

Hamiltonians defined by biorthogonal sets

Fabio Bagarello

Dipartimento di Energia, Ingegneria dell'Informazione e Modelli Matematici,
Università di Palermo, and INFN, Torino
e-mail: fabio.bagarello@unipa.it
Tel: +390912389722; Fax: +39091427258
home page: www.unipa.it/fabio.bagarello

Giorgia Bellomonte

Dipartimento di Matematica e Informatica,
Università di Palermo, I-90123 Palermo, Italy
e-mail: giorgia.bellomonte@unipa.it

Abstract

In some recent papers, the studies on biorthogonal Riesz bases has found a renewed motivation because of their connection with pseudo-hermitian Quantum Mechanics, which deals with physical systems described by Hamiltonians which are not self-adjoint but still may have real point spectra. Also, their eigenvectors may form Riesz, not necessarily orthonormal, bases for the Hilbert space in which the model is defined. Those Riesz bases allow a decomposition of the Hamiltonian, as already discussed in some previous papers. However, in many physical models, one has to deal not with o.n. bases or with Riesz bases, but just with biorthogonal sets. Here, we consider the more general concept of \mathcal{G} -quasi basis and we show a series of conditions under which a definition of non self-adjoint Hamiltonian with purely point real spectra is still possible.

I Introduction and preliminaries

For several years physicists have devoted their studies to those systems which were described by self-adjoint Hamiltonians. This choice have been led by the fact that the eigenvalues of a Hamiltonian describing a physical system represent the energy of that system hence they must be real to have a physical meaning and self-adjoint Hamiltonians have real eigenvalues. This is important to ensure that the dynamics of the system is unitary, so that the probability described by the wave-function is preserved during the time evolution. In recent years many physicists (as Bender and his collaborators) first, and mathematicians after, started to consider with more and more interest non self-adjoint Hamiltonians with real spectra because they described some physical system. The beginning of the story goes probably back to the paper [11], in which the eigenstates of the manifestly non self-adjoint Hamiltonian $H = p^2 + ix^3$ were deduced and found to be real. Here x and p are the position and momentum operators, satisfying the Weyl algebra. A very recent book on this and related topics is [2].

The key objects in that analysis were the so-called PT-symmetric Hamiltonians with real point spectra. A PT-symmetric Hamiltonian is an operator such that

$$PTH(PT)^{-1} = PTHPT = H,$$

where P and T are respectively the operators of parity and time-reversal transformations, usually defined¹ according to $PxP = -x$, $PpP = TpT = -p$, $Ti\mathbb{1}T = -i\mathbb{1}$, where $x, p, \mathbb{1}$ are respectively the position, momentum, and identity operators acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ and i is the square root of -1 . Later was understood that PT-symmetry can be replaced by more general requirements, still getting the same conclusion: at the beginning of this century Mostafazadeh introduced the concept of pseudo-hermitian (also called pseudo-symmetric or quasi-Hermitian) operators as those operators satisfying, in some sense, an intertwining relation of the form $AG = GA^\dagger$. Here A is the pseudo-hermitian operator, while G is a certain positive operator. Notice that, quite often, both A and G are unbounded, so that the equality $AG = GA^\dagger$ is only formal. However, this equality was recently made rigorous in [1], where A and G are defined as those operators A , with dense domain $D(A)$ for which there exists a positive operator G , with dense domain $D(G)$ in Hilbert space \mathcal{H} such that $D(A) \subset D(G)$ and

$$\langle A\xi, G\eta \rangle = \langle G\xi, A\eta \rangle, \quad \xi, \eta \in D(A). \quad (1.1)$$

¹In the physical literature the definition of P and T really depends on the particular model under consideration, and can change quite a bit, from model to model.

Some recent results show that, even if reality of the eigenvalues of a certain Hamiltonian is ensured, the basis property of its eigenstates is, in many cases, lost. In other words, if a non self-adjoint operator H in the Hilbert space \mathcal{H} has purely punctual real spectrum, and if H is PT-symmetric or, more generally, pseudo-symmetric, it is not necessarily true that the set of eigenstates of H , $\mathcal{F}_\varphi = \{\varphi_n \in \mathcal{H}\}$ and of its adjoint H^\dagger , $\mathcal{F}_\Psi = \{\Psi_n \in \mathcal{H}\}$, are biorthogonal bases for \mathcal{H} . Indeed, this feature was already discussed in two papers by Davies, [17, 18], and then in several papers by one of us (FB), see [3] for a recent review, in [24], and in other recent papers. The importance of the basis property is obvious. For our present purposes, it mainly lays in the fact that the operators which has those bases as sets of eigenvectors can be decomposed in terms of those sets. Still, quite often, the sets of eigenvectors (although are not a basis of the Hilbert space) are *complete* in \mathcal{H} , meaning that the only vector which is orthogonal to all the φ_n 's or to all the Ψ_n 's is the zero vector.

Recent literature has dealt with, in a certain sense, the inverse problem than that before. In [7, 8, 10], the problem of considering some particular biorthogonal sets of vectors to define non self-adjoint operators, has been considered, leading to a number of interesting results as factorizability of Hamiltonians by some kind of lowering and raising operators. In particular, in [8] Riesz bases of a given Hilbert space have been used, while in [10] and, later, in [7], the interest was focused again on a generalization of the notion of Riesz basis, living in a rigged Hilbert space. In [4], the notion of \mathcal{G} -quasi bases is considered. As we will see in the following section, two biorthogonal sets $\mathcal{F}_\varphi = \{\varphi_n \in \mathcal{H}, n \geq 0\}$ and $\mathcal{F}_\Psi = \{\Psi_n \in \mathcal{H}, n \geq 0\}$, satisfying $\langle \varphi_n, \Psi_m \rangle = \delta_{n,m}$, are called *\mathcal{G} -quasi bases* if, for all $f, g \in \mathcal{G}$, the following holds:

$$\langle f, g \rangle = \sum_{n \geq 0} \langle f, \varphi_n \rangle \langle \Psi_n, g \rangle = \sum_{n \geq 0} \langle f, \Psi_n \rangle \langle \varphi_n, g \rangle, \quad (1.2)$$

being \mathcal{G} a dense subset of the Hilbert space \mathcal{H} . The physical relevance of these families of vectors is that in some physical system driven by some Hamiltonian H , see [3], the set of eigenstates of H and H^\dagger turn out to be \mathcal{G} -quasi bases, even if they are and bases.

The problem of generalizing those results to biorthogonal \mathcal{G} -quasi bases raises naturally, and in fact this is the main content of this paper, where we will discuss how some of the general ideas introduced in [8] (we will work on a given Hilbert space \mathcal{H} , leaving a possible extension to rigged Hilbert spaces to a future analysis) still work even if Riesz bases are replaced by \mathcal{G} -quasi bases. This generalization requires us to go into two parallel directions: first of all, since \mathcal{G} -quasi bases have only be introduced recently, [4], and since several unusual features are related to these sets of vectors, we will describe in some details three examples of them. Secondly, physical applications of \mathcal{G} -quasi bases show that unbounded operators (and intertwining operators in

particular, see Section IV) become relevant, in this context. So we will take care of this aspect and we will also discuss in some details what happens when one deals with \mathcal{G} quasi-bases and, therefore, when a resolution of the identity can only be introduced in a weak form, as in (1.2).

The paper is organized as follows: in Section II we introduce some useful definitions on bases and biorthogonal sets and we shortly review the results in [8] on Hamiltonians defined by Riesz bases. In Section III we give examples of \mathcal{G} -quasi bases, and we start considering a few physical consequences, which are analyzed further in Section IV. Section V contains our conclusions.

II Some useful definitions and results

Let us begin by recalling some well known definitions. Let \mathcal{H} be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ linear in the second entry. We recall the following definitions.

Definition 1 *The sequence $\mathcal{X} = \{\xi_n \in \mathcal{H}, n \geq 0\}$ is said to be complete if $\langle \varphi, \xi_n \rangle = 0$ for every $n \in \mathbb{N}$, with $\varphi \in \mathcal{H}$, then $\varphi = 0$.*

This means that the set \mathcal{X} is if the only vector $\varphi \in \mathcal{H}$ which is orthogonal to all the ξ_n 's is necessarily the zero vector. Sometimes in the literature, rather than *complete* the word *total* is adopted.

In our analysis the following, slightly modified, version of completeness will be also used.

Definition 2 *Let $\mathcal{F} = \{f_n \in \mathcal{H}, n \geq 0\}$ be a set and $\mathcal{V} \subseteq \mathcal{H}$ a subspace, we will say that \mathcal{F} is complete in \mathcal{V} if, taken $\varphi \in \mathcal{V}$ such that $\langle \varphi, f_n \rangle = 0$ for all $n \geq 0$, then $\varphi = 0$.*

In particular, if $\mathcal{V} = \mathcal{H}$, then we will simply say that \mathcal{F} is complete.

Definition 3 *A set $\mathcal{E} = \{e_n \in \mathcal{H}, n \geq 0\}$ is said to be a (Schauder) basis for \mathcal{H} if for every $f \in \mathcal{H}$ there exists a unique sequence $\{c_n(f)\}$ of complex numbers (depending on the vector f) such that*

$$f = \sum_{n=0}^{\infty} c_n(f) e_n. \quad (2.1)$$

Definition 4 *A (Schauder) basis for \mathcal{H} , $\mathcal{E} = \{e_n \in \mathcal{H}, n \geq 0\}$, is said to be an orthonormal (o.n.) basis for \mathcal{H} if $\langle e_n, e_m \rangle = \delta_{n,m}$, for every $n, m \geq 0$.*

If $\mathcal{E} = \{e_n \in \mathcal{H}, n \geq 0\}$ is an orthonormal (o.n.) basis for \mathcal{H} , then the coefficient $c_n(f)$ in (2.1) can be written as $c_n(f) = \langle e_n, f \rangle$, for every $n \geq 0$ and for every $f \in \mathcal{H}$.

Definition 5 A set $\mathcal{F} = \{f_n \in \mathcal{H}, n \geq 0\}$ is said to be a Riesz basis for \mathcal{H} if there exists a bounded operator T on \mathcal{H} , with bounded inverse, and an orthonormal basis $\mathcal{E} = \{e_n \in \mathcal{H}, n \geq 0\}$ of \mathcal{H} , such that $f_n = Te_n$, for all $n \geq 0$.

Remark 6 It is clear that if \mathcal{F} is a Riesz basis, then in general $\langle f_n, f_m \rangle \neq \delta_{n,m}$, for every $n, m \geq 0$. Moreover, if T and \mathcal{E} are as in Definition 5, the set of vectors $\mathcal{L} = \{l_n = (T^{-1})^\dagger e_n, n \geq 0\}$ is a Riesz basis for \mathcal{H} as well (called dual basis), and $\langle f_n, l_m \rangle = \delta_{n,m}$ for all $n, m \geq 0$, i.e. \mathcal{F} and \mathcal{L} are biorthogonal. Here the symbol \dagger indicates the adjoint with respect to the natural scalar product² $\langle \cdot, \cdot \rangle$ in \mathcal{H} . In these hypotheses, any vector $h \in \mathcal{H}$ can be expanded as follows:

$$h = \sum_{n=0}^{\infty} \langle f_n, h \rangle l_n = \sum_{n=0}^{\infty} \langle l_n, h \rangle f_n. \quad (2.2)$$

Here and in the remainder of the paper the convergence of the various series is always assumed to be unconditional. Of course, any o.n. basis is a Riesz basis, with $T = \mathbb{1}$ the identity operator on \mathcal{H} . In this case the three sets above just collapse: $\mathcal{E} = \mathcal{F} = \mathcal{L}$.

The same expansion as in (2.2) holds when \mathcal{F} and \mathcal{L} are biorthogonal bases, but not necessarily of the Riesz kind. Notice that each basis \mathcal{F} in \mathcal{H} possesses a unique biorthogonal set \mathcal{V} which is also a basis: hence the expansion in (2.2) is unique, [12, Theorem 3.3.2]. If $\mathcal{F} = \{f_n \in \mathcal{H}, n \geq 0\}$ is a basis (in any of the senses considered so far), then \mathcal{F} is complete, while the converse is not true, in general. In fact, while for o.n. sets completeness is equivalent to the basis property, for non o.n. sets this is false, see e.g. [3, Section 3.2.1].

Remark 7 It should be observed that equation (2.2), as many others results in the rest of the paper, make sense, in principle, also in the context of frame theory, see for instance [12, 14, 15, 21]. However, we will not discuss the relation between \mathcal{G} -quasi bases and frames here. In fact, we are more interested to those sets having no redundancy, in contrast with what happens in frame theory, since, while this aspect is surely important for signal analysis, it is not so important for quantum mechanics, which is our main interest here. We will comment something more on this aspect along the paper.

For convenience of the reader we also recall the following definition.

Definition 8 Let $\mathcal{F} = \{f_n \in \mathcal{H}, n \geq 0\}$ and $\mathcal{L} = \{l_n \in \mathcal{H}, n \geq 0\}$ be two biorthogonal sets. The projection operator P_k is defined as $P_k f = \langle f_k, f \rangle l_k$.

²The word *natural* is used since, quite often, in Physics literature on \mathcal{PT} -quantum mechanics other scalar products are also introduced.

Remark 9 If \mathcal{F} is a basis, both P_k and $\sum_{k=1}^N P_k$ are uniformly bounded. Viceversa, if \mathcal{F} is complete, and if $\sum_{k=1}^N P_k$ is uniformly bounded, then \mathcal{F} is a basis, [16, Lemma 3.3.3]. This fact will be used later on. In particular, the norm of P_k and those of f_k and l_k are related by the following relation: $1 \leq \|P_k\| = \|f_k\| \|l_k\|$, for all k .

Another absolutely non trivial difference between o.n. and not o.n. sets has to do with the possibility of extending the basis property from a dense subset of \mathcal{H} to the whole Hilbert space. Let $\mathcal{E} = \{e_n \in \mathcal{H}, n \geq 0\}$ be an o.n. set and let \mathcal{V} be a dense subspace of \mathcal{H} . Suppose that each vector $f \in \mathcal{V}$ can be written as follows: $f = \sum_n \langle e_n, f \rangle e_n$, then \mathcal{E} is an o.n. basis for \mathcal{V} and, furthermore, all the vectors in \mathcal{H} , and not only those in \mathcal{V} , admit a similar expansion: $\hat{f} = \sum_n \langle e_n, \hat{f} \rangle e_n, \forall \hat{f} \in \mathcal{H}$. Hence \mathcal{E} is also an o.n. basis for \mathcal{H} [20, Theorem 3.4.7]. Let us now replace the o.n. set \mathcal{E} with a second, no longer o.n., set $\mathcal{X} = \{x_n \in \mathcal{H}, n \geq 0\}$, and let us again assume that every $f \in \mathcal{V}$ can be written, in a unique way, as $f = \sum_n c_n(f) x_n$, for certain coefficients $c_n(f)$ depending on f . Then, explicit counterexamples show that a similar expansion does not hold in general for all vectors in \mathcal{H} , see [3, Section 3.2.1]. This will be evident also in the examples considered later on in this paper. Then we see once again that losing orthonormality³ produces new and often undesired mathematical consequences. Moreover, the lack of orthonormality has physical consequences as well, [3, 17, 18, 24]. In fact, when a non self-adjoint Hamiltonian H with only punctual real spectrum is considered, the set of its eigenstates, $\mathcal{F}_\varphi = \{\varphi_n \in \mathcal{H}\}$, is not an o.n. one, in general. Therefore, we are forced to deal with problems similar to those discussed above. In particular, even if \mathcal{F}_φ were complete, it would not be necessarily a basis for \mathcal{H} . Also, even if each vector in a dense subspace of \mathcal{H} can be linearly expanded in terms of the φ_n 's, a similar expansion may not be true in all of \mathcal{H} . For this reason, in connection with several recent applications, see [3, 4, 5] and references therein, the notion of \mathcal{G} -quasi basis was introduced and used heavily in the analysis of the eigenvectors and the eigenvalues of certain non self-adjoint Hamiltonians.

Definition 10 Let \mathcal{G} be a dense subspace of Hilbert space \mathcal{H} . Two biorthogonal sets $\mathcal{F}_\varphi = \{\varphi_n \in \mathcal{H}, n \geq 0\}$ and $\mathcal{F}_\Psi = \{\Psi_n \in \mathcal{H}, n \geq 0\}$ ($\langle \varphi_n, \Psi_m \rangle = \delta_{n,m}$, for every $n, m \geq 0$), are called \mathcal{G} -quasi bases if, for all $f, g \in \mathcal{G}$, the following holds:

$$\langle f, g \rangle = \sum_{n \geq 0} \langle f, \varphi_n \rangle \langle \Psi_n, g \rangle = \sum_{n \geq 0} \langle f, \Psi_n \rangle \langle \varphi_n, g \rangle. \quad (2.3)$$

When $\mathcal{G} = \mathcal{H}$, i.e. when (2.3) holds for all $f, g \in \mathcal{H}$, then \mathcal{F}_φ and \mathcal{F}_Ψ are simply called quasi bases.

³please recall that we are not considering any redundancy here, as we have already stressed above.

From formula (2.3), it follows immediately that, if $f \in \mathcal{G}$ is orthogonal to all the Ψ_n 's (or to all the φ_n 's), then f is necessarily zero and, as a consequence, \mathcal{F}_Ψ (or \mathcal{F}_φ) is complete in \mathcal{G} . Indeed, using (2.3) with $g = f \in \mathcal{G}$, if $\langle \Psi_n, f \rangle = 0$ (or $\langle f, \varphi_n \rangle = 0$) for all n , we find $\|f\|^2 = \sum_{n \geq 0} \langle f, \varphi_n \rangle \langle \Psi_n, f \rangle = 0$. Therefore $\|f\| = 0$, so that $f = 0$.

When \mathcal{F}_φ and \mathcal{F}_Ψ are quasi bases, then they are complete. We refer to [3] for more details on \mathcal{G} -quasi bases. Here we just want to observe that equation (2.3) can be seen as a weak version of a resolution of the identity, which turns out to be very important, if not essential, in several physical applications, as, just to cite one, in quantization problems, see Part 2 of [19], i.e. when one wants to replace a classical system with its quantum counterpart⁴

Remark 11 *It could happen that it is easier to check that f is orthogonal to, say, all the φ_{2n} 's and to all the Ψ_{2n+1} 's. Then, again, $f = 0$, for similar reasons. Of course, this implies in turns that f is also orthogonal to all the φ_{2n+1} 's and to all the Ψ_{2n} 's.*

II.1 Working with Riesz bases

We now briefly review what is the role of Riesz bases in our scheme. Let $\mathcal{F}_\phi = \{\phi_n \in \mathcal{H}, n \geq 0\}$ be a Riesz basis in the Hilbert space \mathcal{H} and let $\mathcal{F}_\psi = \{\psi_n \in \mathcal{H}, n \geq 0\}$ be its dual biorthogonal Riesz basis. In [8] the following operators have been introduced and their properties have been studied in details:

$$\begin{cases} D(H_{\phi,\psi}^\alpha) = \{f \in \mathcal{H}; \sum_{n=0}^{\infty} \alpha_n \langle \psi_n, f \rangle \phi_n \text{ exists in } \mathcal{H}\} \\ H_{\phi,\psi}^\alpha f = \sum_{n=0}^{\infty} \alpha_n \langle \psi_n, f \rangle \phi_n, f \in D(H_{\phi,\psi}^\alpha) \end{cases}$$

and

$$\begin{cases} D(H_{\psi,\phi}^\alpha) = \{f \in \mathcal{H}; \sum_{n=0}^{\infty} \alpha_n \langle \phi_n, f \rangle \psi_n \text{ exists in } \mathcal{H}\} \\ H_{\psi,\phi}^\alpha f = \sum_{n=0}^{\infty} \alpha_n \langle \phi_n, f \rangle \psi_n, f \in D(H_{\psi,\phi}^\alpha) \end{cases}$$

⁴For instance, a classical harmonic oscillator with energy (in suitable units) $E = \frac{1}{2}(p^2 + x^2)$ can be *quantized* replacing the time-depending functions $x(t)$ and $p(t)$ with the operators $\hat{X} := \sum_{n,m=1}^{\infty} \left(\int_{\mathbb{R}} \overline{e_n(x)} x e_m(x) dx \right) P_{n,m}$ and $\hat{P} := \sum_{n,m=1}^{\infty} \left(\int_{\mathbb{R}} \overline{e_n(x)} \left(-i \frac{d}{dx}\right) e_m(x) dx \right) P_{n,m}$. Here $e_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-\frac{x^2}{2}}$, $H_n(x)$ being the n -th Hermite polynomial, and $P_{n,m}$ is the operator defined by $(P_{n,m} f)(x) = \langle e_m, f \rangle e_n(x)$, for all $f(x) \in \mathcal{L}^2(\mathbb{R})$. Of course, one should pay attention to the convergence of the series appearing in the definition of \hat{X} and \hat{P} . We refer to [19] for more details.

Here $\alpha = \{\alpha_n\}$ is any sequence of complex numbers. Once again we recall that the convergence of the series is, as everywhere in this paper, the unconditional one.

Once we put $\mathcal{D}_\phi := \text{span}\{\phi_n\}$ and $\mathcal{D}_\psi := \text{span}\{\psi_n\}$, it is clear that $\mathcal{D}_\phi \subset D(H_{\phi,\psi}^\alpha)$, $\mathcal{D}_\psi \subset D(H_{\psi,\phi}^\alpha)$ and that $H_{\phi,\psi}^\alpha \phi_k = \alpha_k \phi_k$, and $H_{\psi,\phi}^\alpha \psi_k = \alpha_k \psi_k$, $k \geq 0$, so that ϕ_k and ψ_k are eigenstates of $H_{\phi,\psi}^\alpha$ and $H_{\psi,\phi}^\alpha$ respectively, with the same eigenvalues. Hence, in particular, since \mathcal{F}_ϕ and \mathcal{F}_ψ are bases for \mathcal{H} , $H_{\phi,\psi}^\alpha$ and $H_{\psi,\phi}^\alpha$ are densely defined, closed, and $(H_{\phi,\psi}^\alpha)^\dagger = H_{\psi,\phi}^{\bar{\alpha}}$, where $\bar{\alpha} = \{\bar{\alpha}_n\}$. Moreover $H_{\phi,\psi}^\alpha$ is bounded if and only if $H_{\psi,\phi}^\alpha$ is bounded and this is true if and only if α is a bounded sequence. In particular $H_{\phi,\psi}^{\mathbf{1}} = H_{\psi,\phi}^{\mathbf{1}} = \mathbb{1}$, where $\mathbf{1}$ is the sequence constantly equal to 1. Moreover, the spectra of $H_{\phi,\psi}^\alpha$ and $H_{\psi,\phi}^\alpha$ are real if and only if each α_n is real.

Remark 12 *It is worth to notice that the operators $H_{\phi,\psi}^\alpha$ and $H_{\psi,\phi}^\alpha$ introduced above looks quite similar to the so called multipliers, see [32, 33], which are operators of the form*

$$M_{m,\Phi,\psi} f = \sum_n m_n \langle \psi_n, f \rangle \Phi_n,$$

where $\{\Phi_n\}$ and $\{\Psi_n\}$ are fixed sequences in the Hilbert space, while $\{m_n\}$ is a sequence of scalars. The interest in [32, 33] was on mathematical aspects of these multipliers, like, for instance, their unconditional convergence. On the other hand, we are more interested in the role of \mathcal{G} -quasi bases in the definition of our (physically-motivated) multipliers.

In a similar way, introducing a second sequence of complex numbers, $\beta := \{\beta_n\}$, the following operators can be also defined:

$$\begin{cases} D(S_\phi^\beta) = \{f \in \mathcal{H}; \sum_{n=0}^\infty \beta_n \langle \phi_n, f \rangle \phi_n \text{ exists in } \mathcal{H}\} \\ S_\phi^\beta f = \sum_{n=0}^\infty \beta_n \langle \phi_n, f \rangle \phi_n, f \in D(S_\phi^\beta) \end{cases}$$

and

$$\begin{cases} D(S_\psi^\beta) = \{f \in \mathcal{H}; \sum_{n=0}^\infty \beta_n \langle \psi_n, f \rangle \psi_n \text{ exists in } \mathcal{H}\} \\ S_\psi^\beta f = \sum_{n=0}^\infty \beta_n \langle \psi_n, f \rangle \psi_n, f \in D(S_\psi^\beta). \end{cases}$$

It is clear that

$$\mathcal{D}_\psi \subset D(S_\phi^\beta) \quad \text{and} \quad S_\phi^\beta \psi_k = \beta_k \phi_k, \quad k \geq 0; \quad (2.4)$$

$$\mathcal{D}_\phi \subset D(S_\psi^\beta) \quad \text{and} \quad S_\psi^\beta \phi_k = \beta_k \psi_k, \quad k \geq 0. \quad (2.5)$$

Hence, in particular, S_ϕ^β and S_ψ^β are also densely defined, again due to the fact that \mathcal{F}_ϕ and \mathcal{F}_ψ are bases, and, see [8, Proposition 2.2], they are closed and self adjoint if each $\beta_n \in \mathbb{R}$. Furthermore, S_ϕ^β is bounded if and only if S_ψ^β is bounded and this is true if and only if β is a bounded sequence. Moreover, if $\beta_n = 1$ for all $n \geq 0$, then $S_\phi := S_\phi^1$ and $S_\psi := S_\psi^1$ are bounded positive self-adjoint operators on \mathcal{H} and they are inverses of each other⁵, that is $S_\phi = (S_\psi)^{-1}$. Also, see [8, Proposition 2.3] $S_\psi H_{\phi,\psi}^\alpha = H_{\psi,\phi}^\alpha S_\psi = S_\psi^\alpha$, and $S_\phi H_{\psi,\phi}^\alpha = H_{\phi,\psi}^\alpha S_\phi = S_\phi^\alpha$, which are useful intertwining relations. More details on these operators and their domains can be found in [8].

Remark 13 *The relevance of intertwining relations and of the intertwining operators is discussed in detail, for instance, in [6, 25, 26, 31]. They turn out to be very useful in the deduction of the eigenvalues and eigenvectors for certain pairs of Hamiltonians, obeying suitable intertwining relations. When this happens, the Hamiltonians turn out to be isospectral, and their eigenvectors are mapped ones into the others by the intertwining operator itself.*

Going back to the general case, let $\alpha = \{\alpha_n\}$ be a sequence of complex numbers. We define the operators $h_{\phi,\psi}^\alpha$ and $h_{\psi,\phi}^\alpha$ as follows:

$$\begin{cases} h_{\phi,\psi}^\alpha = S_\psi^{1/2} H_{\phi,\psi}^\alpha S_\phi^{1/2}, \\ h_{\psi,\phi}^\alpha = S_\phi^{1/2} H_{\psi,\phi}^\alpha S_\psi^{1/2}. \end{cases} \quad (2.6)$$

Then, see [8, Proposition 2.4], $D(h_{\phi,\psi}^\alpha) = \{S_\psi^{1/2} f; f \in D(H_{\phi,\psi}^\alpha)\}$, $D(h_{\psi,\phi}^\alpha) = \{S_\phi^{1/2} f; f \in D(H_{\psi,\phi}^\alpha)\}$ and these are both dense in \mathcal{H} . Moreover $(h_{\phi,\psi}^\alpha)^* = h_{\psi,\phi}^{\bar{\alpha}}$. Finally, and very important, if $\{\alpha_n\} \subset \mathbb{R}$, then $h_{\phi,\psi}^\alpha$ is self-adjoint.

More results can be found in [8, Section III], where ladder (i.e. raising and lowering) operators are also introduced in terms of the Riesz bases \mathcal{F}_ϕ and \mathcal{F}_ψ . In the rest of the paper we will discuss, both from a general point of view and considering particular examples, how much of the above structure can be recovered when the biorthogonal sets are no longer Riesz bases. In particular, we will see under what conditions some relevant operators we are going to introduce are, in fact, densely defined.

⁵ S_ψ and S_ϕ are usually called *frame operators* in the literature of frames, or *metric operators* in quantum mechanics, while the operators S_ϕ^β and S_ψ^β are particular cases of *Riesz bases multipliers*, [33].

III Some examples when orthonormality is lost

In this section we discuss how, and to which extent, loosing orthonormality can give rise to certain mathematical and physical consequences which do not appear whenever one uses o.n. bases. Of course, we will also give up the assumption that the set of vectors we consider is a Riesz basis, since this case is completely under control.

For making these differences evident, we discuss in some details three examples, the last one being directly physically motivated, whereas the first two are interesting mainly from a mathematical point of view, but not only.

III.1 First example

Let $\mathcal{F}_e := \{e_n, n \geq 1\}$ be an o.n. basis of a Hilbert space \mathcal{H} , and let us introduce the set $\mathcal{F}_x = \{x_n = \sum_{k=1}^n \frac{1}{k} e_k, n \geq 1\}$, see [21]. Of course we can write x_n as follows: $x_1 = e_1$, and $x_n = x_{n-1} + \frac{1}{n} e_n, n \geq 2$. It is clear that $x_n \in \mathcal{H}$ for all $n \geq 1$. Also, \mathcal{F}_x is complete in \mathcal{H} . In fact, $\langle f, x_n \rangle = 0$ for all $n \geq 1$ implies that $\langle f, e_n \rangle = 0$ for all $n \geq 1$ as well, so that $f = 0$, necessarily. Let us now introduce \mathcal{G} as the linear span of the e_n 's. This set is dense in \mathcal{H} . It is possible to see that \mathcal{F}_x is a basis for \mathcal{G} , [3], but not for \mathcal{H} . In particular, if on one hand it is easy to check that each vector $f = \sum_{k=1}^N c_k e_k, N < \infty$, can be written as a linear combination of the x_n 's, on the other hand it is also possible to check that $h := \sum_{k=1}^{\infty} \frac{1}{k} e_k$, which is a non zero vector in \mathcal{H} , cannot be written as $\sum_{k=1}^{\infty} \alpha_k x_k$, for any choice of the complex numbers α_k . In fact, assume that this is possible. Then, we should have

$$\left\{ \begin{array}{l} \langle h, e_1 \rangle = 1, \text{ and } \langle h, e_1 \rangle = \sum_{k=1}^{\infty} \alpha_k \quad \Rightarrow \quad \sum_{k=1}^{\infty} \alpha_k = 1 \\ \langle h, e_2 \rangle = \frac{1}{2}, \text{ and } \langle h, e_2 \rangle = \frac{1}{2} \sum_{k=2}^{\infty} \alpha_k \quad \Rightarrow \quad \sum_{k=2}^{\infty} \alpha_k = 1 \\ \langle h, e_3 \rangle = \frac{1}{3}, \text{ and } \langle h, e_3 \rangle = \frac{1}{3} \sum_{k=3}^{\infty} \alpha_k \quad \Rightarrow \quad \sum_{k=3}^{\infty} \alpha_k = 1, \end{array} \right.$$

and so on. Hence, we should have $\alpha_1 = \alpha_2 = \alpha_3 = \dots = 0$, which implies that $h = 0$, which is absurd.

The set which is biorthogonal to \mathcal{F}_x is $\mathcal{F}_y := \{y_n = n e_n - (n+1) e_{n+1}, n \geq 1\}$. Indeed we can prove that $\langle x_k, y_l \rangle = \delta_{k,l}, \forall k, l \in \mathbb{N}$. \mathcal{F}_y is complete in \mathcal{G} , but not in \mathcal{H} . Indeed, let $f \in \mathcal{G}$ be orthogonal to all the y_n 's. The vector f can be written as $f = \sum_{k=1}^N c_k e_k$, for some finite N . Of course, $c_k = \langle e_k, f \rangle$. Now, condition $\langle f, y_1 \rangle = 0$ implies that $\langle f, e_1 \rangle = 2 \langle f, e_2 \rangle$. Also, from $\langle f, y_2 \rangle = 0$, it follows that $2 \langle f, e_2 \rangle = 3 \langle f, e_3 \rangle$, and so on. However, since $\langle f, y_N \rangle = 0$, we deduce that $N \langle f, e_N \rangle = (N+1) \langle f, e_{N+1} \rangle = 0$. Then, $\langle f, e_k \rangle = 0$ for all $k = 1, 2, \dots, N$, so that $f = 0$. Therefore, as stated, \mathcal{F}_y is complete in \mathcal{G} . To prove that \mathcal{F}_y is not complete in \mathcal{H} , it is

sufficient to observe that the vector h , already introduced, is orthogonal to all the y_n 's, but it is not zero.

Contrarily to \mathcal{F}_x , it can be shown that the set \mathcal{F}_y is not a basis for \mathcal{G} . Therefore, a fortiori, \mathcal{F}_y is not a basis for \mathcal{H} . We can further prove that, even though \mathcal{F}_x and \mathcal{F}_y are not quasi-bases, they are still \mathcal{G} -quasi bases. To prove these claims we first observe that, taking again h as above, on one hand we have $\|h\|^2 = \langle h, h \rangle = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, whereas on the other hand we have $\sum_{n=1}^{\infty} \langle h, x_n \rangle \langle y_n, h \rangle = 0$, since $\langle y_n, h \rangle = 0$ for all n . Hence, at least for this h , $\langle h, h \rangle \neq \sum_{n=1}^{\infty} \langle h, x_n \rangle \langle y_n, h \rangle$, and our first assertion is proved. Of course, this is in agreement with the fact that \mathcal{F}_y is not complete in \mathcal{H} , as it should be, if they were quasi-bases. However, they are \mathcal{G} -quasi bases because, taking f and g in \mathcal{G} , it is just a straightforward computation to check that

$$\sum_{n=1}^{\infty} \langle f, x_n \rangle \langle y_n, g \rangle = \sum_{n=1}^{\infty} \langle f, y_n \rangle \langle x_n, g \rangle = \langle f, g \rangle,$$

which is what we had to prove.

III.2 Second example

Let, as before, $\mathcal{F}_e := \{e_n, n \geq 1\}$ be an o.n. basis of a Hilbert space \mathcal{H} , \mathcal{G} its linear span, and consider the sets

$$\mathcal{F}_x = \left\{ x_n = \sum_{k=1}^n (-1)^{n+k} e_k, n \geq 1 \right\} \quad \text{and} \quad \mathcal{F}_y = \{y_n = e_n + e_{n+1}, n \geq 1\}.$$

Then $\langle y_n, x_k \rangle = \delta_{n,k}$ for all $k, n \geq 1$, [12]. \mathcal{F}_x is complete in \mathcal{H} and, interestingly enough, it is a basis for \mathcal{G} . Indeed, let $f \in \mathcal{G}$, then f can be written as a finite linear combination of the e_n 's. Let us assume, to begin with, that $f = \sum_{k=1}^{2N} c_k e_k$. Now we will show that there exist $\alpha_j \in \mathbb{C}$, $j = 1, 2, \dots, 2N$ such that $f = \sum_{k=1}^{2N} \alpha_k x_k$. In fact, equating the two expansions, we deduce that

$$\underline{c}_{2N} = T_{2N} \underline{\alpha}_{2N},$$

where

$$\underline{c}_{2N} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \cdot \\ \cdot \\ c_{2N-1} \\ c_{2N} \end{pmatrix}, \quad \underline{\alpha}_{2N} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdot \\ \cdot \\ \alpha_{2N-1} \\ \alpha_{2N} \end{pmatrix}, \quad T_{2N} = \begin{pmatrix} 1 & -1 & 1 & \cdot & \cdot & 1 & -1 \\ 0 & 1 & -1 & \cdot & \cdot & -1 & 1 \\ 0 & 0 & 1 & \cdot & \cdot & 1 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $\det(T_{2N}) = 1$ for all N , T_{2N}^{-1} surely exists, and therefore $\underline{\alpha}_{2N} = T_{2N}^{-1}\underline{c}_{2N}$. Then we recover the coefficients α_n 's of the expansion of f in terms of x_n 's. Exactly the same conclusion we find if $f = \sum_{k=1}^{2N+1} c_k e_k$. Hence, \mathcal{F}_x is a basis for \mathcal{G} , as stated. What is more, one also can prove that \mathcal{F}_x is, in fact, a basis also for \mathcal{H} . This is not particularly surprising, since the determinant of T_{2N} (and of T_{2N+1}), i.e. the possibility of inverting those matrices, is independent of N . A consequence is the completeness of \mathcal{F}_x in \mathcal{H} . As for \mathcal{F}_y , this set is complete in \mathcal{H} . Indeed if $f \in \mathcal{H}$ is orthogonal to every y_n , then we easily deduce that $|\langle f, e_j \rangle|$ is independent of j so that $\|f\|^2 = \sum_{j=1}^{\infty} |\langle f, e_j \rangle|^2$ can be finite if only if $\langle f, e_j \rangle = 0$ for all j . Hence $f = 0$. However, it turns out that \mathcal{F}_y is not a basis for \mathcal{H} , [12]. For instance, one can notice that e_1 cannot be written in terms of y_n 's. This incidentally also implies that, as in the previous example, \mathcal{F}_y cannot even be a basis for \mathcal{G} , since $e_1 \in \mathcal{G}$.

It is now interesting to see that, even though \mathcal{F}_y is not a basis for \mathcal{G} , \mathcal{F}_x and \mathcal{F}_y are \mathcal{G} -quasi bases. This can be proved with a direct computation:

$$\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, y_n \rangle \langle x_n, g \rangle = \sum_{n=1}^{\infty} \langle f, x_n \rangle \langle y_n, g \rangle = \sum_{n=1}^{\text{Min}(N,M)} \overline{f_n} g_n,$$

for all $f, g \in \mathcal{G}$ such that $f = \sum_{n=1}^N f_n e_n$, $g = \sum_{n=1}^M g_n e_n$ with $f_i, g_j \in \mathbb{C}$, $i = 1, \dots, N$ and $j = 1, \dots, M$. It is evident, therefore, that as in Section III.1, also here it is possible to recover a (weak) resolution of the identity, even though we are working with biorthogonal sets which are not bases.

III.3 Third example, with Hamiltonians

This example is, in a certain sense, more physically-motivated, since it is directly linked to a quantum harmonic oscillator. Moreover, exactly for this reason, it is also relevant because

it will suggest how to enrich our previous examples by adding some physical insight to their original mathematical aspects. We start defining the following functions of $\mathcal{S}(\mathbb{R})$, the set of C^∞ , fast decreasing, functions:

$$x_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-\frac{x^2}{4}}, \quad y_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-\frac{3x^2}{4}},$$

for $n = 0, 1, 2, 3, \dots$. Here $H_n(x)$ is the n -th Hermite polynomial. It is easy to check that

$$\langle x_m, y_n \rangle = \langle e_m, e_n \rangle = \delta_{m,n},$$

where $e_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-\frac{x^2}{2}}$ is the n -th function of the set $\mathcal{F}_e = \{e_n(x)\}$, which is the well known o.n. basis of $\mathcal{L}^2(\mathbb{R})$ consisting of eigenvectors of the self-adjoint Hamiltonian of the quantum harmonic oscillator, [27]. We continue to call \mathcal{G} the linear span of the e_n 's. We have

$$h e_n = \left(n + \frac{1}{2} \right) e_n,$$

$n = 0, 1, 2, 3, \dots$. It is clear that, for all such n 's, $y_n = T e_n$, $x_n = T^{-1} e_n$, where T is the multiplication operator defined as $(Tf)(x) = e^{-\frac{x^2}{4}} f(x)$, for all $f(x) \in \mathcal{L}^2(\mathbb{R})$. Of course, T is bounded and self-adjoint. It is also invertible, but its inverse is unbounded. However, T^{-1} is densely defined since its domain, $D(T^{-1})$, contains e.g. the set $D(\mathbb{R})$ of the compactly supported C^∞ -functions, which is dense in $\mathcal{L}^2(\mathbb{R})$. Of course, this is a proper inclusion since each $e_n(x)$ belongs to $D(T^{-1})$, but does not belong to $D(\mathbb{R})$, for any n .

A standard argument, see [23], shows that $\mathcal{F}_y = \{y_n(x), n \geq 0\}$ and $\mathcal{F}_x = \{x_n(x), n \geq 0\}$ are both complete in $\mathcal{L}^2(\mathbb{R})$. They are also $D(\mathbb{R})$ -quasi bases: in fact, let $f, g \in D(\mathbb{R})$, then

$$\begin{aligned} \langle f, g \rangle &= \langle T^{-1} T f, g \rangle = \langle T f, T^{-1} g \rangle = \sum_{n=0}^{\infty} \langle T f, e_n \rangle \langle e_n, T^{-1} g \rangle = \\ &= \sum_{n=0}^{\infty} \langle f, T e_n \rangle \langle T^{-1} e_n, g \rangle = \sum_{n=0}^{\infty} \langle f, y_n \rangle \langle x_n, g \rangle. \end{aligned}$$

Analogously we can check that $\langle f, g \rangle = \sum_{n=0}^{\infty} \langle f, x_n \rangle \langle y_n, g \rangle$. On the other hand, \mathcal{F}_y and \mathcal{F}_x are not bases for $\mathcal{L}^2(\mathbb{R})$ because, as we have already recalled in Remark 9, a necessary condition for \mathcal{F}_y and \mathcal{F}_x to be bases is that $\sup_n \|y_n\| \|x_n\| < \infty$, but this is not the case. In fact, (see the integral 2.20.16. nr. 2 in [30]), we get

$$\|y_n\|^2 = \sqrt{\frac{2}{3}} \frac{2}{3^{n/2}} P_n \left(\frac{2}{\sqrt{3}} \right), \quad \|x_n\|^2 = \sqrt{2} 3^{n/2} P_n \left(\frac{2}{\sqrt{3}} \right),$$

where $P_n(x)$ is the n -th Legendre polynomial. Now, using the asymptotic behavior in n of these polynomials, for $x > 1$, see [34], we see that, for large n ,

$$\|y_n\|^2 \|x_n\|^2 \simeq \frac{2}{\sqrt{3}\pi} \frac{3^n}{n},$$

which diverges with n . Hence $\sup_n \|y_n\| \|x_n\| = \infty$. Therefore, neither \mathcal{F}_y nor \mathcal{F}_x can be bases for $\mathcal{L}^2(\mathbb{R})$. Nevertheless, they have interesting physical properties, since they are eigenstates of the following manifestly non self-adjoint operators

$$H_1 = \frac{1}{2} \left[-\frac{d^2}{dx^2} - x \frac{d}{dx} + \frac{1}{2} \left(\frac{3x^2}{2} - 1 \right) \right]$$

and

$$H_2 = \frac{1}{2} \left[-\frac{d^2}{dx^2} + x \frac{d}{dx} + \frac{1}{2} \left(\frac{3x^2}{2} + 1 \right) \right].$$

In fact, since the Hermite polynomial $H_n(x)$ satisfies the differential equation $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$, for all $n \in \mathbb{N}$, a direct computation shows that

$$H_1 y_n = E_n y_n, \quad H_2 x_n = E_n x_n,$$

where $E_n = n + \frac{1}{2}$ and $n = 0, 1, 2, 3, \dots$

Then the three Hamiltonians H_1 , H_2 and h are all isospectral. This is in agreement with the fact that they are all (*strongly*) similar, i.e. that, for every $f \in D(\mathbb{R})$, we have $ThT^{-1}f = H_1f$ and $T^{-1}hTf = H_2f$. This also suggests that, for all those functions, $H_2f = H_1^\dagger f$, as one also can explicitly check.

Quite often, when isospectral Hamiltonians appear related by some (possibly extended, as in this case) similarity operator, it is a standard procedure to introduce the following operators:

$$D(S_x) = \left\{ g \in \mathcal{H} : \sum_{n=0}^{\infty} \langle x_n, g \rangle x_n \text{ exists in } \mathcal{H} \right\}, \quad D(S_y) = \left\{ f \in \mathcal{H} : \sum_{n=0}^{\infty} \langle y_n, f \rangle y_n \text{ exists in } \mathcal{H} \right\},$$

and $S_x g = \sum_{n=0}^{\infty} \langle x_n, g \rangle x_n$, $S_y f = \sum_{n=0}^{\infty} \langle y_n, f \rangle y_n$, for $g \in D(S_x)$ and $f \in D(S_y)$. Using the continuity of T , it is clear that S_y is everywhere defined and that $S_y = T^2$. In fact, taken $f \in D(S_y)$, we have

$$S_y f = \sum_{n=0}^{\infty} \langle y_n, f \rangle y_n = \sum_{n=0}^{\infty} \langle T e_n, f \rangle T e_n = T \left(\sum_{n=0}^{\infty} \langle e_n, T f \rangle e_n \right) = T(Tf).$$

Of course, since T^2 is bounded, this equality can be extended to the whole \mathcal{H} . Notice that, in particular, $S_y x_n = y_n$ for all n .

Of course, such a simple argument does not hold for S_x . This is because T^{-1} is unbounded and, therefore, not continuous on \mathcal{H} . However, we can still prove that $S_x = T^{-2}$, but in a *weak form*, i.e. we can prove that

$$\langle \xi, (T^{-2} - S_x) g \rangle = 0,$$

for all $\xi \in D(\mathbb{R})$, and for all $g \in D(\mathbb{R}) \cap D(S_x)$, which we assume here to be a sufficiently rich set. This is a reasonable assumption since, recalling that $S_x y_n = x_n$ for all n , we see that the linear span of the y_n 's, \mathcal{D}_y , is a subset of $D(S_x)$. Moreover, we observe that \mathcal{D}_y is the image of the dense set \mathcal{G} , via the bounded operator T . Since \mathcal{F}_y is complete, its linear span \mathcal{D}_y is dense in \mathcal{H} . So, $D(S_x)$ is surely a rather rich set. Of course, what is not evident is that $D(\mathbb{R}) \cap D(S_x)$ is also rich, and will only be assumed here.

IV Hamiltonians defined by \mathcal{G} -quasi bases: theory and examples

The examples in literature, together with those introduced in Section III, show that there exist several biorthogonal sets of vectors with different characteristics, most of which are harder to deal with than o.n bases, but which are still interesting (both in mathematics and in physics) and sufficiently well behaved. In what follows, somehow inspired by what we have done in Section III.3, we discuss more mathematical and physical facts related to the biorthogonal sets considered in Sections III.1 and III.2.

In particular, we use the general results on \mathcal{F}_x and \mathcal{F}_y deduced in those sections to define some manifestly non self-adjoint operators, which we still call Hamiltonians, having x_n and y_n as eigenstates. In this way we will significantly extend what was first proposed in [8], starting from biorthogonal Riesz bases and then in [7, 9] in the more general settings of rigged Hilbert spaces. Interestingly enough, we will see that many of the results deduced in [8] also can be recovered here, in a situation in which \mathcal{F}_x and \mathcal{F}_y are not even bases, but just \mathcal{G} -quasi bases, with \mathcal{G} some dense subset of \mathcal{H} .

To make our results model-independent, we devote the first part of this section to discuss some general results which extend those in [8] (see also Section II.1) to the present settings. In the remaining part of the section, we will go back to the particular choices considered in Sections III.1 and III.2, and we will see what can be said in those cases.

IV.1 Some general results

Let $\alpha := \{\alpha_n, n \in \mathbb{N}\}$ be a sequence of complex numbers, \mathcal{F}_x and \mathcal{F}_y biorthogonal \mathcal{G} -quasi bases for some dense subset $\mathcal{G} \subset \mathcal{H}$, and $H_{x,y}^\alpha$ and $H_{y,x}^\alpha$ two operators defined as follows:

$$D(H_{x,y}^\alpha) = \left\{ f \in \mathcal{H} : \sum_{n=1}^{\infty} \alpha_n \langle y_n, f \rangle x_n \text{ exists in } \mathcal{H} \right\},$$

$$D(H_{y,x}^\alpha) = \left\{ g \in \mathcal{H} : \sum_{n=1}^{\infty} \alpha_n \langle x_n, g \rangle y_n \text{ exists in } \mathcal{H} \right\},$$

and

$$H_{x,y}^\alpha f := \sum_{n=1}^{\infty} \alpha_n \langle y_n, f \rangle x_n, \quad H_{y,x}^\alpha g := \sum_{n=1}^{\infty} \alpha_n \langle x_n, g \rangle y_n, \quad (4.1)$$

for all $f \in D(H_{x,y}^\alpha)$ and $g \in D(H_{y,x}^\alpha)$. In analogy with what has been discussed in Section II.1, we easily see that

$$\mathcal{D}_y := \text{span}\{y_n\} \subseteq D(H_{y,x}^\alpha) \quad \mathcal{D}_x := \text{span}\{x_n\} \subseteq D(H_{x,y}^\alpha); \quad (4.2)$$

$$H_{y,x}^\alpha y_k = \alpha_k y_k, \quad H_{x,y}^\alpha x_k = \alpha_k x_k, \quad k \geq 1. \quad (4.3)$$

Therefore, the x_n 's and the y_n 's are eigenstates respectively of $H_{x,y}^\alpha$ and $H_{y,x}^\alpha$, and the complex numbers α_n 's are their (common) eigenvalues. So, from this point of view, not much has changed with respect to what we have summarized in Section II.1. What is really different here is that, since neither \mathcal{F}_x nor \mathcal{F}_y are (Riesz) bases, neither $H_{y,x}^\alpha$ nor $H_{x,y}^\alpha$ need to be densely defined, in general. However, when this is true, more can be deduced⁶. We will assume, in the remaining part of this section, that the operators $H_{y,x}^\alpha$ and $H_{x,y}^\alpha$ are densely defined. For this reason we will see that suitable conditions exist which make these assumptions verified in concrete situations, as the examples in Sections IV.2 and IV.3 show.

Proposition 14 *Let $\{\alpha_n\} \subset \mathbb{R}$, then $(H_{y,x}^\alpha)^\dagger \supseteq H_{x,y}^\alpha$.*

Proof. We have to check that each $h \in D(H_{x,y}^\alpha)$ also belongs to $D((H_{y,x}^\alpha)^\dagger)$, and that for such h 's, $(H_{y,x}^\alpha)^\dagger h = H_{x,y}^\alpha h$. If $h \in D(H_{x,y}^\alpha)$, the series $\sum_{n=1}^{\infty} \alpha_n \langle y_n, h \rangle x_n$ is norm convergent to

⁶It is clear that having densely defined Hamiltonians does not necessarily imply that their eigenvectors do form Riesz bases. In fact, in Section III.3 we have seen an explicit example in which H_1 and H_2 are densely defined even though their eigenstates are not Riesz bases.

$H_{x,y}^{\alpha}h$. Hence, using the continuity of the scalar product, we have

$$\begin{aligned}\langle f, H_{x,y}^{\alpha}h \rangle &= \left\langle f, \sum_{n=1}^{\infty} \alpha_n \langle y_n, h \rangle x_n \right\rangle = \sum_{n=1}^{\infty} \alpha_n \langle y_n, h \rangle \langle f, x_n \rangle = \\ &= \left\langle \sum_{n=1}^{\infty} \alpha_n \langle x_n, f \rangle y_n, h \right\rangle = \langle H_{y,x}^{\alpha}f, h \rangle,\end{aligned}$$

for all $f \in D(H_{y,x}^{\alpha})$. This means that $h \in D((H_{y,x}^{\alpha})^{\dagger})$ and that $(H_{y,x}^{\alpha})^{\dagger}h = H_{x,y}^{\alpha}h$, which is what we had to prove. \square

Remark 15 *If $\{\alpha_n\} \subset \mathbb{R}$, in some cases the two operators $(H_{y,x}^{\alpha})^{\dagger}$ and $H_{x,y}^{\alpha}$ do coincide. One of these cases is when the sets \mathcal{F}_x and \mathcal{F}_y are Riesz bases, see e.g. [8]. Another, even simpler, situation is when $H_{x,y}^{\alpha}$ and $H_{y,x}^{\alpha}$ are bounded operators, which is the case, for instance, if the sequence $\{|\alpha_n| \|x_n\| \|y_n\|\}$ belongs to $l^1(\mathbb{R})$.*

Let us define now, in analogy e.g. with [8], the following lowering and raising operators which can be used to factorize the Hamiltonians introduced before. To this aim, we assume that the sequence α satisfies the following condition: $0 = \alpha_1 \leq \alpha_2 \leq \dots$. Then we introduce the following operators:

$$\begin{cases} D(A_{x,y}) = \{f \in \mathcal{H}; \sum_{n=2}^{\infty} \sqrt{\alpha_n} \langle y_n, f \rangle x_{n-1} \text{ exists in } \mathcal{H}\} \\ A_{x,y}f = \sum_{n=2}^{\infty} \sqrt{\alpha_n} \langle y_n, f \rangle x_{n-1}, \quad f \in D(A_{x,y}) \end{cases}$$

$$\begin{cases} D(A_{y,x}) = \{f \in \mathcal{H}; \sum_{n=2}^{\infty} \sqrt{\alpha_n} \langle x_n, f \rangle y_{n-1} \text{ exists in } \mathcal{H}\} \\ A_{y,x}f = \sum_{n=2}^{\infty} \sqrt{\alpha_n} \langle x_n, f \rangle y_{n-1}, \quad f \in D(A_{y,x}) \end{cases}$$

$$\begin{cases} D(B_{x,y}) = \{f \in \mathcal{H}; \sum_{n=1}^{\infty} \sqrt{\alpha_{n+1}} \langle y_n, f \rangle x_{n+1} \text{ exists in } \mathcal{H}\} \\ B_{x,y}f = \sum_{n=1}^{\infty} \sqrt{\alpha_{n+1}} \langle y_n, f \rangle x_{n+1}, \quad f \in D(B_{x,y}) \end{cases}$$

$$\begin{cases} D(B_{y,x}) = \{f \in \mathcal{H}; \sum_{n=1}^{\infty} \sqrt{\alpha_{n+1}} \langle x_n, f \rangle y_{n+1} \text{ exists in } \mathcal{H}\} \\ B_{y,x}f = \sum_{n=1}^{\infty} \sqrt{\alpha_{n+1}} \langle x_n, f \rangle y_{n+1}, \quad f \in D(B_{y,x}). \end{cases}$$

These operators behave as some sort of ladder operators. In fact we have, for instance: $x_n \in D(A_{x,y})$ and $A_{x,y}x_n = \sqrt{\alpha_n}x_{n-1}$, for all $n \geq 2$. Also, $x_n \in D(B_{x,y})$ and $B_{x,y}x_n = \sqrt{\alpha_{n+1}}x_{n+1}$, for all $n \geq 1$, and so on.

From now on we will assume that (at least) \mathcal{F}_x is a basis for some dense subset $\mathcal{G} \subseteq \mathcal{H}$. This hypothesis holds true, for instance, in Sections III.1 and III.2, where, we recall, \mathcal{G} is the linear span of the vectors e_n of a given o.n. basis. Hence $A_{x,y}$, $B_{x,y}$ and $H_{x,y}^\alpha$ are all densely defined. Of course, similar results also can be deduced for $A_{y,x}$, $B_{y,x}$ and for $H_{y,x}^\alpha$, at least if \mathcal{F}_y is a basis for \mathcal{G} . However, this is not what happens in our examples, unless we impose some extra conditions to α , as we will see later. When these extra conditions are satisfied, all our operators turn out to be densely defined.

Now we can prove the following Proposition, which allows us to factorize both $H_{x,y}^\alpha$ and $H_{y,x}^\alpha$ respectively on \mathcal{D}_x and \mathcal{D}_y .

Proposition 16 *Let \mathcal{F}_x and \mathcal{F}_y be biorthogonal sets. Assume that the sequence α satisfies the following condition: $0 = \alpha_1 \leq \alpha_2 \leq \dots$, then the following statements hold.*

- i) *The operators $H_{x,y}^\alpha$ and $B_{x,y}A_{x,y}$ coincide on \mathcal{D}_x ;*
- ii) *the operators $H_{y,x}^\alpha$ and $B_{y,x}A_{y,x}$ coincide on \mathcal{D}_y .*

Proof. i) Let $f \in \mathcal{D}_x$. For every $m \in \mathbb{N}$ we have

$$\langle y_m, A_{x,y}f \rangle = \langle y_m, \sum_{n=2}^{\infty} \sqrt{\alpha_n} \langle y_n, f \rangle x_{n-1} \rangle = \sum_{n=2}^{\infty} \sqrt{\alpha_n} \langle y_n, f \rangle \langle y_m, x_{n-1} \rangle = \sqrt{\alpha_{m+1}} \langle y_{m+1}, f \rangle$$

because of the continuity of the inner product. Then, recalling that $\alpha_1 = 0$, we have

$$\begin{aligned} B_{x,y}(A_{x,y}f) &= \sum_{m=1}^{\infty} \sqrt{\alpha_{m+1}} \langle y_m, A_{x,y}f \rangle x_{m+1} = \\ &= \sum_{m=1}^{\infty} \sqrt{\alpha_{m+1}} \sqrt{\alpha_{m+1}} \langle y_{m+1}, f \rangle x_{m+1} = \\ &= \sum_{m=1}^{\infty} \alpha_{m+1} \langle y_{m+1}, f \rangle x_{m+1} = \sum_{n=2}^{\infty} \alpha_n \langle y_n, f \rangle x_n \\ &= \sum_{n=1}^{\infty} \alpha_n \langle y_n, f \rangle x_n = H_{x,y}^\alpha f, \end{aligned}$$

which is what we had to prove. Of course, our assertion ii) can be proved in the same way. □

Of course, factorizability of the Hamiltonians could be true not just on \mathcal{D}_x and \mathcal{D}_y , but also on larger sets. In other words, Proposition 16 does not exclude that, for instance, $H_{x,y}^\alpha \hat{f} = B_{x,y} A_{x,y} \hat{f}$ for some \hat{f} belonging to $\mathcal{H} \setminus \mathcal{D}_x$. This is the case when $D(H_{x,y}^\alpha) \supset \mathcal{D}_x$, and when the set of vectors \hat{f} such that $A_{x,y} \hat{f} \in D(B_{x,y})$ is larger than \mathcal{D}_x too.

Remark 17 *The possibility of factorizing the Hamiltonian of a physical system is quite useful in concrete applications, both for general reasons, and in connection with pseudo-hermitian and with supersymmetric quantum mechanics (SUSY-QM). This, in fact, simplifies the computation of the eigenstates and also can produce more exactly solvable models, i.e. quantum mechanical Hamiltonians with known eigenvalues and eigenvectors. In fact, if H can be written (at least formally) in a factorized form $H = BA$, using SUSY-QM we can deduce how and when the eigenvectors and the eigenvalues of the new Hamiltonian $H_1 := AB$ can be found, [13, 22]. Sometimes it happens that the procedure can be iterated, and in this case one is able to deduce a full family of solvable models. Moreover, the same operators used to factorize the Hamiltonians are often used in connection with bi-coherent states, [5].*

As in Section II.1, we can further introduce two sequences of strictly positive real numbers $\beta := \{\beta_n > 0, n \in \mathbb{N}\}$ and $\gamma := \{\gamma_n > 0, n \in \mathbb{N}\}$, and two related operators S_x^β and S_y^γ as follows:

$$D(S_x^\beta) = \left\{ f \in \mathcal{H} : \sum_{n=1}^{\infty} \beta_n \langle x_n, f \rangle x_n \in \mathcal{H} \right\}, \quad D(S_y^\gamma) = \left\{ g \in \mathcal{H} : \sum_{n=1}^{\infty} \gamma_n \langle y_n, g \rangle y_n \in \mathcal{H} \right\},$$

and

$$S_x^\beta f = \sum_{n=1}^{\infty} \beta_n \langle x_n, f \rangle x_n, \quad S_y^\gamma g = \sum_{n=1}^{\infty} \gamma_n \langle y_n, g \rangle y_n, \quad (4.4)$$

for all $f \in D(S_x^\beta)$ and $g \in D(S_y^\gamma)$. These operators are positive and, if densely defined, are symmetric too. For instance, this is true if \mathcal{F}_x and \mathcal{F}_y are bases for \mathcal{G} , or when some suitable conditions on β or γ are satisfied as we will see in the following. Whenever $D(S_x^\beta)$ and $D(S_y^\gamma)$ are dense, both S_x^β and S_y^γ admit a self-adjoint (Friedrichs) extension, see [29]. Definition (4.4) and the biorthogonality of the families \mathcal{F}_x and \mathcal{F}_y imply that, for all n , $y_n \in D(S_x^\beta)$, $x_n \in D(S_y^\gamma)$, $S_x^\beta y_n = \beta_n x_n$ and $S_y^\gamma x_n = \gamma_n y_n$. Hence these operators map \mathcal{F}_x into \mathcal{F}_y and viceversa, with some extra normalization factor which we cannot get rid of.

Let us now see in details what happens when we consider the sets of vectors introduced in Sections III.1 and III.2.

IV.2 Back to the first example

As we have shown in Section III.1, each vector of \mathcal{G} can be written as a finite linear combination of the x_n 's. Then, being \mathcal{G} dense in \mathcal{H} , $H_{x,y}^\alpha$ is densely defined. In fact, from (4.1), we see that $\mathcal{G} \subseteq D(H_{x,y}^\alpha)$. On the other hand, in general we cannot say, using the same argument, that \mathcal{G} is also contained in $D(H_{y,x}^\alpha)$, since \mathcal{F}_y is not a basis for \mathcal{G} . Therefore, we cannot conclude that $H_{y,x}^\alpha$ is densely defined, in general. However, it is possible to prove that, if α is such that $\{n\alpha_n\}$ belongs to the Hilbert space $l^2(\mathbb{N})$, i.e. if $\sum_{n=1}^{\infty} n^2 |\alpha_n|^2 < \infty$, then $\mathcal{G} \subseteq D(H_{y,x}^\alpha)$, so that $H_{y,x}^\alpha$ is densely defined too. In fact, let $f \in \mathcal{G}$, then f can be written as a finite linear combination $f = \sum_{l=1}^M c_l e_l$, for some M with $c_l = \langle e_l, f \rangle$. Then, after few computations,

$$H_{y,x}^\alpha f = \sum_{l=1}^M \frac{c_l}{l} \sum_{n=l}^{\infty} \alpha_n y_n.$$

It is clear that the series on the right-hand side converges if and only if $\sum_{l=1}^M \frac{c_l}{l} \sum_{n=1}^{\infty} \alpha_n y_n =: \tilde{c}_f \sum_{n=1}^{\infty} \alpha_n y_n$ converges, since the two series differ for a finite number of terms. Here we have introduced $\tilde{c}_f = \sum_{l=1}^M \frac{c_l}{l}$, which is clearly well defined. Then $f \in D(H_{y,x}^\alpha)$ if $\sum_{n=1}^{\infty} \alpha_n y_n$ converges in \mathcal{H} . Recalling now that for every $n \in \mathbb{N}$, $y_n = n e_n - (n+1) e_{n+1}$, one can check that

$$\left\| \sum_{n=1}^{\infty} \alpha_n y_n \right\|^2 = \sum_{n=1}^{\infty} |\alpha_n|^2 (n^2 + (n+1)^2) - \sum_{n=1}^{\infty} ((n+1)^2 \overline{\alpha_n} \alpha_{n+1} + c.c.).$$

Here *c.c.* stands for complex conjugate. Using now our assumption on α , and the Schwartz inequality, we conclude that $\|\sum_{n=1}^{\infty} \alpha_n y_n\|$ is finite, which is what we had to prove. From now on we will assume that α is such that $\{n\alpha_n\} \in l^2(\mathbb{N})$.

Let us now see what can be said for the operators S_y^γ and S_x^β . Because of the properties of \mathcal{F}_x , S_y^γ is densely defined, symmetric and positive for any possible choice of $\gamma = \{\gamma_n\}$, with $\gamma_n > 0$. In contrast, the fact that each y_n belongs to $D(S_x^\beta)$ does not ensure us that S_x^β is densely defined as well, in general. However, in analogy with what we have done for $H_{y,x}^\alpha$, we can see that, taken $f = \sum_{l=1}^M c_l e_l \in \mathcal{G}$, then $S_x^\beta f = \sum_{l=1}^M \frac{c_l}{l} \sum_{n=l}^{\infty} \beta_n x_n$. It is clear that the series on the right-hand side converges if and only if $\sum_{n=1}^{\infty} \beta_n x_n$ converges. Now, since for every $n \in \mathbb{N}$, $\|x_n\| = \left(\sum_{k=1}^n \frac{1}{k^2}\right)^{1/2} < \left(\frac{\pi^2}{6}\right)^{1/2}$,

$$\left\| \sum_{n=1}^{\infty} \beta_n x_n \right\| \leq \sum_{n=1}^{\infty} \beta_n \|x_n\| < \frac{\pi}{\sqrt{6}} \sum_{n=1}^{\infty} \beta_n,$$

which is convergent if $\beta \in l^1(\mathbb{N})$. Then, when this happens, $\mathcal{G} \subseteq D(S_x^\beta)$, and therefore S_x^β is also densely defined. Of course, taking $\beta_n = 1$ for every $n \in \mathbb{N}$ (as it is sometimes found in the literature, see [3, 8, 9] for instance), does not appear to be a good choice here, since in this case $D(S_x^\beta)$ is not a dense set, in principle. So we will not make this choice. Now, if we fix $\gamma_n = \frac{1}{\beta_n}$, we deduce that

$$S_x^\beta S_y^\gamma x_n = x_n, \quad \text{and} \quad S_y^\gamma S_x^\beta y_n = y_n,$$

for all $n \in \mathbb{N}$. This, of course, does not imply that S_x^β is the inverse of S_y^γ . In fact, neither \mathcal{F}_x nor \mathcal{F}_y are bases for \mathcal{H} , so that the above equalities cannot both be extended automatically to the whole Hilbert space. Nevertheless, they can be extended on some large sets, i.e. on \mathcal{D}_x and on \mathcal{D}_y , which are in fact rather rich sets.

As in [8, Proposition 2.3], these operators produce interesting intertwining relations:

$$\left(H_{x,y}^\alpha S_x^\beta - S_x^\beta H_{y,x}^\alpha \right) y_n = 0, \quad \left(H_{y,x}^\alpha S_y^\gamma - S_y^\gamma H_{x,y}^\alpha \right) x_n = 0, \quad (4.5)$$

for all $n \in \mathbb{N}$, which are related to the fact that $H_{x,y}^\alpha$ and $H_{y,x}^\alpha$ share the same eigenvalues, and that S_x^β and S_y^γ map \mathcal{F}_x into (multiples of) \mathcal{F}_y and viceversa. As before, the second equality in (4.5) can be extended to all vectors of \mathcal{G} , while the first one, with no further extra assumption, cannot. Finally, as stated at the end of Section IV.1, due to the fact that S_x^β and S_y^γ are positive and symmetric, they admit self-adjoint extensions which we still indicate with the same symbols, and which are also positive. Hence, they admit positive square roots, which can be used to introduce, at least formally,

$$\hat{e}_n := \frac{1}{\sqrt{\beta_n}} \left(S_x^\beta \right)^{1/2} y_n, \quad h_{x,y} = \left(S_y^\gamma \right)^{1/2} H_{x,y}^\alpha \left(S_x^\beta \right)^{1/2}.$$

Then we can, again formally, check that $\hat{e}_n := \frac{1}{\sqrt{\gamma_n}} \left(S_y^\gamma \right)^{1/2} x_n$ and that $h_{x,y} \hat{e}_n = \alpha_n \hat{e}_n$. This shows that $h_{x,y}$ has the same eigenvalues $\{\alpha_n\}$ as $H_{x,y}^\alpha$, for instance. It worths to stress that these last claims are based on some subtle mathematical assumptions, which are not necessarily satisfied, in general, if \mathcal{F}_x and \mathcal{F}_y are not Riesz or o.n. bases. We refer to [8] for more details on this kind of problems in the easiest situation, i.e. when \mathcal{F}_x and \mathcal{F}_y are Riesz bases indeed. Here we just want to say that, while it is easy to see that $\mathcal{F}_{\hat{e}} = \{\hat{e}_n\}$ is an o.n. set, we can conjecture that $\mathcal{F}_{\hat{e}}$ is not a basis for \mathcal{H} . In fact, this set is the image of \mathcal{F}_x (and \mathcal{F}_y), none of which is a basis for \mathcal{H} .

IV.3 Back to the second example

Similar considerations as those discussed in Section IV.2 can also be worked out by considering the two sets introduced in Section III.2, where we have shown, among other things, that the x_n 's form a basis for \mathcal{G} and for \mathcal{H} too. Then, being \mathcal{G} dense in \mathcal{H} , $H_{x,y}^\alpha$ is densely defined. In fact, from (4.1), we see that $\mathcal{G} \subseteq D(H_{x,y}^\alpha)$. On the other hand, in general we cannot say, using the same argument, that \mathcal{G} is also contained in $D(H_{y,x}^\alpha)$, since \mathcal{F}_y is not a basis for \mathcal{G} .

However, it is possible to prove that, if α is such that $\{\alpha_n\}$ belongs to $l^2(\mathbb{N})$, i.e. if $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$, then $\mathcal{G} \subseteq D(H_{y,x}^\alpha)$, so that $H_{y,x}^\alpha$ is densely defined too. In fact, let $f \in \mathcal{G}$ and, as before, let $f = \sum_{l=1}^M c_l e_l$, for some M , with $c_l = \langle e_l, f \rangle \in \mathbb{C}$. Then

$$\begin{aligned} H_{y,x}^\alpha f &= H_{y,x}^\alpha \left(\sum_{k=1}^M c_k e_k \right) = \sum_{k=1}^M c_k H_{y,x}^\alpha e_k = \sum_{k=1}^M c_k \sum_{n=1}^{\infty} \alpha_n \langle x_n, e_k \rangle y_n \\ &= \sum_{k=1}^M c_k \sum_{n=k}^{\infty} (-1)^{n+k} \alpha_n y_n \end{aligned}$$

being

$$\langle x_n, e_k \rangle = (-1)^n \langle -e_1 + e_2 + \cdots + (-1)^n e_n, e_k \rangle = \begin{cases} (-1)^{n+k}, & \text{for } n \geq k; \\ 0, & \text{otherwise.} \end{cases} \quad (4.6)$$

Now,

$$\begin{aligned} \|H_{y,x}^\alpha f\| &= \left\| \sum_{k=1}^M c_k \sum_{n=k}^{\infty} (-1)^{n+k} \alpha_n y_n \right\| = \left\| \sum_{k=1}^M (-1)^k c_k \sum_{n=k}^{\infty} (-1)^n \alpha_n y_n \right\| \\ &\leq \sum_{k=1}^M |c_k| \left\| \sum_{n=k}^{\infty} (-1)^n \alpha_n y_n \right\| = L_f \left\| \sum_{n=1}^{\infty} (-1)^n \alpha_n y_n \right\| \end{aligned}$$

with $L_f = \sum_{k=1}^M |c_k|$. Since $\|\sum_{n=k}^{\infty} (-1)^n \alpha_n y_n\| < \infty$ if and only if $\|\sum_{n=1}^{\infty} (-1)^n \alpha_n y_n\| < \infty$, and since

$$\left\| \sum_{n=1}^{\infty} (-1)^n \alpha_n y_n \right\|^2 = 2 \left[\sum_{n=1}^{\infty} |\alpha_n|^2 - \Re \left(\sum_{n=1}^{\infty} \alpha_n \overline{\alpha_{n+1}} \right) \right]$$

(where $\Re z$ stays for real part of z) after some calculations, we have:

$$\left\| \sum_{n=1}^{\infty} (-1)^n \alpha_n y_n \right\|^2 \leq 3 \sum_{n=1}^{\infty} |\alpha_n|^2$$

and the r.h.s. converges since $\{\alpha_n\} \in l^2(\mathbb{N})$.

Consider now the operators defined as in (4.4). Because of the properties of \mathcal{F}_x , S_y^γ is densely defined, symmetric and positive. On the other hand, the fact that each y_n belongs to $D(S_x^\beta)$ does not ensure us that S_x^β is densely defined as well, as we have already observed in the previous section. However, also in this case it is possible to give some sufficient conditions in order S_x^β to be densely defined. For example, it is enough to require that $\{\beta_n\sqrt{n}\} \in l^1(\mathbb{N})$. If this is the case, then, recalling (4.6),

$$\begin{aligned} \|S_x^\beta f\| &= \left\| S_x^\beta \left(\sum_{k=1}^M c_k e_k \right) \right\| = \left\| \sum_{k=1}^M c_k \sum_{n=1}^{\infty} \beta_n \langle x_n, e_k \rangle x_n \right\| = \left\| \sum_{k=1}^M c_k \sum_{n=k}^{\infty} \beta_n (-1)^{n+k} x_n \right\| \\ &\leq \sum_{k=1}^M |c_k| \left\| \sum_{n=k}^{\infty} \beta_n (-1)^n x_n \right\|, \end{aligned}$$

which converges if and only if $\sum_{k=1}^M |c_k| \left\| \sum_{n=1}^{\infty} \beta_n (-1)^n x_n \right\| = L_f \left\| \sum_{n=1}^{\infty} \beta_n (-1)^n x_n \right\|$ converges, which is trivially true if $\{\beta_n\sqrt{n}\} \in l^1(\mathbb{N})$, with L_f the same constant as before.

Here it is possible to repeat the same consideration as in Section IV.2: the fact that $H_{y,x}^\alpha$ and $H_{x,y}^\alpha$ share the same eigenvalues, and that S_y^γ and S_x^β map \mathcal{F}_y into (multiple of) \mathcal{F}_x and viceversa is reflected by the following weak form of the intertwining relations:

$$(H_{y,x}^\alpha S_y^\gamma - S_y^\gamma H_{x,y}^\alpha) x_n = 0, \quad (H_{x,y}^\alpha S_x^\beta - S_x^\beta H_{y,x}^\alpha) y_n = 0, \quad (4.7)$$

for all $n \in \mathbb{N}$. Since \mathcal{F}_x is a basis for \mathcal{G} we can conclude that the first equality in (4.7) holds on \mathcal{G} , because every vector $f \in \mathcal{G}$ can be written as a finite combination of the x_n 's. In contrast, the second equality holds on the whole \mathcal{G} only under additional assumptions.

Remark 18 *Going back to the example in Section III.3, it is clear that we can adopt the same representation for the Hamiltonians H_1 and H_2 than in this section and write them as in (4.1). Quite often, see [7, 8, 9], one adopts the more compact expressions:*

$$H_1 = \sum_{n=0}^{\infty} E_n y_n \otimes \bar{x}_n, \quad H_2 = \sum_{n=0}^{\infty} E_n x_n \otimes \bar{y}_n,$$

where, for instance, $(y_n \otimes \bar{x}_n)(f) = \langle x_n, f \rangle y_n$, for all $f \in \mathcal{H}$. Of course, these equations must be completed with some information on the domains of H_1 and H_2 . For instance $D(H_1) = \{f \in \mathcal{H} : H_1 f \text{ exists in } \mathcal{H}\}$. For what we have seen before, this is surely dense in $\mathcal{L}^2(\mathbb{R})$ since $D(\mathbb{R}) \subset D(H_1)$. The same result can be shown for H_2 .

V Conclusions

In this paper we have shown how and when some particular biorthogonal sets, the so-called \mathcal{G} -quasi bases, can be used to define manifestly non self-adjoint operators with known eigenvectors and simple punctual spectra, even when these eigenvectors do not form bases for the Hilbert space where the model is defined. In particular, we have devoted a part of the paper to analyze in some details the properties of three \mathcal{G} -quasi bases, and another part to show how these sets can be used to define Hamiltonians and ladder operators, and how the latter ones can be used to factorize the Hamiltonians themselves.

Our paper can be seen as another step toward a better comprehension of the role of biorthogonal sets in physical contexts where self-adjointness of the observables is not required. Also, from a more mathematical side, the paper suggests to undertake a deeper analysis of \mathcal{G} -quasi bases and of their relations with frames and this is, in fact, one of our future project.

Acknowledgements

This work was partially supported by the University of Palermo, by the Gruppo Nazionale per la Fisica Matematica (GNFM) and by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

- [1] J.-P. Antoine and C. Trapani, *Metric operators, generalized hermiticity and lattices of hilbert spaces*, in *Non-selfadjoint operators in quantum physics*, F. Bagarello, J.P. Gazeau, F.H. Szafraniec and M. Znojil Eds., 345-402, John Wiley and Sons, (2015)
- [2] F. Bagarello, J. P. Gazeau, F. Szafraniec and M. Znojil Eds, *Non-selfadjoint operators in quantum physics: Mathematical aspects*, J. Wiley and Sons, (2015)
- [3] F. Bagarello, *Deformed canonical (anti-)commutation relations and non hermitian hamiltonians*, in *Non-selfadjoint operators in quantum physics: Mathematical aspects*, F. Bagarello, J. P. Gazeau, F. Szafraniec and M. Znojil Eds, J. Wiley and Sons, (2015)
- [4] F. Bagarello, *More mathematics for pseudo-bosons*, J. Math. Phys., **54**, 063512 (2013)

- [5] F. Bagarello, *Pseudo-bosons, Riesz bases and coherent states*, J. Math. Phys., **50**, 023531 (2010)
- [6] F. Bagarello *Mathematical aspects of intertwining operators: the role of Riesz bases*, J. Phys. A, **43**, 175203 (2010) (12pp)
- [7] F. Bagarello, G. Bellomonte, *On non-self-adjoint operators defined by Riesz bases in Hilbert and rigged Hilbert spaces*, Proceedings of the 8th International Conference on Topological Algebras and their Applications (ICTAA-2014), to appear.
- [8] F. Bagarello, A. Inoue, C. Trapani, *Non-self-adjoint hamiltonians defined by Riesz bases*, J. Math. Phys., **55**, 033501, (2014)
- [9] G. Bellomonte, *Bessel sequences, Riesz-like bases and Operators in triplets of Hilbert Spaces*, Springer Proceedings in Physics, to appear
- [10] G. Bellomonte, C. Trapani, *Riesz-like bases in rigged Hilbert spaces*, submitted to Z. Anal. Anwendungen.
- [11] C. M. Bender, S. Boettcher, *Real Spectra in Non-. Hermitian Hamiltonians Having PT-Symmetry*, Phys. Rev. Lett. **80**, 5243-5246, (1998)
- [12] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, (2003)
- [13] F. Cooper, A. Khare and U. Sukhatme, *Supersymmetry and quantum mechanics*, World Scientific, Singapore (2001)
- [14] I. Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, Philadelphia (1992)
- [15] I. Daubechies, A. Grossmann, Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., **27**, 12711283 (1986)
- [16] E.B. Davies, *Linear operators and their spectra*, Cambridge University Press, Cambridge (2007)
- [17] E.B. Davies, *Pseudospectra, the harmonic oscillator and complex resonances*, Proc. Roy. Soc. London A, **455**, 585-599, (1999)

- [18] E.B. Davies, B. J. Kuijlaars, *Spectral asymptotics of the non-self-adjoint harmonic oscillator*, J. London Math. Soc., **70**, 420-426, (2004)
- [19] J.P. Gazeau, *Coherent States in Quantum Physics*, Wiley-VCH, Berlin 2009
- [20] V.L. Hansen, *Functional analysis: entering Hilbert space*, World Scientific, Singapore (2006)
- [21] C. Heil, *A basis theory primer: expanded edition*, Springer, New York, (2010)
- [22] G. Junker, *Supersymmetric methods in quantum and statistical physics*, Springer-Verlag, Berlin Heidelberg (1996)
- [23] A. Kolmogorov and S. Fomine, *Eléments de la théorie des fonctions et de l'analyse fonctionnelle*, Mir (1973)
- [24] D. Krejcirik, P. Siegl, M. Tater, J. Viola, *Pseudospectra in non-Hermitian quantum mechanics*, arXiv:1402.1082 [math-SP].
- [25] Kuru S., Tegmen A., Vercin A., *Intertwined isospectral potentials in an arbitrary dimension*, J. Math. Phys, **42**, No. 8, 3344-3360, (2001)
- [26] Kuru S., Demircioglu B., Onder M., Vercin A., *Two families of superintegrable and isospectral potentials in two dimensions*, J. Math. Phys, **43**, No. 5, 2133-2150, (2002)
- [27] A. Messiah, *Quantum mechanics*, vol. 1, North Holland Publishing Company, Amsterdam, (1961)
- [28] A. Mostafazadeh, *Pseudo-Hermitian representation of Quantum Mechanics*, Int. J. Geom. Methods Mod. Phys. **7**, 1191-1306 (2010)
- [29] G. K. Pedersen, *Analysis now*, Springer-Verlag, New York (1989)
- [30] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Integrals and series*, vol. 2, Special functions, Opa, Amsterdam, (1986)
- [31] Samani K. A., Zarei M., *Intertwined hamiltonians in two-dimensional curved spaces*, Ann. of Phys., **316**, 466-482, (2005).
- [32] D.T. Stoeva, P. Balasz, *Canonical forms of unconditionally convergent multipliers*, Jour. Math. Anal. Appl., **399**, 252-259 (2013)

- [33] D.T. Stoeva, P. Balasz, *Riesz bases multipliers*, in *Concrete Operators, Spectral Theory, Operators in Harmonic Analysis and Approximation*, M. Cepedello Boiso, H. Hedenmalm, M. A. Kaashoek, A. Montes Rodriguez, and S. Treil, eds., Operator Theory: Advances and Applications, Springer Basel, **236**, 475-482 (2014)
- [34] G. Szegő, *Orthogonal Polynomials*, AMS, Providence, (1939)
- [35] R.M. Young, *On complete biorthogonal bases*, Proceedings of the American Mathematical Society, **83**, No. 3, 537-540, (1981)
- [36] M. Znojil, *Three-Hilbert-Space formulation of Quantum Mechanics*, SIGMA **5**, 1-19, (2009)