Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

# Best approximation and variational inequality problems involving a simulation function

Fairouz Tchier[1], Calogero Vetro[2*] and Francesca Vetro[3]

*Correspondence: calogero.vetro@unipa.it
[2]Department of Mathematics and Computer Sciences, University of Palermo, Via Archirafi 34, Palermo, 90123, Italy
Full list of author information is available at the end of the article

**Abstract**

We prove the existence of a $g$-best proximity point for a pair of mappings, by using suitable hypotheses on a metric space. Moreover, we establish some convergence results for a variational inequality problem, by using the variational characterization of metric projections in a real Hilbert space. Our results are applicable to classical problems of optimization theory.

**MSC:** 41A65; 47J20

**Keywords:** best proximity point; metric projection; proximal $\mathcal{Z}$-contraction; variational inequality

## 1 Introduction

Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$ and $T : A \to B$ be a non-self-mapping. The equation $Tx = x$ is known as a general fixed point equation and its solution is related to the solution of many practical situations arising in pure and applied sciences. For instance, it is well known that many problems involving differential equations may be solved by searching for the existence of a fixed point of an integral operator. But for the existence of a fixed point of $T$, we need that $T(A) \cap A \neq \emptyset$, otherwise $d(x, Tx) > 0$ for all $x \in A$. In such a situation, it is natural to search a point $x \in A$ such that $x$ is closest to $Tx$ in some sense. To clarify and support this assertion, we recall the following best approximation theorem due to Ky Fan [1], in a metric version.

**Theorem 1.1** ([1]) *Let $A$ be a nonempty compact convex subset of a normed linear space $X$ and $T : A \to X$ be a continuous mapping. Then there exists $x \in A$ such that $\|x - Tx\| = d(Tx, A)$.*

This result is related to the existence of an approximate solution to the equation $Tx = x$. Theoretical and practical aspects of this theorem have been discussed by various mathematicians; we refer the reader to [2–11].

On the other hand, very recently Khojasteh *et al.* [12] introduced the concept of $\mathcal{Z}$-contraction, by using a notion of simulation function. Consequently, fixed point results involving a $\mathcal{Z}$-contraction are established in [12]. This approach has been of great importance to discuss various fixed point problems from an unifying point of view; see for instance [13–15] and the references therein. For more contributions on the development of fixed point theorems see [16–18].

We generalize and extend many results in the existing literature, by establishing some best proximity point theorems involving $\mathcal{Z}$-proximal contractions; see [19, 20]. In particular, we prove the existence of a unique $g$-best proximity point, which is a point $x \in A$ such that $d(gx, Tx) = d(A, B)$, where $g : A \to A$ is a self-mapping. As an application, we give sufficient conditions to ensure the existence of a unique solution for a variational inequality problem and propose a convergent iterative algorithm to approximate this solution, by using metric projections. Our results are applicable to some classical problems of optimization theory.

## 2 Preliminaries

Let $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Q}$ denote the sets of all real numbers, positive integers and rational numbers, respectively. Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. By using the usual notation in nonlinear analysis, we recall the following notions:

$$A_0 = \{x \in A : d(x, y) = d(A, B), \text{for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B), \text{for some } x \in A\}.$$

Kirk *et al.* gave sufficient conditions to ensure that $A_0$ and $B_0$ are nonempty sets; see [8]. On the other hand, Sadiq Basha and Veeramani proved that $A_0$ is contained in the boundary of $A$; see [20].

In the sequel, we are interested in establishing results involving new types of proximal contraction and hence we recall the fundamental definitions in this direction; see [21, 22].

**Definition 2.1** Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. A non-self-mapping $T : A \to B$ is said to be a contraction if

$$d(Tx, Ty) \le kd(x, y),$$

for all $x, y \in X$, where $k \in [0, 1[$.

**Definition 2.2** Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. A non-self-mapping $T : A \to B$ is said to be a proximal contraction of the first kind if

$$\left.\begin{array}{l} d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{array}\right\} \implies d(u, v) \le kd(x, y),$$

for all $u, v, x, y \in A$, where $k \in [0, 1[$.

**Definition 2.3** Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. A non-self-mapping $T : A \to B$ is said to be a proximal contraction of the second kind if

$$\left.\begin{array}{l} d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{array}\right\} \implies d(Tu, Tv) \le kd(Tx, Ty),$$

for all $u, v, x, y \in A$, where $k \in [0, 1[$.

Many authors generalized these concepts and proved their best approximation theorems; see for instance [23–25].

In 2011, Sankar Raj [10] introduced the notion of $P$-property as follows.

**Definition 2.4** ([10], Definition 3) Let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$ with $A_0 \neq \emptyset$. Then the pair $(A,B)$ is said to have the $P$-property if and only if $d(x_1,y_1) = d(A,B) = d(x_2,y_2)$ implies $d(x_1,x_2) = d(y_1,y_2)$ where $x_1,x_2 \in A_0$ and $y_1,y_2 \in B_0$.

By using Definition 2.4, Sankar Raj in [10] gave an extended version of the contraction mapping principle in [21]. Of course, for every nonempty subset $A$ of $X$, the pair $(A,A)$ has the $P$-property. We shall consider this property in a remark of the next section.

**Definition 2.5** Let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$. Let $g: A \to A$ be a self-mapping and $T: A \to B$ a non-self-mapping. Then

(i) $g \in \mathcal{G}_A$ if $g$ is continuous and $d(x,y) \leq d(gx,gy)$ for all $x,y \in A$;
(ii) $T \in \mathcal{T}_g$ if $d(Tx,Ty) \leq d(Tgx,Tgy)$ for all $x,y \in A$.

Finally, Khojasteh *et al.* in [12] defined a simulation function as follows.

**Definition 2.6** A simulation function is a mapping $\zeta : [0,+\infty[ \times [0,+\infty[ \to \mathbb{R}$ satisfying the following conditions:

($\zeta_1$) $\zeta(0,0) = 0$;
($\zeta_2$) $\zeta(t,s) < s - t$, for all $t,s > 0$;
($\zeta_3$) if $\{t_n\}$, $\{s_n\}$ are sequences in $]0,+\infty[$ such that $\lim_{n\to\infty} t_n = \lim_{n\to+\infty} s_n = \ell \in ]0,+\infty[$, then $\limsup_{n\to+\infty} \zeta(t_n,s_n) < 0$.

Consequently, they established the existence and uniqueness of fixed point for a self-mapping defined in a complete metric space.

**Theorem 2.1** ([12]) *Let $(X,d)$ be a complete metric space and $f: X \to X$ be a $\mathcal{Z}$-contraction with respect to a certain simulation function $\zeta$, that is,*

$$\zeta\big(d(fx,fy),d(x,y)\big) \geq 0, \quad \text{for all } x,y \in X. \tag{1}$$

*Then $f$ has a unique fixed point. Moreover, for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges to this fixed point.*

Successively, Argoubi *et al.* [13] point out the fact that condition ($\zeta_1$) is not mentioned in the proof of Theorem 2.1. Moreover, by putting $x = y$ in (1), it follows that $\zeta(0,0) \geq 0$ and hence, if $\zeta(0,0) < 0$, the set of mappings $f: X \to X$ satisfying condition (1) is an empty set.

Consequently, Argoubi *et al.* proposed a slight modification of Definition 2.6, by removing the condition ($\zeta_1$) and retaining the rest.

**Remark 2.1** Every simulation function of Khojasteh *et al.* is also a simulation function of Argoubi *et al.* However, the converse is not true.

**Example 2.1** ([13], Example 2.4) Let $\zeta_\lambda : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ be the function defined by

$$\zeta_\lambda(t,s) = \begin{cases} 1 & \text{if } (s,t) = (0,0), \\ \lambda s - t & \text{otherwise,} \end{cases}$$

where $\lambda \in ]0,1[$. Then $\zeta_\lambda$ satisfies $(\zeta_2)$ and $(\zeta_3)$ with $\zeta_\lambda(0,0) > 0$.

In order to avoid confusion, we refer to the following definition.

**Definition 2.7** A simulation function is a mapping $\zeta : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ satisfying the conditions $(\zeta_2)$ and $(\zeta_3)$.

## 3 Best proximity point theorems

In view of Definition 2.7, we consider the following notions of proximal contractions.

**Definition 3.1** Let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$. A non-self-mapping $T : A \to B$ is said to be a $\mathcal{Z}$-proximal contraction of the first kind if there exists a simulation function $\zeta : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ such that

$$\left.\begin{array}{l} d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\} \quad \Longrightarrow \quad \zeta\big(d(u,v), d(x,y)\big) \geq 0,$$

for all $u,v,x,y \in A$.

**Remark 3.1** If $T : A \to B$ is a $\mathcal{Z}$-proximal contraction of the first kind and $(A,B)$ has the $P$-property, then $T$ is a $\mathcal{Z}$-contraction.

**Example 3.1** Let $X = \mathbb{R}$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$. Consider $A = [0,\frac{1}{2}]$ and $B = [0,1]$ so that $d(A,B) = 0$. Define a mapping $T : A \to B$ by

$$Tx = \frac{x}{1+x}, \quad \text{for all } x \in A,$$

and the simulation function $\zeta : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ by

$$\zeta(t,s) = \begin{cases} s - \frac{t}{1-t} & \text{if } t \in [0,\frac{1}{2}], \\ s - 2t & \text{otherwise.} \end{cases}$$

It is easy to show that $T$ is a $\mathcal{Z}$-proximal contraction of the first kind, but not a proximal contraction of the first kind.

Indeed, from $d(u,Tx) = d(v,Ty) = 0 = d(A,B)$, we get $(x,y) = (\frac{u}{1-u}, \frac{v}{1-v})$, with $u,v \in [0,\frac{1}{3}]$, and hence

$$\begin{aligned} &\zeta\big(d(u,v), d(x,y)\big) \\ &\quad = \zeta\left(d(u,v), d\left(\frac{u}{1-u}, \frac{v}{1-v}\right)\right) \\ &\quad = \left|\frac{u}{1-u} - \frac{v}{1-v}\right| - \frac{|u-v|}{1-|u-v|} \geq 0, \end{aligned}$$

since $u + v \geq |u-v| + uv$. Thus, $T$ is a $\mathcal{Z}$-proximal contraction of the first kind.

On the other hand, there does not exist $k \in [0,1[$ such that

$$d(u,v) = |u-v| \nleq k\frac{|u-v|}{1-u-v+uv} = kd\left(\frac{u}{1-u}, \frac{v}{1-v}\right) = kd(x,y),$$

for all $u,v \in [0,\frac{1}{3}]$, and hence $T$ is not a proximal contraction of the first kind.

**Definition 3.2** Let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$. A non-self-mapping $T : A \to B$ is said to be a $\mathcal{Z}$-proximal contraction of the second kind if there exists a simulation function $\zeta : [0,+\infty[ \times [0,+\infty[ \to \mathbb{R}$ such that

$$\left.\begin{array}{l} d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B) \end{array}\right\} \implies \zeta\bigl(d(Tu,Tv), d(Tx,Ty)\bigr) \geq 0,$$

for all $u,v,x,y \in A$.

**Example 3.2** Let $X = \mathbb{R}$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$, and $A = B = [0,1]$. Define a mapping $T : [0,1] \to [0,1]$ by

$$Tx = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 1 & \text{otherwise.} \end{cases}$$

Now, consider the function $\zeta_\lambda : [0,+\infty[ \times [0,+\infty[ \to \mathbb{R}$ given in Example 2.1. It is easy to show that $T$ is a $\mathcal{Z}$-proximal contraction of the second kind, but not a $\mathcal{Z}$-proximal contraction of the first kind.

The following lemma is useful to show that a given sequence is Cauchy; see Lemma 2.1. in [26]; see also Lemma 2.1. in [17].

**Lemma 3.1** *Let $(X,d)$ be a metric space and $\{x_n\}$ a given sequence in $X$. Suppose that*

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0.$$

*If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that*

(i) $n_k > m_k \geq k$, $k \in \mathbb{N}$;
(ii) $d(x_{n_k}, x_{m_k}) \geq \varepsilon$, $d(x_{n_k-1}, x_{m_k}) < \varepsilon$, $k \in \mathbb{N}$;
(iii) $\lim_{k\to+\infty} d(x_{n_k}, x_{m_k}) = \varepsilon = \lim_{k\to+\infty} d(x_{n_k+1}, x_{m_k+1})$.

On this basis, we construct our results. Precisely, we establish some theorems of $g$-best proximity point for $\mathcal{Z}$-proximal contractions and deduce some corollaries.

**Theorem 3.1** *Let $A$ and $B$ be two nonempty subsets of a complete metric space $(X,d)$. Suppose that $A_0$ is nonempty and closed. Assume also that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions*:

(a) *$T$ is a $\mathcal{Z}$-proximal contraction of the first kind*;
(b) *$g \in \mathcal{G}_A$*;
(c) *$T(A_0) \subseteq B_0$*;
(d) *$A_0 \subseteq g(A_0)$*.

*Then there exists a unique point $x \in A$ such that $d(gx, Tx) = d(A, B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for every $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$.*

*Proof* Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B).$$

Clearly, for $x_1 \in A_0$, there exists $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B).$$

By repeating this process, for $x_n \in A_0$, we can find $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Tx_n) = d(A, B), \quad \text{for all } n \in \mathbb{N}.$$

In the constructive process of $\{x_n\}$, if for some $m > n$, we have $Tx_m = Tx_n$, then we choose $x_{m+1} = x_{n+1}$. Also, if there exists $m \in \mathbb{N}$ such that $d(gx_{m+1}, gx_m) = 0$, then $x_{m+1} = x_m$, and hence $Tx_{m+1} = Tx_m$ and $x_{m+2} = x_{m+1}$. It follows that $x_n = x_m$ for all $n \in \mathbb{N}$ with $n \geq m$ and so the sequence $\{x_n\}$ converges to $x_m \in A$. We also have $d(gx_m, Tx_m) = d(A, B)$.

Then we suppose that $0 < d(x_{n+1}, x_n) \leq d(gx_{n+1}, gx_n) \neq 0$ for all $n \in \mathbb{N}$. Since $T$ is a $\mathcal{Z}$-proximal contraction of the first kind and $g \in \mathcal{G}_A$, we write

$$\begin{aligned} 0 &\leq \zeta\bigl(d(gx_{n+1}, gx_n), d(x_n, x_{n-1})\bigr) \\ &< d(x_n, x_{n-1}) - d(gx_{n+1}, gx_n) \\ &\leq d(x_n, x_{n-1}) - d(x_{n+1}, x_n), \end{aligned} \tag{2}$$

for every $n \in \mathbb{N}$. This implies that the sequence $\{d(x_n, x_{n-1})\}$ is decreasing and hence there exists $r \geq 0$ such that $d(x_n, x_{n-1}) \to r$. Suppose $r > 0$. From (2), we deduce also that

$$d(gx_{n+1}, gx_n) \leq d(x_n, x_{n-1}), \quad \text{for all } n \in \mathbb{N}.$$

On the other hand $g \in \mathcal{G}_A$ and hence

$$d(x_{n+1}, x_n) \leq d(gx_{n+1}, gx_n) \leq d(x_n, x_{n-1}), \quad \text{for all } n \in \mathbb{N}.$$

Consequently,

$$\lim_{n \to +\infty} d(gx_{n+1}, gx_n) = r.$$

Now, using the property $(\zeta_3)$ of a simulation function, we write

$$0 \leq \limsup_{n \to +\infty} \zeta\bigl(d(gx_{n+1}, gx_n), d(x_n, x_{n-1})\bigr) < 0,$$

which is a contradiction and hence $r = 0$.

The next step is to show that the sequence $\{x_n\}$ is Cauchy. By contradiction, assume that $\{x_n\}$ is not a Cauchy sequence. Then, by Lemma 3.1, there exist an $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $n_k > m_k \geq k$ and $d(x_{n_k}, x_{m_k}) \geq \varepsilon$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to+\infty} d(x_{n_k}, x_{m_k}) = \varepsilon = \lim_{k\to+\infty} d(x_{n_k+1}, x_{m_k+1}).$$

Then we can assume that $d(x_{n_k+1}, x_{m_k+1}) > 0$ for all $k \in \mathbb{N}$. Since $T$ is a $\mathcal{Z}$-proximal contraction of the first kind and $d(gx_{n_k+1}, Tx_{n_k}) = d(A,B) = d(gx_{m_k+1}, Tx_{m_k})$, we obtain

$$\begin{aligned} 0 &\leq \zeta\big(d(gx_{n_k+1}, gx_{m_k+1}), d(x_{n_k}, x_{m_k})\big) \\ &< d(x_{n_k}, x_{m_k}) - d(gx_{n_k+1}, gx_{m_k+1}), \end{aligned}$$

for all $k \in \mathbb{N}$. Thus, the previous inequality and $g \in \mathcal{G}_A$ ensure that

$$\lim_{k\to+\infty} d(gx_{n_k+1}, gx_{m_k+1}) = \varepsilon.$$

By using the property $(\zeta_3)$ of a simulation function, with $t_k = d(gx_{n_k+1}, gx_{m_k+1})$ and $s_k = d(x_{n_k}, x_{m_k})$, we obtain

$$0 \leq \limsup_{k\to+\infty} \zeta\big(d(gx_{n_k+1}, gx_{m_k+1}), d(x_{n_k}, x_{m_k})\big) < 0,$$

which is a contradiction. We conclude that the sequence $\{x_n\}$ is Cauchy. Since $(X,d)$ is complete and $A_0$ is closed, then $A_0$ is complete and hence there exists $x \in A_0$ such that $x_n \to x$. Moreover, by the continuity of $g$, we have $gx_n \to gx$ and thus $gx \in A_0$, since $gx_n \in A_0$ for all $n \in \mathbb{N}$ and $A_0$ is closed. On the other hand, since $x \in A_0$ and $T(A_0) \subseteq B_0$, there exists $z \in A_0$ such that $d(z, Tx) = d(A,B)$.

Now, if $z = gx_n$ for infinite $n \in \mathbb{N}$, then $z = gx$. Hence we assume that $z \neq gx_n$ for all $n \in \mathbb{N}$. Also there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \neq x$ for all $k \in \mathbb{N}$. Again, since $T$ is a $\mathcal{Z}$-proximal contraction of the first kind, we get

$$\zeta\big(d(z, gx_{n_k+1}), d(x, x_{n_k})\big) < d(x, x_{n_k}) - d(z, gx_{n_k+1}),$$

and hence

$$d(z, gx_{n_k+1}) < d(x, x_{n_k}), \quad \text{for all } k \in \mathbb{N}.$$

Letting $k \to +\infty$, we obtain $d(z, gx_{n_k+1}) \to 0$ and then $z = gx$. This implies that

$$d(gx, Tx) = d(A,B).$$

To prove the uniqueness, let $x^* \neq x$ be another point in $A_0$ such that

$$d\big(gx^*, Tx^*\big) = d(A,B).$$

Since $g \in \mathcal{G}_A$ and $T$ is a $\mathcal{Z}$-proximal contraction of the first kind, we write

$$\begin{aligned} 0 &\leq \zeta\left(d\left(gx, gx^*\right), d\left(x, x^*\right)\right) \\ &< d\left(x, x^*\right) - d\left(gx, gx^*\right) \\ &\leq d\left(x, x^*\right) - d\left(x, x^*\right) = 0, \end{aligned}$$

which leads to $x = x^*$, a contradiction. □

We get the following corollary, by setting $g$ as the identity mapping on $A$ in Theorem 3.1.

**Corollary 3.1** *Let $A$ and $B$ be two nonempty subsets of a complete metric space $(X,d)$. Suppose that $A_0$ is nonempty and closed. Assume also that the mapping $T : A \to B$ satisfies the following conditions*:

(a) *$T$ is a $\mathcal{Z}$-proximal contraction of the first kind*;
(b) *$T(A_0) \subseteq B_0$.*

*Then there exists a unique point $x \in A$ such that $d(x, Tx) = d(A,B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(x_{n+1}, Tx_n) = d(A,B)$ for every $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$.*

**Example 3.3** Let $X$, $A$, $B$, $d$, $T$, and $\zeta$ be as in Example 3.1. Notice that $A_0 = A = B_0$ is closed and $T(A_0) \subseteq B_0$. Thus, by an application of Corollary 3.1, the mapping $T : A \to B$ has a unique point $x \in A$ such that $d(x, Tx) = 0 = d(A,B)$; here $x = 0$.

From Theorem 3.1, we obtain the following corollary which is a generalization of Theorem 3.1 of [6].

**Corollary 3.2** *Let $A$ and $B$ be two nonempty subsets of a complete metric space $(X,d)$. Suppose that $A_0$ is nonempty and closed. Assume also that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions*:

(a) *$T$ is a proximal contraction of the first kind*;
(b) *$g \in \mathcal{G}_A$*;
(c) *$T(A_0) \subseteq B_0$*;
(d) *$A_0 \subseteq g(A_0)$.*

*Then there exists a unique point $x \in A$ such that $d(gx, Tx) = d(A,B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(gx_{n+1}, Tx_n) = d(A,B)$ for every $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$.*

*Proof* Note that a proximal contraction of the first kind is a $\mathcal{Z}$-proximal contraction of the first kind with respect to the simulation function $\zeta : [0,+\infty[ \times [0,+\infty[ \to \mathbb{R}$ defined by $\zeta(t,s) = ks - t$ for all $t, s \in [0,+\infty[$, where $k \in [0,1)$. □

The following theorem establishes a result of existence of a $g$-best proximity point for a $\mathcal{Z}$-proximal contraction of the second kind.

**Theorem 3.2** *Let $A$ and $B$ be two nonempty subsets of a complete metric space $(X,d)$. Suppose that $T(A_0)$ is nonempty and closed. Assume also that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions*:

(a) *$T$ is a $\mathcal{Z}$-proximal contraction of the second kind*;
(b) *$T$ is injective on $A_0$*;
(c) $T \in \mathcal{T}_g$;
(d) $T(A_0) \subseteq B_0$;
(e) $A_0 \subseteq g(A_0)$.

*Then there exists a unique point $x \in A$ such that $d(gx, Tx) = d(A,B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(gx_{n+1}, Tx_n) = d(A,B)$ for every $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$.*

*Proof* By following a similar reasoning to that in the proof of Theorem 3.1, one can construct a sequence $\{x_n\} \subseteq A_0$ such that $d(gx_{n+1}, Tx_n) = d(A,B)$ for all $n \in \mathbb{N}$. Moreover, in the constructive process of $\{x_n\}$ if $Tx_m = Tx_n$ for some $m > n$, then we choose $x_{m+1} = x_{n+1}$. This condition ensures that if, for some $m \in \mathbb{N}$, we have $x_m = x_{m+1}$, then $x_n = x_m$ for all $n \geq m$. So the sequence $\{x_n\}$ converges to $x_m$ and also $d(gx_m, Tx_m) = d(A,B)$. Thus, we can suppose that $d(x_{n+1}, x_n) \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since $T$ is a $\mathcal{Z}$-proximal contraction of the second kind, we have

$$\zeta\left(d(Tgx_{n+1}, Tgx_n), d(Tx_n, Tx_{n-1})\right) \geq 0, \quad \text{for all } n \in \mathbb{N}.$$

From $T \in \mathcal{T}_g$ and $T$ being injective on $A_0$, we deduce that $d(Tgx_{n+1}, Tgx_n) > 0$ and $d(Tx_n, Tx_{n-1}) > 0$ for all $n \in \mathbb{N}$. By using the property $(\zeta_2)$ of a simulation function, we get

$$\begin{aligned} 0 &\leq \zeta\left(d(Tgx_{n+1}, Tgx_n), d(Tx_n, Tx_{n-1})\right) \\ &< d(Tx_n, Tx_{n-1}) - d(Tgx_{n+1}, Tgx_n) \\ &\leq d(Tx_n, Tx_{n-1}) - d(Tx_{n+1}, Tx_n), \end{aligned} \tag{3}$$

for every $n \in \mathbb{N}$. This implies that the sequence $\{d(Tx_n, Tx_{n-1})\}$ is decreasing and hence there exists $r \geq 0$ such that $d(Tx_n, Tx_{n-1}) \to r$. Suppose $r > 0$. From (3) we deduce also that

$$d(Tgx_{n+1}, Tgx_n) < d(Tx_n, Tx_{n-1}), \quad \text{for all } n \in \mathbb{N}.$$

On the other hand $T \in \mathcal{T}_g$ and hence

$$d(Tx_{n+1}, Tx_n) \leq d(Tgx_{n+1}, Tgx_n) < d(Tx_n, Tx_{n-1}), \quad \text{for all } n \in \mathbb{N}.$$

Consequently,

$$\lim_{n\to+\infty} d(Tgx_{n+1}, Tgx_n) = r.$$

Now, using the property $(\zeta_3)$ of a simulation function, we write

$$0 \leq \limsup_{n\to+\infty} \zeta\left(d(Tgx_{n+1}, Tgx_n), d(Tx_n, Tx_{n-1})\right) < 0,$$

which is a contradiction and hence $r = 0$.

Next step is to show that the sequence $\{Tx_n\}$ is Cauchy. By contradiction, assume that $\{Tx_n\}$ is not a Cauchy sequence. Then, by Lemma 3.1, there exist an $\varepsilon > 0$ and two subsequences $\{Tx_{n_k}\}$ and $\{Tx_{m_k}\}$ of $\{Tx_n\}$ such that $n_k > m_k \geq k$ and $d(Tx_{n_k}, Tx_{m_k}) \geq \varepsilon$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to+\infty} d(Tx_{n_k}, Tx_{m_k}) = \varepsilon = \lim_{k\to+\infty} d(Tx_{n_k+1}, Tx_{m_k+1}).$$

Then we can assume that $d(Tx_{n_k+1}, Tx_{m_k+1}) > 0$ for all $k \in \mathbb{N}$. Since $T$ is a $\mathcal{Z}$-proximal contraction of the second kind and $d(gx_{n_k+1}, Tx_{n_k}) = d(A,B) = d(gx_{m_k+1}, Tx_{m_k})$, we obtain

$$\begin{aligned} 0 &\leq \zeta\big(d(Tgx_{n_k+1}, Tgx_{m_k+1}), d(Tx_{n_k}, Tx_{m_k})\big) \\ &< d(Tx_{n_k}, Tx_{m_k}) - d(Tgx_{n_k+1}, Tgx_{m_k+1}), \end{aligned}$$

for all $k \in \mathbb{N}$. Thus the previous inequality and $T \in \mathcal{T}_g$ ensure that

$$\lim_{k\to+\infty} d(Tgx_{n_k+1}, Tgx_{m_k+1}) = \varepsilon.$$

By using the property $(\zeta_3)$ of a simulation function, with $t_k = d(Tgx_{n_k+1}, Tgx_{m_k+1})$ and $s_k = d(Tx_{n_k}, Tx_{m_k})$, we obtain

$$0 \leq \limsup_{k\to+\infty} \zeta\big(d(Tgx_{n_k+1}, Tgx_{m_k+1}), d(Tx_{n_k}, Tx_{m_k})\big) < 0,$$

which is a contradiction. We conclude that the sequence $\{Tx_n\}$ is Cauchy.

By the completeness of $(X,d)$ and since $T(A_0)$ is closed, we have $Tx_n \to Tu \in B_0$. Moreover, there exists $z \in A_0$ such that

$$d(z, Tu) = d(A,B).$$

Since $A_0 \subseteq g(A_0)$, we obtain $z = gx$ for some $x \in A_0$, and hence

$$d(gx, Tu) = d(A,B).$$

Clearly, if $x = x_n$ for infinite $n \in \mathbb{N}$, then $Tx = Tu$. Therefore, we assume that $x \neq x_n$ for all $n \in \mathbb{N}$. Also there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \neq Tu$ for all $k \in \mathbb{N}$. Again, since $T$ is a $\mathcal{Z}$-proximal contraction of the second kind, we get

$$\begin{aligned} 0 &\leq \zeta\big(d(Tgx, Tgx_{n_k+1}), d(Tu, Tx_{n_k})\big) \\ &< d(Tu, Tx_{n_k}) - d(Tgx, Tgx_{n_k+1}) \end{aligned}$$

and hence

$$d(Tx, Tx_{n_k+1}) \leq d(Tgx, Tgx_{n_k+1}) < d(Tu, Tx_{n_k}),$$

for all $k \in \mathbb{N}$, since $T \in \mathcal{T}_g$. Letting $k \to +\infty$, we obtain $d(Tx, Tx_{n_k+1}) \to 0$ and hence $Tx = Tu$. This implies that

$$d(gx, Tx) = d(A,B).$$

To prove the uniqueness, let $x^* \neq x$ be another point in $A_0$ such that

$$d\big(gx^*, Tx^*\big) = d(A,B).$$

Since $T \in \mathcal{T}_g$ is injective on $A_0$ and $T$ is a $\mathcal{Z}$-proximal contraction of the second kind, we write

$$\begin{aligned} 0 &\leq \zeta\big(d\big(Tgx, Tgx^*\big), d\big(Tx, Tx^*\big)\big) \\ &< d\big(Tx, Tx^*\big) - d\big(Tgx, Tgx^*\big) \\ &\leq d\big(Tx, Tx^*\big) - d\big(Tx, Tx^*\big) = 0, \end{aligned}$$

which leads to contradiction; we conclude that $Tx = Tx^*$ and hence $x = x^*$. □

We get the following corollary, by setting $g$ as the identity mapping on $A$ in Theorem 3.2.

**Corollary 3.3** *Let $A$ and $B$ be two nonempty subsets of a complete metric space $(X,d)$. Suppose that $T(A_0)$ is nonempty and closed. Assume also that the mapping $T : A \to B$ satisfies the following conditions*:

(a) *$T$ is a $\mathcal{Z}$-proximal contraction of the second kind*;
(b) *$T$ is injective on $A_0$*;
(c) *$T(A_0) \subseteq B_0$*.

*Then there exists a unique point $x \in A$ such that $d(x, Tx) = d(A,B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(x_{n+1}, Tx_n) = d(A,B)$ for every $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$.*

**Example 3.4** Let $X = \mathbb{R}$ be endowed with the usual metric $d(x,y) = |x - y|$ for all $x, y \in X$. Consider $A = [-3,-1]$, $B = [0,1]$ so that $d(A,B) = 1$ and define $T : A \to B$ by

$$Tx = \begin{cases} 3 + x & \text{if } x \in [-3,-2], \\ -1 - x & \text{if } x \in ]-2,-1]. \end{cases}$$

We have

$$\begin{aligned} A_0 &= \big\{x \in A : d(x,y) = d(A,B) = 1, \text{for some } y \in B\big\} = \{-1\}, \\ B_0 &= \big\{y \in B : d(x,y) = d(A,B) = 1, \text{for some } x \in A\big\} = \{0\}, \end{aligned}$$

and hence $T(A_0) = \{0\} = B_0$.

It is easy to show that $T$ is a $\mathcal{Z}$-proximal contraction of the second kind, where the function $\zeta_\lambda : [0,+\infty[ \times [0,+\infty[ \to \mathbb{R}$ is given in Example 2.1.

Indeed, from $d(u, Tx) = d(v, Ty) = 1 = d(A,B)$, we get $(u,v) = (-1,-1)$ for $x, y \in \{-3,-1\}$ and hence

$$\zeta\big(d(Tu, Tv), d(Tx, Ty)\big) = \zeta\big(d(0,0), d(0,0)\big) = \zeta(0,0) = 1.$$

Therefore all the conditions of Corollary 3.3 hold true and $x = -1$ is the unique point such that $d(-1, T(-1)) = 1 = d(A,B)$.

## 4 Variational inequality problems

Let $H$ be a real Hilbert space, with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $K$ be a nonempty, closed, and convex subset of $H$. We consider a monotone variational inequality problem as follows; see [27, 28].

**Problem 4.1** Find $u \in K$ such that $\langle Su, v-u\rangle \geq 0$ for all $v \in K$, where $S : H \to H$ is a monotone operator (*i.e.*, $\langle Su - Sv, v-u\rangle \geq 0$ for all $u, v \in K$).

The interest for variational inequalities theory is due to the fact that a wide class of equilibrium problems, arising in pure and applied sciences, can be treated in an unified framework [29]. Now, we recall the metric projection, say $P_K : H \to K$, which is a powerful tool for solving a variational inequality problem. Referring to classical books on approximation theory in inner product spaces, see [30], we recall that for each $u \in H$, there exists a unique nearest point $P_K u \in K$ such that

$$\|u - P_K u\| \leq \|u - v\|, \quad \text{for all } v \in K.$$

The theoretical background of projection and related approximation methods can be found in [30], too. Here, we need the following crucial lemmas, relating the existence of a solution for a variational inequality problem and the existence of a fixed point of a certain mapping.

**Lemma 4.1** *Let $z \in H$. Then $u \in K$ satisfies the inequality $\langle u - z, y - u\rangle \geq 0$, for all $y \in K$ if and only if $u = P_K z$.*

**Lemma 4.2** *Let $S : H \to H$ be monotone. Then $u \in K$ is a solution of $\langle Su, v-u\rangle \geq 0$, for all $v \in K$, if and only if $u = P_K(u - \lambda Su)$, with $\lambda > 0$.*

On this basis, we give some general convergence results on the solution of Problem 4.1.

**Theorem 4.1** *Let $K$ be a nonempty, closed, and convex subset of a real Hilbert space $H$ and $I_K$ be the identity operator on $K$. Assume that the monotone operator $S : H \to H$ satisfies the following condition*:

(a) $P_K(I_K - \lambda S) : K \to K$ *is a $\mathcal{Z}$-contraction, with $\lambda > 0$.*

*Then there exists a unique point $u \in K$ such that $\langle Su, v-u\rangle \geq 0$ for all $v \in K$. Moreover, for every $u_0 \in K$, there exists a sequence $\{u_n\} \subseteq K$ such that $u_{n+1} = P_K(u_n - \lambda Su_n)$ for every $n \in \mathbb{N} \cup \{0\}$ and $u_n \to u$.*

*Proof* Define $T : K \to K$ by $Tx = P_K(x - \lambda Sx)$ for all $x \in K$ so that, by Lemma 4.2, $u \in K$ is a solution of $\langle Su, v-u\rangle \geq 0$ for all $v \in K$ if and only if $u = Tu$. Clearly, the operator $T$ satisfies all the hypotheses of Theorem 2.1 by setting $A = B = K$. We deduce that the conclusions of Theorem 4.1 hold true as an immediate consequence of Theorem 2.1. □

Inspired by Theorem 4.1, one can consider the following algorithm to solve Problem 4.1.

**Variational inequality problem solving algorithm**

**Step 1** (**Initialization**): Select an arbitrary starting point $u_0 \in K$.

**Step 2** (**Iteration**): Given the current approximation point $u_n \in K$, $n \in \mathbb{N} \cup \{0\}$, compute

$$u_{n+1} = P_K(u_n - \lambda S u_n),$$

satisfying Theorem 4.1(a).

In view of the proof of Theorem 4.1 (and hence Theorem 2.1), this algorithm generates sequences converging to a unique solution of Problem 4.1.

From Corollary 3.3, we obtain the following result for the solution of a variational inequality problem.

**Theorem 4.2** *Let $K$ be a nonempty, closed, and convex subset of a real Hilbert space $H$ and $I_K$ be the identity operator on $K$. Assume that the monotone operator $S : H \to H$ satisfies the following conditions*:

(a) $P_K(I_K - \lambda S) : K \to K$ *is a $\mathcal{Z}$-proximal contraction of the second kind, with* $\lambda > 0$;
(b) $P_K(I_K - \lambda S)$ *is injective on* $K$;
(c) $P_K(I_K - \lambda S)(K)$ *is closed*.

*Then there exists a unique point $u \in K$ such that $\langle Su, v - u\rangle \geq 0$ for all $v \in K$. Moreover, for every $u_0 \in K$, there exists a sequence $\{u_n\} \subseteq K$ such that $u_{n+1} = P_K(u_n - \lambda S u_n)$ for every $n \in \mathbb{N} \cup \{0\}$ and $u_n \to u$.*

Inspired by Theorem 4.2, one can consider the following algorithm to solve Problem 4.1.

**Variational inequality problem solving algorithm**

**Step 1** (**Initialization**): Select an arbitrary starting point $u_0 \in K$.
**Step 2** (**Iteration**): Given the current approximation point $u_n \in K$, $n \in \mathbb{N} \cup \{0\}$, compute

$$u_{n+1} = P_K(u_n - \lambda S u_n),$$

satisfying Theorem 4.2(a)-(c).

Our results apply to some fundamental problems of optimization theory. In fact, as a special case of Problem 4.1, we retrieve the following constrained minimization problem.

**Problem 4.2** Find $u \in K$ such that $\langle \nabla f u, v - u\rangle \geq 0$ for all $v \in K$, where $f : H \to \mathbb{R}$ is a continuously differentiable function which is convex on $K$ with $\nabla f$ denoting the gradient of $f$.

A second special case of Problem 4.1, is the following hierarchical variational inequality problem.

**Problem 4.3** Let $\mathrm{Fix}(g) := \{x \in K : x = gx\}$, where $g : K \to K$ is such that $\|gx - gy\| \leq \|x - y\|$ (*i.e.*, $g$ is nonexpansive). Find $u \in \mathrm{Fix}(g)$ such that $\langle Su, v - u\rangle \geq 0$ for all $v \in \mathrm{Fix}(g)$, where $S : K \to K$ is a monotone continuous operator.

Finally, by using the Gâteaux directional derivative of a metric projection, we denote

$$\Pi_K\bigl(x, -S(x)\bigr) := \lim_{t \to 0^+} \frac{P_K(x - tS(x)) - x}{t},$$

where $S : K \to H$ is continuous.

Now, consider the initial value problem:

$$\frac{dx(t)}{dt} = \Pi_K\big(x(t), -S\big(x(t)\big)\big), \qquad x(0) = x_0 \in K, \quad t \in [0, +\infty[, \tag{4}$$

whose critical points satisfy $\frac{dx(t)}{dt} = 0$.

In [31], the authors proved that the set of critical points of (4) coincides with the set of solutions of a monotone variational inequality problem involving the operator $S$. Thus, our theory is applicable to the study of (4), which is associated to various economic problems; see again [31].

## 5 Conclusions

Best approximation and fixed point theories are continuously expanding topics due to their applications in many fields of pure and applied mathematics. Thus, we gave new theorems of $g$-best proximity point by using a notion of simulation function. This approach is useful to cover existing results in the literature from an unifying point of view. A discussion of the solvability of monotone variational inequality problems and related optimization problems supports the new theory.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

**Author details**
[1]Mathematics Department, College of Science (Malaz), King Saud University, P.O. Box 22452, Riyadh, Kingdom of Saudi Arabia. [2]Department of Mathematics and Computer Sciences, University of Palermo, Via Archirafi 34, Palermo, 90123, Italy. [3]Department of Energy, Information Engineering and Mathematical Models (DEIM), University of Palermo, Viale delle Scienze, Palermo, 90128, Italy.

**Acknowledgements**
This research project was supported by a grant from the 'Research Center of the Center for Female Scientific and Medical Colleges', Deanship of Scientific Research, King Saud University.

Received: 3 November 2015 Accepted: 28 February 2016 Published online: 09 March 2016

**References**

1. Fan, K: Extensions of two fixed point theorems of F.E. Browder. Math. Z. **122**(3), 234-240 (1969)
2. Abkar, A, Gabeleh, M: Proximal quasi-normal structure and a best proximity point theorem. J. Nonlinear Convex Anal. **14**(4), 653-659 (2013)
3. Al-Thagafi, MA, Shahzad, N: Convergence and existence results for best proximity points. Nonlinear Anal. **70**(10), 3665-3671 (2009)
4. Di Bari, C, Suzuki, T, Vetro, C: Best proximity points for cyclic Meir-Keeler contractions. Nonlinear Anal. **69**(11), 3790-3794 (2008)
5. Eldred, AA, Veeramani, P: Existence and convergence of best proximity points. J. Math. Anal. Appl. **323**(2), 1001-1006 (2006)
6. Fernández-León, A: Best proximity points for proximal contractions (2012). arXiv:1207.4349v1 [math.FA]
7. Karpagam, S, Agrawal, S: Best proximity point theorems for *p*-cyclic Meir-Keeler contractions. Fixed Point Theory Appl. **2009**, Article ID 197308 (2009)
8. Kirk, WA, Reich, S, Veeramani, P: Proximinal retracts and best proximity pair theorems. Numer. Funct. Anal. Optim. **24**(7-8), 851-862 (2003)
9. Sadiq Basha, S: Best proximity points: global optimal approximate solutions. J. Glob. Optim. **49**(1), 15-21 (2011)
10. Sankar Raj, V: A best proximity point theorem for weakly contractive non-self-mappings. Nonlinear Anal. **74**(14), 4804-4808 (2011)
11. Vetro, C: Best proximity points: convergence and existence theorems for *p*-cyclic mappings. Nonlinear Anal. **73**(7), 2283-2291 (2010)
12. Khojasteh, F, Shukla, S, Radenović, S: A new approach to the study of fixed point theorems via simulation functions. Filomat **29**(6), 1189-1194 (2015)
13. Argoubi, H, Samet, B, Vetro, C: Nonlinear contractions involving simulation functions in a metric space with a partial order. J. Nonlinear Sci. Appl. **8**(6), 1082-1094 (2015)

14. Nastasi, A, Vetro, P: Fixed point results on metric and partial metric spaces via simulation functions. J. Nonlinear Sci. Appl. **8**(6), 1059-1069 (2015)
15. Roldán, A, Karapınar, E, Roldán, C, Martínez-Moreno, J: Coincidence point theorems on metric spaces via simulation functions. J. Comput. Appl. Math. **275**, 345-355 (2015)
16. Khojasteh, F, Karapınar, E, Radenović, S: $\theta$-Metric spaces: a generalization. Math. Probl. Eng. **2013**, Article ID 504609 (2013)
17. Radenović, S, Kadelburg, Z, Jandrlić, D, Jandrlić, A: Some results on weakly contractive maps. Bull. Iran. Math. Soc. **38**(3), 625-645 (2012)
18. Van An, T, Van Dung, N, Kadelburg, Z, Radenović, S: Various generalizations of metric spaces and fixed point theorems. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **109**(1), 175-198 (2015)
19. Sadiq Basha, S: Best proximity point theorems: an exploration of a common solution to approximation and optimization problems. Appl. Math. Comput. **218**(19), 9773-9780 (2012)
20. Sadiq Basha, S, Veeramani, P: Best proximity pair theorems for multifunctions with open fibres. J. Approx. Theory **103**(1), 119-129 (2000)
21. Banach, S: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3**(1), 133-181 (1922)
22. Sadiq Basha, S: Best proximity points: optimal solutions. J. Optim. Theory Appl. **151**(1), 210-216 (2011)
23. Alghamdi, MA, Shahzad, N, Vetro, F: Best proximity points for some classes of proximal contractions. Abstr. Appl. Anal. **2013**, Article ID 713252 (2013)
24. Gabeleh, M: Best proximity points for weak proximal contractions. Bull. Malays. Math. Sci. Soc. **38**(1), 143-154 (2015)
25. Kadelburg, Z, Radenović, S: A note on some recent best proximity point results for non-self mappings. Gulf J. Math. **1**(1), 36-41 (2013)
26. Jleli, M, Ćojbašić Rajić, V, Samet, B, Vetro, C: Fixed point theorems on ordered metric spaces and applications to nonlinear elastic beam equations. J. Fixed Point Theory Appl. **12**(1-2), 175-192 (2012)
27. Censor, Y, Gibali, A, Reich, S, Sabach, S: Common solutions to variational inequalities. Set-Valued Var. Anal. **20**(2), 229-247 (2012)
28. Kinderlehrer, D, Stampacchia, G: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York (1980)
29. Patriksson, M: Nonlinear Programming and Variational Inequality Problems: A Unified Approach. Kluwer Academic, Dordrecht (1999)
30. Deutsch, F: Best Approximation in Inner Product Spaces. Springer, New York (2001)
31. Dupuis, P, Nagurney, A: Dynamical systems and variational inequalities. Ann. Oper. Res. **44**(1), 7-42 (1993)