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Mean-field game modeling the bandwagon effect with activation costs

Fabio Bagagiolo · **Dario Bauso** · **Raffaele Pesenti**

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Abstract This paper provides a mean-field game theoretic model of the bandwagon effect in social networks. This effect can be observed whenever individuals tend to align their own opinions to a mainstream opinion. The contribution is three-fold. First, we describe the opinion propagation as a mean-field game with local interactions. Second, we establish *mean-field equilibrium* strategies in the case where the mainstream opinion is constant. Such strategies are shown to have a threshold structure. Third, we extend the use of threshold strategies to the case of time-varying mainstream opinion and study the evolution of the macroscopic system.

1 Introduction

In the last years, there were many examples of situations where social networks had an effect on political and socio-economic events. Thus a rigorous study of the mutual influence between individuals' opinions and population's mainstream opinion has attracted attention of scientists in different disciplines such as engineering, economics, finance and game theory, just to name a few.

A common observation is that in most cases opinions evolve following so-called averaging processes [3,5,11, 20,16,24]. It has been shown, in [10], that when agents have local interactions, that is, agents talk only with those "who think similarly" (in the parlance of social science this is called *homophily*), the macroscopic behavior yields clusters of opinions, representing separate groups, parties, or communities. On the other hand, when the interaction is global, that is, every agent interacts with all the other agents, the opinions may converge to a unique consensus-value, see, e.g., Result 1 in [16]. The literature offers a variety of Lagrangian and Eulerian models to model opinion dynamics [1]. Within the framework of the Eulerian approach, this paper provides a

F. Bagagiolo is with Dipartimento di Matematica, Università di Trento, Via Sommarive 14, 38050 Povo-Trento, Italy `bagagiol@science.unitn.it`, tel: +39 0461 281638, fax: +39 0461 281624. Corresponding author.

D. Bauso is with Department of Automatic Control and Systems Engineering, The University of Sheffield, Mappin Street Sheffield, S1 3JD, United Kingdom, and with Dipartimento di Ingegneria Chimica, Gestionale, Informatica, Meccanica, Università di Palermo, V.le delle Scienze, 90128 Palermo, Italy, `dario.bauso@unipa.it`

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R. Pesenti is with the Dipartimento di Management, Università "Ca'Foscari" Venezia, Cannaregio I-30121 Venezia, Italy `pesenti@unive.it`

Address(es) of author(s) should be given

mean-field game theoretic perspective on the problem. We focus on the *bandwagon effect*, that is the situation in which the tendency of "any individual adopting an opinion increases with the proportion of those who have already done so" [12]. We show that the presence of an activation cost attenuates the bandwagon effect as the agents face a trade-off between maintaining their opinion (and paying for "thinking differently") and tracking the mainstream opinion at a given cost. Specifically, we show that the agents, whose opinion is far from the mainstream one, may be still willing to change their opinions toward this latter opinion; on the contrary the agents, whose opinions are already close to the mainstream one, may consider not worth facing the activation cost to get closer to that target.

Note that a mean-field game model arises as the cost of each agent depends also on the mode (the mainstream) of the actual distribution of the agents, which, in turns depends on the optimal choices of the agents.

In the context of opinion dynamics, an Eulerian description is used when the focus is on how the distribution of the players over a set of n opinions $X = \{x_i : i = 1, \dots, n\}$ evolves over time and, hence, a vector function $m(.) = \{m_i(.) : i = 1, \dots, n\}$ is to be determined, where $m_i(.) : [t, T] \to [0, 1]$ indicates the percentage $m_i(s)$ of the players whose opinion is x_i at time $s \in [t, T]$. An Eulerian description can be associated to a *network of opinions* $G = (X, A_1, A_2)$. The network G includes a node for each opinion x_i and has two sets A_1 and A_2 of arcs. The set A_1 includes an arc (x_i, x_j) if the players with opinion x_i may change their opinion into x_j and vice versa. The set A_2 includes an arc (x_i, x_j) if the players with opinion x_i may be influenced in changing their opinion by the presence of players with opinion x_j and vice versa. Weight coefficients may be associated to the arcs in A_2. The weight $h(x_i, x_j)$ measures the influence of players with opinion x_j on the players with opinion x_i and vice versa.

Figure 1 depicts three opinion networks of respectively 4, 8, 16 opinions. The arcs in A_1 are drawn in the upper part of the figure, while the arcs in A_2 are shown in the lower part. The weights of the arcs in A_2 are represented by a different width of the arcs. Specifically, the networks in Fig. 1 model situations in which the players change their opinion gradually, they move at most from an opinion to the closest one at each time instant, and are homophile, namely they put a larger weight on the players with an opinion close to theirs.

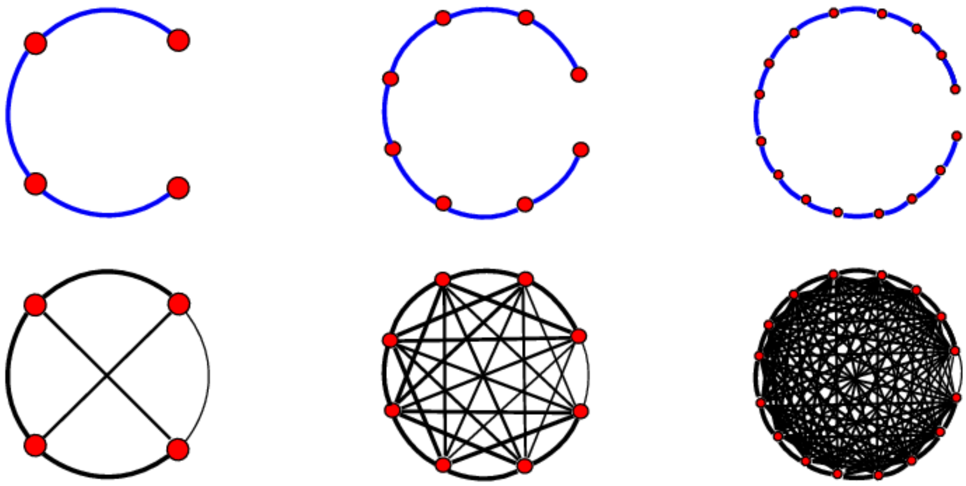

Fig. 1 Examples of three opinion networks of respectively 4, 8, 16 opinions.

The problem dealt with in this paper considers the asymptotic limit situation when the set of opinions X can be seen as the union of continuous intervals of smaller and smaller length. This problem can be cast within the area of mean-field game theory. One aspect that the game at hand shares with many other models available in the literature is the "crowd-seeking" attitude of the players. Indeed, while targeting a common value, in the specific case this is the mainstream opinion, the players move toward states characterized by a higher density of opinions. In some cases, at the end of the horizon we may also note that opinions concentrate in one point thus generating a Dirac impulse in the plot of the density function. However, two elements distinguish the game at hand from other classical models available in the literature. First, the target value is the mode of the density distribution, and as such it results from every player searching the maximum in a suitable neighborhood of their opinion. As a consequence of such a local maximization procedure, the time evolution of the target (the mode computed locally by each player) is in general nonlinear and may also be discontinuous. This is in contrast with most literature which usually considers the local mean rather than the mode which results in a smooth trajectory for the target. A second distinction is in the structure of the cost function. This involves a fixed cost on control which makes the problem highly nonlinear. Other works of the authors feature fixed costs in the context of decision making with large number of players [9,23]. We deal with this issue using an approach based on the partition of the horizon window in two parts (one when the control is non-null and the other when the control is null). This is only possible under certain conditions which we will analyze and highlight throughout the paper.

Description of the contents. Initially, we provide a mean-field game type model that describes the opinion propagation under local interaction (Section 2).

As in a standard mean-field game, under the assumption of a population of infinitely many homogeneous agents/players, we are interested in determining the agents' optimal strategies as functions of their state, possibly of the time, and of the macroscopic behavior of the whole population. Such optimal strategies have to induce a so-called mean-field equilibrium. In a mean-field equilibrium no player can benefit from a unilateral deviation similarly to the definition of Nash equilibrium strategies in noncooperative n-player games [7].

The classical structure of mean-field games consists of two partial differential equations (PDEs), that define the necessary optimality/equilibrium conditions. The first PDE is the Hamilton-Jacobi-Bellman equation (HJB), which returns the optimal individual response to the population mean-field behavior. The second PDE is the Fokker-Planck equation (FP) (just a transport equation in the deterministic case, as in the present paper) which describes the density evolution of the players as a result of the implementation of the individually optimal strategies.

Unfortunately, given the structure of the agents' cost considered in this paper (which takes into account an activation cost and the mode of the population's density function, see below for definitions and comments), the derivation of the corresponding couple HJB-FP equations seems to be not immediate (and also not tractable with standard techniques for uniqueness and regularity). Indeed, up to our knowledge, there are not similar cases studied in the literature.

Hence, our approach consists in three steps.

First step. We show the optimality of a given class of strategies (threshold ones) under specific conditions on the population distribution. More specifically, we show that, supposing the mainstream to be constant, each agent, also due to the presence of the activation cost, will implement a so-called "non-idle" strategy, that is, a control that can switch from non-null to null only but not vice-versa. Moreover, such strategies are also shown to be "threshold strategies", that is, the control is non-null only if the target point (the mainstream opinion) is sufficiently far from the current opinion. This corresponds to saying that for a control to be non-null the distance between the mainstream and the player's opinion must be greater than a given time-varying threshold. The search for the optimal control strategies, conducted in this step, builds upon the notion of *dominance*. A strategy is dominated if there exists at least one other better strategy in the sense that the value of the cost function associated to this latter strategy is strictly less than the one associated to the former strategy. Obviously, a dominated solution is not optimal and can be excluded from the search of the optimum. By analogy, a solution for which no strictly better solution exists will be called *non-dominated*.

Second step. Our interest is in the particular mean-field equilibria for which the distribution of the population remains constant (see the end of the next session for a detailed definition). The threshold strategies of the previous step are then mean-field equilibrium strategies for any initial distribution of the population for which the corresponding mode, when supposed constant, induces an optimal threshold strategy with control always null. This means that no one of the agents is worth changing its state, hence nothing moves and we get an equilibrium. We exhibit some of such initial distributions. Moreover, we also analyze the case where the agents have a distorted perception of the mode.

Third step. In the previous step, we presented situations in which threshold strategies are optimal and establish a mean-field equilibrium. Here, we argue that these strategies, even if they are not optimal, are of interest even in some more general contexts. Specifically, the can be implemented by players with bounded rationality and who use myopic descriptive models of the “world”. This means that, at each time instant, they make a decision assuming that the population macroscopic behavior will freeze from that time instant on.

In this situation, for some initial distributions, we describe the possible time evolution of the population distribution in terms of weak solutions of the associated FP equations (the transport equation). Due to the threshold feature of the optimal control, the proposed solution of the equation may immediately present discontinuities which also evolve in time. This is a common feature of the theory of scalar conservation laws, however, we point out that in our case the situation is quite different. Indeed, such discontinuities derive from the presence of the threshold which evolves in the opposite direction with respect to the flow of the agents, whereas in the classical theory of shocks for scalar conservation laws, the discontinuities and the flow evolve in the same direction. This seems to be an interesting feature of this equation which may deserve a deeper study. Finally, we investigate some convergence results.

Related literature on mean-field games. The theory of mean-field games originates in the paper by Lasry and Lions [22] and independently in that of M.Y. Huang, P. E. Caines and R. Malhamé [17–19]. This theory studies interactions between indistinguishable individuals and the population around them. Among the foundations of the theory is the assumption that the population influences the individuals' strategies through mean-field parameters. Several application domains such as economics, physics, biology, and network engineering accommodate mean-field game theoretic models (see [2, 15, 21, 28]). For instance: in multi-inventory systems mean field games may be useful to capture coordination of reordering policies when set up costs depend on the number of active players, the latter being all those who place joint-orders to a warehouse [23]; decision problems with mean-field coupling terms have also been formalized and studied in [9]; applications to power grids management are recently studied in [4].

The literature provides explicit solutions in the case of linear-quadratic structure, see [6]. In most other cases, a variety of solution schemes have been proposed which make use of discretization techniques and numerical approximations [2]. Robustness and risk-sensitivity are also open lines of research as evidenced in [8, 27]. We refer the reader to [14] for a recent survey.

We finally observe that, in the different case of the presence of a stochastic noise in the dynamics, some partial results were already published in [26].

2 The game

Consider a population of homogeneous players, each one characterized by an opinion $x(s) \in \mathbb{R}$ at time $s \in [0, T]$, where $[0, T]$ is the time horizon window.

The control variable is a measurable function of time $u(\cdot)$, $s \mapsto u(s) \in \mathbb{R}$ and establishes the rate of variation of the player's opinion. It turns out that, for a fixed initial time $t \in [0, T]$ and initial status $x \in \mathbb{R}$ the opinion dynamics can be written in the form

$$\begin{cases} x'(s) = u(s), \ \ s \in [t, T], \\ x(t) = x_0. \end{cases} \tag{1}$$

Consider a probability density, in the most favorable case given by a function $m : \mathbb{R} \times [t, T] \to \mathbb{R}$, $(x, s) \mapsto m(x, s)$, representing the percentage of players in state x at time s, which satisfies $\int_{\mathbb{R}} m(x, s)dx = 1$ for every $s \in [t, T]$. Let us also denote the *mainstream opinion* at time s (i.e. the mode of the distribution) as $\overline{m}(s) = \arg\max_y \{m(y, s)\}$, or, more generally the *mainstream opinion as perceived by* a player in state x at time s, as $\overline{m}[x](s) = \arg\max_y \{h(|x - y|)m(y, s)\}$. The latter, as formally defined in the following section, is a distorted mode, as perceived by the players that use a function $h(.)$ to weigh the other opinions based on a distance measure defined in the space of opinions.

The objective of a player with opinion x is to adjust its opinion based on the (possibly perceived) mainstream opinion $\overline{m}[x](s)$. In other words, a player feels more and more rewarded when its and the mainstream opinion get closer and closer. Then, we assume that the players

consider a running cost $g : \mathbb{R}^3 \to [0, +\infty)$, $(x, \overline{m}, u) \mapsto g(x, \overline{m}, u)$ of the form:

$$g(x, \overline{m}, u) = \frac{1}{2}\left[q\Big(\overline{m} - x\Big)^2 + ru^2\right] + K\delta(u), \tag{2}$$

where q, r and K are constant positive values; and $\delta : \mathbb{R} \to \{0, 1\}$ is defined as

$$\delta(u) = \begin{cases} 0 & \text{if } u = 0 \\ 1 & \text{otherwise} \end{cases}. \tag{3}$$

When $\overline{m}$ represents the (perceived) mainstream opinion by an agent with opinion x, then the running cost $g(\cdot)$ given by (2)-(3) penalizes the square deviation of the player's opinion from the mainstream opinion and involves also a penalty term on the energy of control (a quick change of opinion has a greater cost than a slow change), and a fixed cost on control. The latter term captures the level of stubbornness of the players, in that a greater fixed cost increases the inertia of the players, namely, the tendency to maintain their original opinion in spite of the different inputs received from everybody else with a different opinion in the population.

At the end of the horizon, every player pays a quantity that equals the square deviation of its opinion from the mainstream. This is modeled by a terminal penalty $\Psi : \mathbb{R}^2 \to [0, +\infty)$, $(\overline{m}, x) \mapsto \Psi(\overline{m}, x)$ of the form

$$\Psi(\overline{m}, x) = \frac{1}{2} S\Big(\overline{m} - x\Big)^2, \tag{4}$$

where S is scalar and positive.

The problem in its generic form is the following:

Problem 1 Given a finite horizon $T > 0$, an initial distribution of opinions m_0, a running cost g as in (2), a final cost Ψ as in (4), and given the dynamics for $x(\cdot)$ as in (1), minimize over $\mathcal{U}$ the following cost functional,

$$J(x_0, t, u(\cdot)) = \int_t^T g(x(s), \overline{m}[x(s)](s), u(s))ds + \Psi(\overline{m}[x(T)](T), x(T)) \tag{5}$$

where $\overline{m}[x(\cdot)](\cdot)$ as time-dependent function is the evolution of the mainstream opinion as a function of the players' strategies and $\mathcal{U}$ is the set of all measurable functions from $[t, T]$ to $\mathbb{R}$.

In the above problem, the players show a crowd-seeking behavior in that they target a common value, the mainstream opinion. While doing this, the players move toward states characterized by a higher density of opinions. This may result in a Dirac impulse, a concentration of opinions in one point of the opinion space, at the end of the game. In this respect, a first issue is that the considered target value or mainstream opinion is the mode of the density distribution which must be computed via local maximization on the part of each player. This implies that the mainstream opinion is, in general, discontinuous in time and space. A second challenging aspect is that the cost functional includes a fixed cost on control which makes our problem different from classical linear quadratic ones.

In general, one would like to use tools from dynamic programming, such as the Hamilton-Jacobi-Bellman equation. However, we highlight next a few issues. First, besides the already mentioned possible discontinuity in time, the mainstream may not be well defined when the maximum of the density function m is reached in more than one point; in such a case it is not immediate to detect which value to insert in the costs g and Ψ. In addition, when the density function is a discontinuous function, it is not clear what is the mode of it. For example, if the maximum of m is a point x where m has a large negative "jump", may such x be considered as the mode? To deal with the above issues, in the following, we provide an ad-hoc definition of mode.

For a further clarification of the language used in the sequel of the paper, hereinafter, we give the following definition.

Definition 1

- a player is *active* (*inactive*) if its control is *non-null* (*null*)
- a control u is *non-null* (*null*) in a interval $[t_1, t_2] \subseteq [t, T]$ if $\int_{t_1}^{t_2} \delta(u) d\tau = t_2 - t_1$, i.e. if it is non-null almost everywhere ($\int_{t_1}^{t_2} \delta(u) d\tau = 0$, i.e. if it is null almost everywhere), that is, if the measure of the set in which $u = 0$ ($u \neq 0$) in $[t_1, t_2]$ is zero
- a control u is *non-idle* if there are not two intervals $[\hat{t}_1, \hat{t}_2], [t_1, t_2] \subset [t, T]$ such that $\hat{t}_2 \leq t_1$ and u is null in $[\hat{t}_1, \hat{t}_2]$ and non-null in $[t_1, t_2]$. A player using non-idle control is i) always active, ii) always inactive or iii) first active and then inactive. In no case a player can switch from being inactive to active. We call *switching time instant* the time when a player turns to be inactive, with the understanding that if the player is always active as in case i) then $\hat{s} = t$; if the player is always inactive as in case ii) then $\hat{s} = T$; and finally, if the player is first active and then inactive as in case iii) then $\hat{s} = \inf\{s \in [t, T] : u(s) = 0\}$
- In contrast, we call *idle* all the policies that do not match the above criterion, i.e., policies that admit a switch from a null to a non-null control.
- a control u^a is *dominated* by u^b, for a given pair $(x_0, t) \in \mathbb{R} \times [0, T]$, if

$$J(x_0, t, u^a) - J(x_0, t, u^b) > 0$$

- a control u is *optimal*, for a given pair $(x_0, t) \in \mathbb{R} \times [0, T]$, if it is not dominated
- a strategy u (i.e. a control in a feedback form) induces an equilibrium at time t if it is optimal for any x_0 in the support of $m(x, t) = m_0(x)$ and if its application by the agents implies that $m(x, s) = m_0(x)$ for all $t \leq s \leq T$, i.e. the distriubtion remains constant (and so its mode too.). Specifically, we say that the couple (u, m_0) defines an *equilibrium*.

We stress that the above definition delimits the scope of this work. Specifically, it indicates the particular kind of equilibrium that we consider. This kind of equilibrium requires that the population distribution remains constant over time. As we pointed out in the introduction, more general equilibria may possibly exist that just require that no player can benefit from a unilateral deviation from its current behavior.

3 Constant mainstream and global interaction

In this section, we assume that the mainstream opinion is constant all over the horizon window. This allows us to introduce the notion of threshold strategy which will play a role throughout the paper. This situation is of particular interest for two reasons. On the one hand, it allows to identify the structure of an equilibrium policy. Indeed, in equilibrium the population distribution $m(.,.)$ remains constant over time by definition and then also the mainstream opinion perceived by the agents. On the other hand, in the second part of this work, we will study the consequences of applying such threshold strategies to a more general setting, where the perceived mainstream is possibly time-varying.

Assuming a time-invariant mainstream implies the equality $\overline{m}(\cdot) = \overline{m}$, where $\overline{m}$ is constant. We are then interested in determining the solution of the following auxiliary problem.

Problem 2 Considering a dynamics for x as in (1), minimize over $\mathcal{U}$ the following cost functional

$$J(x(t), t, u(\cdot)) = \int_t^T \left(\frac{1}{2} q \Big(\overline{m} - x \Big)^2 + \frac{1}{2} r u^2 + K\delta(u) \right) ds + \frac{1}{2} S \Big(\overline{m} - x(T) \Big)^2. \tag{6}$$

Note that Problem 2 would be a classical linear quadratic tracking problem with constant reference signal, were it not for the presence of the fixed cost $K\delta(u)$. In order to restrict the set of candidate optimal control strategies, in the following we introduce some additional properties that characterize a non-dominated solution.

3.1 Properties of non-dominated control

There are control policies that can immediately be labeled as non optimal as they are *trivially dominated*. These policies include any control u^a non-null in a measurable interval $[t_1, t_2] \subseteq [t, T]$ that induces a trajectory such that $\Big(\overline{m} - x(t_1)\Big)^2 \leq \Big(\overline{m} - x(t_2)\Big)^2$. Actually, such a control would induce a fixed cost $K\delta(u^a)$ in addition to an increased cost derived from pushing the player's state away from the target $\bar{m}$. Control policies that are not trivially dominated are called *non-trivially dominated.*

A second class of policies that do not fit in the set of optimal solutions involves all policies that are idle. Actually, the following lemma proves that a control that is idle is dominated.

Lemma 1 *Consider a non-trivially dominated control $u \in \mathcal{U}$. If u is idle then u is dominated.*

Proof This proof is based on a direct comparison of the cost induced by u with the cost of another solution, say u_b. If u is idle then there exist three time instants t_1, $t_1 + \Delta t$ and t_2, with $t \leq t_1 < t_1 + \Delta t < t_2 \leq T$ such that $[t_1, t_1 + \Delta t]$ and $[t_1 + \Delta t, t_2]$ are intervals and u is null in $[t_1, t_1 + \Delta t]$ and is non-null in $[t_1 + \Delta t, t_2]$. We prove that u is dominated by a control u_b defined as follows:

$$u^b(t) = \begin{cases} u(t), & t \notin [t_1, t_2] \\ u(t + \Delta t), & t \in [t_1, t_2 - \Delta t) \\ 0, & t \in [t_2 - \Delta t, t_2] \end{cases}.$$

Under both controls u and u^b, the player state evolves so that $x(t_1) = \int_t^{t_1} u(\tau)d\tau + x_0 = \int_t^{t_1} u^b(\tau)d\tau + x_0$, and $x(t_2) = \int_t^{t_2} u(\tau)d\tau + x_0 = \int_t^{t_2} u^b(\tau)d\tau + x_0$. Consequently, the costs induced by the two controls are equal for $0 \leq t \leq t_1$ and $t_2 \leq t \leq T$, as in such intervals the two controls assume the same values and induce the same states for the player.

Then consider the interval $[t_1, t_2]$, the cost paid by u is

$$\int_{t_1}^{t_1+\Delta t} \frac{1}{2}q\Big(\overline{m} - x(t_1)\Big)^2 dt + \int_{t_1+\Delta t}^{t_2} \frac{1}{2}q\Big(\overline{m} - x(t)\Big)^2 dt + \int_{t_1+\Delta t}^{t_2} \frac{1}{2}ru(t)^2 dt + \int_{t_1+\Delta t}^{t_2} K dt.$$

Differently, the cost paid by u^b is (here $x^b(\cdot)$ is the trajectory under the control u^b)

$$\int_{t_1}^{t_2-\Delta t} \frac{1}{2}q\Big(\overline{m} - x^b(t)\Big)^2 dt + \int_{t_2-\Delta t}^{t_2} \frac{1}{2}q\Big(\overline{m} - x(t_2)\Big)^2 dt + \int_{t_1}^{t_2-\Delta t} \frac{1}{2}ru(t + \Delta t)^2 dt + \int_{t_1}^{t_2-\Delta t} K dt.$$

Now, by the almost obvious equalities $\int_{t_1+\Delta t}^{t_2} K dt = \int_{t_1}^{t_2-\Delta t} K dt$, $\int_{t_1+\Delta t}^{t_2} \frac{1}{2}ru(t)^2 dt = \int_{t_1}^{t_2-\Delta t} \frac{1}{2}ru(t + \Delta t)^2 dt$, and $\int_{t_1+\Delta t}^{t_2} \frac{1}{2}q\Big(\overline{m} - x(t)\Big)^2 dt = \int_{t_1}^{t_2-\Delta t} \frac{1}{2}q\Big(\overline{m} - x^b(t)\Big)^2 dt$, we get that

$$J(x_0, t, u) - J(x_0, t, u^b) = \frac{1}{2}q\left(\Big(\overline{m} - x(t_1)\Big)^2 - \Big(\overline{m} - x(t_2)\Big)^2\right)\Delta t > 0,$$

which is positive, as u is non-trivially dominated. Hence, $J(x_0, t, u) > J(x_0, t, u^b)$, which proves the lemma. □

Consider, for each $\hat{t} \in [t, T]$, the following cost functional, which is defined as in (6) but only on control strategies with switching time $\hat{t}$, namely, a control strategy characterized by a non-null control only up to time $\hat{t}$, after which the control is set to zero:

$$\hat{J}(x_0, t, \hat{t}, u(\cdot)) = \int_t^{\hat{t}} \left(\frac{1}{2}q\Big(\overline{m} - x(\tau)\Big)^2 + \frac{1}{2}ru(\tau)^2 + K\right) d\tau + \frac{1}{2}(S + q(T - \hat{t}))\Big(\overline{m} - x(\hat{t})\Big)^2. \tag{7}$$

Due to the quadratic structure of (7), there always exists a unique optimal solution in $\mathcal{U} \times [t,T]$ for the problem $\hat{J}^*(x_0,\hat{t}) = \min_{(u,\hat{t})\in\mathcal{U}\times[t,T]}\{\hat{J}(x_0,t,\hat{t},u(\cdot)) \; s.t. \; \dot{x}=u\}$.

In the light of these considerations, and from Lemma 1, we have the following theorem.

Theorem 1 (Threshold optimal policy) *Given (x_0,t), there is a unique solution*

$$(u^*,\hat{t}^*) = arg \min_{(u,\hat{t})\in\mathcal{U}\times[t,T]} \hat{J}(x_0,t,\hat{t},u(\cdot))$$

where the control u^* is given by (in a form depending on s and on the actual state $x(s)$):

$$u^*[(x_0,t)](s) = \begin{cases} a[(x_0,t)](s)(\bar{m}-x(s)) & for\ t \le s < \hat{t}^* \\ 0 & for\ \hat{t}^* \le s < T, \end{cases} \tag{8}$$

where $a[(x_0,t)] : [t,T) \to [0,+\infty)$ is a non negative time-varying function strictly decreasing in $[t,\hat{t}^*]$ whose structure depends on the initial choice (x_0,t). Furthermore, given Problem 2, no control in $\mathcal{U}$ dominates u^*, that is Problem 2 has an optimal control which is non-idle.

In addition, there exists a time-varying threshold function $\lambda : [t,T) \to \mathbb{R}$ such that $u^*[(x_0,t)](s)$ can be rewritten as time-varying state-feedback strategy

$$u^*[\phi(x(s),s),t](s) = \begin{cases} 0 & if\ |\overline{m}-x(s)| \le \lambda(s), \\ a[\phi(x(s),s),t](s)(\bar{m}-x(s)) \neq 0 & otherwise, \end{cases} \tag{9}$$

where $\phi : \mathbb{R}\times[t,T] \to \mathbb{R}$ is a function that, given a time s and a state $x(s)$, returns the unique initial state x_0 from which we can reach $x(s)$ at time s.

Function $\lambda(\cdot)$ is increasing in time and equal to

$$\lambda(s) = \frac{\sqrt{2Kr}}{q(T-s)+S}\ , \quad t \le s \le T. \tag{10}$$

Proof We determine the unique solution of problem $\hat{J}^*(x_0,t) = \min_{(u,\hat{t})\in\mathcal{U}\times[t,T]}\{\hat{J}(x_0,t,\hat{t},u(\cdot)) \; s.t. \; \dot{x}=u\}$, where $\hat{J}(\cdot)$ is defined as in (7), through the application of the maximum principle that leads to the following conditions:

$$\dot{x}(s) = u(s) \tag{11a}$$
$$\dot{\xi}(s) = q(x(s)-\bar{m}) \tag{11b}$$
$$\xi(s) = -ru(s) \tag{11c}$$
$$x(t) = x_0 \tag{11d}$$
$$\xi(\hat{t}^*) = -(S+q(T-\hat{t}^*))(x(\hat{t}^*)-\bar{m}) \tag{11e}$$
$$\frac{1}{2}ru^2(\hat{t}^*) = K \tag{11f}$$

where $\xi(s)$ represents the generic player costate.

The structure of the optimal control as in (8) is a straightforward consequence of the solution of the conditions (11a) - (11e). The existence of function $\phi(\cdot)$ is a consequence of the uniqueness of the optimal control. Indeed, this property implies that a state $x(s)$ can be reached at time s starting from a unique initial state x_0 at time t if the optimal control is applied.

Finally, we observe that through conditions (11c), (11e) and (11f), we have an expression for the player state $x(\hat{t}^*)$, where $\hat{t}^*$ is the switching time. In particular, we obtain $|x(\hat{t}^*)-\bar{m}| = \frac{\sqrt{2Kr}}{q(T-\hat{t}^*)+S}$ and, hence (9) and (10). □

Figure 2 depicts a qualitative plot of the state trajectories and of the threshold function for the optimal control strategy given in (8).

Note that if $|\overline{m} - x_0| \leq \frac{\sqrt{2Kr}}{q(T)+S}$ then the corresponding control strategy yields controls constantly null over the horizon. On the other hand, if $|\overline{m} - x(T)| > \frac{\sqrt{2Kr}}{S}$ then the corresponding control strategy yields controls constantly non-null over the horizon.

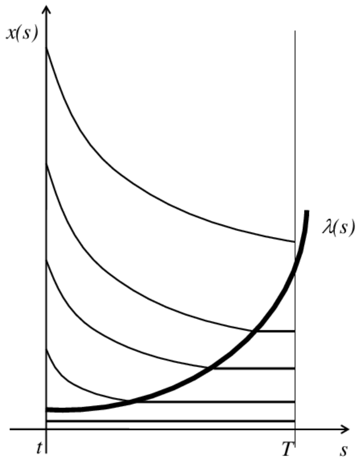

Fig. 2 Example of possible state trajectories for different initial states $x(t)$ and of the threshold function $\lambda(s)$.

Remark 1 Theorem 1 states that the presence of an activation cost attenuates the bandwagon effect and formalizes in which way the attenuation operates:

- the control policy (9) indicates that the players who draw their opinion toward the mainstream opinion, are those whose opinions are away from the mainstream by more than the value indicated by the threshold; differently, the players, whose opinions differ from the mainstream of a value smaller than or equal to that of the threshold, do not pay the activation cost and keep their opinion fixed.
- the structure (10) of threshold $\lambda(.)$ indicates that the smaller the activation cost the smaller the attenuation of the bandwagon effect. The attenuation disappears when the activation cost is null.

In the following section we will also provide some sufficient condition under which the bandwagon effect disappears due to the presence of a high activation cost.

4 Threshold policy and local interactions

In the previous section we focused on individually optimal strategies. That is, we proved that equilibrium strategies are threshold ones exploiting the fact that the mainstream opinion $\overline{m}(\cdot)$ is constant and equal to a given value $\overline{m}$. In this section, we focus on the macroscopic system described by the "population distribution". We first present some sufficient conditions that help to identify initial distributions that together with the threshold policy derived in the previous section define an equilibrium. After doing this, we study the problem faced by partially rational players that are not able to determine an optimal strategy. Specifically, we consider the case in which the players, given the results for the constant mainstream opinion case, behave myopically and make decisions freezing all future values of the mainstream opinion to the last available measured value. In other words, at each time instant, the players act as if the current mainstream opinion would remain constant all over the rest of the horizon window.

In the sequel we make the following assumptions.

Assumptions 1

1. *The density function m is continuous on its support.*
2. *The mainstream opinion as perceived by a player in state $x(s)$ at time s is a (distorted) mode defined as $\overline{m}[x(s)](s) = arg\max_y\{h(|x(s)-y|)m(y,s)\}$, where $m(y,s)$ is the density of the players' states in y at time s and $h:[0,+\infty) \to [0,+\infty)$ is a continuous fading function, non-increasing such that $h(0)=1$. In presence of multiple distorted modes function $arg(\cdot)$ returns the distorted mode closest to $x(s)$. In presence of two closest distorted modes equidistant from $x(s)$ function $arg(\cdot)$ returns the closest distorted mode on the left of $x(s)$.*
3. *The players implement a threshold strategy as in (9)*

$$u^*(x(s)) = \begin{cases} 0 & \text{if } |\overline{m}[x(s)](s) - x(s)| \leq \lambda(s), \\ a(s)(\overline{m}[x(s)](s) - x(s)) \neq 0 & \text{otherwise.} \end{cases} \tag{12}$$

in spite of the possible time-varying nature of $\overline{m}(\cdot)$. In particular, as $\lambda(s) > 0$ for $t \leq s \leq T$, $u^*(x(s))$ switches abruptly to 0 when there exists $\varepsilon > 0$ such that $\overline{m}[x(\tau)](\tau) \neq x(\tau)$ for $s-\epsilon \leq \tau < s$ and $\overline{m}[x(s)](s) = x(s)$ so that

$$m(x(s),s) \leq \sup_{s-\epsilon\leq\tau<s}\{h(|x(\tau) - \overline{m}[x(\tau)](\tau)|)m(\overline{m}[x(\tau)](\tau),\tau)\}. \tag{13}$$

Under the above assumption we are interested in studying the game evolution. To this end, we preliminarily determine under what initial opinion density distribution functions the controls, given by the threshold strategy, are null for all the players, i.e., $u^*(x(s)) = 0$ for all $x(s)$ belonging to the support, $supp(m(.,s))$, of $m(.,s)$. Hereafter, we refer to such distributions as Null Control Inducing (NCI) distributions.

4.1 Null Control Inducing distributions

Lemma 2 *Let a threshold strategy (9) be implemented and Assumptions 1 hold. An initial distribution m_0 is NCI iff*

$$\forall x \in supp(m_0(.))\ \exists y: \ |y-x| \leq \lambda(t),\ m_0(y)h(|y-x|) \geq m_0(z)h(|z-x|), \forall z. \tag{14}$$

Proof By definition of threshold strategy (9), $u^*(x,t) = 0$ iff $|\overline{m}_0[x] - x| \leq \lambda(t)$. Since $m_0(y)h(|y-x|) \geq m_0(z)h(|z-x|)$ for all z then we have $y = \overline{m}_0[x]$ and $|y-x| \leq \lambda(t)$ and the lemma is proved. □

We can restate the above lemma saying that m_0 is NCI, if each player with opinion x thinks that the mainstream opinion is within $I_t(x)$, where $I_t(x)$ is the neighborhood of x with radius $\lambda(t)$. Then, it is immediate to verify that the uniform distribution is NCI as well as, the limit case of a distribution with a single point mass (a Dirac impulse in a single point). In the former case each player considers its opinion the mainstream one. In the latter case all the players' opinions have a same value.

Remark 2 The interest in NCI distributions is given by the fact that, together with a threshold strategy (12), they define an equilibrium (see Definition 1). Indeed, being $u^*(x(s)) = 0$ for all $x(s)$ in the support of $m(.,s)$, then $m(.,s)$, and hence its (distorted) mode $\bar{m}$ too, remains constant in time and, by the result of the previous section, the threshold strategy is then optimal. We recall that the value of $\lambda(s)$ in (12) is independent of $\bar{m}(s)$ and increases over time for $t \leq s \leq T$.

Hence, in this particular case, we almost recover the usually fixed-point definition of an equilibrium for a mean-field game, that is: the pair $(given u^*, m_0)$ given by a the threshold strategy and a NCI distribution is a fixed point solution of the following system

$$\begin{cases} \min_u \int_t^T g(x(s), \overline{m}[x(s)](s), u(s))ds + \Psi(\overline{m}[x(T)](T), x(T)) \\ \partial_t m = -\partial_x(mu^*) = 0 \quad t \le s \le T \\ m(x,t) = m_0(x) \end{cases}$$

Here, we said "we almost recover" because we do not write the HJB equation synthesizing u^*, but we replace it by the minization problem, satisfied by u^*. Moreover, the transport equation (with null right-hand side by the nature of u^*), is the natural transport equation satisfied by the distribution of agents moving subject to the dynamics in (1) with u^* as control. Note that in this case, such partial differential equation is easily deduced by the vanishing of u^*, however, even for rather general discontinuous u^*, we can deduce the equation just by a conservation of mass principle, which only involves integrabilty of the functions (and indeed the equation is also called a conservation law). Obviously in such a more general case, the partial differential equation must be suitably interpreted in a weak sense. We are going to use such a weak sense in the next paragraph 4.2

In the following, we introduce some particular non trivial NCI distributions, in presence of fading functions of the following types:

- piecewise linear: $h(q) = \max\{1 - \alpha q, 0\}$, with $\alpha > 0$;
- exponential: $h(q) = e^{-\alpha q}$, with $\alpha > 0$;
- Gaussian: $h(q) = e^{-\alpha q^2/2}$, with $\alpha > 0$.

We will show that m is NCI if it is either sufficiently smooth or sufficiently "peaky".

4.1.1 Lipschitz $m_0(x)$ function

Let $m_0(x)$ have a compact support $supp(m_0(.)) \subset \mathbb{R}$ and be such that $\inf_{x \in supp(m_0(.))}\{m_0(x)\} > \epsilon > 0$ for a suitable ϵ and $|m_0(y) - m_0(x)| \le L|x - y|$, for some positive constant L, in $supp(m_0(.))$, i.e., $m_0(x)$ is a Lipschitz function in $supp(m_0(.))$. Then, if $h(\cdot)$ is piecewise linear, $m_0(x)$ is NCI if $\alpha \ge \frac{L}{\inf_{x \in supp(m_0(.))}\{m_0(x)\}}$.

Under the above hypotheses, condition (14) holds for $y = x$, i.e., each player considers its opinion the mainstream one. Indeed, for $y = x$, $|y - x| \le \lambda(t)$ trivially holds, and the second condition can be rewritten as

$$m_0(x) \ge m_0(z)h(|z - x|), \quad \forall z. \tag{15}$$

The above condition is trivially true for all $z \notin supp(m_0(.))$ or such that $h(|z - x|) = 0$. Differently for $z \in supp(m_0(.))$ and such that $h(|z - x|) > 0$, that is $h(|z - x|) = 1 - \alpha|z - x|$ the following argument holds. As $m_0(x)$ is a Lipschitz function in $supp(m_0(.))$, we have $m_0(x) \ge m_0(z) - L|z - x|$. Hence, (15) certainly holds because $m_0(z) - L|z - x| \ge m_0(z)h(|z - x|)$. Indeed, for all $z \in supp(m_0(.))$ and such that $h(|z - x|) > 0$,

$$\begin{aligned} &\alpha \ge \frac{L}{\inf_{x \in supp(m_0(.))}\{m_0(x)\}} \quad \Rightarrow \quad m_0(z)\alpha \ge L \quad \Rightarrow \\ &m_0(z)\alpha|z - x| \ge L|z - x| \quad \Leftrightarrow \\ &m_0(z) - L|z - x| \ge m_0(z) - m_0(z)\alpha|z - x| = m_0(z)h(|z - x|) \end{aligned}$$

4.1.2 Lipschitz $\log(m_0(x))$ function

Let $\log(m_0(x))$ be a Lipschitz function, i.e., $|\log(m_0(y)) - \log(m_0(x))| \le L|x - y|$, on the support of m_0. Note that this fact implies that m_0 is strictly positive in all its support. If $h(\cdot)$ is piecewise linear or exponential, then $m_0(x)$ is NCI if $\alpha \geq L$. Indeed, under these hypotheses, for any $x \in supp(m_0(.))$, condition (14) holds for $y = x$. In fact, (15) certainly holds for $z \notin supp(m_0(,))$, but it also holds for $z \in supp(m_0(.))$ because

$$|\log(m_0(x)) - \log(m_0(z))| \leq L|x - z| \quad \Rightarrow \quad m_0(x) \geq m_0(z)e^{-L|z-x|} \tag{16}$$

and $m_0(z)e^{-L|z-x|} \geq m_0(z)h(|z-x|)$. Indeed, the latter condition is trivially true for $h(|z-x|) = 0$. Differently, for $h(|z-x|) > 0$, we have that

$$\begin{aligned} &L \leq \alpha \quad \Rightarrow \quad e^{-L|z-x|} \geq 1 - \alpha|z-x| \text{ and } e^{-L|z-x|} \geq e^{-\alpha|z-x|} \quad \Rightarrow \\ &e^{-L|z-x|} \geq h(|z-x|) \quad \Rightarrow \quad m_0(z)e^{-L|z-x|} \geq m_0(z)h(|z-x|) \end{aligned}$$

Now, if $\log(m_0(x))$ is a Lipschitz function and differentiable, then distribution $m_0(x)$ is NCI also if $h(q)$ is a Gaussian kernel and $L \leq \alpha\lambda(t)$ as, for all x, we have $\overline{m}_0[x] = arg\max\{m_0(y)e^{-\alpha(x-y)^2/2}\} \in I_t(x)$. Indeed, we have

$$\frac{\partial m_0(y)e^{-\alpha(x-y)^2/2}}{\partial y} = e^{-\alpha(x-y)^2/2}(m_0'(y) + \alpha(x-y)m_0(y)) = 0 \quad \Leftrightarrow \quad m_0'(y) + \alpha(x-y)m_0(y) = 0.$$

Hence, $\hat{y} = \overline{m}_0[x]$ is such that $\frac{m_0'(\hat{y})}{m_0(\hat{y})} = \alpha(\hat{y} - x)$

As $\log(m_0(x))$ is a Lipschitz function, we have $\left|\frac{m_0'(y)}{m_0(y)}\right| \leq L$ for all y. Hence, $y \in I_t(x)$ if $|\hat{y} - x| \leq \frac{L}{\alpha} \leq \lambda(t)$.

4.1.3 "Peaky" $m_0(x)$ function

Let $m_0(x)$ be defined on a bounded support and characterized by a set $\Gamma = \{x^1, x^2, \ldots, x^n : x^1 < x^2 < \ldots < x^n\}$ of n local maxima such that:
i) each $x^k \in \Gamma$ is an absolute maximum of function $m_0(x)h(|x - x^k|)$, that is, each player in x^k feels itself a leader and considers its opinion the mainstream one;
ii) for all $x \in \partial I_t(x^k)$, $\overline{m}_0[x] \notin I_t(x^k) \setminus \partial I_t(x^k)$, for all $x^k \in \Gamma$, that is, each player on the frontier of $I_t(x^k)$ thinks that the mainstream opinion is not in $I_t(x^k)$;
iii) $\bigcup_{x^k \in \Gamma} I_t(x^k) \supseteq [x^0, x^{n+1}]$, where $[x^0, x^{n+1}]$, with $x_0 \leq x_1$ and $x^{n+1} \geq x^n$ is the minimum interval including the support set of $m_0(x)$, that is, the neighborhoods of the leaders cover all the possible opinions.
We have that $m_0(x)$ is NCI if $\log h(\cdot)$ is sublinear, that is, for any $p, q \geq 0$ $h(p+q) \leq h(p)h(q)$.

To prove such a result we need to show that the following critical properties hold true:

- *first critical property*: if $x \in [x^k, x^{k+1}]$ then $\overline{m}_0[x] \in [x^k, x^{k+1}]$.
- *second critical property*: if $x \in [x^k, x^{k+1}]$, then either both x and $\overline{m}_0[x]$ are in $I_t(x^k)$ or both of them are in $I_t(x^{k+1})$.

If both properties hold, we have that if $x \in [x^k, x^{k+1}]$ then $|\overline{m}_0[x] - x| \leq \lambda(t)$, as assumption iii) implies that $x^{k+1} - x^k \leq 2\lambda(t)$ and $I_t(x^k)$ (respectively $I_t(x^{k+1})$) has radius $\lambda(t)$ and is centered in x^k (respectively is x^{k+1}). Consequently, if both such properties hold, condition (14) holds true as well since, from assumption iii), all x in the support of $m_0(x)$ belong to some interval $[x^k, x^{k+1}]$. Hence $m_0(x)$ is NCI.

To prove the first critical property, we observe that, given $x \in [x^k, x^{k+1}]$, if $y = \overline{m}_0[x] < x^k$ were true, then it would hold $m_0(y)h(|y-x|) > m_0(x^k)h(|x^k - x|)$ in contradiction with $m_0(y)h(|y-x|) \leq m_0(y)h(|y - x^k|)h(|x^k - x|) \leq m_0(x^k)h(|x^k - x|)$, where the first inequality holds for the logarithmic sublinearity of $h(\cdot)$ and the second inequality for assumption i). Symmetric argument applies to $y > x^{k+1}$.

To prove the second critical property, we observe that $x \in [x^k, x^{k+1}]$ implies that either

1. $x \in I_t(x^k) \setminus I_t(x^{k+1})$ or

2. $x \in I_t(x^k) \cap I_t(x^{k+1})$ or
3. $x \in I_t(x^{k+1}) \setminus I_t(x^k)$.

If condition 2) holds the property is proved since the first critical property implies $\overline{m}_0[x] \in I_t(x^k) \cup I_t(x^{k+1})$. Condition 3) is symmetrical to condition 1), so we need to verify the second critical property only when the latter condition holds.

Let z be the only element on the frontier of $I_t(x^{k+1})$ in $[x^k, x^{k+1}]$ (if z does not exist, that is if $I_t(x^{k+1})$ contains $[x^k, x^{k+1}]$, then the thesis is straightforward), $\hat{z} = \overline{m}_0[z]$, $\hat{x} = \overline{m}_0[x]$. By assumption ii) we have $x^k \leq \hat{z} \leq z$. If condition 1) holds then $x^k \leq x < z$ and we have to prove that also $x^k \leq \hat{x} \leq z$ is true. We prove the latter inequalities by contradiction, i.e., assuming that $z \leq \hat{x} \leq x^{k+1}$.

By definition of $\hat{z}$ we have

$$m_0(\hat{z})h(|z - \hat{z}|) \geq m_0(\hat{x})h(|\hat{x} - z|).$$

If $x^k \leq \hat{z} \leq x$ and $z \leq \hat{x} \leq x^{k+1}$, we also have

$$m_0(\hat{z})h(|x - \hat{z}|) \geq m_0(\hat{z})h(|z - \hat{z}|) \geq m_0(\hat{x})h(|\hat{x} - z|) \geq m_0(\hat{x})h(|\hat{x} - x|)$$

in contradiction with the definition of $\hat{x}$ that implies $m_0(\hat{z})h(|x - \hat{z}|) < m_0(\hat{x})h(|\hat{x} - x|)$.

The proof is complete if also $x^k \leq x < \hat{z}$ and $z \leq \hat{x} \leq x^{k+1}$ lead to a contradiction. Specifically, under these conditions, we can write

$$\begin{aligned} &m_0(\hat{z})h(|\hat{z} - x|) \geq m_0(\hat{z})h(|\hat{z} - x|)h(|z - \hat{z}|) \geq \\ &\geq m_0(\hat{x})h(|\hat{x} - z|)h(|\hat{z} - x|) \geq m_0(\hat{x})h(|\hat{x} - z| + |\hat{z} - x|) \geq m_0(\hat{x})h(|\hat{x} - x|), \end{aligned}$$

where the first inequality holds as $h(|z - \hat{z}|) \leq 1$, the second inequality holds by definition of $\hat{z}$, the third inequality holds by logarithmic sublinearity of the fading function, finally the forth inequality holds since $h(q)$ is non-increasing for $q \geq 0$ and $|\hat{x} - x| = |\hat{x} - z| + |z - \hat{z}| + |\hat{z} - x| \geq |\hat{x} - z| + |\hat{z} - x|$. Again, the above chain of inequalities is in contradiction with the definition of $\hat{x}$ that implies $m_0(\hat{z})h(|\hat{z} - x|) < m_0(\hat{x})h(|\hat{x} - x|)$.

We conclude this section observing that piecewise linear, exponential and Gaussian fading functions are examples of logarithmic sublinear functions. We also observe that the above theorem can be trivially generalized to any time instant different from the initial one.

4.2 Game evolution

In general we cannot prove that a threshold strategy (12) is either optimal or leads to an equilibrium. In this section, we provide some results about the evolution of the distribution $m(.,.)$ over the interval of time $t \leq s \leq T$, when players apply such a threshold strategy in presence of a non-NCI initial distribution. We consider just a simple initial distribution of opinions with just one leading opinion that may represent an initial virgin state which has never experienced evolution. Our aim is to show how threshold strategies may introduce particular "pathological" behavior in the evolution of the distribution $m(.,.)$. Specifically, even distribution initially smooth may evolve to include many discontinuous points.

From a technical point of view it is worth mentioning that in this section we derive some property of the population distribution having fixed the players' strategies, whereas in Section 3 we derived some property of the players' strategies having fixed the statistics of interest of the population distribution. Note that, while in Section 3 we dealt with an optimal control problem, we now deal with the problem of individuating weak solutions of a transport PDE in this section, (see also Remark 2 for the meaning of the transport equation).

We suppose that the following assumptions hold.

Assumptions 2 *At the initial time t the distribution $m(x,t) = m_0(x)$ with support $supp(m_0(.)) \subseteq \mathbb{R}$ is*

1. *a differentiable function in $supp(m_0(.))$ with no point mass;*
2. *a weakly unimodal function in $supp(m_0(.))$.*

We define $m_0(\cdot)$ as weakly unimodal, if there exists a value $\bar{x}$ for which $m_0(\cdot)$ is not decreasing for any $x \leq \bar{x}$ and not increasing for any $x \geq \bar{x}$. For the ease of illustration, we also assume that $\bar{x}$ is the only solution of $\bar{x} = \arg\max\{m_0(x)\}$. However, our arguments trivially generalize if $\arg\max\{m_0(x)\}$ define an interval.

We recall that $I_s(x) = \{y : |y - x| \leq \lambda(s)\}$ is the neighborhood of radius $\lambda(s)$ of the state x.

Theorem 2 *Under Assumptions 1 and 2 the following facts hold:*

1. *$\bar{x} = \arg\max\{m(x,s)\}$ for $t \leq s \leq T$;*
2. *$u^*[x](\tau) = 0$ for all $t \leq s \leq \tau \leq T$ for all $x \in I_s(\bar{x})$;*
3. *$\partial_x u^*[x](s)|_{x=\bar{x}} = 0$.*

Proof We observe that, if $\bar{x} = \arg\max\{m(x,s)\}$ for some s, then the players apply $u^*[x](s) = 0$ for all $x \in I_s(\bar{x})$, Indeed, for all $x \in I_s(\bar{x})$ we have that $\bar{m}[x](s) \in I_s(x)$ as

$$m(y,s)h(|y-x|) \leq m(\bar{x},s)h(|\bar{x}-x|) \leq m(\bar{m}[x](s),s)h(|\bar{m}[x](s)-x|), \quad \forall y \notin I_s(x).$$

Specifically, the first inequality holds due to the maximality of $\bar{x}$ with respect to $m(.,s)$ and as $|\bar{x} - x| \leq \lambda(s) \leq |y - x|$. The second inequality holds due to the maximality of $\bar{m}[x](s)$ with respect to $m(.,s)h(|\cdot - x|)$ that in turn implies $\partial_x u^*[x](s)|_{x=\bar{x}} = 0$.

As $u^*[x](s) = 0$ in a neighborhood of $\bar{x}$, specifically for all $x \in I_s(\bar{x})$ for $t \leq s \leq T$, we have that $dx(s) = 0$ for $x(s) \in I_s(\bar{x})$, hence $\partial_t m(\bar{x},s) = 0$ which in turn implies $m(\bar{x},s) = m_0(\bar{x})$. Then, from the latter result we obtain what follows. First, condition 1), namely, $\bar{x} = \arg\max\{m(x,s)\}$ for $t \leq s \leq T$, is a straightforward consequence of Assumptions 1.2, and more specifically of (13) that prevents any $x(s) \neq \bar{x}$ to be associated with $m(x(s),s) \geq m(\bar{x},s) = m_0(\bar{x})$ for $t \leq s \leq T$. Second, condition 2), namely, $u^*[x](\tau) = 0$ for all $t \leq s \leq \tau \leq T$ for all $x \in I_s(\bar{x})$, holds since $I_s(\bar{x}) \subseteq I_\tau(\bar{x})$ if $t \leq s \leq \tau \leq T$. Third, condition 3), namely, $\partial_x u^*[x](s)|_{x=\bar{x}} = 0$ holds as $I_s(\bar{x})$ is a neighborhood of $\bar{x}$.

□

In addition, in the general case, $m(.,.)$ can present discontinuities even if $m_0(.)$ is continuous. To this end, let us consider the situation in which $h(.) = 1$ for all $x \in \mathbb{R}$. From Assumption 2.1, we can describe $m(.,.)$ in terms of weak solutions of the following problem:

$$\begin{aligned} &\partial_t m = -\partial_x(mu^*) \\ &m(x,t) = m_0(x) \quad \forall x \in supp(m_0(.)). \end{aligned} \tag{17}$$

We recall that, in this setting, "weak solution" means that $\int\int m\varphi_t + (u^*m)\varphi_x dxds + \int m_0\varphi(\cdot,t)dx = 0$ for all C^1 test functions φ with compact support. We also recall (see, e.g., [25]) that when the domain is divided in two parts by a regular curve $s \mapsto \gamma(s) = x$, and when m is of class C^1 in every one of the two parts, and separately in both parts satisfies the transport problem (17), then, in the whole domain, m is a "weak solution" if and only if the possible jump discontinuities of m and of mu^*, say $[m]$ and $[mu^*]$, on the curve γ satisfy the so-called Rankine-Hugoniot condition: $\gamma'(s)[m(\gamma(s),s)] = [(mu^*)(\gamma(s),s))]$.

We will study $m(x,s)$ for $x \geq \bar{x}$ as symmetrical arguments hold for $x \leq \bar{x}$.

Let us introduce the following notation:

– $\tilde{T} = \sup\{s : t \le s \le T, \bar{m}[x](s) = \bar{x}, x \ge \bar{x}\}$
– $\hat{u}[x](s) = a(s)(\bar{x} - x)$ for $t \le s \le \tilde{T}$, that is, $\hat{u}$ can be seen as a "prolongation" of u^* (defined in the first line of (8)) as $\hat{u} = u^*$ where $u^* \ne 0$
– $\hat{m}(x, s)$ the solution of the PDE

$$\partial_t \hat{m} = -\partial_x(\hat{m}\hat{u}), \quad \hat{m}(x, t) = m_0(x)), \tag{18}$$

that is, $\hat{m}$ is the solution of the transportation equation when the control $\hat{u}$ is implemented

We are then in the condition to state the following theorem.

Theorem 3 *For $t \le s \le \tilde{T}$ and for $x \ge \bar{x}$*

$$m(x, s) = \begin{cases} \hat{m}(x, s) & \text{for } t \le s < s_x \\ \min\{\hat{m}(x, s_x) - \frac{\hat{m}(x,s_x)\hat{u}(x,s_x)}{\lambda'(s_x)}, m_0(\bar{x})\} & \text{for } s_x \le s \le \tilde{T} \end{cases} \tag{19}$$

is a weak solution of equation (17) with possibly a jump discontinuity at (x, s_x), where

$$s_x = \begin{cases} t & \text{if } x \le \bar{x} + \lambda(0), \\ \hat{s} : \lambda(\hat{s}) = x & \text{if } \bar{x} + \lambda(0) < x \le \bar{x} + \lambda(\tilde{T}), \\ \tilde{T} & \text{if } x > \bar{x} + \lambda(\tilde{T}), \end{cases} \tag{20}$$

and $\hat{m}$ is the solution of (18).

Proof We initially observe that solution (19) is equal to $m(x, t)$ for all $x \in I_t(\bar{x})$. In addition, solution (19) is equal to $\hat{m}(x, t)$ for $x > \bar{x} + \lambda(\tilde{T})$. For the remaining points, if $\hat{m}(x, s_x) - \frac{\hat{m}(x,s_x)\hat{u}(x,s_x)}{\lambda'(s_x)} < m_0(\bar{x})$, this theorem is an immediate consequence of the Rankine-Hugoniot Theorem (see, e.g., [25] p.79). Indeed, we note that all the conditions of the Rankine-Hugoniot Theorem are satisfied. Specifically, for $t \le s \le \tilde{T}$ and for each x, $m_1(x, s) = \hat{m}(x, s)$ and $m_2(x, s) = \hat{m}(x, s_x) - \frac{\hat{m}(x,s_x)\hat{u}(x,s_x)}{\lambda'(s_x)} = const$ are classical solutions of problem (17) respectively for $t \le s < s_x$ and for $s_x < s \le \tilde{T}$. Both $m_1()$ and $m_2()$ are differentiable where defined. In addition, (20) defines a monotonic curve that divides the open right quadrant (x, s) for $x \ge \bar{x}$ and $s \ge t$ in two pieces lying to the right and to the left of the curve. Finally, $\lim_{(y,s)\to(x,s_x^+)} u^*[y](s)m_2(y, s) - \lim_{(y,s)\to(x,s_x)^-} u^*[y](s)m_1(y, s) = \lambda'(s_x)(\Delta m)(x, s_x)$, where $(\Delta m)(x, s_x)$ is the "jump" that (19) shows in each (x, s_x). (See Figure 3.)

Consider now $\hat{m}(x, s_x) - \frac{\hat{m}(x,s_x)\hat{u}(x,s_x)}{\lambda'(s_x)} \ge m_0(\bar{x})$. The condition $m(x, s_x) = l < m_0(\bar{x})$ cannot realize as it is in contradiction with the fact that one can define a control u° such that $u^*[x](s) < u^\circ[x](s) < 0$ for $t \le s \le \tilde{T}$ and $x \ge \bar{x} + \lambda(\tilde{T})$ which induces a value $l < m(x, s_x) < m_0(\bar{x})$. In plain words, one can find a less intense control that induces a greater jump. A value $m(x, s_x) \ge m_0(\bar{x})$ implies $s_x = \tilde{T}$, then the Rankine-Hugoniot Theorem does not apply and, on the contrary, any value $m(x, s_x) \ge m_0(\bar{x})$, and hence $m(x, s_x) = m_0(\bar{x})$, corresponds to a weak solution for problem (17). Indeed, $m_1(x, s) = \hat{m}(x, s)$ is a classical solution for $t \le s < s_x$ whereas the interval $[s_x, \tilde{T}]$ has a null measure. This latter condition holds even in the pathological situation in which the interval $[t, s_x)$ is empty.

□

It is worth noting that in the above theorem the choice of setting $m(x, s_x) = m_0(\bar{x})$ when $\hat{m}(x, s_x) - \frac{\hat{m}(x,s_x)\hat{u}(x,s_x)}{\lambda'(s_x)} > m_0(\bar{x})$ needs a stronger justification. To this purpose, it is worth noting that such a choice is coherent with the fact that the control applied by a player with state equal to x drops abruptly to 0 when x becomes an absolute maximum.

The iterative application of the above theorem allows to describe $m(x, s)$ for all x and for all $t \le s \le T$ in terms of the composition of successive weak solutions of (17) as described in the following three-step recursive procedure. In this context, we define $F(s) = \{x \ge \bar{x} : m(x, \tau) = m(x, s), \tau \ge s\}$ as the set of the states x for which the value of $m(x, \tau)$ does not change from s on.

1. We know that $m(x,s)$, as defined in (19), is a weak solution of (17) $t \le s \le \tilde{T}$. Then, $F(t) \supseteq I_t(\bar{x})$ provided that $\max\{m(x,s); s \ge t\} = m_0(\bar{x})$. If $\tilde{T} = T$ the problem is solved otherwise, if $\tilde{T} < T$, we must define $m(x,s)$ for $\tilde{T} \le s \le T$ and we go to the next step.
2. We observe that, if $\tilde{T} < T$, $m(\bar{x}+\lambda(\tilde{T}),\tilde{T}) = m(\bar{x},t)$ must hold, as, for all $x > \bar{x}+\lambda(\tilde{T})$, $m(\bar{x},\tilde{T})$ is certainly less than $m(\bar{x},t)$ since $\partial_s \hat{m}(x,s) \le 0$ for all $x > \bar{x}$ and $t \le s \le \tilde{T}$. This fact in turn implies that $\bar{m}[x][\tilde{T}] = \bar{x}+\lambda(\tilde{T})$ for all $x \ge \bar{x}+\lambda(\tilde{T})$ and hence $m(x,s) = m(x,\tilde{T})$ for $\tilde{T} < s \le T$ and $x \in I_{\tilde{T}}(\bar{x}+\lambda(\tilde{T}))$. Then $F(s) \supseteq F(t) \cup I_{\tilde{T}}(\bar{x}+\lambda(\tilde{T}))$ will also hold provided that $\max\{m(x,s); s \ge \tilde{T}\} = m_0(\bar{x})$. For the definition of $m(x,s)$ for $s > \tilde{T}$ and $x > \bar{x}+2\lambda(\tilde{T})$ we go to the next step.
3. We recursively repeat the above steps after having applied at each recursion the arguments of Theorem 3 by renominating $\bar{x}+\lambda(\tilde{T})$ with $\bar{x}$ and $\tilde{T}$ with t.

We first note that the above procedure defines a weak solution whose maximum value is always $m_0(\bar{x})$. Hence the recursive inclusion $F(s) \supseteq F(t) \cup I_{\tilde{T}}(\bar{x}+\lambda(\tilde{T}))$ always holds. Then, let us denote by $\tilde{T}_k$ the value of $\tilde{T}$ at the end of the k-recursion of the above procedure and set $\tilde{T}_0 = t$. The above recursive procedure surely terminates in a number of steps less than or equal to $\frac{|supp(m_0(.))|}{\lambda(t)}$ if the measure of $supp(m_0(.))$ is finite since $supp(m_0(.)) \supseteq supp(m(.,s))$ for $t \le s \le \tilde{T}$ and since at the end of the generic k recursion we have that $F(\tilde{T}_s) \supseteq \{x : \bar{x} \le x \le \bar{x}+k\lambda(t)\}$.

If there exists no finite value R such that $\mathbb{B}(0,R) \supseteq supp(m_0(.))$, then we can have at most a countable number of values of $\tilde{T}_k$, some of which may also coincide. A particular pathological situation is the one in which $\hat{m}(x,t) - \frac{\hat{m}(x,t)\hat{u}(x,t)}{\lambda'(t)} = m_0(x) - \frac{m_0(x)\hat{u}(x,t)}{\lambda'(t)} \ge m_0(\bar{x})$ for all $x \in supp(m_0(.)) = \mathbb{R}$. In this case the application of the above algorithm leads to the following weak solution with a countable number of discontinuous points:

$$m(x,s) = \begin{cases} m_0(x) & \text{for all } x \ne \pm k\lambda(t) \text{ with } k \in \mathbb{N} \\ m_0(\bar{x}) & \text{for all } x = \pm k\lambda(t) \text{ with } k \in \mathbb{N} \end{cases} \quad \text{for } t < s \le T. \tag{21}$$

Remark 3 Equation (17) is a discontinuous scalar conservation law in the unknown m, where the discontinuity is due to the discontinuity of u^* with respect to x (it suddenly becomes equal to 0 whenever x is suitably near to the mode of m (threshold policy)). This fact brings several problems in order to define what a solution is and further to give some uniqueness results. Here we only comment some facts.

First of all, as explained in Theorem 3, and qualitatively shown in Figure 3 (1), the solution is expected to be discontinuous. This is indeed a common feature of (even continuous) scalar conservation laws. In our case, however, we have a discontinuity threshold ($\lambda(\cdot)$) which is moving in the opposite direction with respect to the transportation given by the field $f(m) := u^*m$, and hence we do not recover the standard concept of "shock" for conservation laws (as, for example in the case of the Riemann problem (see [25])). Moreover, the discontinuity curve is not generated by the solution (as it is the case when we have a shock in a continuous scalar law), but instead it is a discontinuity in the coefficient u^* of the equation.

In general, the Rankine-Hugoniot condition does not guarantee the uniqueness of the solution. One usually adds the so-called Lax-shock condition which, in the scalar case, requires that the characteristics impinge on the discontinuity curve, which is not our case, where instead (see Figure 3 (3)), the characteristics cross the discontinuity curve. Indeed, if we search for a solution satisfying the usual Lax-shock condition, then we should arrive at a discontinuity line non-consistent with the definition of u^*. This is also the reason why, in the characteristics portrait Figure 3 (3) we explicitly require to consider a jump in the solution when crossing the discontinuity line (as given in (19)). This seems to be the unique way to define a weak solution of (17). Also note that we have another discontinuity curve: the vertical line $\gamma(s) = \lambda(0)$ (see Figure 3(1)). This is because at the point $(\lambda(0), 0)$ the solution immediately shows a discontinuity. However, for this curve, the derivative (with respect to s) is zero and u^* is also zero on both sides. Hence, the Rankine-Hugoniot condition is trivially satisfied.

The above discussion refers to the case when the jump across the discontinuity curve is such that the new value of m remains lower than the maximum $m_0(\overline{x})$ of the initial distribution, and so the mode does not change. Indeed, in this case, even if m is discontinuous, we do not have any problem in defining $\overline{x}$ as the mode of $m(\cdot,s)$

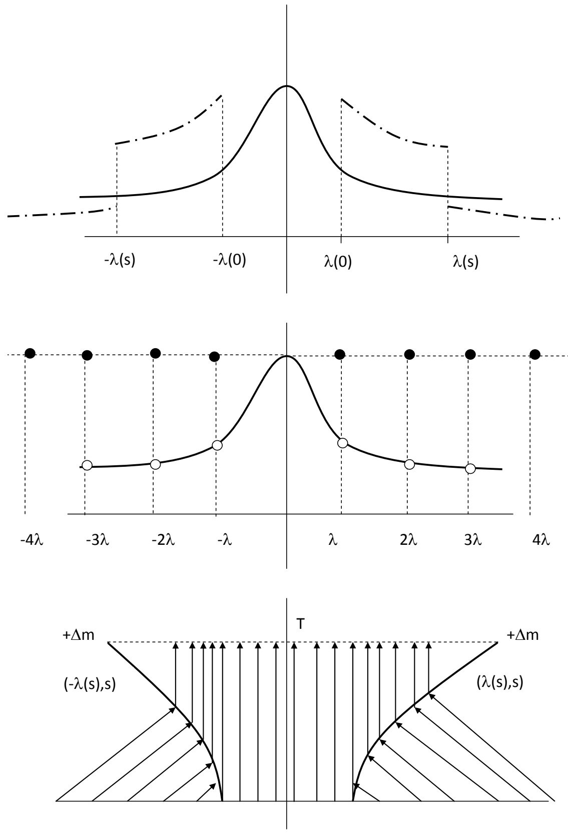

Fig. 3 From top to bottom: 1) the filled curve represents the initial distribution m_0 at the initial time, here assumed $t = 0$; the point-dashed curve represents a qualitative modification of the profile of the solution at time s when $m(x, s_x) < m_0(\overline{x})$ for all x; 2) a possible representation of the solution m in the case as in (21), as well as in the case of constant threshold $\lambda(s) \equiv \lambda$; 3) qualitative representation of the characteristics for the solution in the previous case 1); in particular, the solution is subject to a jump (here generically indicated by Δm) when crossing the discontinuity curve $(\lambda(s), s)$.

too. This is because a natural generalization of the concept of mode, as the point where the maximum is reached, is the one for which the mode of an integrable function f is the point x where $\lim_{\varepsilon\to 0^+}(1/\varepsilon)\int_{x-\varepsilon}^{x+\varepsilon} f(\xi)d\xi$ has the maximum value.

Now, we consider the case where the discontinuity threshold is constant $\lambda(s) \equiv \lambda$, and then the discontinuity curve in the plane (x, s) should be a vertical line (however, this case is also quite similar to the case explained before (and in) (21).) Let us focus only on the first quadrant of Figure 3 (2). All the agents in $[0, \lambda]$ do not move, and the agents immediately to the right of λ start to move left. Now, the threshold is not moving right, as in the previous case, and hence, in principle, such agents do arrive at the value λ. However, if some of them arrive at λ then, in $x = \lambda$ we immediately get a delta function. Using the control u^*, this is impossible. Indeed any delta function has a weight much larger than any other value of the continuous part of m (see the possible generalized definition of mode as before), and hence, whenever the delta function arises, then it is immediately recognized as the mainstream and no other agents can reach it anymore. In conclusion a delta function cannot arise. The only possibility is then that all agents do not move, which implies that the solution satisfies $m(x, s) = m_0(x)$ for all (x, s). But, for making this consistent with the definition of u^* (which in this case must be equal to zero everywhere), the only possibility is that the agents recognize the mainstream as a position sufficiently close to theirs (by no more than λ). To this end, we have to consider a function as in Figure 3 (2), where the mode is still considered as the point where the maximum is reached, even if such a procedure is almost meaningless, due to isolated discontinuities and because of the fact that a single point has zero density. However, the function in Figure 3 (2) satisfies the equality $m(x, s) = m_0(x)$ for all s and for almost every x, and so, in any L^p spac es they are exactly the same function.

Also note that the situation in Figure 3 (2) cannot be recovered as limit of the picture (1) as well as of the characteristic picture (3), when the moving threshold $\lambda(\cdot)$ tends to become constant (and so the discontinuity line tends to become vertical). Indeed, in such a limit procedure, $\lambda'(\cdot)$ tends to become zero and hence the jump tends to become larger and larger and the new values of m, after crossing the discontinuity line, are no more smaller then $m_0(\overline{x})$.

Further studies of this kind of discontinuous conservation law, in connection with some possible weaker definition of the mode, are left for future studies.

5 Conclusions and future directions

Well-established phenomena in social networks, which also characterize the bulk of the literature on opinion dynamics, are the "bandwagon effect" and the so-called "homophily". The bandwagon effect consists in all players seeking to align their opinions to the mainstream opinion, this being the opinion shared by the majority. On the other hand, homophily consists in every player interacting only with those whose opinion is not too distant from its own. This paper has shown how mean-field game theory can successfully accommodate both phenomena. Individuals act in a competitive scenario modeled as a game with a large number of players. Consequently the propagation of the opinions is a process resulting from all players adopting equilibrium strategies. To capture stubbornness on the part of the players, we have considered an additional fixed cost in the cost functional. This has lead us to the notion of threshold strategy, consisting in every single player changing its opinion only if the mainstream opinion is significantly different. The work has shed light on the implications that such a threshold strategy has on the macroscopic time evolution of the density distribution of the opinion within the population.

This study is part of the current research activity on opinion dynamics conducted by the authors. Future directions involve the analysis of scenarios where the players present different opinion changing rates, as it is often observed in real cases. Such a complex scenario can be dealt with considering a multi-population game. Another challenge is represented by more sophisticated interrelation between the control and the opinion variation. It may happen that the control has no direct effect on the variation of the opinion and this results in a highly nonlinear controlled dynamics. We aim at extending the results provided in this paper to such a complex scenario. Finally, we will argue on the importance of game theory as a design tool. If the current study uses such a theory to explain observed phenomena, it is our belief that the role of game theory can go much further, providing tools for the analysis and design of information mechanisms leading to pre-specified opinion distributions. The analysis of the "strategic thinking" which characterizes the modern media will be our starting point.

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