RADON-NIKODÝM THEOREMS FOR FINITELY ADDITIVE MULTIMEASURES

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Abstract. In this paper we deal with interval multimeasures. We show some Radon-Nikodým theorems for such multimeasures using multivalued Henstock or Henstock-Kurzweil-Pettis derivatives. We do not use the separability assumption in the results.

1. Introduction.

One of the most fascinating problems arising when we deal with multimeasures is the represention of a multimeasure as an integral, i.e. the existence of a Radon-Nikodým derivative.

Several papers concerning this question appeared since the 1970's where pioneering results have been established amongst others by Artstein [1], Costé [6], Costé and Pallu de la Barrière [7]. These papers deal with countably additive multimeasures and use classical notions of integral existing in literature.

In the 1990's other results dealing with finitely additive multimeasures have been obtained by A. Martellotti, K. Musiał and A. R. Sambucini (see [15, 16]). In particular, they have been extended the trattation beyond the Banach spaces (in particular, to locally convex spaces), but also in this case classical integrals are used for the representation.

In general the results existing in literature use multimeasures defined on a σ-algebra. Moreover, most of them uses the separability assumption.

In this paper we deal with the Radon-Nikodým problem for multimeasures defined on the family $\mathcal{I}$ of all non trivial closed subintervals of $[0,1]$ and consequently we look for Radon-Nikodým derivatives of Henstock type. This is the first paper where that problem has been undertaken.

The Henstock integral has been introduced in the 1960's independently by Henstock [11] and Kurzweil [14], by a simple modification of Riemann's method. It is a non absolutely convergent integral more general than Lebesgue's one, integrates all derivatives and its primitive is a finitely additive interval

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function.

Our starting point is the remarkable recent article of B. Cascales, V. Kadets and J. Rodríguez [5], where they obtain two Radon-Nikodým theorems for countably additive multimeasures without any separability assumption.

Here we go on in such kind of investigation and we consider finitely additive multimeasures defined on $\mathcal{I}$, taking convex compact values or convex weakly compact values, in an arbitrary Banach space X.

The paper is organized as follows. In Section 2 we give necessary notations, definitions and preliminaries.

In Section 3 we extend to the multivalued case the notion of variational measure already known for vector valued interval measure. This measure is a useful tool for our investigation.

In Section 4 we prove the main results. In the convex compact case we obtain a Radon-Nikodým theorem for dominated interval multimeasures (see Theorem 4.2) that improves Theorem 3.1 of [5]. To get our goal we use an extension of a finitely additive multimeasure to a countably additive multimeasure defined in the σ-algebra of the Borel subsets of $[0,1]$ (see Proposition 4.1).

In the more general context of convex weakly compact valued multimeasures we find an *HKP*-integrable derivative under the hypothesis of absolute continuity for the associated variational measure (see Theorem 4.6). Also in such a case we do not require the separability of the target Banach space X, but we assume that X possesses the Radon-Nikodým property (shortly RNP).

2. NOTATIONS AND PRELIMINARY DEFINITIONS.

Let $[0,1]$ be the unit interval of the real line, endowed with the usual topology and the Lebesgue measure λ. We denote by $\mathcal{L}$ the family of all measurable subsets of $[0,1]$, by $\mathcal{A}$ the ring generated by the subintervals $[a,b) \subseteq [0,1]$ and by $\mathcal{I}$ the family of all non trivial subintervals of $[0,1]$.

A *partition in* $[0,1]$ is a finite collection of pairs $\{(I_j,t_j)\}_{j=1}^q$, where $I_1,\dots,I_q$ are non-overlapping subintervals of $[0,1]$ and $t_1,\dots,t_q \in [0,1]$. Given a subset E of $[0,1]$, we say that the partition $\{(I_j,t_j)\}_{j=1}^q$ is *anchored* on E if $t_j \in E$ for each $j=1,\dots,q$. If $\bigcup_{j=1}^q I_j = [0,1]$, we say that $\{(I_j,t_j)\}_{j=1}^q$ is a *partition of* $[0,1]$.

A *gauge* on $B \subset [0,1]$ is a positive function on B. Given a gauge δ, we say that a partition $\{(I_j,t_j)\}_{j=1}^q$ is *δ-fine* if $I_j \subset (t_j-\delta(t_j), t_j+\delta(t_j))$ for every $j=1,\dots,q$.

Let denote by X a real Banach space, with dual X^*. The closed unit ball of X (resp. X^*) is denoted by $B(X)$ (resp. $B(X^*)$).

Definition 2.1. A function $f:[0,1]\to X$ is said to be *Henstock integrable* (or simply *H-integrable*) on $[0,1]$ if there exists $x \in X$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that

$$\left\| \sum_{j=1}^{q} f(t_j)|I_j| - x \right\| < \varepsilon,$$

for every δ-fine partition $\{(I_j, t_j)\}_{j=1}^{q}$ of $[0,1]$.
We call x the *Henstock integral* of f on $[0,1]$ and we set $(H)\int_0^1 f\, d\lambda := x$.

It is well known that if $f : [0,1] \to X$ is Henstock integrable on $[0,1]$ and $I \in \mathcal{I}$, also the function $f\chi_I$ is Henstock integrable on $[0,1]$ [19, Theorem 3.3.4]. We say in such a case that f is Henstock integrable on I. If $X = \mathbb{R}$, then f is said to be *Henstock-Kurzweil integrable* or simply *HK-integrable* on $[0,1]$ and we denote by $(HK)\int_0^1 f\, d\lambda$ the corresponding interval.

Definition 2.2. A function $f : [0,1] \to X$ is said to be *scalarly HK-integrable* (resp. *scalarly integrable*) if for every $x^* \in X^*$ the real function $\langle x^*, f(\cdot)\rangle$ is *HK*-integrable (resp. integrable).
A scalarly *HK*-integrable (resp. scalarly integrable) function $f : [0,1] \to X$ is said to be *Henstock-Kurzweil-Pettis integrable* or simply *HKP-integrable* (resp. *Pettis integrable*) on $[0,1]$ if for every interval $I \in \mathcal{I}$ (resp. for every $E \in \mathcal{L}$), there exists $x_I \in X$ (resp. $x_E \in X$) such that

$$\langle x^*, x_I\rangle = (HK)\int_I \langle x^*, f\rangle\, d\lambda, \text{ for every } x^* \in X^*$$

$$\left(\text{resp. } \langle x^*, x_E\rangle = \int_E \langle x^*, f\rangle\, d\lambda, \text{ for every } x^* \in X^* \right).$$

We call x_I (resp. x_E) the *HKP-integral* of f on I (resp. *Pettis integral* of f on E) and we write $(HKP)\int_I f\, d\lambda := x_I$ (resp. $(P)\int_E f\, d\lambda := x_E$).

For more details about vector valued functions see [8], [18] or [19].
The class of all non-empty subsets of X is 2^X. By $cl(X)$, $cc(X)$, $cbc(X)$, $ck(X)$, $cwk(X)$ we denote respectively the subfamilies of 2^X of all closed, closed convex, closed convex bounded, convex compact and convex weakly compact subsets of X.
For every $C \in 2^X$, the *support function* of C is denoted by $s(\cdot, C)$ and defined on X^* by $s(x^*, C) := \sup\{\langle x^*, x\rangle : x \in C\}$, for each $x^* \in X^*$.
We denote by d_H the *Hausdorff distance* in 2^X:

$$d_H(C, C') := \max\{e(C, C'), e(C', C)\},\ C, C' \in 2^X,$$

where $e(C, C') = \sup\{d(x, C') : x \in C\}$ is the *excess* of C over C', while $d(x, C) = \inf\{||x - y|| : y \in C\}$ is the *distance* of x from C. For $A \in 2^X$, we set $||A|| := \sup\{||x|| : x \in A\}$ and we call it the *radius* of A.
A map $F : [0,1] \to cl(X)$ is called a *multifunction.* A function $f : [0,1] \to X$ is called a *selection* of F if for every $t \in [0,1]$ one has $f(t) \in F(t)$.

A multifunction F is said to be *scalarly integrable* (resp. *scalarly HK-integrable*) if for every $x^* \in X^*$ the map $s(x^*, F(\cdot))$ is integrable (resp. HK-integrable). More information concerning Pettis integrability of multifunctions can be found in [17].

Definition 2.3. A multifunction $F : [0,1] \to cbc(X)(ck(X), cwk(X))$ is said to be *Henstock-Kurzweil-Pettis integrable* or simply *HKP-integrable* (resp. *Pettis integrable*) in $cbc(X)(ck(X), cwk(X))$ if F is scalarly HK-integrabile (resp. scalarly integrable) and for every interval $I \in \mathcal{I}$ (resp. for every $E \in \mathcal{L}$), there exists $C_I \in cbc(X)(ck(X), cwk(X))$ (resp. $C_E \in cbc(X)(ck(X), cwk(X))$) such that

$$s(x^*, C_I) = (HK)\int_I s(x^*, F)\, d\lambda, \text{ for every } x^* \in X^*$$

$$\left(\text{resp. } s(x^*, C_E) = \int_E s(x^*, F)\, d\lambda \text{ for every } x^* \in X^*\right).$$

We call C_I (resp. C_E) the *HKP-integral* of F over I (resp. *Pettis integral* of F over E) and we set $(HKP)\int_I F\, d\lambda := C_I$ (resp. $(P)\int_E F\, d\lambda := C_E$).

More information concerning Pettis integrability of multifunctions can be found in [17].

Definition 2.4. A multifunction $F : [0,1] \to cbc(X)(ck(X), cwk(X))$ is said to be *Henstock integrable* if there exists $W \in cbc(X)(ck(X), cwk(X))$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that for every δ-fine partition $\{(I_j, t_j)\}_{j=1}^p$ of $[0,1]$ we have

$$d_H\left(W, \sum_{j=1}^{p} F(t_j)|I_j|\right) < \varepsilon.$$

W is called the *Henstock integral* of F and we write $(H)\int_I F\, d\lambda := W$.

A multifunction $M : \mathcal{L} \to cl(X)$ is said to be a d_H*-multimeasure* if for every sequence $(A_n)_{n\geq 1} \subset \mathcal{L}$ of pairwise disjoint sets with $A = \bigcup_{n\geq 1} A_n$, we have $d_H(M(A), \sum_{k=1}^n M(A_k)) \to 0$ as $n \to +\infty$.
A multifunction $M : \mathcal{L} \to cl(X)$ is said to be a *weak multimeasure* or simply a *multimeasure* if for every $x^* \in X^*$, the map $A \mapsto s(x^*, M(A))$ is a real valued measure.
It is known that every $cl(X)$-valued d_H-multimeasure is a multimeasure (see [13, Proposition 8.4.7]).
The two notions coincide whenever the multimeasure takes its values in $cwk(X)$ (see [13, Theorem 8.4.10]).
We say that the multimeasure $M : \mathcal{L} \to 2^X$ is λ*-continuous* and we write $M << \lambda$, if $\lambda(A) = 0$ yields $M(A) = \{0\}$.
Given a multimeasure $M : \mathcal{L} \to 2^X$, a vector measure $m : \Sigma \to X$ such that

$m(A) \in M(A)$ for every $A \in \mathcal{L}$ is called a *selection* of M.
Moreover, the *variation* of M is the extended non-negative function $|M|$ whose value on a set $A \in \mathcal{L}$ is given by

$$|M|(A) := \sup \sum_i ||M(A_i)||,$$

where the supremum is taken over all finite partitions $(A_i)_i$ of A in $\mathcal{L}$.
If $|M|([0,1]) < +\infty$, then M is called of *finite variation.*
If there exists a sequence $(A_n)_n \subset \mathcal{L}$ of pairwise disjoint sets covering $[0,1]$ and such that $|M|(A_n) < +\infty$ for every n, then M is called of *σ-finite variation*

3. INTERVAL MULTIMEASURES AND THEIR SELECTIONS. VARIATIONAL MEASURES.

We start with the following definitions.

Definition 3.1. An interval multifunction $\Phi : \mathcal{I} \to cwk(X)$ is said to be *finitely additive* if for every non-overlapping intervals $I_1, I_2 \in \mathcal{I}$ such that $I_1 \cup I_2 \in \mathcal{I}$ we have $\Phi(I_1 \cup I_2) = \Phi(I_1) + \Phi(I_2)$.

Remark 3.2. The primitives of Henstock or Henstock-Kurzweil-Pettis integrable multifunctions are interval multimeasures. Moreover, it is known that if $F : [0,1] \to cwk(X)$ is Pettis integrable in $cwk(X)$, then its primitive is σ-additive (see [18]). If we set $\Phi(I) := \nu(I)$, $I \in \mathcal{I}$, then Φ is an interval multimeasure.

Definition 3.3. A multifunction $\Psi : \mathcal{A} \to cwk(X)$ is said to be a *finitely additive multimeasure* if for every $A_1, A_2 \in \mathcal{A}$ whose interiors are disjoint we have $\Psi(A_1 \cup A_2) = \Psi(A_1) + \Psi(A_2)$.

Remark 3.4. In the following we identify a finitely additive interval multifunction $\Phi : \mathcal{I} \to cwk(X)$ with the finitely additive multimeasure $\Psi : \mathcal{A} \to cwk(X)$ defined by $\Psi(A) := \sum_{j=1}^{q} \Phi(I_j)$, where $A = \bigcup_{j=1}^{q} I_j$ and $I_1, \ldots, I_q$ are pairwise disjoint subintervals of $[0,1]$. We use a similar identification for the corresponding selections.
Hence we call *interval multimeasure* every finitely additive interval multifunction and *interval measure* every finitely additive interval function.
Moreover, we observe that if $\Phi : \mathcal{I} \to cwk(X)$ is an interval multimeasure, then for every $x^* \in X^*$, $s(x^*, \Phi(\cdot))$ is a real-valued interval measure.
An interval measure $\phi : \mathcal{I} \to X$ is said to be a *selection* of an interval multimeasure Φ if $\phi(I) \in \Phi(I)$ for every $I \in \mathcal{I}$.

We recall that for $\emptyset \neq K \subset X$, an element $x \in K$ is called an *exposed point* of K if there exists $x^* \in X^*$ such that $\langle x^*, x\rangle > \langle x^*, y\rangle$ for every $y \in K \setminus \{x\}$; an exposed point x of K is called a *strongly exposed point* of K if for every $(x_n)_n \subset K$ with $\langle x^*, x_n\rangle \to \langle x^*, x\rangle$, then $||x_n - x|| \to 0$, where x^* is the functional that exposes x. We denote by $exp(K)$ (resp. $str\,exp(K)$) the set of the exposed points (resp. strongly exposed points) of K.

It is known that if $K \in ck(X)$ (resp. $K \in cwk(X)$), then $exp(K) \neq \emptyset$ (resp. $str\,exp(K) \neq \emptyset$) and $K = \overline{co}(exp(K))$ (resp. $K = \overline{co}(str\,exp(K))$) (see [4, Theorem 3.6.1]).

The following result is well known in case of countably additive multimeasures (see [12]). We omit its proof, since it is very similar to that in [12, Proposition 2.1].

Proposition 3.5. *Let $\Phi : \mathcal{I} \to ck(X)$ be an interval multimeasure. If $x_0 \in exp(\Phi([0,1]))$ then there exists a selection $\phi : \mathcal{I} \to X$ of Φ such that $\phi([0,1]) = x_0$ and $\phi(I) \in exp(\Phi(I))$ for every $I \in \mathcal{I}$.*

With similar argument, we obtain

Proposition 3.6. *Let $\Phi : \mathcal{I} \to cwk(X)$ be an interval multimeasure. If $x_0 \in str\,exp(\Phi([0,1]))$ then there exists a selection $\phi : \mathcal{I} \to X$ of Φ such that $\phi([0,1]) = x_0$ and $\phi(I) \in str\,exp(\Phi(I))$ for every $I \in \mathcal{I}$.*

Now we extend the notion of variational measure to additive interval multimeasures (cf. [2] or [9]). This notion is a useful tool to study the primitives of real valued or, more in general, vector valued integrable functions.

Given an interval multimeasure $\Phi : \mathcal{I} \to cwk(X)$, a gauge δ and a set $E \subset [0,1]$, we define

$$Var(\Phi, \delta, E) = \sup\left\{\sum_{j=1}^{p} ||\Phi(I_j)|| : \{(I_j, t_j)\}_{j=1}^{p} \ \delta\text{-fine partition anchored on } E\right\}.$$

Then we set

$$V_\Phi(E) := \inf\left\{Var(\Phi, \delta, E) : \delta \text{ gauge on } E\right\}.$$

V_Φ is called *the variational measure generated by* Φ.

Remark 3.7. If Φ is an interval multimeasure, then V_Φ is the variational measure generated by the single valued map $R \circ \Phi$, where $R : cwk(X) \to l_\infty(B(X^*))$ is the Rådstrom Embedding defined by $R(C) := s(\cdot, C)$, for $C \in cwk(X)$. In fact, for $I \in \mathcal{I}$ we obtain:

$$||R(\Phi(I))||_{l_\infty} = ||s(\cdot, \Phi(I))||_{l_\infty} = \sup_{x^* \in B(X^*)} |s(x^*, \Phi(I))|$$

$$= \sup_{x^* \in B(X^*)} |s(x^*, \Phi(I)) - s(x^*, \{0\})| = d_H(\Phi(I), \{0\}) = ||\Phi(I)||.$$

Consequently, $Var(\Phi, \delta, E) = Var(R(\Phi), \delta, E)$ for any gauge δ and any set $E \subset [0,1]$, and $V_\Phi(E) = V_{R\circ\Phi}(E)$ for any set $E \subset [0,1]$.
Therefore, as in the X-valued case, V_Φ is a metric outer measure on $[0,1]$ (see [2]) and a measure over all Borel sets of $[0,1]$.

We say that the variational measure V_Φ is *σ-finite* if there exists a sequence of (pairwise disjoint) sets $(E_n)_{n\geq 1}$ covering $[0,1]$ and such that $V_\Phi(E_n) < \infty$, for every $n \geq 1$. Moreover, we say that V_Φ is *absolutely continuous* with respect to λ (or *λ-continuous*) and we write $V_\Phi << \lambda$ if for every $E \in \mathcal{L}$ with $\lambda(E) = 0$ we have $V_\Phi(E) = 0$.

Remark 3.8. Taking into account that $V_\Phi = V_{R\circ\Phi}$ and using [2, Corollary 2.3], we have σ-finiteness of every λ-continuous variational measure.

4. MAIN RESULTS.

4.1. **The $ck(X)$ case.** We start by proving an extension result.

Proposition 4.1. *Let $\Phi : \mathcal{I} \to ck(X)$ be an interval multimeasure such that there exists a set $Q \in ck(X)$ with $\Phi(I) \subseteq |I|Q$ for every $I \in \mathcal{I}$.*
Then Φ can be extended to a multimeasure $M : \sigma(\mathcal{A}) \to ck(X)$ such that $M(B) \subseteq \lambda(B)Q$ for every $B \in \sigma(\mathcal{A})$.

Proof. We observe that for every $x^* \in X^*$, $s(x^*, \Phi)$ is a real-valued measure and

$$-s(-x^*, Q)|I| \leq s(x^*, \Phi(I)) \leq s(x^*, Q)|I|, \text{ for every } I \in \mathcal{I}.$$

Fix $x^* \in X^*$. Then $s(x^*, \Phi)$ can be extended to $\mathcal{A}$, the ring generated by $\mathcal{I}$. Hence for every $A \in \mathcal{A}$,

$$-s(-x^*, Q)\lambda(A) \leq s(x^*, \Phi(A)) \leq \lambda(A)s(x^*, Q).$$

Conesquently

$$|s(x^*, \Phi(A))| \leq |s(x^*, Q)|\lambda(A) + |s(-x^*, Q)|\lambda(A).$$

Since $A \mapsto \lambda(A)s(x^*, Q)$ is σ-additive on $\mathcal{A}$ and bounded, $s(x^*, \Phi(\cdot))$ can be extended to a measure $\mu_{x^*} : \sigma(\mathcal{A}) \to \mathbb{R}$, where $\sigma(A)$ consists of all Borel subsets of $[0,1]$.
Now let $B \in \sigma(\mathcal{A})$ and consider a sequence $(A_n)_n \subset \mathcal{A}$ such that $\lambda(B \triangle A_n) \to 0$. We prove that $(\Phi(A_n))_n$ is a Cauchy sequence in $(ck(X), d_H)$.

In fact, for every natural numbers n, m, we have

$$\begin{aligned}
&d_H(\Phi(A_n), \Phi(A_m)) = \sup_{x^* \in B(X^*)} |s(x^*, \Phi(A_n)) - s(x^*, \Phi(A_m))| \\
&= \sup_{x^* \in B(X^*)} |s(x^*, \Phi(A_n \setminus A_m)) - s(x^*, \Phi(A_m \setminus A_n))| \\
&\leq \sup_{x^* \in B(X^*)} |s(x^*, \Phi(A_n \setminus A_m))| + \sup_{x^* \in B(X^*)} |s(x^*, \Phi(A_m \setminus A_n))| \\
&\leq 2 \sup_{x^* \in B(X^*)} |s(x^*, Q)| \lambda(A_n \setminus A_m) + 2 \sup_{x^* \in B(X^*)} |s(x^*, Q)| \lambda(A_m \setminus A_n) \\
&= k\lambda(A_n \setminus A_m) + k\lambda(A_m \setminus A_n) = k\lambda(A_n \triangle A_m),
\end{aligned}$$

where $k = 2||Q||$.
Since $\lambda(A_n \triangle A_m) \to 0$, also $d_H(\Phi(A_n), \Phi(A_m)) \to 0$. Since $(ck(X), d_H)$ is a complete metric space, we obtain that $(\Phi(A_n))_n$ is d_H-convergent to an element of $ck(X)$.
At this point let us define $M(B) := (d_H) \lim_n \Phi(A_n)$ for $B \in \sigma(\mathcal{A})$. The multifunction M is well defined. In fact, if $(A'_n) \subset \mathcal{A}$ is another sequence such that $\lambda(A'_n \triangle B) \to 0$, then also $\lambda(A'_n \triangle A_n) \to 0$. Consequently,

$$d_H(\Phi(A'_n), \Phi(A_n)) \leq k\lambda(A'_n \triangle A_n) \to 0.$$

Thus

$$(d_H) \lim_n \Phi_n(A'_n) = (d_H) \lim_n \Phi_n(A_n).$$

Moreover, M is $ck(X)$-valued and is an extension of Φ to $\sigma(\mathcal{A})$.
We claim that $s(x^*, M) = \mu_{x^*}$ for every $x^* \in X^*$. In fact, let fix $x^* \in X^*$. It follows from the definition of M that for every $B \in \sigma(\mathcal{A})$ $s(x^*, \Phi(A_n)) \to s(x^*, M(B))$, where $(A_n)_n$ is one of the above considerated sequence.
On the other hand,

$$\begin{aligned}
&|\mu_{x^*}(B) - s(x^*, \Phi(A_n))| = |\mu_{x^*}(B) - \mu_{x^*}(A_n)| \\
&= |\mu_{x^*}(B \setminus A_n) - \mu_{x^*}(A_n \setminus B)| \leq |\mu_{x^*}(B \setminus A_n)| + |\mu_{x^*}(A_n \setminus B)| \\
&\leq k\lambda(B \triangle A_n) \to 0,
\end{aligned}$$

for every $B \in \sigma(\mathcal{A})$.
Hence $s(x^*, M(B)) = \mu_{x^*}(B)$ for every $B \in \sigma(\mathcal{A})$.
Therefore for each $x^* \in X^*$ $s(x^*, M)$ is a measure. Since M is $ck(X)$-valued, we have that M is a multimeasure (see [13, Theorem 8.4.10]).
Finally for each $B \in \sigma(\mathcal{A})$ and each $x^* \in X^*$

$$s(x^*, M(B)) = \mu_{x^*}(B) \leq s(x^*, Q)\lambda(B) = s(x^*, \lambda(B)Q).$$

Therefore $M(B) \subseteq \lambda(B)Q$ for each $B \in \sigma(\mathcal{A})$. □

The following result improves [5, Theorem 3.1] valid for dominated convex compact valued multimeasures that can be representated by Pettis integrable multifunctions. More precisely we show that a Pettis integrable density can be obtained even considering dominated interval multimeasures.

Theorem 4.2. *Let* $\Phi : \mathcal{I} \to ck(X)$ *be an interval multimeasure such that there exists a set* $Q \in ck(X)$ *with* $\Phi(I) \subseteq |I|Q$ *for every* $I \in \mathcal{I}$*. Then there exists a multifunction* $F : [0,1] \to ck(X)$ *Pettis integrable in* $ck(X)$ *such that:*

(i) *for every finitely additive selection* ϕ *of* Φ *there exists a Pettis integrable selection* f *of* F *with* $\phi(I) = (P)\int_I f(t)\,dt$ *for all* $I \in \mathcal{I}$*;*

(ii) $\Phi(I) = (P)\int_I F\,d\lambda$ *for all* $I \in \mathcal{I}$*.*

Proof. By Proposition 4.1, Φ can be extended to a multimeasure $M : \sigma(\mathcal{A}) \to ck(X)$ such that $M(B) \subseteq \lambda(B)Q$ for every $B \in \sigma(\mathcal{A})$. Therefore by of [5, Theorem 3.1] there exists a Pettis integrable multifunction $F : [0,1] \to ck(X)$ such that

(i) for each countably additive selection m of M, there exists a Pettis integrable selection f of F with $m(B) = (P)\int_B f\,d\lambda$, for each $B \in \sigma(\mathcal{A})$,

(ii) $M(B) = (P)\int_B F\,d\lambda$

We conclude that F satisfies the required properties. □

One can assume on Theorem 4.2 that for each $t \in [0,1]$ there exist $Q_t \in ck(X)$ and $\delta_t > 0$ such that $\Phi(I) \subseteq Q_t|I|$ for every interval I containing t with $|I| < \delta_t$. But a simple topological argument shows that these assumptions imply that $[0,1]$ is a finite union of non-overlapping closed intervals in each of which the assumptions of Theorem 4.2 are fulfilled.

Proposition 4.3. *Let* $\Phi : \mathcal{I} \to ck(X)$ *be an interval multimeasure such that* $V_\Phi << \lambda$*. Assume that there exists a sequence* $(I_n)_n$ *of non-overlapping intervals such that* $\lambda([0,1] \setminus \bigcup_n I_n) = 0$ *and for each natural number* n *there exists a compact set* $Q_n \subset X$ *with the property that* $\Phi(I) \subseteq |I|Q_n$ *for all subinterval* I *of* I_n*.*
Then Φ *is the primitive of a* $ck(X)$*-valued multifunction HKP-integrable in* $ck(X)$*.*

Proof. By Theorem 4.2, for each n there exists a multifunction $G_n : I_n \to ck(X)$, Pettis integrable in $ck(X)$, such that

$$\Phi(I) = (P)\int_I G_n\,d\lambda, \text{ for each interval } I \subseteq I_n.$$

Let us consider now the multifunction $G : [0,1] \to ck(X)$ defined as $G(t) := \sum_n G_n(t)$.
Since $V_\Phi << \lambda$, we have also $V_{s(x^*,\Phi)} << \lambda$ for every $x^* \in X^*$. Therefore by [3, Theorem 3] for every $x^* \in X^*$ there exists $g_{x^*} \in HK([0,1])$ such that

$$s(x^*, \Phi(I)) = (HK)\int_I g_{x^*}\,d\lambda, \text{ for all } I \in \mathcal{I}.$$

Fix $x^* \in X^*$. For each n and each interval $I \subset I_n$ we have

$$s(x^*, \Phi(I)) = (HK) \int_I g_{x^*} \, d\lambda.$$

But for the same n and I we have also

$$s(x^*, \Phi(I)) = (HK) \int_I s(x^*, G_n) \, d\lambda.$$

Therefore we obtain $(HK) \int_I s(x^*, G_n) \, d\lambda = (HK) \int_I g_{x^*} \, d\lambda$ for each n and each interval $I \subset I_n$. It follows by [10, Theorem 9.12] that for every n, $s(x^*, G_n) = g_{x^*}$ almost everywhere on I_n (and the exceptional set depends only on x^*).

By the definition of G we have that $s(x^*, G) = g_{x^*}$ almost everywhere on $[0, 1]$ (and the exceptional set depends only on x^*). Therefore, by [10, Theorem 9.10] $s(x^*, G)$ is *HK*-integrable. Since x^* is arbitrary, then G is scalarly *HK*-integrable.

Finally, if $I \in \mathcal{I}$ and $x^* \in X^*$, we have

$$s(x^*, \Phi(I)) = (HK) \int_I g_{x^*} \, d\lambda = (HK) \int_I s(x^*, G) \, d\lambda.$$

We conclude that G is *HKP*-integrable in $ck(X)$ and that Φ is its *HKP*-primitive. □

4.2. **The** $cwk(X)$ **case.** Now we are going to consider the more general case of $cwk(X)$-valued multifunctions.

Proposition 4.4. *Let* $\Phi : \mathcal{I} \to cwk(X)$ *be an interval multimeasure such that* $V_\Phi << \lambda$. *Assume that* $s(x^*, \Phi(I)) \geq 0$ *for every* $x^* \in X^*$ *and for every* $I \in \mathcal{I}$. *Then* Φ *can be extended to a* σ*-additive multimeasure* $M : \mathcal{L} \to cwk(X)$ *of* σ*-finite variation and with* $M << \lambda$.

Proof. Since $V_\Phi << \lambda$, we have also $V_{s(x^*,\Phi)} << \lambda$ for each $x^* \in X^*$. By [3, Theorem 3], for every $x^* \in X^*$ there exists $g_{x^*} \in HK([0, 1])$ such that

$$s(x^*, \Phi(I)) = (HK) \int_I g_{x^*} \, d\lambda, \text{ for every } I \in \mathcal{I}.$$

Since $s(x^*, \Phi) \geq 0$, it follows by [10, Theorem 9.12] that $g_{x^*} \geq 0$ almost everywhere on $[0, 1]$. By [10, Theorem 9.13], g_{x^*} is Lebesgue integrable for every $x^* \in X^*$. Moreover, $V_{s(x^*,\Phi)}$ is a measure over all Borel sets of $[0, 1]$. By [9, Theorem 2] $V_{s(x^*,\Phi)}(B) = \int_B g_{x^*} \, d\lambda$ for every $B \in \sigma(\mathcal{A})$. Now let us consider the family

$$\mathcal{B} := \left\{ B \in \sigma(\mathcal{A}) : \exists\, C_B \in cwk(X) \text{ such that } \forall\, x^* \in X^*,\ s(x^*, C_B) = \int_B g_{x^*} \, d\lambda \right\}.$$

Notice that for every $B \in \mathcal{B}$, $s(x^*, C_B) \leq \int_0^1 g_{x^*} \, d\lambda = s(x^*, \Phi([0, 1]))$ for every $x^* \in X^*$. Hence $C_B \subseteq \Phi([0, 1])$ for every $B \in \mathcal{B}$.

It is clear that $\mathcal{B}$ contains $\mathcal{A}$. We claim that $\mathcal{B}$ is a monotone class. In fact, let $(B_n)_n$ be a monotone increasing sequence of $\mathcal{B}$ and let $C_{B_n} \in cwk(X)$ such that $s(x^*, C_{B_n}) = \int_{B_n} g_{x^*}\, d\lambda$ for every $x^* \in X^*$. By the Monotone Convergence Theorem (see [10, Theorem 3.21]) $\lim_n \int_{B_n} g_{x^*}\, d\lambda = \int_{\bigcup_n B_n} g_{x^*}\, d\lambda$. Moreover, also $(C_{B_n})_n$ is a monotone increasing sequence, in fact, for every n and every $x^* \in X^*$, $s(x^*, C_{B_n}) = \int_{B_n} g_{x^*}\, d\lambda \le \int_{B_{n+1}} g_{x^*}\, d\lambda = s(x^*, C_{B_{n+1}})$. Hence $C_{B_n} \subseteq C_{B_{n+1}}$ for every n.

Consequently, $\lim_n s(x^*, C_{B_n}) = s(x^*, \bigcup_n C_{B_n}) = s(x^*, \overline{\bigcup_n C_{B_n}})$. In fact, the first equality follows from the fact that $\lim_n s(x^*, C_{B_n}) = \sup_n s(x^*, C_{B_n}) = s(x^*, \bigcup_n C_{B_n})$, the second equality is a property of the support function. Since $\bigcup_n C_{B_n} \subseteq \Phi([0,1]) \in cwk(X)$, we have $\overline{\bigcup_n C_{B_n}} \in cwk(X)$. Hence $s(x^*, \overline{\bigcup_n C_{B_n}}) = \int_{\bigcup_n B_n} g_{x^*}\, d\lambda$ and therefore $\bigcup_n B_n \in \mathcal{B}$.

Let $(B_n)_n$ be a monotone decreasing sequence of $\mathcal{B}$, and let $C_{B_n} \in cwk(X)$ such that $s(x^*, C_{B_n}) = \int_{B_n} g_{x^*}\, d\lambda$ for every $x^* \in X^*$. Clearly $\lim_n \int_{B_n} g_{x^*}\, d\lambda = \int_{\bigcap_n B_n} g_{x^*}\, d\lambda$. Moreover, also $(C_{B_n})_n$ is a monotone decreasing sequence. Thus $\lim_n s(x^*, C_{B_n}) = s(x^*, \bigcap_n C_{B_n}) = s(x^*, \overline{\bigcap_n C_{B_n}})$. Moreover, $\overline{\bigcap_n C_{B_n}} \in cwk(X)$, because $\bigcap_n C_{B_n} \subseteq \Phi([0,1]) \in cwk(X)$. Hence $s(x^*, \overline{\bigcap_n C_{B_n}}) = \int_{\bigcap_n B_n} g_{x^*}\, d\lambda$ for every $x^* \in X^*$. Therefore $\bigcap_n B_n \in \mathcal{B}$.

By the Monotone Class Theorem (see [20, p. 15]), $\mathcal{B}$ contains the smallest σ-algebra containing $\mathcal{A}$. Hence $\mathcal{B} = \sigma(\mathcal{A})$.

Let us define $M : \sigma(\mathcal{A}) \to cwk(X)$ as follows: $M(B) = C_B$, $B \in \sigma(\mathcal{A})$.

M is a multimeasure, because for every $x^* \in X^*$, $s(x^*, M(\cdot))$ is a Lebesgue integral. Since M is $cwk(X)$-valued, M is also a d_H-multimeasure.

We prove that $M << \lambda$. In fact, if $B \in \sigma(\mathcal{A})$ and $\lambda(B) = 0$, then for every $x^* \in X^*$, $s(x^*, M(B)) = \int_B g_{x^*}\, d\lambda = 0$. Consequently, $||M(B)|| = \sup_{x^* \in B(X^*)} |s(x^*, M(B))| = 0$, hence $M(B) = \{0\}$.

It remains to prove that M is of σ-finite variation. Since $V_\Phi << \lambda$, we have that V_Φ is σ-finite. Let $(B_n)_n \subseteq \sigma(\mathcal{A})$ be a partition of $[0,1]$ such that $V_\Phi(B_n) < +\infty$ for every n. Fix n and let $\{B_{n,1}, \ldots, B_{n,k}\} \subseteq \sigma(\mathcal{A})$ be a partition of B_n. Then for every $x^* \in B(X^*)$ and every $j = 1, \ldots, k$ we obtain $s(x^*, M(B_{n,j})) = V_{s(x^*,\Phi)}(B_{n,j}) \le V_\Phi(B_{n,j})$. Hence for every $j = 1, \ldots, k$, $||M(B_{n,j})|| \le V_\Phi(B_{n,j})$ and therefore $\sum_{j=1}^k ||M(B_{n,j})|| \le V_\Phi(B_n)$. Finally, $|M|(B_n) \le V_\Phi(B_n) < +\infty$.

Since $M << \lambda$, we can extend M to $\mathcal{L}$, because any measurable set is the union of a Borel set and a set of zero Lebesgue measure. The proof is complete. □

Remark 4.5. The condition $s(x^*, \Phi(I)) \ge 0$ for every $x^* \in X^*$ and every $I \in \mathcal{I}$ implies that $0 \in \Phi(I)$ for every $I \in \mathcal{I}$.

Theorem 4.6. *Let X be a Banach space with the RNP and let $\Phi : \mathcal{I} \to cwk(X)$ be an interval multimeasure such that $V_\Phi << \lambda$. Then Φ admits a cbc(X)-valued density F which is HKP-integrable in cwk(X).*

Proof. Let us consider first the case when $s(x^*, \Phi) \geq 0$ for every $x^* \in X^*$. By Proposition 4.4, Φ can be extended to a σ-additive multimeasure $M : \mathcal{L} \to cwk(X)$ such that M is of σ-finite variation and $M << \lambda$.
Let $(A_n)_n$ be a sequence of pairwise disjoint sets of $\mathcal{L}$ such that $\bigcup_n A_n = [0,1]$ and $|M|(A_n) < +\infty$ for all n. Let us denote by M_n the restriction of M to all measurable subsets of A_n. Each M_n is a $cwk(X)$-valued (hence $cbc(X)$-valued) multimeasure of finite variation. Moreover, since $M << \lambda$, also $M_n << \lambda$, for all n.
Since X has the RNP, by [5, Theorem 4.1] we have that, for all n, M_n has a density $F_n : A_n \to cbc(X)$ which is Pettis integrable in $cbc(X)$.
Now let us define the multifunction $F : [0,1] \to cbc(X)$ as follows:

$$F(t) := F_n(t), \text{ if } t \in A_n.$$

We check that F is scalarly integrable. Let us fix $x^* \in X^*$. Since M is $cwk(X)$-valued, $s(x^*, M)$ is a positive (by construction) real-valued measure absolutely continuous with respect to λ. Therefore by the classic Radon-Nikodým Theorem, there exists $h_{x^*} \in L^1([0,1])$ such that

$$s(x^*, M(A)) = \int_A h_{x^*}\, d\lambda, \text{ for every } A \in \mathcal{L}.$$

Moreover, for each n, F_n is a Pettis integrable density of M_n, hence

$$s(x^*, M_n(A)) = \int_A s(x^*, F_n)\, d\lambda, \text{ for every } A \in \mathcal{L}, A \subseteq A_n.$$

It follows that for every n, $s(x^*, F_n) = h_{x^*}$ almost everywhere on A_n (and the exceptional set depends only on x^*).
By the definition of F we have also that $s(x^*, F) = h_{x^*}$ (and the exceptional set depends only on x^*). Therefore $s(x^*, F)$ is integrable. Since x^* is arbitrary, then F is scalarly integrable.
Finally we observe that for every $A \in \mathcal{L}$ and every $x^* \in X^*$,

$$s(x^*, M(A)) = \int_A h_{x^*}\, d\lambda = \int_A s(x^*, F)\, d\lambda.$$

Therefore F is a Pettis integrable (in $cwk(X)$) density of M. In particular,

$$\Phi(I) = (P)\int_I F\, d\lambda, \text{ for every } I \in \mathcal{I}.$$

In the general case, let ϕ be a finitely additive selection of Φ (existing by Proposition 3.5) and let consider $\Psi := \Phi - \phi$. It is clear that $s(x^*, \Psi) \geq 0$ for every $x^* \in X^*$. We have also that $V_\Psi << \lambda$, since $V_\Phi << \lambda$ and $V_\phi << \lambda$. Consequently, Ψ has a density $G : [0,1] \to cbc(X)$ Pettis integrable in $cwk(X)$.

Since X has the RNP, by [2, Theorem 3.6] ϕ has a variationally Henstock integrable (and then Henstock integrable) density $f : [0,1] \to X$ (see [2] or [19] for the definition of variational Henstock integral).
Now let us consider the multifunction $F := G + f$. Clearly F is $cbc(X)$-valued. Moreover, $s(x^*, F) = s(x^*, G) + \langle x^*, f\rangle$, for every $x^* \in X^*$. Since each $s(x^*, G)$ is Lebesgue integrable and each $\langle x^*, f\rangle$ is HK-integrable, also $s(x^*, F)$ is HK-integrable. Hence F is scalarly HK-integrable.
Finally for every $x^* \in X^*$ and for every $I \in \mathcal{I}$ we have

$$\begin{aligned} &s(x^*, \Phi(I)) = s(x^*, \Psi(I)) + \langle x^*, \phi(I)\rangle \\ &= \int_I s(x^*, G)\, d\lambda + (HK)\int_I \langle x^*, f\rangle\, d\lambda = (HK)\int_I s(x^*, F)\, d\lambda. \end{aligned}$$

We conclude that F is HKP-integrable in $cbc(X)$ and

$$\Phi(I) = (HKP)\int_I F\, d\lambda, \text{ for every } I \in \mathcal{I}$$

. □

Remark 4.7. In general, under the hypothesis of Theorem 4.6, the density of Φ is only $cbc(X)$ and not $cwk(X)$-valued, as the following example shows (see [6, Exemple 2]).
Let X be the space l_1 and let $(e_n)_{n\geq 0}$ be the canonical base of l_1. Let $(\alpha_n^k)_{n,k\geq 0}$ be a sequence of real numbers such that

$$\sum_{n\geq 0} |\alpha_n^k| = 1 \text{ for every } k \geq 0 \text{ and } \sum_{k\geq 0}\left(\sum_{n\geq 0} |\alpha_n^k|^2\right)^{\frac{1}{2}} < +\infty.$$

Let $(r_n)_{n\geq 0}$ be the sequence of the Rademacher functions. For $k \geq 0$ and $t \in [0,1]$, set $\sigma_k(t) := (\alpha_n^k r_n(t))_{n\geq 0} \in l_1$. Now let define $F(t) := \overline{co}\{\sigma_k(t) : k \geq 0\}$, $t \in [0,1]$. Then, the $cbc(l_1)$-valued multifunction F is Pettis integrable in $cwk(l_1)$, but $F(t) \notin cwk(l_1)$ a.e.

Remark 4.8. If the Banach space is the real line, then $cbc(\mathbb{R}) = ck(\mathbb{R}) = cwk(\mathbb{R})$ and the Henstock integrability coincides with the Henstock-Kurzweil-Pettis integrability. Let us give in such a case a simple proof of Theorem 4.6, using properties of the real line.

Proof of Theorem 4.6 for $X = \mathbb{R}$. Since Φ is $ck(\mathbb{R})$-valued, $\Phi(I)$ is a closed bounded interval of the real line for all $I \in \mathcal{I}$.
Let us consider the real functions $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ defined respectively by $\varphi(I) := \min \Phi(I)$ and $\psi(I) := \max \Phi(I)$.
Of course, φ and ψ are selections of Φ. Since $V_\Phi << \lambda$, we have $V_\varphi << \lambda$ and $V_\psi << \lambda$. So by [2, Theorem 3.6], φ and ψ are differentiable almost everywhere in $[0,1]$ and there exist $f, g \in \mathcal{HK}([0,1])$ such that $\varphi(I) = (HK)\int_I f\, d\lambda$

and $\psi(I) = (HK)\int_I g\, d\lambda$ for each $I \in \mathcal{I}$. Moreover, $\varphi' = f$ and $\psi' = g$ a.e. Since $\varphi \leq \psi$, we have $(HK)\int_I f\, d\lambda \leq (HK)\int_I g\, d\lambda$ for all $I \in \mathcal{I}$. Consequently, $f \leq g$ a.e.

Now let consider the multifunction F defined by

$$F(t) := \begin{cases} [f(t), g(t)] & \text{if } f(t) \leq g(t) \\ \{0\} & \text{elsewhere.} \end{cases}$$

It is clear that F is $ck(\mathbb{R})$-valued. Let γ be a selection of Φ. Since $V_\Phi << \lambda$, also $V_\gamma << \lambda$. Therefore by [3, Theorem 3], γ is differentiable almost everywhere in $[0,1]$ and there exists $h \in \mathcal{HK}([0,1])$ such that $\gamma(I) = (HK)\int_I h\, d\lambda$. Moreover, $\gamma' = h$ a.e.

Since $\varphi \leq \gamma \leq \psi$, then we get also that $f \leq h \leq g$ a.e. Consequently, $h(t) \in F(t)$ for almost every $t \in [0,1]$. So, changing eventually the values in a negligible set, we have that h is a selection of F.

Since $f, g \in \mathcal{HK}([0,1])$, for each $\varepsilon > 0$, there exists a gauge δ on $[0,1]$ such that

$$\left| (HK)\int_0^1 f\, d\lambda - \sum_{j=1}^p f(t_j)|I_j| \right| < \varepsilon/2$$

and

$$\left| (HK)\int_0^1 g\, d\lambda - \sum_{j=1}^p g(t_j)|I_j| \right| < \varepsilon/2,$$

for every δ-fine partition $\{(I_j, t_j)\}_{j=1}^p$ of $[0,1]$. Thus

$$\begin{aligned} & d_H\left(\Phi([0,1]), \sum_{j=1}^p F(t_j)|I_j|\right) \\ & = d_H\left(\left[(HK)\int_0^1 f\, d\lambda, (HK)\int_0^1 g\, d\lambda\right], \left[\sum_{j=1}^p f(t_j)|I_j|, \sum_{j=1}^p g(t_j)|I_j|\right]\right) \\ & \leq \left|(HK)\int_0^1 f\, d\lambda - \sum_{j=1}^p f(t_j)|I_j|\right| + \left|(HK)\int_0^1 g\, d\lambda - \sum_{j=1}^p g(t_j)|I_j|\right| < \varepsilon, \end{aligned}$$

for every δ-fine partition $\{(I_j, t_j)\}_{j=1}^p$ of $[0,1]$. Therefore F is Henstock integrable and $(H)\int_0^1 F\, d\lambda = \Phi([0,1])$.

Finally, using Hausdorff distance we infer for each $I \in \mathcal{I}$,

$$d_H\left(\Phi(I), (H)\int_I F\, d\lambda\right) \leq \left|\varphi(I) - (HK)\int_I f\, d\lambda\right| + \left|\psi(I) - (HK)\int_I g\, d\lambda\right| = 0.$$

Hence $\Phi(I) = (H)\int_I F\, d\lambda$ for every $I \in \mathcal{I}$ and the proof is over. $\square$

In [21] it has been proved the following result.

Theorem 4.9. *Let* X *be a separable Banach space with the RNP. Assume that also* X^* *has the RNP. Let* M *be a* $cwk(X)$*-valued multimeasure of* σ*-finite variation and such that* $M << \lambda$*. Then* M *admits a unique density* $F : [0,1] \to cwk(X)$ *which is Pettis integrable in* $cwk(X)$*.*

Under the same assumptions for the Banach space X, by previous theorem we have the following result for interval multimeasures.

Theorem 4.10. *Let* X *be a separable Banach space with the RNP. Assume that also* X^* *has the RNP. Let* $\Phi : \mathcal{I} \to cwk(X)$ *be an interval multimeasure such that* $V_\Phi << \lambda$*. Then* Φ *admits a cwk(X)-valued density* F *which is HKP-integrable in cwk(X).*

Proof. First let us consider the particular case when $s(x^*, \Phi) \geq 0$ for every $x^* \in X^*$. By Proposition 4.4, Φ can be extended to a σ-additive multimeasure $M : \mathcal{L} \to cwk(X)$ such that M is of σ-finite variation and $M << \lambda$. By hypothesis, X is separable, has the RNP and also its dual X^* has the RNP. Therefore by Theorem 4.9, M has a density $F : [0,1] \to cwk(X)$ which is Pettis integrable in $cwk(X)$. Consequently, we have

$$\Phi(I) = (P)\int_I F\, d\lambda, \text{ for every } I \in \mathcal{I}.$$

In the general case, let ϕ be a finitely additive selection of Φ and let consider $\Psi := \Phi - \phi$. It is clear that $s(x^*, \Psi) \geq 0$ for every $x^* \in X^*$. We have also that $V_\Psi << \lambda$, since $V_\Phi << \lambda$ and $V_\phi << \lambda$. Consequently, Ψ has a density $G : [0,1] \to cwk(X)$ Pettis integrable in $cwk(X)$. By [2, Theorem 3.6] ϕ has a variationally Henstock (then a Henstock) integrable density $f : [0,1] \to X$ (see [2] or [19] for the definition of variational Henstock integral).

Now let consider the multifunction $F := G + f$. Clearly F is $cwk(X)$-valued. Moreover, it is easy to check that $s(x^*, F) = s(x^*, G) + \langle x^*, f \rangle$, for every $x^* \in X^*$. Since each $s(x^*, G)$ is Lebesgue integrable and each $\langle x^*, f\rangle$ is HK-integrable, also $s(x^*, F)$ is HK-integrable. Hence F is scalarly HK-integrable. Finally for every $x^* \in X^*$ we have

$$\begin{aligned} &s(x^*, \Phi(I)) = s(x^*, \Psi(I)) + \langle x^*, \phi(I)\rangle \\ &= \int_I s(x^*, G)\, d\lambda + (HK)\int_I \langle x^*, f\rangle\, d\lambda = (HK)\int_I s(x^*, F)\, d\lambda, \end{aligned}$$

for every $I \in \mathcal{I}$. We conclude that F is HKP-integrable in $cwk(X)$ and

$$\Phi(I) = (HKP)\int_I F\, d\lambda, \text{ for every } I \in \mathcal{I}.$$

□

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