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Abstract

The aim of this paper is to study relationships among “gauge integrals” (Henstock, Mc Shane, Birkhoff) and Pettis integral of multifunctions whose values are weakly compact and convex subsets of a general Banach space, not necessarily separable. For this purpose, we prove the existence of variationally Henstock integrable selections for variationally Henstock integrable multifunctions. Using this and other known results concerning the existence of selections integrable in the same sense as the corresponding multifunctions, we obtain three decomposition theorems (Theorems 3.2, 4.2, 5.3). As applications of such decompositions, we deduce characterizations of Henstock (Theorem 3.3) and $\mathcal{H}$ (Theorem 4.3) integrable multifunctions, together with an extension of a well-known theorem of Fremlin [22, Theorem 8].

Keywords (separated by '-')	Multifunction - Gauge integral - Decomposition theorem for multifunction - Pettis integral - Selection
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Relations among Gauge and Pettis integrals for $cwk(X)$-valued multifunctions

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1 Introduction

A large amount of work about measurable and integrable multifunctions was done in the last decades. Some pioneering and highly influential ideas and notions around the matter were inspired by problems arising in Control Theory and Mathematical Economics. But the topic is interesting also from the point of view of measure and integration theory, as showed in the papers [2,3,8,9,11,12,18–20,29,31–34,37,38]. In particular, comparison of different generalizations of Lebesgue integral is, in our opinion, one of the milestones of the modern theory of integration. Inspired by [6,7,10,12,13,19,24,39], we continue in this paper the study on this subject and we examine relationship among "gauge integrals" (Henstock, Mc Shane, Birkhoff) and Pettis integral of multifunctions whose values are weakly compact and convex subsets of a general Banach space, not necessarily separable.

The name "gauge integrals" refers to integrals defined through partitions controlled by a positive function, traditionally named gauge. J. Kurzweil in 1957 and then R. Henstock in 1963 were the first who introduced a definition of a gauge integral for real-valued functions, called now the Henstock–Kurzweil integral. Its generalization to vector-valued functions or to multivalued functions is called in the literature the Henstock integral. In the family of the gauge integrals, there is also the McShane integral and the versions of the Henstock and the McShane integrals when only measurable gauges are allowed ($\mathcal{H}$ and $\mathcal{M}$ integrals, respectively), and the variational Henstock and the variational McShane integrals. Moreover according to [41] and [39, Remark 1], the Birkhoff integral is a gauge integral too and it turns out to be equivalent to the $\mathcal{M}$ integral.

The main results of the paper are the existence of variationally Henstock integrable selections (Theorem 5.1), which solves the problem of the existence of variationally Henstock integrable selection for a $cwk(X)$-valued variationally Henstock integrable multifunction ([6, Question 3.11]) and three decomposition theorems (Theorems 3.2, 4.2, 5.3). The first one says that each Henstock integrable multifunction is the sum of a McShane integrable multifunction and a Henstock integrable function. The second one describes each $\mathcal{H}$-integrable multifunction as the sum of a Birkhoff integrable multifunction and an $\mathcal{H}$-integrable function, and the third one proves that each variationally Henstock integrable multifunction is the sum of a variationally Henstock integrable selection of the multifunction and a Birkhoff integrable multifunction that is also variationally Henstock integrable. As applications of such decomposition results, characterizations of Henstock (Theorem 3.3) and $\mathcal{H}$ (Theorem 4.3) integrable multifunctions are presented as extensions of the result given by Fremlin, in the remarkable paper [22, Theorem 8], and of more recent results given in [6,19]. 2

Finally, we want to point out that in order to obtain the decomposition theorems and also the extension of the Fremlin result is not enough simply to apply the embedding theorem of Rådström, but more sophisticated techniques are required.

2 Preliminary facts

Let $[0, 1] \subset \mathbb{R}$ be endowed with the usual topology and Lebesgue measure λ. The family of all Lebesgue measurable subsets of $[0, 1]$ is denoted by $\mathcal{L}$, while $\mathcal{I}$ is the collection of all closed subintervals of $[0, 1]$. If $I \in \mathcal{I}$, then its Lebesgue measure will be denoted by $|I|$.

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A finite partition $\mathcal{P}$ *in* [0, 1] is a collection $\{(I_1, t_1), \dots, (I_m, t_m)\}$, where $I_1, \dots, I_m$ are nonoverlapping (i.e., the intersection of two intervals is at most a singleton) closed subintervals of [0, 1], t_i is a point of [0, 1], $i = 1, \dots, m$. If $\cup_{i=1}^m I_i = [0, 1]$, then $\mathcal{P}$ is *a partition of* [0, 1].

If $t_i \in I_i$, $i = 1, \dots, m$, we say that $\mathcal{P}$ is a *Perron partition of* [0, 1].

A countable partition $(A_n)_n$ of [0, 1] in $\mathcal{L}$ is a collection of pairwise disjoint $\mathcal{L}$-measurable sets such that $\cup_n A_n = [0, 1]$; we admit empty sets.

A *gauge* on [0, 1] is any strictly positive map on [0, 1]. Given a gauge δ, we say that a partition $\{(I_1, t_1), \dots, (I_m, t_m)\}$ is δ*-fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1, \dots, m$. Π_δ and Π_δ^P are the families of δ-fine partitions, and δ-fine Perron partitions of [0, 1], respectively.

X is an arbitrary Banach space with its dual X^*. The closed unit ball of X^* is denoted by B_{X^*}. As usual $cwk(X)$ denotes the family of all nonempty convex weakly compact subsets of X; on this hyperspace, the usual Minkowski addition and the multiplication by positive scalars are considered, together with the Hausdorff distance d_H. Moreover, $\|A\| := \sup\{\|x\| : x \in A\}$. The *support function* $s: X^* \times cwk(X) \to \mathbb{R}$ is defined by $s(x^*, C) := \sup\{\langle x^*, x\rangle : x \in C\}$.

Definition 2.1 A map $\Gamma : [0, 1] \to cwk(X)$ is called a *multifunction*. Γ is *simple* if there exists a finite collection $\{A_1, ..., A_p\}$ of measurable pairwise disjoint subsets of [0, 1] such that Γ is constant on each A_j.

A map $\Gamma : \mathcal{I} \to cwk(X)$ is called an *interval multifunction*. A multifunction $\Gamma : [0, 1] \to cwk(X)$ is said to be *scalarly measurable* if for every $x^* \in X^*$, the map $s(x^*, \Gamma(\cdot))$ is measurable.

Γ is said to be *Bochner measurable* if there exists a sequence of simple multifunctions $\Gamma_n : [0, 1] \to cwk(X)$ such that $\lim_{n\to\infty} d_H(\Gamma_n(t), \Gamma(t)) = 0$ for almost all $t \in [0, 1]$.

It is well known that Bochner measurability of a $cwk(X)$-valued multifunction yields its scalar measurability. The reverse implication in general fails, even if X is separable (see [6, p. 295 and Example 3.8]).

If a multifunction is a function, then we use the traditional name of strong measurability instead of Bochner measurability.

A function $f : [0, 1] \to X$ is called a *selection of* Γ if $f(t) \in \Gamma(t)$, for every $t \in [0, 1]$.

Definition 2.2 A multifunction $\Gamma : [0, 1] \to cwk(X)$ is said to be *Birkhoff integrable* on [0, 1], if there exists a set $\Phi_\Gamma([0, 1]) \in cwk(X)$ with the following property: For every $\varepsilon > 0$, there is a countable partition $\mathcal{P}_0$ of [0, 1] in $\mathcal{L}$ such that for every countable partition $\mathcal{P} = (A_n)_n$ of [0, 1] in $\mathcal{L}$ finer than $\mathcal{P}_0$ and any choice $T = \{t_n : t_n \in A_n, n \in \mathbb{N}\}$, the series $\sum_n \lambda(A_n)\Gamma(t_n)$ is unconditionally convergent (in the sense of the Hausdorff metric) and

$$d_H\left(\Phi_\Gamma([0, 1]), \sum_n \Gamma(t_n)\lambda(A_n)\right) < \varepsilon . \tag{1}$$

(see for example [11, Proposition 2.6]).

Definition 2.3 A multifunction $\Gamma : [0, 1] \to cwk(X)$ is said to be *Henstock* (resp. *McShane*) *integrable* on [0, 1], if there exists $\Phi_\Gamma([0, 1]) \in cwk(X)$ with the property that for every $\varepsilon > 0$ there exists a gauge δ on [0, 1] such that for each $\{(I_1, t_1), \dots, (I_p, t_p)\} \in \Pi_\delta^P$ (resp. $\in \Pi_\delta$) we have

$$d_H\left(\Phi_\Gamma([0, 1]), \sum_{i=1}^p \Gamma(t_i)|I_i|\right) < \varepsilon . \tag{2}$$

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Γ is said to be *Henstock* (resp. McShane) *integrable* on $I \in \mathcal{I}$ ($E \in \mathcal{L}$) if $\Gamma 1_I$ ($\Gamma 1_E$) is integrable on [0, 1] in the corresponding sense.

In case the multifunction is a single-valued function, and X is the real line, the corresponding integral is called *Henstock–Kurzweil integral* (or HK-*integral*) and it is denoted by the symbol $(HK)\int_I$.

Remark 2.4 If the gauges above considered are taken to be measurable, then we speak of $\mathcal{H}$ (resp. $\mathcal{M}$)-integrability on [0, 1].

Given $\Gamma : [0, 1] \to cwk(X)$, it is known that the property of integrability is inherited on every $I \in \mathcal{I}$ if Γ is Henstock ($\mathcal{H}$) integrable on [0, 1], while the same is true for every $E \in \mathcal{L}$ when Γ is McShane ($\mathcal{M}$) integrable on [0, 1] (see, e.g., [19]).

As pointed out before, in case of single-valued functions, according to [41] and [39, Remark 1], $\mathcal{M}$-integrability is equivalent to the Birkhoff integrability.

Definition 2.5 A multifunction $\Gamma : [0; 1] \to cwk(X)$ is said to be *Henstock–Kurzweil–Pettis integrable* (or HKP-integrable) on [0, 1] if for every $x^* \in X^*$ the map $s(x^*, \Gamma(\cdot))$ is HK-integrable and for each $I \in \mathcal{I}$ there exists a set $W_I \in cwk(X)$ such that $s(x^*, W_I) = (HK)\int_I s(x^*, \Gamma)$, for every $x^* \in X^*$. The set W_I is called the Henstock–Kurzweil–Pettis integral of Γ over I, and we set $W_I := (HKP)\int_I \Gamma$.

In the previous definition, if HK-integral is replaced by Lebesgue integral and intervals by Lebesgue measurable sets, then we get the definition of the Pettis integral.

For more detailed properties of the integrals involved and for all that is unexplained in this paper, we refer to [12,18,19,26,35–38].

Definition 2.6 An interval multifunction $\Phi : \mathcal{I} \to cwk(X)$ is said to be *finitely additive*, if $\Phi(I_1 \cup I_2) = \Phi(I_1) + \Phi(I_2)$ for every nonoverlapping intervals $I_1, I_2 \in \mathcal{I}$ such that $I_1 \cup I_2 \in \mathcal{I}$. In this case, Φ is said to be an *interval multimeasure*.

A map $M : \mathcal{L} \to cwk(X)$ is said to be a *multimeasure* if for every $x^* \in X^*$, the map $\mathcal{L} \ni A \mapsto s(x^*, M(A))$ is a real-valued measure (cf. [28, Theorem 8.4.10]).

$M : \mathcal{L} \to cwk(X)$ is said to be a d_H*-multimeasure* if for every sequence $(A_n)_{n\geq 1}$ in $\mathcal{L}$ of pairwise disjoint sets with $A = \bigcup_{n\geq 1} A_n$, we have

$$d_H\left(M(A), \sum_{k=1}^{n} M(A_k)\right) \to 0 \quad \text{as } n \to +\infty.$$

A multimeasure $M : \mathcal{L} \to cwk(X)$ is said to be λ*-continuous*, and we write $M \ll \lambda$, if $M(A) = \{0\}$ for every $A \in \mathcal{L}$ such that $\lambda(A) = 0$.

Remark 2.7 It is well known that M is a d_H-multimeasure if and only if it is a multimeasure (cf. [28, Theorem 8.4.10]). Observe moreover that this is a multivalued analogue of Orlicz–Pettis Theorem. It is also known that the indefinite integrals of Henstock or $\mathcal{H}$ integrable multifunctions are interval multimeasures, while the indefinite integrals of Pettis (hence also McShane or Birkhoff) integrable multifunctions are multimeasures.

Definition 2.8 A multifunction $\Gamma : [0, 1] \to cwk(X)$ is said to be *variationally Henstock* (*McShane*) integrable, if there exists an interval multimeasure $\Phi_\Gamma : \mathcal{I} \to cwk(X)$ with the following property: For every $\varepsilon > 0$ there exists a gauge δ on [0, 1] such that for each $\{(I_1, t_1), \ldots, (I_p, t_p)\} \in \Pi_\delta^P$ (resp. Π_δ), we have

$$\sum_{j=1}^{p} d_H\left(\Phi_\Gamma(I_j), \Gamma(t_j)|I_j|\right) < \varepsilon\,. \tag{3}$$

We write then $(vH)\int_0^1 \varGamma\,dt := \varPhi_\varGamma([0,1])$ $((vMS)\int_0^1 \varGamma\,dt := \varPhi_\varGamma([0,1]))$. The set multifunction $\varPhi_\varGamma$ will be called the *variational Henstock* (*McShane*) *primitive* of $\varGamma$.

The variational integrals on a set $I \in \mathcal{I}$ can be defined in an analogous way, and they are uniquely determined. It has been proven in [6, Proposition 2.8] that each variationally Henstock integrable multifunction $\varGamma : [0,1] \to cwk(X)$ is Bochner measurable.

Important tools for the study of multifunctions are embeddings and variational measures. Let $l_\infty(B_{X^*})$ be the Banach space of bounded real-valued functions defined on B_{X^*} endowed with the supremum norm $||\cdot||_\infty$. The Rådström embedding $i : cwk(X) \to l_\infty(B_{X^*})$, given in [6,30] by the relation $cwk(X) \ni W \longrightarrow s(\cdot, W)$, allows to consider G-integrable multifunctions $\varGamma : [0,1] \to cwk(X)$ as G-integrable functions $i \circ \varGamma : [0,1] \to l_\infty(B_{X^*})$. Thanks to the embedding, a multifunction $\varGamma$ is G-integrable if and only if its image $i \circ G$ in $l_\infty(B_{X^*})$ is G-integrable (G stands for any of the gauge integrals).

For what concerns the variational measure we recall that

Definition 2.9 The *variational measure* $V_\varPhi : \mathcal{L} \to \mathbb{R}$ generated by an interval multimeasure $\varPhi : \mathcal{I} \to cwk(X)$ is defined by

$$V_\varPhi(E) := \inf_\delta \{Var(\varPhi, \delta, E) : \delta \text{ is a gauge on } E\},$$

where

$$Var(\varPhi,\delta,E) = \sup\left\{\sum_{j=1}^p \|\varPhi(I_j)\| : \{(I_j,t_j)\}_{j=1}^p \in \varPi_\delta^P \ \text{ and } t_j \in E,\ j=1,\ldots,p.\right\}$$

For other properties, we refer to [5,6,20].

We also remember that for a Pettis integrable mapping $G : [0,1] \to cwk(X)$, its integral J_G is a multimeasure on the σ-algebra $\mathcal{L}$ (cf. [13, Theorem 4.1]) that is λ-continuous. As also observed in [13, section 3], this means that the *embedded* measure $i(J_G)$ is a countably additive measure with values in $l_\infty(B_{X^*})$.

We recall that

Definition 2.10 [39, Definition 2] A function $f : [0,1] \to X$ is said to be *Riemann measurable* on $[0,1]$ if for every $\varepsilon > 0$, there exist an $\eta > 0$ and a closed set $F \subset [0,1]$ with $\lambda([0,1]\setminus F) < \varepsilon$ such that $\left\| \sum_{i=1}^p \left\{f(t_i) - f(t_i')\right\} |I_i| \right\| < \varepsilon$ whenever $\{I_i\}$ is a finite collection of pairwise nonoverlapping intervals with $\max_{1\le i\le p} |I_i| < \eta$ and $t_i,\ t_i' \in I_i \bigcap F$.

According to [39, Theorem 4], each $\mathcal{H}$-integrable function is Riemann measurable on $[0,1]$. Moreover in [10, Theorem 9] it was proved that a function $f : [0,1] \to X$ is $\mathcal{M}$-integrable if and only f is both Riemann measurable and Pettis integrable. So we get the following characterization, that is parallel to Fremlin's description [22]:

Theorem 2.11 *A function $f : [0,1] \to X$ is Birkhoff integrable if and only if it is $\mathcal{H}$-integrable and Pettis integrable.*

Proof The only if part is trivial. For the converse observe that $\mathcal{H}$-integrability implies Riemann measurability by [39, Theorem 4]. Moreover by [22, Theorem 8] f is Mc Shane integrable, and Riemann measurability together with Mc Shane integrability implies $\mathcal{M}$-integrability by [39, Theorem 7]. □

We denote by $\mathcal{S}_P(\varGamma), \mathcal{S}_{MS}(\varGamma), \mathcal{S}_\mathcal{H}(\varGamma), \mathcal{S}_H(\varGamma), \mathcal{S}_{Bi}(\varGamma) = \mathcal{S}_\mathcal{M}(\varGamma)$ and $\mathcal{S}_{vH}(\varGamma)$, the collections of all selections of $\varGamma : [0,1] \to cwk(X)$, which are, respectively, Pettis, McShane, $\mathcal{H}$, Henstock, Birkhoff and variationally Henstock integrable.

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3 Henstock and McShane integrability of $cwk(X)$-valued multifunctions

Proposition 3.1 *Let* $\Gamma : [0, 1] \to cwk(X)$ *be such that* $\Gamma(\cdot) \ni 0$ *a.e. If* Γ *is Henstock integrable (resp.* $\mathcal{H}$*-integrable) on* $[0, 1]$*, then it is also McShane (resp. Birkhoff, i.e.,* $\mathcal{M}$*) integrable on* $[0, 1]$.

Proof Let i be the Rådström embedding of $cwk(X)$ into $l_\infty(B_{X^*})$. If Γ is Henstock integrable, then we just have to prove that $i \circ \Gamma$ is McShane integrable. By the hypothesis, we have that $i \circ \Gamma$ is Henstock integrable. Then, thanks to [22, Corollary 9 (iii)], it will be sufficient to prove convergence in $l_\infty(B_{X^*})$ of all series of the type $\sum_n (H) \int_{I_n} i \circ \Gamma$, where $(I_n)_n$ is any sequence of pairwise nonoverlapping subintervals of $[0, 1]$.

But Γ is HKP-integrable and $s(x^*, \Gamma) \geq 0$ a.e. for every $x^* \in X^*$. It follows from [18, Lemma 1] that Γ is Pettis integrable. Consequently, the range of the indefinite Pettis integral of Γ via the Rådström embedding is a vector measure. This fact guarantees the convergence of the series $\sum_n (H) \int_{I_n} i \circ \Gamma$, since $(P) \int_I \Gamma = (H) \int_I \Gamma$ and $i \circ ((H) \int_I \Gamma) = (H) \int_I i \circ \Gamma$, for every $I \in \mathcal{I}$.

As said before, thanks to [22, Corollary 9 (iii)], $i \circ \Gamma$ is McShane integrable. Consequently, Γ is McShane integrable.

If Γ is $\mathcal{H}$-integrable, then $i \circ \Gamma$ is $\mathcal{H}$-integrable and being already McShane integrable, it is also Pettis integrable [22, Theorem 8]. Applying now Theorem 2.11, we obtain Birkhoff integrability of $i \circ \Gamma$. This yields Birkhoff integrability of Γ . □

Observe that from this proposition it follows that if Γ is Henstock integrable and $\Gamma(\cdot) \ni 0$ a.e., then $i \circ \Gamma$ is Pettis. We remember that the relation between Pettis integrability of Γ and $i \circ \Gamma$ is delicate question and it is examined, for example, in [12].

Theorem 3.2 *Let* $\Gamma : [0, 1] \to cwk(X)$ *be a multifunction. Then the following conditions are equivalent:*

(i) Γ *is Henstock integrable;*
(ii) $\mathcal{S}_H(\Gamma) \neq \emptyset$ *and for every* $f \in \mathcal{S}_H(\Gamma)$ *the multifunction* $\Gamma - f$ *is McShane integrable;*
(iii) *there exists* $f \in \mathcal{S}_H(\Gamma)$ *such that the multifunction* $G := \Gamma - f$ *is McShane integrable.*

Proof $(i) \Rightarrow (ii)$ According to [19, Theorem 3.1] $\mathcal{S}_H(\Gamma) \neq \emptyset$. Let $f \in \mathcal{S}_H(\Gamma)$ be fixed. Then $\Gamma - f$ is also Henstock integrable (in $cwk(X)$) and $0 \in \Gamma - f$ for every $t \in [0, 1]$. By Proposition 3.1, the multifunction $\Gamma - f$ is McShane integrable. Since each McShane integrable multifunction is also Henstock integrable, $(ii) \Rightarrow (iii)$ is trivial, $(iii) \Rightarrow (i)$ follows at once. □

The next result generalizes [19, Theorem 3.4], proved there for $cwk(X)$-valued multifunctions with compact valued integrals.

Theorem 3.3 *Let* $\Gamma : [0, 1] \to cwk(X)$ *be a multifunction. Then the following conditions are equivalent:*

(i) Γ *is McShane integrable;*
(ii) Γ *is Henstock integrable and* $\mathcal{S}_H(\Gamma) \subset \mathcal{S}_{MS}(\Gamma)$.
(iii) Γ *is Henstock integrable and* $\mathcal{S}_H(\Gamma) \subset \mathcal{S}_P(\Gamma)$*;*
(iv) Γ *is Henstock integrable and* $\mathcal{S}_P(\Gamma) \neq \emptyset$.
(v) Γ *is Henstock and Pettis integrable.*

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Proof $(i) \Rightarrow (ii)$ Pick $f \in \mathcal{S}_H(\Gamma)$; then, according to Theorem 3.2, $\Gamma = G + f$ for a McShane integrable G. But as Γ is Pettis integrable, also f is Pettis integrable (cf. [37, Corollary 1.5], [13, Corollary 2.3]). In view of [22, Theorem 8], f is McShane integrable.

$(ii) \Rightarrow (iii)$ is valid, because each McShane integrable function is also Pettis integrable ([23, Theorem 2C]).

$(iii) \Rightarrow (iv)$ In view of [19, Theorem 3.1] $\mathcal{S}_H(\Gamma) \neq \emptyset$ and so ***(iii)*** implies $\mathcal{S}_P(\Gamma) \neq \emptyset$.

$(iv) \Rightarrow (v)$ Take $f \in \mathcal{S}_P(\Gamma)$. Since Γ is Henstock integrable, it is also HKP-integrable and so applying [18, Theorem 2], we obtain a representation $\Gamma = G + f$, where $G : [0, 1] \to cwk(X)$ is Pettis integrable in $cwk(X)$. Consequently, Γ is also Pettis integrable in $cwk(X)$ and so ***(v)*** holds.

$(v) \Rightarrow (i)$ In virtue of [19, Theorem 3.1] Γ has a McShane integrable selection f. It follows from Theorem 3.2 that the multifunction $G : [0, 1] \to cwk(X)$ defined by $\Gamma(t) = G(t) + f(t)$ is McShane integrable. □

4 Birkhoff and $\mathcal{H}$-integrability of $cwk(X)$-valued multifunctions

A quick analysis of the proof of [19, Theorem 3.1] proves the following:

Proposition 4.1 *If $\Gamma : [0, 1] \to cwk(X)$ is $\mathcal{H}$-integrable, then $\mathcal{S}_{\mathcal{H}}(\Gamma) \neq \emptyset$. If $\Gamma : [0, 1] \to cwk(X)$ is Pettis and $\mathcal{H}$-integrable, then $\mathcal{S}_{Bi}(\Gamma) \neq \emptyset$.*

As a consequence, we have the following result:

Theorem 4.2 *Let $\Gamma : [0, 1] \to cwk(X)$ be a multifunction. Then the following conditions are equivalent:*

(i) *Γ is $\mathcal{H}$-integrable;*
(ii) *$\mathcal{S}_{\mathcal{H}}(\Gamma) \neq \emptyset$ and for every $f \in \mathcal{S}_{\mathcal{H}}(\Gamma)$ the multifunction $\Gamma - f$ is Birkhoff integrable;*
(iii) *there exists $f \in \mathcal{S}_{\mathcal{H}}(\Gamma)$ such that the multifunction $\Gamma - f$ is Birkhoff integrable.*

Proof $(i) \Rightarrow (ii)$ Instead of [19, Theorem 3.1] we apply Proposition 4.1. The remaining implications are trivial. □

Applying Theorems 4.2 and 2.11, we have the following:

Theorem 4.3 *Let $\Gamma : [0, 1] \to cwk(X)$ be a multifunction. Then the following conditions are equivalent:*

(i) *Γ is Birkhoff integrable;*
(ii) *Γ is $\mathcal{H}$-integrable and $\mathcal{S}_{\mathcal{H}}(\Gamma) \subset \mathcal{S}_{Bi}(\Gamma)$.*
(iii) *Γ is $\mathcal{H}$-integrable and $\mathcal{S}_{\mathcal{H}}(\Gamma) \subset \mathcal{S}_{MS}(\Gamma)$.*
(iv) *Γ is $\mathcal{H}$-integrable and $\mathcal{S}_{\mathcal{H}}(\Gamma) \subset \mathcal{S}_P(\Gamma)$;*
(v) *Γ is $\mathcal{H}$-integrable and $\mathcal{S}_P(\Gamma) \neq \emptyset$.*
(vi) *Γ is Pettis and $\mathcal{H}$-integrable.*

Proof $(i) \Rightarrow (ii)$ If $f \in \mathcal{S}_{\mathcal{H}}(\Gamma)$, then, according to Theorem 4.2, $\Gamma = G + f$ for a Birkhoff integrable G. But as Γ is Pettis integrable, also f is Pettis integrable (cf. [13, Corollary 2.3], [37, Corollary1.5]). In view of Theorem 2.11, f is Birkhoff integrable.

$(ii) \Rightarrow (iii) \Rightarrow (iv)$ are valid, because each Birkhoff integrable function is McShane integrable ([21, Proposition 4]) and each McShane integrable function is also Pettis integrable ([23, Theorem 2C]).

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$(iv) \Rightarrow (v)$ In view of Proposition 4.1 $\mathcal{S}_{\mathcal{H}}(\Gamma) \neq \emptyset$ and so ***(iii)*** implies $\mathcal{S}_P(\Gamma) \neq \emptyset$.

$(v) \Rightarrow (vi)$ Take $f \in \mathcal{S}_P(\Gamma)$. Since Γ is $\mathcal{H}$-integrable, it is also HKP-integrable and so applying [18, Theorem 2], we obtain a representation $\Gamma = G + f$, where $G : [0, 1] \to cwk(X)$ is Pettis integrable in $cwk(X)$. Consequently, Γ is also Pettis integrable in $cwk(X)$ and so ***(v)*** holds.

$(vi) \Rightarrow (i)$ In virtue of Proposition 4.1, Γ has a Birkhoff integrable selection f. It follows from Theorem 4.2 that the multifunction $G : [0, 1] \to cwk(X)$ defined by $G := \Gamma - f$ is Birkhoff integrable. □

5 Variationally Henstock integrable selections

Now, in order to examine [6, Question 3.11], we are going to consider the existence of variationally Henstock integrable selections for a variationally Henstock integrable multifunction $\Gamma : [0, 1] \to cwk(X)$. In particular, we extend [6, Theorem 3.12] which gives only a partial answer, and we remove the hypothesis that X has the Radon–Nikodým property or the hypothesis $\mathcal{S}_{vH} \neq \emptyset$ in the theorems of decomposition arising from the previous quoted result; so we give a complete answer to the open question.

First of all we give the following result which extends [6, Theorem 3.12].

Theorem 5.1 *Let $\Gamma : [0, 1] \to cwk(X)$ be any variationally Henstock integrable multifunction. Then $\mathcal{S}_{vH} \neq \emptyset$ and every strongly measurable selection of Γ is also variationally Henstock integrable.*

Proof Let us notice first that Γ is Bochner measurable and so it possesses strongly measurable selections [6, Proposition 3.3] (the quoted result is a consequence of [27]). Let f be a strongly measurable selection of Γ. Then f is Henstock–Kurzweil–Pettis integrable, and the mapping G defined by $G := \Gamma - f$ is Pettis integrable: see [18, Theorem 1]. Since Γ is vH-integrable, then Γ is Bochner measurable ([6, Proposition 2.8]). As the difference of $i(\Gamma)$ and $i(\{f\})$, the function $i(G)$ is strongly measurable, together with G. Therefore, G has essentially d_H-separable range (that is, there is $E \in \mathcal{L}$, with $\lambda([0, 1]\setminus E) = 0$ and $G(E)$ is d_H-separable) and so $i(G)$ is also Pettis integrable (see [11, Theorem 3.4 and Lemma 3.3 and their proofs]).

Now, since Γ is variationally Henstock integrable, the variational measure V_Φ associated with the vH-integral of Γ is absolutely continuous (see [40, Proposition 3.3.1]). If V_ϕ is associated with the Henstock–Kurzweil–Pettis integral of f, then $V_\phi \leq V_\Phi$ and so it is also absolutely continuous with respect to λ. Since $\|G\| \leq \|\Gamma\| + \|f\|$, it is clear that also V_G is λ-continuous.

Then, $i(G)$ satisfies all the hypotheses of [5, Corollary 4.1], and therefore, it is variationally Henstock integrable. But then $i(\{f\})$ is too, as the difference of $i(\Gamma)$ and $i(G)$, and finally f is variationally Henstock integrable. □

Remark 5.2 At this point, it is worth to observe that the thesis of Theorem 5.1 holds true only for strongly measurable selections of Γ. In general, Γ may have scalarly measurable selections which are neither strongly measurable nor even Henstock integrable (see [6, Proposition 3.2] and [1, Theorem 3.7]).

A decomposition result, similar to Theorem 4.2, can be formulated now. It is also given in [7, Corollary 3.5] but with a different proof.

Theorem 5.3 ([7, Corollary 3.5]) *Let $\Gamma : [0, 1] \to cwk(X)$ be a variationally Henstock integrable multifunction. Then Γ is the sum of a variationally Henstock integrable selection f*

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and a Birkhoff integrable multifunction $G : [0,1] \to cwk(X)$ that is variationally Henstock integrable.

Proof Let f be any variationally Henstock integrable selection of Γ. Then, as previously proved, Γ is Bochner measurable, f is strongly measurable and the variational measures associated with their integral functions are λ-continuous. Moreover, f is HKP-integrable, and, according to [18, Theorem 1], the multifunction G, defined by $G := \Gamma - f$, is Pettis integrable. Since Γ and f are variationally Henstock integrable, the same holds true for G. Hence, also $i(G)$ is variationally Henstock integrable and, consequently, by [6, Proposition 4.1], G is also Birkhoff integrable. □

Remark 5.4 There is now an obvious question: Let $\Gamma : [0,1] \to cwk(X)$ be a variationally Henstock integrable multifunction. Does there exist a variationally Henstock integrable selection f of Γ such that $G := \Gamma - f$ is variationally McShane integrable?

Unfortunately, in general, the answer is negative. The argument is similar to that applied in [17]. Assume that X is separable and g is the X-valued function constructed in [15] that is vH (and so strongly measurable by [6, Proposition 2.8]), Pettis but not vMS-integrable (see [15]). Let $\Gamma(t) := \text{conv}\{0, g(t)\}$. Then, Γ is vH-integrable (see [6, Example 4.7]), but it is not vMS-integrable ([6, Theorem 3.7] or [6, Example 4.7]) and possesses at least one vH-integrable selection by Theorem 5.1 . Let now $f \in \mathcal{S}_{vH}(\Gamma)$ and consider the multifunction $G = \Gamma - f$. Clearly G is vH-integrable and $G(t) = conv\{-f(t), g(t) - f(t)\}$ for all $t \in [0,1]$. If we suppose that G is variationally McShane integrable, then its selections $-f, g - f$ will be Bochner integrable since they are strongly measurable and dominated by $\|G\|$, but that would mean that g is Bochner integrable, contrary to the assumption. □

The next theorems 5.5 extend [6, Theorems 4.3, 4.4]. In fact we can remove the hypothesis $\mathcal{S}_{vH}(\Gamma) \neq \emptyset$ thanks to Theorem 5.1 and [6, Proposition 3.6]. Its proof is the same of the quoted results in [6].

Theorem 5.5 *Let $\Gamma : [0,1] \to cwk(X)$ be a vH-integrable multifunction. Then the following equivalences hold true:*

$$\mathcal{S}_{vH}(\Gamma) \subset \mathcal{S}_{MS}(\Gamma) \iff \mathcal{S}_{vH}(\Gamma) \subset \mathcal{S}_{P}(\Gamma) \iff \mathcal{S}_{P}(\Gamma) \neq \emptyset \iff$$
$$\Gamma \textit{ is Pettis integrable} \iff \Gamma \textit{ is McShane integrable}$$

Moreover if Γ is also integrably bounded, then all the previous statements are equivalent to the variational McShane integrability of Γ.

So, in particular

Corollary 5.6 *A function $f : [0,1] \to X$ is variationally McShane integrable (= Bochner integrable, cf. [16]) if and only if it is variationally Henstock integrable and integrably bounded.*

6 Variational $\mathcal{H}$-integral

Recently, Naralenkov introduced stronger forms of Henstock and McShane integrals of functions and called them $\mathcal{H}$ and $\mathcal{M}$ integrals. We apply that idea to variational integrals. Since the variational McShane integral of functions coincides with Bochner integral, the same holds true for the $\mathcal{M}$-integral. In case of the variational $\mathcal{H}$-integral, the situation is not as obvious,

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but we shall prove in this section that the variational $\mathcal{H}$-integral coincides with the variational Henstock integral. We begin with the following strengthening of the Riemann measurability, due to [39].

Definition 6.1 We say that a function $f : [0, 1] \to X$ is *strongly Riemann measurable*, if for every $\varepsilon > 0$, there exist a positive number η and a closed set $F \subset [0, 1]$ such that $\lambda([0, 1]\setminus F) < \varepsilon$ and $\sum_{k=1}^{K} \| f(t_k) - f(t'_k)\| \cdot |I_k| < \varepsilon$ whenever $\{I_1, \ldots, I_K\}$ is a nonoverlapping finite family of subintervals of $[0, 1]$ with $\max_k |I_k| < \eta$ and, all points t_k, t'_k are chosen in $I_k \cap F$, $k = 1, \ldots, K$.

Lemma 6.2 *If $f : [0, 1] \to X$ is strongly measurable, then f is strongly Riemann measurable.*

Proof Fix $\varepsilon > 0$. Then there exists a closed set $F \subset [0, 1]$ such that $\lambda([0, 1]\setminus F) < \varepsilon$ and $f_{|F}$ is continuous. Since F is compact, then $f_{|F}$ is uniformly continuous, and so there exists a positive number $\delta > 0$ such that, as soon as t, t' are chosen in F, with $|t - t'| < \delta$, then $\| f(t) - f(t')\| < \varepsilon$. Now, fix any finite family $\{I_1, \ldots, I_K\}$ of nonoverlapping intervals with $\max_k |I_k| < \eta$, and choose arbitrarily points t_k, t'_k in $I_k \cap F$ for every k: Then we have

$$\sum_{k=1}^{K} \| f(t_k) - f(t'_k)\| \cdot |I_k| < \sum_{k=1}^{K} \varepsilon |I_k| < \varepsilon.$$

□

Now, in order to prove that each variationally Henstock function $f : [0, 1] \to X$ is also variationally $\mathcal{H}$-integrable, we shall follow the lines of the proof of [39, Theorem 6], with $E = [0, 1]$.

Another preliminary result is needed, concerning *interior* Perron partitions.

Definition 6.3 Let $\delta : [0, 1] \to \mathbb{R}^+$ be any gauge on $[0, 1]$, and let $P := \{(t_1, I_1), (t_2, I_2), \ldots, (t_K, I_K)\} \in \Pi^P_\delta$. P is said to be an *interior* Perron partition if $t_k \in int(I_k)$ for all k, except when I_k contains 0 or 1, in which case $t_k \in int(I_k)$ or $t_k \in I_k \cap \{0, 1\}$.

We can observe that the result given by Naralenkov in [39, Lemma 3] can be expressed in the following way:

Lemma 6.4 [39, Lemma 3] *Let δ be a gauge on $[0, 1]$, and let $P := \{(t_1, I_1), \ldots, (t_K, I_K)\}$ be any δ-fine Perron partition of $[0, 1]$, where the tags $t_1, \ldots, t_K$ are all distinct. Then, for each function $\phi : [0, 1] \to X$ and each $\varepsilon > 0$ there exists a δ-fine interior Perron partition of $[0, 1]$, $P' := \{(t_1, I'_1), (t_2, I'_2), \ldots, (t_K, I'_K)\}$ such that $\sum_{k=1}^{K} \|\phi(t_k)\| \cdot \big| |I_k| - |I'_k| \big| < \varepsilon$.*

Thanks to this Lemma we can obtain, for variationally Henstock integrable functions, the following result:

Lemma 6.5 *Let $f : [0, 1] \to X$ be any variationally Henstock integrable mapping, and denote by Φ its primitive, i.e., $\Phi(I) = \int_I f$, for all intervals I. Suppose that δ is a gauge on $[0, 1]$, and $P := \{(t_1, I_1), (t_2, I_2), ..., (t_K, I_K)\} \in \Pi^P_\delta$ has all the tags $t_1, \ldots, t_K$ distinct. Then, for each $\varepsilon > 0$ there exists a δ-fine interior Perron partition $P' := \{(t_1, I'_1), (t_2, I'_2), \ldots, (t_K, I'_K)\}$ of $[0, 1]$, such that $\sum_{k=1}^{K} \| f(t_k)\| \cdot \big| |I_k| - |I'_k| \big| < \varepsilon$, and $\sum_{k=1}^{K} \|\Phi(I_k) - \Phi(I'_k)\| \le \varepsilon$.*

Proof Since f is variationally Henstock integrable, the function $t \mapsto \Phi([0, t])$ is continuous with respect to the norm topology of X. □

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We are now ready to present the announced result.

Theorem 6.6 *Let Γ:[0, 1] $\to cwk(X)$ be any variationally Henstock integrable multifunction. Then it is also variationally $\mathcal{H}$-integrable.*

Proof Thanks to Rådström embedding Theorem we may assume that Γ is a function taking values in a Banach space. Denote it by f. First of all, we observe that f is strongly measurable, and therefore strongly Riemann measurable. Fix $\varepsilon > 0$. Then there exists a sequence of pairwise disjoint closed sets $(F_n)_n$ in $[0, 1]$ and a decreasing sequence $(\eta_n)_n$ in $\mathbb{R}^+$ tending to 0, such that the set $N := \bigcap_n([0, 1]\setminus F_n)$ has Lebesgue measure 0, and moreover such that for every integer n

$$\sum_{k=1}^{K} \left\| f(t_k) - f(t'_k) \right\| \cdot |I_k| \le \frac{\varepsilon}{2^n}$$

holds, as soon as $(I_k)_{k=1}^K$ is any nonoverlapping family of subintervals with $\max_k |I_k| < \eta_n$ and the points t_k, t'_k are taken in $F_n \cap I_k$. Now, choose any bounded gauge δ_0, corresponding to ε in the definition of variational Henstock integral of f, and set $\delta(t) = \theta_n(t)$, when $t \in F_n$ for some index n, and $\delta(t) = \delta_0$ if $t \in N$, where

$$\theta_n(t) = \min\left\{\eta_n, \frac{1}{2}\max\{\delta_0(t), \limsup_{F_n \ni \tau \to t} \delta_0(\tau)\}\right\}.$$

δ is measurable, as proved in [39, Theorem 6]. We shall prove now that the gauge $\delta/2$ can be chosen in correspondence with ε in the notion of variational integrability of f. To this aim, fix any partition $P := \{(t_1, I_1), \ldots, (t_K, I_K)\} \in \Pi^P_{\delta/2}$. Without loss of generality, we may assume that all tags t_k are distinct. Indeed, if a tag t is common to two intervals I, J of P, then

$$\left\| f(t)|I| - \int_I f \right\| + \left\| f(t)|J| - \int_J f \right\| \le 2\max\left\{ \left\| f(t)|I| - \int_I f \right\|, \left\| f(t)|J| - \int_J f \right\| \right\}$$

and therefore the sum

$$\sum_k \left\| f(t_k)|I_k| - \int_{I_k} f \right\|$$

is dominated by twice the analogous sum evaluated on a (possibly partial) partition with distinct tags.

Thanks to Lemma 6.5, there exists an *interior* Perron partition $P' := \{(t_k, J_k), k = 1, \ldots, K\} \in \Pi^P_{\delta/2}$ such that

$$\max\left\{ \sum_{k=1}^{K} \|f(t_k)\| \cdot \big||I_k| - |J_k|\big|, \quad \sum_{k=1}^{K} \left\| \int_{I_k} f - \int_{J_k} f \right\| \right\} \le \varepsilon\,. \tag{4}$$

Now, we shall suitably modify the tags of P'; fix k and consider the tag t_k.

If $t_k \in F_n$ for some n and $\limsup_{F_n \ni s \to t_k} \delta_0(s) \ge \delta_0(t_k)$, then we pick t'_k in the set $int(I_k) \cap F_n$ in such a way that $\delta_0(t'_k) > \delta(t_k)$. This is possible since then we have $\limsup_{F_n \ni s \to t_k} \delta_0(s) \ge 2\delta(t_k)$.

If $t_k \in F_n$ for some n and $\limsup_{F_n \ni s \to t_k} \delta_0(s) < \delta_0(t_k)$ or if $t_k \in N$, then we set $t'_k = t_k$. From this, it follows that the partition $P'' := \{(t'_k, I_k) : k = 1, \ldots, K\}$ is a δ_0-fine interior Perron partition. Summarizing, we have

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$$\sum_k \left\| f(t_k)|I_k| - \int_{I_k} f \right\| \le \sum_k \|f(t_k)\| \cdot \big||I_k| - |J_k|\big| + \sum_k \|f(t_k) - f(t'_k)\| \cdot |J_k| +$$

$$+ \sum_k \left\| f(t'_k)|J_k| - \int_{J_k} f \right\| + \sum_k \left\| \int_{I_k} f - \int_{J_k} f \right\|.$$

Now,

$$\sum_k \|f(t_k)\| \cdot \big||I_k| - |J_k|\big| + \sum_k \left\| \int_{I_k} f - \int_{J_k} f \right\| \le 2\varepsilon$$

thanks to (4), and

$$\sum_k \left\| f(t'_k)|J_k| - \int_{J_k} f \right\| \le \varepsilon$$

because P'' is δ_0-fine. Finally, thanks to the strong Riemann measurability,

$$\sum_k \|f(t_k) - f(t'_k)\| \cdot |J_k| = \sum_{t_k \in N^c} \|f(t_k) - f(t'_k)\| \cdot |J_k| \le \sum_n \frac{\varepsilon}{2^n} = \varepsilon,$$

and so

$$\sum_k \left\| f(t_k)|I_k| - \int_{I_k} f \right\| \le 4\varepsilon$$

which concludes the proof. □

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