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Game Theoretic Decentralized Feedback Controls in Markov Jump Processes

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Abstract This paper studies a decentralized routing problem over a network, using the paradigm of mean-field games with large number of players. Building on a state space extension technique, we turn the problem into an optimal control one for each single player. The main contribution is an explicit expression of the optimal decentralized control which guarantees the convergence both to local and global equilibrium points. Furthermore, we study the stability of the system also in the presence of a delay which we model using an hysteresis operator. As a result of the hysteresis, we prove existence of multiple equilibrium points and analyze convergence conditions. The stability of the system is illustrated via numerical studies .

Keywords Optimal control · Mean field games · Inverse control problem · Decentralized routing policies · Hysteresis

Mathematics Subject Classification (2000) 91A13 · 91A25 · 49N25 · 49L20 · 47J40

1 Introduction

In recent years, dynamics on networks have sparked interest in different application domains such as data transmission, traffic flows and consensus (see [1–3]) just to name a few. In this paper, we investigate a routing problem defined over a network. The problem involves a population of individuals, referred to as players. As main contribution, we provide convergence conditions to an equilibrium point, characterized by uniform distribution over all the vertices of the network. At the equilibrium, each player plays its best response, given the observed distribution of the other players. To prove such a convergence result, we recast the problem within the framework of optimal control theory. A similar problem is studied in [4], in which the authors consider a centralized control and a density flow for each edge dependent on the density of the whole population. This implies that each player minimizes a common cost functional which depends on the whole population's density distribution. Differently from [4], we consider a decentralized control (as in [5–7]), in which the density of each node is controlled locally. We highlight next three distinct approaches relating to routing/jump problems. The first one consists in controlling the probability to jump from a node to another one (or to flow along the edges) [4]. The second one consists in controlling the transition rate from nodes (or edges) [8], and the last one in assigning the product among the probability and the relative transition rate. As in [9], in this paper we use the last approach, in particular we control the product between the probability to jump from one node to an adjacent one and the relative transition rate. In the same spirit as in inverse control problems, [10], we provide an explicit expression of the running cost function in order to obtain our desired optimal feedback control as solution of the optimal control problem.

In this paper, we consider the problem of stabilizing the system under the assumption that each agent ignores both the controls of the far agents and the network topology. We formulate the problem as follows: from a microscopic point of view, each player jumps from a node to an adjacent one according to a continuous-time Markov process. From a macroscopic point of view, each node is characterized by a dynamics describing the time-evolution of the density. Such dynamics depends on a decentralized control. We rearrange the problem as a mean-field game and then via a state-space extension approach as an optimal control one. The state space extension procedure is reminiscent of the McKaen-Vlasov control problem, in which the statistical distribution is encoded by our density. Similarities and differences between the McKean-Vlasov and the Mean-Field framework are analyzed in [11].

Furthermore, we prove convergence to a local equilibrium which is characterized by an equal density on the neighbor nodes. Finally, we prove a similar convergence result for the global equilibrium, characterized by a uniform distribution of the density over all nodes. We then introduce a hysteresis operator acting on the optimal feedback decentralized control. A similar model was already discussed in [12]. The authors make a rigorous treatment of continuous-time average consensus dynamics with uniform quantization in communications. The consensus is reached by quantized measurement which are transmitted using a delay thermostat. In contrast to this, we use a different hysteresis operator, the play operator, that can be considered as a concatenation of delayed thermostats. Moreover, we apply such an operator to our control, and this results in a nonlinear dynamics. We use an hysteresis to capture a scenario where the players have distorted information on the density distribution in neighbor nodes. We prove that the problem has multiple equilibrium points, and we prove their stability.

1.1 Related Literature

The mean-field game theory was developed in the work of M.Huang, R. Malhamé and P. Caines [13, 14] and independently in that of J. M. Lasry and P.L. Lions [15,16], where the new standard terminology of Mean Field Games (MFG) was introduced. This theory includes methods and techniques to study differential games with a large population of rational players, and it is based on the assumption that the population influences the individuals' strategies through mean-field parameters. In addition to this theory, the notion of Oblivious Equilibria for large population dynamical game was introduced by G. Weintraub, C. Benkard, and B. Van Roy [17] in the framework of Markov Decision Processes. Several application domains, such as economic, physics, biology and network engineering accommodate mean-field game theoretical models (see [16,18–20]). Decision problems with mean-field coupling terms have also been formalized and studied in [21], and application to power grid management are recently provided in [22]. The literature provides explicit solutions in the case of linear quadratic structure. In most cases, a variety of solution schemes have been recently proposed, based on discretization and or numerical approximations (see [18, 23]). Computing an explicit solution in the nonlinear case is difficult, and therefore in this paper, in spirit with [24, 25], we reformulate the problem as an inverse optimal control problem.

Regarding hysteresis, the concept of hysteretic operator is due to Krasnoselskii and his co-worker [26]. There are several physical and natural phenomena in which hysteresis occurs such as in filtration through porous media, phase transition, superconductivity, shape memory and communication delay (see [12,27] for more details).

1.2 Structure of the Paper

This paper is organized as follows: a mean-field game formulation of the problem is provided in Sect. 2. In Sect. 3, we introduce a state-space extension solution approach which is an alternative method to the classical fixed point one and exhibits the optimal decentralized feedback control under a suitable assumption. In Sect. 4 we study the convergence to and the stability of a local Wardrop equilibrium and then its extension to a global equilibrium. In Sect. 5 we carry out numerical studies. Finally, in Sect. 6 we introduce the play operator which acts on the control function, and study both the global equilibrium and the stability of the density equation subject to this operator.

2 Model and Problem Set-up

In this section, we provide a model of a pedestrian density flow over a network with dynamics defined on each node, and using a line graph as topology. Let G be a graph with h nodes, e edges, and vertex degree d_i for $i = 1, \dots, h$. We define the line graph $L(G) = (V, E)$ to be the graph with $n = e$ nodes and $m = \frac{1}{2} \sum_{i=1}^{h} d_i^2$ edges. In particular, the graph is obtained by associating a vertex to each edge of the original graph and connecting two vertices with an edge if and only if the corresponding edges of G have a vertex in common. Hence, instead of considering a flux on the edges, from now on we will consider jumps between vertices. Now, let a connected line graph $L(G) = (V, E)$ be given, where $V = \{1, \dots, n\}$ is the set of vertices and $E = \{1, \dots, m\}$ is the set of edges. For each node $i \in V$, let us denote by $N(i)$ the set of neighbor nodes of i:

$$N(i) = \{j \in V : \{i, j\} \in E\}.$$

We consider a large population of players and each of them is characterized by a time-varying state $X(t) \in V$ at time $t \in [0, T]$, where $[0, T]$ is the time horizon window. Players represent pedestrians and jump across the nodes of the graph according to a decentralized routing policy described by the matrix-valued function

$$u(\cdot) : \mathbb{R}^+ \longrightarrow \mathbb{R}_+^{n \times n}, \quad t \longmapsto u(t). \tag{1}$$

Note that u takes value in $\mathbb{R}_+^{n \times n}$ because each component u_{ij} is the product between the probability to jump from one node to an adjacent one and the relative transition rate.

Let $i \in V$ be the player's initial state. The state evolution of a single player is then captured by the following continuous-time Markov process:

$$\begin{aligned} &\{X(t), t \geq 0\} \\ &q_{ij}(u_{ij}) = \begin{cases} u_{ij}, & j \in N(i), j \neq i, \\ -\sum_{k \in N(i), k \neq i} u_{ik}, & i = j, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{2}$$

where q_{ij} is the microscopic dynamics from i to j.

Denote by ρ the vector whose components are the densities on vertices. This implies that the sum of the components is equal to one. Thus we have

$$\rho \in D := \{\hat{\rho} \in [0,1]^n : \sum_{i \in V} \hat{\rho}_i = 1\}.$$

The density evolution can be described by the following forward Kolmogorov Ordinary Differential Equation (ODE)

$$\begin{cases} \dot{\rho}(t) = \rho(t) A(u), \\ \rho(0) = \rho_0, \end{cases} \tag{3}$$

where ρ is a row vector, ρ_0 is the initial condition and the matrix-valued function $A : \mathbb{R}_+^{n\times n} \to \mathbb{R}^{n\times n}$ is given by

$$A_{ij}(u) = \begin{cases} u_{ij} & \text{if } j \in N(i), j \neq i, \\ -\sum_{j \in N(i), j \neq i} u_{ij} & \text{if } i = j, \\ 0 & \text{if } j \notin N(i). \end{cases}$$

Equation (3) establishes that the density variation on each node balancing densities on neighbor nodes.

It is well known that the uniform distribution of the density on a graph corresponds to a Wardrop equilibrium [28]. Since we are considering a line graph, our aim is to achieve a uniform distribution of the density over all nodes. Indeed in traffic network the Wardrop equilibrium corresponds to equidistribution of agents along edges. Therefore on its line graph we view the equilibrium as uniform distribution on nodes. We start by proving convergence to a local equilibrium, i.e. a uniform density on the nodes adjacent to i.

For each player, consider a running cost $\ell(\cdot) : V \times [0,1]^n \times \mathbb{R}_+^{n\times n} \to [0,+\infty[$, and an exit cost $g(\cdot) : V \times [0,1]^n \to [0,+\infty[$ of the form given below

$$\ell(i,\rho,u) = \sum_{j\in N(i), j\neq i} \frac{u_{ij}^2}{2} \left(\gamma_{ij}(\rho)\right)^+, \tag{4}$$

$$g(i,\rho) = dist(\rho, \hat{M}_i). \tag{5}$$

where γ_{ij} is a suitable coefficient yet to be designed and $(\cdot)^+$ is the positive part operator.

In (5) the $dist(\rho, \hat{M}_i)$ denotes the distance of the vector ρ from the manifold $\hat{M}_i$, where $\hat{M}_i$ is the local consensus manifold/local Wardrop equilibrium set for the player i defined as

$$\hat{M}_i = \{\xi \in \mathbb{R}^n : \xi_j = \xi_i \ \forall \ j \in N(i)\}. \tag{6}$$

Therefore, the choice of the exit cost $g(i,\rho)$ describes the difference between the number of agents in the node i and the local equidistribution of agents among the adjacent nodes.

The problem in its general form is then the following:

Problem 1: Design a decentralized routing policy to minimize the output disagreement, i.e., each player solves the following problem:

$$\begin{cases} \inf_{u(\cdot)} J(i, u(\cdot), \rho[\cdot](\cdot), \cdot), \\ J(\cdot) = \int_t^T \ell(X(\tau), \rho(\tau), u(\tau))d\tau + g(X(T), \rho(T)), \\ \{X(t), t \geq 0\} \text{ as in (2)}, \\ X(t) = i, \end{cases} \tag{7}$$

where u is the control (1) taking value in $\mathbb{R}_+^{n\times n}$ for any $t \in [0,T]$ and ρ evolves as in (3). Note that every player minimizes a cost functional which depends on the density of his neighbours. Thus, the microscopic (2) and macroscopic (3) representations of the system are strongly intertwined which makes the problem different from classical optimal control.

2.1 Mean-Field Formulation

This subsection presents a mean-field formulation of problem (7). Let $v(i,t)$ be the value function of the optimization problem (7) starting from time t in state i. We can establish the following preliminary result.

Lemma 2.1 *The mean-field system for the decentralized routing problem in Problem 1 takes the form:*

$$\begin{cases} \dot{v}(i,t) + H(i,\Delta(v),t) = 0 \ \textit{in}\ V \times [0,T[, \\ v(i,T) = g(i,\rho(T)), \forall x \in V, \\ \dot{\rho}(t) = \rho(t)A(u^*), \\ \rho(0) = \rho_0, \end{cases} \tag{8}$$

where

$$H(i,\Delta(v),t) = \inf_u \left\{ \sum_{j\in N(i)} q_{ij}(v(j,t) - v(i,t)) + \ell(i,\rho,u) \right\}, \tag{9}$$

and g is given as in (5).

In the expression above, $\Delta(v)$ denotes the difference of the value function computed in two successive vertices, q_{ij} is given in (2) *and $\ell(i,\rho,u)$ as in* (4). *The* *optimal time-varying control* $u^*(i,t)$ *is given by*

$$u^*(i,t) \in \arg\min_u \Big\{ \sum_{j\in V} q_{ij}(v(j,t) - v(i,t)) + \ell(i,\rho,u) \Big\}. \tag{10}$$

Proof.: To prove the first equation of (8) we know from dynamic programming that

$$\dot{v}(i,t) + \inf_u \{ \sum_{j\in N(i)} q_{ij}(v(j,t) - v(i,t)) + \ell(i,\rho,u) \} = 0 \quad \text{in } V \times [0,T[.$$

We obtain the first equation, by introducing the Hamiltonian in (9). Since (2) depends on the routing policy u, then the latter is obtained minimizing the Hamiltonian as expressed by (10). The second equation is the boundary condition on the terminal cost. The third and fourth equation are the forward Kolmogorov equation and the corresponding initial condition. □

The mean-field game (8) appears in the form of two coupled ODEs linked in a forward-backward way. The first equation in (8) is the *Hamilton-Jacobi-Bellman* (HJB) equation with variable $v(i,t)$ and parametrized in $\rho(\cdot)$. Given the boundary condition on final state and assuming a given population density behaviour captured by $\rho(\cdot)$, the HJB equation is solved backwards and returns the value function and the optimal control (10). The Kolmogorov equation is defined on variable $\rho(\cdot)$ and parametrized in $u^*(i,t)$. Given the initial condition $\rho(0) = \rho_0$ and assuming a given individual behaviour described by u^*, the density equation is solved forward and returns the population time evolution $\rho(t)$.

3 State Space Extension

We solve Problem 1 and the related mean-field game (8) through state space extension, in spirit with [4]; namely we review ρ as an additional state variable. Then the resulting problem is of the form

$$\begin{aligned} &\inf_{u(\cdot)} \;\; J(i,u(\cdot),\rho[\cdot](\cdot),\cdot), \\ \text{subject to} \;\; &\{X(t), t \leq 0\} \text{ as in (2)}, \\ &\dot{\rho}(t) = \rho(t)A(u). \end{aligned}$$

We are looking for a value function $\tilde{V}(i,\rho,t)$ which depends on i and on the density vector ρ as a state variable, rather than as a parameter as in (7). The problem can be rewritten as follow.

Lemma 3.1 *The mean-field system for the decentralized routing problem in Problem 1 takes the form:*

$$\begin{cases} \partial_t \tilde{V}(i,\rho,t) + \tilde{H}(i,\rho,\Delta(\tilde{V}),\partial_\rho \tilde{V},t) = 0 \;\; in\; V \times [0,1]^n \times [0,T[, \\ \tilde{V}(i,\rho,T) = g(i,\rho(T)), \end{cases} \tag{11}$$

where for the Hamiltonian we have

$$\tilde{H}(i,\rho,\Delta(\tilde{V}),\partial_\rho \tilde{V},t) = \inf_u \left\{ \sum_{j\in N(i)} q_{ij}(\tilde{V}(j,\rho,t) - \tilde{V}(i,\rho,t)) + \partial_\rho \tilde{V}(i,\rho,t)(\rho A(u))^T + \ell(i,\rho,u) \right\}, \tag{12}$$

and the optimal time-varying control $u^*(i,\rho,t)$ is given by

$$u^*(i,\rho,t) \in \arg\min_u \left\{ \sum_{j\in N(i)} q_{ij}(\tilde{V}(j,\rho,t) - \tilde{V}(i,\rho,t)) + \partial_\rho \tilde{V}(i,\rho,t)(\rho A(u))^T + \ell(i,\rho,u) \right\}. \tag{13}$$

Proof: From dynamic programming we obtain

$$\partial_t \tilde{V}(i,\rho,t)+\inf_u\left\{\sum_{j\in V} q_{ij}(\tilde{V}(j,\rho,t)-\tilde{V}(i,\rho,t))+\partial_\rho \tilde{V}(i,\rho,t)(\rho A(u))^T+\ell(i,\rho,u)\right\}=0.$$

By introducing the Hamiltonian $\tilde{H}(i,\rho,\Delta(\tilde{V}),\partial_\rho\tilde{V},t)$ given in (12), the first equation is proven. To prove (13), observe that the optimal control is the minimizer in the computation of the extended Hamiltonian. Finally, the second equation in (11) is the boundary condition. □

Remark 3.1 *The use of the state space extension approach reduces our initial problem to an optimal control problem. Therefore from now on we will no longer consider the mean field formulation.*

Now, our aim is to review the optimal control problem as an inverse problem. Our aim is to find a suitable γ_{ij} (see (4)) such that the optimal control u^*_{ij}, which is the *argmin* of the extended Hamiltonian, is

$$u^*_{ij}=\begin{cases}\rho_i(t)-\rho_j(t) & \rho_i(t)>\rho_j(t),\ j\in N(i),\\ 0 & \text{otherwise.}\end{cases} \tag{14}$$

In [4] for the infinite horizon problem, the authors take the value functions as $V(\rho)=dist(\rho,M)$, where M is the global equilibrium manifold. Therefore in our finite horizon problem we assume that

$$V(i,\rho)=dist(\rho,M_i)=\sqrt{\sum_{j\in N(i)}\left(\rho_j-\frac{\sum_{k\in N(i)}\rho_k}{\#N(i)}\right)^2}. \tag{15}$$

Note that the above satisfies the boundary condition in (11), according to our choice of the exit cost g (see (5)).

We can write (3) for the generic component i as

$$\dot{\rho}_i(t) = \sum_{j \in N(i), j \neq i} \rho_j(t) u_{ji} - \sum_{j \in N(i), j \neq i} \rho_i(t) u_{ij}.$$

Starting from the Hamiltonian (12) (see also (4)) we assume that if $\rho_i \neq \rho_j$, γ_{ij} is

$$\gamma_{ij}(\rho) = \left(\frac{\rho_i^2 - \rho_i\rho_j - dist(\rho, \hat{M}_j)dist(\rho, \hat{M}_i) + dist(\rho, \hat{M}_i)^2}{(\rho_i - \rho_j)dist(\rho, \hat{M}_i)} \right). \quad (16)$$

We want to prove that, using (16), the correspondent running cost (4) is such that our control (14) is the optimal one. We have the following cases:

a) $\boxed{\gamma_{ij} > 0}$

The Hamiltonian (12) is strictly convex in u_{ij}. Therefore the optimal control u_{ij} is the solution of

$$\frac{\partial \tilde{H}}{\partial u_{ij}} = u_{ij}\gamma_{ij}(\rho) + \frac{\rho_i\rho_j - \rho_i^2}{dist(\rho, \hat{M}_i)} + dist(\rho, \hat{M}_j) - dist(\rho, \hat{M}_i) = 0. \quad (17)$$

Namely if $\rho_i > \rho_j$, $u_{ij} = \rho_i - \rho_j$, instead if $\rho_i < \rho_j$ since we are supposing that $u_{ij} \in \mathbb{R}^+$ we have that the optimal control is $u_{ij} = 0$.

b) $\boxed{\gamma_{ij} \leq 0}$

The Hamiltonian (12) is linear in u_{ij} and is increasing or decreasing depending on the sign of

$$\beta_{ij} = \frac{\rho_i\rho_j - \rho_i^2}{dist(\rho, \hat{M}_i)} + dist(\rho, \hat{M}_j) - dist(\rho, \hat{M}_i) = -\gamma_{ij}(\rho)(\rho_i - \rho_j)$$

- if $\rho_i > \rho_j$ the Hamiltonian is increasing in u_{ij}, hence it admits minimum at $u_{ij} = 0$,
- if $\rho_i < \rho_j$ the Hamiltonian is decreasing in u_{ij}, therefore it takes smaller and smaller values as $u_{ij} \to +\infty$.

Now note that, if the densities are converging in time to the same value, which is the case if we use the control u^*_{ij}, the function (16) is never negative and thus case b) before cannot occur. Simulations will show this phenomenon and also suggest that

$$\lim_{\rho_i \to \rho_j, j \in N(i) \setminus \{i\}} \gamma_{ij} = +\infty. \quad (18)$$

which is coherent with the constraint $u^*_{ij} = 0$. Therefore, using function (16), the corresponding running cost given by

$$\ell(i, \rho, u) = \sum_{j \in N(i),\, j \neq i\, \rho_i > \rho_j} \frac{u_{ij}^2}{2} \underbrace{\left(\frac{\rho_i^2 - \rho_i \rho_j - dist(\rho, \hat{M}_j) dist(\rho, \hat{M}_i) + dist(\rho, \hat{M}_i)^2}{(\rho_i - \rho_j) dist(\rho, \hat{M}_i)} \right)^+}_{\gamma_{ij}(\rho)}, \quad (19)$$

leads the optimal feedback control to take the same value as in (14). Moreover, when using control (14), the Hamiltonian (12) also converges to zero as t tends to infinity. Hence, the function $V(i, \rho)$ as defined in (15), is almost a solution of the Hamilton-Jacobi-Bellman problem (11). Such a consideration leads to the fact that, at least when time becomes large, the control (14) is optimal. The fact that the Hamiltonian (12) converges to zero derives from (17) where the second addendum of the right-hand side is bounded (the distance from the manifold is larger than $|\rho_i - \rho_j|$ up to a multiplicative constant). This boundedness leads to $(\rho_i - \rho_j)^2 \gamma_{ij}(\rho) \to 0$ and hence the conclusion, because inside the Hamiltonian we almost have (17) multiplied by $(\rho_i - \rho_j)$.

Now with the control (14), we can rewrite the evolution of ρ as

$$\dot{\rho}_i(t) = \sum_{j \neq i, j \in N(i): \rho_j > \rho_i} \rho_j(t)(\rho_j(t) - \rho_i(t)) - \sum_{j \in N(i): \rho_i > \rho_j} \rho_i(t)(\rho_i(t) - \rho_j(t)) \quad \forall i \quad (20)$$

Now our aim is to study the stability properties of dynamical system (20). In other words if using the optimal control u^*_{ij} the system converges to an equilibrium.

4 Wardrop Equilibrium

In this section, we will show how to obtain a uniform distribution of the density ρ, at first on a neighborhood of a node and then throughout the graph.
The right-hand side of equation (20) is zero only when $\rho_i = \rho_j$ $\forall i \in V$ and $j \in N(i)$, which leads to a uniform density over the nodes.
The following assumption establishes that for a given feasible target manifold, there always exists a decentralized routing policy $u(t)$ which drives the density ρ toward the relative manifold $\hat{M}_i$ (see (6)).
This assumption will be used later on to prove the convergence to a local Wardop equilibrium.

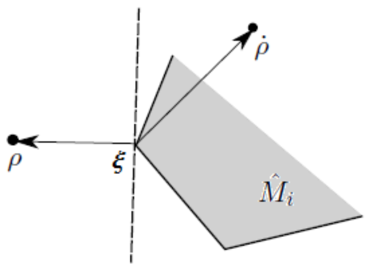


Fig. 1 Geometric illustration of the Attainability condition.

Assumption 1 (Attainability condition)

Let $\hat{M}_i$ be given by (6), $r > 0$ and $S_i = \{\rho : dist(\rho, \hat{M}_i) < r\}$. For all $\rho \in S_i \backslash \hat{M}_i$ there exists an element in the projection, $\xi(i, \rho) \in \Pi_{\hat{M}_i}\rho$, such that the value $val[\lambda_i]$ is negative for every $\lambda_i = (\rho(t) - \xi(i, \rho))$, namely

$$\forall i,\ val[\lambda_i] = \inf_u \{\lambda_i \cdot [(I - \partial_\rho(\xi(i, \rho)))\dot{\rho}^T + \sum_{j \in N(i)} (\xi(j, \rho) - \xi(i, \rho))q_{ij}]\} < 0, \quad (21)$$

where $\partial_\rho \xi(i, \rho)$ is a constant matrix since $\xi(i, \rho)$ is a linear function of ρ.

We point out that, as we will show in Section 5 (see (25)), assumption (21) is satisfied by our optimal control u^*_{ij} (14).

Assumption (21) represents the trend of the agents in node i to be influenced by the choices of the neighbor agents. Agents can act in order to reach the same density as in the adjacent nodes.

In the proof of the next theorem, we review the value function of (11) as a Lyapunov function.

Theorem 4.1 *Let Assumption 1 hold true. Then, $\rho(t)$ converges asymptotically to $\hat{M}_i$, i.e.*

$$\lim_{t \to \infty} dist(\rho, \hat{M}_i) = 0. \quad (22)$$

Proof : Let ρ be a solution of (3) with initial value $\rho(0) \in S_i \setminus \hat{M}_i$.

Set $\tau = \{\inf t > 0 : \rho(t) \in \hat{M}_i\} \leq \infty$ and let $V(i(t), \rho(t)) = dist(\rho(t), \hat{M}_i)$. For all $t \in [0, \tau]$ and $\xi \in \Pi_{\hat{M}_i}(\rho(t))$. We wish to compute $\dot{V}(i(t), \rho(t))$ as the limit of the incremental ratio, thus at first we write its numerator, where $X(t)$ is the Markov process giving the evolution of the index $i(t)$, that is:

$$V(i(t),\rho(t+dt)) - V(i(t),\rho(t)) + V(i(t+dt),\rho(t)) - V(i(t),\rho(t)) =$$

$$\|\rho(t+dt) - \xi(\rho(t+dt),X(t))\| - \|\rho(t) - \xi(\rho(t),X(t))\| +$$

$$\|\rho(t) - \xi(\rho(t),X(t+dt))\| - \|\rho(t) - \xi(\rho(t),X(t))\| =$$

$$\|\rho(t) + \dot{\rho}(t)dt - \xi(\rho(t),X(t)) - \nabla_\rho \xi(\rho(t),X(t))\dot{\rho}(t)dt\| -$$

$$\|\rho(t) - \xi(\rho(t),X(t))\| + |dt|\varepsilon(dt) +$$

$$\|\rho(t) - \xi(\rho(t),X(t)) - \partial_X \xi(\rho(t),X(t))\dot{X}(t)dt + o(dt)\| - \|\rho(t) - \xi(\rho(t),X(t))\|$$

where $\lim_{dt\to 0}\varepsilon(dt) = 0$ and $\lim_{dt\to 0} o(dt) = 0$. Hence

$$
\begin{aligned}
&\dot V(i(t),\rho(t)) = \\
&\lim_{dt\to 0}\frac{1}{dt}\Bigg(\frac{\|\rho(t)+\dot\rho(t)dt-\xi(\rho(t),X(t))-\nabla_\rho\xi(\rho(t),X(t))\dot\rho(t)dt\|^2}{\|\rho(t)+\dot\rho(t)dt-\xi(\rho(t),X(t))-\nabla_\rho\xi(\rho(t),X(t))\dot\rho(t)dt\|}- \\
&\frac{\|\rho(t)-\xi(\rho(t),X(t))\|^2}{\|\rho(t)-\xi(\rho(t),X(t))\|}+|dt|\varepsilon(dt)+ \\
&\frac{\|\rho(t)-\xi(\rho(t),X(t))-\partial_X\xi(\rho(t),X(t))\dot X(t)dt\|^2}{\|\rho(t)-\xi(\rho(t),X(t))-\partial_X\xi(\rho(t),X(t))\dot X(t)dt\|}- \\
&\frac{\|\rho(t)-\xi(\rho(t),X(t))\|^2}{\|\rho(t)-\xi(\rho(t),X(t))\|}+o(dt)\Bigg) = \\
&\lim_{dt\to 0}\frac{1}{dt}\Bigg(\frac{\|\rho(t)+\dot\rho(t)dt-\xi(\rho(t),X(t))-\nabla_\rho\xi(\rho(t),X(t))\dot\rho(t)dt\|^2}{\|\rho(t)-\xi(\rho(t),X(t))\|+O(\sqrt{dt})}- \\
&\frac{\|\rho(t)-\xi(\rho(t),X(t))\|^2}{\|\rho(t)-\xi(\rho(t),X(t))\|}+|dt|\varepsilon(dt)+ \\
&\frac{\|\rho(t)-\xi(X(t))-\partial_X\xi(\rho(t),X(t))\dot X(t)dt\|^2}{\|\rho(t)-\xi(\rho(t),X(t))\|+O(\sqrt{dt})}- \\
&\frac{\|\rho(t)-\xi(\rho(t),X(t))\|^2}{\|\rho(t)-\xi(\rho(t),X(t))\|}+o(dt)\Bigg) = \\
&\frac{1}{\|\rho(t)-\xi(\rho(t),X(t))\|}\frac{d}{dt}\Bigg(\|\rho(t)-\xi(\rho(t),X(t))\|^2\Bigg) \le \\
&\frac{2}{\|\rho(t)-\xi(i,\rho)\|}\Bigg[(\rho(t)-\xi(i,\rho))\cdot \\
&\Bigg((I-\nabla_\rho(\xi(i,\rho)))\dot\rho(t)^T+\sum_{j\in N(i)}(\xi(j,\rho)-\xi(i,\rho))q_{ij}\Bigg)\Bigg].
\end{aligned}
$$

Using now Assumption 1 we have that the second factor of the last product is strictly negative, hence $\dot V(i(t),\rho(t))<0$. This proves not only that a Wardrop equilibrium but also that the solution ρ of the dynamics (3) is locally asymptotically stable for the Lyapunov theorem. □

The next step is to prove the asymptotic convergence of ρ, solution of (3), to the global consensus manifold M defined as follows

$$
M=\{\rho\in D:\rho=\mathbf{1}\frac{1}{n}\}, \tag{23}
$$

where n is the number of nodes.

Corollary 4.1 *Let Assumption 1 hold true, then*

$$\lim_{t\to+\infty} d(\rho(t), M) = 0.$$

Proof: We are in the hypothesis of Theorem (4.1), then

$$\lim_{t\to\infty} dist(\rho, \hat{M}_i) = 0.$$

It follows that for any sequence $(t_m)_{m\in\mathbb{N}}$ such that $t_m \to +\infty$ we have that

$$\begin{aligned}
&\rho_i \to \beta \\
&\rho_j \to \beta \ \forall\ j \in N(i) \\
&\rho_k \to \beta \ \forall\ k \in N(j) \text{ s.t } j \in N(i) \\
&\vdots
\end{aligned} \tag{24}$$

By doing this, since the graph is connected, we can conclude that

$$\rho_i(t_m) \to \beta = \frac{1}{n} \quad \forall i \in V.$$

Then, there exists a subsequence $(t_{m_\ell})_{\ell\in\mathbb{N}}$ such that

$$\rho_i(t_{m_\ell}) \to \frac{1}{n} \quad \forall i \in V.$$

This proves that $\rho(t) \to \frac{1}{n}$ for $t \to +\infty$ and thus $\lim_{t\to+\infty} d(\rho(t), M) = 0$. □

5 Numerical Example

In this section, numerical simulation show that on a graph with seven nodes, the provided distributed routing policy (14) provides convergence to the equilibrium.

Consider the following network consisting of 7 nodes and 8 edges.

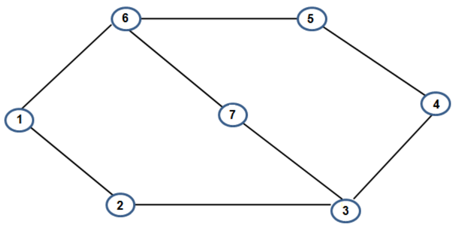


Fig. 2 Network system with seven nodes.

Solving the Kolmogorov equation (20) with the following initial conditions

$$\rho_1(0) = 0.15, \quad \rho_2(0) = 0.2, \quad \rho_3(0) = 0.1, \quad \rho_4(0) = 0.3,$$
$$\rho_5(0) = 0.1, \quad \rho_6(0) = 0.15, \quad \rho_7(0) = 0,$$

we obtain the density evolution as shown in Fig. 3

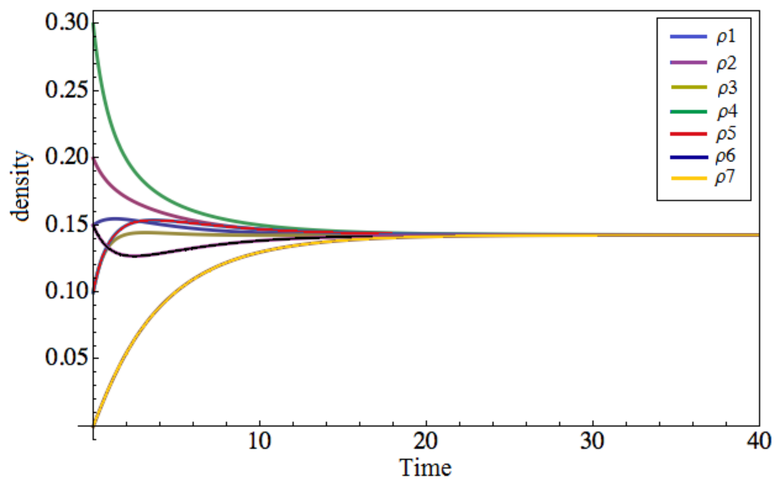


Fig. 3 Simulation of the density.

As expected the density converges to the global equilibrium in which all the ρ_i are equal.

In Fig. 4 we can see that the function γ_{ij} (16) is positive, in accordance with our statements in Section 3.

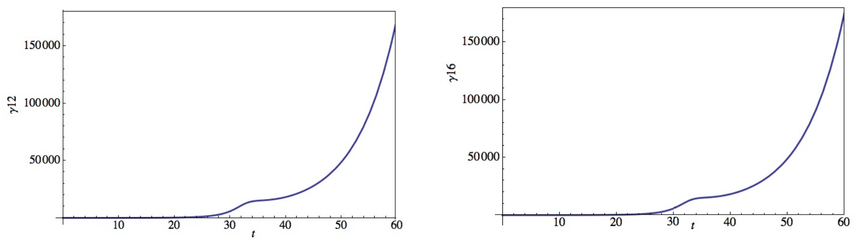


Fig. 4 Evolution of γ_{12} and γ_{16} along the trajectories obtained using control u^*_{ij} (14).

Note that the optimal control $u^*_{ij} = (\rho_i - \rho_j)^+$ satisfies Assumption 1 as by defining

$$\alpha_i = \lambda_i \cdot [(I - \partial_\rho(\xi(i,\rho)))\dot{\rho}(t)^T + \sum_{j \in N(i)} (\xi(j,\rho) - \xi(i,\rho))q_{ij}], \quad \forall i = 1, \cdots, 7,$$

we have that the maximum values of α_i are

$$\begin{aligned}
&\max_\rho\{\alpha_1\} = -6.1489 \cdot 10^{-7} \quad && \max_\rho\{\alpha_2\} = -2.1462 \cdot 10^{-6} \\
&\max_\rho\{\alpha_3\} = -3.1123 \cdot 10^{-9} \quad && \max_\rho\{\alpha_4\} = -6.7065 \cdot 10^{-7} \\
&\max_\rho\{\alpha_5\} = -8.0771 \cdot 10^{-7} \quad && \max_\rho\{\alpha_6\} = -2.1169 \cdot 10^{-6} \\
&\max_\rho\{\alpha_7\} = -7.4670 \cdot 10^{-7}.
\end{aligned} \tag{25}$$

Then, function α_i is negative for all i, for our choice of the control. According to Theorem (4.1), in Fig. 4 we show that the distance of ρ_i from the relative $\hat{M}_i$, $\forall i = 1, \ldots, 7$, converges to zero.

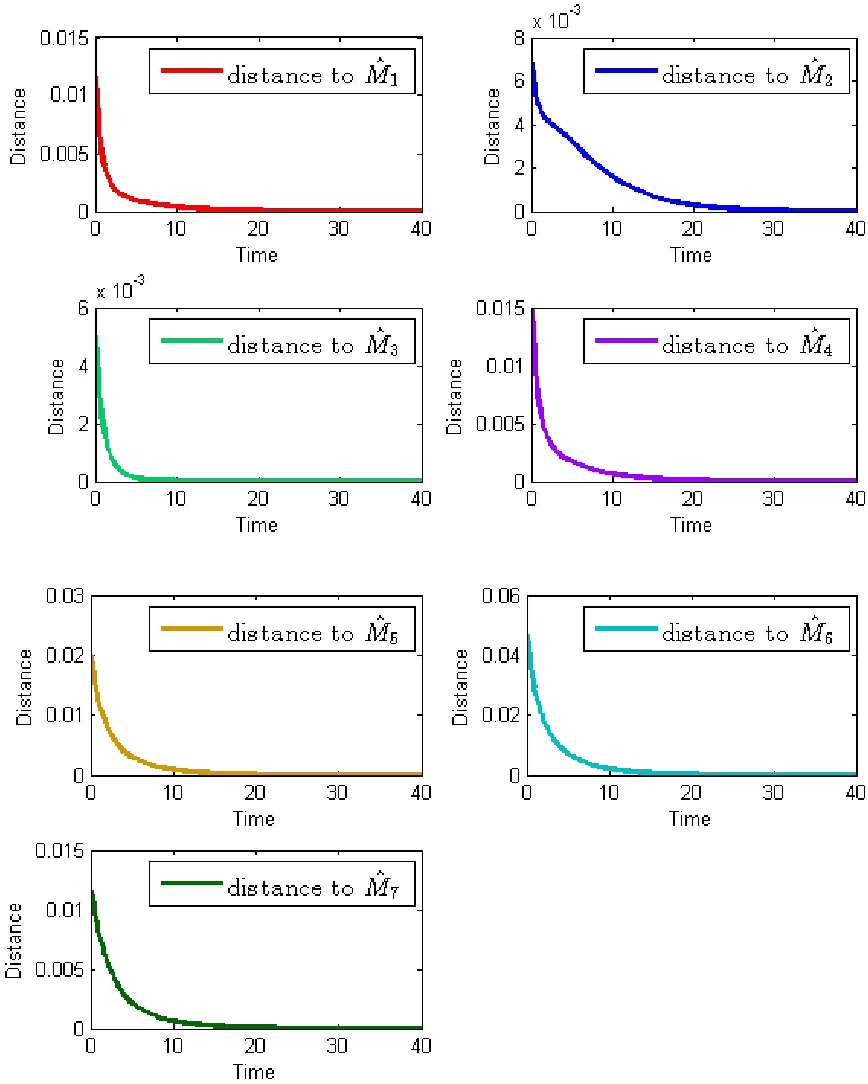


Fig. 5 Distances to the consensus manifolds.

6 Stability with Hysteresis

In the following section we study stability of the macroscopic dynamics of the vector ρ when the optimal decentralized feedback control (14) is affected by a hysteresis phenomena modeled by a scalar play operator. We study how the evolution of the macroscopic equation changes when we apply the play operator to the control u^* obtained from (14). Furthermore, we characterize the set of equilibrium points as union of several manifolds. Finally, we provide convergence condition for the resulting dynamics.

After introducing the play operator the controlled dynamical system is given by

$$\begin{cases} \dot{\rho}(t) = \rho(t)A(w), \\ w(t) = P[u^*]^+(t), \\ \rho(0) = \rho_0, \\ w(0) = w_0, \end{cases} \tag{26}$$

where $P[\cdot](\cdot)$ is the play operator whose behavior is explained in the following subsection and $\hat{}+$ is the positive part.

6.1 The Play operator

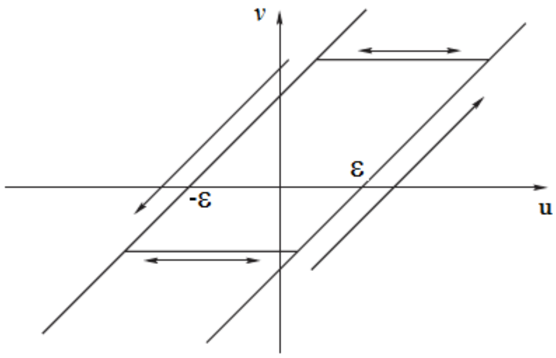


Fig. 6 Hysteresis play operator

Let $\varepsilon > 0$ be a parameter which characterizes the Play operator and define

$$\Omega_\varepsilon := \left\{(u,v) \in \mathbb{R}^2 : u - \varepsilon < v < u + \varepsilon\right\}.$$

The behavior of the scalar play operator $v(\cdot) := P[u](\cdot)$, with its typical hysteresis loops, can be described using Fig. 6. For instance, supposing that u is piecewise monotone, if $(u(t), v(t)) \in \Omega_\varepsilon$ then v is constant in a neighborhood of t; if $v(t) = u(t) - \varepsilon$ and u is non increasing in $[t, t+\tau]$ (with small τ) then v stays constant in $[t, t+\tau]$; if $v(t) = u(t) - \varepsilon$ and u is non decreasing in $[t, t+\tau]$ then $v = u(t) - \varepsilon$ in $[t, t+\tau]$. A similar argument holds when replacing $u(t) - \varepsilon$ by $u(t) + \varepsilon$.

The same explanation of the play operator behavior can be extended to continuous inputs [26, 27].

With reference to system (26), we consider as input to matrix A the positive part of the play operator, applied to the control $u^*_{ij} = (\rho_i - \rho_j)^+$, i.e. $w_{ij}(t) = P[(\rho_i - \rho_j)]^+(t)$.

Remark 6.1 *Since* $(\rho_i(0) - \rho_j(0)) = -(\rho_j(0) - \rho_i(0))$*, then*

$(\rho_i(t) - \rho_j(t)) = -(\rho_j(t) - \rho_i(t))\ \forall t.$

Thus it is not a restriction as

$P[(\rho_i - \rho_j)](0) = -P[(\rho_j - \rho_i)](0)$*, hence* $P[(\rho_i - \rho_j)](t) = -P[(\rho_j - \rho_i)](t)\ \forall t.$

Moreover since we are taking the positive part of the play, we will have that if $w_{ij} > 0$ *then* $w_{ji} = 0$.

6.2 Equilibria

We are looking for the equilibrium points of the first equation of (26) considering the simple case of a network with four nodes as the one depicted in Fig. 7

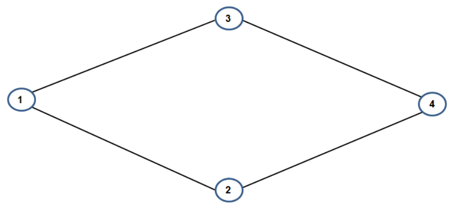


Fig. 7 Network system with four nodes.

The evolution of the vector ρ is given by

$$\begin{cases} \dot\rho_1(t) = & -\big(w_{12}+w_{13}\big)\rho_1(t) + w_{21}\rho_2(t) + w_{31}\rho_3(t), \\ \dot\rho_2(t) = & w_{12}\rho_1(t) - \big(w_{21}+w_{24}\big)\rho_2(t) + w_{42}\rho_4(t), \\ \dot\rho_3(t) = & w_{13}\rho_1(t) - \big(w_{31}+w_{34}\big)\rho_3(t) + w_{43}\rho_4(t), \\ \dot\rho_4(t) = & w_{24}\rho_2(t) + w_{34}\rho_3(t) - \big(w_{42}+w_{43}\big)\rho_4(t). \end{cases} \tag{27}$$

Case 1

Assume that $w_{12}>0, w_{31}>0, w_{24}>0, w_{43}>0$. If

$$|\varepsilon| > \max\big\{\rho_4\big(\frac{w_{43}}{w_{12}} - \frac{w_{43}}{w_{24}}\big), \rho_4\big(\frac{w_{43}}{w_{24}} - 1\big), \rho_4\big(\frac{w_{43}}{w_{31}} - \frac{w_{43}}{w_{12}}\big), \rho_4\big(1 - \frac{w_{43}}{w_{31}}\big)\big\},$$

then the system to solve is

$$\begin{cases} \dot\rho_1(t) = & -w_{12}\rho_1(t) + w_{31}\rho_3(t), \\ \dot\rho_2(t) = & w_{12}\rho_1(t) - w_{24}\rho_2(t), \\ \dot\rho_3(t) = & -w_{31}\rho_3(t) + w_{43}\rho_4(t), \\ \dot\rho_4(t) = & w_{24}\rho_2(t) - w_{43}\rho_4(t), \end{cases} \tag{28}$$

that is zero in

$$\Big(\rho_4\frac{w_{43}}{w_{12}}, \rho_4\frac{w_{43}}{w_{24}}, \rho_4\frac{w_{43}}{w_{31}}, \rho_4, w_{12}, w_{24}, w_{31}, w_{43}\Big). \tag{29}$$

In the following we consider only the values of $w_{12}, w_{24}, w_{31}, w_{43}$ because their symmetric $w_{21}, w_{42}, w_{13}, w_{34}$ are always zero according to Remark 6.1.

Case 2

Assume that $w_{12} > 0, w_{31} > 0, w_{24} > 0, w_{43} = 0$. If $|\varepsilon| > 1$, then the system is zero in

$$\big(0, 0, 0, 1, w_{12}, w_{31}w_{24}, 0\big). \tag{30}$$

Case 3

For $w_{12} > 0, w_{31} > 0, w_{24} = 0, w_{43} = 0$. If $|\varepsilon| > \max\{\rho_4, 1 - \rho_4\}$, then the system is zero in

$$\big(0, 1 - \rho_4, 0, \rho_4, w_{12}, w_{31}, 0, 0\big). \tag{31}$$

Case 4

For $w_{12} > 0, w_{31} = 0, w_{24} = 0, w_{43} = 0$. if $|\varepsilon| > \max\{\rho_4,\ \rho_3, 1 - \rho_4 - \rho_3\}$ then the system is zero in

$$\big(0, 1 - \rho_4 - \rho_3, \rho_3, \rho_4, w_{12}\big). \tag{32}$$

Case 5

Assume that all $w_{ij} = 0\ \forall j \in N(i)$. If

$$|\varepsilon| > \max\{\rho_1 - \rho_2, \rho_2 - \rho_4, \rho_3 - \rho_1, \rho_4 - \rho_3\},$$

then the equilibrium point of the system is

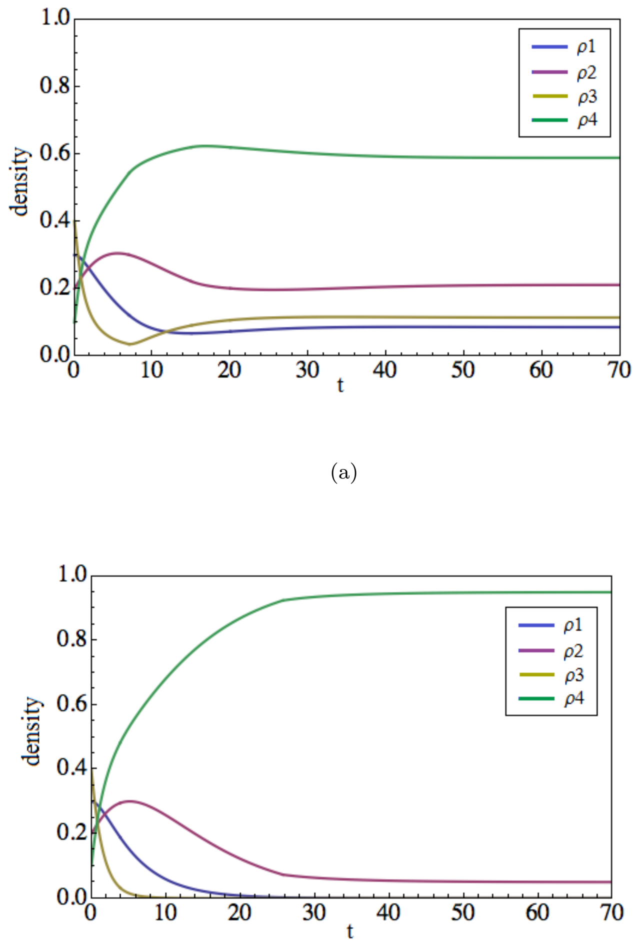
$$(1 - \rho_2 - \rho_3 - \rho_4, \rho_2, \rho_3, \rho_4, \mathbf{w}) = (\rho_1, \rho_2, \rho_3, \rho_4, \mathbf{0}), \tag{33}$$

where $\mathbf{w}$ denotes the vector of all eight w_{ij}.

Remark 6.2 *Note that the equilibria in cases* 2, 3, 4, 5 *can be obtained as limits of the equilibrium in case* 1. *Indeed if we let* $w_{43} \to 0$ *we end up with equilibrium* (30) *and since* $\sum_{i=1}^{4} \rho_i = 1$, $\rho_4 = 1$. *If* $w_{43} \to 0$ *and* $w_{24} \to 0$ *we obtain equilibrium* (31) *where we denoted by* $1 - \rho_4$ *the indeterminate form* $\rho_4 \frac{w_{43}}{w_{24}}$, *taking into account the conservation of mass. Furthermore if* $w_{31} \to 0$, $w_{24} \to 0$ *and* $w_{43} \to 0$ *we get equilibrium* (32), *where we call the indeterminate forms* $\rho_4 \frac{w_{43}}{w_{31}}$ *and* $\rho_4 \frac{w_{43}}{w_{24}}$ *respectively* ρ_3 *and* $1 - \rho_4 - \rho_3$ *for the same reason as before.*

Finally letting all $w_{ij} \to 0$ *we end up with equilibrium* (33), *in which* ρ_2, ρ_3, *and* $1 - \rho_2 - \rho_3 - \rho_4$ *denote the indeterminate forms* $\rho_4 \frac{w_{43}}{w_{24}}$, $\rho_4 \frac{w_{43}}{w_{31}}$, *and* $\rho_4 \frac{w_{43}}{w_{12}}$ *that respect the conservation of mass.*
Moreover, our choice of taking $w_{12} > 0, w_{31} > 0, w_{24} > 0, w_{43} > 0$ *and not other* w_{ij} *is completely arbitrary, indeed taking any* 4 *non symmetric* $w_{ij} > 0$ *we will end up with an equilibrium of the same type of* (29).

In the following numerical simulations we show the behavior of the system for two different choices of the parameter ε



(a)

(b)

Fig. 8 Numerical simulations of the system converging to the equilibria in case 1 (Fig. 8(a)) and case 3 (Fig. 8(b))

In Fig. 8(a) we take $\varepsilon = 0.5$. We can see that the densities converge to the equilibrium (29). Instead in the Fig. 8(b), using $\varepsilon = 0.95$, the system converges to equilibrium (31).

6.3 Stability

In the following subsection we show that also in the presence of the play operator we converge to the equilibrium for $t \to \infty$. Before doing this we make a further assumption for the manifold as defined next.

The global equilibrium manifold $\overline{M}$ in this case is the union of different equilibrium manifolds

$$\overline{M} = \bigcup_{z=1}^{5} \overline{M}_z, \tag{34}$$

where $\bar{M}_z$ denotes the manifold whose points are equilibria relative to the z-th case.

Assumption 2

Let $\overline{M}$ be given as in (34), $s > 0$ and $S = \{\overline{\rho} : dist(\overline{\rho}, \overline{M}) < s\}$. For all $\overline{\rho} = (\rho, w) \in S \setminus \overline{M}$, there exists $\overline{\xi} \in \Pi_{\overline{M}}\overline{\rho}$ such that the value $val[\lambda]$ is negative for every $\lambda = (\overline{\rho} - \overline{\xi})$, namely

$$val[\lambda] = \inf_{u}\{\lambda \cdot (I - \partial_{\overline{\rho}}\overline{\xi}(\overline{\rho}(t)))\dot{\overline{\rho}}(t)^T\} < 0. \tag{35}$$

This assumption is analogous to the attainability (21) in the presence of hysteresis. Note that here the term involving q_{ij} in (21) is not present, since depending on whether we are in the node i or in the node j the projection on the global manifold $\overline{\xi}$ is the same. Moreover, at the end of this section we will stress the fact that (35) is satisfied under control $w_{ij} = P[(\rho_i - \rho_j)]^+$.

Theorem 6.1 *Let Assumption 2 hold true. Then $\overline{\rho}(t)$ converges asymptomatically to $\overline{M}$, namely*

$$\lim_{t\to+\infty} dist(\overline{\rho}, \overline{M}) = 0. \tag{36}$$

Proof.: Let $\overline{\rho}$ a solution of (26) with initial value $\overline{\rho}(0) \in S \setminus \overline{M}$. Set $\tau = \{\inf t > 0 : \overline{\rho}(t) \in \overline{M}\} \le \infty$ and let $V(\overline{\rho}(t)) = dist(\overline{\rho}, \overline{M})$. We compute:

$$\begin{aligned}\dot{V}(\overline{\rho}(t)) =& \frac{d}{dt}\left(\|\overline{\rho}(t) - \overline{\xi}(\overline{\rho}(t))\|\right) = \\ & \frac{1}{\|\overline{\rho}(t) - \overline{\xi}(\overline{\rho}(t))\|}\left[\big(\overline{\rho}(t) - \overline{\xi}(\overline{\rho}(t))\big)\big(I - \partial_{\overline{\rho}}\overline{\xi}(\overline{\rho}(t))\big)\dot{\overline{\rho}}(t)^T\right] < 0\end{aligned}$$

by (35). Then the solution $\overline{\rho}$ of (26) is asymptotically stable and we have a global equilibrium. □

In the following we deal with some examples of convergence to the equilibria in different $\overline{M}_z$ using the decentralized control $u^*_{ij} = (\rho_i - \rho_j)^+$.

At first we suppose that $\varepsilon > 1$ thus for all t, $w(t)$ satisfies the conditions in case 2. The system to study is

$$\begin{cases} \dot{\rho}_1(t) = -w_{12}(t)\rho_1(t) + w_{31}(t)\rho_3(t), \\ \dot{\rho}_2(t) = w_{12}(t)\rho_1(t) + w_{24}(t)\rho_2(t), \\ \dot{\rho}_3(t) = -w_{31}(t)\rho_3(t), \\ \dot{\rho}_4(t) = w_{24}(t)\rho_2(t). \end{cases} \tag{37}$$

From the assumption on the w_{ij} we have

$$\exists c > 0 : w_{ij}(t) > c \; \forall t \ge 0.$$

Then considering the third equation of (37) we have that

$\rho_3(t) \le e^{-ct}\rho_3(0) \to 0$ for $t \to +\infty$. By contradiction, we suppose that

$\rho_1(t) \to \bar{\rho}_1$ with $\bar{\rho}_1 > 0$. Thus,

$$\lim_{t\to+\infty} \dot{\rho}_1(t) = \lim_{t\to+\infty} -w_{12}(t)\bar{\rho}_1 + \lim_{t\to+\infty} w_{31}(t)\rho_3(t) \neq 0. \tag{38}$$

This is a contradiction as the left hand side should be equal to zero. Hence $\lim_{t\to+\infty} \rho_1(t) = 0$. With similar argument also $\lim_{t\to+\infty} \rho_2(t) = 0$. For the mass conservation $\rho_4(t) \to 1$ for $t \to +\infty$ hence we obtain the equilibrium point (30).

Assuming now that $\varepsilon > \max\{\rho_4(0), 1-\rho_4(0)\}$ and $w(0)$ satisfies the conditions in case 3, the system becomes

$$\begin{cases} \dot{\rho}_1(t) = -w_{12}(t)\rho_1(t) + w_{31}(t)\rho_3(t), \\ \dot{\rho}_2(t) = w_{12}(t)\rho_1(t), \\ \dot{\rho}_3(t) = -w_{31}(t)\rho_3(t), \\ \dot{\rho}_4(t) = 0, \end{cases} \tag{39}$$

for all $t \in [0, \bar{t}[$ where

$$\bar{t} = \sup\{t \geq 0 : u^*_{12} + \varepsilon > w_{12}(t) \equiv w_{12}(0) > 0, u^*_{31} + \varepsilon > w_{31}(t) \equiv w_{31}(0) > 0, \\ w_{24} \equiv 0, w_{43} \equiv 0\}.$$

We will now prove that $\bar{t} = +\infty$.

Let us suppose by contradiction that $\bar{t} < +\infty$. Obviously $\rho_4(t) \equiv \rho_4(0)$ in $[0, \bar{t}[$. Using the hypothesis over w_{ij} we have that $\rho_3(t) = e^{-w_{31}(0)t}\rho_3(0)$ in $[0, \bar{t}[$ and thus ρ_3 decreases. Moreover ρ_2 is increasing.

Let us now focus on the differences among the densities. Since ρ_4 is constant and $\rho_3 \searrow$ then $\rho_4-\rho_3 \nearrow$. This difference is always less than or equal to ρ_4 and thus it is less than ε. By the continuity of ρ, $\lim_{t\to\bar{t}}(\rho_4(t)-\rho_3(t)) < \varepsilon$. Therefore w_{43} does not change and remains equal to 0 in $[0,\bar{t}]$.

Let us now consider $\rho_2-\rho_4$. By (39), in $[0,\bar{t}[$ $\rho_2 \nearrow$, thus $\rho_2-\rho_4$ increases and is less than $1-\rho_4 < \varepsilon$. By the previous continuity argument $w_{24} \equiv 0$ in $[0,\bar{t}]$. From the last two results we can conclude that $\rho_4(t) \equiv \rho_4(0)$ in $[0,\bar{t}]$.

Considering $\rho_3-\rho_1$ we have that, if $\rho_3-\rho_1 \searrow$ in $[0,\bar{t}[$, the last difference is greater than $-\rho_1 = \rho_4-1+\rho_3+\rho_2 > \rho_4-1 > -\varepsilon$. This implies $\varepsilon > \rho_1$ and thus using the continuity argument $w_{31}(t) = w_{31}(0) > 0$ in $[0,\bar{t}]$. Instead if $\rho_3-\rho_1 \nearrow$ it is always less that $\rho_3 < 1-\rho_4 < \varepsilon$. Then as before $w_{31}(t) = w_{31}(0) > 0$ in $[0,\bar{t}]$. From the last one and $w_{43} \equiv 0$ we conclude $\rho_3(t) = \rho_3(0)e^{-w_{31}(0)t}$ in $[0,\bar{t}]$.

Again if $\rho_1-\rho_2 \searrow$ it is greater than $-\rho_2 > \rho_4-1 > -\varepsilon$. Proceeding as before we conclude that $w_{12}(t) = w_{12}(0) > 0$ in $[0,\bar{t}]$. Instead if $\rho_1-\rho_2 \nearrow$ reasoning as before we reach the same conclusion, i.e, $w_{12}(t) = w_{12}(0) > 0$ in $[0,\bar{t}]$.

Hence we have proven that in $\bar{t}$, the same conditions valid in the interval $[0,\bar{t}[$, hold. Therefore there exists $\delta > 0$ such that in $[0,\bar{t}+\delta]$, $w_{ij}(t)$ are the same as in $t=0$. This is a contradiction as $\bar{t}$ is a supremum, thus we conclude $\bar{t} = +\infty$.

We will now prove that the system converges to equilibrium (31). From the assumption on the w_{ij} we have $\rho_3(t) = e^{-w_{31}(0)t}\rho_3(0) \to 0$ for $t \to +\infty$. By contradiction, we suppose that $\rho_1(t) \to \bar{\rho}_1$ with $\bar{\rho}_1 > 0$. Thus,

$$\lim_{t\to+\infty} \dot{\rho}_1(t) = \lim_{t\to+\infty} -w_{12}(t)\bar{\rho}_1 \neq 0. \tag{40}$$

This is a contradiction as it should be equal to zero. Hence $\lim_{t\to+\infty} \rho_1(t) = 0$. Regarding ρ_4 and ρ_2, the first is constant and $\lim_{t\to+\infty} \rho_2(t) = \bar{\rho}_2 > 0$. From the mass conservation $\bar{\rho}_2 = 1 - \bar{\rho}_4$ hence we obtain an equilibrium point as in (31).

Using similar arguments, if ε is like in case 4 and 5 we will converge to equilibria (32) and (33) respectively.

The above procedure can be extended to the case where ε is such that for all t we have four non symmetric $w_{ij} > 0$ like in case 1.

Note also that the decentralized control $u^*_{ij} = (\rho_i - \rho_j)^+$ satisfies Assumption 2, indeed the function $V(\overline{\rho}(t))$ is strictly decreasing along the trajectories (see Fig. 9).

As a consequence, the distance of $\overline{\rho}$ from the manifold $\overline{M}$ is a Lyapunov function and thus Theorem 6.1 holds true.

The picture below (Fig. 9) displays the distance of $\overline{\rho}$ from the manifold $\overline{M}_1$ as function of time. It is visually clear that the time plot is decreasing in accordance to our expectations.

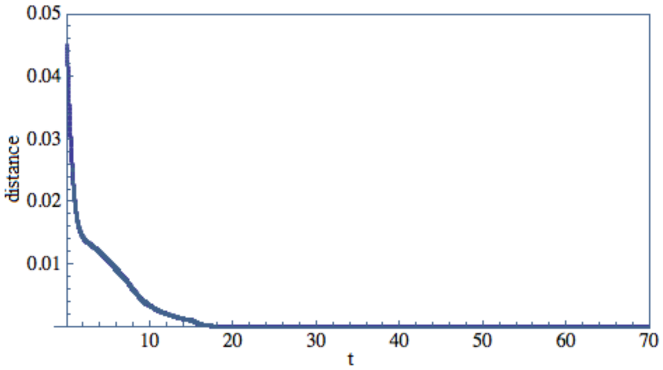


Fig. 9 The distance of $\overline{\rho}$ from the manifold M_1.

7 Conclusions

In this paper we study a decentralized routing problem defined over a network. We show that by reformulating the problem as a mean-field game, we obtain a consensus dynamics on the densities. Using a state space extension approach we recast the problem in the framework of optimal control. We give an explicit expression of a suitable current cost function in order to obtain a preassigned optimal decentralized control. We provide conditions for the convergence to both to local and global consensus. In the presence of a play operator, we prove that the same control does not guarantee convergence to a consensus point. In this case, we characterize the set of equilibrium points for the hysteretic system and prove its asymptotic stability.

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