

Transient Behavior of a Population Dynamical Model

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The transient behavior of an ecosystem with N random interacting species in the presence of a multiplicative noise is analyzed. The multiplicative noise mimics the interaction with the environment. We investigate different asymptotic dynamical regimes and the role of the external noise on the probability distribution of the local field.

§1. Introduction

Population dynamics attracted a lot of attention in recent years and became the object of many studies as well by biologists as by physicists.^{1)–5)} Tools developed in the context of nonequilibrium statistical physics to analyze nonequilibrium nonlinear physical systems provide new insights and at the same time new approaches to study biological systems. Biological population dynamics has many interesting, and still not solved, problems such as pattern formation,^{6)–9)} the role of the noise on complex ecosystem behaviour, and the noise-induced effects, such as stochastic resonance, noise delayed extinction, quasi periodic oscillations etc.^{10)–18)} The dynamical behavior of ecological systems of interacting species evolves towards the equilibrium states through the long, slow and complex process of nonlinear relaxation, which is strongly dependent on the random interaction between the species, the initial conditions and the random interaction with environment. A good mathematical model to analyze the dynamics of N biological species with spatially homogeneous densities is the generalized Lotka-Volterra system with a Malthus-Verhulst modelization of the self regulation mechanism, in the presence of a multiplicative noise.^{19)–21)} By neglecting the fluctuations of the local field we derive a quasi-stationary probability of the populations. We obtain the asymptotic analytical expressions for different nonlinear relaxation regimes, and we analyze the role of the multiplicative noise on the probability distribution of the local field.

§2. The model and results

The dynamical evolution of our ecosystem composed of N interacting species in a noisy environment and in the presence of an absorbing barrier is described by the following Ito stochastic differential equation

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$$dx_i(t) = \left[\left(g_i(x_i(t)) + \sum_{j \neq i} J_{ij} x_j(t) \right) dt + \sqrt{\epsilon} dw_i \right] x_i(t), \quad i = 1, \dots, N, \quad (2.1)$$

where $x_i(t) \geq 0$ is the population density of the i th species at time t and the function $g_i(x_i(t))$

$$g_i(x_i(t)) = \left(\alpha + \frac{\epsilon}{2} \right) - x_i(t) \quad (2.2)$$

describes the development of the i th species without interacting with other species. In Eq. (2.1), α is the growth parameter, the interaction matrix J_{ij} models the interaction between different species ($i \neq j$), and w_i is the Wiener process whose increment dw_i satisfies the usual statistical properties $\langle dw_i(t) \rangle = 0$, and $\langle dw_i(t) dw_j(t') \rangle = \delta_{ij} \delta(t - t') dt$. We consider a random asymmetric interaction matrix J_{ij} , whose elements are independently distributed according to a Gaussian distribution with $\langle J_{ij} \rangle = 0$, $\langle J_{ij} J_{ji} \rangle = 0$ and $\sigma_j^2 = J^2/N$. The term $J_{ij} x_i x_j$ is the loss or the growth rate of species i due to interaction with species j , when $J_{ij} < 0$ or $J_{ij} > 0$ respectively. With this choice of interaction matrix our ecosystem contains 50% of prey-predator interactions ($J_{ij} < 0$ and $J_{ji} > 0$), 25% competitive interactions ($J_{ij} < 0$ and $J_{ji} < 0$), and 25% symbiotic interactions ($J_{ij} < 0$ and $J_{ji} > 0$). We consider all species equivalent so that the characteristic parameters of the ecosystem are independent of the species. The random interaction with the environment (climate, disease, etc.) is taken into account by introducing a multiplicative noise in Eq. (2.1). The solution of the dynamical equation (2.1) is given by

$$x_i(t) = \frac{x_i(0) z_i(t)}{1 + x_i(0) \int_0^t dt' z_i(t')}, \quad (2.3)$$

where

$$z_i(t) = e^{\alpha t + \sqrt{\epsilon} w_i(t) + \int_0^t dt' \eta_i(t')} \quad (2.4)$$

and

$$\eta_i(t) = \sum_{j \neq i} J_{ij} x_j(t) \quad (2.5)$$

is the local field acting on the i th population and represents the influence of other species on the differential growth rate. We note that the dynamical behavior of the i th population depends on the time integral of the process $z_i(t)$ and the time integral of the local field. For a large number of interacting species we can assume that the local field $\eta_i(t)$ is Gaussian with zero mean and variance $\sigma_{\eta_i}^2 = \langle \eta_i^2 \rangle = J^2 \langle x_i^2 \rangle$. As a consequence, in the absence of external multiplicative noise, from the fixed-point equation $x_i(\alpha - x_i + \eta_i) = 0$, the stationary probability distribution of the populations is the sum of a truncated Gaussian distribution at $x_i = 0$ ($x_i > 0$ always) and a delta function for extinct species. The initial values of the populations

$x_i(0)$ have also Gaussian distribution with mean value $\langle x_i(0) \rangle = 1$, and variance $\sigma_{x(0)}^2 = 0.01, 0.03, 0.05$.

The interaction strength between the species J determines two different dynamical behaviors of the ecosystem. Above a critical value $J_c = 1.1$, the system is unstable and at least one of the populations diverges. Below J_c the system is stable and asymptotically reaches an equilibrium state. The equilibrium values of the populations depend both on their initial values and on the interaction matrix. If we consider a quenched random interaction matrix, the ecosystem has a great number of equilibrium configurations, each one with its attraction basin. For an interaction strength $J = 1$ and an intrinsic growth parameter $\alpha = 1$ we obtain: $\langle x_i \rangle = 1.4387$, $\langle x_i^2 \rangle = 4.514$ and $\sigma_{x_i}^2 = 2.44$. These values agree with that obtained from numerical simulation of Eq. (2.1). From the Fokker-Planck equation associated to the Langevin equation (2.1)

$$\frac{\partial}{\partial t} P(x_i, t) = -\frac{\partial}{\partial x_i} \left[\frac{\epsilon}{2} \frac{\partial}{\partial x_i} x_i^2 - \left(\alpha + \frac{\epsilon}{2} - x_i + \eta_i \right) x_i \right] P(x_i, t) \quad (2.6)$$

we obtain a quasi stationary distribution by neglecting the fluctuations of the local field in the asymptotic regime

$$\frac{dP(x_i)}{P(x_i)} = \frac{2}{\epsilon} \left[\alpha + \eta_i - x_i - \frac{\epsilon}{2} \right] \frac{dx_i}{x_i} \quad (2.7)$$

that is

$$P(x_i) = N_{x_i(0)} \exp \left[\frac{2}{\epsilon} \left(\left(\alpha + \eta_i - \frac{\epsilon}{2} \right) \ln x_i - x_i \right) \right] \Theta(x_i), \quad (2.8)$$

where the normalization factor is

$$N_{x_i(0)} = \frac{P_{x_i(0)} e^{\frac{2x_i(0)}{\epsilon}}}{x_i(0)^{\frac{2}{\epsilon}(\alpha + \eta_i - \frac{\epsilon}{2})}}, \quad (2.9)$$

and Θ is the Heaviside unit step function. Now we focus on the statistical properties of the time integral of the i th population $X_i(t)$

$$X_i(t) = \int_0^t dt' x_i(t'), \quad (2.10)$$

in the asymptotic regime. From Eqs. (2.3) and (2.10) we have

$$X_i(t) = \ln \left[1 + x_i(0) \int_0^t dt' \exp [\alpha t' + \sqrt{\epsilon} w_i(t') + \eta_i(t')] \right]. \quad (2.11)$$

We use the same approximation of the mean field interaction,¹⁹⁾ and after differentiating Eq. (2.11), we get the asymptotic solution of $X_i(t)$ as

$$X_i(t) \simeq \ln \left[x_i(o) e^{\sqrt{\epsilon} w_{max_i}(t) + \eta_{max_i}(t)} \int_0^t dt' e^{\alpha t'} \right], \quad (2.12)$$

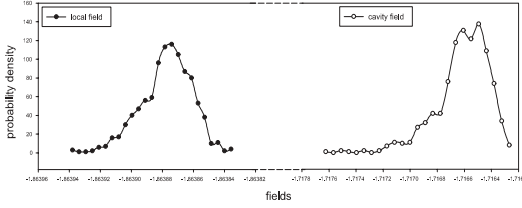


Fig. 1. The local and the cavity fields in the absence of noise, $\epsilon = 0$.

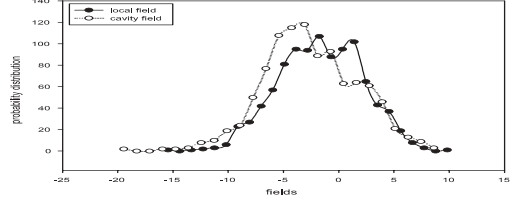


Fig. 2. The local and the cavity fields in the presence of noise, $\epsilon = 0.1$.

where $w_{max_i}(t) = \sup_{0 < t' < t} w(t')$ and $\eta_{max_i}(t) = \sup_{0 < t' < t} \eta(t')$. Equation (2.12) is valid for $\alpha \geq 0$, that is, when the system relaxes toward an equilibrium population and at the critical point. After making the ensemble average, we obtain for the time average of the i th population \bar{X}_i

$$\langle \bar{X}_i \rangle \simeq \frac{1}{t} \left[N_w \sqrt{\epsilon t} + \ln t + \langle \ln [n_i(o)] \rangle \right], \quad \alpha = 0, \quad (2.13)$$

and

$$\langle \bar{X}_i \rangle \simeq \frac{1}{t} \left[N_w \sqrt{\epsilon t} + (\alpha + N_\eta)t + \left\langle \ln \left[\frac{x_i(o)}{\alpha} \right] \right\rangle \right], \quad \alpha > 0, \quad (2.14)$$

where N_w and N_η are variables with a semi-Gaussian distribution¹⁹⁾ and N_η must be determined self-consistently from the variance of the local field (Eq. (2.5)). We obtain, consistently with mean field approximation, the typical long time tail behavior ($t^{-1/2}$) dependence, which characterizes nonlinear relaxation regimes when $\alpha \geq 0$. When the system relaxes toward the absorbing barrier ($\alpha < 0$), the time average of the i th population $\langle \bar{X}_i \rangle$ is a functional of the local field and the Wiener process. We have also analyzed the dynamics of the ecosystem when one species is absent. Specifically, we considered the cavity field, which is the field acting on the i th population when this population is absent. The probability distributions for both local and cavity field have been obtained by simulations for a time $t = 200$ (a. u.) in absence of external noise, and for different species. We found that the probability distributions of the cavity fields differ substantially from those of local fields for the same species, while in the presence of noise the two fields overlap (see Figs. 1 and 2).

This overlap is different for different species and depends on the variance of the initial species distribution. This strange behavior, found for some populations, is reminiscent of the phase-transition phenomenon, and it is related to the following peculiarities of our dynamical system: (i) all the populations are positive; (ii) different initial conditions drive the ecosystem into different attraction basins; and (iii) the complex structure of the attraction basins of our dynamical system. While, in the presence of noise, all the populations seem to be equivalent from the dynamical point of view, some populations, in the absence of external noise, have an asymptotical dynamical behavior such that they significantly influence the dynamics of other

species.

§3. Conclusions

We analyzed the nonlinear relaxation of an ecosystem composed of N interacting species. We obtain the quasi-stationary probability distribution of the population in the presence of multiplicative noise. By using an approximation of the integral equation, which gives the stochastic evolution of the system, we obtained asymptotic behavior for different nonlinear relaxation regimes. We observe an interesting phenomenon: the local and the cavity fields, whose probability distributions are different in the absence of noise, coincide for some populations in the presence of multiplicative noise. This phenomenon can be ascribed to the peculiarity of the dynamical system ($x_i > 0$, always), the influence of different initial conditions on the asymptotic regime, and the complex structure of the attraction basins of our ecosystem.

Acknowledgements

This work was supported by MIUR, INFM and INTAS (project 2001-450).

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