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EXISTENCE AND UNIQUENESS FOR PRANDTL EQUATIONS AND ZERO VISCOSITY LIMIT OF THE NAVIER-STOKES EQUATIONS

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The existence and uniqueness of the mild solution of the boundary layer (BL) equation is proved assuming analyticity of the data with respect to the tangential variable. Moreover we use the well-posedness of the BL equation to perform an asymptotic expansion of the Navier-Stokes equations on a bounded domain.

1   Introduction

Prandtl's boundary layer equations were first formulated in 1904 in order to solve the differences between the viscous and inviscid theory of fluids. In particular inviscid flow does not account for the total drag on a body. Moreover, in presence of a boundary, a perfect flow allows only vanishing normal velocity, while a viscous flow imposes all the components of the velocity to be zero on the surface of a stationary object.

Introducing the proper scaling to make the viscous forces to be of the same order of magnitude of the inertial forces, one derives Prandtl equations. They hold inside a narrow "boundary layer" region of thickness $O\left(\sqrt{\nu}\right)$ where viscous drag and no-slip boundary conditions occur. The BL equations are:

$$\left(\partial_t - \partial_{YY}\right) u^P + u^P \partial_x u^P + v^P \partial_Y u^P + \partial_x p^P = 0 \,, \tag{1}$$

$$\partial_Y p^P = 0 \,, \tag{2}$$

$$\partial_x u^P + \partial_Y v^P = 0 \,, \tag{3}$$

$$u^P(x, Y = 0, t) = 0 \,, \tag{4}$$

$$u^P(x, Y \to \infty, t) \longrightarrow U(x, t) \,, \tag{5}$$

$$p^P(x, Y \to \infty, t) \longrightarrow p^E(x, y = 0, t) \,, \tag{6}$$

$$u^P(x, Y, t = 0) = u^P_{in} \,. \tag{7}$$

In the above equations (u^P, v^P) and p^P represent the components of the fluid velocity and the pressure inside the BL, Y is the rescaled normal variable $Y = y/\sqrt{\nu}$. Equation (5) is the matching condition between the velocity of inside the BL and the outer Euler flow; $U(x, t)$ is the tangential component of the Euler flow at the boundary. Up to date the well-posedness of the

above system of equations is an open question and an exhaustive theory of the Prandtl equations is far from being achieved (see [1]). Relevant results have been obtained by Oleinik and coworkers (see [5] for an update review) but they have to require quite restrictive monotonicity assumptions on the initial data.

Existence and uniqueness of the Prandtl system Eqs. (1)-(7) was proved by Caflisch and Sammartino in [6] without any monotonicity restriction but imposing analitycity on the initial data with respect to both the tangential and the normal component of the velocity. The main tool of their proof was the abstract formulation of the Cauchy Kowalewski theorem (ACK) in the Banach spaces of analytic functions.

In this paper we extend the results of [6], proving existence and uniqueness for the Prandtl equations requiring analyticity only with respect to the tangential variable. The well-posedness of the Boundary Layer equations is then used to address the study of the zero viscosity limit of the Navier-Stokes equations on a bounded domain. The results of this paper are valid in $3D$ as well as in $2D$. To simplify the notation we shall restrict to the $2D$ case.

2 Function spaces

In this section we introduce the function spaces used in the proof of existence and uniqueness of the Prandtl equations.

Definition 2.1 $K^{l,\rho}$ is the space of the analytic functions $f(x)$ defined in $\{x \in \mathbb{C} : \Im x \in (-\rho, \rho)\}$ such that:

- if $\Im x \in (-\rho, \rho)$ and $0 \le j \le l$, then $\partial_x^j f(\Re x + i\Im x) \in L^2(\Re x)$;

- $|f|_{l,\rho} \equiv \sum_{j=0}^{l} \sup_{\Im x \in (-\rho,\rho)} \|\partial_x^j f(\cdot + i\Im x)\|_{L^2(\Re x)} < \infty$.

Definition 2.2 $K^{l,\rho,\mu}$, with $\mu > 0$, is the space of the functions $f(Y,x)$ such that:

- $e^{\mu Y} \partial_x^i \partial_Y^j f \in L^\infty(\mathbb{R}^+, K^{0,\rho})$ when $i + j \le l$ and $j \le 2$;

- $|f|_{l,\rho,\mu} \equiv \sum_{j \le 2} \sum_{i \le l-j} \sup_{Y \in \mathbb{R}^+} e^{\mu Y} |\partial_Y^j \partial_x^i f(Y,\cdot)|_{0,\rho} < \infty$.

Definition 2.3 $K_{\beta,T}^{l,\rho}$, with $\beta > 0$, is the space of the functions $f(x,t)$ such that:

- $\partial_t^i \partial_x^j f(x,t) \in K^{l,\rho-\beta t}$ $\quad \forall 0 \le t \le T$, where $0 \le i + j \le l$ and $0 \le i \le 1$;

- $|f|_{l,\rho,\beta,T} \equiv \sum_{0 \le j \le 1} \sum_{i \le l-j} \sup_{0 \le t \le T} |\partial_t^j \partial_x^i f(\cdot,t)|_{0,\rho-\beta t} < \infty$.

Definition 2.4 $K^{l,\rho,\mu}_{\beta,T}$ is the space of the functions $f(x,Y,t)$ such that:

- $f \in K^{l,\rho-\beta t,\mu-\beta t}$; $\partial_t \partial_x^i f \in K^{0,\rho-\beta t,\mu-\beta t}$ $\forall 0 \leq t \leq T$, for $0 \leq i \leq l-2$;

- $|f|_{l,\rho,\mu,\beta,T} \equiv \sum_{0 \leq j \leq 2} \sum_{i \leq l-j} \sup_{0 \leq t \leq T} |\partial_Y^j \partial_x^i f(\cdot,\cdot,t)|_{0,\rho-\beta t,\mu-\beta t}$
  $+ \sum_{i \leq l-2} \sup_{0 \leq t \leq T} |\partial_t \partial_x^i f(\cdot,\cdot,t)|_{0,\rho-\beta t,\mu-\beta t} < \infty$.

3   A parabolic equation

In this section we shall be concerned with the following equations:

$$(\partial_t - \partial_{YY})u + \alpha(x,t)\,Y\,\partial_Y u = f, \tag{1}$$

$$u(x,Y=0,t) = g, \tag{2}$$

$$u(x,Y,t=0) = u_0. \tag{3}$$

To solve the above system, we first introduce the following kernel:

$$F_\alpha(x,Y,t) = \frac{1}{\sqrt{4\pi}} \frac{1}{\left(\int_0^t d\tau\, e^{-2A(x,\tau)}\right)^{1/2}} \exp\left(\frac{Y^2 e^{-2A(x,t)}}{4\left(\int_0^t d\tau\, e^{-2A(x,\tau)}\right)}\right), \tag{4}$$

where $A(\tau)$ and E_α are defined as:

$$A(x,\tau) = \int_0^\tau d\theta\,\alpha(x,\theta)\ ;\ E_\alpha(x,Y,t) = \int_0^\infty dY'\,[F_\alpha(x,Y-Y',t) - F_\alpha(x,Y+Y',t)].$$

We can introduce the following operators:

$$M_0 u_0 = \int_0^\infty dY'\,[F_\alpha(Y-Y',t) - F_\alpha(Y+Y',t)]\,u_0(x,Y'), \tag{5}$$

$$M_2 f = \int_0^t ds \int_0^\infty dY'\,[F_\alpha(Y-Y',t-s) - F_\alpha(Y+Y',t-s)]\,f(x,Y',s), \tag{6}$$

$$M_1 g = 2\int_0^t ds \left(-2\frac{\partial F_\alpha}{\partial Y} + 2Y\alpha F_\alpha - \alpha E_\alpha\right)(x,Y,t-s)\,g(x,s), \tag{7}$$

$$M_3 h = \int_0^t ds \int_0^\infty dY'\partial_Y\,[F_\alpha(Y-Y',t-s) - F_\alpha(Y+Y',t-s)]\,f(x,Y',s). \tag{8}$$

Notice that, if $h(x,Y=0,t)=0$, then, integrating by parts, one gets $M_3 h \equiv M_2 \partial_Y h$. Using the operators M_0, M_1 and M_2 given by (5), (6) and (7), one can write the explicit expression of the initial boundary value problem (1)-(3), and prove the following Proposition:

Proposition 3.1 *Suppose* $\alpha, g \in K_{\beta,T}^{l,\rho}$, $f \in K_{\beta,T}^{l,\rho,\mu}$ *and* $u_0 \in K^{l,\rho,\mu}$. *Moreover let the compatibility condition* $g(x, t = 0) = 0$ *be satisfied. Then the solution* u *of Eqs. (1)-(3) is in* $K_{\beta,T}^{l,\rho,\mu}$, *and the following estimate holds:*

$$|u|_{l,\rho,\mu,\beta,T} \leq c \left(|\alpha|_{l,\rho,\beta,T} + |f|_{l,\rho,\mu,\beta,T} + |g|_{l,\rho,\beta,T} \right) .$$

We will also need the following estimates for the operators M_2 and M_3:

Lemma 3.1 *Suppose* $\alpha \in K_{\beta,T}^{l,\rho}$ *and* $f \in K_{\beta,T}^{l,\rho,\mu}$ *with* $f(x, Y = 0, t) = 0$. *If* $\rho' < \rho - \beta t$ *and* $\mu' < \mu - \beta t$ *then the following estimate holds:*

$$|M_2 f|_{l,\rho',\mu'} \leq c \int_0^t ds |f(\cdot, \cdot, s)|_{l,\rho',\mu'} \leq c |f|_{l,\rho,\mu,\beta,T} .$$

Lemma 3.2 *Suppose* $\alpha \in K_{\beta,T}^{l,\rho}$, $h \in K_{\beta,T}^{l,\rho,\mu}$ *with* $h|_{Y=0} = 0$, $\partial_Y h|_{Y=0} = 0$. *If* $0 < \mu' < \mu(s) < \mu - \beta s$ *then* $M_3 h \in K^{l,\rho,\mu'}$ *for each* $0 < t < T$ *and the following estimate holds:*

$$|M_3 h|_{l,\rho,\mu'} \leq c \int_0^t ds \left(\frac{|h(\cdot, \cdot, s)|_{l,\rho,\mu'}}{\sqrt{t-s}} + \frac{|h(\cdot, \cdot, s)|_{l,\rho,\mu(s)}}{\mu(s) - \mu'} \right) .$$

For the details of the proofs of the above statements see [3].

4   The mild form of the Prandtl equations

Following [3], we introduce the new variable $u = u^P - U$, so that using the Euler equation at the boundary, Eqs. (1)-(7) can be written in the form:

where:
$$u = F(u, t) , \tag{9}$$

$$F(u, t) = M_2 K_1(u, t) + M_3 K(u, t) + \mathcal{C} , \tag{10}$$

$$K_1(u, t) = - \left(2u \, \partial_x u + U \, \partial_x u + u \, \partial_x U \right) , \tag{11}$$

$$K(u, t) = u \int_0^Y dY' \, \partial_x u, \tag{12}$$

where $\mathcal{C}$ takes into account the initial and boundary conditions and we have identified the α appearing in the kernel F_α with the Euler datum at the boundary U. We shall call the Eq. (9) the mild form of the Prandtl equations.

5   The main result

To prove the existence and the uniqueness of the mild solution of the Prandtl equations, we shall use a slightly modified version of the Abstract Cauchy-Kowalewski Theorem (ACK) as given in [6]. We refer to [2] and [3] for the formal

statement and the proof of the Modified ACK theorem while here we shall only give an informal version.

Theorem 5.1 (Modified ACK) *Let $\{X_\rho : 0 < \rho \le \rho_0\}$ be a Banach scale with norms $|\ |_\rho$, such that $X_{\rho'} \subset X_{\rho''}$ and $|\ |_{\rho''} \le |\ |_{\rho'}$ when $\rho'' \le \rho' \le \rho_0$.*

For t in $[0, T]$, consider the equation

$$u + F(t, u) = 0. \tag{13}$$

Suppose that the function $F(t, u) : [0, \tau] \mapsto X_{\rho'}$ is continuous and $|F(t, 0)|_{\rho_0 - \beta_0 t} \le R_0 < R$. Moreover $\forall 0 < \rho' < \rho(s) \le \rho_0 - \beta_0 s$ and $\forall u^1$ and $u^2 \in \{u \in X_\rho : \sup_{0 \le t \le T} |u(t)|_{\rho - \beta_0 t} \le R\}$,

$$|F(t, u^1) - F(t, u^2)|_{\rho'} \le C \int_0^t ds \left(\frac{|u^1 - u^2|_{\rho(s)}}{\rho(s) - \rho'} + \frac{|u^1 - u^2|_{\rho'}}{\sqrt{t - s}} \right). \tag{14}$$

Then $\exists \beta > \beta_0$ and $0 < T_1 < T$ such that Eq. (13) has a unique solution $u(t) \in X_{\rho_0 - \beta t}$ with $t \in [0, T_1]$; moreover $\sup_{0 \le t \le T_1} |u(t)|_{\rho_0 - \beta t} \le R$.

To apply the ACK theorem to the Prandtl equation, we have to prove that the right hand side of Eq. (10) satisfies an estimate like (14). Using Proposition 3.1, Lemma 3.1, Lemma 3.2 and through the Cauchy estimate for the derivative of an analytic function (see also [6]), one can easily prove the following Proposition.

Proposition 5.1 *Suppose that u^1 and u^2 are in $K_{\beta_0, T}^{l, \rho_0, \mu_0}$. Suppose $0 < \rho' < \rho(s) < \rho_0 - \beta_0 s$ and $0 < \mu' < \mu(s) < \mu_0 - \beta_0 s$. Then the following estimate holds:*

$$\left| F(u^1, t) - F(u^2, t) \right|_{l, \rho', \mu'}$$

$$\le c \int_0^t ds \left(\frac{|u^1 - u^2|_{l, \rho(s), \mu}}{\rho(s) - \rho'} + \frac{|u^1 - u^2|_{l, \rho, \mu(s)}}{\mu(s) - \mu'} + \frac{|u^1 - u^2|_{l, \rho', \mu'}}{\sqrt{t - s}} \right). \tag{15}$$

Therefore using the above Proposition and the ACK Theorem, the main result of this paper can be proved.

Theorem 5.2 *Suppose $U \in K_{\beta_0, T}^{l, \rho_0}$ and $u_{in}^P - U \in K^{l, \rho_0, \mu_0}$. Moreover let the compatibility conditions $u_{in}^P(x, Y = 0) = 0$; $u_{in}^P(x, Y \to \infty) - U \to 0$ hold. Then there exists a unique mild solution u^P of Eqs. (1)-(7), which can be written as $u^P(x, Y, t) = u(x, Y, t) + U$ where $u \in K_{\beta_1, T_1}^{l, \rho_1, \mu_1}$ with $0 < \rho_1 < \rho_0$, $0 < \mu_1 < \mu_0$, $\beta_1 > \beta_0 > 0$ and $0 < T_1 < T$.*

6 Zero Viscosity Limit on Bounded Domains

In this section we shall show how one could address the problem of the zero-viscosity limit of the Navier-Stokes equations in domains with curved bound-

aries. We define adapted coordinates (s, n) and, as we can allow non-analytic initial data with respect to the normal variable, we introduce a C^∞ cut-off function $m(n/\varepsilon^\alpha)$, with $0 < \alpha < 1$, such that:

$$m(n/\varepsilon^\alpha) = \begin{cases} 1 & \text{for } 0 < n < \varepsilon^\alpha, \\ 0 & \text{for } n > 2\,\varepsilon^\alpha. \end{cases}$$

We thus construct an approximate solution to the Navier-Stokes equations of the form:

$$u^\varepsilon = (1 - m)\left(u_0^E + \varepsilon\, u_1^E + \varepsilon^2\, u_2^E\right) + m\left(U_0^P + \varepsilon\, U_1^P + \varepsilon^2\, U_2^P\right),$$

where the u_i^E terms satisfy the $i-th$ order Euler equations and the U_i^P terms satisfy the $i-th$ order Prandtl equations with proper boundary and matching conditions (see [4]). Then the solution to the Navier-Stokes equation can be written as:

$$u^{NS} = u^\varepsilon + \varepsilon^2 w,$$

where the correction term w satisfies a Navier-Stokes-type equation with bounded source term. The possibility of giving a rigorous estimate of the norm of w in the appropriate function space is under current investigation.

Acknowledgments

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