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VARIATIONAL HENSTOCK INTEGRABILITY OF BANACH SPACE VALUED FUNCTIONS

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Abstract. We study the integrability of Banach space valued strongly measurable functions defined on $[0,1]$. In the case of functions f given by $\sum_{n=1}^{\infty} x_n\chi_{E_n}$, where x_n are points of a Banach space and the sets E_n are Lebesgue measurable and pairwise disjoint subsets of $[0,1]$, there are well known characterizations for Bochner and Pettis integrability of f. The function f is Bochner integrable if and only if the series $\sum_{n=1}^{\infty} x_n|E_n|$ is absolutely convergent. Unconditional convergence of the series is equivalent to Pettis integrability of f. In this paper we give some conditions for variational Henstock integrability of a certain class of such functions.

Keywords: Kurzweil-Henstock integral; variational Henstock integral; Pettis integral

MSC 2010: 26A39

1. Introduction

In this paper we study the variational Henstock integrability of strongly measurable functions. It is well known (cf. [5], Lemma 5.1) that each strongly measurable Banach valued function, defined on a measurable space, can be written as $f = g + \sum_{n=1}^{\infty} x_n\chi_{E_n}$, where g is a bounded strongly measurable function, x_n are vectors of the given Banach space and E_n are measurable and pairwise disjoint sets. As each bounded strongly measurable function is Bochner integrable, it is enough to study

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integrability only for functions of the form $\sum_{n=1}^{\infty} x_n\chi_{E_n}$. In the case of Bochner and Pettis integrals, a necessary and sufficient condition for integrability of a function given by $\sum_{n=1}^{\infty} x_n\chi_{E_n}$ is, respectively, the absolute and the unconditional convergence of the series $\sum_{n=1}^{\infty} x_n|E_n|$ (see Theorem A). In the case of Kurzweil-Henstock or variational Henstock integrals, in general the series $\sum_{n=1}^{\infty} x_n|E_n|$ is only conditionally convergent. So the conditions for integrability depend on the order of the terms $x_n|E_n|$. In [1], [3] and [4] conditions for the Kurzweil-Henstock integrability of functions of the form $\sum_{n=1}^{\infty} x_n\chi_{E_n}$ are given. Here we go a bit further in this investigation. We give another characterization of the Kurzweil-Henstock integrability (see Theorem 3.1). The main results are Proposition 4.1 and Theorem 4.1. In the latter, a necessary and sufficient condition for the variational Henstock integrability of a special type of such functions is given. It needs a particular order of the sets E_n.

2. Basic facts

Let $[0,1]$ be the unit interval of the real line equipped with the usual topology and Lebesgue measure. If a set $E \subset [0,1]$ is Lebesgue measurable, then $|E|$ denotes its Lebesgue measure. $\mathcal{I}$ denotes the family of all closed subintervals of $[0,1]$.

A *partition in* $[0,1]$ is a finite collection of pairs $\mathcal{P} = \{(I_1,t_1),\dots,(I_p,t_p)\}$, where $I_1,\dots,I_p$ are nonoverlapping subintervals of $[0,1]$ and $t_i \in I_i$, $i = 1,\dots,p$. If $\bigcup_{i=1}^{p} I_i = [0,1]$, we say that $\mathcal{P}$ is a *partition of* $[0,1]$. A *gauge* on $E \subset [0,1]$ is a positive function on E. For a given gauge δ, we say that a partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ is δ*-fine* if $I_i \subset (t_i-\delta(t_i), t_i+\delta(t_i))$, $i = 1,\dots,p$.

Throughout this paper, X is a Banach space with dual X^*. We recall the following definitions:

Definition 2.1. A function $f\colon [0,1] \to X$ is said to be *Kurzweil-Henstock integrable* (or simply KH-*integrable*) on $[0,1]$ if there exists $w \in X$ with the following property:

For every $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that

$$\Bigl\|\sum_{i=1}^{p} f(t_i)|I_i| - w\Bigr\| < \varepsilon,$$

for each δ-fine partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ of $[0,1]$. We set $(\mathrm{KH})\int_0^1 f := w$.

Definition 2.2. A function $f\colon [0,1] \to X$ is said to be *variationally Henstock integrable* (briefly vH-*integrable*) on $[0,1]$, if there exists an additive function $F\colon \mathcal{I} \to X$, satisfying the following condition:

Given $\varepsilon > 0$ there exists a gauge δ such that if $\mathcal{P} = \{(I_i, t_i)\colon i = 1, \dots, p\}$ is a δ-fine partition in $[0,1]$, then

$$\sum_{i=1}^{p} \|f(t_i)|I_i| - F(I_i)\| < \varepsilon.$$

It is obvious that each vH-integrable function is KH-integrable. It is also well known that in the case of real-valued functions the variational Henstock and the Kurzweil-Henstock integrals are equivalent.

We recall the following classical result for the Bochner and Pettis integrals:

Theorem A ([2], page 55). *Let $f = \sum\limits_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and the sets E_n are Lebesgue measurable and pairwise disjoint subsets of $[0,1]$. Then*

(1) *f is Pettis integrable if and only if the series $\sum\limits_{n=1}^{\infty} x_n|E_n|$ is unconditionally convergent;*

(2) *f is Bochner integrable if and only if the series $\sum\limits_{n=1}^{\infty} x_n|E_n|$ is absolutely convergent.*

In both cases $\int_E f = \sum\limits_{n=1}^{\infty} x_n|E_n \cap E|$, for every measurable set E.

3. Kurzweil-Henstock integrability

In [1], Theorem 1, a necessary condition for the Kurzweil-Henstock integrability of the function $f = \sum\limits_{n=1}^{\infty} x_n \chi_{E_n}$ is given. Here we prove that the condition is also sufficient.

Theorem 3.1. *Let $f\colon [0,1] \to X$ be defined by $f = \sum\limits_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and the sets E_n are Lebesgue measurable and pairwise disjoint. Then the following conditions are equivalent:*

(A) *f is Kurzweil-Henstock integrable with*

$$(\mathrm{KH})\int_I f(t)\,\mathrm{d}t = \sum_{n=1}^{\infty} x_n|E_n \cap I|,$$

for every interval $I \in \mathcal{I}$;

(B) *for every* $\varepsilon > 0$ *there exist a gauge* δ *and* $k_0 \in \mathbb{N}$ *such that given a* δ*-fine partition* $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ *of* $[0,1]$ *and given* $s > r > k_0$ *we have*

$$\Bigl\|\sum_{k=r}^{s} x_k \Bigl|\bigcup_{t_j \in E_k} I_j\Bigr|\Bigr\| < \varepsilon.$$

Proof. (B) $\Rightarrow$ (A) was proved in [1].

(A) $\Rightarrow$ (B) We assume that f is Kurzweil-Henstock integrable with

$$(\mathrm{KH})\int_0^1 f(t)\,\mathrm{d}t = \sum_{n=1}^{\infty} x_n |E_n|.$$

According to [3], Theorem 2, for every $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that if $\mathcal{P} := \{(i_1, t_1), \ldots, (I_p, t_p)\}$ is a δ-fine partition of $[0,1]$, then there exists $n_{\mathcal{P}} \in \mathbb{N}$ such that

$$\Bigl\|\sum_{n=1}^{n} x_k \Bigl(\Bigl|\bigcup_{t_i \in E_k} I_i\Bigr| - |E_k|\Bigr)\Bigr\| < \frac{\varepsilon}{3} \quad \text{for all } n > n_{\mathcal{P}}.$$

Since the series $= \sum\limits_{n=1}^{\infty} x_n |E_n|$ is convergent, there is $n_1 > n_{\mathcal{P}}$ such that if $s > r > n_1$, then

$$\Bigl\|\sum_{i=r}^{s} x_i |E_i|\Bigr\| < \frac{\varepsilon}{3}.$$

Hence, if $s > r > n_1$, then

$$\begin{aligned}\Bigl\|\sum_{k=r}^{s} x_k \Bigl|\bigcup_{t_j \in E_k} I_j\Bigr|\Bigr\| &\leqslant \Bigl\|\sum_{k=1}^{s} x_k \Bigl|\bigcup_{t_j \in E_k} I_j\Bigr| - \sum_{k=1}^{s} x_k |E_k|\Bigr\| \\ &\quad + \Bigl\|\sum_{k=1}^{r-1} x_k \Bigl|\bigcup_{t_j \in E_k} I_j\Bigr| - \sum_{k=1}^{r-1} x_k |E_k|\Bigr\| + \Bigl\|\sum_{i=r}^{s} x_i |E_i|\Bigr\| < \varepsilon.\end{aligned}$$

□

4. Variational Henstock integrability

The aim of this section is to formulate conditions for the variational Henstock integrability of a certain class of strongly measurable functions.

Proposition 4.1. *Let $\{a_n\}$ be a decreasing sequence converging to zero such that $a_1 = 1$. Let $\{x_n\} \subset X$ be arbitrary and define $f\colon [0,1] \to X$ by $f = \sum\limits_{n=1}^{\infty} x_n \chi_{E_n}$, where each $E_n \subseteq [a_{n+1}, a_n)$ is Lebesgue measurable. Then the following conditions are equivalent:*

(i) *the series $\sum\limits_{n=1}^{\infty} x_n |E_n|$ is convergent;*

(ii) *f is* vH-*integrable;*

(iii) *f is* KH-*integrable.*

In each case

$$(\mathrm{vH}) \int_I f = \sum_{n=1}^{\infty} x_n |E_n \cap I| \quad \text{for every } I \in \mathcal{I} \tag{4.1}$$

and the series $\sum\limits_{n=1}^{\infty} x_n |E_n \cap I|$ is uniformly convergent on $\mathcal{I}$.

Proof. (i) $\Rightarrow$ (ii) Assume that the series $\sum\limits_{n=1}^{\infty} x_n |E_n|$ is convergent. Notice then that the series $\sum\limits_{n=1}^{\infty} x_n |E_n \cap I|$ is convergent for every $I \in \mathcal{I}$. Let $F(I) = \sum\limits_{n=1}^{\infty} x_n |E_n \cap I|$. Now we show that f is vH-integrable. Without loss of generality we may assume that $f(0) = 0$.

Let $\varepsilon > 0$. Since the series $\sum\limits_{n=1}^{\infty} x_n |E_n|$ is convergent, there is $K \in \mathbb{N}$ such that for $s \geqslant n \geqslant K$,

$$\left\| \sum_{k=n}^{s} x_k |E_n| \right\| < \frac{\varepsilon}{4}.$$

Moreover, for each $n \in \mathbb{N}$, let $\delta_n\colon [a_{n+1}, a_n] \to (0, \infty)$ be a gauge such that if $\mathcal{P} = \{(I_i, t_i),\ i = 1, \ldots, p\}$ is a δ_n-fine partition of $[a_{n+1}, a_n]$, then

$$\sum_{i=1}^{p} \| f(t_i) |I_i| - F(I_i) \| < \frac{\varepsilon}{2^{n+1}}.$$

We may assume that $\delta_{n+1}(a_{n+1}) = \delta_n(a_{n+1})$.

Define $\delta(t)$ on $[0,1]$ as follows:

$$\delta(t) = \begin{cases} \delta_n(t) & \text{if } t \in (a_{n+1}, a_n), \\ \min\{\delta_n(a_n), \delta_{n-1}(a_n)\} & \text{if } t = a_n, \\ a_K & \text{if } t = 0. \end{cases}$$

Let us consider now a δ-fine partition $\mathcal{P} = \{(I_i, t_i),\ i = 1, \ldots, p\}$ of $[0,1]$ and the corresponding sum

$$\sum_{i=1}^{p} \|f(t_i)|I_i| - F(I_i)\|.$$

If $q \geqslant K$ is the largest integer such that $I_1 \subset [0, a_q)$, then

$$\begin{aligned}
(4.2) \qquad \|f(t_1)|I_1| - F(I_1)\| &= \left\|\sum_{n=1}^{\infty} x_n |E_n \cap I_1|\right\| \\
&= \left\|\sum_{k=q}^{\infty} x_k |E_k \cap I_1|\right\| = \left\|x_q |E_q \cap I_1| + \sum_{k=q+1}^{\infty} x_k |E_k|\right\| \\
&\leqslant \|x_q\| |E_q \cap I_1| + \left\|\sum_{k=q+1}^{\infty} x_k |E_k|\right\| < \frac{\varepsilon}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{i=1}^{p} \|f(t_i)|I_i| - F(I_i)\| \\
&\qquad = \|f(t_1)|I_1| - F(I_1)\| \\
&\qquad\quad + \sum_{n=1}^{\infty} \sum_{t_i \in (a_{n+1}, a_n]} \|f(t_i)|I_i \cap [a_{n+1}, a_n]| - F(I_i \cap [a_{n+1}, a_n])\| \\
&\qquad \leqslant \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \varepsilon,
\end{aligned}$$

which proves the vH-integrability of f and equality (4.1) for $I = [0,1]$.

(iii) $\Rightarrow$ (i) If f is KH-integrable, its primitive $F(t) = (\mathrm{vH}) \int_0^t f$ is continuous on $[0,1]$. Let $F(I)$ be the additive interval function associated to $F(t)$. We have

$$F([0,1]) = \sum_{k=1}^{n} F([a_{k+1}, a_k)) + F([0, a_{n+1}]) = \sum_{k=1}^{n} x_k |E_k| + F([0, a_{n+1}]).$$

Letting $n \to \infty$, the convergence of the series $\sum\limits_{n=1}^{\infty} x_n |E_n|$ follows.

In the same way, setting $F_I(t) := (\mathrm{vH}) \int_\alpha^t f$ if $t \in I = [\alpha, \beta]$, we obtain (4.1).

Now we are going to prove that the series $\sum\limits_{n=1}^{\infty} x_n |E_n \cap I|$ is uniformly convergent on $\mathcal{I}$.

Since F is uniformly continuous, there is $n_0 \in \mathbb{N}$ such that if $I \subset [0, a_{n_0}]$, then

$$(4.3) \qquad \left\|\sum_{n=1}^{\infty} x_n |E_n \cap I|\right\| = \|F(I)\| \leqslant \varepsilon.$$

Now, if $I \in \mathcal{I}$ and $m > n_0$, then applying (4.1) and (4.3), we have the following inequalities:

$$
\begin{aligned}
&\left\|F(I) - \sum_{n=1}^{m} x_n |E_n \cap I|\right\| \\
&\qquad \leqslant \left\|F(I \cap [0, a_m]) - \sum_{n=1}^{m} x_n |E_n \cap I \cap [0, a_m]|\right\| \\
&\qquad\quad + \left\|F(I \cap [a_m, 1]) - \sum_{n=1}^{m} x_n |E_n \cap I \cap [a_m, 1]|\right\| \\
&\qquad \leqslant \left\|F(I \cap [0, a_m]) - \sum_{n=1}^{\infty} x_n |E_n \cap I \cap [0, a_m]|\right\| \\
&\qquad\quad + \left\|\sum_{n=m+1}^{\infty} x_n |E_n \cap I \cap [0, a_m]|\right\| \\
&\qquad\quad + \left\|F(I \cap [a_m, 1]) - \sum_{n=1}^{\infty} x_n |E_n \cap I \cap [a_m, 1]|\right\| \\
&\qquad\quad + \left\|\sum_{n=m+1}^{\infty} x_n |E_n \cap I \cap [a_m, 1]|\right\| \\
&\qquad \overset{(4.1)}{=} \left\|\sum_{n=m+1}^{\infty} x_n |E_n \cap I \cap [0, a_m]|\right\| + \left\|\sum_{n=m+1}^{\infty} x_n |E_n \cap I \cap [a_m, 1]|\right\| \\
&\qquad = \left\|\sum_{n=1}^{\infty} x_n |E_n \cap I \cap [0, a_m]|\right\| \overset{(4.3)}{\leqslant} \varepsilon \quad \text{for every } I \in \mathcal{I}.
\end{aligned}
$$

The last equality follows from the fact that $E_n \cap [a_m, 1] = \emptyset$ if $n > m$. □

Reordering the sets E_n in a suitable way, we obtain the following more general result:

Theorem 4.1. *Let $\{a_n\}$ and $\{b_n\}$ be decreasing sequences converging to zero such that $a_1 = 1$ and $a_{n+1} \leqslant b_n \leqslant a_n$, for every $n \in \mathbb{N}$. Let $\{x_n\} \subset X$ be arbitrary and define $f\colon [0,1] \to X$ by $f = \sum_{k=1}^{\infty} x_k \chi_{E_k}$, where $\{E_k\colon k \in \mathbb{N}\}$ is a sequence of pairwise disjoint Lebesgue measurable sets of positive measure with the following properties:*

(j) $\lim_k \operatorname{diam}(E_k) = 0$;

(jj) *for each $n \in \mathbb{N}$, the set $\{E_k\colon E_k \subset [a_{n+1}, a_n]\}$ is split into two disjoint collections (one of them may be empty):*

$$\{E_{2n-1,p_i}\colon \forall\, i \in \mathbb{N} \ \sup E_{2n-1,p_{i+1}} \leqslant \inf E_{2n-1,p_i}\} \subset [a_{n+1}, b_n]$$

and

$$\{E_{2n,q_i}\colon \forall\, i\in\mathbb{N}\ \inf E_{2n,q_{i+1}} \geqslant \sup E_{2n,q_i}\} \subset [b_n,a_n];$$

(jjj) *for each* $n\in\mathbb{N}$, $\lim_i d_H(\{a_{n+1}\},E_{2n-1,p_i}) = 0 = \lim_i d_H(\{a_n\},E_{2n,q_i})$, *where* $d_H(\cdot,\cdot)$ *is the Hausdorff distance between two sets.*

Let $c_{2n-1,i}(I) := x_n|E_{2n-1,p_i}\cap I|$ *and* $c_{2n,i}(I) := x_n|E_{2n,q_{i+1}}\cap I|$, $n\in\mathbb{N}$. *We order the series* $\sum\limits_{k=1}^{\infty} x_k|E_k\cap I|$ *in the following way:*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} c_{i,n+1-i}(I). \tag{4.4}$$

Then, the following conditions are equivalent:

(a) *the series* (4.4) *is uniformly convergent on the family* $\mathcal{I}$;

(b) f *is* vH-*integrable;*

(c) f *is* KH-*integrable.*

In each case

$$(\mathrm{vH})\int_I f = \sum_{n=1}^{\infty}\sum_{i=1}^{n} c_{i,n+1-i}(I) \qquad \text{for every } I\in\mathcal{I}. \tag{4.5}$$

Proof. Without loss of generality, we may assume that if for some $n\in\mathbb{N}$ one has $\{E_{n_{p_i}}\colon i\in\mathbb{N},\ \text{for all } i\in\mathbb{N}\}=\emptyset$, then $a_{n+1}=b_n$, and if $\{E_{n_{q_i}}\colon i\in\mathbb{N},\ \text{for all } i\in\mathbb{N}\}=\emptyset$, then $a_n=b_n$. We may assume also that each interval $[a_{n+1},a_n]$ contains infinitely many sets E_k and $f(0)=0$.

(a) $\Rightarrow$ (b) It follows from Proposition 4.1 that if for every $I\in\mathcal{I}$ the series $\sum\limits_{n=1}^{\infty}\sum\limits_{i=1}^{n} c_{i,n+1-i}(I)$ is convergent, then f is vH-integrable on every interval $[a_{n+1},b_n]$ and $[b_n,a_n]$. Consequently, f is vH-integrable on $[a_{n+1},a_n]$ and

$$(\mathrm{vH})\int_{a_{n+1}}^{a_n} f = \sum_{n=1}^{\infty}\sum_{i=1}^{n} c_{i,n+1-i}([a_{n+1},a_n]).$$

Now let $F(I)=\sum\limits_{n=1}^{\infty}\sum\limits_{i=1}^{n} c_{i,n+1-i}(I)$ for every $I\in\mathcal{I}$ and let $\varepsilon>0$. For each $n\in\mathbb{N}$ there exists a gauge $\delta_n\colon [a_{n+1},a_n]\to(0,\infty)$ with the property that for each δ_n-partition $\{(J_1,s_1),\dots,J_p,s_p)\}$ of $[a_{n+1},a_n]$ one has

$$\sum_{j=1}^{p}\left\| f(s_j)|J_j| - (\mathrm{vH})\int_{J_j} f\right\| \leqslant \frac{\varepsilon}{2^{n+2}}.$$

Taking $\min\{\delta_{n+1}(a_{n+1}),\delta_n(a_{n+1})\}$, one may assume that $\delta_{n+1}(a_{n+1})=\delta_n(a_{n+1})$.

Let $k_0 \in \mathbb{N}$ be such that $k \geqslant k_0$ yields

$$\left\| \sum_{n=k}^{\infty} \sum_{i=1}^{n} c_{i,n+1-i}(I) \right\| < \frac{\varepsilon}{4} \quad \text{for every } I \in \mathcal{I}.$$

Then, let $n_0 \in \mathbb{N}$ be such that all sets E_j built into some $c_{i,n+1-i}(I)$ with $n \leqslant k_0$ are contained in $(a_{n_0}, 1]$.

Define $\delta(t)$ on $[0,1]$ as follows:

$$\delta(t) = \begin{cases} \delta_n(t) & \text{if } t \in [a_{n+1}, a_n],\ n \in \mathbb{N}, \\ a_{n_0} & \text{if } t = 0. \end{cases}$$

Let $\mathcal{P} = \{(I_i, t_i),\ i = 1, \dots, p\}$ be a δ-fine partition of $[0,1]$ and let us consider the sum

$$\sum_{i=1}^{p} \|f(t_i)|I_i| - F(I_i)\|.$$

Without loss of generality, one may assume that the right end point of I_1 is equal to a point a_m with $m > n_0$.

It follows that

$$\sum_{j=2}^{p} \left\| f(s_j)|J_j| - (\mathrm{vH}) \int_{J_j} f \right\| \leqslant \frac{\varepsilon}{2}.$$

Then,

$$\left\| f(0)|J_1| - \sum_{n=1}^{\infty} \sum_{i=1}^{n} c_{i,n+1-i}(J_1) \right\| = \left\| \sum_{n=1}^{\infty} \sum_{i=1}^{n} c_{i,n+1-i}(J_1) \right\| = \left\| \sum_{k=k_0}^{\infty} \sum_{i=1}^{n} c_{i,n+1-i}(J_1) \right\| < \frac{\varepsilon}{4}$$

and so f is vH-integrable.

(c) $\Rightarrow$ (a) Assume the KH-integrability of f and let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ be such that $I \subset [0, a_{n_0}]$ yields $\|F(I)\| < \varepsilon/2$. In virtue of Proposition 4.1, the series $\sum\limits_{n=1}^{\infty} \sum\limits_{i=1}^{n} c_{i,n+1-i}(J)$ is uniformly convergent to $F(J)$ on the family $\mathcal{I} \cap [a_{n_0}, 1]$. Let $k_0 > n_0$ be such that if $m > k_0$, then $E_m \cap (a_{n_0}, 1] = \emptyset$ and

$$\left\| F(J \cap [a_{n_0}, 1]) - \sum_{n=1}^{m} \sum_{i=1}^{n} c_{i,n+1-i}(J \cap [a_{n_0}, 1]) \right\| \leqslant \frac{\varepsilon}{2} \quad \text{for every } J \in \mathcal{I}.$$

If $I \in \mathcal{I}$ is fixed and $m > k_0$, then

$$\left\| \sum_{n=m+1}^{\infty} \sum_{i=1}^{n} c_{i,n+1-i}(I \cap [0, a_{n_0}]) \right\| = \|F(I \cap [0, a_{n_0}])\| \leqslant \frac{\varepsilon}{2}$$

and so, taking into account (4.5), we have

$$\begin{aligned}\Big\|F(I\cap[0,a_{n_0}])-\sum_{n=1}^{m}\sum_{i=1}^{n}c_{i,n+1-i}(I\cap[0,a_{n_0}])\Big\| \\ \leqslant \Big\|F(I\cap[0,a_{n_0}])-\sum_{n=1}^{\infty}\sum_{i=1}^{n}c_{i,n+1-i}(I\cap[0,a_{n_0}])\Big\| \\ +\Big\|\sum_{n=m+1}^{\infty}\sum_{i=1}^{n}c_{i,n+1-i}(I\cap[0,a_{n_0}])\Big\| \leqslant \frac{\varepsilon}{2}.\end{aligned}$$

As a result, if $m > k_0$, then

$$\Big\|F(I)-\sum_{n=1}^{m}\sum_{i=1}^{n}c_{i,n+1-i}(I)\Big\| \leqslant \varepsilon \quad \text{for every } I\in\mathcal{I},$$

which proves the uniform convergence of the series (4.5) on $\mathcal{I}$. □

R e m a r k 4.1. In the same way as Theorem 4.1 was deduced from Proposition 4.1, one can obtain subsequent generalizations of Theorem 4.1.

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