OPINION DYNAMICS IN SOCIAL NETWORKS THROUGH MEAN–FIELD GAMES*

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Abstract. Emulation, mimicry, and herding behaviors are phenomena that are observed when multiple social groups interact. To study such phenomena, we consider in this paper a large population of homogeneous social networks. Each such network is characterized by a vector state, a vector-valued controlled input and a vector-valued exogenous disturbance. The controlled input of each network is to align its state to the mean distribution of other networks' states in spite of the actions of the disturbance. One of the contributions of this paper is a detailed analysis of the resulting mean field game for the cases of both polytopic and $\mathcal{L}_2$ bounds on controls and disturbances. A second contribution is the establishment of a *robust mean-field equilibrium*, that is, a solution including the worst-case value function, the state feedback best-responses for the controlled inputs and worst-case disturbances, and a density evolution. This solution is characterized by the property that no player can benefit from a unilateral deviation even in the presence of the disturbance. As a third contribution, microscopic and macroscopic analyses are carried out to show convergence properties of the population distribution using stochastic stability theory.

Key words. Opinion dynamics, mean field games, stochastic stability

1. Introduction.

As social networks are gaining increasing grounds and popularity, thus influencing political and socio-economic realities, a rigorous understanding of the relationships between macroscopic and microscopic opinion propagation is sparking interest among scientists in different disciplines, from engineering to economics, from finance to game theory, just to name a few. It has been observed that the evolution of opinions as a result of the interactions among agents has the flavor of an averaging process (Ayesel 2008, Castellano et al. 2009, Krause 2000, Hegselmann and Krause 2002, Pluchino et al. 2006). While with global interaction consensus can be reached, with local interactions only clusters of consensus opinions may emerge (Blondel et al. 2010). More specifically, such a clustering occurs when agents interact only with their neighbors, these being those carrying a similar opinion, and avoiding any contact with those who think differently. To capture the differences in the asymptotic values of the opinions, the literature offers a Lagrangian model of "dissensus" which makes use of graph theory and the theory of stochastic stability (Acemoğlu et al. 2013, Arnold 1974).

In this paper, rather than one single network, we consider a large number of homogeneous social networks; see Fig. 1.1. Therefore we have two layers: individual

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networks and a global network. In the last part of the work, we add a third layer as we consider a multi-population model, each population involving a large number of homogenous networks.

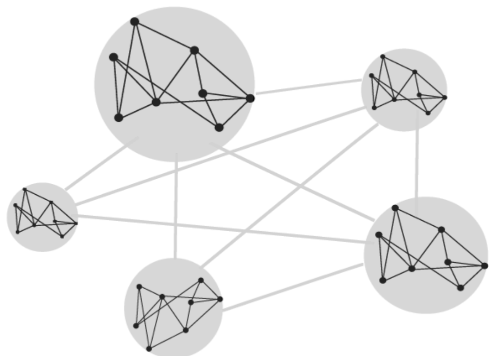

Fig. 1.1. *Network of networks.*

We consider two different scenarios, depending on whether the controls and the disturbances are polytopic or $\mathcal{L}_2$ bounded. In the first case, the daily interactions among the players are captured by a vector payoff, one component per player. The state of the network is the cumulative payoff up to the current time. The perspective we adopt is a worst-case one, in the sense that each network state is affected by an adversarial disturbance beyond the presence of a controlled input. Thus the resulting game on each network is a two-player repeated one, with vector payoffs. The first player selects the controlled input, while the second player chooses the adversarial disturbance. In the second case, we model the game as a differential one between the controller and the adversary where both the controls and the disturbances are $\mathcal{L}_2$-gain bounded. The dynamics is a typical fluid flow over a network, and thus the state represents the buffer level at each node.

The difference between the two settings, characterized by polytopic and $\mathcal{L}_2$-gain bounds, is that the disturbance has bounded instantaneous fluctuations in the former setting, while the disturbance may have unbounded instantaneous fluctuations but has bounded energy in the latter setting.

The interaction in the "global network" is based on the common knowledge of the average, which is *global knowledge*. In particular, in both cases, that is under both polytopic and $\mathcal{L}_2$-gain bounds, given a current state distribution over the entire population (of networks), the controlled input of a single network attempts to steer the state to the average of the state distribution. This resembles herd behavior or crowd-seeking attitudes in that certain social groups tend to mimic the behavior of other social groups. In addition to this, adversarial disturbances may accommodate *imperfect or partial information*, as well as *irrational behaviors on the part of some players*. In the last part of the paper, we also study *local interactions*, in the case of multiple populations.

A similar emulation behavior can be observed in financial markets under the name of "stock market bubbles", which sees investors to emulate other investors. A third application area is in everyday decision making in that decisions are made on the basis of the observed information (past decisions), thus influencing successive decisions. Such a phenomenon is known as "cascaded information" (Banerjee 1992, Bauso et al. 2012).

Highlights of the results. A first result in this paper involves providing a mean field game framework that captures the interactions among a large number of networks where herding behavior is rewarding. Mean-field games provide better insights on the *strategic nature* of the interactions between the players. Players are not "programmed" to behave in a certain way, but they have their own criteria (cost or utility functionals) and their behaviors are captured by their best-responses to the population behavior. We consider the cases of both polytopic and $\mathcal{L}_2$ bounds on controls and disturbances. The novelty of the proposed model is in the two-layer structure. At the lower layer, we have two-player repeated games, one for each network in the polytopic case, and a differential game with network flow dynamics in the $\mathcal{L}_2$ case. At the higher layer, a large number of networks interact according to a mean field game model in the case of b oth polytopic and $\mathcal{L}_2$ bounds.

For the mean field game at hand, a second result is that we establish a *robust mean-field equilibrium*, that is, a solution including the worst-case value function, the state feedback best-responses for the controlled inputs and worst-case disturbances and a density evolution. This solution is characterized by the property that no player can benefit from a unilateral deviation even in the presence of the disturbance. The robust mean-field equilibrium is an extension to the case of infinite players of the robust Nash equilibrium concept in differential games already existing in the literature (Başar and Olsder 1999, Başar 1992).

As a third contribution, microscopic and macroscopic analyses are carried out to show convergence properties of the population distribution. This is accomplished by resorting to Markov chain stability tools and stochastic stability (Arnold 1974, Gard 1988, Thygesen 1997). Under suitable assumptions, we show that the population state converges in both mean and variance. As a further contribution, we emphasize the structural features of the networks by establishing a relation between the Fiedler eigenvalue or algebraic connectivity of the Laplacian matrix of the controlled network and the maximal eigenvalue of the Laplacian matrix of the uncontrolled network.

The results in this paper constitute substantial generalizations of those in the two earlier conference papers (Bauso and Başar 2012) and (Bauso et al. 2013). Specifically, first the earlier scalar results (briefly summarized in Section 4.5) have been generalized to the vector case. Second, a study of polytopic bounds has been included (Section 3) in addition to the $\mathcal{L}_2$-gain bounds. Third, local interactions and network topologies have been studied (Section 5) and these new results have been supported by additional numerical studies (as Examples 2 and 3 in Section 6).

Related literature on mean field games. Mean field games originated in the works of J. M. Lasry and P. L. Lions (Lasry and Lions 2006a,b, 2007) and independently in those of M.Y. Huang, P. E. Caines and R. Malhamé (Huang et al. 2003, 2006, 2007). The featuring aspect is that the strategy choices of a single agent are influenced by the mass behavior of the other agents. In addition to this, the closely related notion of Oblivious Equilibrium for large population dynamic games was introduced by G. Weintraub, C. Benkard, and B. Van Roy (Weintraub et al. 2005) in the framework of Markov Decision Processes. Mean-field games find applications in many diverse domains, including economics, physics, biology, and network and production engineering (Achdou et al. 2012, Bauso et al. 2012, Gueant et al. 2010, Huang et al. 2007, Lachapelle et al. 2010, Pesenti and Bauso 2013).

The mathematical characterization of a mean field game consists of a system of two partial differential equations (PDEs). The first PDE is the *Hamilton-Jacobi-Isaacs* (HJI) equation whose solution is the *value function* which is parametrized in the population distribution (Bauso et al. 2012). The HJI equation is coupled with a second PDE, known as *Fokker-Planck-Kolmogorov (FPK)* equation, defined on variable population distribution and parametrized in the value function (Achdou and Capuzzo Dolcetta 2010, Lasry and Lions 2007, Tembine et al. 2011, 2014, Zhu et al. 2011).

Explicit closed-form expressions for mean field equilibria are available only for a limited class of problems, among which is the class of linear-quadratic mean field games; see (Bardi 2012) among others. As an alternative to explicit solutions, a variety of numerical approaches are available which hinge on discretization and finite difference approximations (Achdou and Capuzzo Dolcetta 2010).

Evolutionary games constitute another stream of literature strongly connected to mean field games and large population games (Selten 1970, Jovanovic and Rosenthal 1988, Tembine et al. 2009). A first attempt to introduce dynamics in such games can be seen in (Jovanovic and Rosenthal 1988) for a discrete time version of anonymous stochastic games, which represent precursors of mean field games.

More recently, and in the spirit of the present paper, the notion of robustness has been brought into the picture. Robust mean field games aim to achieve robust performance and/or stability in the presence of unknown disturbances when there is a large number of players; see (Tembine et al. 2011, 2014), where relations with risk-sensitive games and risk-neutral games have been studied.

The balance of this paper is organized as follows. In Section 2 we formulate the problem and introduce the model. In Section 3 we analyze the case of polytopic bounds on control and disturbance. In Section 4 we consider the case of $\mathcal{L}_2$ bounds on control and disturbance. In Section 6 we provide numerical examples. Finally, in Section 7 we draw some conclusions. Even though the proofs of the results are not auxiliary material but constitute an essential part and contribution of the paper, we find it more convenient for readability purposes to collect them all in an appendix, toward the end of the paper.

Notation We let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We let $\mathcal{B}$ be a finite-dimensional standard Brownian motion process defined on this probability space. We define $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, its natural filtration augmented by all the $\mathbb{P}-$null sets (sets of measure-zero with the respect $\mathbb{P}$). We write ∂_x and ∂_{xx}^2 to stand respectively for the gradient vector operator and the Hessian matrix operator with respect to x. We write *div* to mean the divergence operator. Given any matrix A, we write $Tr(A)$ to denote its trace. We denote by $[0, T]$ the time interval (horizon) over which the game evolves. Given a stochastic process $X_{[0,T]}$ over $[0, T]$, and its value at $t \in [0, T]$ denoted by $X(t)$, and given a scalar ξ, the expression $\mathbb{P}(X(t) \leq \xi)$ stands for "the probability" that the event $\{X(t) \leq \xi\}$ occurs. Given a set U we denote by $|U|$ its cardinality, when it is appropriate. We denote by I the identity matrix.

2. Formal statement of the problem. We consider a "large population" of *networks*, i.e., social groups characterized by a controlled time-varying behavior/state/characteristics. By "large population" we mean a large number of *homogeneous networks*, namely, the scenario where the players' interactions in each network are similar. Each network is subject to controlled inputs and adversarial disturbances.

The characteristic of each network is an n-dimensional opinion vector $X(t) \in \mathbb{R}^n$ at time $t \in [0, T]$. The control variable, which is the rate of variation of the group's opinion, is a measurable function of time, $u(\cdot) \in U$, where $U \subseteq \mathbb{R}^p$ is the control set and p is a positive integer. An adversary (persuader or stubborn agent) attempts to disturb the group's opinions in a way that is proportional to his advertisement efforts $w(\cdot) \in W$, where $W \subseteq \mathbb{R}^q$ is the control set of the adversary, which we will call disturbance set. In a nutshell, we have

- $X : [0, T] \to \mathbb{R}^n$, $t \mapsto X(t)$, group's opinion at time t, and x its initial opinion at time $t = 0$
- $u : [0, T] \to U$, $t \mapsto u(t)$, control at time t
- $w : [0, T] \to W$, $t \mapsto w(t)$, disturbance at time t

In accordance with the above, the opinion dynamics of each group can be written in general terms as

$$\left\{ \begin{array}{l} dX(t) = f(X(t), u(t), w(t))dt + \sigma d\mathcal{B}(t), \ t > 0 \\ X(0) = x, \end{array} \right. \tag{2.1}$$

where $f : \mathbb{R}^n \times U \times W \to \mathbb{R}^n$, $\sigma > 0$ is a scalar weighting coefficient and $\mathcal{B}(t)$ is a standard vector Brownian motion. For the sake of notational simplicity, and since groups are homogeneous, we do not index X, u, w, $\mathcal{B}$ by i even though they are associated with a generic group i. Note also that the Brownian motion processes are independent across groups.

Consider a probability density function $m : \mathbb{R}^n \times [0, +\infty[\to \mathbb{R}$, $(x, t) \mapsto m(x, t)$, representing the density of the groups whose collective opinion is x at time t, which satisfies $\int_{\mathbb{R}^n} m(x, t)dx = 1$ for every t. Let us also define the mean opinion over groups at time t as $\overline{m}(t) := \int_{\mathbb{R}^n} xm(x, t)dx$.

The objective of a group is to adjust its vector opinion based on the average opinion of the other groups. This reflects a typical crowd-seeking behavior in that emulating others makes an agent more comfortable and at ease.

Accordingly consider, for each group, a running cost $g : \mathbb{R}^n \times \mathbb{R}^n \times U \times W \to [0, +\infty[$, $(x, \overline{m}, u, w) \mapsto g(x, \overline{m}, u, w)$ of the form:

$$g(x, \overline{m}, u, w) = \frac{1}{2} \left[(\overline{m} - x)^T Q (\overline{m} - x) + u^T C u + w^T \Gamma w \right], \tag{2.2}$$

where Q and C are positive definite, and Γ is negative definite.

The above cost describes i) the (weighted) square deviation of an individual's state from the mean state computed over the entire population, ii) the (weighted) energy of the control, and iii) the (weighted) energy of the disturbance. Such a cost reflects the willingness of the individuals to mimic the mean population behavior as it happens in herd behavior or crowd-seeking attitudes.

Also consider a terminal cost $\Psi : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[$, $(x, \overline{m}) \mapsto \Psi(x, \overline{m})$ of the form

$$\Psi(x, \overline{m}) = \frac{1}{2} (\overline{m} - x)^T S (\overline{m} - x), \tag{2.3}$$

where S is positive definite. The problem in its generic form is then the following:

Problem 1.

Corresponding to a generic group, let $\mathcal{B}$ be a standard Brownian Motion, independent across components, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the natural filtration generated by $\mathcal{B}$. Let the initial state $X(0)$ be independent of $\mathcal{B}$ and with density $m_0 : \mathbb{R}^n \to \mathbb{R}$. Given a finite horizon $T > 0$, the initial density m_0, a suitable running cost: $g : \mathbb{R}^n \times \mathbb{R}^n \times U \times W \to [0, +\infty[$, $(x, \overline{m}, u, w) \mapsto g(x, \overline{m}, u, w)$, as in (2.2); a *terminal cost $\Psi : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[$, $(\overline{m}, x) \mapsto \Psi(\overline{m}, x)$, as in (2.3), and given a suitable dynamics $f : \mathbb{R}^n \times U \times W \to \mathbb{R}^n$ for X as in (2.1), solve the minmax problem*

$$\inf_{u(\cdot)\in\mathcal{U}} \sup_{w(\cdot)\in\mathcal{W}} \Big\{ J(x, u(\cdot), w(\cdot), \overline{m}(\cdot)) = \mathbb{E}\Big[\int_0^T g(X(t), \overline{m}(t), u(t), w(t))dt + \Psi(X(T), \overline{m}(T))\Big] \Big\}, \quad (2.4)$$

where $\bar{m}$ is the first moment of m, as introduced earlier, and $\mathcal{U}$ and $\mathcal{W}$ are the sets of all measurable functions $u(\cdot)$ and $w(\cdot)$ from $[0, T]$ to U and W, respectively.

2.1. Turning the problem into a mean field game. Let us denote by $v(x, t)$ the (upper) value of the robust optimization problem defined above, under worst-case disturbance starting from time t at state x. The first step is to show that the problem results in the following mean field game system for the unknown scalar functions $v(x, t)$, and $m(x, t)$ when each group behaves according to (2.4):

$$\left\{ \begin{array}{l} \partial_t v(x,t) + \inf_{u\in U} \sup_{w\in W} \{\partial_x v(x,t) f(x,u,w) + g(x, \overline{m}, u, w)\} \\ + \frac{\sigma^2}{2} Tr\Big(\partial^2_{xx} v(x,t)\Big) = 0 \text{ in } \mathbb{R}^n \times [0, T[, \\ \\ v(x,T) = \Psi(x, \overline{m}) \ \forall\ x \in \mathbb{R}^n, \\ \\ \partial_t m(x,t) + div(m(x,t) \cdot f(x, u^*, w^*)) - \frac{\sigma^2}{2} Tr(\partial^2_{xx} m(x,t)) = 0, \\ \\ m(x,0) = m_0(x). \end{array} \right. \quad (2.5)$$

In the above, $u^*(t, x)$ and $w^*(t, x, u)$ are the optimal time-varying state-feedback controls and disturbances, respectively, obtained as

$$\left\{ \begin{array}{l} u^*(t,x) \in \arg\min_{u\in U} \{\partial_x v(x,t) f(x,u,w^*) + g(x, \overline{m}, u, w^*)\}, \\ \\ w^*(t,x,u) \in \arg\max_{w\in W} \{\partial_x v(x,t) f(x,u,w^*) + g(x, \overline{m}, u, w)\}, \end{array} \right. \quad (2.6)$$

where we have implicitly assumed that in the HJI equation in (2.5), inf and sup can be replaced by min and max, respectively.

The mean field game system (2.5) appears in the form of two coupled PDEs intertwined in a forward-backward way. The first equation in (2.5) is the HJI equation with variable $v(x, t)$ and parametrized in $m(\cdot)$. Given the boundary condition on final state (second equation in (2.5)), and assuming a given population behavior captured by $m(\cdot)$, the HJI equation is solved backwards and returns the value function and best-response behavior of the individuals (first equation in (2.6)) as well as the worst adversarial response (second equation in (2.6)). The HJI equation is coupled with a second PDE, known as *Fokker-Planck-Kolmogorov (FPK)* (third equation in (2.5)), defined on variable $m(\cdot)$ and parametrized in $v(x, t)$. Given the boundary condition on initial density $m(x, 0) = m_0(x)$ (fourth equation in (2.5)), and assuming a given individual behavior described by u^*, the FPK equation is solved forward and returns the population behavior time evolution $m(x, t)$.

Noting that the distribution enters the cost through its mean, we can simplify the above system through model reduction. Indeed, we can replace the FPK equation by an ordinary differential equation in the variable mean distribution and obtain the following system:

$$\begin{cases} \partial_t v(x,t) + \inf_{u\in U} \sup_{w\in W} \Big\{ \partial_x v(x,t) f(x,u,w) + g(x,\overline{m},u,w) \Big\} \\ + \frac{\sigma^2}{2} Tr(\partial^2_{xx} v(x,t)) = 0 \text{ in } \mathbb{R}^n \times [0,T[, \\ \\ v(x,T) = \Psi(x,\overline{m}) \;\forall\; x \in \mathbb{R}^n \\ \\ \dfrac{d}{dt}\overline{m}(t) = \displaystyle\int f(x,u^*,w^*) m(x,t) dx, \\ \\ \overline{m}(0) = \overline{m}_0. \end{cases} \tag{2.7}$$

Any solution of the above system of equations along with (2.6) is referred to as *worst-disturbance feedback mean-field equilibrium* as no individual benefits arise from deviating from the adopted strategy.

3. Polytopic bounds on control and disturbances. We specialize the above model to the case where i) the control and the disturbances are probability distributions over Euclidean spaces, ii) the dynamics is "normalized" and bilinear in control and disturbance, and iii) the running cost does not depend on the control and disturbance. More formally,

- the control set $U = \Delta(\mathbb{R}^p)$, where $\Delta(\mathbb{R}^p)$ is the simplex in $\mathbb{R}^p$
- the disturbance set $W = \Delta(\mathbb{R}^q)$, where $\Delta(\mathbb{R}^q)$ is the simplex in $\mathbb{R}^q$
- function $f : U \times W \to [-1,1]^n$ is bilinear
- function $g : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty[$ depends only on state x and mean distribution $\overline{m}$

Each player then solves the following special case of Problem 1:

$$\begin{cases} \inf_{u(\cdot)\in\mathcal{U}} \sup_{w(\cdot)\in\mathcal{W}} \quad \Big\{ J(x,u(\cdot),w(\cdot),\overline{m}(\cdot)) = \mathbb{E}\Big[\int_0^T g(X(t),\overline{m}(t))dt \\ \qquad\qquad\qquad\qquad + \Psi(X(T),\overline{m}(T)) \Big] \Big\} \\ \\ \qquad\qquad dX(t) = f(u(t),w(t))dt + \sigma dB(t), \quad X(0) = x. \end{cases} \tag{3.1}$$

The above dynamics describes the cumulative payoff in a two player repeated game with vector payoffs, which is studied next.

3.1. Repeated two player game with vector payoffs. Consider a two player repeated game where player 1 plays $u(t) \in U \equiv \Delta(\mathbb{R}^p)$, player 2 plays $w \in W \equiv \Delta(\mathbb{R}^q)$, and $f : U \times W \to [-1,1]^n$ is the payoff. We assume that f is bilinear in u and w. We denote by G the one-shot vector payoff game $(U,W,f(t))$. With respect to the above game, in the spirit of attainability (Lehrer et al. 2011), we aim at analyzing convergence properties of the disturbed cumulative payoff $X(t)$ obtained by integrating (in the sense of Itô) the stochastic differential equation in (3.1),

$$X(t) = x + \int_0^t f(u(t),w(t))dt + \sigma\mathcal{B}(t), \quad X(0) = x. \tag{3.2}$$

Toward this goal, we will make use of the notion of *projected game* which we recall next. Let us consider $\lambda \in \mathbb{R}^n$ and denote by $\langle \lambda, G\rangle$ the one-shot zero-sum game whose set of players and their actions are as in game G, and the amount player 2 pays to player 1 is $\lambda^T f(u,w)$). Note that, as a zero-sum one-shot game, the game $\langle \gamma, G\rangle$ has a *value*, denoted by $val[\lambda]$, obtained as

$$val[\lambda] := \inf_{u\in\Delta(U)}\sup_{w\in\Delta(W)}\{\lambda^T f(u,w)\}.$$

ASSUMPTION 3.1. ***(Attainability condition)*** *The value of the projected game, $val[\lambda]$, is negative for every $\lambda \in \mathbb{R}^n$, i.e.,*

$$val[\lambda] := inf_{u\in\Delta(U)} sup_{w\in\Delta(W)}\{\lambda^T f(u,w)\} < 0, \quad \forall\, \lambda \in \mathbb{R}^n. \tag{3.3}$$

It is worth noting that, denoting the set of all possible payoffs for a fixed mixed action u of player 1 by $D_1(u) = \{f(u,w) \colon w \in \Delta(\mathbb{R}^q)\}$, then the attainability condition (3.3) implies that for every $\lambda \in \mathbb{R}^n$ there always exists a u such that $D_1(u)$ is contained in the open half space $H := \{x \in \mathbb{R}^n | \, \lambda^T x < 0\}$ in $\mathbb{R}^n$ (Lehrer et al. 2011).

3.2. Uncertain network flow control problems. Consider the continuous time dynamic system (3.4) where $X(t) \in \mathbb{R}^n$ is the state variable, and $F \in \mathbb{R}^{n\times\mathbf{p}}$ and $G \in \mathbb{R}^{n\times\mathbf{q}}$ are the controlled matrices (Blanchini et al. 2000):

$$\begin{cases} dX(t) = (F\mathbf{u}(t) - E\mathbf{w}(t))dt + \sigma d\mathcal{B}(t), \\ X(0) = x. \end{cases} \tag{3.4}$$

We can think of $F \in \{-1,0,1\}^{n\times\mathbf{p}}$ and $E\{-1,0,1\}^{n\times\mathbf{q}}$ as incidence matrices of hypergraphs. Here, $t \mapsto \mathbf{u}(t)$ is the measurable control, taking values, for all $t \geq 0$, in the set of constant controls (3.5) with preassigned vectors $\mathbf{u}^-$ and $\mathbf{u}^+$, and $t \mapsto \mathbf{w}(t)$ is the measurable unknown disturbance, taking values, for all $t \geq 0$, in the set of constant disturbances (3.6) with preassigned vectors $\mathbf{w}^-$ and $\mathbf{w}^+$:

$$\mathbf{U} = \left\{\mu \in \mathbb{R}^{\mathbf{p}} \middle| \mathbf{u}_i^- \leq \mu_i \leq \mathbf{u}_i^+, \, \forall i = 1, \ldots, \mathbf{p}\right\}, \tag{3.5}$$

$$\mathbf{W} = \left\{\omega \in \mathbb{R}^{\mathbf{q}} \middle| \mathbf{w}_i^- \leq \omega_i \leq \mathbf{w}_i^+, \, \forall j = 1, \ldots, \mathbf{q}\right\}. \tag{3.6}$$

The above dynamics tell us that matrix F combines the controls $\mathbf{u}(t)$ in order to counterbalance the disturbance $\mathbf{w}(t)$. A state variable $X(t)$ accumulates up to time t any discrepancies in the counterbalancing action. Such a dynamics often arises in network flow (Bauso et al. 2010, Blanchini et al. 1997).

In (Blanchini et al. 2000), in the absence of Brownian motion, and for the corresponding ordinary differential equation of (3.4) (where $\sigma = 0$), the authors provide a solution to the problem of finding a control strategy that drives the state $X(t)$ from any initial point in a preassigned set $\mathcal{Q}_0$ to any point of a given set $\mathcal{Q} = \{\xi \in \mathbb{R}^n : \xi_i^- \leq \xi \leq \xi_i^+\}$, with assigned bounds $\xi_i^-, \xi_i^+ \in \mathbb{R}^n$, strictly including $\mathcal{Q}_0$. This problem goes under the name of reachability control problem, and its formal statement is adapted from (Blanchini et al. 2000) and reproduced below.

DEFINITION 3.1. ***(Robust reachability problem)*** *Find a control strategy $\mathbf{u}(t) = \psi(X(t))$ such that for all $x \in \mathcal{Q}_0$ and for any $\mathbf{w}(t) \in \mathbf{W}$ there exists a $\tau > 0$ and a solution $X(t)$ to equation $\dot{X}(t) = F\mathbf{u}(t) + E\mathbf{w}(t)$ satisfying $X(t) \in \mathcal{Q}$, for all $t \geq \tau$ and $\mathbf{u}(t) \in \mathbf{U}$, for all $t \geq 0$.*

Whenever the above problem is feasible we say that the set $\mathcal{Q}$ is reachable in a robust sense, i.e., for any $\mathbf{w} \in \mathbf{W}$ (Bertsekas and Rhodes 1971).

It turns out that a necessary and sufficient condition for the above problem to be feasible is the "dominance" condition (3.7) as proved in (Blanchini et al. 1997, 2000). Formally, suppose that matrix E is full row rank; then the reachability control problem is feasible for the continuous time system

$$\begin{cases} \dot{X}(t) = F\mathbf{u}(t) - E\mathbf{w}(t), \\ X(0) = x \in \mathcal{Q}_0, \end{cases}$$

if and only if the set $E\mathbf{W} := \left\{E\omega \middle| \omega \in \mathbf{W}\right\}$ is in the interior of the set $F\mathbf{U} := \left\{F\mu \middle| \mu \in \mathbf{U}\right\}$, that is,

$$E\mathbf{W} \subseteq int\{F\mathbf{U}\}. \tag{3.7}$$

In the rest of this section, we first show how to turn the network system (3.4) into the stochastic differential equation in (3.1) and then we describe the fundamental relations between the reachability condition (3.7) and the attainability condition (3.3) developed in the previous subsection for the repeated game.

To do this, let us look at $X(t)$ as the vector valued payoff function of a two player game as in (3.2), where one player, the controller, selects action $u(t) \in U$ and the other player, nature, selects the disturbance $w(t) \in W$.

In addition to this, let us denote by $vert\{\mathbf{U}\}$ the set of vertices of $\mathbf{U}$ and similarly for $vert\{\mathbf{W}\}$, and let $p = |vert\{\mathbf{U}\}|$ (number of vertices in $\mathbf{U}$). Likewise, let us take $q = |vert\{\mathcal{W}\}|$. Also, let us use the symbol $u^{(i)}$, $i = 1, \ldots, p$ to denote the generic ith vertex in $\mathbf{U}$ and similarly for $w^{(j)}$, $j = 1, \ldots, q$.

We now formalize a game where both players select vertices from their respective bounding polytopic sets. More formally, consider the set of actions $vert\{\mathcal{U}\}$ and $vert\{\mathcal{W}\}$ for the control and the disturbance, and observe that mixed actions lie in the simplices in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, i.e., $u \in \Delta(\mathbb{R}^p)$ and $w \in \Delta(\mathbb{R}^q)$. Let us take as cumulative payoff at time t the function

$$X(t) = \int_0^t f(u(t), w(t))dt = \int_0^t \Big(F \sum_{i=1}^{p} u_i(t)u^{(i)} + E \sum_{j=1}^{q} w_j(t)w^{(j)}\Big)dt + X(0), \tag{3.8}$$

where u_i is the probability assigned to control $u^{(i)}$, w_j the probability assigned to disturbance $w^{(j)}$. We are now in a position to state one of the main results of this paper.

THEOREM 3.2. *If condition (3.7) holds, then the payoff function $f(u, w)$ in (3.8) satisfies the attainability condition (3.3).*

Proof. Given in the appendix. □

To see how players' interactions over networks captured by the network system (3.4) turn into the stochastic differential equation used in (3.1), consider the following example borrowed from (Lehrer et al. 2011).

EXAMPLE 1.

Consider the network depicted in Fig. 3.1 where nodes represent two opposite states of the characteristics of a population, such as political opinions (left-right), social behavior (cooperative-noncooperative) or marketing strategies (aggressive-nonaggressive). Suppose that a central planner or persuasor can influence the flow of people entering each state, and let us represent this by three controlled flows. A unit of flow f_1 produces a unitary increase of state X_1 per time unit. Similarly, flow f_2 represents a migration of people from state X_1 to state X_2 per time unit. A unit of flow f_3 produces a unitary increase of people in state X_2 per time unit. Uncontrolled flows w_1 and w_2

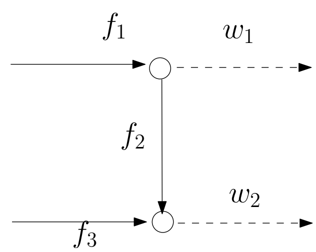

FIG. 3.1. *Network system.*

represent migrations from X_1 *and* X_2 *driven by exogenous forces. The associated dynamics then read:*

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} f_1^t \\ f_2^t \\ f_3^t \end{bmatrix}}_{\mathbf{u}(t)} - \underbrace{\begin{bmatrix} w_1^t \\ w_2^t \end{bmatrix}}_{\mathbf{w}(t)}.$$

Now, suppose that migrations occur only in batches and therefore take for instance $f_i \in \{-5, -2, 1, 6\}$*, and* $w_i \in \{-3, 2\}$*.*

Let us enumerate all the actions of players 1 and 2, so that we have $vert\{\mathbf{U}\} = \{u^{(1)}, \ldots, u^{(p)}\}$ *and* $vert\{\mathbf{W}\} = \{w^{(1)}, \ldots, w^{(q)}\}$ *with* $p = 4^3$ *and* $q = 2^2$*.*

The complete matrix of vector payoffs is then obtained from the following table, where each entry represents a possible vector payoff $f(u, w)$*:*

$u^{(i)}/w^{(j)}$	$w^{(1)}$	$\ldots$	$w^{(q)}$
$u^{(1)}$	$Fu^{(1)} - w^{(1)}$	$\ldots$	$Fu^{(1)} - w^{(q)}$
$\vdots$	$\vdots$		$\vdots$
$u^{(p)}$			

For sake of conciseness, we can simply extract from the above table the rows corresponding to the following four actions of player 1:

$$u^{(1)} = (1, -2, 6), u^{(2)} = (1, -2, -5),$$
$$u^{(3)} = (-5, 1, -5),\ u^{(4)} = (-5, 1, 6).$$

For player 2, we consider the following four actions:

$$w^{(1)} = (-3, -3), w^{(2)} = (2, -3),$$
$$w^{(3)} = (-3, 2),\ w^{(4)} = (2, 2).$$

Control and disturbance pure action sets are then $vert\{\mathbf{U}\} = \{u^{(1)}, \ldots, u^{(4)}\}$ *and* $vert\{\mathbf{W}\} = \{w^{(1)}, \ldots, w^{(4)}\}$*, respectively.*

The following 4×4 *matrix includes all possible vector payoffs corresponding to pure actions of both players:*

$$\begin{pmatrix} (6,7) & (1,7) & (6,2) & (1,2) \\ (6,-4) & (1,-4) & (6,-9) & (1,-9) \\ (-3,-1) & (-8,-1) & (-3,-6) & (-8,-6) \\ (-3,10) & (-8,10) & (-3,5) & (-8,5) \end{pmatrix}. \tag{3.9}$$

Function $f(u,w)$ is the bilinear extension to mixed strategies $u,w \in \Delta(\mathbb{R}^4)$ of such a vector payoff matrix,

$$\left[\begin{array}{c} \dot{X}_1 \\ \dot{X}_2 \end{array}\right] = f(u,w) = \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & 1 \end{array}\right] \sum_{i=1}^{4} u_i u^{(i)} - \sum_{j=1}^{4} w_j w^{(j)}.$$

Note that we can always normalize $f(\cdot)$ so as to have $(\Delta(\mathbb{R}^4), \Delta(\mathbb{R}^4)) \mapsto [-1,1]^n$.

3.3. A solution for the polytopic case. Let Assumption 3.1 hold true. Now, for given x, take for λ the value $\lambda(\partial_x v) = \frac{\partial_x v(x,t)}{\|\partial_x v(x,t)\|}$ which is the gradient direction on x. Then, we can introduce the value of the *projected anti-gradient game*

$$val[\partial_x v(x,t)] := \lambda(\partial_x v)^T f(u^*, w^*). \tag{3.10}$$

Also let us introduce

$$A_{ij} := f(\mathbf{1}_i, \mathbf{1}_j) \in [-1,1]^n, \tag{3.11}$$

where $\mathbf{1}_i \in \mathbb{R}^p$ is a unit vector with all zero components except the ith one which is equal to 1, and similarly for $\mathbf{1}_j \in \mathbb{R}^q$.

We can then establish the following result.

THEOREM 3.3. *The mean-field game for the crowd-seeking opinion propagation with polytopic bounds is*

$$\left\{\begin{array}{l} \partial_t v(x,t) + \|\partial_x v(x,t)\| val[\partial_x v(x,t)] + \frac{1}{2}(\overline{m}(t)-x)^T Q(\overline{m}(t)-x) \\ +\frac{1}{2}\sigma^2 Tr(\partial^2_{xx} v(x,t)) = 0, \quad in\ \mathbb{R}^n \times [0,T[, \\ \\ v(x,T) = \Psi(\overline{m}(T), x),\ \ in\ \mathbb{R}^n, \\ \\ \partial_t m(x,t) + div(m(x,t)\cdot A_{i^*j^*}) - \frac{\sigma^2}{2} Tr(\partial^2_{xx} m) = 0,\ \ in\ \mathbb{R}^n \times [0,T[, \\ \\ m(x,0) = m_0(x)\ \ in\ \mathbb{R}^n, \end{array}\right. \tag{3.12}$$

where $\|\partial_x v(x,t)\|$ is the 2-norm of the vector $\partial_x v(x,t)$ and $val[\partial_x v(x,t)]$ is the value of the projected anti-gradient game as in (3.10). Furthermore, the optimal control and worst-case disturbance are

$$\left\{\begin{array}{l} u^*(x,t) = i^* = \arg\min_{i\in I}\sup_{j\in J} \lambda(\partial_x v)^T A_{ij} \\ w^*(x,t) = j^* = \arg\max_{j\in J} \lambda(\partial_x v)^T A_{ij}. \end{array}\right. \tag{3.13}$$

Proof. Given in the appendix. □

In principle, to find the optimal control input we need to solve the two coupled PDEs in (3.12) in v and m with given boundary conditions (second and last conditions).

Note that since the HJI equation depends explicitly on the mean of the mean field and not on the higher moments, one can reduce the mean field system to a lower dimensional one. The reduced mean-field system of the robust mean-field game can be written as

$$
\begin{cases}
\partial_t v(x,t) + \|\partial_x v\| val[\partial_x v] + \frac{1}{2}(\overline{m}(t)-x)^T Q(\overline{m}(t)-x) \\
+\frac{1}{2}\sigma^2 Tr(\partial^2_{xx} v(x,t)) = 0, \quad \text{in } \mathbb{R}^n \times [0,T[, \\
\\
v(x,T) = \Psi(\overline{m}(T), x), \text{ in } \mathbb{R}^n, \\
\frac{d}{dt}\overline{m}(t) = \int A_{i^* j^*} m(x,t) dx \text{ in } \mathbb{R}^n, \\
\overline{m}(0) = \overline{m}_0 = \int_{\mathbb{R}^n} x m_0(x) dx, \\
m(x,0) = m_0(x) \text{ in } \mathbb{R}^n.
\end{cases} \tag{3.14}
$$

Also note that since the function f is bounded and the cost functional is strictly convex in x and u, the existence and uniqueness of a solution to the above is a direct consequence of the theorem in (Øksendal 2003) chapter 5.2. Indeed, jumps of j are finite when the trajectory is far from the switching surfaces, and once the trajectory hits a switching surface either it leaves it or it evolves along it in which case one needs to refer to a solution at least in the sense of (Filippov 1964). From a numerical perspective a way to regularize the RHS of the dynamics is to introduce a low-pass filter and therefore to turn jumps into smooth transitions from one vertex to another.

3.3.1. Microscopic model. In this subsection we provide a microscopic description of the system based on a linearized approximation of the model and a finite set of players $\{1,\ldots,\nu\}$. Let us denote by $Y_i(t)$ for all $i=1,\ldots,\nu$, the corresponding state of each player.

To linearize the system, suppose that we can replace $f(u^*,w^*) = A_{i^*,j^*}$ by $f(u,w) = -\delta\partial_x v$, for a scalar $\delta > 0$. Also, let us take $v(Y_i,t) = Y_i^T\phi(t)Y_i$ for an opportune positive definite matrix ϕ and for all $Y_i \in \mathbb{R}^n$. Then the resulting linear dynamics are captured by the stochastic differential equation (SDE)

$$
dY_i(t) = \delta\phi(t)(\overline{m} - Y_i(t))dt + \sigma d\mathcal{B}_i(t), \tag{3.15}
$$

where $\mathcal{B}_i(t)$ is a vector Brownian motion affecting the dynamics of player i.

A first observation is that the mean opinion $\overline{m}(t) = \frac{1}{\nu}\sum_{i=1}^{\nu} Y_i(t)$ is a stochastic process which evolves according to

$$
\begin{array}{rcl}
d\overline{m}(t) & = & \frac{1}{\nu}\sum_{i=1}^{\nu} dY_i(t) \\
& = & \frac{1}{\nu}\sum_{i=1}^{\nu}\Big(\delta\phi(t)(\overline{m} - Y_i(t))dt + \sigma d\mathcal{B}_i(t)\Big) \\
& = & \frac{1}{\nu}\sum_{i=1}^{\nu}\Big(\delta\phi(t)(\frac{1}{\nu}\sum_{j=1}^{\nu} Y_j(t) - Y_i(t))dt + \sigma d\mathcal{B}_i(t)\Big) \\
& = & \frac{1}{\nu}\sum_{i=1}^{\nu}\sigma d\mathcal{B}_i(t).
\end{array} \tag{3.16}
$$

and with 1st moment always equal to zero, i.e.,

$$
\mathbb{E} d\overline{m}(t) = 0.
$$

Since we wish to analyze convergence of the players' opinions to their average, let us observe that for the ith error we can write the expression below, which relates $e_i(t)$ to $Y_i(t)$:

$$
e_i(t) = \overline{m}(t) - Y_i(t).
$$

The next result establishes that the errors are stochastically bounded with respect to the mean opinion.

THEOREM 3.4. *For each $\pi > 0$, there exists an $\varepsilon(\pi) > 0$ such that*

$$\mathbb{P}(\|e_i(t)\|_\infty \leq \varepsilon(\pi)) > 1 - \pi. \tag{3.17}$$

Proof. Given in the appendix. ☐

4. $\mathcal{L}_2$-gain bounds. We next study the second model, where i) the control and the disturbances are bounded in $\mathcal{L}_2$, ii) the dynamics are "normalized" and bilinear in control and disturbance, and iii) the running cost penalizes both the control and the disturbance in square norm. More formally,

- the control set $U = \mathbb{R}^p$
- the disturbance set $W = \mathbb{R}^q$
- function $f : U \times W \to \mathbb{R}^n$ is linear and of the form

$$f(u, w) = Fu + Ew, \tag{4.1}$$

  where $F \in \{-1, 0, 1\}^{n\times p}$ and $E \in \{-1, 0, 1\}^{n\times q}$, and represent incidence matrices of hypergraphs
- the running cost $g : \mathbb{R}^n \times \mathbb{R}^n \times U \times W \to [0, +\infty[$, $(x, \overline{m}, u, w) \mapsto g(x, \overline{m}, u, w)$ is of the form:

$$g(x, \overline{m}, u, w) = \frac{1}{2}\left[(\overline{m} - x)^T Q\,(\overline{m} - x) + u^T Cu - \gamma^2 w^T w\right]$$

  where $Q > 0$, $C > 0$ and we take Γ from (2.2) equal to $-\gamma^2 I$, where I is the identity matrix of appropriate dimensions. The above cost describes i) the square deviation of an individual's state from the mean state computed over the whole population, ii) the energy of the control, and iii) the energy of the disturbance.

Each player solves the following special case of Problem 1:

$$\begin{cases} \inf_{u(\cdot)\in\mathcal{U}} \sup_{w(\cdot)\in\mathcal{W}} & \Big\{J(x, u(\cdot), w(\cdot), \overline{m}(\cdot)) = \mathbb{E}\Big[\int_0^T \frac{1}{2}[(\overline{m} - X(t))^T Q\,(\overline{m} - X(t)) \\ & +u(t)^T Cu(t) - \gamma^2 w(t)^T w(t)]dt + \Psi(X(T), \overline{m}(T))\Big]\Big\}, \\ & dX(t) = [Fu(t) + Ew(t)]dt + \sigma dB(t), \quad X(0) = x. \end{cases} \tag{4.2}$$

4.1. The mean field game for the $\mathcal{L}_2$-gain bounds case. Let the Hamiltonian (without disturbance w) be given by

$$H(x, p, \overline{m}) = \inf_u \left\{\tilde{g}(x, \overline{m}, u) + \Pi^T Fu\right\},$$

where Π is the co-state and

$$\tilde{g}(x, \overline{m}, u) = \frac{1}{2}\left[(\overline{m} - x)^T Q\,(\overline{m} - x) + u^T Cu\right].$$

The robust Hamiltonian is then

$$\tilde{H}(x, p, \overline{m}) = H(x, p, \overline{m}) + \sup_w \left\{\Pi^T Ew - \frac{1}{2}\gamma^2 w^T w\right\}.$$

After solving for w we obtain

$$w^* = \frac{1}{\gamma^2} E^T \partial_x v(x,t).$$

Introducing the Hamiltonian and the expression for w^* in the mean-field system (2.5) we obtain

$$\begin{cases} \partial_t v(x,t) + H(x,p,\overline{m}) + \dfrac{1}{2\gamma^2}(\partial_x v(x,t)^T E E^T \partial_x v(x,t)) \\ +\frac{\sigma^2}{2} Tr(\partial^2_{xx} v(x,t)) = 0, \text{ in } \mathbb{R}^n \times [0,T[, \\ v(x,T) = \Psi(\overline{m}(T), x) \text{ in } \mathbb{R}^n, \\ \partial_t m(x,t) + div\Big(m(x,t)\partial_p H(x,p,\overline{m})\Big) + \dfrac{1}{\gamma^2} div\Big(m(x,t) E E^T \partial_x v(x,t)\Big) \\ -\frac{\sigma^2}{2} Tr(\partial^2_{xx} m(x,t)) = 0, \text{ in } \mathbb{R}^n \times [0,T[, \\ m(x,0) = m_0(x) \text{ in } \mathbb{R}^n, \end{cases} \tag{4.3}$$

We are now in a position to specialize the results obtained above to the case of a crowd-seeking opinion propagation.

THEOREM 4.1. *The mean-field system of the robust mean field game for the crowd-seeking $\mathcal{L}_2$ bounded opinion propagation system is*

$$\begin{cases} \partial_t v(x,t) + \frac{1}{2}\partial_x v(x,t)^T \Big(-FC^{-1}F^T + \frac{1}{\gamma^2} E E^T\Big)\partial_x v(x,t) \\ +\frac{1}{2}(\overline{m}(t) - x)^T Q(\overline{m}(t) - x) + \frac{1}{2}\sigma^2 Tr(\partial^2_{xx} v(x,t)) = 0, \ \textit{in } \mathbb{R}^n \times [0,T[, \\ v(x,T) = \Psi(\overline{m}(T), x), \ \textit{in } \mathbb{R}^n, \\ \partial_t m(x,t) + div\Big(m(x,t)(-\frac{1}{2}C^{-1}FF^T + \frac{1}{2\gamma^2} E E^T)\partial_x v(x,t)\Big) \\ -\frac{1}{2}\sigma^2 Tr(\partial^2_{xx} m(x,t)) = 0, \ \textit{in } \mathbb{R}^n \times [0,T[, \\ m(x,0) = m_0(x) \ \textit{in } \mathbb{R}^n. \end{cases} \tag{4.4}$$

Furthermore, the optimal control and worst-case disturbance are

$$\begin{cases} u^*(x,t) = -C^{-1}F^T \partial_x v(x,t) \\ w^*(x,t) = \frac{1}{\gamma^2} E^T \partial_x v(x,t). \end{cases} \tag{4.5}$$

Proof. Given in the appendix. □

The significance of the above result is that to find the optimal control input we need to solve the two coupled PDEs in (4.4) in v and m with given boundary conditions (second and last conditions). This is usually done by iteratively solving the HJI equation for fixed m and by entering the optimal u obtained from (4.5) in the FPK equation in (4.4) until a fixed point in v and m is reached.

Note that since the HJI equation depends explicitly on the mean of the mean field and not on the other moments, one can reduce the mean field system to a lower dimensional system. The reduced mean-field system of the robust mean-field game is

$$\left\{\begin{array}{l}\partial_t v(x,t)+\frac{1}{2}\partial_x v(x,t)^T\Big(-FC^{-1}F^T+\frac{1}{\gamma^2}EE^T\Big)\partial_x v(x,t)\\ +\frac{1}{2}(\overline{m}(t)-x)^T Q(\overline{m}(t)-x)+\frac{1}{2}\sigma^2 Tr(\partial^2_{xx}v(x,t))=0,\ \text{in } \mathbb{R}^n\times[0,T[,\\ \\ v(x,T)=\Psi(\overline{m}(T),x),\ \text{in } \mathbb{R}^n,\\ \\ \frac{d}{dt}\overline{m}(t)=F\overline{u}(t)^*+E\overline{w}(t)^* \text{ in } \mathbb{R}^n,\\ \\ \overline{m}(0)=\overline{m}_0=\int_{\mathbb{R}^n}xm_0(x)dx,\\ \\ m(x,0)=m_0(x) \text{ in } \mathbb{R}^n,\end{array}\right. \tag{4.6}$$

where $\overline{u}(t)^*$ and $\overline{w}(t)^*$ are the mean of the optimal individual state feedback control and disturbance, respectively.

4.2. Approximate computation of the mean-field equilibrium. We borrow from Bauso et al. (2014) the idea of studying the problem in the extended state space involving both the vector state of the group and the average state distribution. In the mean-field system (4.6) the gradient $\partial_x v(x,t)$ depends implicitly on the average distribution $\overline{m}(t)$, which we can see as a parameter, and which evolves according to a nonlinear differential equation. Then, we consider a new variable $\tilde{m}_t$ whose dynamics approximates the nonlinear dynamics of $\overline{m}(t)$. In the extended state space, the state variable evolves according to the equations

$$\left\{\begin{array}{l} dX(t)=Fu(t)^*+Ew(t)^*dt+\sigma dB(t), \quad X(0)=x,\\ \\ \frac{d}{dt}\overline{m}(t)=F\overline{u}(t)^*+E\overline{w}(t)^*, \quad \overline{m}(0)=m_0,\end{array}\right. \tag{4.7}$$

which can be rewritten in matrix form as

$$\begin{array}{r}\left[\begin{array}{c} dX(t)\\ d\overline{m}(t)\end{array}\right]=\Big(\left[\begin{array}{cc} F & 0\\ 0 & F\end{array}\right]\left[\begin{array}{c} u(t)^*\\ \overline{u}(t)^*\end{array}\right]\\ +\left[\begin{array}{cc} E & 0\\ 0 & E\end{array}\right]\left[\begin{array}{c} w(t)^*\\ \overline{w}(t)^*\end{array}\right]\Big)dt+\left[\begin{array}{c}\sigma d\mathcal{B}(t)\\ 0\end{array}\right].\end{array} \tag{4.8}$$

For this system we introduce an assumption on the rate of convergence of the state $\overline{m}(t)$.

Assumption 4.1. *There exists θ such that*

$$\frac{d}{dt}\overline{m}(t)=F\overline{u}(t)^*+E\overline{w}(t)^*\geq-\theta\overline{m}(t),\ \textit{for all}\ t\in[0,T].$$

The above assumption implies that there exists a variable $\tilde{m}(t)$ which approximates the average distribution from below, that evolves according to

$$\left\{\begin{array}{l}\frac{d}{dt}\tilde{m}(t)=-\theta\tilde{m}(t), \quad \text{for all } t\in[0,T],\\ \tilde{m}_0=\overline{m}_0.\end{array}\right. \tag{4.9}$$

By substituting the current average distribution $\bar{m}_t$ by its estimate $\tilde{m}_t$ the extended state dynamics takes the form

$$\begin{array}{l}\left[\begin{array}{c} dX(t)\\ d\tilde{m}(t)\end{array}\right]=\Big(\left[\begin{array}{cc} 0 & 0\\ 0 & -\theta\end{array}\right]\left[\begin{array}{c} X(t)\\ \tilde{m}(t)\end{array}\right]\\ +\left[\begin{array}{c} F\\ 0\end{array}\right]u(t)^*+\left[\begin{array}{c} E\\ 0\end{array}\right]w(t)^*\Big)dt+\left[\begin{array}{c}\sigma d\mathcal{B}(t)\\ 0\end{array}\right].\end{array} \tag{4.10}$$

Given the above dynamics, we summarize the problem at hand as

$$\begin{cases} \inf_{u(\cdot)\in\mathcal{U}} \sup_{w(\cdot)\in\mathcal{W}} \Big\{ J(x,u(\cdot),w(\cdot),\overline{m}(\cdot)) = \mathbb{E}\Big[\int_0^T \frac{1}{2}[(\overline{m}-X(t))^T Q\,(\overline{m}-X(t)) \\ \qquad +u(t)^T Cu(t) - \gamma^2 w(t)^T w(t)]dt + \Psi(X(T),\overline{m}(T))\Big]\Big\}, \\ \\ \begin{bmatrix} dX(t) \\ d\tilde{m}(t) \end{bmatrix} = \Big(\begin{bmatrix} 0 & 0 \\ 0 & -\theta \end{bmatrix} \begin{bmatrix} X(t) \\ \tilde{m}(t) \end{bmatrix} \\ + \begin{bmatrix} F \\ 0 \end{bmatrix} u(t)^* + \begin{bmatrix} E \\ 0 \end{bmatrix} w(t)^* \Big) dt + \begin{bmatrix} \sigma d\mathcal{B}(t) \\ 0 \end{bmatrix}. \end{cases}$$

Reformulating the problem in terms of the extended state

$$\mathcal{X}(t) = \begin{bmatrix} X(t) \\ \tilde{m}(t) \end{bmatrix},$$

yields the linear quadratic problem:

$$\begin{cases} \inf_{u(\cdot)\in\mathcal{U}} \sup_{w(\cdot)\in\mathcal{W}} \int_0^T \Big[\frac{1}{2}\Big(\mathcal{X}(t)^T\tilde{Q}\mathcal{X}(t) + u(t)^T Cu(t) - w(t)^T \Gamma w(t)\Big)\Big]dt + \Psi(\mathcal{X}(T)) \\ \qquad d\mathcal{X}(t) = \Big(\tilde{A}\mathcal{X}(t) + \tilde{B}\tilde{u}(t) + \tilde{C}w(t)\Big)dt + \tilde{E}d\mathcal{B}(t), \end{cases}$$

where

$$\tilde{Q} = \begin{bmatrix} -I \\ I \end{bmatrix} Q \begin{bmatrix} -I & I \end{bmatrix}, \quad \Gamma = 2\gamma^2 I, \quad \tilde{A} = \begin{bmatrix} 0 & 0 \\ 0 & -\theta \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} F \\ 0 \end{bmatrix}, \qquad \tilde{C} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \qquad \tilde{E} = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}.$$

The idea is therefore to consider a new value function $\mathcal{V}(X,\tilde{m},t)$ (in compact form $\mathcal{V}(\mathcal{X},t)$) in the extended state space, which satisfies

$$\begin{cases} \partial_t \mathcal{V}(\mathcal{X},t) + H(\mathcal{X}, \partial_{\mathcal{X}}\mathcal{V}(\mathcal{X},t)) + \left(\frac{\sigma}{2\gamma}\right)^2 |\partial_x \mathcal{V}_t(X)|^2 \\ +\frac{1}{2}\sigma^2\partial^2_{xx}\mathcal{V}(\mathcal{X},t) = 0, \text{ in } \mathbb{R}^{2n}\times[0,T[, \\ \mathcal{V}(\mathcal{X},T) = \Psi(\mathcal{X}) \text{ in } \mathbb{R}^{2n}. \end{cases}$$

Given that $\mathcal{V}(\mathcal{X},t)$ can be expressed in the quadratic form

$$\mathcal{V}(\mathcal{X},t) = [X^T(t) \quad \overline{m}^T(t)] \underbrace{\begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix}}_{P(t)} \begin{bmatrix} X(t) \\ \overline{m}(t) \end{bmatrix},$$

where the matrix $P(t)$ is the solution of the differential Riccati equation

$$\begin{aligned} &\dot{P}(t) + P(t)\tilde{A} + \tilde{A}^T P(t) + \tilde{Q}/2 + W \\ &-2P(t)(\tilde{B}C^{-1}\tilde{B}^T - \tilde{C}\Gamma^{-1}\tilde{C}^T)P(t) = 0, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} &\tilde{B}C^{-1}\tilde{B}^T - \tilde{C}\Gamma^{-1}\tilde{C}^T = \begin{bmatrix} FC^{-1}F^T - \frac{1}{\gamma^2}EE^T & 0 \\ 0 & 0 \end{bmatrix}, \\ &W = \begin{bmatrix} \sigma^2 P_{11}(t) & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Note that in the stationary case the above differential equation simplifies to

$$P\tilde{A} + \tilde{A}^T P - 2P(\tilde{B}C^{-1}\tilde{B}^T - \tilde{C}\Gamma^{-1}\tilde{C}^T)P + \tilde{Q}/2 + W = 0. \tag{4.12}$$

Let $P(t)$ be the solution of the differential Riccati equation (4.11). Then, the optimal control is given by

$$\begin{aligned}
&\tilde{u}(t) = -2C^{-1}\tilde{B}^T P(t)\mathcal{X}(t) \\
&= -2C^{-1}[F\ 0]\begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix}\begin{bmatrix} X(t) \\ \overline{m}(t) \end{bmatrix} \\
&= -2C^{-1}\beta(P_{11}(t)X(t) + P_{12}(t)\overline{m}(t)),
\end{aligned} \tag{4.13}$$

and the worst disturbance is

$$\begin{aligned}
&\tilde{w}(t) = 2\Gamma^{-1}\tilde{C}^T P(t)X_t \\
&= \frac{1}{\gamma^2} I[E\ 0]\begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix}\begin{bmatrix} X(t) \\ \overline{m}(t) \end{bmatrix} \\
&= \frac{1}{\gamma^2} IE(P_{11}(t)X(t) + P_{12}(t)\overline{m}(t)).
\end{aligned} \tag{4.14}$$

The next subsection analyzes conditions for the existence of a mean field equilibrium for problem (4.6) in the cases of stationarity and zero-mean symmetric distribution.

4.3. Considerations on stationarity. A first step is the analysis of the HJI equation in the stationary case. In this context, we look for a function Ψ that satisfies the stationary HJI equation. Note that stationarity is obtained from (4.6) by dropping the term $\partial_t v(x,t)$ and introducing the new variable Ψ_x:

$$\frac{1}{2}\Psi_x^T\Big(-FC^{-1}F^T + \frac{1}{\gamma^2}EE^T\Big)\Psi_x + \frac{1}{2}(\overline{m}(t) - x)^T Q(\overline{m}(t) - x) + \frac{1}{2}\sigma^2 Tr(\Psi_{xx}^2) = 0, \text{ in } \mathbb{R}^n. \tag{4.15}$$

We want to analyze the influence of the disturbance on the value function Ψ. Toward this end, for sake of simplicity, we take $\overline{m} = 0$, and $\sigma = 0$. Then the HJI equation in (4.15) turns out to be

$$\frac{1}{2}\Psi_x^T\Big(-FC^{-1}F^T + \frac{1}{\gamma^2}EE^T\Big)\Psi_x + \frac{1}{2}x^T Q x = 0, \text{ in } \mathbb{R}^n, \tag{4.16}$$

which is valid as long as $-FC^{-1}F^T + \frac{1}{\gamma^2}EE^T < 0$ (see e.g. Appendix of (Başar and Bernhard 1995) on the theory of conjugate points). Taking $C = I$ for sake of simplicity, the above condition becomes

$$-FF^T + \frac{1}{\gamma^2}EE^T < 0, \tag{4.17}$$

which establishes a relation between the *Fiedler eigenvalue* or *algebraic connectivity* of the Laplacian matrix FF^T and the maximal eigenvalue of the Laplacian matrix EE^T.

4.4. Symmetric zero-mean distribution. Let us start by noting that a stationary mean distribution $\overline{m}_0(x)$ together with a value function $v(x,t)$ can be regarded as a mean-field game equilibrium if they constitute a fixed point for (4.6). This requires that, given the corresponding mean-field optimal control $\overline{u}^*(t)$ and worst disturbance $\overline{w}^*(t)$, we have

$$F\overline{u}^*(t) = -E\overline{w}^*(t). \tag{4.18}$$

Let us suppose that $\overline{m}_0 = 0$ is a mean-field game equilibrium and that m_0 is symmetric with respect to zero (for instance is a Gaussian distribution or uniform distribution with mean zero). Then, the HJI part of (4.6) becomes

$$\left\{\begin{array}{l} \partial_t v(x,t) + \frac{1}{2}\Psi_x^T\Big(- FC^{-1}F^T + \frac{1}{\gamma^2}EE^T\Big)\Psi_x + \frac{1}{2}x^TQx \\ +\frac{1}{2}\sigma^2 Tr(\partial^2_{xx}v_t(x)) = 0, \text{ in } \mathbb{R}^n \times [0,T[, \\ \\ v(x,T) = \Psi(0,x), \text{ in } \mathbb{R}^n. \end{array}\right. \tag{4.19}$$

Now, our goal is to obtain a suitable terminal cost Ψ such that there exists a solution with $\bar{m} \equiv 0$.

Note that (4.19) has the form leading to a differential Riccati equation. To see this, let us take $v(x,t) = \frac{1}{2}x^T\phi(t)x + \chi(t)$, and note that (4.19) can be rewritten as

$$\begin{array}{r} \dot{\chi}(t) + \frac{1}{2}\dot{\phi}(t)x^2 + \frac{1}{2}x^T\phi(t)^T\Big(- FC^{-1}F^T + \frac{1}{\gamma^2}EE^T\Big)\phi(t)x + \frac{1}{2}x^TQx + \frac{1}{2}\sigma^2\phi(t) = 0 \\ \text{in } \mathbb{R}^n \times [0,T[,\ \phi(T) = S. \end{array} \tag{4.20}$$

Since this is an identity in x, it reduces to two equations:

$$\left\{\begin{array}{l} \dot{\phi}(t) + \phi(t)^T\Big(- FC^{-1}F^T + \frac{1}{\gamma^2}EE^T\Big)\phi(t) + Q = 0 \text{ in } \mathbb{R}^n \times [0,T[, \\ \quad \phi(T) = S, \\ \\ \dot{\chi}(t) + \frac{1}{2}\sigma^2\phi(t) = 0, \text{ in } [0,T[, \\ \quad \chi(T) = 0. \end{array}\right. \tag{4.21}$$

The corresponding optimal control and worst-case disturbance are

$$\left\{\begin{array}{l} u^*(x,t) = -C^{-1}F^T\phi(t)x \\ w^*(x,t) = \frac{1}{\gamma^2}E^T\phi(t)x. \end{array}\right. \tag{4.22}$$

Given a symmetric distribution centered at zero, we have $F\overline{u}(t) = E\overline{w}(t) = 0$, and therefore (4.18) holds true. We can then conclude that the value function obtained from a primitive of ϕ in (4.21), together with optimal control and worst-case disturbance in (4.22) and a zero-mean symmetric distribution m, constitute a fixed point for the reduced mean-field game (4.6) and as such they provide a mean-field game equilibrium. A main question we will address in the following sub-sections is whether this equilibrium is reachable dynamically, at least in the scalar case.

4.5. The scalar case. In this sub-section we provide a detailed analysis of the scalar version of the earlier problem. Consider a population of homogeneous agents (players), each one characterized by an opinion $X(t) \in \mathbb{R}$ at time $t \in [0,T]$, where $[0,T]$ is the time horizon window. Let U be the control set. The control variable be a measurable function of time, $u(\cdot) \in U$, defined as $t \mapsto \mathbb{R}$ and establishing the rate of variation of an agent's opinion. A persuader tries to perturb the opinions of the agents in a way that is proportional to his advertisement efforts $w(\cdot) \in W$, where W is the control set of the persuader.

For this scalar case, the opinion dynamics can be written in the form (2.1) with $f : \mathbb{R} \times U \times W \to \mathbb{R}$ affine:

$$\left\{\begin{array}{l} dX(t) = (u(t) + w(t))dt + \sigma d\mathcal{B}(t),\ t > 0 \\ X(0) = x, \end{array}\right. \tag{4.23}$$

where $\sigma > 0$ is a weighting coefficient and $\mathcal{B}(t)$ is a scalar Brownian motion process. For the agents, consider a running cost $g : \mathbb{R}\times\mathbb{R}\times U \to [0, +\infty[$, $(x, \overline{m}, u) \mapsto g(x, \overline{m}, u)$ of the form:

$$g(x, \overline{m}, u) = \frac{1}{2}\left[a(\overline{m} - x)^2 + cw^2\right]. \tag{4.24}$$

Also consider a terminal cost $\Psi : \mathbb{R} \times \mathbb{R} \to [0, +\infty[$, $(\overline{m}, x) \mapsto \Psi(\overline{m}, x)$ of the form

$$\Psi(\overline{m}, x) = \frac{1}{2}S(\overline{m} - x)^2.$$

The problem is then the following:

$$\min_{u(\cdot)} \max_{w(\cdot)} \mathbb{E} \int_0^T \left[g(X(t), \overline{m}(t), u_1(t)) - \frac{\gamma^2}{2}u_2(t)^2\right]dt + \Psi(\overline{m}(T), X(T)). \tag{4.25}$$

The following result then follows from Theorem 4.1 for this special scalar case.

COROLLARY 4.2. *The mean-field system of the robust mean field game for the crowd-seeking opinion propagation system is*

$$\left\{\begin{array}{l} \partial_t v(x,t) + \left(-\frac{1}{2c_1} + \frac{1}{2\gamma^2}\right)|\partial_x v(x,t)|^2 + \frac{1}{2}a(\overline{m}(t) - x)^2 + \frac{1}{2}\sigma^2\partial^2_{xx}v(x,t) = 0, \\ \qquad\qquad\qquad\qquad \textit{in } \mathbb{R}\times[0,T[, \\ \\ v(x,T) = \Psi(\overline{m}(T), x), \;\; \textit{in } \mathbb{R}, \\ \\ \partial_t m(x,t) + \left(-\frac{1}{2c_1} + \frac{1}{2\gamma^2}\right)\partial_x\Big(m(x,t)\partial_x v(x,t)\Big) - \frac{1}{2}\sigma^2\partial^2_{xx}m(x,t) = 0, \\ \\ m(x,0) = m_0(x) \;\; \textit{in } \mathbb{R}. \end{array}\right. \tag{4.26}$$

Furthermore, the optimal control and worst-case disturbance are

$$\left\{\begin{array}{l} u^*(x,t) = -\frac{1}{c_1}\partial_x v(x,t), \\ w^*(x,t) = \frac{1}{\gamma^2}\partial_x v(x,t). \end{array}\right. \tag{4.27}$$

The significance of the above result is that to find the optimal control input we need to solve the two coupled PDEs in (4.26) in v and m with given boundary conditions (second and last conditions). This is usually done by iteratively solving the HJI equation for fixed m and by entering the optimal u obtained from (4.27) in the FPK equation in (4.26) until a fixed point in v and m is reached.

Note that the first part, namely, the HJI equation for fixed m, can be solved explicitly by assuming the value function being quadratic plus affine in x. Computation of $v(x,t)$ involves an offline computation as detailed next.

Isolating the HJI part of (4.26) for fixed m, we have

$$\left\{\begin{array}{l} \partial_t v(x,t) + \left(-\frac{1}{2c_1} + \frac{1}{2\gamma^2}\right)|\partial_x v(x,t)|^2 + \frac{1}{2}a(\overline{m}(t) - x)^2 + \frac{1}{2}\sigma^2\partial^2_{xx}v(x,t) = 0, \\ \qquad \text{in } \mathbb{R}\times[0,T[, \\ \\ v(x,T) = \Psi(\overline{m}(T), x), \;\; \text{in } \mathbb{R}. \end{array}\right. \tag{4.28}$$

Let us consider the following value function

$$v(x,t) = \frac{1}{2}\phi(t)x^2 + h(t)x + \chi(t),$$

so that (4.28) can be rewritten as

$$\begin{cases} \frac{1}{2}\dot{\phi}(t)x^2 + \dot{h}(t)x + \dot{\chi}(t) + \left(-\frac{1}{2c_1} + \frac{1}{2\gamma^2}\right)[\phi(t)^2x^2 + h(t)^2 + 2\phi(t)h(t)x] \\ \qquad + \frac{1}{2}a(\overline{m}(t)^2 + x^2 - 2\overline{m}(t)x) + \frac{1}{2}\sigma^2\phi(t) = 0 \text{ in } \mathbb{R}\times[0,T[, \\ \phi(T) = S, \quad h(T) = -S\overline{m}(T), \quad \chi(T) = \frac{1}{2}S\overline{m}(T). \end{cases} \tag{4.29}$$

Again, since this is an identity in x, it reduces to three equations:

$$\begin{cases} \dot{\phi}(t) + \left(-\frac{1}{c_1} + \frac{1}{\gamma^2}\right)\phi(t)^2 + a = 0 \text{ in } [0,T[,\ \phi(T) = S, \\ \dot{h}(t) + \left(-\frac{1}{2c_1} + \frac{1}{2\gamma^2}\right)2\phi(t)h(t) - a\overline{m}(t) = 0 \text{ in } [0,T[,\ h(T) = -S\overline{m}(T), \\ \dot{\chi}(t) + \left(-\frac{1}{2c_1} + \frac{1}{2\gamma^2}\right)h(t)^2 + \frac{1}{2}a\overline{m}(t)^2 + \frac{1}{2}\sigma^2\phi(t) = 0 \text{ in } [0,T[,\ \chi(T) = \frac{1}{2}S\overline{m}(T). \end{cases} \tag{4.30}$$

For the optimal control and worst-case disturbance we have

$$\begin{cases} u^*(x,t) = -\frac{1}{c_1}(\phi(t)x + h(t)) \\ w^*(x,t) = \frac{1}{\gamma^2}(\phi(t)x + h(t)). \end{cases} \tag{4.31}$$

4.5.1. Microscopic model. Consider a finite set of players $\{1,\dots,\nu\}$ and let $Y_i(t)$ for all $i = 1,\dots,\nu$, be the corresponding states. In order to provide a microscopic description of the system evolution in vector form let us collect all states into a state vector $Y(t) = [Y_1(t),\dots,Y_n(t)]^T$. Given the optimal controls $u^*(x,t)$ and $w^*(x,t)$ as computed above, the evolution of the state vector is captured by the SDE

$$dY(t) = \left(\frac{1}{c_1} - \frac{1}{\gamma^2}\right)\phi(\overline{m} - Y(t))dt + \sigma d\mathcal{B}(t). \tag{4.32}$$

For future use, it is convenient to rewrite (4.32) making use of a stochastic matrix. To do this, let us introduce

$$W = -\left(\frac{1}{c_1} - \frac{1}{\gamma^2}\right)\phi L + I,$$

where L is the Laplacian of a fully connected network, i.e., its ith row appears as

$$L_{i\bullet} = \begin{bmatrix} -\frac{1}{\nu} & -\frac{1}{\nu} & \dots & \frac{\nu-1}{\nu} & \dots & -\frac{1}{\nu} \end{bmatrix}.$$

Note that

$$W = W^T \quad W\mathbf{1} = \mathbf{1}. \tag{4.33}$$

Then, we can rewrite (4.32) as

$$dY(t) = [(W - I)\,Y(t)]\,dt + \sigma d\mathcal{B}(t). \tag{4.34}$$

The above equation is useful as it allows us to analyze the evolution of the stochastic properties of the mean opinion $\overline{m}(t)$. Indeed, observe that $\overline{m}(t)$ is a stochastic process with first-order moment satisfying

$$\mathbb{E}d\overline{m}(t) = 0.$$

However, the realization follows the law

$$\begin{array}{rcl} d\overline{m}(t) & = & \frac{1}{\nu}\mathbf{1}^T dY(t) \\ & = & \frac{1}{\nu}\mathbf{1}^T (W-I)Y(t)dt + \frac{1}{\nu}\mathbf{1}^T \sigma d\mathcal{B}(t) \\ & = & \frac{1}{\nu}\mathbf{1}^T \sigma d\mathcal{B}(t). \end{array} \tag{4.35}$$

Now, our aim is to analyze convergence of the agents to their average. Toward this goal, define the averaging matrix $\mathcal{M} = \frac{1}{n}\mathbf{1} \otimes \mathbf{1}$. Then for a given vector $Y(t)$ we have $\mathcal{M}Y(t) = (\frac{1}{n}\mathbf{1} \otimes \mathbf{1})Y(t) = \frac{1}{n}\mathbf{1}\mathbf{1}^T Y(t) = \overline{m}(t)\mathbf{1}$. In other words, $\mathcal{M}Y(t)$ is the vector all of whose components are the average of the entries of $Y(t)$. The averaging matrix is useful to introduce the error vector $e(t)$ describing the deviations of the components of $Y(t)$ from their average values. For the error vector we can write the expression below, which relates $e(t)$ to $Y(t)$:

$$\begin{aligned} e(t) &= Y(t) - \frac{1}{n}\mathbf{1} \otimes \mathbf{1}^T Y(t) \\ &= Y(t) - \overline{m}(t)\mathbf{1} \\ &= (I - \mathcal{M})Y(t). \end{aligned}$$

The next result establishes that the error vector converges to zero, which implies that all opinions converge to the mean opinion.

THEOREM 4.3. *For each $\pi > 0$, there exists an $\varepsilon(\pi) > 0$ such that*

$$\mathbb{P}(\|e(t)\|_\infty \leq \varepsilon(\pi)) > 1 - \pi. \tag{4.36}$$

Proof. Given in the appendix. □

5. Local interactions. The analysis conducted so far can be extended to the case where p populations of players interact according to a predefined topology. The probability density function is now indexed by $k \in \{1, \ldots, p\}$. In other words we consider a probability density function $m_k : \mathbb{R}^n \times [0, +\infty[\to \mathbb{R}$, $(x,t) \mapsto m_k(x,t)$, which satisfies $\int_{\mathbb{R}^n} m_k(x,t)dx = 1$ for every t. Similarly, we define the local *mean state* of population k at time t as $\overline{m}_k(t) := \int_{\mathbb{R}} x m_k(x,t)dx$.

Let a graph $G = (V, E)$ be given where $V = \{1, \ldots, p\}$ is the set of vertices, one per each population, and $E = V \times V$ the set of edges. Let us denote the set of neighbors of k by $N(k) = \{j \in V | \, (k,j) \in E\}$. Mean states of *neighbor* populations are related by the following *local interaction* rule

$$\frac{d}{dt}\rho_k(t) = \frac{d}{dt}\overline{m}_k(t) + \sum_{j \in N(k)} l_{jk}(\rho_j(t) - \rho_k(t)) \tag{5.1}$$

By introducing L as the corresponding Laplacian matrix, and letting $\rho = (\rho_1, \rho_2, \ldots, \rho_p)^T$, $\mu = \frac{d}{dt}(\overline{m}_1, \overline{m}_2, \ldots, \overline{m}_p)^T$, (5.1) can be rewritten in a compact form as

$$\dot{\rho}(t) = -L\rho(t) + \mu(t). \tag{5.2}$$

In other words, local interactions involve a local averaging (the term including the Laplacian) and a local adjustment. The local adjustment involving μ describes the evolution of the average opinion as a result of the interactions among players in the same population. We henceforth refer to ρ as the vector of *aggregate states*.

Equation (5.2) is general enough to include second-order consensus, in which case the vector ρ is extended to include the derivatives $\dot{\rho}_k(t)$, i.e.,

$$\rho = (\rho_1, \rho_2, \ldots, \rho_p, \dot{\rho}_1, \dot{\rho}_2, \ldots, \dot{\rho}_p)^T$$

ν	x_{min}	x_{max}	dt	$std(m_0)$	T	$\overline{m}_0$
10^3	-50	50	0.01	{8,10,15}	40	0

TABLE 6.1
Simulation parameters.

and similarly for matrix L. Second-order consensus will be considered in the simulation analysis to highlight potential inter-cluster oscillations.

The objective of an agent is to adjust his state based on the aggregate kth state. This reflects a typical crowd-seeking behavior based on local interaction.

Now, for the agents, consider a running cost $g : \mathbb{R} \times \mathbb{R} \times U \to [0, +\infty[$, $(x, \rho_k, u) \mapsto g(x, \rho_k, u)$ of the form (cf. (2.2)):

$$g(x, \rho_k, u, w) = \frac{1}{2}\left[(\rho_k - x)^T Q\,(\rho_k - x) + u^T C u + w^T \Gamma w\right], \tag{5.3}$$

$Q > 0$, $C > 0$, $\Gamma < 0$, and analogously for the terminal penalty that now takes the form

$$\Psi(\rho_k, x) = \frac{1}{2}(\rho_k - x)^T S(\rho_k - x), \quad S > 0. \tag{5.4}$$

The analysis carried out in the preceding sections still holds the only difference being that every population is now tracking a distinct signal ρ_k and signals between neighbor populations follow a consensus dynamics. As a result we may observe multi-scale phenomena. On the one hand we have the synchronization of the agents belonging to different populations to the local tracking signals ρ_k, which justifies the formation of clusters. On the other hand, we may also observe, on a different time-scale, the synchronization of the tracking signals ρ_k, $k \in \{1, \ldots, p\}$ which may lead to consensus or polarization of the opinions.

6. Simulation examples. Numerical studies show a typical evolution leading to an ε-consensus on opinions. This means that the agents' opinions deviate from one another by no more than ε. They have been conducted considering a number of players $\nu = 10^3$ and a discretized set of states $\mathcal{X} = \{x_{min}, x_{min} + 1, \ldots, x_{max}\}$ where $x_{min} = 50$ (minimum temperature) and $x_{max} = 50$ (maximum temperature). The simulation parameters are listed in Table 6.1. We assume that the step size for the simulation is $dt = 0.01$. The horizon length (number of iterations) is $T = 40$, large enough to show convergence of the population regimes.

As regards the initial distribution, we assume m_0 to be Gaussian with mean $\overline{m}_0$ equal to 0. For the examples, the standard deviation $std(m_0)$ is taken to be equal to $8, 10$, and 15.

Example 1. The first simulation example has been carried out using the algorithm displayed in Table 6.2. Figure 6.1, left, from top to bottom, shows the time plot of the microscopic evolution of each agent's opinion. The initial distribution m_0 has mean zero, $\overline{m}_0 = 0$, and standard deviation $std(m_0) = 8$ (top), $std(m_0) = 10$ (middle), $std(m_0) = 15$ (bottom). The graphs on the right column display the time plot $\overline{m}(t)$ (solid line and y-axis labeling on the left) and the evolution of the standard deviation $std(m(t))$ (dashed line and y-axis labeling on the right). Note that, the mean distribution $\overline{m}(t)$ is fixed to zero, and at approximately $t = 20$ the standard deviation $std(m_t)$ drastically approaches a neighborhood of zero, which means that all the agents' opinions have reached ε-consensus around the persuader opinion.

Algorithm

Input: Set of parameters as in Table 6.1
Output: Distribution function m_t, mean $\overline{m}_t$ and standard deviation $std(m_t)$.

```
Initialize. Generate x_0 from Gaussian distribution
    with mean m̄_0 and standard deviation std(m_0),
for time t = 0, 1, ..., T − 1 do
    if t > 0, then compute m_t, m̄_t, and std(m_t),
    end if
    for player i = 1, 2, ..., n do
        compute X(t + 1) by executing (4.23),
    end for
end for
STOP
```

TABLE 6.2
Algorithm used for the simulations

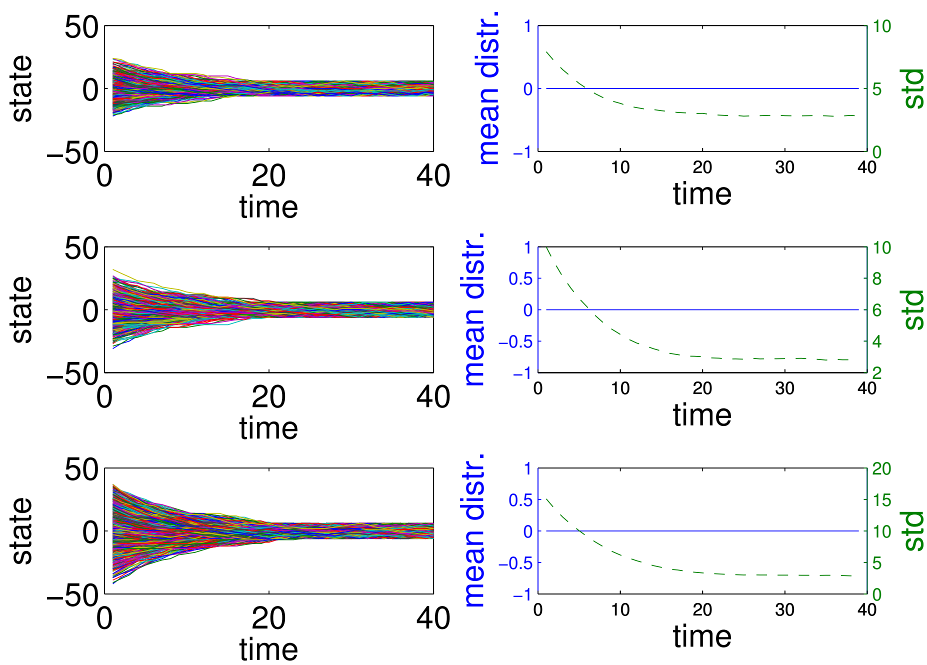

FIG. 6.1. *Microscopic time plot (left) and time plot of mean distribution and standard deviation (right).*

Example 2. In a second example, we consider $p = 5$ distinct populations and analyze the influence on the time plot of the opinions, of the local interaction topology. In particular, we consider a chain directed topology which has one single connected component and a second-order consensus dynamics for ρ as in (5.1). Figure 6.2 shows the microscopic time plot (left) and time plot of the standard deviation (right) for increasing damping constant (from top to bottom). As a result all tracking signals ρ_k, $k = 1, \ldots, 5$ reach consensus on zero in case 2 (middle) and 3 (bottom), but not in case 1 due to a too small damping constant. Inter-cluster oscillations are clearly visible. It can also be observed that the agents within each population synchronize at a faster rate than the one characterizing the consensus dynamics for ρ. Figure 6.3

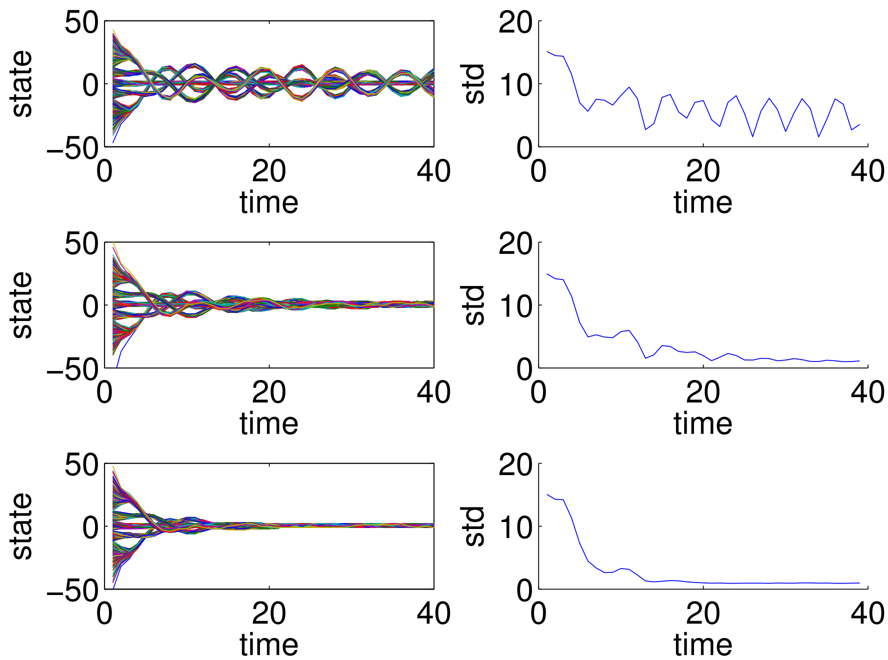

FIG. 6.2. *Microscopic time plot (left) and time plot of standard deviation (right) with local interaction.*

shows the inter-cluster oscillations under a small damping constant. The damping constant depends on the capability on the part of the players to predict the future evolution of others' opinions. Each individual uses not only the current values of others' opinions but also their derivatives. Farsighted players have a higher damping constant. As in the mass-spring-damper dynamics, a small damping constant makes convergence slow. If we take it asymptotically to zero, oscillations become persistent and no convergence occurs. The plot highlights the fast synchronization of the trajectories within each cluster.

Example 3. A third example is carried out in order to investigate the influence of the topology on the consensus values for ρ. The example involves three different topologies as illustrated in Fig. 7.1: directed chain with the top cluster as leader (left), directed chain where the bottom cluster is leader (middle), topology with multiple connected components (right). The time plot associated with the three cases is in Fig. 7.2 from

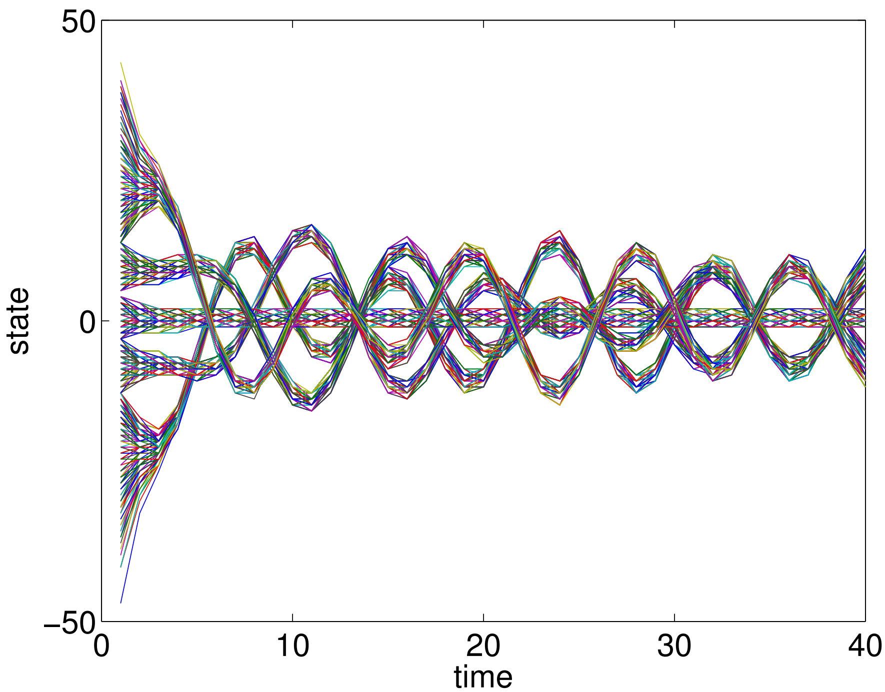

FIG. 6.3. *Microscopic time plot of the opinion for a small damping constant.*

top to bottom. Note that in cases 1 and 2 we have consensus on opinions as the clusters converge to a common value; the transient is characterized by inter-cluster oscillations. Case 3 shows polarization of opinions due to the presence of multiple connected components in the topology. Polarization implies that groups of clusters converge to different values.

7. Conclusions. This paper has shown how repeated games, differential games and mean field games can be intertwined to capture interactions among homogeneous social groups when herding behavior is rewarding for the groups. For the selected games, we have established a mean field equilibrium and studied state feedback best-response strategies as well as worst-case adversarial disturbances. Future directions of research involve the extension of the framework to other social behaviors (other types of cost functions), as well as social dynamics. The impact that existing results and techniques from repeated and population game literatures can have on social networks is still a broad and open field of enquiry.

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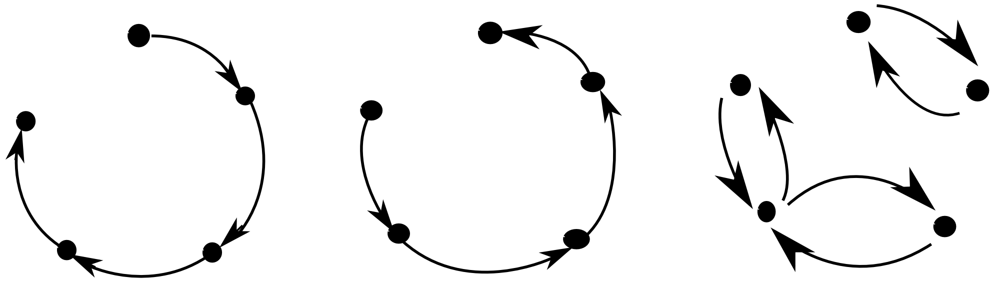

Fig. 7.1. *Three different topologies for the multi-population example: directed chain with the top cluster as leader (left), directed chain where the bottom cluster is leader (middle), topology with multiple connected components (right).*

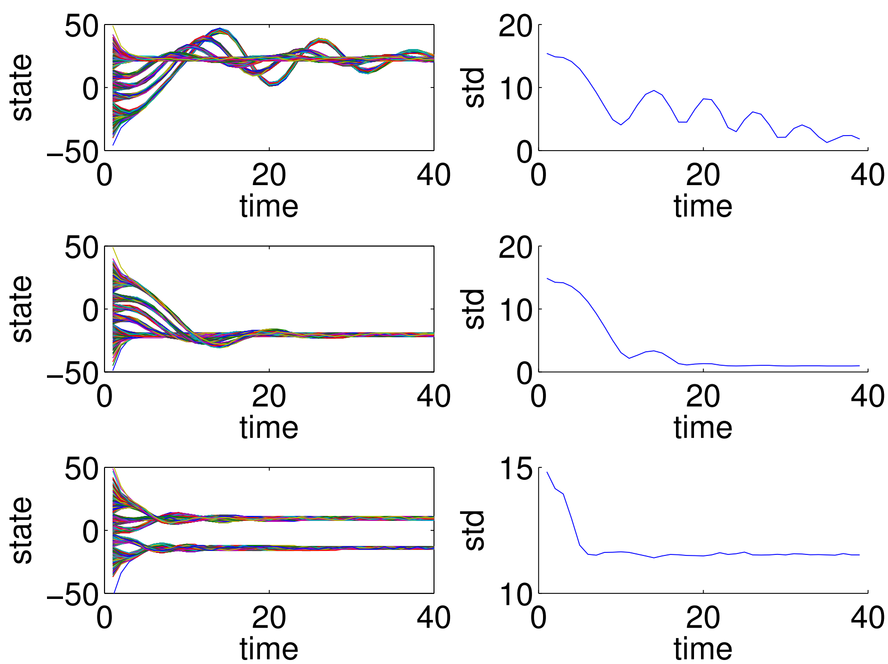

Fig. 7.2. *Microscopic time plot (left) and time plot of mean distribution and standard deviation (right).*

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APPENDIX. In this appendix, we provide proofs for Theorems 3.2-3.4 and 4.1-4.3, which constitute the main results of the paper. We also recall a few definitions from stochastic stability (Arnold 1974, Loparo and Feng 1996).

Proof of Theorem 3.2. If condition (3.7) holds, then for all $\eta \in \mathbb{R}^n$,

$$\exists \mathbf{u} \in \mathcal{U} \,|\quad \eta^T \mathcal{F}\mathbf{u} > \max_{\mathbf{w}\in\mathcal{W}} \eta^T \mathcal{E}\mathbf{w}.$$

Recalling that any point in a convex set can be expressed as a convex combination of its vertices, we can use in the above equation the expression $\mathbf{u} = \sum_{i=1}^{p} u_i u^{(i)}$ and similarly $\mathbf{w} = \sum_{j=1}^{q} w_j w^{(j)}$. The above condition then corresponds to saying that for all $\eta \in \mathbb{R}^n$ there exists $u \in \Delta(\mathbb{R}^p)$ such that

$$\eta^T F \sum_{i=1}^{p} u_i u^{(i)} > \max_{w\in\Delta(\mathbb{R}^q)} \eta^T \sum_{j=1}^{q} E w_j w^{(j)}.$$

From the above condition, we can derive equivalently that for all $\eta \in \mathbb{R}^n$ there exists $u \in \Delta(\mathbb{R}^p)$ such that for all $w \in \Delta(\mathbb{R}^q)$

$$\eta^T \Big(F \sum_{i=1}^{p} u_i u^{(i)} + E \sum_{j=1}^{q} w_j w^{(j)} \Big) > 0.$$

Using the definition of $f(u, w)$ provided in (3.8), the above condition can be rewritten as, for all $\eta \in \mathbb{R}^n$

$$\exists u \in \Delta(\mathbb{R}^p) \,|\quad \eta^T f(u, w) > 0, \, \forall w \in \Delta(\mathbb{R}^q).$$

Recalling the definition $D_1(u) = \{f(u, w)\colon w \in \Delta(\mathbb{R}^q)\}$, the above condition implies that

$$\exists u \in \Delta(\mathbb{R}^p) \,|\quad D_1(u) \subseteq H = \{x \in \mathbb{R}^n |\, \eta^T x > 0\}.$$

We conclude our proof by taking $\eta = -\lambda$ and observing that this coincides with the attainability condition (3.3).

Proof of Theorem 3.3. Due to the bilinear structure of f, we can deduce that the best-response strategy u^* and worst adversarial disturbance w^* are on a vertex of the associated simplices in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. This corresponds to saying that both strategies are *pure strategies*. We recall here that pure strategies are such that each player chooses as a result a single predetermined action, in contrast with *mixed strategies* where players select probabilities on actions and at time of play a random mechanism consistent with the selected probability distribution determines the actual action. A consequence of this is that the mean field equilibrium, if exists, is in pure strategies as well.

From (3.11) we can we can rewrite the value of the anti-gradient projected game as

$$val[\partial_x v] = \inf_{i\in I} \sup_{j\in J} A_{ij} \lambda(\partial_x v),$$

where $I = \{1, \ldots, p\}$ and $J = \{1, \ldots, q\}$ are opportune sets of indices. Best responses and adversarial strategies are then

$$(u^*, w^*) = \arg \min_{i\in I} \max_{j\in J} A_{ij} \lambda(\partial_x v).$$

With the above definition of $val[\partial_x v]$ in mind, the Hamilton-Jacobi part of (2.7) can be rewritten as

$$v_t + \|v_x\| val[v_x] + \frac{1}{2}\left(\overline{m}(t) - x(t)\right)^T Q \left(\overline{m}(t) - x(t)\right) + \frac{\sigma^2}{2} Tr(v_{xx}^2) = 0 \text{ in } \mathbb{R}^n \times [0,T[,$$
$$v(x,T) = \Psi(x) \ \forall \ x \in \mathbb{R}^n. \tag{7.1}$$

It is left to observe that $f(u^*, w^*) = A_{i^*j^*}$ and proves the third equation (FPK equation).

Proof of Theorem 3.4. This proof mirrors in many parts the proof of Theorem 4.3 which contains more details. Let us start by observing that for the error vector we have

$$\begin{aligned} de_i(t) &= d\overline{m}(t) - dY_i(t) \\ &= \frac{1}{\nu}\sum_{j=1}^{\nu} \sigma d\mathcal{B}_j(t) - \delta\phi(t)(\overline{m} - Y_i(t))dt - \sigma d\mathcal{B}_i(t)dt \\ &= -\delta\phi(t)e_i(t)dt + \frac{1}{\nu}\sum_{j=1}^{\nu} \sigma d\mathcal{B}_j(t) - \sigma d\mathcal{B}_i(t). \end{aligned}$$

Note that the second term goes to zero for $\nu \to \infty$ so we can ignore it. The above SDE is linear and the corresponding stochastic process can be studied in the framework of stochastic stability theory (Loparo and Feng 1996). To do this, consider the infinitesimal generator

$$\mathcal{L} = \frac{1}{2}\sigma^2 \frac{d^2}{de_i(t)^2} - \delta\phi(t)e_i(t)\frac{d}{de_i(t)}. \tag{7.2}$$

Note that by definition $\phi(t)$ must be positive definite for all $t \in [0,T]$, i.e., $\xi^T \phi(t)\xi > 0$ for all $\xi \in \mathbb{R}^n$. We use this fact to study the infinitesimal generator of the Lyapunov function $V(e) = \frac{1}{2}e^T e$.

Our aim now is to prove that there exists a finite scalar κ and a neighborhood of zero of size κ, denoted by $\mathcal{N}_\kappa = \{e \in \mathbb{R}^n | \, V(e_i) \leq \kappa\}$, such that $\mathcal{L}V(e_i(t)) < 0$ for all $e_i(t) \notin \mathcal{N}_\kappa$, where $\mathcal{L}$ is the infinitesimal generator of the process $e_i(t)$.

Actually, the following lemma, borrowed from (Gard 1988), p. 129, and reported also in (Thygesen 1997), Theorem 2, establishes that if the former condition holds true, which is $\mathcal{L}V(e_i(t)) < 0$, then $V(e_i(t))$ is a supermartingale whenever $e_i(t)$ is not in $\mathcal{N}_\kappa$ and therefore by the martingale convergence theorem the system is stochastically sample path bounded.

Lemma 7.1. *(Stochastic sample path boundedness, cf. (Thygesen 1997), Theorem 2) Let there exist a proper* $\mathbb{C}^2$ *(twice differentiable) function* V *and a number* $\kappa > 0$ *such that for* $\|\zeta\|_\infty > \kappa$ *we have* $\mathcal{L}V \leq 0$. *Let* $\tau = \tau(\kappa)$ *be the first exit time from* $\{\zeta | \|\zeta\|_\infty > \kappa\}$ *for the solution* $\zeta(t)$ *where* $\|\zeta(0)\|_\infty > \kappa$. *Then for each* $\pi > 0$ *there exists a* $\varepsilon(\pi)$ *such that*

$$\mathbb{P}(\sup_{0 \leq t \leq \tau} \|\zeta(t)\|_\infty \leq \varepsilon(\pi)) > 1 - \pi.$$

Proof. Sketch from (Thygesen 1997), proof of Theorem 2: the underlying idea is to stop the process whenever ζ exits the region $\{\zeta | \, \kappa \leq \|\zeta\|_\infty \leq K\}$ for a properly chosen $K > \kappa$. Then V applied to the stopped process is a super-mantigale. When we let $K \to \infty$, we have that the process stops for $\|\zeta\|_\infty = \kappa$ w.p.1. From the super-martingale inequality applied to V we can conclude the proof. □

With the above lemma at hand, let us first consider the SDE for the error vector, $de_i(t) = -\delta\phi(t)e_i(t)dt - \sigma d\mathcal{B}_i(t)$, and rewrite $(I - \mathcal{M})\sigma d\mathcal{B}(t) = \sum b_i d\mathcal{B}_i(t)$, where

$$b_i = \sigma \begin{bmatrix} -\frac{1}{n} \\ \vdots \\ 1 - \frac{1}{n} \\ \vdots \\ -\frac{1}{n} \end{bmatrix} \quad i\text{th row.} \tag{7.3}$$

Then, for the infinitesimal generator of the Lyapunov function, we have

$$\mathcal{L}V(e_i) = -\delta e_i(t)^T \phi(t) e_i(t) + \frac{1}{2}\sigma^2 \sum_{i=1}^{n} \phi_{ii}(t),$$

where $\phi_{ii}(t)$ is the iith entry of matrix $\phi(t)$.

Now, consider the level sets $\mathcal{N}_\kappa = \{e_i(t) \in \mathbb{R}^n |\, V(e_i(t)) \le \kappa\}$, and note that there always exists a sufficiently large but finite $\hat{\kappa}$ such that for every $e_i(t) \notin \mathcal{N}_{\hat{\kappa}}$, i.e., $\frac{1}{2}e_i(t)^T e_i(t) > \hat{\kappa}$, we have $-\delta e_i(t)^T \phi e_i(t) + \frac{1}{2}\sigma^2 \sum_{i=1}^{n} \phi_{ii}(t) < 0$. The latter means $\mathcal{L}V(e_i(t)) < 0$ for all $e_i(t) \notin \mathcal{N}_{\hat{\kappa}}$, which proves that every level set $\mathcal{N}_\kappa$ where $\kappa \ge \hat{\kappa}$ is contractive.

A value for $\hat{k}$ can be obtained from the solution of the optimization problem

$$\begin{cases} \hat{k} := \min k \\ \{e_i |\, V(e_i) \le k\} \supset \left\{e_i | -\delta e_i(t)^T \phi e_i(t) + \frac{1}{2}\sigma^2 \sum_{i=1}^{n} \phi_{ii}(t)\right\}. \end{cases} \tag{7.4}$$

This concludes the proof.

Proof of Theorem 4.1. We first prove condition (4.5). To do this, let us write the Hamiltonian as:

$$H(x, \partial_x v(x,t), \overline{m}) = \inf_u \Big\{ \frac{1}{2} \left[(\overline{m} - x)^T Q (\overline{m} - x) + u^T C u \right] + \partial_x v(x,t)^T F u \Big\}.$$

The robust Hamiltonian is then

$$\tilde{H}(x, \partial_x v(x,t), \overline{m}) = H(x, \partial_x v(x,t), \overline{m}) + \sup_w \left\{ \partial_x v(x,t)^T E w - \frac{1}{2}\gamma^2 w^T w \right\}.$$

Differentiating with respect to u and w, we obtain

$$\begin{cases} Cu + F^T \partial_x v(x,t) = 0 \\ -\gamma^2 w + E^T \partial_x v(x,t) = 0, \end{cases} \tag{7.5}$$

from which we can derive (4.5).

We now prove (4.4). First notice that the second and last equations are the boundary conditions and follow straightforwardly from HJI equations and the evolution of the law of states.

To prove the first equation, which is a PDE corresponding to the HJI, let us replace u^* appearing in the Hamiltonian (7.5) by its expression (4.5):

$$H(x, \partial_x v(x,t), \overline{m}) = \frac{1}{2}\left[(\overline{m}-x)^T Q(\overline{m}-x) + u^{*T} C u^*\right] + \partial_x v(x,t) F u^*$$
$$= \frac{1}{2}(\overline{m}-x)^T Q(\overline{m}-x) + \frac{1}{2}\partial_x v(x,t)^T F C^{-1} F^T \partial_x v(x,t) - \partial_x v(x,t)^T F C^{-1} F^T \partial_x v(x,t)$$
$$= \frac{1}{2}(\overline{m}-x)^T Q(\overline{m}-x) - \frac{1}{2}\partial_x v(x,t)^T F C^{-1} F^T \partial_x v(x,t).$$

Using the above expression of the Hamiltonian in the HJI equation in (4.3), we obtain the HJI equation in (4.4).

To prove the third equation, which is a PDE representing the FPK equation, we simply plug (4.5) into the FPK in (4.3), and this concludes the proof.

Proof of Theorem 4.3. The time evolution of the error vector is given by the SDE

$$\begin{aligned}
de(t) &= (I-\mathcal{M})dY(t) \\
&= (I-\mathcal{M})\left[(W-I)Y(t)\right]dt + (I-\mathcal{M})\sigma d\mathcal{B}(t) \\
&= (W-\mathcal{M})Y(t)dt - (I-\mathcal{M})Y(t)dt + (I-\mathcal{M})\sigma d\mathcal{B}(t) \\
&= (W-\mathcal{M})(I-\mathcal{M})Y(t)dt - e(t)dt + (I-\mathcal{M})\sigma d\mathcal{B}(t) \\
&= \underbrace{(W-\mathcal{M}-I)}_{A} e(t)dt + (I-\mathcal{M})\sigma d\mathcal{B}(t).
\end{aligned}$$

We obtain the second equality from the equation of $dY(t)$ in (4.34). In the third equality we use the fact $\mathcal{M}W = \mathcal{M}$. In the fourth equality we use the property

$$\begin{aligned}
(W-\mathcal{M})(-\mathcal{M}) &= -W\mathcal{M} + \mathcal{M}^2 \\
&= -W\frac{1}{n}\mathbf{1}\mathbf{1}^T + \frac{1}{n^2}\mathbf{1}\mathbf{1}^T\mathbf{1}\mathbf{1}^T \\
&= -\frac{1}{n}\mathbf{1}\mathbf{1}^T + \frac{1}{n}\mathbf{1}\mathbf{1}^T = 0.
\end{aligned}$$

Also, let $A = W - \mathcal{M} - I$ and rewrite the dynamics for error vector as $de(t) = Ae(t)dt + (I-\mathcal{M})\sigma d\mathcal{B}(t)$.

The above SDE is linear and the corresponding stochastic process can be studied in the framework of stochastic stability theory (Loparo and Feng 1996). To do this, consider the infinitesimal generator

$$\mathcal{L} = \frac{1}{2}\sigma^2 (I-\mathcal{M})^T (I-\mathcal{M})\frac{d^2}{de(t)^2} + Ae(t)\frac{d}{de(t)}. \tag{7.6}$$

The above equation is obtained from

$$\frac{1}{2}\mathbb{E}\left(de^T \frac{d^2}{de^2} de\right) + \mathbb{E}\left(de\frac{d}{de}\right) = \frac{1}{2}\left[\mathbb{E}\left(e(t)^T A^T A e(t) dt^2\right) + \mathbb{E}\left(\sigma^2 (I-\mathcal{M})^T (I-\mathcal{M}) d\mathcal{B}(t)^2\right)\right.$$
$$\left. + \mathbb{E}\left(2e(t)^T A^T dt \sigma (I-\mathcal{M}) d\mathcal{B}(t)\right)\right]\frac{d^2}{de(t)^2} + \left[\mathbb{E}\left(Ae(t)dt\right) + \mathbb{E}\left(\sigma(I-\mathcal{M})d\mathcal{B}(t)\right)\right]\frac{d}{de(t)}.$$

Now, recalling that for a Brownian motion, $\mathbb{E}d\mathcal{B}(t) = 0$ and $\mathbb{E}d\mathcal{B}(t)^2 \to 0$ and ignoring the second-order terms (in dt^2 or $dtd\mathcal{B}(t)$) we obtain (7.6).

Note that from (4.33) we have that $\|W - \mathcal{M}\| < 1$ which in turn implies that A is negative definite, i.e., $\xi^T A \xi < 0$ for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$. We use this fact to study the infinitesimal generator of the Lyapunov function $V(e) = \frac{1}{2} e^T e$.

Our aim now is to prove that there exists a finite scalar κ and a neighborhood of zero of size κ, denoted by $\mathcal{N}_\kappa = \{e \in \mathbb{R}^n | \, V(e) \leq \kappa\}$, such that $\mathcal{L}V(e(t)) < 0$ for all $e(t) \notin \mathcal{N}_\kappa$, where $\mathcal{L}$ is the infinitesimal generator of the process $e(t)$.

Actually, Lemma 7.1, borrowed from (Gard 1988), p. 129, and reported also in (Thygesen 1997), Theorem 2, establishes that if the former condition holds, which is $\mathcal{L}V(e(t)) < 0$, then $V(e(t))$ is a supermartingale whenever $e(t)$ is not in $\mathcal{N}_\kappa$ and therefore by the martingale convergence theorem the system is stochastically sample path bounded.

With the above lemma at hand, let us first consider the SDE for the error vector, $de(t) = Ae(t)dt + (I - \mathcal{M})\sigma d\mathcal{B}(t)$, and rewrite $(I - \mathcal{M})\sigma d\mathcal{B}(t) = \sum b_i d\mathcal{B}_i(t)$, where

$$b_i = \sigma \begin{bmatrix} -\frac{1}{n} \\ \vdots \\ 1 - \frac{1}{n} \\ \vdots \\ -\frac{1}{n} \end{bmatrix} \quad i\text{th row.} \tag{7.7}$$

Then, for the infinitesimal generator of the Lyapunov function, we have

$$\begin{aligned} \mathcal{L}V(e) &= e(t)^T A e(t) + \frac{1}{2} \sum_{i=1}^{n} \Sigma_{ii} \\ &= e(t)^T A e(t) + \frac{1}{2} n \sigma^2 [(n-1)\frac{1}{n^2} + (1 - \frac{1}{n})^2], \end{aligned}$$

where $\Sigma = \sum_{k=1}^{n} b_k b_k^T \in \mathbb{R}^{n \times n}$, whose elements in the principal diagonal are $\Sigma_{ii} = \sigma^2[(n-1)\frac{1}{n^2} + (1 - \frac{1}{n})^2]$.

Now, consider the level sets $\mathcal{N}_\kappa = \{e(t) \in \mathbb{R}^n | \, V(e(t)) \leq \kappa\}$ and note that there always exists a $\hat{\kappa}$ big enough and finite such that for every $e(t) \notin \mathcal{N}_{\hat{\kappa}}$, i.e., $\frac{1}{2} e(t)^T e(t) > \hat{\kappa}$, we have $e(t)^T A e(t) + \frac{1}{2} n \sigma^2 [(n-1)\frac{1}{n^2} + (1 - \frac{1}{n})^2] < 0$. The latter means $\mathcal{L}V(e(t)) < 0$ for all $e(t) \notin \mathcal{N}_{\hat{\kappa}}$, which proves that every level set $\mathcal{N}_\kappa$ where $\kappa \geq \hat{\kappa}$ is contractive.

In other words, for every $e(t) \in \partial \mathcal{N}_{\hat{\kappa}}$, $e(t + dt) \in \mathcal{N}_{\hat{\kappa}}$. The same reasoning leads to the conclusion that every level set $\mathcal{N}_\kappa$ where $\kappa \geq \hat{\kappa}$ is contractive. Thus, we can conclude that for every $\kappa \geq \hat{\kappa}$ there exists an $\varepsilon = \sqrt{2\kappa}$ for which the level set $\{\overline{m} \in \mathbb{R} | \, \|\overline{m}\| \leq \varepsilon\}$ is contractive. A value for $\hat{k}$ can be obtained from the solution of the optimization problem

$$\left\{ \begin{array}{l} \hat{k} := \min k \\ \{e | \, V(e) \leq k\} \supset \left\{e | \, e(t)^T A e(t) \, + \, \frac{1}{2} n \sigma^2 [(n-1)\frac{1}{n^2} + (1 - \frac{1}{n})^2] < 0 \right\}. \end{array} \right. \tag{7.8}$$

This concludes the proof.