Writing*)

Pupils are asked to read a chapter of their textbook and, thereafter, to search for and make a list of words and verbs in the vehicular language that are "difficult", discuss about their meaning and translate them into the foreign languages spoken in the classroom, thus producing a *micro-dictionary*.

Pupils are then asked to search for and make a list of words and verbs that are proper of the mathematical language, compare them to the same words and verbs in the natural language, discuss about and write their possible different meaning and translate the words and verbs into the foreign languages spoken in the classroom, thus producing a *mathematics glossary* and a *mathematics dictionary*.

All groups are asked to re-write the analysed pages of the textbook in the vehicular language and minority pupils are asked to translate the most significant sentences into their own mother tongues.

- **Analysis of a "word problem" from a National standard assessment test** (*Reading* and *Writing*)

The teacher chooses a "word problem" from a National standard assessment test that is meaningful as to the language used. Pupils are then given the same tasks as in the first activity.

- **Natural language and mathematics language**

Pupils are asked to identify possible conflicts originated by different meaning of words and verbs that are common to both the natural and mathematical languages, and to write the two different meaning in their own mother tongues.

- **Writing a "word problem"**

Pupils, still working in groups, are asked to write in the vehicular language a word problem. The problems are presented to the whole class for discussion about their clarity as to the language used and the mathematical notions required. Greater attention is paid to minority students.

- **"Writing a textbook"**

Students, still working in groups, are asked to write in the vehicular language a "page of a textbook" about a mathematical topic chosen by the teacher. The "pages" are presented to the whole class for discussion about their clarity as to the language used and the mathematical notions involved. Greater attention is paid to minority students.

REFERENCES

Abreu, G. (1998). The mathematics learning in sociocultural contexts: the mediating role of social valorisation, *Learning and Instruction*, 8(6), 567-572.

Barwell, R., Clarkson, P., Planas, N., & Setati, M. (2011). Researching the teaching and learning of mathematics in multilingual classrooms. In *Proceedings of the 33rd Psychology of Mathematics Education Conference*. Thessaloniki, Greece.

Bergsten, C., Grevholm, B., & Favilli, F. (2009). Learning to Teach Mathematics: Expanding the Role of Practicum as an Integrated Part of a Teacher Education Programme. In *The Professional Education and Development of Teachers of Mathematics - Study 15 of the International Commission for Mathematics Instruction of the International Mathematics Union* (pp. 57-70). New York, NY: Springer.

Bishop, A. J. (1988). Mathematics Education in its cultural context, *Educational Studies in Mathematics*, 19 (pp. 179-191).

Ellerton, N.F., & Clarkson, P.C. (1996). Language Factors in Mathematics Teaching and Learning. In Bishop A.J. et al. (eds.), *International Handbook of Mathematics Education* (pp. 987-1033). Dordrecht: Kluwer Academic Publishers.

Favilli, F. (2013). Globalization in mathematics education: integrating indigenous and academic knowledge. In *Proceedings of CIEAEM 65 – Mathematics Education in a Globalized Environment* (pp. 49-67). Retrieved from http://math.unipa.it/~grim/quaderno23_suppl_1.htm

Kaiser, G. (2003). Learning mathematics within the context of linguistic and cultural diversity – an empirical study. In *Proceedings of CERME 3*. Retrieved from http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG10/TG10_Kaiser_cerme3.pdf

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Rôle de l'histoire des mathématiques dans l'enseignement-apprentissage des mathématiques : le point de vue socioculturel

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Résumé: Plusieurs auteurs proposent des arguments appuyant la présence de l'histoire dans la classe de mathématiques. Leurs considérations théoriques entretiennent des positions variées, et parfois divergentes, concernant à la fois le statut ontologique et épistémologique du savoir mathématique, des fondements philosophiques de l'éducation mathématique et le statut même de l'histoire de la discipline. Ces positions teintent inextricablement la teneur et l'orientation de leurs arguments. Cela dit, il a été mainte fois question dans ce champ de recherche du véritable « fossé » entre études empiriques et spéculatives. Ainsi apparait le besoin de mettre en évidence les postures épistémologiques sous-jacentes aux différents discours théoriques afin de pouvoir mieux les opérationnaliser dans la recherche de terrain. Comme contribution, cette communication tente d'éclairer spécifiquement le point de vue socioculturel en éducation mathématique sur l'apport de l'histoire, et ce, à travers quelques éléments conceptuels de la théorie de l'objectivation.

Abstract: Several authors bring arguments supporting the presence of history of mathematics in mathematics classroom. Their theoretical considerations show various, and sometimes divergent, positions concerning; ontological and epistemological status of mathematical knowledge, philosophical principles for mathematics education and status of the history of the discipline itself. Of course, these positions taint inextricably the content and the orientation of their arguments. This being said, there is now a recurrent problem in the field of study concerning an important "gap" between empirical and speculative researches. In this problematic, the need to reveal the epistemological position of those different theoretical discourses becomes crucial. Indeed, this could lead to a better operationalization of the empirical research. This paper is an attempt to clarify specifically the sociocultural point of view on the potential of history for the learning of mathematics, and this, trough conceptual elements from the theory of objectivation.

Histoire et enseignement-apprentissage des mathématiques

Depuis plusieurs décennies, de nombreux penseurs, chercheurs et enseignants se sont penchés sur le « comment » et le « pourquoi » du recours à l'histoire des mathématiques dans la classe de mathématiques. Dès le début du 20e siècle, pédagogues, philosophes et mathématiciens s'y sont intéressés. Jusqu'à récemment, il semble que tous, enseignants et chercheurs, s'entendaient pour dire que l'histoire est bénéfique et se veut d'emblée un outil cognitif efficace et motivant dans l'apprentissage des mathématiques (Charbonneau 2006). Cet engouement a donné lieu à de nombreuses études concernant l'utilisation de l'histoire des mathématiques dans l'enseignement des mathématiques.

Cependant, le champ de recherche semble aujourd'hui essoufflé quant à la production et au design d'activités d'apprentissage, de situations problèmes ou de séquences d'enseignement. Les activités des chercheurs du milieu se déplacent dorénavant vers la recherche en termes de fondements didactiques et pédagogiques à partir desquels il serait possible de mieux penser le rôle de l'histoire des mathématiques dans l'enseignement-apprentissage des mathématiques (Kjeldsen, 2012). À ce titre, les travaux de Barbin (1997, 2006, 2012) situés dans la tradition de l'épistémologique historique française, ceux de Fried (2001, 2007, 2008) qui présente une posture humaniste et ceux de Jankvist (2009, 2010) en prises davantage pragmatiques sont de vibrants exemples

De plus, la nécessité d'articuler cette recherche spéculative aux études de terrain apparait cruciale dans ce champ de recherche. Dans un article mainte fois cité, et encore aujourd'hui fortement pertinent, Guliker et Blom (2001) mettaient déjà en évidence, au début des années 2000, le véritable « fossé » qui sépare les recherches de terrain et les recherches théoriques du domaine. En effet, on observe, d'une part, une recherche théorique importante fournissant des conceptualisations profondes, riches et fécondes et, d'autre part, une recherche empirique qui tente de « mettre à l'épreuve » le développement de certains outils d'introduction de l'histoire en classe de mathématiques sans prendre en compte et y articuler les développements théoriques du domaine. Ces deux formes de la recherche marchent côte à côte et ont peine à se stimuler et s'orienter mutuellement. Non seulement le manque d'études de terrain sérieuses se fait sentir, mais apparait aussi le besoin de mettre en évidence les postures épistémologiques sous-jacentes aux différents discours théoriques, afin de les voir mieux s'opérationnaliser dans la recherche de terrain.

Dans un effort allant dans ce sens, nous tenterons d'éclairer ici plus spécifiquement le point de vue socioculturel en éducation mathématique, et ce, à travers quelques éléments conceptuels de la théorie de l'objectivation, une théorie émergente en didactique des mathématiques (Radford, 2011, 2012, 2013).

Un point de vue socioculturel sur l'éducation mathématique

D'inspiration vygotskienne, la théorie de l'objectivation est une théorie socioculturelle contemporaine de l'enseignement-apprentissage qui plaide pour une conception non mentaliste de la pensée. S'opposant au courant rationaliste et idéaliste, elle propose une conception de la pensée que serait à la fois sensible et historique. D'une part, elle est sensible, car elle s'enracine dans le corps, les sens et l'affectivité, lesquels sont invoqués dans la saisie des objets de la réalité. Le corps, la perception, mais aussi les gestes et les signes sont donc considérés comme des parties constitutives de la pensée elle-même. D'autre part, elle se veut historique, car tout aussi enracinée dans l'histoire et la culture, étant perçue comme une forme sociale de réflexion et d'action historiquement constituée. On parlera ainsi de la pensée comme d'une *praxis cogitans* (Radford, 2011).

Concernant les objets mathématiques, la position ontologique de la théorie de l'objectivation s'éloigne du discours réaliste qui les considère comme indépendants de l'époque et de la culture, précédant l'activité humaine. L'attraction de cette ontologie réaliste réside dans son pouvoir explicatif du miracle de l'applicabilité des mathématiques au monde phénoménal. Cependant, les penseurs réalistes font un acte de foi en croyant que l'accès aux objets véritables est possible. La théorie de l'objectivation suggère plutôt que les objets mathématiques sont « générés historiquement au cours de l'activité mathématique par les individus » et constituent des « schèmes fixes d'activités réflexives incrustés dans le monde changeant de la pratique sociale » (*id*., p. 7). Comme tous les objets mathématiques, le concept de cercle par exemple est une réflexion du monde dans la forme de l'activité des individus. Cette dernière forme la racine génétique de l'objet conceptuel, laquelle renferme une dimension expressive variée qui se décline sous des aspects rationnels, esthétiques et fonctionnels liés à la culture.

Pour les penseurs du socioculturel, le problème théorique central est donc d'expliquer comment se réalise l'acquisition du savoir ainsi déposé dans la culture. Avant tout, selon la théorie de l'objectivation, l'apprentissage n'est pas un processus de construction ou de reconstruction personnel de la connaissance. L'apprentissage résulte plutôt de notre contact avec le monde des artefacts culturels de notre environnement (objets, instruments, productions littéraires et scientifiques, etc.) et de l'interaction sociale.

Dans cette perspective, l'apprentissage, entendu comme objectivation, est précisément un processus social de prise de conscience progressive de l'*eidos* homérique, c'est-à-dire de quelque chose qui se dresse devant nous, une figure, une forme, quelque chose dont nous percevons continuellement la généralité en même temps que nous lui donnons sens. L'objectivation signifie littéralement la rencontre avec quelque chose, quelque chose qui s'(ob)jecte, qui se donne à voir, s'affirme en tant qu'altérité et qui se présente à nous petit à petit (Radford, 2002). Elle est « le perçu qui se dévoile dans l'intention qui elle-même s'exprime dans le signe ou dans l'action que médiatise l'artefact au cours de l'activité pratique sensorielle […] quelque chose susceptible de se convertir en une action reproductible, dont le sens vise à ce schème eidétique culturel qui est l'objet conceptuel lui-même » (Radford, 2011, p. 12).

Allons rapidement un peu plus loin dans l'exploration de ce processus d'apprentissage dit d'objectivation, processus au cœur même des approches socioculturelles.

L'intelligence des artefacts et la dimension sociale : deux sources de l'apprentissage

En premier lieu, la théorie de l'objectivation est sensible à l'influence des artefacts chez l'être humain. Au contact du monde des objets et des signes, l'être humain restructure ses actions (Baudrillard, 1968/1990) et forme des capacités d'actions et des capacités intellectuelles nouvelles. Dans ce sens, les travaux de Vygotsky et Luria (1994) ont montré leur influence, entre autres, sur les capacités liées à la perception et à l'anticipation. Certes, les artefacts ont une importance considérable dans le processus d'apprentissage, mais ils ne se suffisent pas à eux-mêmes. La rencontre avec les objets et leur contenu historique, symbolique et signifiant ne peut se rendre claire d'un seul tenant. L'intelligence des artefacts se doit d'être mise en œuvre dans des activités partagées avec d'autres personnes qui savent déchiffrer ces contenus intellectuels. Par exemple, l'objet que constitue le langage de l'algèbre ne peut être porteur de sens pour l'élève que lorsqu'il est identifié par l'autre[8] au travers de l'activité sociale ayant cours à l'école. Sans l'apport d'un autre, le code symbolico-algébrique serait réduit à des hiéroglyphes dont le sens serait totalement à élaborer.

C'est pourquoi la dimension sociale constitue, pour la théorie de l'objectivation, la seconde source essentielle de l'apprentissage. Encore là, il ne faut pas réduire la dimension sociale de l'apprentissage à la négociation de signifié (arrière-plan socioconstructiviste) ou comme une simple ambiance qui offre des possibilités d'adaptation supportant le développement cognitif des apprenants (point de vue cognitiviste ou behavioriste). La classe ne peut non plus être perçue comme un espace neutre et fermé, un monde clos à l'intérieur duquel se négocient les normes, les formes et les valeurs du savoir, car, comme expliqué précédemment, les modes d'activité qui y prennent place sont médiatisés par les objets et la culture, lesquels sont imprégnés de valeurs scientifiques, esthétiques, éthiques, etc., et ces modalités sont, bien entendu, partout présentes dans l'activité de classe. La classe n'est donc pas fondamentalement un lieu neutre dans lequel les apprenants agissent selon des mécanismes invariables d'adaptation générale. En effet, l'interaction sociale ne remplit pas une fonction adaptative, catalytique ou facilitante, elle est plutôt, et c'est là la radicalité du propos, « consubstantielle de l'apprentissage » (Radford, 2011, p. 10).

Ainsi, l'apprentissage comme objectivation culturelle du savoir est à la fois une prise de conscience, entendue comme une (re)connaissance, d'éléments culturels et un processus de développement de nos capacités humaines. Autrement dit, apprendre des mathématiques n'est pas simplement apprendre à « faire » des mathématiques (encore moins à résoudre des problèmes[9] dits

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[8] Ici l'autre est perçu dans un sens très large, c'est-à-dire que la lecture d'un livre est entendue comme activité sociale, la méditation dans un contexte des plus reculés l'est aussi. Comme le dit Bakhtine : « le centre nerveux de toute énonciation, de toute expression, n'est pas intérieur, mais extérieur : il est situé dans le milieu social qui entoure l'individu. Seul le cri animal, inanalysable, procède de l'intérieur de l'appareil physiologique de l'individu isolé. C'est une réaction physiologique pure et non marquée idéologiquement » (1929/1977, p. 132).

[9] Sans enlever le mérite aux problèmes dans l'acquisition de connaissances mathématiques, la théorie de l'objectivation ne considère simplement pas la résolution de problème comme une fin en soi, mais un moyen pour atteindre ce type de *praxis cogitans* appelé la pensée mathématique.

mathématiques), mais à « être-en-mathématiques », l'activité mathématique n'étant rien d'autre qu'une manière d'« être-avec-les-autres » (Radford, 2012). Ce basculement de la sphère du « faire » à la sphère de l'« être » en ce qui concerne l'apprentissage aura bien entendu des conséquences importantes sur l'organisation de la classe et des activités, mais aussi sur le rôle de l'enseignant et de l'apprenant.

Avant de retourner à l'histoire des mathématiques et à son potentiel pour l'enseignement-apprentissage de la discipline, abordons rapidement cette thématique de l'être-en-mathématique dont les réflexions théoriques et philosophiques qui en émergent se fondent sur une perspective éthique fondamentale.

La classe et le concept du je-communautaire

La classe est le lieu de la rencontre entre le sujet et l'objet de savoir et l'objectivation qui permet cette rencontre est un processus éminemment social. Cependant, cette dimension sociale ne peut être réduite à un marché de la connaissance ou le savoir est transmis, partagé ou négocié dans une optique pragmatique de recherche de satisfaction personnelle, de jeu entre adversaires où chacun s'investit dans l'espoir d'obtenir une plus value, dans le repli de la sphère privée. Même s'il nous faut ramener quelque chose de la classe vers un chez-soi, cela n'implique pas nécessairement de faire de l'Autre un Même pour soi, la possession étant la forme par excellence sous laquelle l'Autre devient un Même (Levinas, 1971/2010). La relation au monde « qui se joue avec l'être, qui se joue comme ontologie, consiste à neutraliser l'étant pour le comprendre ou pour le saisir […] elle n'est donc pas une relation avec l'Autre comme tel, mais la réduction de l'Autre au Même. » (*id*., p. 36-37.) Dans ce cadre où règne l'objet et où s'exalte la souveraineté des pouvoirs techniques, la liberté consiste à « se maintenir contre l'Autre, malgré toute relation avec l'Autre, assurer l'autarcie d'un moi » (*ibid*.).

A contrario, la socialité du processus d'apprentissage signifie la formation et la transformation de la conscience, qui est justement (con)science, c'est-à-dire « savoir en commun » ou « savoir-avec-d'autres » (Radford, 2011, p. 12). Transformation des consciences qui est subjectivation, formation de la subjectivité, c'est-à-dire, d'un devenir. Devenir, puisque justement l'apprenant est individu (qui est indivisible, ne peut être réduit, réifié, chosifié par la question du « qu'est-ce que? »). C'est dans cet ordre d'idées que s'inscrit le concept du je-communautaire et se développe celle d'autonomie dans la théorie de l'objectivation. Ici, la théorie de l'objectivation s'éloigne de la conception d'un sujet autorégulé et autoéquilibrant, replié dans un moi-carapace dont la perméabilité se règle aux détours de logiques internes, et à travers laquelle le sujet serait doté des capacités de réfléchir à l'instar d'un scientifique ou d'un enquêteur méticuleux, autrement dit, du sujet souverain kantien autonome.

La sociabilité des élèves est ici pensée autrement. La perspective de la théorie de l'objectivation peut être comprise à travers le rôle de l'élève qui est d'apprendre à être avec les autres, à « être-avec-autres » (Radford, 2009). Cette forme d'être constitue l'essence de ce qui est appelé le je-communautaire (Radford, 2006) et dont l'arrière-plan éthique est celui de l'engagement envers autrui. C'est ici que la thématique éthique de la théorie de l'objectivation s'installe. Radford (2006, 2008, 2009), à travers Bakhtine et Levinas, insiste sur le fait qu'une relation proprement éthique à l'autre, ainsi que l'acceptation d'une véritable responsabilité personnelle, impliquent la présence d'une conscience aimante et l'absence d'un regard réifiant et intéressé. La contemplation abstraite et à distance du monde supplante notre participation active et incarnée au monde sous un horizon commun de valeur et de sens. Arraché au contexte interactif qui met en relation le Moi, l'Autre et le Monde, le sujet succombe au solipsisme. Dans cette mouvance, comme le dit Bakhtine, il « perd pied, devient vide, arrogant, dégénère et meurt » (Bakhtine, 1978/1997, p. 40). Ainsi, un élève qui est capable d'utiliser les outils mathématiques pour résoudre les problèmes proposés, dans la classe ou ailleurs, sans comprendre ou s'intéresser aux solutions des autres ou aider ceux-ci à comprendre la sienne, n'est pas à mi-chemin de ce qu'on peut appeler réussite de l'apprentissage dans la perspective de la théorie de l'objectivation (Radford, 2011).

Dès lors, le rôle de l'enseignant doit être repensé. Aussi, ce dernier se devra de disposer « d'actions d'inclusion », c'est-à-dire d'actions qui sont dirigées vers l'inclusion de chaque élève dans la communauté. Un élève qui arrive à résoudre des problèmes à sa façon sera donc accompagné par l'enseignant petit à petit à gagner son espace à l'intérieur de la communauté pour à la fois s'y inclure et la transformer.

C'est pourquoi la classe ne peut être perçue comme un milieu clos où les élèves développent des compétences ou une certaine adaptabilité à travers un processus de négociation capitaliste et antagoniste, mais plutôt comme « l'espace de collaboration et de coopération avec l'élève pour que celui-ci se transforme en élément du collectif » (*id.*, p. 14). L'enseignant n'a donc pas comme rôle de promouvoir l'idée individualiste d'autonomie associée à la philosophie rationaliste, mais plutôt celle conçue comme engagement social. Son rôle est celui de développer une (con)science, c'est-à-dire un savoir-avec, les hommes ne pouvant se libérer qu'ensemble par l'intermédiaire du monde (Freire, 1974). Dans ses travaux sur les cultures anciennes grecque et romaine, Hannah Arendt (1961/1989) a illustré comment, en opposition aux idées de la modernité, la notion d'autonomie avait une connotation civico-sociale encapsulée dans l'idée de citoyenneté chez les Grecs et les Romains. En effet, cette autonomie était intimement rattachée à la sphère publique, étant la caractéristique propre de l'existence humaine dans le monde. Or, toujours selon Arendt, notre tradition philosophique semble avoir plutôt épousé l'idée que la liberté ne commence pas lors de notre association avec les autres, mais quand les individus ont abandonné la sphère politique et, qu'au contraire, elle commence dans l'isolement avec soi-même. C'est justement à l'encontre de cette perspective que s'inscrit l'idée du je-communautaire et se développe celle d'autonomie dans la théorie de l'objectivation.

C'est pourquoi le rôle de l'enseignant est de promouvoir l'idée de la classe comme une « forme de vie » (Radford, 2011, p. 14), idée qui s'écarte de la conception instrumentale de la classe de mathématiques empruntée au mouvement d'efficacité industrielle et d'une conception bancaire du savoir (Freire, 1974). L'activité de classe comme forme de vie ne peut donc pas être perçue comme un contrôle de variables dans l'optique d'une optimisation de ressources cognitives et matérielles. Cette perspective offre plutôt « des manières d'être et de connaitre selon la façon dont les élèves s'engagent en groupe dans leur quête du savoir culturel visé » (Radford, 2011, p. 15). L'irréductibilité de la classe, sa résistance, est la manifestation de sa contingence, de son immanence, de son « organicité ». Elle ne peut donc ici être vue que dans une perspective radicalement systémique.

La théorie de l'objectivation offre donc une perspective profonde, sensible et fort cohérente de l'enseignement-apprentissage des mathématiques. Ses thématiques et ancrages épistémologiques, d'ailleurs très clairement détaillés et mis en évidences, la place de manière indubitable à l'intérieur de ce qu'on peut appeler le point de vue « socioculturel ». Cependant, la thématique de l'altérité qui fonde le concept de je-communautaire, thématique centrale à la perspective radfordienne, apporte une dimension particulière importante, et complète un ensemble dont la profondeur et la cohésion frappe et étonne. Mais *ad rem*, ayant maintenant exploré l'arrière-plan épistémologique de la théorie de l'objectivation, tâchons de mettre en relief, de ce point de vue, les arguments qui appuient la présence de l'histoire des mathématiques dans la classe de mathématiques du primaire et du secondaire.

Un point de vue sur le rôle de l'histoire des mathématiques

Dans cette perspective, le sens particulier attribué aux objets mathématiques est circonscrit aux limites de notre propre expérience. Cette limite ne peut être franchie que par la rencontre avec une forme étrangère de compréhension, car « le sens ne s'approfondit véritablement que par la rencontre et le contact avec un autre sens, une culture étrangère. Il s'engage alors une forme de dialogue qui surmonte la fermeture et la partialité » (Bakhtine, 1986, cité dans Radford, Furinghetti et Katz, 2007, p. 108, traduction libre).

Dans ce sens, l'histoire des mathématiques se veut un possible endroit où il est possible de surmonter la particularité de notre propre compréhension des objets mathématiques limitée à nos expériences personnelles. Elle « fournit les instruments de dialogues avec d'autres compréhensions […] avec celles de ceux qui nous ont précédés » (Radford, Furinghetti et Katz, 2007, p. 109, traduction libre).

L'histoire apparait ici comme la toile de fond ou le lieu rendant possibles l'introspection, la confrontation et la réflexion critique autour de ses propres conceptions et connaissances. Radford et *al*. (2000) soulignent que l'histoire des mathématiques est « un endroit merveilleux, où il est possible de reconstruire et de réinterpréter le passé dans le but d'ouvrir de nouvelles possibilités pour les futurs enseignants » (p. 165, traduction libre).

Or, notons que le regard est ici porté non pas sur un individu rencontrant des possibilités d'émancipation personnelles, dans un mouvement plus ou moins appuyé d'autosuffisance et d'autoréférence, mais vers la possibilité pour les apprenants de découvrir de nouvelles manières d'être-en-mathématiques, d'ouvrir, avec les autres, l'espace des possibles de l'activité mathématique. En effet, la classe de mathématique est ici perçue comme un espace communautaire, politique et éthique, ouvert à la nouveauté et à la subversion (Radford, 2006, 2008, 2011).

On s'éloigne ici d'une conception de l'étudiant comme propriétaire privé de la connaissance. L'apprenant, comme discuté précédemment est plutôt perçu comme un sujet éthique. En d'autres termes, on souhaite éviter que l'éducation produise des individus émancipés côte à côte, à la fois libres et isolés. La théorie de l'objectivation met plutôt de l'avant le commitment, l'answerability et le caring (Bakhtine, 1929/1977; Heidegger, 1927/1986; Levinas, 1971/2010), ce qui implique une toute autre conception de l'enseignement-apprentissage, de la classe et du rôle de l'élève et de l'enseignant.

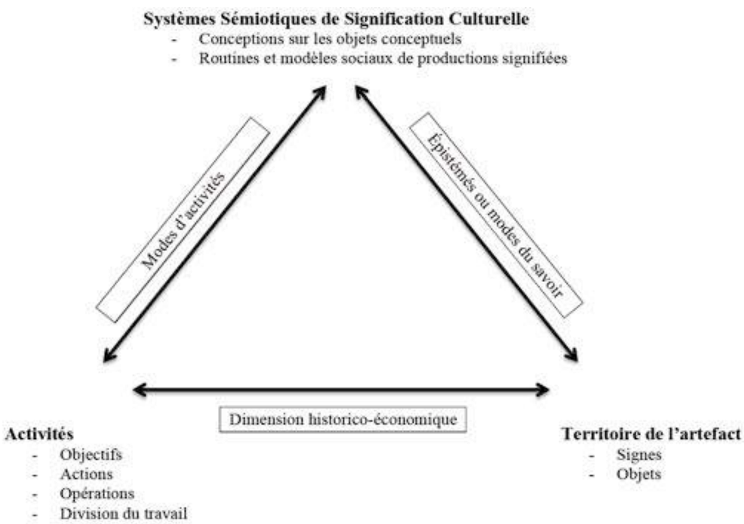
Dans cette perspective, la rencontre avec l'histoire des mathématiques offre des expériences particulières de l'altérité en mathématique, expériences dont les enjeux pour l'apprentissage et le rapport aux savoirs mathématiques des apprenants restent à mettre en évidence. En effet, quelles manières d'être-en-mathématiques peuvent survenir lors de cette rencontre-évènement avec l'histoire ou certains éléments de nature historique en mathématiques? Et quelles manières d'être-avec-les-autres-en-mathématiques peuvent survenir? Quelles sont les modalités d'êtres associées à ces expériences particulières? Ces questions se doublent lorsqu'on se questionne sur les manières de rendre compte en recherche de ces expériences. La prochaine section proposera quelques pistes de réflexion à partir d'une étude menée récemment dans le contexte de la formation des maitres en mathématiques au Canada.

Avant cela tâchons d'entrée plus en profondeur sur ce que peut vouloir dire manière d'être en mathématique. La pensée étant considérée comme « une réflexion médiatisée du monde en accord avec le mode de l'activité des individus » (Radford, 2011, p. 4), elle a d'abord une nature réflexive, c'est-à-dire en mouvement dialectique entre une réalité construite historiquement et culturellement et un individu qui la réfléchit et la modifie selon ses interprétations et sa propre subjectivité. La pensée n'est donc pas une simple assimilation de la réalité externe (point de vue empiriste) ou la construction *ex nihilo* de cette dernière (point de vue du constructivisme radical). Autrement dit, c'est l'individu qui crée la pensée et ses objets, mais tout individu est inséré dans sa réalité culturelle et historique, c’est pourquoi la pensée est dite en accord avec le mode de l’activité des individus. La pensée étant médiatisée par le corps, des signes et des artefacts et, d’autre part, par des signifiés culturels. Ces deux niveaux de médiations laissent une empreinte sur la forme et le contenu de la pensée elle-même.

Émerge alors un champ dynamique de réalisation de la pensée appelé territoire des artefacts (Bakhtine, 1929/1977), champ à l’intérieur duquel la subjectivité de l’individu et l’objectivité culturelle s’imbriquent mutuellement et que la pensée rencontre son « espace d’action » (Radford, 2011, p. 4). L’introduction d’artefacts « éloignés » historiquement et culturellement, comme une source primaire (textes historiques, images, objets anciens…) engendre dans la communauté d’apprenants une intrusion violente sur ce territoire. Violence qui est potentiellement violence métaphysique puisqu’elle peut éventuellement forcer une réorganisation de champ de réalisation et un redéploiement de l’espace d’action. Autrement dit, l’histoire des mathématiques bouscule et dépayse, et porte en elle le germe d’actions inédites et un fort potentiel de réorientation de la production de la subjectivité des apprenants.

Pour ce qui est du second niveau de médiation, celui de la culture. Un des rôles de cette dernière est de fournir à l’apprenant des manières de percevoir la réalité et ses manifestations. La culture permet ainsi d’accéder à des modes de visée comme dirait Merleau-Ponty (1945/2010) ou, pour Husserl (1913/1998), à des formes de recours à l’intuition. Les signifiés culturels orientent l’activité sensible de la pensée en lui donnant une certaine forme, agissent comme des liaisons médiatisantes entre la pensée individuelle et la réalité culturelle et, enfin, s’établissent comme des prérequis et des conditions de possibilité de la pensée individuelle (Ilienkov, 1977). Ces signifiés culturels soutiennent l’activité sensible de la pensée. Il peut s’agir de conceptions autour des objets mathématiques (nature, mode d’existence, applicabilité, etc.) ou autour des modèles sociaux de production de signifié. L’ensemble de cette superstructure symbolique est nommé Système Sémiotique de Signification Culturelle (Radford, 2003).

En interaction avec l’activité (Leontiev, 1984) des individus (objectifs, actions, opérations, distribution du travail, etc.) et ce qui a été appelé plus haut le territoire des artefacts, ce Système Sémiotique de Signification Culturelle génère des modes d’activités spécifiques et, d’autre part, des modes de savoir ou épistémès (Foucault, 1966/1990). La figure suivante montre l’interaction entre ces trois composantes qui permettent de mieux comprendre l’idée de la pensée comme *praxis cogitans* :



Interaction entre le Système Sémiotique de Signification Culturelle, l’Activité et le Territoire de l’artefact (tirée de Radford, 2011, p. 6).

Dans cette perspective, le sens accordé à l'apprentissage est celui de l'« acquisition ». L'apprentissage est compris comme l'« acquisition par l'apprenant de formes culturelles de réflexions sensibles et d'actions instrumentales qui constituent la pensée » (Radford, 2011, p. 3). Cependant, l'apprentissage ne se réduit pas ici à une simple acculturation, à la réception passive et mystérieuse du savoir contenu dans la culture. En effet, le mot acquisition doit être pris dans son sens étymologique, c'est-à-dire du latin *adquaerere* qui signifie « chercher ». Le mot acquisition doit donc être entendu en tant que processus d'ouverture, attitude ou manière d'être. L'apprentissage n'est donc pas une soumission à une culture ambiante, encore moins une possession d'un contenu culturel, mais plutôt un mouvement d'ouverture sur le monde et sur les autres, un processus d'élaboration active de signifiés soit le processus d'objectivation.

Avec l'introduction de l'histoire des mathématiques, cette ouverture sur le monde se voit radicaliser et prend un tournant inédit. En effet, les manières de penser et d'agir se multiplient autour des apprenants au contact de l'histoire des mathématiques. Ces éléments divergents de la culture ambiante invitent à l'introspection, à la prise de consciences de ses ancrages historiques, et l'ensemble rend possible la confrontation constructive de ses conceptions avec les autres en mathématiques. L'histoire place les apprenants en mode recherche.

Ouverture et invitation

Cela dit, il nous faut maintenant mettre en évidence, à partir de ce point de vue socioculturel, différentes manières d'opérationnaliser dans la recherche l'investigation de ces considérations théoriques sur le rôle de l'histoire dans la formation des enseignants. En effet, de quelle manière est-il possible d'investiguer sur le terrain le rôle que peut jouer l'histoire des mathématiques ou la présence d'éléments historiques ou culturels dans la classe de mathématiques dans une telle perspective sur l'éducation mathématiques? Quelques éléments de réponse seront proposés à partir de diverses réflexions d'ordre méthodologiques déployées dans le cadre de notre récente étude doctorale (Guillemette, 2015).

Notre étude avait pour objet une idée récurrente dans le champ de recherche, celle du dépaysement épistémologique. En effet, les chercheurs soulignent que l'histoire des mathématiques dépayse et bouscule les perspectives coutumières des étudiants sur la discipline en mettant en évidence sa dimension historico-culturelle. Globalement, l'étude de l'histoire amènerait un regard critique sur l'aspect social et culturel des mathématiques et pousserait les futurs enseignants à reconsidérer leur rapport à la discipline. Cela dit, bien que commentées par de nombreuses études, ces considérations sur le dépaysement épistémologique ne semblent pas encore avoir fait l'objet de recherches systématiques de terrain qui donneraient véritablement la voix aux acteurs des milieux de formation.

Avant tout, trois points de vue sur le dépaysement épistémologique sont présentés dans la thèse : celui de l'épistémologie historique, de l'humanisme et des approches socioculturelles en éducation mathématique. Habitée par les questions que soulèvent ces positions et appuyée conceptuellement par la théorie de l'objectivation, notre étude s'est donnée pour objectif de décrire le dépaysement épistémologique vécu par les futurs enseignants de mathématiques du secondaire dans le cadre d'activités de formation où intervient l'histoire des mathématiques.

Pour ce faire, une approche phénoménologique a été adoptée et adaptée à la perspective bakhtinienne que porte la théorie de l'objectivation. Concernant l'approche phénoménologique, elle vise à décrire le vécu intime et subjectif des participants et à expliciter le sens de leurs expériences. Quant à la perspective bakhtinienne, elle souligne qu'une œuvre scientifique ou littéraire se doit d'être « polyphonique », c'est-à-dire offrir une pluralité de discours et de compréhensions du monde, afin que la réalité perde de son statisme et de son naturalisme. Habité par cette perspective compréhensive et critique, nous proposons une description du vécu du dépaysement épistémologique qui prend la forme d’un roman polyphonique.

La sélection des participants de l’étude a été faite parmi les futurs enseignants du secondaire inscrits à un cours d’histoire des mathématiques offert à l’Université du Québec à Montréal. Sept activités de lecture de textes historiques ont été vécues en classe. Six étudiants ont été recrutés. Des captations vidéo des activités de classe, des entretiens individuels et un entretien de groupe ont été réalisés et ont fourni les données de l’étude. Pour les captations vidéo, une analyse séance par séance a permis de décrire les processus d’apprentissage ayant eu cours en classe. Pour les entretiens individuels, inspirée par les procédures de plusieurs chercheurs phénoménologues en sciences humaines, le traitement et l’analyse des données ont mené à l’obtention d’une description spécifique du vécu du dépaysement épistémologique pour chaque participant. Le roman polyphonique a ensuite été construit à partir d’extraits de l’entretien de groupe, et peaufiné à partir des phases précédentes d’analyse.

La description obtenue fournit plusieurs regards, lesquels, mis en tensions, sont porteurs d’un discours fécond sur le vécu du dépaysement épistémologique. Spécifiquement, l’étude montre que celui-ci amène; la perception des mathématiques comme fragiles, débutantes et précaires, le vécu d’une forte adversité dans l’interprétation des textes et le déploiement d’une empathie envers l’auteur. Le roman polyphonique suggère que cette empathie se déploie aussi vers la classe de mathématique des futurs enseignants, et insuffle une attention et une valorisation plus grande envers les raisonnements marginaux de leurs élèves. Ces éléments invitent à penser que ces activités de formation, par le biais du dépaysement épistémologique qu’elles suscitent, supportent une éducation mathématique non violente.

RÉFÉRENCES

Arendt, H. (1989). *La crise de la culture*. Paris : Gallimard. (Œuvre originale publiée en 1961)

Bakhtine, M. (1977) [paru sous le nom d'emprunt V. N. Volochinov]. *Le marxisme et la philosophie du langage*. Paris : Minuit. (Œuvre originale publiée en 1929)

Bakhtine, M. (1986). *Speech genres and other late essays*. Austin : University of Texas Press.

Bakhtine, M. (1997). *Esthétique et théorie du roman*. Paris : Gallimard. (Œuvre originale publiée en 1978)

Barbin, E. (1997). Histoire et enseignement des mathématiques : Pourquoi? Comment? *Bulletin de l’Association mathématique du Québec*, *37*(1), 20–25.

Barbin, E. (2006). Apport de l’histoire des mathématiques et de l’histoire des sciences dans l'enseignement. *Tréma*, *26*(1), 20–28.

Barbin, E. (2012). L’histoire des mathématiques dans la formation : une perspective historique (1975-2010). In J.-L. Dorier & S. Coutat (Eds.), *Acte du congrès de l’Espace mathématiques francophone 2012* (pp. 546–554). Genève : Université de Genève.

Baudrillard, J. (1990). *Le système des objets*. Paris : Gallimard. (Œuvre originale publiée en 1968)

Charbonneau, L. (2006). Histoire des mathématiques et les nouveaux programmes au Québec : un défi de taille. In N. Bednarz & C. Mary (Eds.), *Actes du colloque de l’Espace mathématiques francophone 2006* (pp. 11–21). Sherbrooke : Éditions du CRP et Faculté d’éducation de l'Université de Sherbrooke.

Foucault, M. (1990). *Les mots et les choses*. Paris : Gallimard. (Œuvre originale publiée en 1966)

Freire, P. (1974). *Pédagogie des opprimées*. Paris : Maspero.

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, *10*(4), 391–408.

Fried, M. N. (2007). Didactics and history of mathematics : Knowledge and Self-Knowledge. *Educational Studies in Mathematics*, *66*(2), 203–223.

Fried, M. N. (2008). History of mathematics in mathematics education : a saussurean perspective. *The Montana Mathematics Enthusiast*, *5*(2), 185–198.

Guillemette, D. (2015). *L'histoire des mathématiques et la formation des enseignants du secondaire : sur l'expérience du dépaysement épistémologique des étudiants.* Thèse de doctorat inédite, Université du Québec à Montréal, Montréal, Québec.

Gulikers, I., & Blom, K. (2001). « A historical angle » : survey of recent literature on the use and value of history in geometrical education. *Educational Studies in Mathematics*, *47*(2), 223–258.

Heidegger, M. (1986). *Être et temps*. Paris : Gallimard. (Œuvre originale publiée en 1927)

Husserl, E. (1998). *Idées directrices pour une phénoménologie*. Paris : Gallimard. (Œuvre originale publiée en 1913)

Ilienkov, E. (1977). The concept of the Ideal. In *Philosophy in the U.S.S.R. : problem of dialectical materialism* (pp. 71–99). Moscou : Progress Publishers.

Jankvist, U. T. (2009). *Using history as a « goal » in mathematics education*. Thèse de doctorat inédite, Université de Roskilde, Roskilde, Danemark.

Jankvist, U. T. (2010). An empirical study of using history as a « goal ». *Educational Studies in Mathematics*, *74*(1), 53–74.

Kjeldsen, T. H. (2012). Uses of history for the learning of and about mathematics : towards a theorical framework for integrating history of mathematics in mathematics education. In S. Choi & S. Wang (Eds.), *Proceedings of the International Study Group on the Relations between the History and Pedagogy of Mathematics (HPM 2012) satellite meeting of ICMI 12* (pp. 1–21). Daejeon, Corée du Sud : Korean Society of Mathematical Education et Korean Society for History of Mathematics.

Leontiev, A. (1984). *Activité, conscience, personnalité*. Moscou : Les Éditions du Progrès.

Levinas, E. (2010). *Totalité et infini : essai sur l'extériorité*. Paris : Librairie Générale Française. (Œuvre originale publiée en 1971)

Merleau-Ponty, M. (2010). *Phénoménologie de la perception*. Paris : Gallimard. (Œuvre originale publiée en 1945)

Radford, L. (2002). The seen, the spoken and the written : a semiotic approach to the problem of objectification of mathematical knowledge. *For the Learnings of Mathematics*, *22*(2), 14–23.

Radford, L. (2003). On culture and mind : a post-vygotskian semiotic perspectiv with an example from greek mathematical thought. In M. Anderson, A. Sàenz-Ludlow, S. Zellweger & V. Cifarelli (Eds.), *Educational perspectives on mathematics as semiosis : from thinking to interpreting to knowing* (pp. 49–79). Ottawa : Legas Publishing.

Radford, L. (2006). Communication, apprentissage et formation du je-communautaire. In B. D’Amore & S. Sbaragli (Eds.), *Proceedings of the 20th National Italien Conference « Incontri con la Matematica »* (pp. 65–72). Bologne : Université de Bologne.

Radford, L. (2008). The ethics of being and knowing : towards a cultural theory of learning. In L. Radford, G. Schubring & F. Seeger (Eds.), *Semiotics in mathematics education : epistemology, history, classroom and culture* (pp. 215–234). Rotterdam : Sense Publishers.

Radford, L. (2009). L’altérité comme problème éducatif. In J. Boissonneault, R. Corbeil & A. Hein (Eds.), *Actes de la 15e journée Sciences et Savoirs* (pp. 11–27). Sudbury : Université Laurentienne.

Radford, L. (2011). Vers une théorie socioculturelle de l’enseignement-apprentissage : la théorie de l'Objectivation. *Éléments*, *1*, 1–27.

Radford, L. (2012). Education and the illusions of emancipation. *Educational Studies in Mathematics*, *80*(1), 101–118.

Radford, L. (2013). Three key concepts of the theory of objectification : knowledge, knowing, and learning. *Journal of Research in Mathematics Education*, *2*(1), 7–44.

Radford, L., Bartolini Bussi, M. G., Bekken, O., Boero, P., Dorier, J.-L., Katz, V. & Vasco, C. (2000). Historical formation and student understanding of mathematics. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education : the ICMI study* (pp. 143–167). Dordrecht : Kluwer Academic Publishers.

Radford, L., Furinghetti, F. & Katz, V. (2007). Introduction : the topos of meaning or the encounter between past and present. *Educational Studies in Mathematics*, *66*(1), 107–110.

Vygotsky, L. & Luria, A. (1994). Tool and symbol in child developement. In R. van der Veer & J. Valsiner (Eds.), *The Vygotsky reader* (pp. 99–174). Oxford : Blackwell.

Ready-made materials or teachers' flexibility? What do we need in culturally and linguistically heterogeneous mathematics classrooms?

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Abstract: The paper focuses on the issue of coping with the increasing cultural and linguistic heterogeneity in mathematics classrooms across Europe. The authors come out of a teaching unit developed by Barbro Grevholm within the project Multiculturalism, Migration, Mathematics Education and Language. The goal of the project was development of teaching units supporting linguistic and cultural diversity in mathematics classrooms. The authors of the paper argue that teachers of mathematics do not need detailed teaching units (although a survey among them shows this is what they are convinced they need as support) but they need topics with different cultural origins which they then adapt to suit the needs of the particular group of learners, their abilities, skills, age and language competence. In the presentation and the final paper the authors will show how the same teaching unit was grasped by different teachers and what the outcomes of their approach was.

Résumé: L'article concerne l´adaptation à l'hétérogénéité culturelle et linguistique de plus en plus présente dans les classes de mathématiques dans toute l´Europe. Les auteurs partent du concept de l´unité d´enseignement développé par Barbro Grevholm dans le projet «Multiculturalism, Migration, Mathematics Education and Language». L´objectif de ce projet est le développement des unités supportant la diversité linguistique et culturelle dans l´enseignement des mathématiques. Les auteurs affirment que les professeurs de mathématiques n´ont pas besoin d´unités détaillées (quoiqu´ils soient convaincus que c´est ce dont ils ont besoin pour les réaliser), mais que, par contre, ils ont besoin des thèmes provenant des cultures différentes qu´ils pourraient adapter pour les besoins de divers groupes des élèves, c´est-à-dire pour leurs capacités, leur âge et savoir-faire, pour leurs compétences linguistiques. Dans la présentation, tout comme dans la version finale de l´article, les auteurs montreront comment une unité d´enseignement a été modifiée par les différents professeurs et quels ont été les conséquences de leur procédé.

Introduction

The paper focuses on one of the much discussed issues in mathematics education, which is inclusion of multicultural elements into mathematics lessons. The paper presents partial results of the project 526333-LLP-1-2012-1-IT-COMENIUS-CMP Multiculturalism, Migration, Mathematics Education and Language (M^3EaL), whose main goal is to make teachers aware that pupils' culture (including their language) plays a significant role in the teaching/learning processes, also in mathematics. This awareness helps teachers to pay more attention to the different cultures (and languages) in the classroom and give them a higher value in the educational process. Conditions for an intercultural dialogue in the classroom as well as for better inclusion of pupils from different cultures are thus created.

The project aims to design and implement, in each of the partner countries, teaching materials for mathematics, which take into account situations or activities typical for specific (or even a variety of) cultural areas, as well as the role played by language in the teaching/learning of mathematics within multicultural and multilingual classes. The experimental implementation of these modules is expected to lead to the identification of good practices to be exchanged inside and outside of the partnership. The proposed teaching materials for the maths classroom, inspired by practical problems and situations from everyday life and from different cultures, encourage teachers to take into consideration the different cultures in the classroom, highlighting their positive aspects and establishing intercultural dialogue in the classroom, thus promoting better inclusion of minority pupils.

The presented paper here discusses what form the developed teaching units should have – should they be developed with respect to a particular age group or school level, or should they only outline the possible multicultural content and leave it up to the teacher to elaborate it for the needs of their classroom and curricula?

Research in multicultural elements in mathematics education

It is generally accepted (e.g. Barton, Barwell and Setati, 2007; Bishop, 1988; César and Favilli, 2005) that mathematics teachers feel the necessity for training and materials which reflect the needs of their students in terms of linguistic and cultural differences. Their pupils from minority cultures and/or those with a migrant background encounter more difficulties than their native classmates in acquiring fundamental mathematics skills. Moll et al. (1992) claim that these different cultural backgrounds also provide "funds of knowledge" (i.e. "historically accumulated and culturally developed bodies of knowledge and skills") that can support the learning process. However, these "funds of knowledge" have only limited applications in many European classrooms, since they require a close connection and collaboration between teachers, parents and the minority community, and are mainly applicable if there is only one, fairly homogenous minority culture present, which is not the case in most European school backgrounds. Mostly however, learning a new language and culture at the same time as you learn mathematics places additional burden and challenges on migrant and minority pupils (Norén, 2010; Steinhardt & Ulovec, 2013).

Two years ago Ulovec et al. (2013) spoke in their paper of the lack of attention paid to multicultural aspects of teaching mathematics in contrast to a relatively large amount of research relating to multiculturalism in general without really making a distinction between subjects. Only some research focused on the difficulties in relation to the teaching of a particular subject and, moreover, it mainly covered teaching of language or natural sciences (McDermott & Varenne, 1995); research on mathematics teaching was rarer.

However, the changing reality in the classrooms across Europe started to attract attention of mathematics educators and now one can come across a variety of researches focusing directly on the specifics of teaching a culturally and linguistically heterogeneous group of learners in mathematics. A whole working group Multiculturalism and reality at the last CIEAEM conference in Lyon was trying to approach the issue from various perspectives (Aldon, Di Paola and Fazio, 2015). Attention was paid to the difficulties an individual with minority background faces in mathematics classrooms, in problem solving in exams, to the issue of what mathematics actually means to different groups of people and what value they attach to it, to the patterns of parental involvement in their child's mathematics learning in different sociocultural groups, to stereotypes in mathematics assignments and how these may affect minority pupils, to relations between experiences, languages, culture and power in multilingual mathematics classrooms and study the concepts of discourse and agency. Attention was also paid to the issue of how to make the mathematics curriculum more meaningful to minority (Roma) children and how a more meaningful curriculum could improve their participation and performance. This is in line with Meany and Lange (2013), who discuss the issue of learners' transition between contexts and warn of the additional difficulties for learners if their experience of home context is very different from contexts they come across at school.

This shows that the number of perspectives is considerable, moving attention from one student and their background and obstacles to discourse and power in general, to meaning of mathematics to different learners, to cultural obstacles and to how to construct more meaningful curricula to allow learners from different sociocultural background to get involved. Santomé (2009) warns that if schools are to contribute to increasing justice and equity, they will have to analyze to what extent the curriculum is respectful of people's different cultures.

The authors of this paper are convinced that culturally heterogeneous learning environments in mathematics will allow learners to get acquainted with other cultures and their values. Moreover, they offer them novel, innovative ways of solving a problem, can offer new tools and procedures that are used in other countries and cultures and may develop their creativity and originality of methods used. Obviously, inclusion of elements from other cultures in mathematics will be of benefit both to majority and minority learners in the classroom.

A questionnaire survey conducted within the project M^3EaL showed that teachers feel an urgent lack of materials that they could easily use in heterogeneous classrooms. They ask for elaborated teaching units that would be ready for their use. This attitude is understandable, the workload teachers in countries across Europe face is enormous. However, taking account the variety across Europe, this means developing thousands of different teaching units.

The project partners agreed they would develop teaching units ready to be used in mathematics classrooms without any further modifications, i.e. in a way that teachers ask for. However, piloting of these materials has shown an interesting conclusion – considering the variety of classrooms, equipment, needs, abilities and skills of learners and teachers across Europe, it seems what is really needed are topics that can be adapted individually by different teachers for different learners and conditions. Instead of thousands of teaching units, we should look for rich sources of mathematics and alternative solving procedures that can then be adapted by individual teachers.

This paper shows one such multicultural topic (finger multiplication as a method of multiplication of a different cultural origin – though it is not really clear where exactly the method comes from – various sources speak of Russian, Gypsy or Chinese origin), which can easily be adapted for mathematics classrooms at very different levels, primary, upper secondary, teacher training.

A teaching unit with multicultural topic

The original topic was developed by the Norwegian partner of the project (Barbro Grevholm). The area of study is Multiplication from different approaches (history, culture, traditions, use of tools and books), the use of concrete tools in calculations, the use of early algebra for formulation of rules for multiplication and for proving mathematical results, different ways of proving in mathematics and mathematical reasoning. The aim of the unit was to make pupils reflect about the process of multiplication, realise the properties of multiplication and see links between multiplication and other areas of mathematics. Pupils may also reflect upon what they need to know by heart in mathematics and what can be reproduced with different tools or aids and may notice that mathematics is constructed and used by ordinary people in many parts of the world.

This teaching unit was piloted in two very different settings – 3rd grade of primary school in the Czech Republic using CLIL (Content and Language Integrated Learning, i.e. the lessons were conducted in English, thus making the language a "barrier" to everybody in the class), 18 year old upper secondary students in Austria and pre-service teachers of mathematics in the Czech Republic. The teachers studied the Norwegian unit, analysed it and adapted it to suit the needs of their classrooms.

This paper only gives an overview of the piloting process. Its scope does not allow the full description of how the teachers conducted the lessons and to describe the course of the lessons and learners’ activity and involvement; this will be presented in the oral presentation at the conference. It shows that the proposed activity can be successfully used in different settings, for different age groups, for both heterogeneous and homogeneous environment and different languages of instruction (e.g. CLIL – learning content through an additional language).

Piloting in Austria

The teaching unit was piloted by a female mathematics teacher with five years teaching experience working in an upper secondary school near Vienna. The Austrian project team sent the material to the teacher approximately 3 weeks before the planned piloting activity. The teacher had a 5th (age 14-15 years), 6th (15-16) and 8th (17-18) grade available for piloting. After a meeting with the project team, she chose to conduct the piloting during a regular mathematics class (50 minutes) in the 8th grade. Eight students (age 17-18), three of which are migrant students, attended the class, which was video recorded and observed by a member of the Austrian project team.

The teacher conducted session 1 as described in the Norwegian material by handing out sheets containing a multiplication table from the year 1601 and started a group discussion about it. This discussion lasted about 12 minutes. The students were particularly interested in the aspects of why there was a need for such tables, whether such tables existed in their own cultures’ history, and (mathematically) why these (shortened) tables were sufficient and contain the same basic data as the traditional, square-matrix shaped full multiplication tables they know. The information about the various aspects was partly given by the teacher, partly the students used internet resources to retrieve additional information.

Session 2 started with the introduction of the method of finger multiplication (5 minutes). Students were then asked to try the method out and find an explanation why it works (15 minutes). Students came up with several explanations and wanted to find out whether the method can be extended for numbers with more than one digit. They also were interested in whether this or other hand calculations were used historically. Two of the migrant students (from Turkey) reported about finger-based calculation methods from their own culture (12 minutes).

After the class, students were asked by the teacher about their experience with this teaching unit. Both the migrant and non-migrant students responded very positively. The migrant students particularly mentioned the chance of giving background information about their own culture that the other students did not know before. The non-migrant students commented positively on the various historical and cultural references that they not usually get during regular mathematics lessons.

Also, an interview was conducted with the teacher after the class. She particularly welcomed the possibility of having various anchor points for cultural references, and the opportunity to have the migrant students not only participate, but being a source of information for the other students.
The piloting clearly showed that students are interested in mathematics content from different cultures, and that the active participation of migrant students and the introduction of their cultural backgrounds can enrich the learning situation.

Piloting in Czech Republic

The teaching unit was piloted directly by one of the members of the project team and a co-author of this paper who, apart from being involved in research in the field of education, is a teacher. The teaching unit was piloted in the 3rd grade (9 year old pupils) in a primary school in Prague. It was piloted in a sequence of 4 lessons that were taught in the period of four months (about once in 6 weeks). Some of the lessons were video recorded and all the lessons were open to other teachers of the school as the school is now experimenting with the potential of CLIL in teaching in general and also in mathematics.

The teaching experiment was conceived as a sequence of lessons over a longer period of time. The multicultural background of the original Norwegian unit seemed to be the perfect environment for introducing the concept of learning mathematics through English to the learners who had had no former experience with it. Because the teaching unit was a CLIL unit and because it was planned for several lesson, the original Norwegian unit was supplemented by other activities – two different kinds of line multiplications whose origin is reported to be Chinese and games and other activities aiming at developing language skills (games with numbers, songs with numbers, What number am I?) or calculation skills (number centipedes). All the activities had two objectives – developing language and mathematical competence.

The lesson on finger multiplication was taught in September, 4 weeks after the beginning of the school year. The advantage at that stage was that the pupils had already mastered multiplication tables up to five but had not learned multiplication tables from 6 to 10. Thus it was an ideal situation for introduction of finger multiplication. Children who have memorized multiplication tables will find finger multiplication unnecessarily too difficult and time demanding. The teacher started by demonstrating the principle. The children were explained what number was represented by which finger and then shown by the teacher how the system works using fingers and whiteboard. The teacher then asked the children to do it but it turned out that at this point only very few understood. The teacher decided to demonstrate two other problems in front of the whiteboards but this time a pupil was always invited to assist and be showing it on their fingers. The whole class was saying the numbers out loud. After this the pupils were asked to work on their own. The teacher was monitoring, assisting individually to those pupils who needed help. One by one the pupils eventually grasped the principle. The teacher could see the "aha" effect when the pupils finally grasped the principle. In the subsequent lesson the teacher came back to the principle and could observe that the pupils found it relatively easy (but also inefficient as they know multiplication tables already). It was time to move to line multiplications of two and three digit numbers. The whole sequence of lessons, its outcomes and pupils' attitude are described in detail in (Moraová, Novotná, 2015).

It can be concluded that the teaching unit with cultural background proved to be very motivating and suitable for CLIL lesson. The uniqueness of the context contributed to the pupils' motivation and interest to be working on mathematics in English.

Figure 1. Finger multiplication

Conclusion

The fact is that mathematics classrooms are growing increasingly multicultural and multilingual. This growing diversity can be a chance to increase the quality of teaching (Slavin, 1994), but teachers will have to be trained to handle the situation. The differences in cultures and languages make the maths teaching-learning process harder for migrant and minority pupils than it is for majority pupils.

The presented study of adaptations of one topic (which could be understood as substantial learning environment in Wittmann's (1995) sense) clearly shows that it is possible to use the same environment in working with learners of different ages, knowledge of mathematics and sociocultural backgrounds. It is not needed to develop hundreds of teaching units in which teachers would painstakingly look for the one they can use without adaptations. On the contrary, pre-service and in-service teacher training should develop mathematics teachers' ability to work with materials creatively, to look for interesting topics and methods and modify them in such a way that makes the materials tailor-made for the needs of their own classroom.

The teaching experiment also proved that introduction of innovative solving methods whose roots come from different cultures is very motivating for the learners, brings fun into lessons and also develops flexibility of learners' thinking processes and awareness of the value of other cultures. The 3rd graders in the Czech Republic enjoyed the lesson, were active and involved. The teaching unit was piloted in 3 consecutive lessons and the teacher reported on the pupils' delight when they were told the teaching experiment would continue and not end after the first lesson. Doing something magic, learning a "trick", getting introduced to a way of work from other cultures was challenging, entertaining and effective. The Austrian students appreciated the chance to inquire into mathematics and its different cultural backgrounds, to look for the principles behind and for justification of the procedures. Although their mathematics levels were way beyond simple multiplications, they still found it interesting and satisfying to find out why these procedures work and what their limits are. Czech pre-service teachers grew aware of the potential of substantial learning environments as a rich source of teaching materials and experimented with how to adapt a teaching unit to meet different purposes.

REFERENCES

Aldon, G, Di Paola, B., &, Fazio, C. (2015). Mathematics and reality, Proceedings of CIEAEM 66, *Quaderni di Ricerca in didattica*, *24*(1).

Barton, B., Barwell, R. & Setati, M. (Eds.) (2007). Multilingualism in mathematics education. *Special Issue of Educational Studies in Mathematics*, Vol. 64, no. 2.

Bishop, A. J. (1988). Mathematics Education in its cultural context, *Educational Studies in Mathematics*, 19, 179-191.

César, M. & Favilli, F. (2005). Diversity seen through teachers' eyes: discourses about multicultural classes. In *Proceedings of the 4th Conf. of the European Society for Research in Mathematics Education,* pp. 1153-1164.

McDermott R., & Varenne H. (1995). Culture as disability. *Anthropology & Education quarterly, 26*(3), 324-348.

Moll, L., Amanti, C., Neff, D. & Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory Into Practice, XXXI* (2), 132-141.

Moraová, H. & Novotná, J. (2015). Teaching maths in English at primary school level –

Utopis, nightmare or reality. In *Proceedings of SEMT '15*, in print.

Norén, E. (2008). Bilingual students' mother tongue: a resource for teaching and learning mathematics. *Nordic Studies in Mathematics Education*, *13*(4), 29–50.

Wittmann, E. Ch. (1995). Mathematics education as a "Design Science". *Educational Studies in Mathematics, 29*(4), 355-374.

Meany, T. and Lange, T. (2013) 'Learners in Transition between Contexts', *The Third International Handbook of Mathematics Education,* Vol. 27. Springer

Santomé, J.T. (2009). The Trojan Horse of Curricular Contents. In Apple, M.W., Au, W. & Gandin, L.A. (Eds.), *The Routledge International Handbook of Critical Education* (pp. 64-80). New York: Routledge.

Slavin, R.E. (1994). Using student team learning. Baltimore: Johns Hopkins University, Center for Research on Elementary Middle Schools.

Steinhardt, N., Ulovec, A. (2013). First and second language learners: Differences at the Austrian standardized final exam in mathematics. *Proceedings of IAC-ETeL 2013*. Praha: MAC Prague consulting Ltd.

Ulovec, A., Moraová, H., Favilli, F., Grevholm, B., Novotná, J., & Piccione, M. (2013). Multiculturalism in theory and teachers' practice. In Novotná, J., & Moraová, H. (Eds.), *Proceedings of SEMT ´13* (pp. 297-305). Praha: UK-PedF.

Wittmann, E. Ch. (1995). Mathematics education as a "Design Science". *Educational Studies in Mathematics, 29*(4), 355-374.

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Re-approaching the perceived proximities amongst mathematics education theories and methods

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Abstract: In this paper, we discuss the role of perceived proximity in Mathematics Education as a crucial factor determining the relevance of theoretical and empirical tools in mathematics teaching-learning research. It is posited that by topologically re-defining the proximity of various till now distant research tools, the phenomena may be researched as complex wholes thus deepening our understandings. A research project about proof is discussed as an exemplar of such a perspective.

Résumé : Dans cet article, nous discutons le rôle de la proximité perçue dans l'enseignement des mathématiques comme un facteur déterminant de la pertinence des outils théoriques et empiriques dans la recherche sur l'enseignement et l'apprentissage. Il est posé que par redéfinissant topologiquement la proximité de différents jusqu'à maintenant outils de recherche, les phénomènes peut être étudié comme des ensembles complexes, approfondissant ainsi notre compréhension. Un projet de recherche sur la preuve est discuté comme un exemple d'une telle perspective.

Introduction

Mathematics education research often describes the teaching-learning phenomena through conceptual dipoles, including relational/instrumental understanding (Skemp, 1976), conceptual/procedural knowledge (Hiebert & Lefevre, 1986), deep/surface approaches (Marton & Säljö, 1976), process/object (Dubinsky, 1991, Sfard, 1991). Though the value of these perspectives is well-documented, researchers have acknowledged the need for developing more complex approaches, either by extending the dipoles to n-dimensional models (for example, process/object/precept, Gray & Tall, 1994; deep/surface/achieving, Biggs, 2001) or by conceptually re-visiting the phenomena with a completely new theoretical perspective (for example, a systems theory approach that focuses on the relationships amongst the constituting agents of a phenomenon; Davis & Simmt, 2003; Moutsios-Rentzos & Kalavasis, 2012).

At the crux of each of these approaches is the researchers' *perceived proximity* amongst the phenomenon, the theoretical approach and the employed methods: each of the researchers' decisions is determined by whether or not they consider a theory or a method relevant, close, to the under investigation phenomenon. In this paper, we attempt to topologically re-approach the notion of proximity in mathematics education, considering three aspects: a) *socio-cultural proximity*, b) *mind-brain proximity*, c) *cognitive-affective proximity*.

It is posited that by topologically re-defining proximity, multiple perspectives, theories and methods maybe considered to be relevant to the phenomenon, thus providing the opportunity to gain deeper understandings of the complexity of the phenomenon. Nevertheless, mathematics education theorists have emphasised the dangers that such attempts entail, as they may lead to a bricolage of theories, lacking meaningful conceptual coherence (see, for example, Wedege, 2010; Bikner-Ahsbahs & Prediger, 2014). Thus, it is argued that the re-defined proximity should draw upon a theory linked with the under investigation phenomenon that may act as a meaningful attractor of the different theoretical perspectives concentrated in the different aspects of the phenomenon, thus allowing the *networking* (Bikner-Ahsbahs & Prediger, 2014) of different perspectives.

The paper concludes with a brief presentation of the implications of such an approach to research project focussed on proving in goal-oriented exam-type tasks (Moutsios-Rentzos, submitted). In this example, Skemp’s theory of social survival and internal consistency (1979) is utilised as the meaningful attractor that facilitates the appropriate networking of the diverse perspectives.

Proximities: socio-cultural, mind/brain, cognitive/affective

In this paper, we discuss the proximities within two research loci in accordance with two of the major strands of educational research: the sociocultural locus and the psychological locus.

Socio-cultural proximities

The contemporary societies are characterised by socio-cultural diversity, linked with the socio-cultural permeability offered by the technological and financial feasibility to travel. This is more evident in the big urban areas within a country and/or in the so-called ‘developed’ countries where the opportunities (about, amongst others, career and entrainment) appear to be greater and more diverse, thus gathering the interest of larger and more heterogeneous parts of the population.

Mathematics education researchers have noted that socio-cultural proximity has been re-defined within the same country and even the same city, acknowledging the multifaceted challenges that mathematics education encounters in a globalised society in their research reports (Atweh et al, 2007) and in their meetings (see the relevant sub-theme in CIEAEM 66).

Within the same school unit, the time-space is expanded to include virtual social networks, re-defining the traditional power relationships amongst the protagonists (including, amongst others, students, teachers, principals, parents). Though the expanded school time-space maximises the permeability of the school unity system, the protagonists’ constructions of the new reality seem to qualitatively differ with respect to the relevance of the virtual social networks with the in-school mathematics teaching and learning phenomena (Moutsios-Rentzos, Kalavasis & Sofos, 2013).

Moreover, within the same city, the students attending a ‘multi-cultural’ school unit have been found to differ in their perceived parental involvement about mathematics (Kafousi, Moutsios-Rentzos & Chaviaris, 2014). For example, the students attending a ‘multi-cultural’ school appear to experience stronger perceived parental involvement, which appears to be relatively stable with respect to the students’ age and to the different calendar years (Moutsios-Rentzos, Chaviaris & Kafoussi, submitted).

Furthermore, the socio-cultural diversity appear to become less evident as the countries adopt international standards (International Baccalaureate, credit units) and evaluations (PISA, TIMSS). Nevertheless, the attempted convergence of the curricula seems not to be in line with the characteristics of educational systems evaluated as ‘excellent’ (Sahlberg, 2011).

Moreover, in countries that are geographically and socio-culturally ‘distant’, aspects of the ways that mathematics are conceptualised appear to be common. For example, the findings of a comparative study (Moutsios-Rentzos, da Costa, Prado & Kalavasis, 2012) conducted in Brazil and in Greece investigating the views that school principals hold about mathematics (both as a discipline amongst disciplines and as a school course amongst other courses) included convergences in the identified epistemic views, which are also evident in their views about mathematics as a school course.

Consequently, the socio-cultural proximity as a factor with important implications in the teaching and learning mathematics seems to be in need of a topological re-definition in order to regain its analytical and descriptive power. Such a re-definition includes the assignment of not necessarily geographical aspects (including cultural identity, technological abilities, social networking etc) to the traditionally considered as mere geographical characterisations (for example, the classroom, the school unit, a city, a country), thus transforming them in topological characterisation of the contemporary expanded socio-cultural time-space.

Brain/mind proximities

Neuroscience has identified invariant characteristics of the human species, revealing links between brain activity and mathematical thinking, with respect to both its specific and its more general aspects. For example, considering the calculation processes, 'exact arithmetic' (such as a single arithmetic operations) appears to be linked with word-associated brain activity, whilst 'approximate arithmetic' seems to require bilateral visuo-spatial processing (Dehaene, Spelke, Pinel, Stanescu & Tsivkin, 1999). On the other hand, considering broader aspects of mathematical thinking, left hemisphere activity has been linked with logico-mathematical reasoning and problem-solving (Bear, Connors & Paradiso, 2007; Rayner, 1998).

Furthermore, research evidence appear to support the innate character of various cognitive processes or affective processes. For example, subitising has been suggested to be 'built-in' the humans' mental artillery (Lakoff & Núñez, 2000). Moreover, seven of the human's affective responses and alertness to a stimulus –identified as the seven basic emotions, namely sadness, anger, contempt, fear, happiness, disgust, surprise– have been found to transcend socio-cultural contexts in their manifestation in the human facial expressions, thus appearing to be universally common to humans (Ekman, 1992).

On the other hand, the reported brain universality seems to be in stark contrast with mathematics education research reporting the situated –cognitive and affective– character of mathematical thinking and learning (Brown, Collins & Duguid, 1989; Hannula, 2012), as well as the diverse conceptualisation of mathematics itself (consider, for example, evidence from the ethnomathematics research paradigm; D'Ambrosio, 1985). Furthermore, considerable research evidence seem to question traditionally conceptualised stereotypical views of innate characteristics (such as gender; Walkerdine, 1998).

At this point, it is emphasised that the contrasted research evidence are usually documented with epistemologically 'opposed' methodologies and respective research questions. For example, the reported neuroscience research evidence are based on objectively measured variables, in the sense of a natural magnitude (electricity, movement etc) that can be measured with an instrument, thus condensed to a single number. On the other hand, the aforementioned mathematics education research evidence draw upon softer qualitative techniques, including observations and interviews, the results of which are obtained through interpretative methods and are communicated by diverse means, notably text. These accord with Radford's (2008) triadic view of theory including "a system, *P*, of basic principles […] a *methodology*, *M*, which includes techniques of data collection and data-interpretation as supported by P […] a *set*, *Q*, of paradigmatic *research questions*" (Radford, 2008 , p. 320).

Following these, notwithstanding their conceptual differences, we posit that by co-considering the evolutionary derived brain universality along with the socio-culturally derived noetic locality, by acknowledging the fact that mathematical thinking and learning occurs in and emerges from the interaction of both invariant and local characteristics, till now distant and incongruent theories and methodologies may converge to be meaningfully synthesised in a research project.

Cognitive/affective proximities

Considering the mental processes, the aforementioned can be localised in the well-documented differentiation between *mathematical thinking dispositions* (both about the representation and the processing of the information; Burton, 2001; Duffin & Simpson, 2006) and *actual, task-specific mathematical thinking*. Research evidence (Moutsios-Rentzos, 2009) identify complex links between the two levels, adding a broader third level of *general thinking dispositions* that go beyond mathematics and extend to cover thinking in general.

In a similar vein, *affective dispositions* (values, beliefs, attitudes etc) have been identified with respect to *mathematics* in specific or to *education* in general (Hannula, 2012), which also appear to be in complex relationships with *actual task-specific affective experiences* about mathematics (Moutsios-Rentzos & Kalozoumi-Paizi, 2014).

Hence, it is posited that the affective-cognitive proximity could be re-visited in order to identify topological defined loci within the aspects of the six-dimensional space formed by the affective-cognitive and dispositions-actual interactions spectra.

Instead of a conclusion: re-approaching proximity in proof research

Re-approaching the theoretical-methodological proximity in proof research produced a multi-levelled methodological-theoretical framework (Moutsios-Rentzos, submitted). The proposed framework incorporates the topological proximity of the affective and the cognitive, of the disposition and the actual, of the subjectively perceived/reported and the objectively bodily expressed/measured in the case of thinking about exam-type university proving questions.

It is posited that the aforementioned ostensible dipoles *assume topological proximity* through Skemp’s (1979) theoretical framework about both social and inner survival that the learners strive to achieve in goal oriented activities. Skemp (1979) theorised that the learners need to survive socially by meeting the socially accepted, usually externally set criteria of a task, as well as internally in the sense of surviving a task in ways that are consistent with the learners’ internal reality (that may include affective and cognitive aspects).

Following these, the research project focused on the proving processes of mathematics undergraduate students, when attempting to produce exam acceptable proving questions in the form ‘Given that … Prove that…’ (Moutsios-Rentzos, submitted). The multi-level approach brought together different levels and aspects of the phenomenon, along with diverse methods:

1) Actual task-specific cognitive and affective experiences
   a) The students’ *cognitive strategies*, referring to the students’ actual, task-specific dealing with a proving question. Clinical interviews (in the sense of Ginsburg, 1981) were conducted utilising the A-B-Δ proving strategy classification scheme (Moutsios-Rentzos & Simpson, 2011). The chosen conceptualisation of strategy and methodology were chosen in line with the research question posed: the identification of the qualitatively different thinking strategies employed when the students actually produce an exam-acceptable proof.
   b) The students’ *basic emotions* (as defined by Ekman, 1992) as they produce and present the solution. The identification of the emotions was based on the students’ video-taped facial muscle movements with a researcher trained to utilise the ‘Emotional Facial Action Coding System’ (EMFACS; Ekman, Irwin & Rosenberg, 1994). The chosen theory implies that in this project we are interested in the students’ universal, objectively measured and evolutionary derived affective reactions during their proving process. These emotions are clearly differentiated from the mentally processed, socially situated, affective reactions towards a proving situation (see Hannula, 2012).
2) Cognitive and affective dispositions.
   a) The students’ reported *general thinking dispositions*, as conceptualised by Sternberg’s *thinking styles* and measured by the self-report Thinking Styles Inventory (Sternberg, 1999). With this conceptualisation and methodology, the focus is on the self-reported and experienced cognitive patterns, as identified by each participants.
   b) The students’ *affective dispositions (attitudes and beliefs) towards mathematics and exams*, as reported by the students in the semi-structured interviews that accompanied the

students' answering the proving questions. The semi-structure interview was specifically designed for this project and is focussed on mathematics-specific self-reported and experienced affective patterns.

3) The *socio-cultural effect* as manifested by the students' country of residence, with respect to the teaching and learning mathematical proof and importantly for this project with respect to the assessment realities that the students experience. For this purpose, students from Germany and Greece participated in the study.

The conducted analyses were quantitative and/or qualitative in line with the topologically attracted to be neighbours yet clearly distinct theoretical considerations of each level (in line with Radford, 2008).

Notwithstanding the important epistemological questions that may be raised (extensively discussed in the literature; Johnson & Onwuegbuzie, 2004; Smith & Heshusius, 1986), it is argued that the proposed research synthesis derives from a theoretical framework appropriately chosen to fit a situation that acts as a *meaningful attractor* that allows the appropriate networking (Bikner-Ahsbahs & Prediger, 2014) of diverse, seemingly incongruent, theories and methods, crucially avoiding the syncretism pitfall.

Consequently, within the aforementioned framework, it is possible to study the mathematics teaching-learning phenomenon of dealing with exam-type proving questions as complex whole, without losing the analytic capability to investigate its (partial) aspects.

By topologically re-approaching proximity through a theory (specific to the under investigation phenomenon) acting as a meaningful attractor, the phenomenon re-claims its meaningful research wholeness, in ways that go beyond the mere sum of the studies of its parts.

REFERENCES

Bikner-Ahsbahs, A., & Prediger, S. (2014). *Networking of theories as a research practice in mathematics education*. Springer.

Atweh, B., Barton, A. C., Borba, M. C., Gough, N., Keitel, C., Vistro-Yu, C., & Vithal, R. (Eds.). (2007). *Internationalisation and globalisation in mathematics and science education*. Dordrecht: Springer.

Bear, M. F., Connors, B. W., & Paradiso, M. A. (2007). *Neuroscience: Exploring the Brain*. PA: Lippincott Williams & Wilkins.

Biggs, J. B. (2001). Enhancing learning: A matter of style or approach? In R. J. Sternberg & L. F. Zhang (Eds.), *Perspectives on thinking, learning, and cognitive styles* (pp. 73-102). Mahwah, NJ: Erlbaum.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18*(1), 32-42.

Burton, L. (2001). Research mathematicians as learners – and what mathematics education can learn from them. *British Educational Research Journal, 27(5),* 589-599.

D'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. *For the Learning of Mathematics, 5*(1), 44-48.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education, 34*(2), 137-167.

Dehaene, S., Spelke, E., Pinel, P., Stanescu, R., & Tsivkin, S. (1999) Sources of mathematical thinking: behavioral and brain-imaging evidence. *Science, 284*, 970-974.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Dordrecht: Kluwer Academic Publishers.

Duffin, J. M., & Simpson, A. (2006). The transition to independent graduate studies in mathematics. In Hitt, F., Harel, G. & Selden, A. (Eds.), *Research in collegiate mathematics education VI* (pp. 233-246). Oxford: Oxford University Press.

Ekman, P. (1992). An argument for basic emotions. *Cognition & Emotion*, *6*(3-4), 169-200.

Ekman, P., & Friesen, W. V. (1978). *Facial action coding system: a technique for the measurement of facial movement*. Palo Alto, Calif.: Consulting Psychologists Press.

Ekman, P., Irwin, W., & Rosenberg, E. L. (1994). *The emotional facial action system coding (EMFACS)*. London, UK.

Ginsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: aims, rationales, techniques. *For the Learning of Mathematics, 1*(3), 57-64.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education, 25(2),* 115-141.

Hannula, M. S. (2012). Exploring new dimensions of mathematics-related affect: embodied and social theories. *Research in Mathematics Education*, *14*(2), 137-161.

Johnson, R. B., & Onwuegbuzie, A. J. (2004). Mixed methods research: A research paradigm whose time has come. *Educational Researcher*, *33*(7), 14-26.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (σελ. 1-27). Hillsdale, NJ: Lawrence Erlbaum.

Kafousi, S., Moutsios-Rentzos, A., & Chaviaris, P. (2014). An investigation on the school socio-cultural identity and the perceived parental involvement on mathematics learning in Greece. Proceedings of CIEAEM 66. *Quaderni di Ricerca in Didattica (Mathematics), 24*(1), 312-316.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from.* New York: Basic Books.

Marton, F. & Säljö, R. (1976). On qualitative differences in learning - i: outcome and process. *British Journal of Educational Psychology*, *46*, 4-11.

Moutsios-Rentzos, A. (submitted). *Realities and realisations in proving journeys to proof.*

Moutsios-Rentzos, A. (2009). Styles and strategies in exam-type questions. In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.), *Proceedings of PME 33* (Vol. 4, pp. 145-152). Thessaloniki, Greece: PME.

Moutsios-Rentzos, A., da Costa, N. M. L., Prado, M. E. B. B., & Kalavasis, F. (2012). The interrelationships of mathematics and the school unit as viewed by in-service school principals: a comparative study. Proceedings of CIEAEM 64. *International journal for mathematics in education, 4,* 440-445.

Moutsios-Rentzos, A., Chaviaris, P., & Kafoussi, S. (submitted). *School socio-cultural identity and perceived parental involvement about mathematics learning in Greece.*

Moutsios-Rentzos, A**.**, Kalavasis, F., & Sofos, A. (2013). The mathematics classroom in the expanded open system of a globalised virtual social network. Proceedings of CIEAEM 65. *Quaderni di Ricerca in Didattica (Mathematics), 23*(1), 492-496.

Moutsios-Rentzos, A., & Kalavasis, F. (2012). The interrelationships of mathematics and the school unit as viewed by prospective and in-service school principals: a systems theory approach. Proceedings of CIEAEM 63. *Quaderni di Ricerca in Didattica (Mathematics), 22*(1)*,* 288-292.

Moutsios-Rentzos, A., & Kalozoumi-Paizi, F. (2014). Odysseus' proving journeys to proof: an investigation on cognitive and affective realities. Proceedings of CIEAEM 66. *Quaderni di Ricerca in Didattica (Mathematics), 24*(1), 290-296.

Moutsios-Rentzos, A., & Simpson, A. (2011). University mathematics students and exam-style proving questions: The A-B-Δ strategy classification scheme. Proceedings of CIEAEM 64. *International journal for mathematics in education, 3,* 45-64.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education, 40* (2), 317-327.

Rayner, K. (1998). Eye movements in reading and information processing: 20 years of research. *Psychological Bulletin, 124*, 372–422.

Sahlberg, P. (2011). *Finnish lessons*. NY: Teachers College Press.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77,* 20-26.

Skemp, R. R. (1979). *Intelligence, learning and action*. New York: John Wiley & Sons.

Smith, J. K., & Heshusius, L. (1986). Closing down the conversation: The end of the quantitative-qualitative debate among educational inquirers. *Educational Researcher*, *15*(1), 4-12.

Sternberg, R. J. (1999). *Thinking styles*. Cambridge: Cambridge University Press.

Walkerdine, V. (1998). *Counting girls out. Girls and mathematics.* London: Falmer Press.

Wedege, T. (2010). Connecting theories in mathematics education: From bricolage to professionalism. *Nordic Studies in Mathematics Education, 15*(1), 59-78.

Symmetry in portuguese fishing communities: students critical sense while solving symmetry tasks

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Abstract: This article analyzes the knowledge and the critical thinking level of students from two schools of different cultural contexts, regarding the mathematical topic of symmetries. The primary goal is to find and compare the critical thinking of students from a fishing community with students of a school of an urban type, when faced with tasks on symmetries involving artefacts / geometric motifs related to fishing activity. With a theoretical foundation based on Ethnomathematics, it gives prominence to the cultural and professional context students are from, so tasks were built taking into account the fishing context. Students revealed difficulties in symmetries identification and particularly in the representation of figures with symmetry. It also found that in tasks completion, students of the fishing context school proved to have a more accurate critical sense than students from the urban type school.

Résumée : Cet article analyse les connaissances et le niveau de la pensée critique des élèves de deux écoles de différents contextes culturels, concernant le sujet mathématique de symétries. L'objectif principal est de trouver et de comparer la pensée critique des élèves d'une communauté de pêche avec des étudiants d'une école de type urbain, lorsqu'ils sont confrontés à des tâches sur les symétries, comprenant des artefacts / motifs géométriques liés à l'activité de pêche. Avec une base théorique basée sur Ethnomathématique, il met en évidence le contexte culturel et professionnel d'ou les étudiants viennent, alors les tâches ont été construites en tenant compte du contexte de la pêche. Les étudiants ont révélé des difficultés dans l'identification des symétries et en particulier dans la représentation des figures avec symétrie. On a également constaté que dans l'achèvement des tâches, les élèves de l'école de contexte de pêche ont montré d'avoir un sens critique plus précis que les élèves de l'école de type urbain.

Objectives and main idea

From the most basic act of drawing a line segment in carpentry, to the more complex geometric reasoning in tracing a large bridge, there is geometry knowledge being applied. Research in the context of professionals with low educational level, demonstrate a strong presence of mathematics and the use of geometry in their everyday professional activities (Sousa, 2006; Lucena, 2002; 2005). In this work we will take into account geometric motifs used in two fishing communities routines, with special attention to symmetries. The focus on symmetry relates to the fact that there are investigations on this subject in professional groups (Sousa, 2006; Vieira, Palhares and Sarmento, 2008), but also because curriculum documents recommend work with isometries and symmetries in elementary schools. Still, students reveal many difficulties when faced with situations involving symmetries. Being this work grounded on ethnomathematics, one of the central concerns (besides investigating fishing context situations revealing the application of symmetry), is to assess students critical sense in performing tasks on symmetries, in two communities, one belonging to a fishing community the other not.

In this research work, we highlight the symmetries of reflection, rotation and translation. These are the ones identified, as a result of previous fieldwork, as used in daily life of fishing communities.

Ethnomathematics

Ethnomathematics is the theoretical foundation that frames this work both from a cultural point of view and from mathematics education. Etymologically, the word Ethnomathematics can be understood as the art or technique (techne = tica) to explain or to understand reality (matheme) within a specific cultural context (ethno) (D'Ambrosio, 2012). Ethnomathematics, rather than an association to ethnic groups (D'Ambrosio, 1998), is the research of mathematical practices and conceptions of a social group, including also an educational work (Oliveira, 2004; Miller, 2004) which develops in order to identify and decode the group's knowledge and draw comparisons between knowledge of everyday life and academic knowledge (Knijnik, 2008).

It is multicultural societies like ours that educators should reflect on which culture should be considered in the classroom. The dominant? Of the minority? Or maybe we can create a new culture, made up of all the cultures of all citizens living in a given territory. Knowledge and culture are the two foundations of mathematics. So we can assume that in a given area where there are many cultures, we are not facing the Ethnomathematics of these cultures, but in the presence of several (ethno) mathematics (Sousa, 2006) developed over years by cultures in that territory. From the epistemological relativism and demarcation of cultures as interpretation systems of the world, maybe we can talk about ethnomathematics or multimathematics (Oliveras, 2006).

In this paper we give special emphasis to the Ethnomathematics within the context of fishing cultures and also within classroom contexts. We look for a didactic transposition, so that Ethnomathematics, seen as the math arising from social groups (D'Ambrosio, 2006), help in this contextualization process, but also in humanizing mathematics (Palhares, 2012). Thus, the position assumed is taken from D'Ambrosio (1993; 1998; 2002); Olive (2004); Monteiro (2004); and Knijnik (2008), who evoke an investigative nature on the mathematics present in social minorities and an educational character, aiming to combine the mathematical knowledge of everyday life with the formal / school mathematical knowledge. The investigative nature of this work concerns the collection of everyday (ethno) mathematical aspect within fishing communities. The educational refers to the context and implementation of these (ethno) mathematics in the classroom, after analysis and cultural and curricular consideration.

Methodology

Qualitative research is our methodological framework because it is based on a holistic view of the particular setting to be researched without isolating it from its natural context (Amado, 2013).

One of the specific methods used was the multiple-cases study in ethnographic context. With it was possible to observe in detail individuals in each of the specific contexts (Lessard-Herbert, Goyette & bouti, 1994).
We also used participant observation, unstructured interviews (with the help of audio and video recordings) and document analysis, mainly documents and small teaching experiences in the classroom context.

It is therefore intended, in accordance with the theme "symmetry", to answer the question:

- What level of critical sense do students of the Caxinas fishing community reveal in comparison with the most urban contexts of other students?

We prepared a set of tasks, taking into account the curricular documents, as well as age and cultural background of the students. Its focus is on symmetry, while the context addresses the fishing everyday. We tried to draw up various tasks on this content, asking students different levels of knowledge regarding the symmetry.

The proposed tasks are contextualized tasks arising from the fieldwork carried out in fishing communities of Câmara de Lobos (Madeira) and Caxinas (Vila do Conde).

The cultural context involves boat construction, objects used in daily fishing and tiles used in the homes of fishermen. The mathematical content is symmetries.

Tasks that were used, adopting the knowledge that fishermen use in their work, were validated with regard to the suitability to the specific context of these fishing communities, but also validated by a panel of mathematics education experts.

Each task was applied in the classroom, in two separate phases. Initially, the tasks were applied before the approach to the subject related to symmetry. After teaching symmetry, the same tasks were reapplied to the same students. The tasks were applied in 2 groups of 5 and 6 years of schooling (10-12 years). The tasks were applied in two schools with different cultural and professional contexts: one belongs to the geographical area of a fishing community, and the other, is embedded in an urban school. The tasks were applied with 45 students of fishing context school, and 39 students from another school.

To apply this set of tasks in the classroom, three 90 minutes periods were dedicated in each class. Students individually solved each task to try to understand the concepts / knowledge that each student had about the symmetries and the critical sense of the students in relation to their own resolutions tasks. Their arguments and strategies were recorded on paper and on video.

Symmetries and critical sense

School of the fishing community of Caxinas

Critical sense about the rigour in the construction of artifacts

In some artifacts of this task (Figures 1 and 2), the overwhelming majority of students feel they have symmetry, but they question the fact that the objects were not built with mathematical rigor necessary to have symmetry. In other words, students admit that the craftsman who built each artifact intended to build with the symmetry and therefore consider themselves to have symmetry, but mathematically think they should be more "perfect". From the close observation of the represented figures (Figure 1 and Figure 2) and the symmetry axis represented by the students, the objects in all the rigor do not have symmetry of reflection due to imperfections in artisanal construction. Students turn out to be critical of this and despite considering the objects have symmetry (because the craftsman have that intention at the time of manufacture), make the repair that the final product should have more harmony. In this situation the combination of knowledge of everyday life and the formal school knowledge is evident, but also the critical sense with regard to the construction of the artifacts and the presence of symmetry in them.

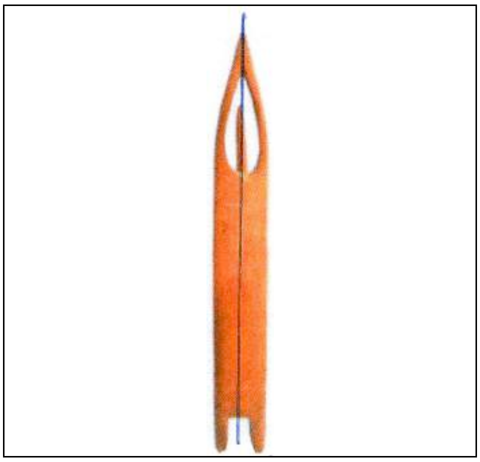

Figure 1. fishing nets sewing needle

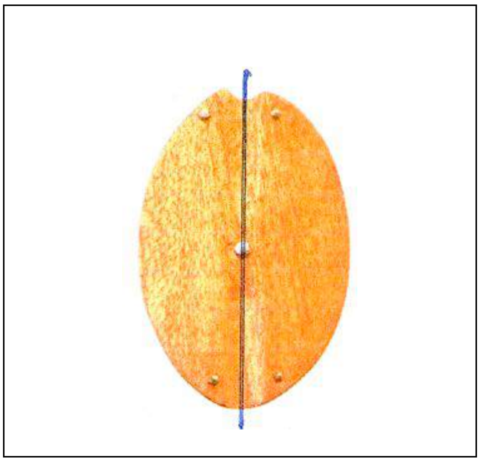

Figure 2. Davit

Critical sense about errors

In this task it is intended that students complete in Figure 3 in accordance with the represented part, obtaining a helix with rotational symmetry. It is found that the students reveal it hard to correctly complete the picture.

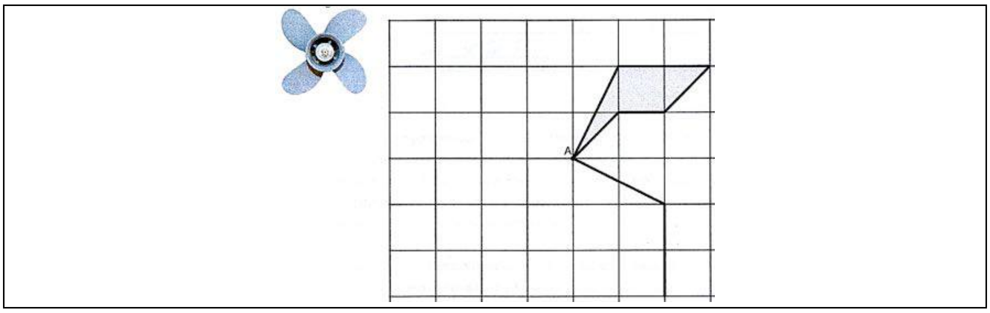

Figure 3. Propeller for students to complete

Taking as its starting point the already represented blade (1st quadrant), the overwhelming majority of students can correctly complete the representation of the 4th quadrant blade, but few can represent all blades correctly as in the representation of one of the students (Figure 4).

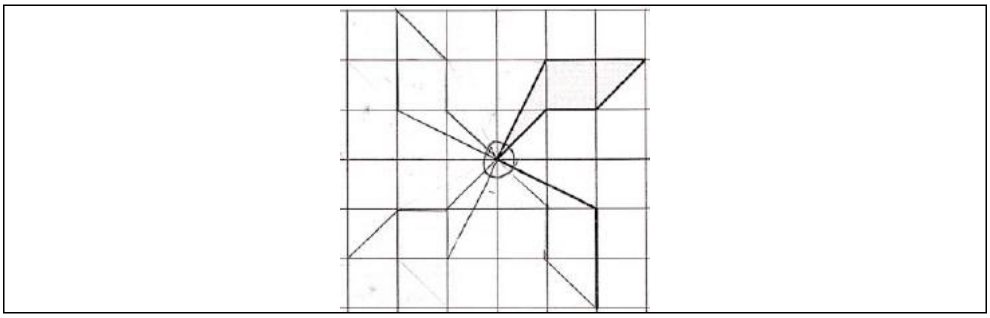

Figure 4. Correct representation of the propeller after several attempts

Although the student has managed to correctly represent the propeller, there were several unsuccessful attempts as seen in the transcript of the field work (Figure 5). One can also verify that the student recognizes that something is not correct, identifies the part of the picture that is not correct, but does not know why it is inaccurate. His keen critical sense in relation to his representations is salient, erasing immediately each blade that he considers not to be visually in harmony with the rest.

Student - Teacher, I have a doubt here. The propeller leaves, is it like this? (Pointing to the propeller blade already represented in the figure).
Student - Ahhh ... wait. (Deletes the part he feels it is wrong and represents again).
Student - I know. (Correctly representing the blade of the 4th quadrant).
Student - This (the blade of the 3rd quadrant) must be reversed.
Researcher - Reversed how?
Student - This part must be here.

Figure 5. Dialogue on propeller representation

The greatest difficulties arise in the 2nd and 3rd quarters. A considerable part of the students complete the picture according to Figure 6.

In this case the student begins to represent the blade in 2nd quadrant by reflecting the blade of the first quadrant. He watched for a few seconds the representation stating, "The blade is in reverse." The student makes several attempts, but fails to correctly represent the propeller. The final product, is given in Figure 6, however, the student said that the representation is not correct, that is, despite failing to represent the propeller with correction, the student is aware that its representation is not correct and is able to identify the parts of the figure to be changed. There is in this situation, strong evidence that the student is critical of his work because he recognizes that something is not right in the representation. The student delivers the task saying, "Teacher, this is not right." This is just one example of many students who, although can not properly represent the propeller, reveal quite critical sense and assume that the representation is not correct.

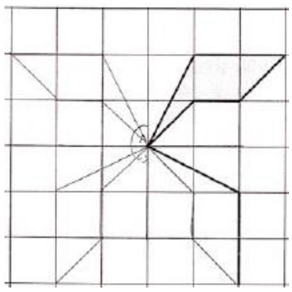

Figure 6. Incorrect propeller representation

School of Calendário

Critical sense about the rigour in the construction of artifacts

In the school of Calendário a large part of the students indicated that the artifacts shown in Figure 7 have only the vertical axis of reflection symmetry. However a considerable number of students said that the artifact to the right (Davit) has vertical and horizontal axis of reflection symmetries; others indicate 4 symmetry axes (Figure 9) and still others believe that the objects do not have symmetry (Figure 8). Only two students point out that both artifacts were not built with due rigor so that mathematically can be considered to have symmetry. The remaining relate nothing regarding this aspect. They do not reveal a critical keen sense as happens in the fishing environment school. In the city context in which the school is located, students externalize little sensitivity to analyze everyday aspects, at least from fishing everyday. They have difficulties in mobilizing fishing everyday knowledge for math classes and connect this knowledge with mathematics learning on the classroom. In the fishing context of the tasks these students are less critical than students from the fishing community of Caxinas. Although many of the students from both schools recognize that artifacts have symmetry, students of the school of Calendário tend to be more mathematically formal (not more able) which may affect the critical sense in relation to the fishing everyday situations.

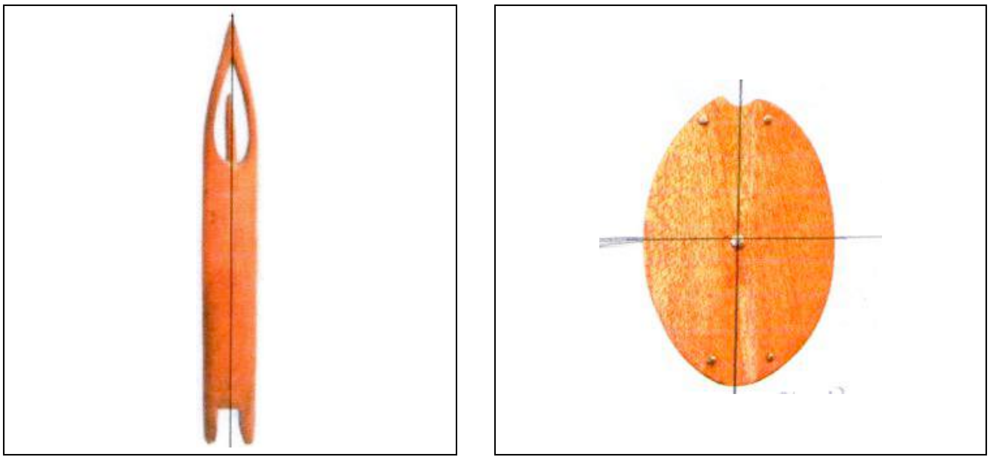

Figure 7. The student believes that the artifacts have one or two axes of symmetry.

Figure 8. Students consider the artifacts have no symmetry

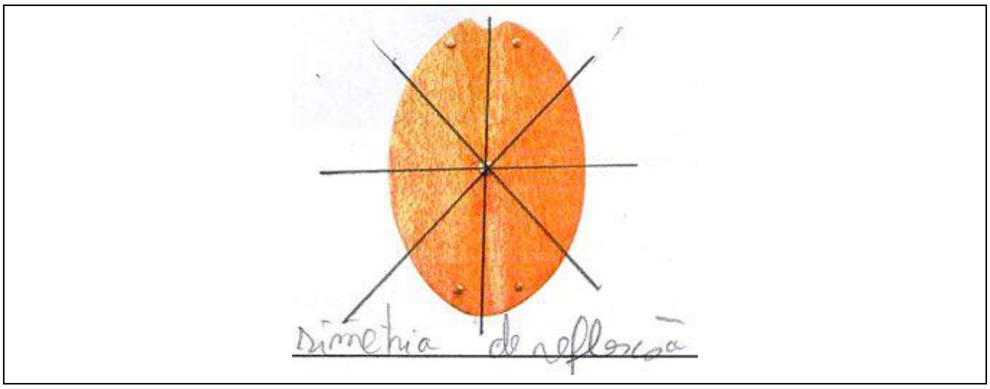

Figure 9. Students consider that the artifact has 4 axes of symmetry

4.1.1. Critical sense about errors

At the school of Calendário most students also failed to properly represent the propeller. Most students completed the propeller as shown in Figure 10. The students completed the helix so that it had vertical axis of reflection symmetry, not realizing that the symmetry involved was rotational symmetry.

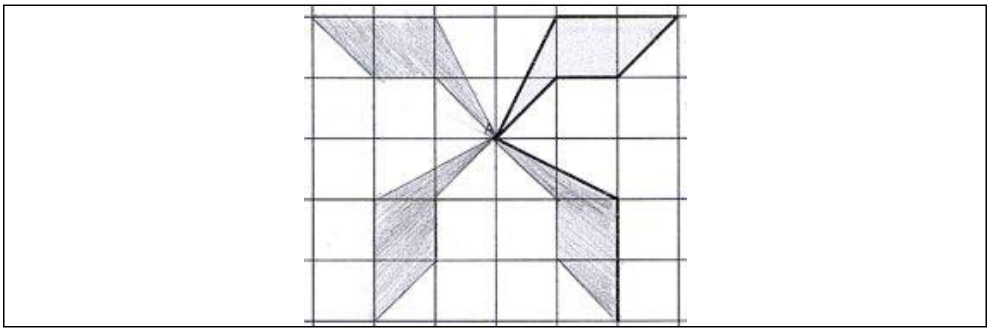

Figure 10. Incorrect representation of the propeller (with vertical axis of reflection symmetry)

In the following cases, the propellers presented do not have any symmetry, although students use isometries in their reproductions. In Figure 11 the student properly represent the blades of 3rd and 4th quadrants using rotation, but in the blade of the 2nd quadrant uses reflection. In Figure 12 and Figure 13 students use rotation on all the blades, but represent incorrectly the blades of 2nd and 3rd quarters. In Figure 14 the student tries to represent the helix so it has vertical axis of reflection symmetry, however, the blade of 4th quadrant is not correct. In addition to this failure it is not correct to build the propeller so that it has symmetry of reflection, as in reality it has only rotational symmetry.

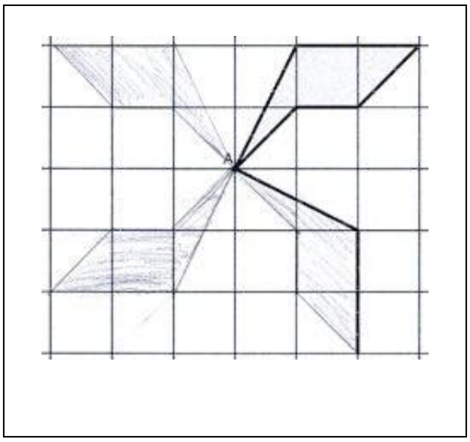

Figure 1. Incorrect representation of the helix (no symmetry)

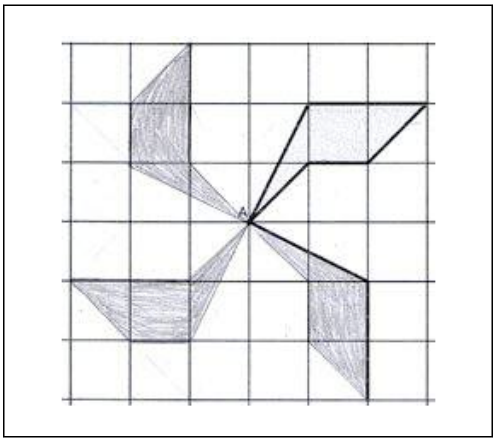

Figure 2. Incorrect representation of the helix (no symmetry)

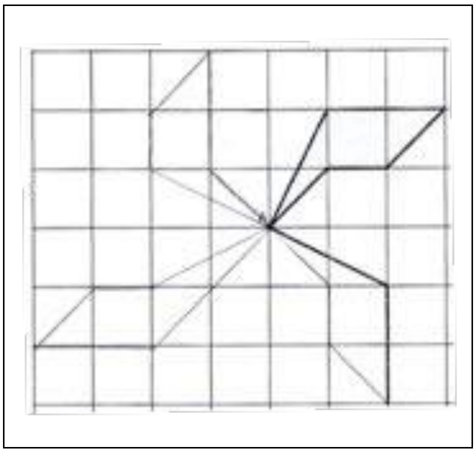

Figure 3. Incorrect representation of the helix (no symmetry)

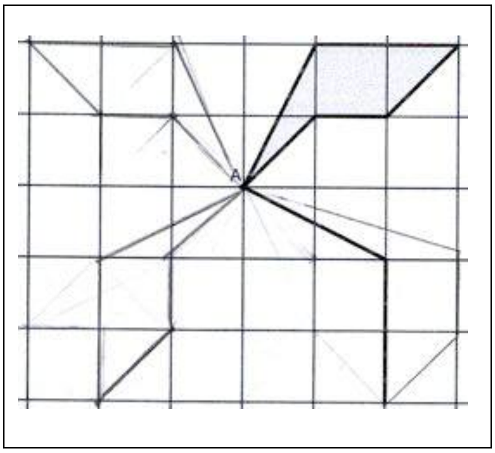

Figure 4. Incorrect representation of the helix (no symmetry)

Most students make mistakes of this kind in their representations, which reveals something unusual. More curious still is that students consider that their representations are correct as shown in the testimony of this one student (Figure 15).

Student - Teacher, I have done.
Researcher - Do you think it is right?
Student - Yes, it's right. It was a little difficult and I had to delete a few times, but I could do it right.

Figure 15. Student opinion on the representation of the figure 10

In this task there was a huge disparity in the critical sense of students. At Caxinas school students sometimes fail in their representations, but are aware that they are wrong and even warn the teacher that something is not right in their representations. In the school of Calendário students find it difficult to complete the propeller, fail in their representation, however they consider their representations correct and do not realize they are wrong. There is a number of situations in this school where the propellers represented not even have symmetry, yet the students feel they have symmetry. These facts reveal that the level of critical sense of students in these two schools are very different realities.

Conclusion

The implementation of the tasks in two schools of different cultural contexts and at different times (before and after teaching of symmetries), allow us to assess the level of critical sense that students have with regard to mistakes and to results in their resolutions. Given that the tasks were built within fishing contexts, the discrepancies in levels of critical thinking among students from both schools are well visible. Students from the fishing community of Caxinas reveal a more refined critical than students of the other school. Students of the fishing community turn out to be more able to detect and censor errors both in resolutions and in obtained results. There has also been no significant changes regarding the performance and the level of judgment when tasks are applied before or after teaching of symmetries. Thus, one can also conclude that students immersion in school/formal mathematics had no influence on their critical thinking ability in the schools involved.

REFERENCES

Amado, J (2013). A investigação em Educação e os seus paradigmas. In João Ama (Coord.). *Manual de Investigação qualitativa em Educação.* Coimbra: Imprensa Universidade de Coimbra.

D'Ambrósio, U. (1993). Etnomatemática: um programa. *A Educação Matemática em Revis 1*(1), 5-11.

D'Ambrósio, U. (1998). *Etnomatemática: Arte ou técnica de explicar e conhecer.* 5ª Ediç São Paulo: Editora Ática.

D'Ambrósio, U. (2002). *Etnomatemática: o elo entre as tradições e a modernidade.* edição. Belo Horizonte. Editora Autêntica.

D'Ambrosio, U. (2006). *Ethnomathematics - Link between traditions and modern* Rotterdam, Netherlands: Sense Publishers.

D'Ambrosio, U. (2012). The program ethnomathematics: theoretical basis and the dynam of cultural encounters. *Cosmopolis. A Journal of Cosmopolitics/Revue de cosmopolitiq* (Ghent), 3-4, pp.13-41.

Gerdes, P. (2012). *Etnogeometria: Cultura e o despertar do pensamento geométrico.* B Horizonte, Boane: ISTEG.

Knijnik, G. (2008). Mathematics education and cultural diversity: peasant's mathematics struggle for land. In P. Palhares (Coord.), *Ethnomathematics: a look at the cultural divers and mathematics learning* (pp. 131-156). V. N. Famalicão: Húmus.

Lessard-Hérbert, M; Goyette, G; Bouti, G. (1994). *Investigação Qualitativa: Fundamento Práticas.* Lisboa: Instituto Piaget.

Lucena, I.C.R. (2002). *Carpinteiros Navais de Abaetetuba: Etnomatemática navega pe rios da Amazónia.* Tese de Mestrado. Natal: Universidade Federal do Rio Grande do Norte.

Lucena, I.C.R. (2005). *Educação Matemática, Ciência e tradição: tudo no mesmo bar* Tese de Doutoramento. Universidade Federal do Rio Grande do Norte.

Monteiro, A. (2004). A Etnomatemática em Cenários de Escolarização: alguns elementos reflexão. In G. Knijnik, F. Wanderer, & C. J. Oliveira (Orgs.). *Etnomatemática: Currícul Formação de Professores* (pp. 432-446). Santa Cruz do Sul: UNISC.

Oliveira, H. D. L. (2004). A Etnomatemática em Cenários de Escolarização: alguns elemen de reflexão. In G. Knijnik, F. Wanderer, C. J. Oliveira (Org.). *Etnomatemática: Currícul Formação de Professores* (pp. 305-322). Santa Cruz do Sul: UNISC.

Oliveras, M. L. (2006). Etnomatemáticas. De la multiculturalidad al mestizaje. In J. M. G Zabala (coord.). *Matemáticas e interculturalidad* (pp. 117-149). N.º 232. Barcelona: Graó.

Palhares, P. (2012). Mathematics Education and ethnomathematics: a connection in need reinforcement. *Redimat–Journal of Research in Mathematics Education, 1*(1), 79-92.

Sousa, F. (2006). *Pescadores e Calafates: a Etnomatemática na baía de Câmara de Lob* Tese de Mestrado. IEC - Universidade do Minho.

Vieira, L., Palhares, P., Sarmento, M. (2008). Etnomatemática: estudo de elemen geométricos presentes na cestaria. In P. Palhares (Coord.). *Etnomatemática: um olhar sobr diversidade cultural e a aprendizagem matemática* (291-315). Vila Nova de Famalic Humus.

WORKSHOPS / ATELIERS

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Which support technology can give to mathematics formative assessment?

The FaSMEd project in Italy and France

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Abstract: This workshop is focused on the role technology may play in supporting the formative assessment process. Different examples from the case studies developed in France and Italy within the European Project FaSMEd will be analysed and discussed. In order to highlight the choices we made in relation to the aim of the project, before discussing the examples we will introduce the methodology adopted in each country and the theoretical frameworks to which we refer for the planning and analysis of the activities.

Resumé: Le but de cet atelier est d'examiner le rôle que la technologie peut jouer dans un processus d'évaluation formative. Des exemples provenant d'études de cas réalisées en France et en Italie dans le cadre du projet européen FaSMEd seront analysés et discutés. Pour mettre en évidence les choix que nous avons faits dans le cadre de ce projet, nous introduirons la méthodologie qui a été adoptée dans les deux pays et les cadres théoriques de référence aussi bien pour la construction que pour l'analyse de ces activités.

Introduction

The idea for this workshop was born from the collaboration of the French team and the Italian team engaged in the European project titled FaSMEd (Improving progress for lower achievers through Formative Assessment in Science and Mathematics Education). The aim of the project is to investigate the role of technologically enhanced formative assessment (FA) methods in raising the attainment levels of low-achieving students. Our hypothesis is that connectivity can support

- *teachers* in collecting data from the students, making timely formative interpretations, and informing their future teaching and, on the other side,
- *students* in exploiting the received feedback to improve their learning.

In line with this hypothesis, FaSMEd investigates: (a) students' use of FA data to inform their learning trajectories; (b) teachers' ways of processing FA data from students using a range of technologies; (c) teachers' ways of using these data to inform their future teaching; and (d) the role played by technology, as a learning tool, in enabling the teachers to become more informed about student understanding.

The research is based on successive cycles of design, observation, analysis and redesign of classroom sequences (Swan, 2014) in order to produce and feed into a set of curriculum materials and methods for teachers, that is called "toolkit". The core of FaSMEd is constituted by the case studies involving different classrooms and feeding little by little the toolkit.

In this paper, after introducing the theoretical frame for the analysis of FA processes, we will present two examples, from our case studies. Both the examples are aimed at investigating the role played by technology in supporting FA processes, and focus in particular on how the teacher plans and implements a lesson, starting from the elaboration of different data from class activities, provided by the technological environment.

Our theoretical framework for Formative Assessment

According to the definition of FA to which the FaSMEd partners refer, FA is conceived as a method of teaching where

> “[...] evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited” (Black & Wiliam, 2009, p. 7).

Such learning evidences can be collected, interpreted and exploited by the teacher in different moments of the learning process and with different purposes. In particular, we focus on three central processes in learning and teaching proposed by Wiliam and Thompson (2007):

a) Establishing where learners are in their learning;
b) Establishing where learners are going;
c) Establishing how to get there.

Different agents are involved in these three processes: the teacher, the learners and their peers. Wiliam and Thompson (2007) conceptualise FA as consisting of five key strategies, that could be activated by these agents:

1) Clarifying and sharing learning intentions and criteria for success;
2) Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding;
3) Providing feedback that moves learners forward;
4) Activating students as instructional resources for one another;
5) Activating students as the owners of their own learning.

The following table (from Wiliam and Thompson, 2007, as quoted in Black and Wiliam, 2009, p. 8) synthetizes how the key strategies could be activated by the three agents, within the three central processes in learning and teaching:

	Where the learner is going	Where the learner is right now	How to get there
Teacher	1 Clarifying learning intentions and criteria for success	2 Engineering effective class-room discussions and other learning tasks that elicit evidence of student understanding	3 Providing feedback that moves learners forward
Peer	Understanding and sharing learning intentions and criteria for success	4 Activating students as instructional resources for one another	
Learner	Understanding learning intentions and criteria for success	5 Activating students as the owners of their own learning	

Fig. 1: FA according to Wiliam and Thompson (2007)

Effective feedback from the different agents involved in these different processes plays a central role in FA. According to Hattie and Timperley (2007), there are four major levels of feedback, influencing its effectiveness. They are:

(1) feedback about the task, which includes feedback about how well a task is being accomplished or performed;
(2) feedback about the processing of the task, which concerns the processes underlying tasks or relating and extending tasks;
(3) feedback about self-regulation, which addresses the way students monitor, direct, and regulate

actions toward the learning goal;

(4) feedback about the self as a person, which expresses positive (and sometimes negative) evaluations and affect about the student.

Analysis of our examples: focus and research questions

We will present and analyse one example from the French case studies and one from the Italian case studies. For each of them, we will introduce the context, give information about the design of the activity, and analyse a brief excerpt from the videos collected in the classroom.

The focus of the analysis will be: (a) the teacher's ways of using technology to foster formative assessment and in particular of referring to feedback from technology to inform and modify her teaching; (b) the students' (in particular low achievers) exploitation of feedbacks given by technology, the teacher and also the classmates, in order to improve their mathematical understanding.

The main research questions that will guide our analyses are:
1) Which aspects of formative assessment can be highlighted?
2) Are there evidences of the teacher's use of feedback to inform and modify her teaching? Are there evidences of the students' exploitation of feedback to improve their understanding?
3) What is the role of technology in supporting the actors involved in these processes in providing feedback to each other?

An example from the French case studies

In France the project is held by the École Normale Supérieure de Lyon and different schools at different levels are involved, from upper primary school to the first year of upper secondary school.

In primary classes (grade 4-5), the focus is on mathematics. Three teachers are working on fractions, using calculators TI-Primaire Plus, an interactive whiteboard, a student response system and a micro document camera. In lower secondary school (grade 6-9) and at the first year of upper secondary school (grade 10) both mathematics and science are involved in a coordinated way. In particular, in one lower secondary school in Lyon, mathematics and science teachers are organising the work around a common theme, namely magnitudes and measure, testing a student response system. They are encouraged to share methodologies and, if possible, activities that could be approached from both perspectives.

In the grade 10 classroom, as well as in another grade 9 classroom out of Lyon, every student is equipped with a tablet. Mathematics and science teachers are using connected classroom technologies. They have the possibility to pose questions to students and collecting the answers, and to check the work done by each student on her tablet in real-time.

These technologies were sometimes already present in the classroom due to school local projects (this is the case for the tablet classrooms) or chosen by the teachers according to their needs at the beginning of the FaSMEd project.

All the classes engaged in the project are mixed ability classes, and some of them include identified lower achievers. In addition, the majority of them are situated in the suburbs where the social context is often source of difficulties.

We consider that formative assessment is a process that is observable over a long period of time. Therefore, our methodology is built in order to catch information over time: the observations as windows open on the classroom at key moments, accompanied by teachers' auto-reflections and description of the whole scenario, from the introduction to the institutionalization of knowledge at stake, in reference to the Theory of Didactic Situations (Brousseau, 2004). Hence, we ask the teachers to fill in a grid of description where the following points have to be considered.

- Before the lesson: the prerequisites, the objectives, the planned organisation of the classroom (which tools, which technologies, individual or collective work,…), in reference to the instrumental orchestration (Trouche, 2004), forecast difficulties of the students and forecast answers to cope with them.
- After the lesson: brief summary of what happened in the classroom, possible gap with the forecast plan for the lesson.

This information is used by the researchers for preparing the observation and by the teachers for enriching their data for the process of formative assessment.

From the different observations carried out in the FaSMEd project, we present a case study in mathematics. It is a grade 9 tablet classroom where each student has his/her own tablet and is responsible for it during school hours. For networking tablets, the teacher (Thomas) uses the NetSupport School software that allows classroom monitoring, management, orchestration and collaboration. As a mathematical platform, Thomas decides to use Maple TA that is an online testing and assessment system designed especially for courses involving mathematics. The classroom is also equipped with an IWB. All the digital equipment has been provided by the school, since the classroom takes part in a school project about the integration of technology in the classrooms. The teacher has to appropriate such technologies also from a technical point of view. Nevertheless, what is completely new for him, from a didactical point of view, is the use of such technologies for formative assessment in his classroom.

The case study leans on a sequence about linear functions, where the following competences are to be acquired, according to the different representations of functions.

(a) Calculating and detecting images.

(b) Calculating and detecting inverse images.

(c) Recognising a linear function.

(d) Shifting from the graphical frame to the algebraic frame and vice versa.

Thomas decides to create a sequence of questionnaires around these four competences, using Maple TA. Following a typical Thomas' teaching sequence with Maple TA, we propose to analyse three specific episodes taken from our observations and referred to the third quiz proposed by Thomas to the students during this learning sequence about linear functions. The first moment concerns a student taking the quiz and the teacher declaring his potential use of the class' results. In the second episode, the teacher comments the quiz results of a student and, during the third excerpt, the teacher comments the whole set of the class' results.

First episode

A student (Mathieu) is working on a question concerning the competence (a): calculating and detecting images. Formulated in the graphical register of representation, the question is: "The curve below represents a linear function. The image of 9 is -2. True/False." Since he is working alone on the mathematical task, Mathieu is *active as the owner of his own learning*. He is reading the task that he has received from the teacher on Maple TA. In this first phase of the work, technology is used as a communication mean for sending tasks to the students.

Mathieu faces the didactic situation devolved by the teacher. After a while, he copies the question, leaves Maple TA, and pastes the question on the interactive environment of his tablet (OneNote) in order to work on the given graphical representation using his previous experience of such an exercise. On his screen, indeed, we can see a previously solved exercise that is very similar to the new one (Fig. 2a). Mathieu starts using the same graphical technique (Fig. 2b), mobilising his knowledge as a reflexive student (Margolinas, 2004). He has transformed the didactic situation into an a-didactic situation where he acts on a reacting milieu.

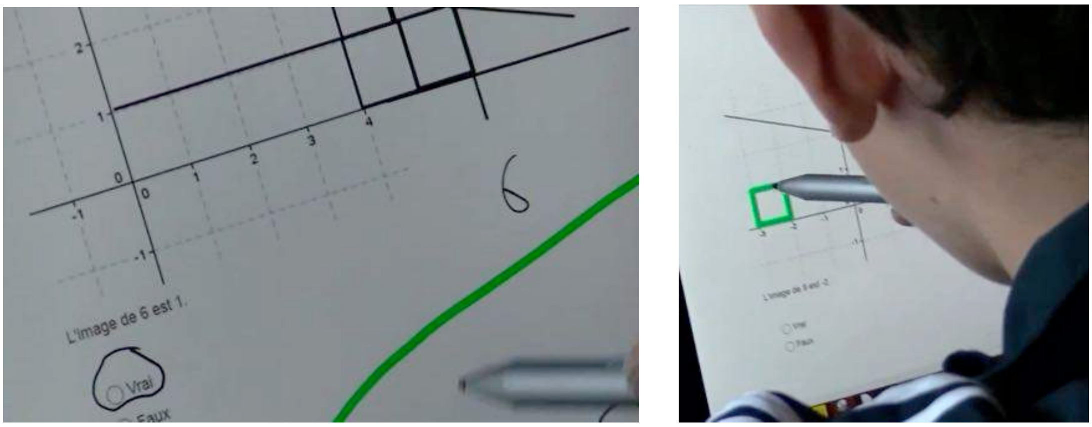

Fig. 2 a and b: Mathieu working on OneNote.

The student starts acting on the technology at his disposal, using it as an interactive environment, and the fact that the teacher has devolved to him the responsibility for solving the mathematical situation (devolution) is at the base of this action. Finally, Mathieu submits his answer, sending it back to the teacher.

To go further in our analysis, we can move on to the teacher's level. This dynamics occurs when the teacher is confronted to the students' answers, and uses technology for analysing such data. In our case, talking to another student, the teacher declares his potential strategies depending on the students' responses.

Thomas : "*I don't know if I'm going to take it into account or not. The idea is that I would like to mark it. If I realise that it doesn't work… I don't know… I'm going to see what's going on… At least I'll know that you don't succeed here. You can skip it if you don't know what to do.*"

The student's results are a feedback for the teacher, who will process and analyse these data. Depending on the student's performance, he may adapt his teaching, for example by choosing another FA strategy, and provide feedback to students. In his words, we detect also a 'feedback about the task' that Thomas gives to the student, by saying "*At least I'll know that you don't succeed here*".

Second episode

Teacher's feedback can be made on the spot, like in the second transcription that we propose to analyse. A student has completed his quiz, submitted his answers and got a 'feedback about the task' from Maple TA: "good answer" or "wrong answer". Then he calls Thomas in order to have further explanations.

Thomas: "*The first one is right, the second one is wrong, the third one is right, and the fourth one is wrong. Finally, I consider that you were right on the two that are easier to explain and you got false on the two that require more mathematical work. That's normal. I consider your result as normal.*"

Both the teacher and the student benefit from the feedback in this episode. The student gets a 'feedback about the processing of the task' and also on his global performance according to the teacher's norm. The teacher, who analyses this quiz result on the spot and considers it as normal, gets information about the student's achievement.

Third episode

Sometimes the teacher's feedback is not given immediately, but during another lesson, as a result of a deep reflection, developed by the teacher, on the data at his disposal and this is the case of the third episode. When all students have completed the quiz, Thomas leads the correction of the questions with the whole classroom using NetSupport School. The lesson after, he proposes a global lecture of the class' results at the three quiz and analyses the whole set of answers stored by Maple TA, by showing them at the IWB and commenting them with the students. In this way, he provides feedback for the whole class on the attained mathematical competences.

Thomas: "*[Here are your results] on several trials. What we can see is that in calculating images you reached 0.778. What does it means? […] about 8 successful students over 10, here. There we had 6 over 10, then 8 over 10. So we are good in calculating images. […] I'm not going further. However, we'll come back on determining the expression of a linear function. 0.1, you see 0.1, 0.3, and here we went down at 0.2. […] I would like to get to realise if I succeed in teaching you two or three things last time, so we are going to work again on these two questions. Open Maple TA, and answer the two questions of the day. Let's go.*"

Thomas analyses the class' results and he *clarifies the learning intentions and criteria for success*. He has worked again with the students on the required competences during the correction phase, and now he wants to test again the competences revealed as not achieved by the analysis of the results, namely competences (b) and (d). Thus, he *engineers other learning tasks* on Maple TA. Two new questions are properly prepared and sent to students as a result of this dynamics. From Thomas' words, we can observe that, as he expected in the first episode, he has adapted his teaching depending on students' progressive achievement.

More generally, relatively to FA strategies, Thomas orchestrates the use of technology in direction of individual students, of the whole classroom or even of himself. Instrumental orchestration helps him in refining his FA strategies. Indeed, analysing students' data in order to share and discuss results in the classroom or to send new learning tasks to the students allows him to choose the most powerful FA strategy according to students' mastering of the competences at stake.

An example from the Italian case studies

In Italy the FaSMEd project involves 19 teachers, from three different clusters of schools located in the North-West of Italy. 12 of them work in primary school (grades 4-5) and the other 7 in lower secondary school (grades 6-7). Within the project, all the teachers work on the same mathematical topic: functions and their different representations (symbolic representations, tables, graphs).

Low-achievers are identified mainly through the teachers' assessment, and attend regular classes with the other students, because (as in France) schooling is based on mixed ability classes.

A research hypothesis of our team is that low achievement is linked not only to lack of basic competences, but also to affective and metacognitive factors. Furthermore, another important assumption is that argumentation can be exploited as a formative assessment tool in the interaction between the teacher and the students. As a consequence, we believe it is important that during class activities students should be guided to: (a) develop ongoing reflections on the teaching-learning processes; (b) make their thinking visible (Collins, Brown and Newmann 1989) and share it with the teacher and the classmates; (c) highlight their affective pathways (De Bellis & Goldin, 2006).

Starting from these assumptions, when we planned our work within the FaSMEd project, we looked for a technology that could support the teachers in the sharing of students' screens and of their ongoing and final written productions and in the collection of students' opinions and reflections both during and at the end of each activity. We chose connected classroom technologies, i.e. networked systems of personal computers or handheld devices specifically designed to be used in a classroom for interactive teaching and learning (Irving, 2006). They both enable to share the ongoing and final productions of the students, and to collect their opinions during the activities and at the end of them (Irving 2006, Roschelle et al. 2004, Shirley et al. 2011). Specifically, we chose the IDM-TClass classroom software, which allows the teacher to: (a) show, to one or more students, the teacher's screen and also other students' screens; (b) distribute documents to students and collect documents from the students' tablets; (c) create different kinds of tests and have a real-time visualization of the correct and the wrong answers; (d) create instant polls and immediately show their results to the whole class. Moreover, the students' written production can be displayed through the data projector or the interactive whiteboard.

Each school has been provided with tablets for the students (who work in pairs), computers for the teachers and, where the interactive whiteboard was not available, a data projector. The students' tablets are connected with the teachers' laptop through the IDM-TClass software. During the teaching-experiments, the teachers use this technology for the first time, and one researcher is present both to collect data and to help the teacher to carry out the activities.

The teaching experiments integrate the connected classroom technologies within activities coming from different sources. Among them, the ArAl Units, which are models of sequences of didactic paths developed within the project "ArAl – Arithmetic pathways towards favouring pre-algebraic thinking" (Cusi, Malara & Navarra 2011). In particular, for each lesson we prepared a set of different worksheets that can be sent by the teacher to the students' tablets. Each lesson is organized with the aim of (a) supporting the students in the verbalisation and the representation of the relations introduced within the lesson; (b) enabling them to compare and discuss their answers; (c) making them reflect at both the cognitive and metacognitive level.

In this paper we analyse an excerpt from a grade 5 class discussion referred to the following worksheet:

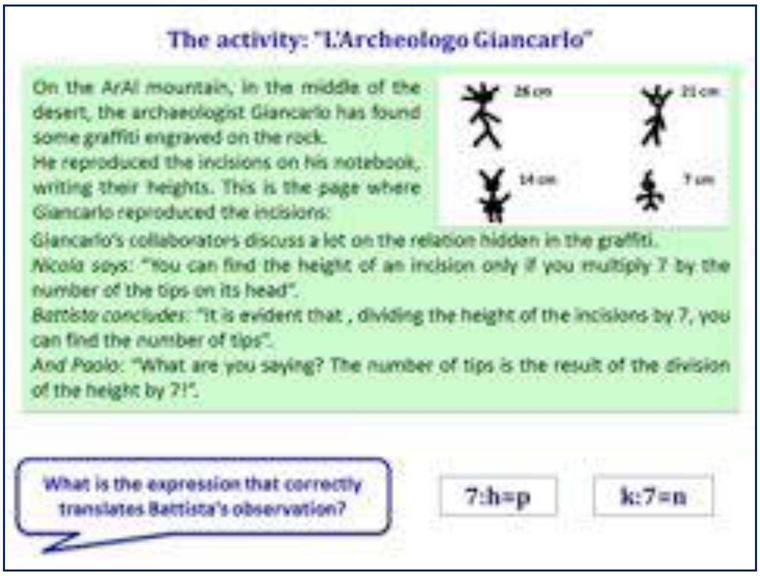

Fig. 3: The worksheet provided to students.

During the lesson reported in this example, the students, who work in pairs, are asked to answer to the question in this worksheet through a poll.

The IDM-TClass software collects all the students' answers and processes them, displaying an analytical as well as a synthetic overview (bar chart) to the teacher. Using the software the teacher can choose to provide or not an immediate automatic correction of students' answers (right/wrong). We (the teacher and the researchers) decided not to provide this correction. The software enables also to choose the time given to students before completing the poll. In this case, students had 6 minutes at disposal.
During the lesson, when all the students answer to the question, the teacher (Monica) shares with them her screen, where the bar chart and the list of students' answers are displayed:

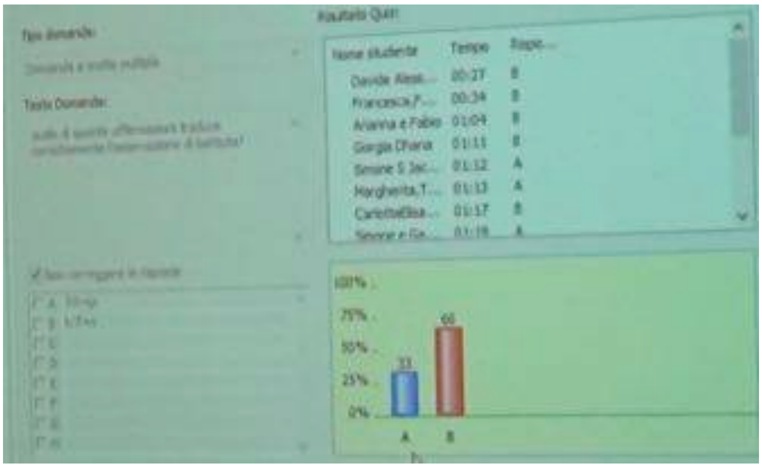

Fig. 4: The teacher's screen shared with the students.

The worksheet is also projected on the interactive whiteboard, next to the poll.

The software's processing of the poll's data enable to highlight that the 33% of the students chose the answer 7:h=p, while the 66% of the students chose the answer k:7=n. The names of the students and the corresponding answers are also displayed.

The teacher chooses not to tell to the students what the right answer is, and asks to the different pairs to explain why they chose a specific answer. The class discusses on the possible strategies that could be used to identify the correct expression, in case the only reading of Battista's observation is not enough. The students are invited to check if the number of tips and the height of every incision verify the two expressions. Some students are asked to substitute, in the two expressions, the different values connected to each incision (4,28; 3,21; 2,14; 1,7). One of them observes that she discarded expression A because the result of the division 7:28 is not 4. Alice, softly, says that 7:7=1. Monica asks her to explain what she means. We report the related excerpt:

Transcript from the class discussion focused on the results of the poll
1. Teacher (*to Alice*): "What were you saying?" 2. Alice: "I was saying that, for example, the figure, the one on the bottom right, is 7cm, so 7:7 is 1, therefore the result is not a decimal number, while with the others (*the other figures*) it is (*the result is a decimal number*). …
Monica focuses on Alice's observation and states that the chosen expression should represent all the incisions, not only the first one. Lisa and Nicolò ask if they can change their mind.
9. Teacher: "Have you changed your mind? That is, Lisa, you chose answer A, but now you have changed your mind. Why?" 10. Lisa: "Ahem … 7 is only that figure. While, if you divide the height by 7, you mean all the figures." …
Another student, Jack, declares that, although h in Italy always stands for the height, in the expression "7:h=p" h does not refer to the height.
14. Teacher: "It does not refer to the height. Is it right, Lisa?" 15. Nicolò (*raising his hand*): "Monica, because h refers only to one (*height*), while k…" 16. Lisa: "Both (*the letters*) … (*Nicolò is speaking*)…no, wait! (*to Nicolò*)"

17. Teacher: "One at a time"

18. Lisa: "Both the letters are always the height, but h is only for one (*height*) … only for this one (*Lisa goes near the interactive whiteboard to indicate the incision 7 cm height*), while k is valid for all (*the incisions*)."

19. Teacher: "k is valid for every incision. (*Stefano is raising his hand*) Stefano?"

20. Stefano: "The first expression … No, I mean: the second expression is more correct than the first. Battista says … where is it? (*Stefano is trying to find Battista's statement*) 'It is evident that dividing by 7'. It is 'Dividing by 7', not 'dividing the height'… that is …

21. Teacher: "Dividing 7 by …the height" …

Dialogue between Monica and Amalia, who observes that Lisa's interpretation of the two expressions is right and declares that, after having listened what Lisa and Nicolò said, she realised that the expression could be interpreted in different ways. Nicolò asks to intervene.

36. Nicolò: "In the first statement (*he is referring to the first expression*) 7 is divided by the height. Instead, in the second (*expression*) the height is divided by 7!"

37. Teacher: "Very good! So … Many times, I realised that many times it is not the same thing. It is necessary to pay attention. It is necessary to think very carefully to what is written. Exchanging, inverting the numbers is not the same thing." …

The FA process '*establishing where the learners are in their learning'* is central in this lesson: the discussion is planned in order to support the students in making the motivations of their choices explicit. This enables to highlight erroneous ways of reasoning and incomplete explanations, but also to highlight the evolution of students' reasoning, together with the way in which it is influenced by the other students' interventions. For example, it is evident how Lisa and Nicolò, two low-achiever students, are *activated as owners of their own learning* during the discussion: they ask to correct their initial answers, effectively motivating their new choice (from line 10). Moreover, it is possible to highlight examples of the *activation of students as instructional resources for one another*. Lisa (lines 10 and 18), for example, refers to Alice's intervention (line 2) and elaborates it to start developing her own argumentation. Also Nicolò (line 36) refers to Stefano's intervention (line 20) and elaborates it.

Another FA process that is central in this lesson is '*establishing what needs to be done to get them there*': the teacher intervenes to highlight the most effective ways of reading symbolic expressions and of identifying the one that better represents the involved relations, providing also guidance on how to read the tasks and the texts of the problems (line 37).

Different kinds of feedback are given during this discussion. In particular, it is possible to highlight feedback related to two of the four categories proposed by Hattie and Timperley (2007): feedback about the task and feedback about the processing of the task. Students' explanations of the reasoning on which their choice was based represent an example of *feedback about the task*, which is given among peers, because of the different levels of effectiveness of these explanations. For example, Stefano's intervention (line 20), which highlights that the expression 7:h=p does not represent Battista's sentence because, in the symbolic expression, 7 is divided by the height and not vice-versa, represents a feedback for Nicolò, who refers to Stefano's statement, clarifying it in an effective way (line 36).

The teacher's meta-level intervention in line 37 aims at sharing criteria to correctly identify the expressions that better represent specific relations among quantities: it can be interpreted as *feedback about the processing of the task*. This is also an example of the teacher's exploitation of feedback from the students, because Nicolò's statement (line 36) provides the teacher the opportunity to discuss the importance of a careful interpretation of symbolic expressions (line 37).

Another example of this kind of feedback is Alice's intervention (line 2), which introduces the special case of the 7cm figure, enabling Lisa to understand her mistake and ask to change her answer, proposing motivations (line 10, line 18) that clearly refer to Alice's observation.

As already stressed, starting from the poll, the teacher has organized a rich discussion, which enables the activation of different FA strategies by the different agents. The technology plays an important role in supporting the agents involved in these processes, in particular in providing feedback to each other. First of all, the software elaboration of the data and the graphical representation of the results of the poll give the teacher the chance to ask for the interpretation of these results and to plan the order of students' interventions during the discussion (Monica decides to start the discussion involving first those who have given the wrong answer).

The teacher's choice of not providing, to students, an immediate automatic correction of their answers may represent a support for students at different levels: (a) it enables to focus on the explanations of the answers, more than on the identification of the correct answer; (b) it pushes the students to motivate their answers; (c) at the affective level, the lack of a written evaluation ensures that the students do not feel worried when they comment upon their choices.

Finally, the long time given (6 minutes) to students to choose their answer enables them to reflect, in pairs, on the motivations on which their choice is based. The moment that precedes the answer to the poll is, therefore, preparatory to the subsequent discussion.

Conclusion

The observations in the classrooms show clearly the contribution of FA in the teaching and learning processes. Moreover, the technology appears as a medium facilitating the different FA strategies but also the dynamics between these strategies. It is possible to '*clarify learning intentions and criteria for success*' also without technology but technology allows to display these intentions and to share with students as a class or with the student as an individual. '*Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding*' is also facilitated by giving the opportunity of '*providing feedback that moves learners forward*' on the spot as well as after reflection. The possibility given by technology to store data and the ease to come back to these data is an important functionality that teachers can use to enhance their teaching strategies. As recognised also by the teachers in the interviews, technology is not at the base of FA, but appears as an essential tool to improve the effects of FA for students and for teachers as well.

Concerning students, it appears that the strategies of '*activating students as the owners of their own learning*' and '*activating students as instructional resources for one another*' constitute the core of FA, since they enable the active involvement of all the participants (teacher, students, peer/group) within the FA process. These strategies are also facilitated by the instrumental orchestration and the possibility given to students to use technology regarding the particular moment of FA at stake.

Our analysis brings to the fore the crucial role of the teacher as a guide in FA lessons with technology. When we began the project, most of the involved teachers stated that FA was present in their practices. However, most of the time, FA was not developed over time and appeared occasionally in the classroom more as a reassuring method than as a teaching strategy. Professional development is surely a big issue of the next years in order to consider technology as a tool enabling the enhancement of teaching strategies including FA.

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REFERENCES

Black, P., & Wiliam, D. (2009). Developing the theory of formative assessment. *Educational Assessment, Evaluation and Accountability,* 21(1), 5-31. [n/a]. 10.1007/s11092-008-9068-5.

Brousseau, G. (2004). *Théorie des Situations Didactiques.* La Pensée Sauvage, Grenoble.

Collins, A., Brown, J.S., & Newman, S.E. (1989). Cognitive Apprenticeship: Teaching the Crafts of Reading, Writing and Mathematics! In L.B. Resnick (Ed.), *Knowing, Learning, and Instruction: Essays in Honor of Robert Glaser* (pp. 453-494). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cusi, A., Malara, N. A., & Navarra, G. (2011). Early Algebra: Theoretical Issues and Educational Strategies for Bringing the Teachers to Promote a Linguistic and Metacognitive approach to it. In J. Cai, & E. J. Knuth (Eds.), *Early Algebraization: Cognitive, Curricular, and Instructional Perspectives* (pp. 483-510). Berlin Heidelberg: Springer.

De Bellis, V. A., & Goldin, G. A. (2006). Affect and meta-affect in mathematical problem solving: a representational perspective. *Educational Studies in Mathematics*, 63,131–147.

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research*, 77(1), 81–112.

Irving, K.I. (2006). The Impact of Educational Technology on Student Achievement: Assessment of and for Learning. *Science Educator*, 15(1), pp. 13-20.

Roschelle, J., Penuel, W.R., & Abrahamson, L. (2004). The networked classroom. *Educational Leadership*, 61(5), 50-54.

Shirley, M., Irving, K.E., Sanalan, V.A., Pape, S.J., & Owens, D. (2011). The practicality of implementing connected classroom technology in secondary mathematics and science classrooms. *International Journal of Science and Mathematics Education*, 9, 459-481.

Swan, M. (2014). Design Research in Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education*. Dordrecht: Springer.

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for mathematical learning, 9*(3), 281-307.

Wiliam, D., & Thompson, M. (2007). Integrating assessment with instruction: What will it take to make it work? In C. A. Dwyer (Ed.), The future of assessment: Shaping teaching and learning (pp. 53–82). Mahwah, NJ: Erlbaum.

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The Tangram Chinese Puzzle in Context: Using Language as a Resource to Develop Geometric Reasoning in a Collaborative Environment

Le tangram casse-tête chinois en contexte : utiliser le langage comme ressource pour l'élaboration de raisonnements géométriques dans un environnement collaboratif

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Abstract: Geometric thinking and geometric measurement is a large part of the curricula in most elementary, middle, and high school systems across the globe. The aim of this workshop is to help teachers and teacher educators consider the role of language in the classroom as a tool for mediating meaning through word choice for meaningful mathematical communication. In this workshop we focus on language as a central resource for negotiating meaning in mathematics as we consider the teachers' language choices that influence the ways that mathematics is presented to learners, in particular, how language influences student understanding of geometric thinking and measurement. The workshop will incorporate hands-on activities and discussions about language as a resource for mathematical meaning. The workshop will span across several of the conference themes, including mathematical content and curriculum, teacher education, classroom practices, and consideration of students' first language(s) as a resource for communicating, understanding, and learning mathematics.

Résumé : La pensée géométrique et la mesure géométrique constituent une grande partie des programmes d'enseignement dans la plupart des écoles primaires, des collèges et des lycées dans le monde entier. Le but de cet atelier est d'aider les enseignants et les formateurs à considerer le rôle du langage dans la salle de classe comme outil pour médier le sens à travers le choix des mots pour une communication mathématique riche de signification. Dans cet atelier, nous nous concentrons sur le langage comme une ressource centrale pour la négociation de signification en mathématiques puisque nous considérons que les choix langagiers des enseignants influencent les façons dont les mathématiques sont présentées aux élèves, en particulier, et la façon dont le langage influence la compréhension de la pensée et de la mesure géométriques des élèves. L'atelier comprendra des activités pratiques et des discussions sur le langage comme ressource pour la signification en mathématiques. L'atelier s'inscrit dans plusieurs thèmes de la conférence, y compris le contenu mathématique et les programmes, la formation des enseignants, les pratiques en salle de classe, et la prise en compte de la langue maternelle des élèves comme une ressource pour la communication, la compréhension et l'apprentissage des mathématiques.

Workshop Activities and Methods of Delivery

Creation of the Tangram Puzzle

This workshop on consideration for language as a resource for developing geometric thinking and measurement will be interactive and hands-on in that participants will create the ancient Chinese Tangram puzzle using plain paper (with no pre-printed pattern) and scissors. Participants will work in small collaborative teams with individual and team accountability to create the Tangram puzzle pieces (see Figure 1). The goal is to ensure that all team members successfully create the puzzle. The puzzle pieces include, relative to one another in size, two large isosceles right triangles, one medium isoscles right triangle, two small isosceles right triangles, on small square, and a medium size parallelogram. Directions for creating the puzzle will be provided orally and in writing.

As we consider language as a resource for the activities with which we will engage, the workshop will provide opportunities for participants to discuss the language(s) that students bring to the classroom as a resource for working collaboratively with other students to accomplish team mathematical goals. Participants of this workshop will take on two roles: one that will allow them to engage with the mathematical content and consider the curricular influence of the activities, and one that will create a space for discussion about the mathematical activities for teacher education that will focus on student participation, collaborative communication, and mathematical language development.

Creation of Polygons

Once the puzzle creation process is completed, participants will collaborate in teams to create various polygons including re-creating the square, and newly creating a trapezoid, parallelogram, triangle, and rectangle. During this process, special attention will be given to the role of language in communication of the position and orientations of the puzzle pieces in the formation of the various polygons. The teams will be encouraged to have each team member create a different polygon so that the team creates all five of the polygons. When the five polygons are created, we will move forward with the next activity using the five polygons to explore concepts of area and perimeter.

Exploration of Area and Perimeter Concepts

Area and perimeter are typically a focus in curriculum as students are going into the upper grades of elementary years and early secondary years in the study of geometry. Questions to guide exploration of area and perimeter concepts will be considered using the Tangram puzzle pieces within each large polygon shape. The teams will work collaboratively to compare and contrast the five large polygons' (square, trapezoid, parallelogram, triangle, and rectangle) areas and perimeters.

Initially, participants will predict which perimeter measurements are greater – the perimeter of each of the large polygons or the sum of the perimeter measurements of each of the smaller pieces that make up the puzzle. The teams will need to find the perimeters of the large polygons and will be asked to reason to make a conclusion about the various perimeter measurements regarding the various polygons. These conclusions may be made by discussing specific measurements of the polygons and then asking the participants to consider the same conclusion through generalized equations.

Following the perimeter measurements and generalizations, participants will explore area of each of the puzzle pieces by comparing the pieces through relative fractional sizes. For example, one of the two larger peices of the puzzle is 1/4 of the area of the whole puzzle, while the five smaller pieces vary from 1/8 to 1/16 of the total area (see Figure 2). Given the total area of the Tangram Puzzle (64 inches), the area of each puzzle piece will be discussed among the teams; this will be followed by discussions about student exploration and learning about areas and fractional parts of areas with respect to knowing the full area of the original area formed by all seven Tangram puzzle pieces. Additionally, exploration and discussions about the formula equations for each of the various small puzzle polygon pieces will bring synthesis and closure to the activities on the concept of area and perimeter specific to the Tangram puzzle.

Creativity in Designing Images

The participants will be given the opportunity to create an image of their choice using all seven pieces from the Tangram puzzle. Ideas will be provided as an inspiration point for personal creativity (see Figure 3). Some images we will share are from literature books, such as Grandfather

Tang's Story by Ann Tompert (1990), in addition to images made from Tangram puzzle pieces from websites, such as Tangram Puzzle Patterns (http://patterns2.othermyall1.net/tangram-puzzle-patterns) and Activity Village (http://www.activityvillage.co.uk/tangrams).

The Role of Informal Language in the Development of Formal Mathematical Language

Initally, participants will be shown various designs and images created with the Tangram pieces, some of which come from children's literature, Grandfather Tang's Story by Tompert (1990). From these, participants will be given the opportunity to create their own image with their Tangram puzzle pieces in secrecy. Participants will be asked to discuss the role of informal language for teaching and learning of mathematical concepts that lead to formal mathematical language. In this part of the workshop, participants will utilize a combination of informal and formal language to describe specific orientaiton and position of specific puzzle pieces to their teams in a game format. The activity is explained below.

Mathematically Speaking is an activity in which participants take turns in taking a leadership role in creating a design with their puzzle pieces in secrecy by concealing their design behind a visual barrier. The design can be created from an image taken from ideas provided (on cards) or created by the individual. Then, the participants will listen for descriptions of the position and orientation of the individual puzzle pieces as described by the leader. The goal is for the participants to re-create the same image by following the description as the leader verbalizes. A list of formal mathematical terms will be provided to both the leader and the participants as a reference. The list of terms will include the terms such as right angle, parallel, adjacent, triangle, rectangle, square, parallelogram, 90 degrees, etc. Once the descriptions of the various pieces are finished, the leader's visual barrier will be removed to reveal the image intended for everyone to create for comparison to that of each of the team members' images.

This activity is designed to give the participants the opportunity to reflect and discuss the purpose for a focus on the deliberate use of mathematical terms in addition to informal language descriptions and gestures to illustrate how the informal language assists the development of formal mathematical language in a particular target language. This activity will emphasize meaning through mathematical communication, which incorporates gestures and first or second languages for students.

We draw from a theoretical framework (Moschkovich 2012) that takes into consideration language as a socio-cultural-historical activity and resource that allows us to communicate mathematical ideas. The literature on the language of specific disciplines provides a more complex view of mathematical language as extended discourse that includes syntax and organization (Crowhurst 1994), the mathematics register (Halliday 1978), and discourse practices (Moschkovich 2007). We aim to focus on language as a resource that allows communicative competence for participation in mathematical discourse practices by all students, including those who are second language learners. These communication forms may take place through learners' first or secondary languages and the use of gestures (Fernandez & McLeman 2012) in which meaning is a central focus.

Engaging students in hands-on activities (such as constructing and manipulating the Tangram puzzle pieces) can provide a platform for students to engage in both informal and formal mathematical language. Expanding talk to include gestures (i.e. pointing, drawing shapes, mimic concepts that have a spatial dimension which cannot be as easily described with speech) can communicate meaning in non-verbal ways. Fernandez & McLeman (2012) suggest that using gestures to make references to features in drawings or activity materials can allow students to create a meaningful argument without using precise academic language.

Expected Outcomes

Expected outcomes for this workshop are for participants to consider the content and curricular connections of this workshop to their own context in their countries for teacher education in upper elementary to middle grades. The particular mathematical content and curricular areas in geometry, geometric measurement, the number system, and algebraic equations involving application of the Pythagorean Theorem in geometric measurement are of particular focus in the Tangram puzzle. Participants will take the opportunity to discuss teacher education in their contexts with respect to supporting teachers in developing suitable knowledge and competencies in specific domains of mathematics in addition to challenges and resources embedded in social dimensions of language development and mathematics learning. We will draw from Brenner's (1998) framework for equitable classroom practices that bring together cultural relevancy (mathematical activities that are relevant to students' lives), social organization (socially productive student participation), and cognitive resources (students' experiences and language), to guide a discussion on how the activities of the workshop can be useful and considered effective practices for diverse student populations.

REFERENCES

Activity Village for Tangrams (http://www.activityvillage.co.uk/tangrams). Retrieved February 1, 2015.

Brenner, M. E. (1998). Adding cognition to the formula for culturally relevant instruction in mathematics. *Anthropology and Education Quarterly, 29*(2), 213-244.

Crowhurst, M. (1994). Language and learning across the curriculum. Scarborough, Ontario: Allyn and Bacon.

Fernandez, A. & McLeman, L. (2012). Interpreting and using gestures of English language learners in mathematics teaching. *Teaching for Excellence and Equity in Mathematics,* 4(1), 15-23.

Halliday, M.A.K. (1978). Language as social semiotic: The social interpretation of language and meaning. London: Longman

Moschkovich, J.N. (2007). Using two lanaguages while learning mathematics. *Educational Studies in Mathematics, 64*(2), 121-144.

Moschkovich, J.N. (2012). Mathematics, the common core, and language: recommendations for mathematics instruction for English Learners aligned with the common core. Understanding Language Project, Standford University School of Education (ell.standord.edu).

Tompert, A. (1990). Grandfather Tang's Story. New York, NY: Crown Publishers.

Tangram Puzzle Patterns (http://patterns2.othermyall1.net/tangram-puzzle-patterns). Retrieved February 1, 2015.

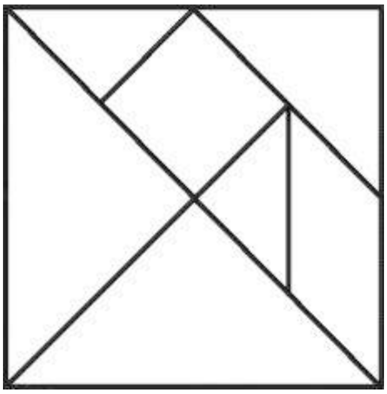

Figure 1. Tangram Puzzle

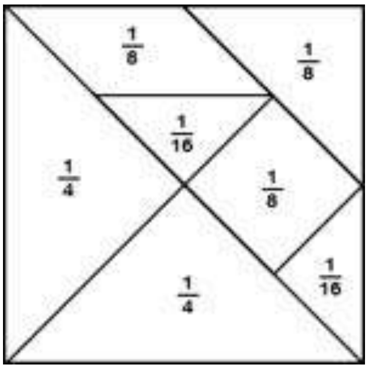

Figure 2. Fractional Areas of Tangram Puzzle

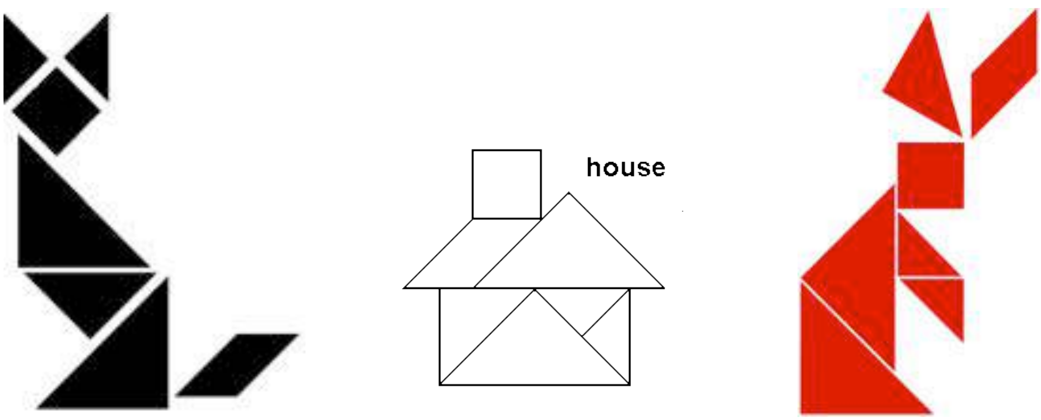

Figure 3. Tangram Images

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Teaching and learning with MERLO: a new challenge for teachers and an opportunity for students

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Abstract: The aim of this paper is to present an Italian research experience developed at the University of Turin and involving both researchers and teachers from secondary school. The group combined different background and experience and shared a common interest: working together in order to improve the quality of mathematics teaching and learning. The focus of the work is the refinement and application of an innovative didactical and pedagogical tool, called Meaning Equivalence Reusable Learning Object (MERLO), that is based on shared meaning of semiotic representations in different sign systems.

Résumé: Le but de cet article est de présenter une expérience de recherche italienne développée près de l'Université de Turin et qui entraîne soit de chercheurs soit des professeurs de l'école secondaire supérieure. Ce groupe représente une variété d'expérience et de disciplines académiques différentes avec un intérêt commun : travailler ensemble pour améliorer la qualité de l'enseignement et de l'apprentissage des mats. Le but principal de l'étude est un instrument didactique et pédagogique innovatif appelé Meaning Equivalence Reusable Learning Object (MERLO) qui est fondé sur la signification partagée des représentations sémiotiques dans plusieurs groups de signes.

Introduction

The application of Meaning Equivalence Reusable Learning Object (MERLO) involves different countries: Canada, Israel, Russia, Italy (Shafrir et al., 2015). In this paper, we focus on the Italian experience, which engages researchers from the University of Turin and teachers of secondary school, coming from Piedmont and Lombardy, who follow a Master program for prospective mathematics teachers' educators at the University of Turin, Department of Mathematics.

As starting point, we will describe the MERLO theoretical framework, which provides a general approach, suitable for different subjects. The MERLO approach was born in a general context and our aim here is to apply it in the field of mathematics education, for the teaching and learning of mathematics at the lower and upper secondary school. The Italian institutional dimension is very important and for this reason some references to Italian national guidelines and to the national assessment tests will be emphasized.

The researchers and the teachers involved in the research experience found in MERLO activities a very useful didactical tool that could be in line with these theoretical directives in Italian schools:

- The coordination of multiple representations of the same object in more than one semiotic register is fundamental for the understanding and learning of the underlying mathematical meaning (Duval, 2006);
- The social aspects are important in human learning processes, because social learning precedes the development of individual competences (Vygotskij, 1934).

Some research results, directed at the design and implementation of MERLO activities in class, will be provided in order to give information to teachers who would like to experience the challenge of using MERLO in their teaching practice.

The paper ends with some final remarks and proposals for the related workshop.

The MERLO theoretical framework

As mentioned, MERLO is an acronym standing for Meaning Equivalence Reusable Learning Objects. It is an innovative didactical and pedagogical tool developed and tested since the 1990s by Uri Shafrir and Masha Etkind at Ontario Institute for Studies in Education (OISE) of University of Toronto, and Ryerson University in Toronto, Canada (Etkind et al, 2010, Shafrir & Etkind, 2014).

They combined in their research the main results related to:

- Cognitive, meta-cognitive and affective aspects in learning processes, also in difficulty contexts;
- Concept science and conceptual thinking;
- Peer cooperation in class (for more about these points see Etkind & Shafrir, 2013).

MERLO is a very adaptable tool and, for this reason, it was applied in different contexts and countries, for several uses and subjects: mathematics, physics, biology, architecture, medicine…

The research (identified here by the name of "MERLO project") involved also Ron Kenett, an expert in the field of statistics and Ferdinando Arzarello, Ornella Robutti and their research group in mathematics education, from the University of Turin in Italy.

MERLO (Arzarello, Kenett, Robutti and Shafrir, to be submitted; Etkind, Kenett, Shafrir, 2010) is a database that allows the sorting and mapping of important concepts through exemplary target statements of particular conceptual situations, and relevant statements of shared meaning.

Each MERLO activity is structured with:

- A target statement TS that encodes different features of an important concept;
- Four other statements from different types - Q2, Q3 or Q4. As shown in the template in the Figure 1, these different types of statements are linked to the target statement by two criteria: *Meaning Equivalence* and *Surface Similarity*. The term *Meaning Equivalence* designates a commonality of meaning across several representations (e.g. the equation $y = x^2$ and the graph of a parabola in the Cartesian representation); while the term *Surface Similarity* means that representations "look similar", sharing the same sign system and being similar only in appearance, but not in the meaning (e.g. $y = x^2$ and $x = x^2$).

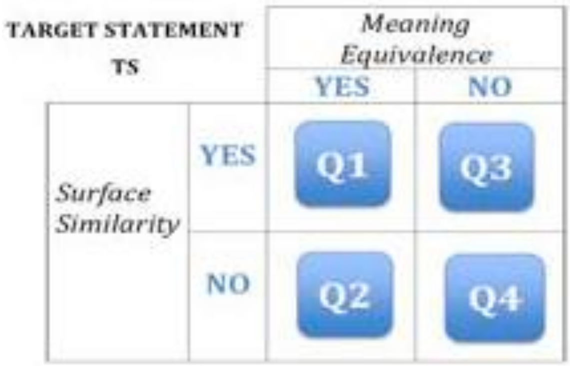


Figure 1: template for constructing a MERLO activity

Experience shows that inclusion of Q1 statements makes the activity too easy (Etkind, Kenett. and Shafrir, 2010); for this reason, we use it only in case of students with great difficulties.

In the MERLO activity for students the type of each statement is not revealed. The students are required to recognize the statements in multiple representations that share the meaning (TS and Q2) and to write the reasons for their decisions. In this way, MERLO activity combines multiple-choice (recognition) and short answers (production).

Application in mathematics education: the Italian research experience

MERLO appears to be a very suitable tool for mathematics education and for the teaching and learning of mathematics in the Italian school context. For this reason, the idea of developing a research experience was born inside the research group in mathematics education of the University of Turin. The Italian experience at the University of Turin involves a Master programme for prospective mathematics teachers' educators. Among the 29 students enrolled in 2015, all are active secondary school teachers. In addition, there is a small group (7 of them) who are also working with researchers on the design, refinement and adaptation of MERLO activities in Italian upper and lower secondary schools. This activity is developed according to the philosophy of a teacher education programme, m@t.abel (see Arzarello et al., 2015), promoted by the Italian Ministry of Education, where the competencies scrutinized by MERLO are deployed.

As an example of MERLO activity consider the following (Figure 2), which was inspired by a question asked in a test of INVALSI, the Italian National Evaluation Institute for the School System (INVALSI, 2012). The test is about recognition of relations and functions in different semiotic systems. It is linked with a real life context and shows:

- A natural language description of two tariff plans, chosen as target statement TS;
- The same tariff plans represented in a different way (Cartesian graph, table and formal language) as Q2 statements, that share meaning, but do not share surface similarity with TS;
- Another Cartesian graph, chosen as Q4 statement, which does not share neither meaning, nor surface similarity with TS.

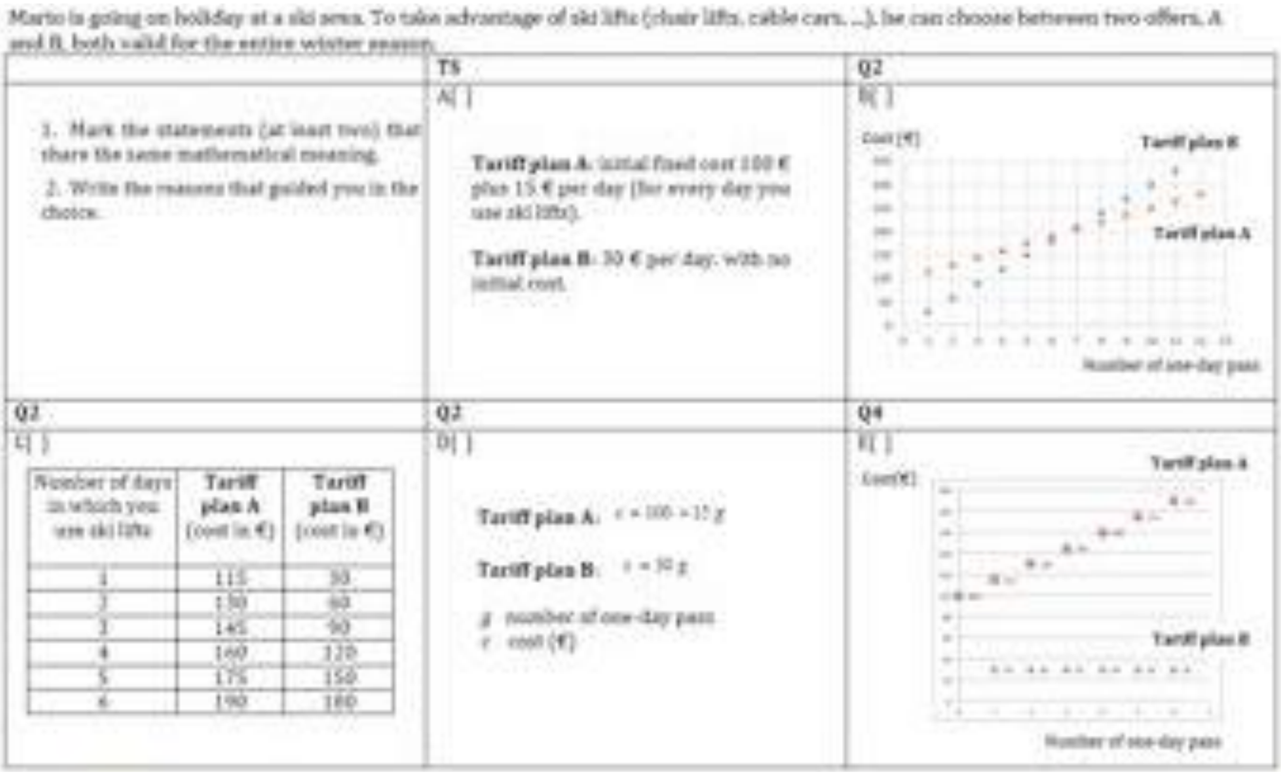


Figure 2: an example of MERLO activity

MERLO activities, such as the one just presented, are consistent with national and international guidelines.

Indeed, we can read from Italian national guidelines (MIUR, 2010):

> Lo studente studierà le funzioni del tipo $f(x) = ax + b$, $f(x) = |x|$, $f(x) = a/x$, $f(x) = x^2$ sia in termini strettamente matematici sia in funzione della descrizione e soluzione di problemi applicativi. [….] Sarà in grado di passare agevolmente da un registro di rappresentazione a un altro (numerico, grafico,

> funzionale), anche utilizzando strumenti informatici per la rappresentazione dei dati.
>
> The student will study functions as $f(x) = ax + b$, $f(x) = |x|$, $f(x) = a/x$, $f(x) = x^2$ both in strictly mathematical terms and in function of the description and solution of applied problems. [....] He/She will be able to shift easily from one register of representation to another one (numerical, graphical, functional), also using computer tools for data representation. (Our translation)

And from PISA 2012 mathematics framework (OECD, 2013):

> Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizen.

In addition, the typical MERLO task for students based on the recognition of a shared meaning in different sign systems is in line with national (INVALSI, in Italy) and international (PISA, TIMSS) assessment tests, where the ability of shifting between representations of the same object into different registers is widely evaluated.

We believe that the use of MERLO activities is particularly appropriate in the institutional context of Italian schools, where nowadays the challenge is to work on students' skills and their assessment at the end of the first two years of upper secondary school.

The use of MERLO could give a contribution in this direction, providing one more tool for this type of evaluation. Furthermore, it may compete, with other types of assessment, in bringing out the best aspects of each student (multiple intelligence) by teachers. In a next section we will show and explain some possible ways for implementing and using MERLO in class.

Results from the UNITO experience

In the following sections we present some results based on the research experience at the University of Turin (UNITO), from September 2014 until May 2015. The results concern in particular some methodological choices about the design and the implementation in class of MERLO activities. The methodological choices were born from the joined work of researchers and teachers, who share some practices after exchanging knowledge and experience.

The design of MERLO activities

Our aim in discussing the design of MERLO activities is to describe the process for creating the final product and its evolution in time. We argue that it is useful to the reader to understand the process and not only the presentation of some examples in their final version. With this objective in mind we start describing the general process that teachers can follow to design a MERLO activity and then we present a specific example.

The first step in creating a new MERLO activity is clarifying the teacher's choice of the mathematical concept or the mathematical knowledge on which to work with the class. Once it has been chosen, the teacher-author creates around it a set of representations that share the same meaning. This set is included within a BoM - Boundary of Meaning, which establishes the boundary of the shared meaning by the representations. Outside the BoM, in a disjoint set, there are representations that have no common meaning with the previous ones and with the chosen mathematical concept. However, some representations that are outside the BoM may present an exterior similarity with those inside the BoM, due to similar words or kind of representation.

A MERLO item is designed with five statements, with at least two of them internal to the BoM and the remaining ones being external to the BoM. The students are required to identify only the internal ones and justify their choice. External elements act as distractors, which can attract a student who does not know well the underlying mathematical concept. A delicate task for the teacher is to produce not too obvious distractors. The teachers' experience in teaching in class with students can get useful information about common errors and spread misconceptions.

An important aspect to highlight about the design of a MERLO activity is the following: all statements, both internal and external to the BoM, are true from the mathematical point of view. This is a particular choice, shared by our group of researchers and teachers, consequent of an extensive internal debate. At the beginning of the experience some MERLO activities were designed taking into account also statements that were mathematically false, following the traditional style of national and international tests. However a reflection on MERLO methodology, based on recognition of shared meaning between different representations, eventually led the group to agree that is most formative to put the student in front of the comparison of items that have true (although different) mathematical meanings.

The members of the group discussed a lot each other about the Italian formulation of the task. This is the original MERLO task for students, formulated in English:

> At least two out of these five statements – but possibly more than two – share equivalence-of-meaning.
>
> 1. Mark all statements – but only those – that share equivalence-of-meaning.
> 2. Write down briefly the reasons that guided you in making these decisions.

It inevitably required a rethinking for adapting to a mathematical context inside an Italian culture and not only a simple literal translation. After various changes, deriving from comparison also with other mathematicians, the task for students assumes the following Italian formulation:

1. Segnare le rappresentazioni (almeno due) che condividono lo stesso significato matematico.
2. Indicare le ragioni che guidano nella scelta.

[1. Mark the statements (at least two) that share the same mathematical meaning.
2. Write the reasons that guided you in the choice]

Finally, teachers have to consider another important aspect during the design process of a MERLO activity and in particular during the design of the representations that are external to the BoM: there is the possibility of some shared meaning between these statements. We want to avoid every possible link among statements outside the BoM, otherwise there might be two different ways to complete the task. We prefer to design MERLO activities with a single answer, aspect we ought to highlight to students, saying them that each MERLO sheet has a single answer.
The methodological choices just described and shared inside the group of researchers and teachers, do not arise by chance, but they are the result of the experience in the design of new MERLO activities inside an Italian mathematical context. These two aspects, that are the design process and the growth of methodological choices, are closely interrelated. As a result, the MERLO activities produced by the group follow various stages, evolving over time (design – revision – re-design).

As an example consider the data and forecast concept and in particular the notion of percentage frequencies. The inspiration for this example is an INVALSI question (INVALSI, 2012), which was transposed into a MERLO activity. The first version was designed at the beginning of the initiative and it is heavily influenced by traditional tasks such as INVALSI tests. This can be seen from the Figure 3 and in particular from square B.

From the first to the second version several changes are observed, not only referred to the square B but also in E and in the formulation of the task for students.

In the third version (Figure 5) there is a change in a graphical representation.

The Figure 6 represents the final version for students, where the type of each statement is not revealed and the teacher can change the position of statements.

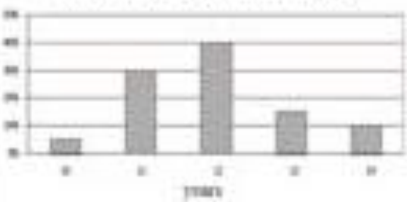
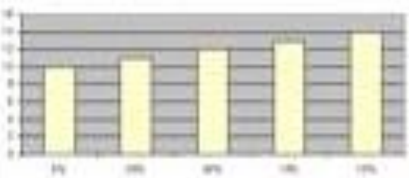
	TS	Q2
At least two out of these five statements – *but possibly more than two* – share equivalence-of-meaning. 1. Mark all statements – but only those – that share equivalence-of-meaning 2. Write down briefly the reasons that guided you in making these decisions	A [ ] Children by age (in percentage) 	B [ ] Read a data distribution, extracts the information, perform operations from the data, given from the exercise.
Q2	**Q3**	**Q4**
C [ ] The percentage of children with 10 or 13 years old is the half of those with 12 years old, and is equal to the difference between the percentage of children with 11 years old and 14 years old.	D [ ] Percentage of the children's age 	E [ ] Every age has on average 20% of the children and the average age of the children is 12 years old.

Figure 3: first version of MERLO activity about percentage frequencies

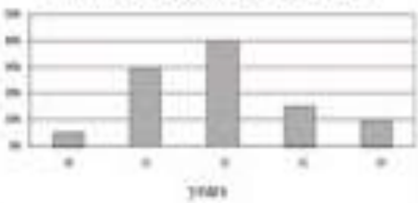
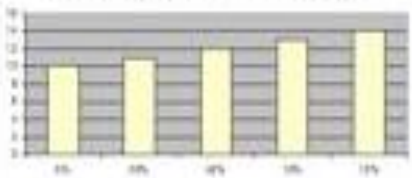
	TS	Q2
1. Mark the statements (at least two) that share the same mathematical meaning. 2. Write the reasons that guided you in the choice.	A [ ] Children by age (in percentage) 	B [ ] A gym is attended by 200 children: 10 of them are 10 years old 60 are 11 years old 80 are 12 years old 20 are 14 years old and the remaining are 13 years old
Q2	**Q3**	**Q4**
C [ ] The percentage of children with 10 or 13 years old is the half of those with 12 years old, and is equal to the difference between the percentage of children with 11 years old and 14 years old.	D [ ] Percentage of the children's age 	E [ ] For calculating the arithmetic average of age of a group of children, you have to add up all their age and then divide the result by the total number of children.

Figure 4: second version of MERLO activity about percentage frequencies

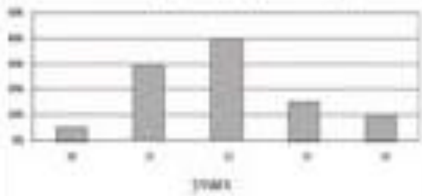
	TS	Q2
1. Mark the statements (at least two) that share the same mathematical meaning. 2. Write the reasons that guided you in the choice.	A [ ] Children by age (in percentage) 	B [ ] A gym is attended by 200 children: 10 of them are 10 years old 60 are 11 years old 80 are 12 years old 20 are 14 years old and the remaining are 13 years old
Q2	**Q3**	**Q4**
C [ ] The percentage of children with 10 or 13 years old is the half of those with 12 years old, and is equal to the difference between the percentage of children with 11 years old and 14 years old.	D [ ] Children by age (in percentage) 	E [ ] For calculating the arithmetic average of age of a group of children, you have to add up all their age and then divide the result by the total number of children.

Figure 5: third version of MERLO activity about percentage frequencies

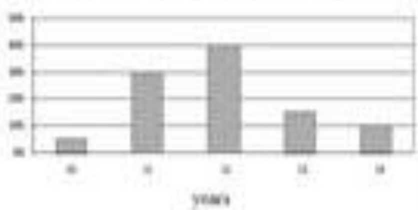
	A [ ]	B [ ]
1. Mark the statements (at least two) that share the same mathematical meaning. 2. Write the reasons that guided you in the choice.	Children by age (in percentage) 	A gym is attended by 200 children: 10 of them are 10 years old 60 are 11 years old 80 are 12 years old 20 are 14 years old and the remaining are 13 years old
C [ ]	D [ ]	E [ ]
For calculating the arithmetic average of age of a group of children, you have to add up all their age and then divide the result by the total number of children.	The percentage of children with 10 or 13 years old is the half of those with 12 years old, and is equal to the difference between the percentage of children with 11 years old and 14 years old.	Children by age (in percentage) 

Figure 6: student version of MERLO activity about percentage frequencies

The implementation of MERLO in class: activities

In this section we present some methodological suggestions about the implementation of MERLO activities in class, in order to help teachers in their work at school. The starting point is the description of the way in which a MERLO item is used in a Canadian context where it was born and in particular in a Department of Architecture. The paper by Etkind, Kenett, Shafrir (2010) describes a MERLO interactive quiz as an in-class procedure that provides learners with opportunities to discuss a PowerPoint display of a MERLO item in small groups, and send their individual responses to the instructor's computer via mobile text messaging, or by using a clicker (CRS – Classroom Response System).

See https://docs.google.com/file/d/0BxJwogdRc6UHYTVaRVN0RWdvQms/edit. The authors write that such a quiz takes 20-30 minutes, and includes the following 4 steps (Etkind, Kenett, Shafrir, 2010):

1. Small group discussion - approximately 5 minutes - following PowerPoint projection of a MERLO item, students are asked to form small discussion groups of 3-5.
2. Individual response - approximately 3 minutes - each student enters the *recognition response* on her clicker (CRS - Classroom Response System) or cell phone, marking at least 2 out of 5 statements in the MERLO item that – in his/her opinion – share equivalence of meaning; then writes down his/her *production response*, briefly describing the concept he/she had in mind while making these decisions, and turns the page upside down on her desk.
3. Feedback on *production response* and class discussion - approximately 5 minutes - PowerPoint projection of the MERLO item, including the teacher's description of the conceptual situation; followed by students' discussion and comparison of their individual production responses.
4. Feedback on *recognition response* and class discussion - approximately 5 minutes - PowerPoint projection of the MERLO item, showing the correct recognition feedback (i.e., correctly marked/unmarked statements); followed by students' discussion and comparison of their individual recognition responses.

The application of MERLO pedagogy in the Italian context of secondary school required a refinement of the methodology just described because most Italian secondary schools do not have technological devices like clickers and sometimes there is not even the possibility of projecting a Power Point slide in class. In addition, some changes were needed for a cultural adaptation and also for an adaptation to secondary school level, that is obviously different from University level.

During our research experience, researchers and teachers were involved in discussions and debates on a theoretical level and in the following experimentation in class, in order to reach a suitable methodology for implementing the MERLO pedagogy in our context. At the end the rethinking leads to a reorganization of the methodology, which remains still divided into various phases, even if times are not strictly defined.

For involving students in a classroom activity that lasts about an hour, the teacher can give two or three MERLO sheets (depending on the level of their difficulty in relation also with the preparation of the class). The printed sheets are delivered to each student who will be involved in the following phases:

1. Individual phase (15-20 minutes): each student, after receiving the MERLO sheets with the activities, is required to identify which boxes are linked by a mathematical concept and to write the reasons for his/her choice. We would like to stress the importance to write the reasons for the choice, in order to develop students' argumentative skills. At the end of this phase the sheets with individual answers are collected by the teacher.
2. Phase in groups of three or four students (15-20 minutes): the teacher divides the students into groups (each group should be composed with pupils of the same level, in order to promote discussion). A blank copy of the MERLO activities that had been solved in the previous individual phase, is returned to each group: the pupils have to compare their personal choices with those of the classmates, discussing for arriving at the ultimate goal of a shared answer.
3. Class discussion (15-20 minutes): the final discussion moderated by the teacher collects the views shared within the groups or the views of those individuals who did not arrive at an agreement with their group. The next metacognition phase is aimed at the clarification and reflection on the personal process of construction of knowledge.

Hence, as described, teachers can use MERLO as a tool to set up activities based on students' working groups and on classroom discussions. The students are also requested to account for their answers and so the discussion among students and between them and their teacher is promoted. The social aspects are important for MERLO activities resolution and then they may foster learning processes. In this perspective, we think MERLO pedagogy is in line with Vygotsky's thought that human learning presupposes a specific social nature (Vygotsky, 1934).

The implementation of MERLO in class: assessment

MERLO can be used for formative assessment in class: teachers can use MERLO activities to check what students really understood and to receive a feedback about the level of comprehension of a mathematical concept in class. The information coming from a MERLO activity and in particular from different kinds of mistakes, gives useful suggestions both to teacher and to student about the kind of deficit in comprehension. Indeed, if a student does not mark a Q2 statement, then it means that he/she has an incomplete understanding because he/she does not recognize the same concept represented in a different way with another sign system; if a student marks a Q3 statement, then it means that his/her understanding is superficial and influenced by similar representations; while the identification of Q4 as connected to the conceptual node is significant when the Q4 is a good distractor, that is "close" in the meaning, although not sharing it with the target statement.

For now we see the MERLO activity just as formative assessment, to improve and promote the connected processes of teaching and learning mathematics at secondary school.

We are also experimenting MERLO for oral questions in class: we propose a MERLO activity and the student is asked to identify statements that share a mathematical meaning, and to say orally which it is. Which may be the differences in oral performances with respect to written? Often in the work only on the paper some students, mainly those who are in greater difficulty, match the right answer but do not write reasons for their choices (even if they have some reasons). During an oral discussion, instead, the teacher can investigate the reasons of the choices, can guide and help these students in making explicit the concepts they have in mind. The teacher has a role of mediator in this case. He/She also can ask which is the meaning of the other boxes (those out of the boundary of shared meaning) and why the students did not chose them.

How could we assess this performance? If a student marks only the statements that share the same meaning and is not able to give the reasons that guided him/her in the choice, then we can say he has a basic level. Instead, if he/she can argue and say the reasons for the choices, then he/she has reached a medium level. Finally, if a student is able to explain exhaustively the meaning of each statement and the relationships among them and gives also the reasons for not a choice, then he/she has reached an advanced level.

Final remarks and proposals for workshop

As final remarks we have the pleasure to quote some sentences (said by the teachers and by the students involved in the research experience) to highlight the didactical potentiality of MERLO.

Teacher A:

> We think it is a useful tool for teachers and students, because it helps and stimulates the arguing, starting from an object to think about and discuss.

Teacher B:

> Using MERLO in oral questions in class, it is easier for me to know students mental processes. Because some of them make a choice but do not write anything about arguing, for several reasons…

Interesting observations emerged from students of the degree course in mathematics (education address, future teachers). They were involved in the resolution and analysis of some MERLO activities and these are the meaningful words that came out in the final discussion:

> I think MERLO has a big usefulness, because it allows to really understand. It takes away the rigidity of mathematics, that the school tends to give (with a textbook or a traditional lecture). Even now at

> University, when ideas are already clear, it makes you see the same thing in different ways.

About the idea of seeing the same mathematical thing in different ways, we can quote the words of a student, who gives these explanations:

> Oh, this is a graphical representation of that definition!

Here we see that the student can recognize the same concept in different registers. In general, MERLO provides activities that should develop this expertise in students, even in the case of pupils with particular difficulties. Here are the observations came out in a problematic class, as answer to the MERLO shown in Figure 6:

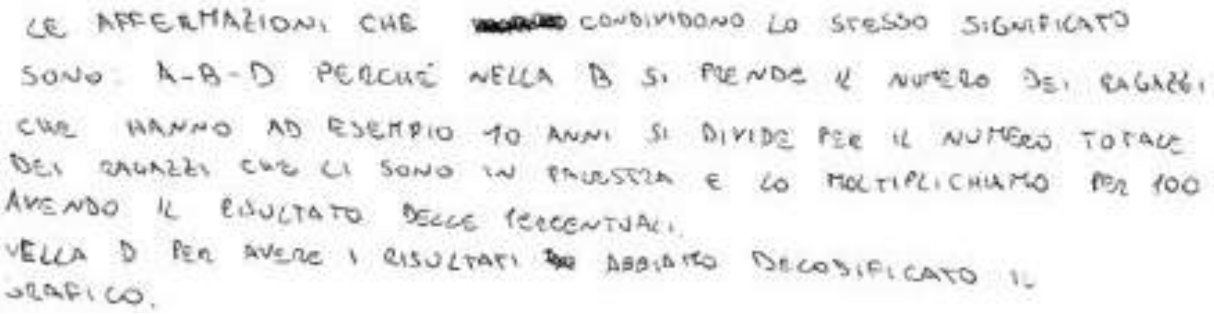

LE AFFERMAZIONI CHE CONDIVIDONO LO STESSO SIGNIFICATO
SONO A-B-D PERCHÉ NELLA B SI PRENDE IL NUMERO DEI RAGAZZI
CHE HANNO AD ESEMPIO 10 ANNI SI DIVIDE PER IL NUMERO TOTALE
DEI RAGAZZI CHE CI SONO IN PALESTRA E LO MOLTIPLICHIAMO PER 100
AVENDO IL RISULTATO DELLE PERCENTUALI.
NELLA D PER AVERE I RISULTATI ABBIAMO DECODIFICATO IL
GRAFICO.

Figure 7: student's answer

> The statements that share the same meaning are: A-B-D because in B we take the number of children with 10 years old, for example, we divide it for the total number of children who are in the gym and multiply by 100, having the result in percentages. In D, to have the results, we have decoded the graphic.

In conclusion, we think that the spread of MERLO pedagogy at schools could improve the quality of the teaching and learning of mathematics. Thinking that an active involvement can be most effective to approach this new kind of pedagogy, we are planning to engage the participants in a workshop. We will propose some MERLO activities about different conceptual nodes (numbers, geometry, relations and functions, data and forecasts), directed to different scholastic levels: the participants will required to solve and analyze them in the perspective of teacher, working and discussing in groups. The workshop will end with a final discussion, which will collect the various points of view and will offer the possibility of a comparison among researchers and teachers. Since it is a workshop, we chose to focus the paper on operational aspects: from the design process of a MERLO activity, to the implementation in class. However if the reader is interested in some more theoretical aspects, he/she can read the plenary in this conference by Robutti: "Mathematics teacher education in the institutions: new frontiers and challenges from research" (Robutti, 2015), where the research experience is analyzed with theoretical lens.

REFERENCES

Arzarello, F., Kenett, R. S., Robutti, O., & Shafrir, U. (to be submitted). The application of concept science to the training of teachers of quantitative literacy and statistical concepts.

Arzarello, F., Garuti, R., and Ricci, R. (2015). The Impact of PISA Studies on the Italian National Assessment System. In: Stacey, K, and Turner, R (Eds.), *Assessing Mathematical Literacy: The PISA Experience*, Springer. 249-260.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103-131.

Etkind, M., Kenett, R., & Shafrir, U. (2010). The evidence based management of learning. Diagnosis and development of conceptual thinking with meaning equivalence reusable learning objects. Invited paper. *Proceeding of the 8th International Conference on Teaching Statistics* (ICOTS8). Ljubljana – Slovenia

Etkind, M., & Shafrir, U. (2013). Teaching and Learning in the Digital Age with Pedagogy for Conceptual Thinking and Peer Cooperation. International Association of Technology, Education and Development. *Proceedings of INTED-2013 Conference*, Valencia, March 2013, pp. 5342-5352.

INVALSI (2012). Rilevazione degli apprendimenti, Anno Scolastico 2011-2012, Prova di Matematica Scuola Secondaria di II grado Classe Seconda.

MIUR (2010). Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali. *Decreto del Presidente della Repubblica 15 marzo 2010, n. 89.* Roma, Italia: MIUR

OECD (2013). PISA 2012 Assessment and Analytical Framework: Mathematics, Reading, Science, Problem Solving and Financial Literacy, *OECD Publishing.* http://dx.doi.org/10.1787/9789264190511-en

Robutti, O. (2015). Mathematics teacher education in the institutions: new frontiers and challenges from research. *Proceedings of CIEAEM 67 Conference*, Aosta, Italy, 20-24 July 2015.

Shafrir, U., & Etkind, M. (2014). Concept Science: Content and Structure of Labeled Patterns in Human Experience. Version 32.0

Shafrir & Kenett (2015). Concept Science Evidence-Based MERLO Learning Analytics*. IGI Global, Handbook of Applied Learning Theory and Design in Modern Education* (in press).

Vygotsky, L. (1934). *Pensiero e linguaggio*. Ed. Laterza.

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Origami: an important resource for the teaching of Geometry

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Abstract: During our workshop, through simple materials manipulation, as paper folding, we introduced several geometrical concepts. Combining the pleasure of make nice objects, we discovered and analysed geometrical properties of polygons. We have found out that even the demonstration of theorems could be simplified using origami.

Résumé: Pendant l'atelier nous avons introduit plusieurs concepts générales de géométrie en utilisant la manipulation de matériels simples, comme le plissage du papier. Le plaisir de créer de petit objets s'est mélangé avec la découverte des propriétés géométriques des polygones. La démonstration des théorèmes aussi, comme nous avons expérimenté,a été simplifié en utilisant les origamis.

Main goals

The main aim of this kind of innovative approach to the teaching of Geometry is to improve student's knowledge of the geometric topics selected for the class and to develop learning skills by exploring and studying the topic while folding the model. (Golan & Jackson, 2009)
Students have a greater involvement in the learning process: from a spectator role to a protagonist role. This deepens their knowledge and motivates them to learn more.

Methodology

Origametria, the approach to Geometry concepts through origami, is a discovery experience. The teacher guides students in this process, stimulating both observation and reflection. Starting from a visualization level, students are brought to reflect on the properties of the geometrical figure (analysis level) and to find relationships between figures and properties (abstraction level) (Van Hiele, 1986), developing logical and sequential thinking. Throughout folding the final object is never revealed: on one side it helps students focusing on geometric properties and on the other side nourishes curiosity and pleasure of discovery.

Observed conclusion

Testing origami approach to different geometrical topics in several schools, through workshops, we had the opportunity to observe that geometric terminology, linked to paper folding experiences, becomes really part of student's own knowledge.
Furthermore paper folding activity helps to involve weaker students too, enhancing their self-esteem thanks to a different approach to Geometry that involves different skills and works on emotional aspects, like curiosity, pleasure an joy. Through a manual activity they can discover geometrical properties themselves, which will improve their self-confidence and the learners become more aware of their mathematical skills.
Both accuracy and the keeping of the rules are picked up in a natural way. Furthermore, this approach increases collaborative behaviour and cooperation among pairs producing a serene working atmosphere.

Theoretical background: Van Hiele level of geometric understanding

Pierre van Hiele theory involves levels of thinking in geometry that students pass through as they progress from merely recognizing a figure to being able to write a formal geometric proof (Mason, 2009).

There are five levels, which are sequential and hierarchical.

Level 1 (Visualization): Students recognize figures by appearance alone, often by comparing them to a known prototype. The properties of a figure are not perceived. At this level, students make decisions based on perception, not reasoning.

Level 2 (Analysis): Students see figures as collections of properties. They can recognize and name properties of geometric figures, but they do not see relationships between these properties. When describing an object, a student operating at this level might list all the properties he knows, but not discern which properties are necessary and which are sufficient to describe the object.

Level 3 (Abstraction): Students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and give informal arguments to justify their reasoning. Logical implications and class inclusions, such as squares being a type of rectangle, are understood. The role and significance of formal deduction, however, is not understood.

Level 4 (Deduction): Students can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. At this level, students should be able to construct proofs such as those typically found in a high school geometry class.

Level 5 (Rigor): Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems. They can understand the use of indirect proof and proof by contrapositive.

According to van Hiele theory, progress from one level to the next level is more dependent on educational experiences than on age or maturation. Some experiences can facilitate (or impede) progress within a level or to a higher level.

A student progresses through each level of geometric understanding as a result of instruction that is organized into five phases of teaching.

Information: Through discussion, the teacher identifies what students already know about a topic and the students become oriented to the new topic.

Guided orientation: Students explore the objects of instruction in carefully structured tasks such as folding, measuring, or constructing. The teacher ensures that students explore specific concepts.

Explicitation: Students describe what they have learned about the topic in their own words. The teacher introduces relevant mathematical terms.

Free Orientation: Students apply the relationships they are learning to solve problems. They can investigate more open-ended tasks.

Integration: Students summarize and integrate what they have learned, developing a new network of objects and relations by reflection

Origametria and the Folding together project

The name *Origametria* is made from the words *origami* and *geometry*. The term was created by the Israeli Origami Centre (IOC), founded by Miri Golan in 1992, to describe its innovative program to teach curriculum geometry through origami. Since 2008, Israeli Ministry of Education has formally approved this program that has become part of the elementary schools geometry curriculum.

The IOC began to teach an origami program with the purpose of developing learning skills. The program was designed to enhance self-esteem and a sense of accomplishment, while developing learning skills such as motor skills, spatial perception, logical and sequential thinking, hand-eye coordination, focusing and concentration, aesthetics and 3D perception.

One of the projects developed by the centre, the Folding together project, brings together Israeli and Palestinian children to make origami. Origami were chosen because of several reasons.
From a relational point of view, it promotes collaboration and co-operation among children. Furthermore, it does not require a special talent, so any child can be involved in the project. Then origami is a relatively inexpensive activity to run, only paper is required.
From a didactical point of view, origami has many educational benefits such as helping to understand Mathematics and Geometry, increasing spatial awareness and fine motor control, as well as concentration. It helps students to understand and interpret verbal and written instructions.

The children are proud of what they make. They enjoy showing their models to friends and family, thus extending the positive message of the project into many social environments beyond it.

REFERENCES

Fuys, D., Geddes, D., & Tischler, R. (1988). The van Hiele model of thinking in Geometry among adolescents. *Journal for research in Mathematics education,* Monograph series, no. 3. Reston, Va: National council of teachers of Mathematics

Golan, M., & Jackson, P. (2009, August). *Origametria: A program to teach geometry and to develop learning skills using the art of origami*. In Origami 4: Fourth International Meeting of Origami Science, Mathematics, and Education (pp. 459-469)

Mason, M. (2009). The van Hiele levels of geometric understanding. Colección Digital Eudoxus, 1(2).

Van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education*. Orlando, FI: Academic Press

Van Hiele, P. M. (1999). Developing Geometric thinking through activities that begin with play. *Teaching Children Mathematics, 5,* 310-316

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Reflective activities upon teaching practices reflexes: grades and errors

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Abstract: In this multi-levelled (the individual, the small sub-group, the whole group of the sub-groups) Reflective Laboratory we focus on the numeric grade as the result of implicit diverse mathematics grading processes that co-exist within and across the levels of the educational system in the teaching and learning mathematics. In order to reveal the sedimented grading processes, the participants are invited to successively assume diverse roles at three levels: class (teacher), school unit (principal), educational system (minister/secretary/inspector of education). From the perspective of each level, we shall reflect upon the systemic interactions amongst and across the students' frequent mathematics errors, the assigned numeric grades, the interpretations, the pedagogic and broader educational actions. The exchange of the successive views is expected to reveal the noematic convergences and divergences of practice in the mathematics teaching and learning that exist in the complex teaching-learning space emerging amongst the classroom, the school unit and the broader educational system.

Résumé: Dans ce réfléchissant laboratoire à plusieurs niveaux (l'individu, le petit sous-groupe, l'ensemble du groupe des sous-groupes), nous nous concentrons sur la note numérique comme le résultat de diverses processus implicites de classement d'élèves en mathématiques qui coexistent au sein et entre les niveaux du système éducatif. Afin de révéler l'aspect interactive de ces classements, les participants sont invités à assumer successivement différents rôles à trois niveaux: la classe (enseignants), l'unité de l'école (directeur de l'école), le système éducatif (ministre / secrétaire / inspecteur de l'enseignement). Du point de vue de chaque niveau et à travers de fréquentes erreurs mathématiques des élèves, nous allons réfléchir sur les interactions systémiques entre les notes numériques attribuées à ces erreurs, leurs interprétations et les initiatives éducatives en vue. Notre objectif est de faire révéler les convergences et divergences des pratiques et/ou des représentations noématiques qui coexistent dans cet espace complexe d'enseignement-apprentissage au sein de la salle de classe, l'unité de l'école et le système éducatif en général.

Errors, grading(s) and grades

The students' errors lie at the heart of the educational processes gathering the interest of mathematics education researchers, practitioners and policy makers (Ruthven, 2000). Mathematics education researchers have focussed on the nature of the students' errors and on the cognitive processes that are linked with these errors (for example, the theme of the CIEAEM 39 meeting in Sherbrooke was "The role of errors in the learning and teaching of mathematics"), identifying a multiplicity of sources that may cause a specific error, thus identifying a multiplicity of 'misconceptions' (or alternative conceptions; Fujii, 2014). Furthermore, the students' errors are present in everyday practice, being an indispensable part of the teaching-learning process. They help in revealing the divergences of the constructed meanings, thus constituting 'sign-posts' indicating 'off-course' (or alternative) learning paths.

By grading these errors, the protagonists of the educational process (including the teachers, the learners, the principals, the families, the policy makers) obtain a measure of the quality of the educational outcome. Considering the multifaceted teaching-learning phenomena and that in most cases this measure is condensed to a simple number, in this workshop we attempt to reveal the complexity hidden within each grade, thus revealing the noematic divergences that may lie within the communications amongst the protagonists. For example, the principals may analyse the grades to obtain a measure of the teachers' quality of their teaching and/or to identify the high (or low) attaining classes or students. This information may be communicated to the Ministry of Education or to the students' families and to the broader community. Furthermore, the policy-makers utilise errors to filter-out the students who have not reached the required level of understanding of the subject-matter: the exams grades are the gate-keepers of the educational system, including the access to higher education degrees. Moreover, international organisations classify countries according to the students' performance in tests (for example, PISA, TIMMS), which in turns affects the educational policies of the classified countries.

Drawing upon a training instrument developed in the Laboratory of Learning Technologies and Didactical Engineering of the University of the Aegean (LTDE; Kalavassis, Kafoussi & Skoumpourdi, 2005), a series of activities are proposed to un-settle the established teaching practices reflexes that someone may hold with the purpose to construct bridges between two constructions of the meaning of a grade:

a) the *top-down approach* (the grade, explicitly or implicitly, is assigned according to external to the class criteria; such as the National Exams), and
b) the *bottom-up approach* (the grade is assigned according to context-situated, teacher-sensitive, class-specific criteria).

Hence, in this workshop, the semiotic polysemy that a grade entails is highlighted to reveal the plethora of pedagogical implications and choices linked with the students' grades.

Systemic interactions: school class, school unit, educational system

The polysemy of a grade is evident in the educational process. For example, during the course "Didactics of Mathematics and Science: Interdisciplinary approach" of the newly founded master's programme *Didactics of Mathematics, Science and I.C.T.: Interdisciplinary Approach* of our Department, the students were asked to grade hypothetical students' responses and to discuss their rationales and the pedagogical implications. It was revealed that the same grade or different grades could be linked with the same or diverse rationales, including a variety of aspects such as the epistemological views of the respondents, the context and the pragmatic implications of the grade (for example, entering university).

This situation can be modelled as a multivariate function with each respondent assigning different weights to each variable, thus defining qualitatively different functions. Hence, the grade acts as a single *semiotic accumulation point* to which different *noematic functions* may converge. The diversity of the converging functions is nevertheless sedimented to a simple sign, thus condensing (even conveniently masking) the complex relationships held by each protagonist about the corpus of the epistemic knowledge, the school unit and the other protagonists.

Moreover, we posit that these functions may act and/or have implications on three distinct, yet interacting, organisational levels:

- on the *micro-level* (for example, the individuals' life or a class),
- on the *meso-level* (for example, a school-unit or a district), and
- on the *macro-level* (for example, a country or networks of countries such as UNESCO).

In order to gain deeper understanding about the process of grading students' errors and the interactions that occur amongst the organisational levels, we propose a *soft systemic approach* according to which we consider the overarching educational system within which we identify the sub-systems of the school unit and the school class. A *system* (Bertalanfy, 1968) can be viewed as an integrated whole, with specific goals, clearly differentiated from its environment, whilst structurally and functionally supersedes its parts and their properties.

Following this systemic approach, the school unit may be viewed as an open system, interacting with the broader systems of the society and the educational system, as well as with the narrower sub-system of the school class (Moutsios-Rentzos & Kalavasis, 2012). Though the educational system sets the broad educational goals, each school unit and school class constitute sub-systems with their own special characteristics that interpret and re-define the broader goals to fit their own goals, which are affected by the protagonists of each sub-system and the specific social context within each school unit is settled.

The workshop: bridging top-down and bottom-up grading approaches

In this workshop, simulate the grading process considering the inter-systemic and intra-systemic interactions sedimented to a single grade. The conceptual structure of the workshop is diagrammatically outlined in Figure 1. Drawing upon the aforementioned training instrument developed in LTDE, upon the micro, meso, macro level differentiation and upon the top-down and bottom-up contrast, the workshop is organised two parts. The structure of the instruments utilised in the workshop is diagrammatically outlined in Figure 2.

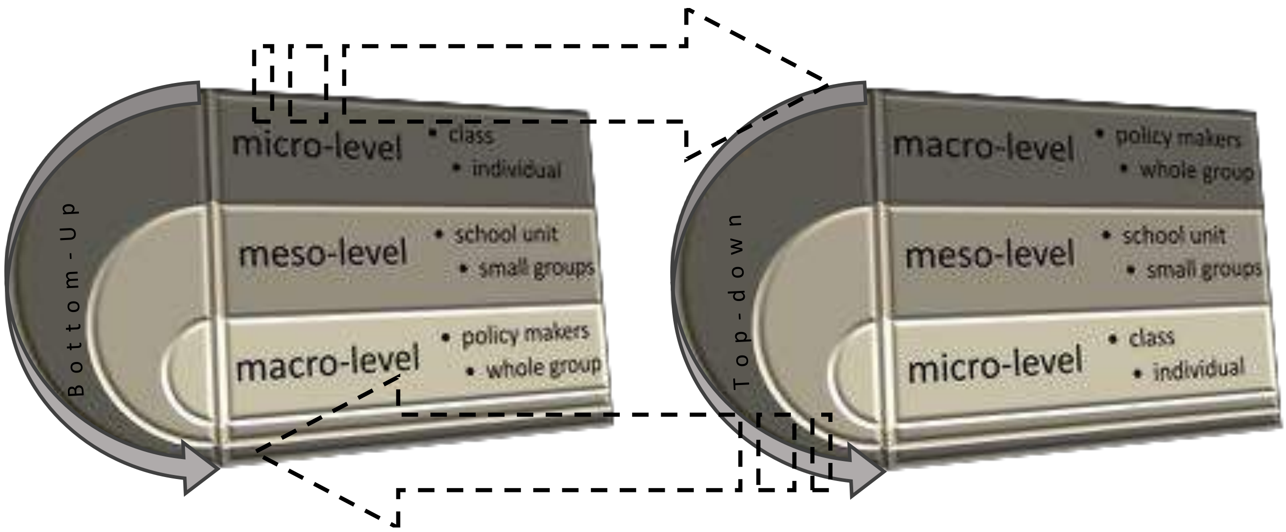
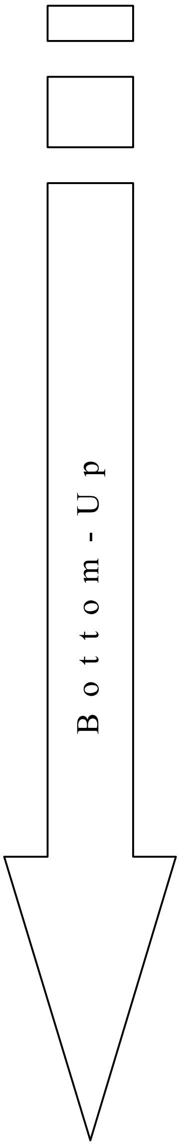


Figure 1. Diagrammatic outline of the structure of the workshop.

Following these, in the first part, simulating a bottom-up grading approach, the participants are presented with sets of students’ responses and they are asked to grade them. Each set may contain responses that are all correct, or all erroneous, or half correct and half erroneous. The participants are asked to first work *individually* (micro-level; simulating the teacher in a single class), subsequently in *small groups* (meso-level; simulating the teachers and the principals in a school unit) and finally as a *whole group* (macro-level; simulating the decisions made by policy makers). During each phase, the participants are asked to consider and to reflect upon: the meaning of the assigned grade; their rationale backing their choice of grade (including teaching-learning experience, epistemology, mathematics, psychology); the appropriate pedagogical actions linked with the assigned grade. The results of each phase will be compared and contrasted in order to un-earth the implicit stereotypes that may affect the grading process. We posit that the bottom-up approach to giving meaning to a grade combined with the reflections upon the meaning transpositions that may occur as the participants assume the expected roles in the different levels-systems allow for our gaining deeper understanding in the complexity that a grade entails.

In the second part, following a top-down grading approach, the participants are asked to discuss a grading scheme provided by an official organisation (for example, the grading scheme provided by the Ministry of Education about a National Exams task) regarding one of the tasks discussed in the first part of the workshop and to consider the implications in the three levels (systems/subsystems): the educational system (macro-level), their school (meso-level) and their class (micro-level).



micro-level (individual-class)				
Teacher 1		Grades	Meanings	Actions
Task A	Response A_1			
	Response A_2			
	…			
Task B	Response B_1			
…	…			
Teacher 2		Grades	Meanings	Actions
Task A	Response A_1			
	Response A_2			
	…			
Task B	Response B_1			
…	…			
Teacher …				
…				

meso-level (small group-school unit)				
Group I		Grades	Meanings	Actions
Task A	Response A_1			
	Response A_2			
	…			
Task B	Response B_1			
…	…			
Group II		Grades	Meanings	Actions
Task A	Response A_1			
	Response A_2			
	Response A_3			
	…			
Task B	Response B_1			
…	…			
Group …				
…				

macro-level (whole group-policy makers)				
Whole Workshop		Grades	Meanings	Actions
Task A	Response A_1			
	Response A_2			
	Response A_3			
	…			
Task B	Response B_1			
…	…			
…				

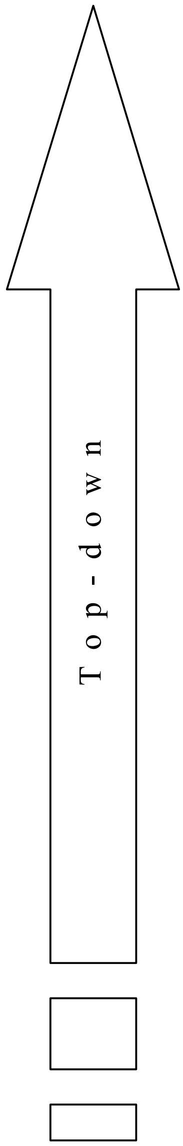


Figure 2. The structure of the instruments utilised in the workshop.

The workshop will conclude with a discussion about the noematic convergences/divergences within/amongst the two grading processes and the three levels, in order to identify aspects that may be meaningfully bridged and aspects that are inherently incongruent noematic constructions. Furthermore, we shall reflect upon the condensed complexity of the space of educational interaction and upon the fragility and unpredictability of its consequences in the environment accentuated by the stresses that the market and the digital era impose on mathematical education.

Overall, in this workshop we expect the participants to meaningfully re-position themselves with respect to the grading processes and the assigned numeric grades. Drawing upon successive reflections upon the interpretations, the constructed meanings and the corresponding actions in terms of the diverse roles and perspectives, we aim to reveal the plethora of realities, the multiplicity of meanings and pedagogical actions and consequences in the students' learning mathematics that co-exist within (and sometimes conveniently masked by) a simple number: the assigned numeric grade to a student's response.

REFERENCES

Bertalanffy, L. V. (1968). *General System Theory: Foundations, Development, Applications*. NY: George Braziller.

Fujii, T. (2014). Misconceptions and alternative conceptions in mathematics education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 453-455). Dordrecht: Springer.

Kalavassis, F., Kafoussi, S., & Skoumpourdi, C. (2005). An instrument for errors' evaluation and decision's prevention in primary school. *Proceedings of CIEAEM 57, Changes in society: A Challenge for Mathematics Education* (pp. 273-277), July23-29, Italy.

Moutsios-Rentzos, A., & Kalavasis, F. (2012). The interrelationships of mathematics and the school unit as viewed by prospective and in-service school principals: a systems theory approach. Proceedings of CIEAEM 63. *Quaderni di Ricerca in Didattica (Mathematics), 22*(1), 288-292.

Ruthven, K. (2000). Knots, Ties and Double-binds: Untangling the cultures of mathematics. In A. Ahmed, J. M. Kraemer & H. Williams (Eds.), *Cultural diversity in Mathematics (Education): CIEAEM 51* (pp. 3-17). Chichester: Horwood Publishing.

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« ENFANTS DE PAPIER » À L'ÉCOLE
La représentation des mathématiques dans la bande dessinée

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Résumé: Des 354 bandes dessinées recensées jusqu'à présent, dont les protagonistes sont tous des enfants à l'école, plus du 70%, c'est-à-dire 250 bandes, étaient au sujet « les enfants de papier et les mathématiques ». Les mathématiques, donc, a été l'un des plus importants thèmes parmi ceux identifiés dans les bandes dessinées.
Ces bandes dessinées concernent des situations problématiques au sens décrit par Zan (2002) et Di Martino (2004), parce que la tâche à laquelle les enfants doivent faire face n'est pas strictement interne aux mathématiques et les caractéristiques de l'environnement contribuent à caractériser cette tâche elle-même. Dans de telles situations, les facteurs qui influencent les processus décisionnels d'un étudiant sont définis, dans l'éducation mathématique, des convictions. Cet affiche veut approfondir l'analyse des interactions entre les différentes convictions dans le domaine des mathématiques qui apparaissent dans les bandes dessinées recensées: *convictions sur la tâche, convictions sur les mathématiques* et *convictions que l'élève porte sur lui-même par rapport aux maths* (Di Martino, Zan, 2002 ; Schoenfeld, 1983).

Introduction

Le rapport entre l'enfance et les bandes dessinées peut être abordé sous différents angles, liés les uns aux autres : BD destinées aux enfants ; la bande dessinée en tant qu'instrument pédagogique ; enfants protagonistes de bandes dessinées.
Comme l'a remarqué Antonio Faeti, les portraits d'enfant les plus croyables – ramenés à une identité autonome et contradictoire, décrite à travers un enregistrement ponctuel et respectueux de multiples données ainsi que des détails – ont été esquissés par un medium – la bande dessinée – qui a été longtemps soustrait au contrôle de la "littérature officielle" (Faeti, 1977 : 243. Traduction par nos soins). En conséquence, l'Université de la Vallée d'Aoste[1], en collaboration avec le Biblio-Musée de la Bande dessinée[2] de Morgex, a entamé une recherche qui vise à mettre en valeur la BD

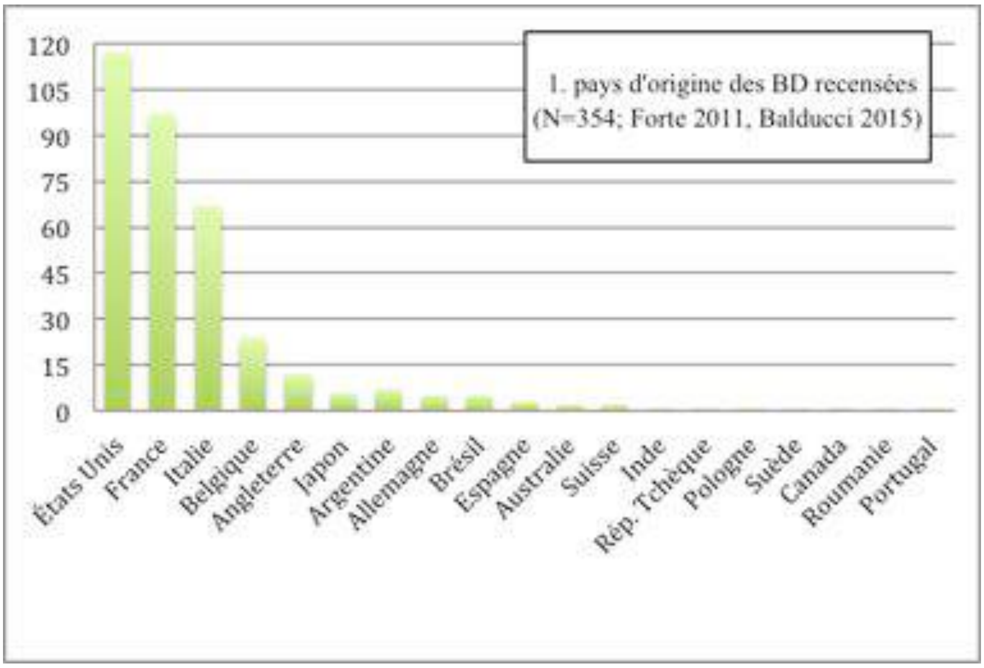
[1] Fondée en 2000, l'Université de la Vallée d'Aoste est un pôle de formation et de recherche caractérisé à la fois par l'ouverture sur l'Europe et par une attention permanente au territoire qui l'accueille. Quinze ans après sa création, l'Université a su élargir son offre de formation et aujourd'hui elle compte 2 Départements et près de 1200 étudiants. Dès son origine, l'Université a toujours mis l'étudiant au cœur de sa stratégie pédagogique : sa petite taille par rapport à celle d'autres universités italiennes permet un contact direct avec les professeurs, un accès aisé aux différents services et favorise les relations entre étudiants. L'ouverture internationale, dont témoignent les nombreuses conventions conclues avec des institutions européennes, se traduit par de multiples possibilités d'échange d'étudiants et de collaboration pédagogique et par des projets de recherche au plan international.

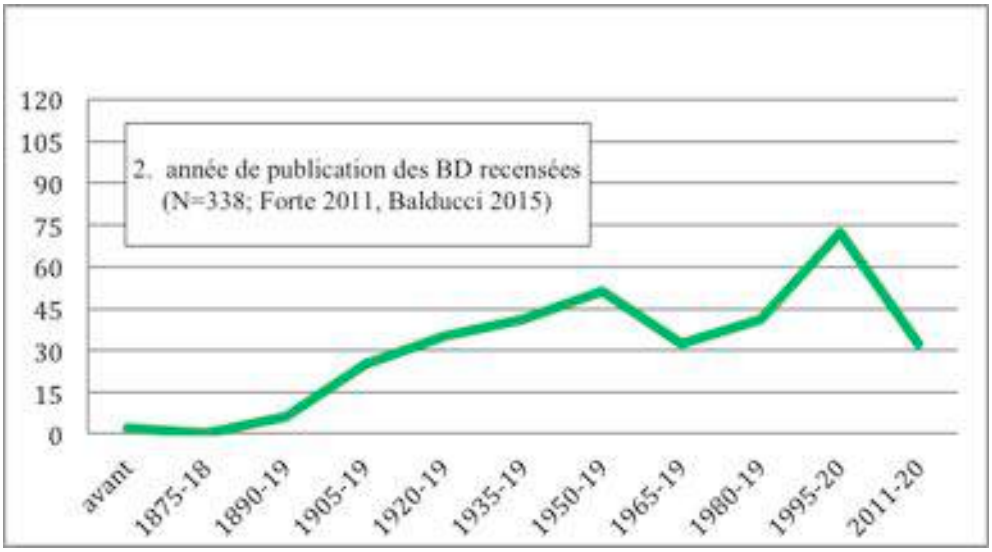
[2] Le Biblio-Musée de la bande dessinée de Morgex est né grâce à la donation à la Fondazione Natalino Sapegno d'une précieuse collection d'environ 35.000 livres et magazines, constituée par Demetrio Mafrica avec la méthode et la rigueur d'un historien et le dévouement d'un passionné. Cette collection – l'une des plus riches de ce genre en Italie – permet de suivre la naissance et l'évolution de la bande dessinée, et de bien connaître l'histoire de la BD italienne : en effet, la Collection Mafrica se caractérise par sa systématicité et son exhaustivité. Le Biblio-Musée a été conçu par la

en tant qu'instrument pédagogique et didactique dans le domaine scolaire et universitaire. Cette recherche porte en particulier sur les *enfants de papier*, c'est-à-dire sur la représentation de l'enfance dans la bande dessinée..

Les enfants de papier à l'école

Les 354 bandes dessinées recensées jusqu'à présent, dont les protagonistes sont tous des enfants (classés de A a Z : de *Ada* et *Junior i pirat* jusqu'à *Zowie*, tout en passant par *Amelia Rules!, Anne et Peter, Bébé 1er, Blondin et Cirage, Boken Dankichi, Boule à Zéro*,…; ill. 1-2), ont constitué la base documentaire pour un dépouillement visé à repérer les matériaux utiles à l'analyse du sujet *les enfants de papier et l'école.*





La lecture des livres et des magazines conservés au BMF, ainsi qu'un élargissement de la recherche sur internet ont permis de repérer environ 2000 vignettes, bandes et planches qui peuvent

Fondation Natalino Sapegno comme un lieu de connaissance du riche et varié univers de la bande dessinée. Ouvert aux passionnés et aux néophytes, le Biblio-Musée permet un premier approche à l'un des plus importants langages du XXe siècle, mais approfondit aussi l'analyse de ses protagonistes et de ses auteurs.

être classées par sujets qui reviennent : le premier jour d'école, le plaisir d'aller à l'école, tout de suite chez le directeur !, les devoirs en classe et le bulletin scolaire, les matières scolaires, dedans et dehors la classe, les devoirs à la maison, succès et échec, le rapport avec les instituteurs, l'harcèlement, les premiers amours…

Parmi ceux-ci, *les enfants de papier et les mathématiques* est l'un des plus importants, avec ses 250 bandes sélectionnées et archivées qui représentent plus du 12% du total.

Ces bandes dessinées des situations que l'étudiant rencontre dans le cadre d'activités mathématiques qui ne sont pas nécessairement la solution d'un problème mathématique : elles concernent aussi, par exemple, des situations telles que le rattrapage d'une note négative dans le bulletin, ou l'obtention de la suffisance dans une interrogation orale. Par conséquent, on peut dire que les enfants de papier vivent des situations problématiques au sens décrit par Zan (2002) et Di Martino (2004), parce que la tâche à laquelle ils doivent faire face n'est pas strictement interne aux mathématiques et les liens et les caractéristiques de l'environnement contribuent à caractériser cette tâche elle-même. Dans de telles situations, les facteurs qui influencent les processus décisionnels d'un étudiant sont définis, dans l'éducation mathématique, des convictions. Elles sont le résultat d'un processus continu d'interprétation des expériences qui concernent les mathématiques et, par conséquent, elles seront utilisées comme modèles pour interpréter et diriger l'expérience future. Comment l'a observé Schoenfeld (1983), les convictions - ou plutôt les systèmes de convictions (Green, 1971) - sont les ressources cognitives sélectionnées et mises en œuvre pour résoudre une certaine situation problématique, c'est-à-dire pour prendre des décisions.

Comme l'a remarqué Di Martino (2004), les convictions interagissent les unes avec les autres au sens où une conviction influence les actions d'un individu de différentes façons en raison des convictions avec lesquelles elle entre en relation. Par exemple, la conviction "pour réussir en maths il faut être doué" va influencer différemment l'action du sujet si elle est associée à la conviction "et moi, je ne le suis pas," ou à la conviction contraire "et moi, je suis doué."

Compte tenu de ces prémisses, nous estimons que la construction théorique *système de convictions* est appropriée à l'analyse de la relation entre les enfants de papier, l'école, les maths et les situations problématiques décrites dans les bandes dessinées sélectionnées.

Les convictions, telles que les BD aussi les représentent, constituent différentes façons d'interpréter sa propre expérience avec les maths et dirigent les processus décisionnels impliqués dans la solution des activités mathématiques.

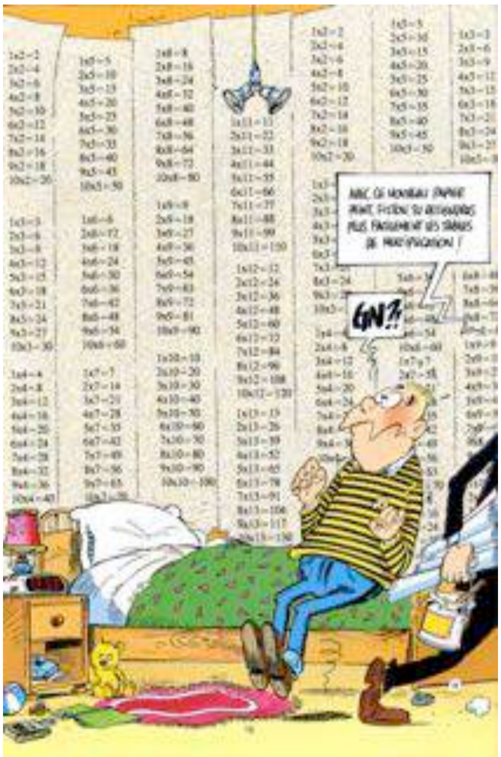
Le but de notre travail est ainsi la mise en évidence de ces aspects par le repérage des bandes et vignettes les plus exemplificatrices.

Bulles de maths et systèmes de convictions

Lire des bandes dessinées n'est pas une opération neutre! L'analyse de vignettes, bandes et planches sur les "mathématiques" a permis de repérer une série de thèmes qui se répètent, et qui sont le résultat d'une contamination consciente entre l'intentionnalité de l'auteur et le point de vue du chercheur.

Au-delà de l'explication détaillée des objectifs, des matériaux et des méthodes de recherche suivies, l'affiche veut approfondir l'analyse des interactions entre les différentes convictions dans le domaine des bandes dessinées: *convictions sur la tâche* qui, comme nous venons de le dire, n'est pas nécessairement interne aux maths, mais peut comprendre l'activité mathématique en général ; convictions des enfants de papier sur ce que réussir en maths signifie, sur les objectifs de l'enseignement et sur les attentes de l'enseignant, sur les causes de succès ou sur les stratégies à activer pour réussir (ceux qu'on appelle « les théories de succès ») ; *convictions sur les mathématiques*, influencées sans doute par les convictions sur le succès en mathématiques ; et, enfin, les convictions que l'élève porte sur lui-même par rapport aux maths (Di Martino, Zan, 2002 ; Schoenfeld, 1983).

Convictions sur la tâche à accomplir. Tâche a ici une connotation générale, de situation problématique. A titre d'exemple, nous pouvons examiner la situation d'un étudiant qui a décidé de rattraper une insuffisance en maths. Les ressources auxquelles il va puiser relèvent de sa conviction sur ce que signifie «réussir en maths» : le sens pourrait être « bien appliquer les règles », ou « apprendre par cœur les règles et savoir les répéter » (ill. 3), ou bien « savoir reconstruire et motiver les règles ». Par conséquent, l'objectif sera poursuivi d'une façon différente et en faisant recours à des ressources et des stratégies multiples.



3. Ducobu utilise des ressources qui font appel à la mémoire (tiré de : Godi et Zidrou, L'élève Ducoubu, Un copieur sachant copier!, p. 16).

Ces ressources dépendent toutefois du contexte dans lequel le sujet se place ; la sélection d'un contexte ou d'un autre dépend à son tour des schémas interprétatifs du sujet lui-même, c'est-à-dire de ses convictions. Ces convictions vont diriger les décisions du sujet dans le contexte dans lequel il s'est placé (ill. 4).

4. *Zoe* fait recours à des ressources qui utilisent l'expérience directe (tiré de : Ernie Bushmiller, *Arturo e Zoe: due bambini allo specchio*, p. 152). [A: Ce problème est très difficile! B: Combien de quarts d'eau il y a-t-il dans une baignoire remplie de soixante litres? C: Arturo au téléphone! /Réponds-lui que je suis en train de faire mon devoir de maths!]

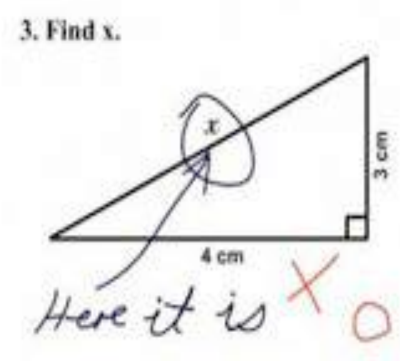
Schoenfeld (1985a) remarque que nombreux sont les étudiants qui ont des convictions générales où implicites sur la résolution de problèmes : ils pensent, par exemple, que les maths (surtout formelles) ont peu ou rien à voir avec la pensée réelle et la résolution de problèmes. En d'autres termes, comme le souligne Rosetta Zan (1998), les enfants ont deux modèles conceptuels distincts et indépendants des situations problématiques : le vrai problème et le problème scolaire (ill. 5).

5. Frazz, de Jeff Mallet (tiré de : www.gocomics.com).

Théorie du succès. C'est-à-dire les convictions des élèves sur les buts de l'enseignement et sur les attentes de l'enseignant, sur ce que signifie avoir succès en mathématiques et quelles sont les stratégies à activer pour réussir (ill. 6).

6. Convinctions "pour avoir succès en mathématiques il faut beaucoup d'exercices" (www.sd67.bc.ca/instruction/mathresources/mathcomicspage.html).

2) "le bon sens ne sert pas" : le «bon sens» a souvent une signification très différente pour l'enfant et pour l'enseignant. L'enseignant, en effet, considère «bon sens» l'utilisation de la rationnalité interne aux mathématiques ; au contraire, l'enfant l'associe au sens commun, dans lequel observation et intuition jouent un rôle clé (ill. 7).



7. Le schéma interprétatif de l'étudiant est biais par rapport à celui de l'enseignant : en effet, il ne se place pas dans le contexte de l'algèbre. (http://knepfle.com/mathcomics/).

3) «Il faut avoir une bonne mémoire»: surtout les étudiants qui ont les plus grandes difficultés en

mathématiques, persuadés de n'être pas en mesure de comprendre, se contentent de la simple mémorisation (ill. 8).

8. Un exemple du rôle de la mémoire en mathématiques (www.glasbergen.com).

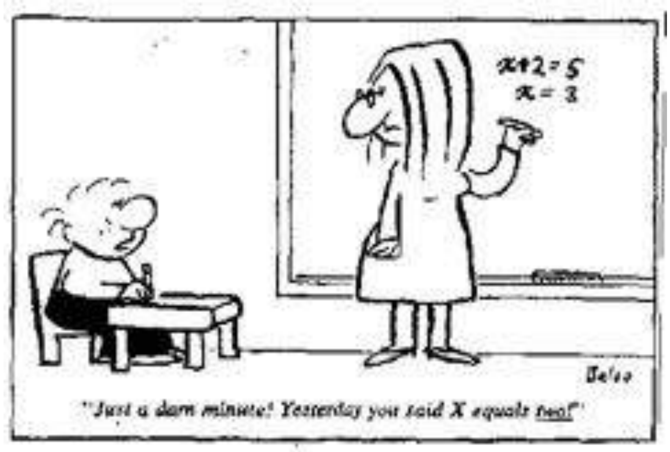
4) «Pour réussir en mathématiques il faut être doué»: d'habitude cette conviction est associée à «... et moi, je ne le suis pas.» Cette particulière conviction trouve un terrain fertile dans notre société, qui considère plus naturel l'échec en mathématiques plutôt que le succès ; en outre, le fait de ne pas être doué est alimenté surtout par la famille et fait partie des convictions que l'étudiant construit sur lui-même (ill. 9).

9. Conviction "Pour réussir en mathématiques il faut être doué" (tiré de : Bill Watterson, Calvin & Hobbes, *Il progresso tecnologico fa boink*, p. 83). [A: Ça va les maths? B: Je ne fais plus des maths, j'ai décidé que je suis une personne "visuelle". C: Bien, visualise toi-même à 45 ans encore en CP. D:On visualise beaucoup de sommes eh? / Je suis en train de te visualiser en traction, veux-tu m'aider ou non?]

Convictions sur les mathématiques. Chaque conviction sur la réussite en mathématiques suggère une vision particulière des maths ou de l'expérience mathématique. Par exemple, l'affirmation «le bon sens en maths ne sert pas» dénote une vision des activités mathématiques dépourvues de sens commun, éloignées de la réalité et, par conséquent, étrangères à l'étudiant.

Par ailleurs, les convictions sur le rôle de la mémoire suggèrent une vision des mathématiques en tant que discipline de produits plutôt que de processus. La conviction que "pour réussir en maths il suffit de faire beaucoup d'exercices" révèle une vision instrumentale de la discipline (Skemp, 1976). A l'opposé, Skemp définit la vision relationnelle des mathématiques, caractérisée par l'application de formules et de règles dont on comprend le sens intrinsèque.

Ors cette double vision des mathématiques entraîne deux différentes interprétations du mot «comprendre»: en effet, dans une vision instrumentale des maths comprendre renvoie à la mémorisation ; en revanche, dans une vision relationnelle comprendre renvoie au raisonnement (ill. 10 et 11).



10. Vision instrumentale des mathématiques (google images).

11. Vision relationnelle des mathématiques où « comprendre » renvoie au raisonnement (www.sd67.bc.ca/instruction/mathresources/mathcomicspage.html).

Convictions sur soi-même. Ces convictions sont surtout liées au sens d'auto-efficacité (Bardura, 1986), c’est-à-dire à la conviction de pouvoir exécuter une tâche spécifique à l’intérieur d’une discipline (ill. 12).

12. Conviction à propos du sens d'auto-efficacité (tiré de : Calvin & Hobbes, Il progresso tecnologico fa boink, p. 9). [A: Je suis bien content de voir que tu es en train de faire tes devoirs. Ça va les maths maintenant ? B: mmm...Très bien. / C'est-à-dire ? Combien? C: Vraiment bien. / As-tu eu la moyenne ? D: J'ai pas dit phenomenal].

Dans le domaine des mathématiques les convictions sur soi-même sont liées tant aux théories du succès et à la vision des mathématiques (par exemple, «Je ne suis pas doué pour les maths"), tant à la perception de l'échec et aux causes que l'élève lie à son propre succès ou à son propre échec.

Par exemple, l'étudiant peut attribuer son échec à des caractéristiques personnelles (causes internes plus ou moins stables, plus ou moins contrôlables) : "en maths l'engagement ne suffit pas, il faut avoir une quelque chose en plus que je n'ai pas" (ill. 13).

13. Conviction négative sur soi-même par rapport aux mathématiques (tiré de : www.comicmath.com).

Conclusions

Dans la bande dessinée la combinaison d'images, graphique et textes produit un langage humoristique: *Qui rit joue le jeu et il se doit de justifier à soi-même pourquoi il a ri. Il peut se reconnaître, s'attendrir, se fâcher ou se vexer, mais il ne peut plus se soustraire au jeu. La satire compromet, et celle-ci est bien sa force* (Tonucci, 1987 : 10; traduction par nos soins).

En raison de ces prémisses, ce travail représente une importante étape du parcours de valorisation de la bande dessinée en tant qu'instrument pédagogique promu par le *BFM* (*Biblio – Museo del Fumetto*) et par l'Université de la Vallée d'Aoste.

L'intuition – voire l'espoir – initiale a été vérifiée : le dépouillement de bandes dessinées de différents auteurs et de plusieurs époques (plus de 15.000 les bandes visionnées grâce à la consultation sur internet, au BMF et dans les collections privées) a confirmé que l'école occupe une partie significative de la quotidienneté des enfants de papier.

Bien que cette analyse doive être approfondie pour aller au-delà des probables stéréotypes, l'école représentée dans la plupart des bandes est un endroit duquel il faut s'enfuir - avec le corps ou par sa propre imagination -, un édifice députe à expérimenter l'échec et l'ennui, et dans lequel les seuls moments agréables semblent être la récréation et la sortie.

L’analyse d’un grand nombre de vignettes portant sur l’école – 1.450 celles qui ont été recueillies, numérisées et classées – a permis de repérer plusieurs *topoi* ; parmi ceux-ci, un sujet prédominant est lié aux aspects motivationnels et affectifs qui entrent en jeu dans le rapport des enfants de papier avec les mathématiques.

Plus de 250 bandes ont été étudiées à partir du modèle théorique “système de convictions” que Zan a formulé dans son ouvrage *Difficoltà in matematica* (2012). Cette analyse a permis de comprendre que le domaine dans lequel les enfants de papier font face aux maths sont des situations problématiques, qui se présentent aux étudiants dans le contexte de l’activité mathématique au sens plus général, et non seulement des problèmes mathématiques. La motivation et l’affectivité de l’étudiant tout au long de l’activité mathématique sont d’ailleurs au cœur même du rapport “OCSE-PISA 2012 *Results : ready to Learn. Students’ engagement, drive and self – beliefs*”, qui attribue en grande partie aux enseignants la responsabilité de l’échec des étudiants en maths. Le rapport souligne en outre l’importance des motivations, le sens d’autoefficacité, la participation des étudiants dans les activités aux tâches mathématiques.

Compte tenu de ces remarques, ainsi que des premiers résultats de notre recherche portant sur les BD, il semble important de prendre en examen le rôle que les BD peuvent jouer en tant qu’instrument apte à soutenir une didactique attentive aux aspects affectifs et motivationnels par rapport aux tâches mathématiques (Zan e Di Martino, 2004). Ce sujet fera l’objet d’une recherche, que nous nous proposons d’approfondir dans les prochaines années.

RÉFÉRENCES BIBLIOGRAPHIQUES

AA.VV. (1990), *La grande avventura dei Fumetti*, Istituto Geografico De Agostini, Novara.

Balducci S. (2015), *A scuola con Lio, Miss Peach, Calvin e tanti loro amici: la rappresentazione della matematica nei fumetti.* Tesi di laurea, Scienze della Formazione primaria, a.a. 2014/2015, relatori prof.ri Fabrizio Bertolino e Elisabetta Robotti.

Bardura A. (1986), *Social foundations of thought and action: A social cognitive theory*. Englewood Cliffs, NJ: Prentice Hall.

Di Martino P. (2004), *Difficoltà in matematica e sistemi di convinzioni*, tesi di dottorato, Dipartimento di Matematica, Università di Pisa.

Di Martino P., Zan R. (2002), A attempt to describe a “negative” attitude toward mathematics. In Di Martino P. (ed.) *Proceedings of the MAVI-XI European Workshop,* Pisa, 22-29.

Faeti A. (1977), *Letteratura per l'infanzia*, La Nuova Italia, Scandicci (FI).

Forte M.C. (2011), Bambini di carta: premessa ad uno studio dell'infanzia attraverso i fumetti. Tesi di laurea, Scienze dell'educazione, Università della Valle d'Aosta, a.a. 2010/2011, relatore prof. Fabrizio Bertolino.

Green T. (1971), *The activities of teaching*. McGraw-Hill, New York.

Schoenfeld A.H. (1983), *Episodes and executive decisions in mathematical Problem-Solving*. In Lesh R., Landau M. (eds), *Acquisition of Mathematics Concepts and Processes.* New York, Academic Press, 345-395.

Shoenfeld A.H. (1985), *Mathematical Problem Solving*. New York, Academic Press.

Skemp R. (1976), *Relational understanding and instrumental understanding* « Mathematics Teaching », 77, 20-26.

Tonucci F. (1987), *Bambini si nasce*, La Nuova Italia, Scandicci (FI).

Zan R. (1998), *Problemi e convinzioni*, Pitagora Editrice, Bologna.

Zan R. (2002), *I comportamenti dei bambini di fronte al problema scolastico standard: alcune riflessioni, « La matematica e la sua didattica »*, 3, 278 – 305.

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The tips of the octopus

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Abstract: An octopus, with its eight tentacles, is the protagonist of a reasoning about teaching probability and statistics. In the text, I will present eight ideas for teachers who want to develop the theme “Data and forecasts”. These ideas are supported by the use of artifacts, digital tools and Web resources.

Introduction

Paul the octopus was a common octopus living in a public aquarium in, Germany, and has become known internationally during the World Cup in 2010 when it was used for groped to "predict" the results of soccer games. Paul correctly predicted all outcomes. What animal best of octopus, with its eight tentacles-tips, lends itself to the introduction of non-deterministic phenomena?

First idea: the different registers

The following quote makes it perfectly the key concept.
“For example, if you throw an ideal dice cube and one wonders what is the probability of getting either 1 or 6, can be answered in different ways, using different registers or different representations within the same register. In the register of native language: "There are two possibilities on six." By conversion to the register fractional "the probability is 2 /6” or by treatment within the fractional registry the probability is 1/3, by conversion to the decimal register “the probability is 0.3”, or yet by conversion to the register proportion "the probability is 33.3%”. (Arrigo, 2010).
It is certainly useful to show these different registers by using, for example, the different types of format available in a spreadsheet (fig.1a, 1b).

Fig. 1a

Register	Rappresentation
Linguistic	2 of 6
Fractional	1/3
Decimal	0,33
Percentage	33,3%

Fig. 1b

Second Idea: the role of combinatorics

One question: the introduction of combinatory calculus in the school has a value mathematical autonomous or is simply an incidental feature to probability?
The discussion would lead away; here we examine only the utilitarian aspect underlining the fundamental principle of the calculation, namely the principle of multiplication.
A typical example is the slot machine and the definition of the number of possible and winning combinations. The ideal would be to have a concrete object (fig. 2). The basic idea is that the arguments should be introduced first by an intuitive point of view, using a variety of examples and, only later, through mathematical formalization.

Fig. 2

Third idea: coins and dice in various probabilistic approaches

All textbooks use examples using coins, dice, cards, marbles, etc. The reason is known: you can build simple examples that will (should?) leave room for more complex random situations.
This approach can be improved by expanding the range of "coins" and "dice" used to make a transition logically motivated by the classical definition to the frequentist probability. As examples (Drivet, 2013), we cite thumbtacks or astragals (fig. 3a, 3b).

Fig. 3a

Fig. 3b

Fourth idea: information technology

"Computer-based assessment gives students the opportunity to work with larger data sets and provides the computational power and data handling capacities they need to work with sets longer available. Students are given the opportunity to choose appropriate tools to manipulate, analyze, and represent date, and to sample from data populations. "(PISA 2012).
In class, we work too often with data invented nor even realistic. "The data presented are not real or even realistic and apply "too" to the model "(Gattuso, 2011).
It will be up to the teacher, based on the age of the students of available resources and skills, decide whether to use a spreadsheet, Geogebra, R or tools available online as Core Math Tools.

Fifth idea: small and big data

A question arises; in the school, it makes sense to do statistics with a few data? A clear answer does not exist, but it is clear that confine itself to processing only a few components ("guys, calculate the arithmetic mean of these five numbers") is limited and often misleading. Without run after the size of the so-called big data is certainly useful to work with a data set sufficient to provide materials for consideration and inferences not trivial.
Anthropometric data of sufficient amplitude are therefore strongly recommended, for example, you can use the data collected from CIRDIS (http://cirdis.stat.unipg.it/index.php?canale=160&lang=ita).

Sixth idea: the probability is not just a game

It is true that the birth of the first probabilistic concepts is closely linked to games of chance and gambling, but it would be wrong to introduce students to this aspect only. Without pretending to go into detail is necessary to present a number of areas of use of stochastic models, for example (fig. 4a, 4b, 4c, 4d): environmental (weather forecast), economic (predictive models), medical (diagnostic), social (insurance).
In addition, in this case it is desirable the use of artefacts (Drivet, 2013), spreadsheet (Drivet, 2006) and Web resources such as mortality tables of Istat.

Fig. 4a

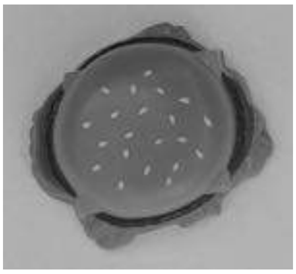

Fig. 4b

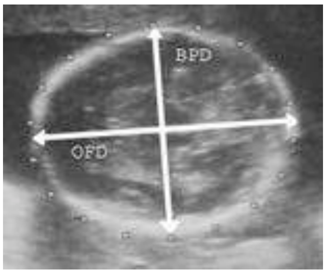

Fig. 4c

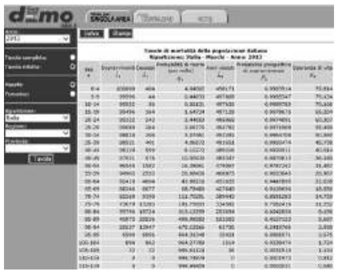

Fig. 4d

Seventh idea: the dynamic statistic

"Many teachers firmly believe that most students hate statistics, perceive it to be a necessary but painful class to teach, and imagine that it will naturally result in poor course evaluations. Unfortunately, these myths can become self-fulfilling prophecies". (Hulsizer-Woolf, 2009).
This assertion is perhaps deliberately extreme, really is not true that students have a difficult relationship with the statistic, often simply are faced to a simulacrum not credible.
Construction of questionnaires, research, and creation of graphics are indispensable, but a real qualitative leap you may have giving dynamism and historical sense to the statistics. A good practice is to make use of resources such as the Interactive maps, for example: IMF DATAMAPPER, GAPMINDER (fig. 5) o MAPPINWORLDS.

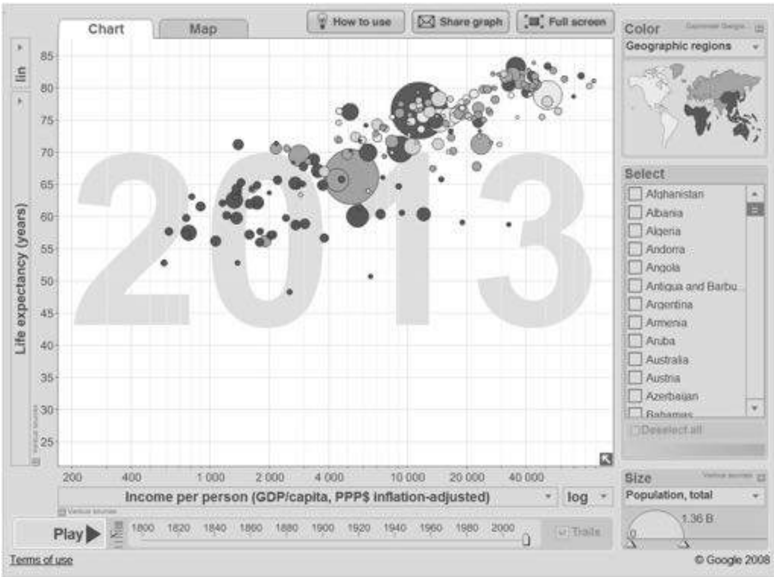

Fig. 5

Eighth idea: modelling

"Only with a large number of single events is lost random and you fall under the control of statistical laws". (Eigen-Winkler, 1986).

To simulate the trend of a population you imagine a game in which black and white pawns are the elements of a population. A chessboard 6x6 is the space where you play the various processes. Two players have randomly put their pieces on the board so that each occupies half of the squares available (Figure 6). Then we introduce the rules of the game that will lead to very different asymptotic probability distributions.
"Drifting"
You flip a coin. If the result is heads, white removes a black pawn and replaces it with a white from its reserves. Otherwise, a white pawn is replaced with a black one.
"Balance"
You roll two dice: the checker from the box corresponding to the coordinates resulting from the shooting is removed and replaced with a spare pawn of the opponent.
"All or nothing"
You roll two dice: the checker from the box corresponding to the coordinates resulting from the roll is doubled at the expense of colour opponent. Therefore, if you have obtained a square with a white pawn removes any black pawn and replace it with another white from the reserve.

Fig. 6

REFERENCES

Arrigo G.(2010), *Le misconcezioni degli allievi di scuola primaria relative al concetto di probabilità matematica*, Bollettino dei docenti di Matematica, Bellinzona

Drivet A. (2006), *Modelli per analisi ecografiche*, Induzioni, n.32

Drivet A. (2013), *La cassetta degli attrezzi*, ilmiolibro.it

Eigen M., Winkler R. (1986), *Il gioco*, Adelphi

Gattuso, L. (2011). *L'enseignement de la statistique : Où, quand, comment pourquoi pas ?* Statistique et Enseignement, 2(1), Société Française de Statistique

Hulsizer M., Woolf L. (2009), *A Guide to Teaching Statistics: Innovations and Best Practices*, Wiley-Blackwell

OECD (2012), *Pisa Mathematics Framework*, OECD Publishing

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Faire des mathématiques à travers le jeu: un exemple sur les compléments de 10

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Introduction

Cette recherche s'intéresse aux jeux mathématiques comme espace non-formel d'apprentissage des mathématiques.

Les jeux sont souvent utilisés par les enseignants pour susciter la motivation des élèves (Peltier, 2000) et travailler le langage mathématique, particulièrement en raison des nombreuses interactions que ceux-ci suscitent entre les élèves (Poirier, 2011). De plus, plusieurs études sur l'utilisation des jeux ont montré une amélioration des compréhensions des élèves en arithmétique, en géométrie et en probabilité, suite à la réalisation de pré- et post-tests et de questionnaires divers (Dumais, 2005; Peltier, 2000; Tourigny, 2004). D'autres études ont montré les apports possibles des jeux mathématiques sur le développement de stratégie de résolutions de problèmes (Bragg, 2003) et le recours au raisonnement de type combinatoire (Caissie, 2007). Toutefois, peu d'études ont décrit et analysé ce qui se produit *durant* le jeu mathématique, soit une analyse des mathématiques travaillées *pendant* que les élèves sont en contexte de jeu mathématique. Cette recherche s'intéresse ainsi à l'élève en action lors des jeux mathématiques, et aborde les questions suivantes : Que font les élèves durant les jeux ? Quelles expériences mathématiques vivent-ils ? Quelles mathématiques explorent-ils ?

Pour aborder ces questions, nous présentons nos réflexions initiales sur la base des observations que nous avons faites suite à l'essai d'un jeu de mémoire abordant le complément de 10 avec des élèves du primaire (6-8 ans).

Qu'entend-on par jeux mathématiques ?

On retrouve plusieurs description des jeux dans la littérature entourant l'enseignement des mathématiques: comme étape obligatoire du développement de l'enfant (Piaget, 1966), comme activité sociale pouvant être utilisée en contexte scolaire (Brougère, 2005), comme activité pédagogique ayant pour but de faire émerger un savoir précis et attendu (Brousseau, 1986) et comme matériel didactique pour amuser et faire avancer les savoirs (Ascher, 1998; Bednarz et al., 2002).

Dans cette lignée de matériel éducatif, Ascher (1998) développe une typologie qui permet d'isoler certaines caractéristiques des jeux pour leur utilisation dans la classe de mathématique. Tout d'abord, elle parle des jeux de hasard (e.g.: bingo, pile ou face, dés) qui sont souvent associés à la chance ou au surnaturel. Pour elle, les jeux de hasard peuvent être abordés en classe puisqu'ils permettent de développer un concept mathématique du curriculum scolaire : la probabilité. Ces jeux permettent d'améliorer les connaissances en probabilité des élèves au niveau de la prise de décision, même si le joueur ne peut pas vraiment « s'améliorer » pour gagner la partie. Ensuite, elle parle des jeux d'énigmes logiques (e.g.: casse-têtes, sodoku) qui ressemblent beaucoup à la résolution de problèmes. En effet, l'élève a un défi à résoudre qui a des implications logiques, un raisonnement pas-à-pas souvent proche de la démonstration. Pour Ascher, ces jeux d'énigmes logiques ne sont pas vus comme des « jeux » à proprement parler puisqu'ils n'ont pas de règles, pas d'adversaire, pas de gagnant, ni de perdant et aucun matériel à manipuler. En fait, ces jeux se résolvent souvent par un seul joueur. Pour sa troisième catégorie, les jeux stratégiques (e.g.: tic-tac-toe, échecs, dames) elle explique que ceux-ci mobilisent des raisonnements de type « implication logique ». Ces jeux sont très près pour elle de la démonstration mathématique, car on peut retrouver un raisonnement qui s'enchaîne pas-à-pas et un raisonnement de type combinatoire (plusieurs scénarios selon les décisions de chaque joueur). Ce type de jeu force le joueur à développer des stratégies afin de prendre en compte l'adversaire.

C'est à cette quatrième catégorie, soit les jeux stratégiques, que se rattachent les jeux mathématiques tel que définis dans ce projet. Ces jeux sont mathématiques, car ils ciblent un contenu mathématique spécifique (et explicité dès le début du jeu) dans le but de le faire travailler et explorer par les élèves ; et non pas dans un but ultime d'aboutir à un contenu ou une connaissance spécifique tel qu'on le retrouve chez Brousseau (1986). Par exemple, dans le cas qui nous concerne ici, le contenu ciblé par le jeu mathématique est le complément de 10.

Fonctionnement du jeu

Le jeu mathématique que nous discutons ici est un jeu de mémoire faisant intervenir la notion de compléments de 10. Dans ce jeu mathématique, à partir d'une trentaine de cartes tournées face contre table, à tour de rôle le joueur retourne une première carte et doit ensuite en tourner une deuxième à associer à la première pour former un total de dix. Les élèves ont pour but est de faire le plus de paires possibles. Le jeu se termine lorsqu'il n'y a plus de paires qui peuvent être faites et le gagnant est celui qui en a accumulé le plus (les élèves jouent à 2 ou 3). Chaque carte possède un nombre de 0 à 12, illustré de différentes façons par des représentations numérales et alphabétiques, des doigts, des cartes à jouer, des « boîtes de dix » et des courtes phrases (voir la Figure 1 ; à noter que les nombres 11 et 12 sont utilisés pour faire les soustractions 11–1 et 12–2).

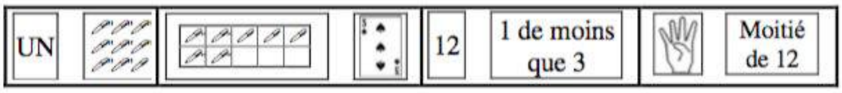

Figure 1 : Exemples de paires de cartes « complément de 10 »

Réflexions sur l'activité mathématique déployée durant le jeu

Les réflexions que nous tirons concernent la nature de l'activité mathématique déployée, soit les stratégies développées en contexte de jeu et tout le potentiel que celles-ci recouvrent pour le sens du nombre et les opérations. À travers les exemples/illustrations de stratégies développées par les élèves de 6 à 8 ans, l'intention est d'explorer la richesse et le potentiel d'une entrée par le jeu mathématique (ici sur les compléments de 10) et de discuter de la nature de l'activité mathématique mobilisée par les élèves. Nous faisons état de nos compréhensions sur leurs façons de s'engager dans la tâche et de la nature de l'activité mathématique déployée pour ce jeu sur les compléments de 10.

Les façons de compter. Dans la classe, nous avons observé différentes façons de compter pour former les paires. Certains élèves comptaient les points ou les dessins illustrés sur les cartes pour trouver la valeur complémentaire manquante pour faire 10. D'autres élèves se dégageaient des dessins sur la carte et transféraient les quantités « lues » sur leurs doigts pour trouver la valeur complémentaire qu'ils devaient rechercher. Enfin, des élèves parvenaient à se dégager de la carte, ainsi que de tout référent physique, en comptant mentalement pour déterminer la valeur complémentaire nécessaire pour faire 10.

Les associations entre les nombres. Nous avons observé que les élèves ont fait de fortes associations entre les nombres et leurs compléments. Certains élèves en tournant une carte savent immédiatement quelle est la carte qu'ils doivent chercher en raison du premier nombre obtenu. Par exemple, lorsqu'un élève tire un 8, tout se produit comme s'il tire aussi un 2 au même moment, alors que le 8 n'est plus considéré seul ou de façon isolée, mais est « connecté » ou associé au 2. Ceci amène plusieurs élèves à anticiper la prochaine carte à trouver en fonction d'une carte faisant office de valeur de référence (par exemple, lorsque le 3 est tiré, ils cherchent le 7 jusqu'à ce qu'ils le trouvent d'un tour à l'autre). Cette anticipation de la carte complément en a même amené certains a modifier le jeu en « trichant » en regardant à travers les cartes (peu opaques) pour trouver le nombre voulu. D'autres élèves, quant à eux, vont plus loin et vont même jusqu'à décider de la valeur à assigner à une des cartes choisies, telles que celles du « joker » ou de cartes blanches (désignées au départ du jeu comme étant de valeur 0), pour que cela fonctionne avec l'autre carte qu'ils ont en main pour arriver à compléter 10.

Flexibilité à travers les représentations. Les diverses représentations des nombres (représentations numérales et alphabétiques, doigts, cartes à jouer, « boîtes de dix », courtes phrases) sont lues, combinées et comprises par les élèves avec beaucoup de fluidité. Même l'utilisation de courtes phrases mathématiques (par exemple, « un de moins que 5 », « deux de plus que 3 ») n'arrêtent pas les élèves dans leurs explorations des compléments de 10. Les élèves passent donc des nombres aux représentations diverses avec une grande flexibilité, la combinaison pour un même complément de deux représentations différentes n'affectant pas leur travail.

Remarques finales

Nos réflexions suite à l'essai de ce jeu de mémoire sur les compléments de 10 nous mènent à discuter de la richesse de l'activité mathématique des élèves durant ce jeu et de l'aisance qu'ils ont manifesté pour travailler la notion de complément de 10 (et tout le succès qui vient avec). Le fait de regarder les élèves dans l'action permet d'apprécier les mathématiques qu'ils vivent et non pas nécessairement ce qu'ils apprennent et conservent suite au jeu : on observe les mathématiques dans leur état vivant, en plein déploiement, et leur viabilité locale pour résoudre les défis posés par le jeu mathématique en question. Nous avons été intéressés n'ont pas par ce qu'ils apprennent *par* le jeu, mais par ce qu'ils font *durant* le jeu, pour y saisir le potentiel mathématique de travailler ces contenus par le jeu. À ce stade préliminaire, la richesse des mathématiques révélées sous forme de flexibilité, d'association entre les nombres et de façon de (se) les représenter montre un potentiel mathématique riche vécu à travers ce jeu. On y décèle une exploration mathématique vaste et importante, qui se démarque des situations usuelles d'apprentissage des compléments de 10 autour de faits numériques et additifs à « apprendre » et réciter.

Ces premières observations et réflexions nous montrent tout l'intérêt de conduire notre étude autour de ces idées, nos pas pour comparer la richesse de ce type d'approche par le jeu d'avec l'enseignement usuel, mais faire ressortir la spécificité et la vivacité des mathématiques travaillées, et leur étendue/richesse, à travers les jeux mathématiques.

REFERENCES

Ascher, M. (1998). *Mathématiques d'ailleurs*. Paris: Éditions du Seuil.

Bednarz, N., Bourdage, N., Charpentier, M., Lartigau, M., Poirier, L., Sauvé, T., Tourigny, C. (2002). *Banque de jeux pour l'apprentissage des mathématiques au primaire.* Mont-Royal, Québec: Modulo.

Bragg, L. (2003). Children's perspectives on mathematics and game playing. *Proceedings of MERGA-26* (pp. 160–167). Geelong, Victoria: MERGA.

Brougère, G. (2005). *Jouer/Apprendre*. Paris: Economica.

Brousseau, G. (1986). *Théorisation des phénomènes d'enseignement des mathématiques*. Thèse d'État, Université de Bordeaux 1, France.

Caissie, C. (2007). *L'apport du jeu pour le développement de compétences en mathématique chez les élèves au premier cycle du secondaire*. Université du Québec à Montréal, Canada.

Douady, R. (1994). Ingénierie didactique et évolution du rapport au savoir. *Repères IREM n° 15,* 25p.

Dumais, S. (2005). *L'utilisation du jeu en classe préscolaire pour viser le développement du concept de nombre*. Thèse de doctorat, Université de Montréal, Canada.

Peltier, M.-L. (2000). Les jeux mathématiques sont-ils la panacée à la démotivation des élèves ? *Grand N*, *67*, 33–40.

Piaget, J. (1966). *La psychologie de l'enfant*. Paris: Presses universitaires de France.

Poirier, L. (2011). Un projet de codéveloppement autour des jeux mathématiques. *Vie Pédagogique*, *158*, 40–41.

Tourigny, C. (2004). *Une intervention en mathématiques en milieu défavorisé s'articulant sur le jeu: contribution au développement de compétences mathématiques chez les enfants*. Mémoire de maîtrise, Université de Montréal, Canada.

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The Tangram Chinese Puzzle: Using Language as a Resource to Develop Geometric Reasoning

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Abstract: The work in our poster showcases geometry-related activities that include modeling desired participation and encouraging engagement in mathematical discourse. Interactive communication of this kind helps prospective teachers of mathematics to become more aware of their own choices of words and grammar, while at the same time grappling with the important geometry content in the Tangram Puzzle activities. A focus on language practices in mathematical discourse is critical to the preparation of mathematics teachers as they prepare to teach diverse populations of students.

Résumé: Le travail dans notre affiche met en vedette les activités liées à la géométrie qui comprennent la modélisation participation souhaitée et encourager l'engagement dans le discours mathématique. Ce genre de communication interactif aide les futurs enseignants de mathématiques à devenir plus conscients de leurs propres choix de mots et de grammaire, alors que dans le même temps aux prises avec la géométrie important contenu dans le Tangram Activités de puzzle. Un accent mis sur les pratiques langagières dans le discours mathématique est essentiel à la préparation des professeurs de mathématiques comme ils se préparer à enseigner à divers groupes d'étudiants et d'étudiantes.

Framework

Language is the principal resource for *making meaning* through mathematical discourse in the classroom (Halliday 1978; Vygotsky 1978). However, there can be obstacles to communicating mathematical ideas due to the complexities of language in the mathematical register. In line with the themes of this conference, we consider the importance of obstacles that prospective teachers face when learning to use the specialized language of the mathematics register. Additionally, we posit that regardless of such obstacles, fluency with the mathematics register is essential to teaching and learning of mathematics because it is the kind of language that allows both learners and teachers to explain methods and strategies used or justify decisions made (Halliday 1978). Therefore, we consider language (verbal and non-verbal communication) as a resource to promote a shift from every day, informal language to one that is more abstract and formal.

The Poster

The presentation of this poster will be of interest to conference attendees as the activities highlighted reflect geometric thinking and measurement spanning from lower and upper grade levels for teacher development. With the perspective that discourse and mathematics learning are closely linked, the goal of our poster is to showcase the ways that problem solving through Tangram activities allow participants to use language in different ways in order to negotiate and connect mathematical meanings with others in a social context. Interactive communication of this kind helps prospective teachers become more aware of their own language and word choices, while engaged in geometric reasoning. The focus of our work reflects the construct of language as a resource to develop communicative competencies for participation in mathematical discourse (Moschkovich 2010).

Our poster contributes to understanding ways of helping prospective teachers develop mathematics language through activities that encourage student-to-student talk and explanations. We find that engagement in the hands-on activities of constructing and using the Tangram pieces helps to foster PTs' understanding of geometric thinking to support collaborative communication and mathematical language development. Specifically, we noticed a shift from using everyday language toward a more formalized use of geometric vocabulary. The work presented places front-and-center the process of doing *and* talking about the mathematics with prospective teachers as they prepare for teaching linguistically diverse student populations.

REFERENCES

Halliday, M. A. K. (1978). *Language as Social Interpretation of Language and Meaning.* University Park Press.

Moschkovich, J. N. (2010). *Language and mathematics education: Multiple perspectives and directions for research.* IAP.

Vygotsky, L. (1978). Interaction between learning and development. *Mind and society* (pp. 79–91). Cambridge, MA: Harvard University Press.

Les mathématiques hors-classe - tradition de la FP UK

Activités hors-classe pour les enfants de 5 a11 ans

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Résumé : Les mathématiques hors-classe (OUTDOOR MATHEMATICS) font, depuis plus de 15 ans, partie des cours a option dispensés à la chaire des mathématiques et de la didactique des mathématiques. Les étudiants d'Erasmus (Grece, Espagne, Portugal, Allemagne Autriche, Slovaquie, Norvaige) ainsi que les enseignants de l'éducation continue (plus de 400 enseignants de la République tcheque et 50 de l'étranger - Etats-Unis, Suède, France, Belgique, Royaume-Uni), et de façon réitérée, les participants de la conférence internationale SEMT ainsi que les étudiants de l'Université de Parme. Les activités sélectionnées, ici présentées, sont orientées vers un phénomene particulier - le traitement des "grandes représentations" dans le milieu mathématique - l'image, le plan, l'ébauche et autres utilisations des informations recueillies. Ces activités constituent une structure didactique (Kaslová SEMT 3, Brno 2014). Ces activités ont été maintes fois vérifiées dans la pratique et les expériences font également partie des solutions proposées par les étudiants dans leurs travaux de sortie. Nous suivons l'efficacité de ces activités qui ne sont en aucun cas conçues comme un simple effort pour rendre le programme plus intéressant ou pour compléter les leçons de mathématique.

Introduction

Nous partons des connaissances en matière de neurologie; un espace important est traité par le cerveau d'une manière autre que le micro- ou mezzo-espace (qui entrant dans un champ de vision donné), en général dans le cadre d'un espace fermé. Pour pouvoir suivre un espace plus grand, il faut tout au moins plus de recul, ce qui permet une expérience préhensible ou tactile, une expérience liée au toucher ou constituer l'image à partir de plusieurs regards partant d'un même lieu et partir de plusieurs endroits dans un espace donné. Il ne s'agit pas d'une simple orientation dans un lieu et dans un temps donnés, notamment quand l'enseignement qui suit repose sur l'expérience en question (Kaslová, M. SEMT 2009). Le regard peut porter également sur un grand espace, en définitive global. Mais pour pouvoir le décrire ou le traiter ultérieurement l'élève doit passer au traitement analytico-synthétique et viser d'une certaine manière la structure (voir neurologie). Il s'agit d'une sélection opérée dans la structure didactique formée par 12 activités. Le traitement d'un grand espace suppose la capacité de s'orienter dans l'espace aussi bien que dans le temps. Chez les enfants plus jeunes, le traitement d'un grand espace s'appuie souvent sur la ligne temps tout comme le fait que l'enfant a suivi l'espace petit à petit, à partir d'un point donné ou comme s'il traversait l'espace. Dans ce cas, nous voyons que son traitement par l'image est "linéaire" ou, en règle générale, l'enfant place dans l'espace des points d'orientation/des objets, le plus fréquemment de gauche à droite (dans le sens de la lecture).

Activités

A1 Salle de gymnastique; âge 5-7 ans: a) Les enfants traversent un <u>parcours a</u> obstacles; b) Ils dessinent le vécu sur une feuille de papier format A3, 10-15 minutes environ après l'exercice; c) Le lendemain, ils racontent le vécu à partir de l'image qu'ils ont dessinée; d) Ils interprètent une image dessinée par quelqu'un d'autre - suit une discussion sur l'univocité du code de communication graphique; e) Ils font un dessin pour permettre la construction du <u>parcours a</u> obstacles dans la même salle de gymnastique mais selon leur propre idée (sans le concours d'une autre personne); f) Ils construisent le parcours à obstacles d'après leur propre dessin; g) Ils construisent le parcours d'après le dessin de quelqu'un d'autre. But: traitement de l'espace par le dessin, compréhension et interprétation, preuve, photo. Photo 1. Gymnastique.

A2 Terrain de sport - sable, âge 8-9 ans: sur le terrain on a placé des points d'orientation (cibles: petit drapeau, ballon, disque F, cerceau,...). Activités similaires à celles de A1, mouvement sur une surface de 3 ares. Groupes d'élèves: 1a) Les groupes se trouvent chacun devant une cible différente. A partir de la, ils doivent placer une autre cible située a 10 mètres de la, dans le sens de leur choix. 1b) Les élèves mesurent la distance. 1c) Ils mesurent 10 m à partir de la nouvelle cible dans une direction différente et ils mesurent une nouvelle fois. 1d) Estimations des distances sur 50 m et 100 m. 2) Ils estiment la hauteur ou la distance des bâtiments en se basant sur leurs expériences précédentes. Chaque estimation se fait à partir de trois différents postes, ensuite chaque groupe se met d'accord sur une des estimations. Ils comparent leur estimation avec celles des autres groupes et avec le résultat (fourni par l'enseignant ou par le groupe qui a choisi les bâtiments en question). La documentation photographique et les fiches de travail servent de base pour le travail en classe. But: fixer l'idée qu'on se fait des dimensions données et créer la base des "grandes idées" et de la réduction des proportions.

A3 Terrain de jeux - âge 4-6 ans; les enfants doivent illustrer leurs jeux après leurs rentrés dans la class. L´activité copie les activité de A1. But : enregistrer un « grand espace ouvert » - la réduction des mésures en respectant les proportion des objets et les relations spatiales.

Photo 2. Les jeux.

Terrain de jeux - neige/sable; âge 9-10 ans, travail en groupes.1) Les élèves délimitent des carrés de 10 m de côté (ils se placent sur leurs sommets). Un autre groupe vérifie les mesures des côtés. Après avoir corrigé le carré, il marque le carré dans le sable (la neige). Il scrute l'école (d'en haut). 2a) Un autre jour, il répète l'activité 1, ensuite il doit définir un rectangle de la même superficie que le carré (a). Il dessine la solution (sur papier blanc ou quadrillé). 2b) Ensuite il définit un triangle toujours de la même superficie. Les dessins, les photos et les méthodes de solution sont discutés en classe. But: créer une idée réelle de l'unité "1 are" comme figure d'une superficie donnée il ne s'agit pas nécessairement d'un carré.

A4 La ville- place/atrium; âge 10-11 ans. Les enfants se déplacent en groupes. Ils font des estimations concernant la surface des façades ou des fenêtres des maisons ou encore de la surface de la place ou des atria. Ils transfèrent leurs expériences de A4 dans un contexte nouveau ayant des formes et des positions différentes (verticales). But: fixer les idées concernant l'are.

Photo 3. Éstimation et correction

Photo 4. Idée d´un are.

A5 Cuisine L´espace dans lequel on produit des différents modèles, dans lequel on découvre une autre signicication de la notion « proportion », où on utilise une vérification non-standart.

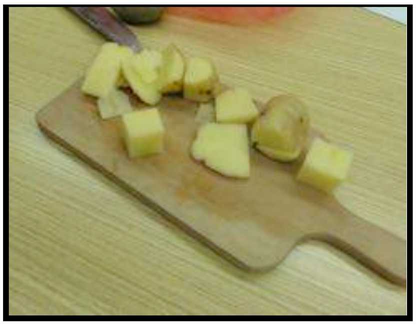

Photo 5. Modèle du cube

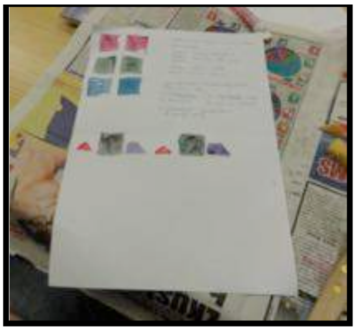

Photo 6. Vérification

Résumé des expériences

A1, A2, A3, A4 et A5 font partie de la structure didactique visant un but à long terme. A1 a prouvé la capacité de l'enfant de "lire" les informations sur son plan. A2 - A4 ont démontré qu'il y avait un important glissement dans la compréhension des notions de l'are et du hectare. Les groupes travaillant hors-classe ont mieux réussi les devoirs portant sur le calcul des surfaces, le transfert des unités et les estimations, mieux que les élèves qui avaient étudié la question uniquement en classe. (Le poster sera accompagné par la photo documentation.)

REFERENCES

KASLOVÁ, M. (1997) *Mathematical trip* - SEMT97. Praha: KMDM UK PDF. (working material for the participants of the conference SEMT, 2 pp.)

KASLOVÁ, M. (2001) *Outdorová matematika*. Texte destiné aux étudiants de UK PEDF. Praha : KMDM (4 pp.)

KASLOVÁ, M. (2001, 2003, 2004) *Outdoor mathematics in Prague*. Texte méthodologique pour la Formation des enseignants étrangers. Praha : KDMD (4 pp.).

KASLOVÁ, M. (2006) *Group mathematical competition in the wood*. Srni: CEIAEM 2006. (Working material, 3 pp.)

KASLOVÁ, M. (2008) *Outdorová matematika* (Outdoor mathematics). Texte pour un cours postuniversitaire à Hradec Králové. Hradec Králové : NIDV (3 pp.)

KASLOVÁ, M. (2009) Outdoor mathematics. In J. Novotná, H. Moraová (Ed.) *Proceedings SEMT09,* pp. 275-276. Praha : UK v Praze, PEDF.

KASLOVÁ, M. (2009) *Outdorová matematika na ZŠ (Outdoor mathematics at school)*. Conférence pour les enseignants des écoles primaires. Litomyšl : JČMF (2pp.).

KASLOVÁ, M. (2010) *Outdorová matematika v regionu* Jihlava (Outdoor mathematics in the region of Jihlava). Texte pour un cours postuniversitaire à Jihlava. Jihlava : NIDV (4 pp.).

KASLOVÁ, M. (2011) *Outdorová matematika v regionu České Budějovice*. Texte pour un

cours postuniversitaire à České Budějovice. České Budějovice : NIDV (10 pp.).
KASLOVÁ, M. (2012) *Oudorová matematika v regionu Liberec*. Texte pour un cours postuniversitaire. (Outodoor mathematics in the region Liberec). Liberec : NIDV (3 pp.)

KASLOVÁ, M.(2013) *Outdorová matematika v regionu Hradec Králové*. Texte pour un cours postuniversitaire à Hradec Králové. Hradec Králové : NIDV (2 pp.).

KASLOVÁ, M. (2014) *Ooutdoorová matematika v mateřské škole*. Texte destiné aux enseignants des écoles maternelles. Jihlava/České Budějovice : Proconsulting (2 pp.)

KASLOVÁ, M. (2015) *Outdoor mathematics*. (working material for Erasmus students). Praha : KMDM (8 pp.)

Resources for teaching and learning mathematics: A new proposal for evaluating their impact

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Abstract : Artefacts, tools and models are generally considered valuable resources for effective mathematics teaching. Their impact in pupils' performance is traditionally evaluated by examining the final result of the teaching intervention. This paper aims to fill a gap in the relevant literature by proposing a more dynamic method of evaluation in accordance with the sociocultural theories of learning. In order to assess the impact of a certain model, the pupils' argumentation is researched by using a rigorous discourse analysis research design. As a result, the model is evaluated based on evidence (actual discourse) and not by inferring reasoning which is not stated. Moreover, its role during the whole course of the teaching is highlighted.

Introduction and theoretical framework

'Models' are widely used as vital resources for effective mathematics teaching. The impact of a model is traditionally evaluated by examining the final result of a teaching intervention. If the result is the one expected then it is inferred that the model was beneficial. This approach is very useful but overlooks the role of the model throughout the intervention and also the fact that sometimes a good result might be due to other factors as well.

This study proposes a new way of assessing the value of a model, by drawing primarily on discourse analysis and sociocultural theories of learning (Gee, 1996). It involved the analysis of pupils' argumentation while working in groups on selected mathematical tasks. The effect of a diversity of models (variety of pictorial representations, concrete representations and the double number line) was investigated. This paper focuses on the 'double number line' model.

A number line is a straight line with distinct points on it that represent a specific scale and a 'double number line' is a number line that affords two sets of such points. Several researchers propose the number line as a valuable tool for mathematical problem solving (e.g. Streefland, 1984). Nevertheless, others warn it is not an obvious model for the pupils and as a result they might not be able to manipulate it effectively (e.g. Michaelidou et al., 2004). Thus, it is essential to be able to efficiently evaluate whether the role of the number line is beneficial or hindering for the pupils'. This paper attempts to provide a novel methodology for doing so.

Methodology

20 groups consisting of three pupils were involved in researcher-guided discussions (lasting 30-40 minutes) working on a specific mathematical problem. Models were selected as resources that would facilitate discussion.

This study adopts the position that the pupils' discourse (combined with their achievement) is a strong indication of their mathematical development (Gee, 1996). It is additionally assumed that a research analysis that focuses on the discursive features of a teaching intervention is based on actual evidence (recorded discourse) rather than on inferences (reasoning inferred by achievement results. Accordingly, in order to provide a robust investigation of the effect of the models, it was decided to research how their presence affected pupils' discourse. The comparison of different 'discourses' was facilitated by adopting Toulmin's (1958) approach. Toulmin (1958) classified the propositions that make up an arguments as data, conclusions, warrants and backings. As a result each argument can be schematically represented by using the same categories.

To sum up, a discourse analysis approach that combined Gee's (1996) and Toulmin's (1958) methodology with the use of Nudist software was used for this study. A significant research result of this approach was the creation of an 'individual discursive path' for each pupil. This is a kind of 'map' that presents the evolution of the pupil's argumentation in the discussion.

The paper focuses on the group discussions about a problem called 'Printing Press' and on one girl's discursive path particularly.

Results

Three groups of pupils discussed the 'Printing Press' problem (Kaput and West, 1994): 'A printing press takes exactly 12 minutes to print 14 dictionaries. How many dictionaries can it print in 30 minutes?' The correct answer is '35' but in test conditions, a high number of pupils gave the answer '32' ('additive strategy': 12+2=14 so 30+2=32). The pupils in each group gave different responses to the 'Printing Press' on a previously administered test (Misailidou and Williams, 2003). It was hypothesized that a double number line might aid the 'adders' towards changing their reasoning.

A member of one of these groups was Charlotte (11 years old). She had given the answer 28 (additive strategy: 14-2=12 so 30-2=28) on the previously administered test. At the beginning of the discussion, Charlotte defended her answer. Thus, the first stage of her discursive path was:

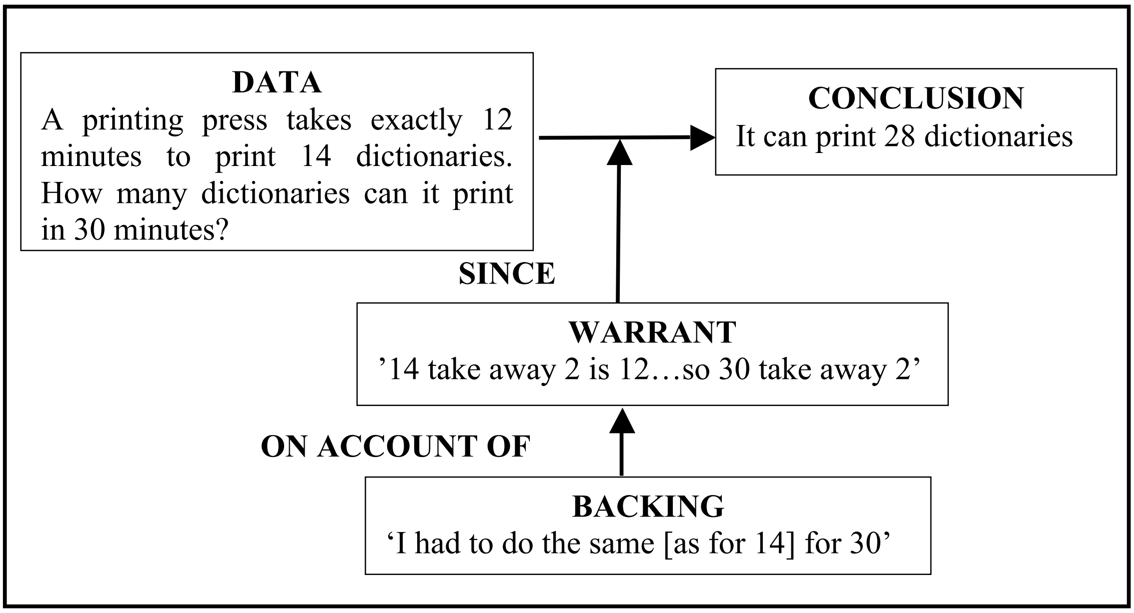

Figure 1: The first stage of Charlotte's discursive path

The discussion continued by comparing methods and at a certain point the researcher provided a piece of the paper with the following double number line on it

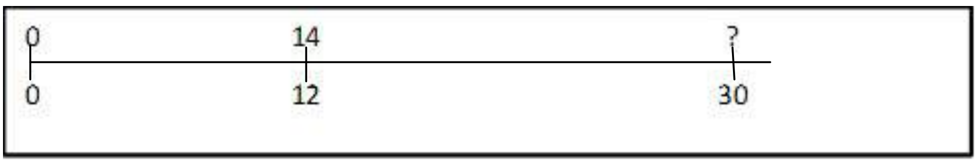

Figure 2: The double number line model

Following the researcher's advice, the pupils started working on the model. Consequently, the quality of the whole group's discourse changed as it incorporated talk about the model and gestures on it. So, they put the point (7 dictionaries, 6 minutes) on it and then the point (14 dictionaries, 12 minutes) on it and then Charlotte led the group to the right answer:

Charlotte: [Points to the gap between 28 and the question mark] It will probably be the same as this gap [points to the gap between 0 and 7]

Researcher: So Charlotte, how long is this gap?

Charlotte: [Points to the gap] 7 dictionaries.

The third and final stage of Charlotte's discursive path is the following:

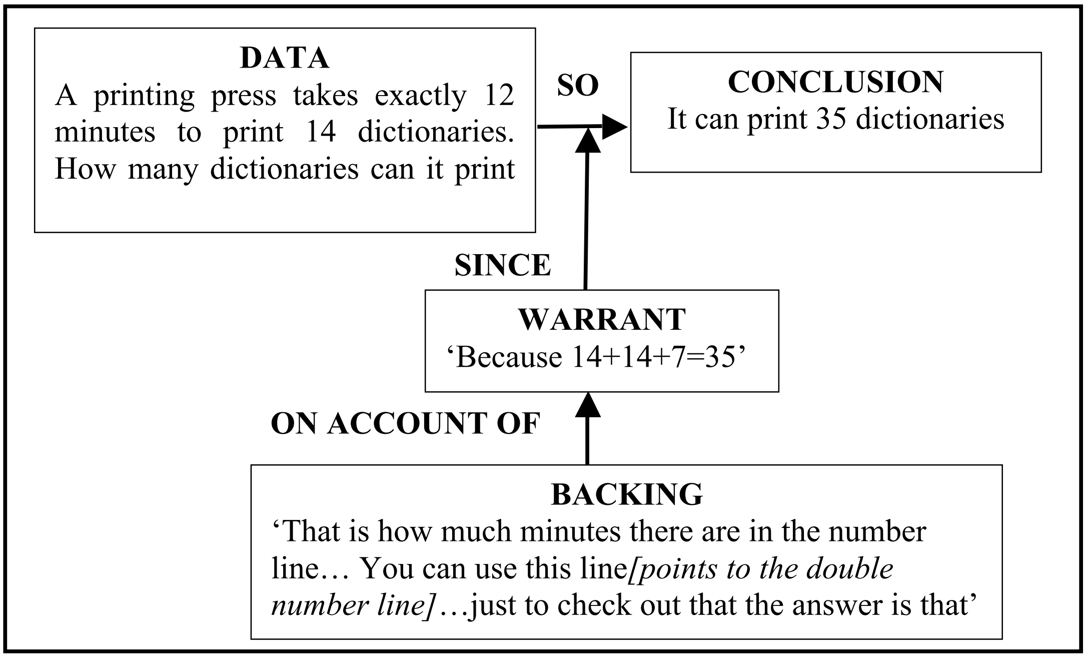

Figure 3: The third stage of Charlotte's discursive path

One can notice, that Charlotte not only gave the right answer '35' at the end of the discussion but also demonstrated an observable change in her mathematical argumentation right after the introduction of the model in the discussion. According to the Nudist analysis, 50% of Charlotte's utterances were coded as 'model-indexed' discourse, that is, discourse directly related to the model.

Conclusion

This paper aims to fill a gap in the mathematics education literature by using a rigorous discourse analysis design, for evaluating the impact of a model.

The pupils' discourse was researched and the difference in argumentation before and after the introduction of the model was coded and analysed. As a result, the effect of the model was judged not only by the pupils' final answer to the problem but also by the change in the quality of their discourse. In other words, the model was evaluated based on evidence (actual discourse) and not by inferring reasoning which is not stated. In addition, the focus was on the role of the model throughout the teaching intervention and not only on the ending result.

REFERENCES

Gee, J. (1996). Social Linguistics and Literacies. London: Taylor and Francis Ltd.

Kaput, J. & Maxwell-West, M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In: G. Harel, & J. Confrey (Eds.), The development of multiplicative reasoning in the learning of mathematics, (pp. 235-287). Albany: State University of New York Press.

Michaelidou, N., Gagatsis, A. & Pitta-Pantazi, D. (2004). The number line as a representation of decimal numbers: a research with sixth grade students. Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 3, 305–312.

Misailidou, C. & Williams, J. (2003). Diagnostic assessment of children's proportional reasoning. Journal of Mathematical Behavior, 22, 335-368.

Streefland, L. (1984). Search for the roots of ratio: Some thoughts on the long term learning process. Part I: Reflections on a teaching experiment. Educational Studies in Mathematics, 15(4), 327-348.

Toulmin, S. (1958). The Uses of Argument. Cambridge: Cambridge University Press.
Toulmin, S. (1958). The Uses of Argument. Cambridge: Cambridge University