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Fixed point results for α-implicit contractions with application to integral equations

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Abstract. Recently, Aydi et al. [On fixed point results for α-implicit contractions in quasi-metric spaces and consequences, *Nonlinear Anal. Model. Control*, 21(1):40–56, 2016] proved some fixed point results involving α-implicit contractive conditions in quasi-b-metric spaces. In this paper we extend and improve these results and derive some new fixed point theorems for implicit contractions in ordered quasi-b-metric spaces. Moreover, some examples and an application to integral equations are given here to illustrate the usability of the obtained results.

Keywords: fixed points, implicit contractions, quasi-b-metric spaces.

1 Introduction and preliminaries

It is always recognized that the contraction mapping principle proved in the Ph.D. dissertation of Banach in 1920, see also [6], is one of the most significant theorems in functional analysis and its applications in other branches of mathematics. In particular, this principle is considered as the source of metric fixed point theory. The study of fixed and common fixed points of mappings satisfying a certain metrical contractive condition has received the attention of many authors, see, for instance, [5, 25, 26, 27, 28, 31, 33, 34, 39].

On the other hand, various authors established fixed and common fixed point results for different classes of mappings defined in some generalized metric spaces [34, 38].

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Following this direction of research, we focus on one of these spaces, namely quasi-b-metric space. We point out that the concept of a b-metric space is strongly related to the papers of Bakhtin [5] and Czerwik [11, 12]; also, some years later, Khamsi [22] and Khamsi and Hussain [23] reintroduced this kind of space, but with the name of metric-type space. Moreover, they gave some fixed point results in such generalized spaces. Here, we use the name b-metric space to denote a symmetric space with some additional properties. Precisely, a b-metric space is a triplet (X, d, s), where (X, d) is a symmetric space and $s \geqslant 1$ is a real number satisfying the condition $d(x, y) \leqslant s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$. The interested reader is referred to [9, 10, 15, 17, 19, 20, 22, 23, 36, 37] for results of fixed and common fixed point in this space.

An interesting situation arises by dropping the requirement that the metric function $d : X \times X \to [0, +\infty)$ satisfies the symmetric condition $d(x, y) = d(y, x)$ for all $x, y \in X$. In this case, formally we pass from a metric space to a quasi-metric space, but this carries some significant consequences to the general theory, see also [2]. Now, to better understand this fact, we will underline the modifications to some fundamental topological notions for a quasi-b-metric space; we will restate the concepts of: limit, continuity, completeness, Cauchyness under left and right approaches, in view of the fact that a quasi-metric is not symmetric. In particular, the uniqueness of limit for a sequence need to be considered carefully, in virtue of the fact that the reader could define a sequence which possesses a left limit and right limit, but these limits are not equal to each other. In the above few lines, we reassume the interest for developing fixed point theory in the new setting of quasi-b-metric spaces. Finally, we recall that Samet et al. [35] introduced the notion of α-ψ-contractive mapping for establishing some fixed point results in the setting of complete metric spaces; this paper is at the basis of an intensive research in fixed point theory in the last years, see, for example, [8, 21, 24, 32]. Thus, in the setting of quasi-b-metric space, we give some fixed point results for a class of self-mappings that satisfy an α-implicit contractive condition. Also, we deduce fixed point results in ordered quasi-b-metric spaces. Our results extend and generalize the results in [4] and many others. Finally, an application to a nonlinear problem involving an integral equation supports the new theory.

Definition 1. Let X be a non-empty set, $s \geqslant 1$ be a real number and let $d : X \times X \to [0, +\infty)$ be a function which satisfies:

(d1) $d(x, y) = 0$ if and only if $x = y$,
(d2) $d(x, y) \leqslant s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a quasi-b-metric and (X, d, s) is called a quasi-b-metric space. Clearly, if $s = 1$, then the pair (X, d) is a quasi-metric space.

Remark 1. Any b-metric space is a quasi-b-metric space, but the converse is not true in general.

Example 1. Let $X = [0, +\infty)$, $d(x, y) = (x - y)^2 + x$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$, then it is clear that $(X, d, 2)$ is a quasi-b-metric space, but it is not a b-metric space. In fact (d2) holds since

$$d(x,y) = (x-y)^2 + x \leqslant 2\big[(x-z)^2 + (z-y)^2 + x + z\big] \\ = 2\big[d(x,z) + d(z,y)\big]$$

for all $x, y, z \in X$, but clearly $d(x,y) \neq d(y,x)$ if $x \neq y$.

Now, we give the topological notions of convergence, completeness and continuity in quasi-b-metric spaces.

Definition 2. Let (X,d,s) be a quasi-b-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if

$$\lim_{n\to+\infty} d(x_n,x) = \lim_{n\to+\infty} d(x,x_n) = 0.$$

Remark 2. In a quasi-b-metric space we have the uniqueness of limit for a convergent sequence.

Definition 3. Let (X,d,s) be a quasi-b-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is

(i) left-Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$ for all $n \geqslant m \geqslant N$,
(ii) right-Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$ for all $m \geqslant n \geqslant N$,
(iii) Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$ for all $m,n \geqslant N$.

Definition 4. Let (X,d,s) be a quasi-b-metric space. We say that (X,d,s) is

(i) left-complete if and only if each left-Cauchy sequence in X is convergent,
(ii) right-complete if and only if each right-Cauchy sequence in X is convergent,
(iii) complete if and only if each Cauchy sequence in X is convergent.

Definition 5. Let (X,d,s) be a quasi-b-metric space. The mapping $T : X \to X$ is continuous if for each sequence $\{x_n\}$ in X convergent to $x \in X$, the sequence $\{Tx_n\}$ converges to Tx, that is,

$$\lim_{n\to+\infty} d(Tx_n,Tx) = \lim_{n\to+\infty} d(Tx,Tx_n) = 0.$$

In recent years, Popa [29,30] initiated the study of fixed points for mappings satisfying an implicit relation; then, many researchers proved interesting fixed point, common fixed point and coincidence point results in various abstract spaces, see, for instance, [3, 7, 18]. Also, it is to mention that these authors used several types of implicit contractions to establish fixed point theorems. Let $s \geqslant 1$ be a real number, we denote by Ψ_s the set of all functions $\psi : [0,+\infty) \to [0,+\infty)$ satisfying:

(ψ1) ψ is nondecreasing,
(ψ2) $\sum_{n=1}^{+\infty} s^n \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

Also, denote by Φ the set of all functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying:

(φ) φ is nondecreasing and such that $\varphi(t) < t$ for all $t > 0$.

Remark 3. It is easy to see that if $\psi \in \Psi_s$, then $\psi(t) < t$ for all $t > 0$.

We introduce the following class of functions.

Definition 6. Let Γ be the set of all functions $F(t_1,\dots,t_6) : [0,+\infty)^6 \to \mathbb{R}$ satisfying:

(F1) F is nondecreasing in variable t_1 and nonincreasing in variable t_5,
(F2) there exists $\psi \in \Psi_s$ such that for all $u, v \geqslant 0$, the condition $F(u, v, v, u, s(u+v), 0) \leqslant 0$ implies $u \leqslant \psi(v)$.

Example 2. Let $s \geqslant 1$ be a real number and let $F : [0,+\infty)^6 \to \mathbb{R}$ be defined by

(i) $F(t_1,\dots,t_6) = t_1 - at_2 - bt_3 - ct_4 - et_5 - Lt_6$, where $a, b, c, e, L \geqslant 0$ with $sa + sb + c + (s + s^2)e < 1$,
(ii) $F(t_1,\dots,t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$, where $k \in [0, (s + s^2)^{-1})$.

In both cases the function F satisfies the conditions (F1) and (F2) and so $F \in \Gamma$.

Definition 7. (See [35].) Let $T : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. One says that T is α-admissible if

$$x, y \in X, \quad \alpha(x,y) \geqslant 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geqslant 1.$$

Definition 8. Let (X, d, s) be a quasi-b-metric space and $T : X \to X$ be a given mapping. We say that T is a right α-implicit contractive mapping if there exist two functions $\alpha : X \times X \to [0,+\infty)$ and $F \in \Gamma$ such that

$$F\big(\alpha(x,y)d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big) \leqslant 0 \qquad (1)$$

for all $x, y \in X$.

Finally, let X be a non-empty set. If (X, d, s) is a quasi-b-metric space and $(X, \preccurlyeq)$ is a partially ordered set, then $(X, d, s, \preccurlyeq)$ is called an ordered quasi-b-metric space. Also, $x, y \in X$ are called comparable if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds. Let $(X, \preccurlyeq)$ be a partially ordered set and let $f, T : X \to X$ be two mappings. T is said to be f-nondecreasing if $fx \preccurlyeq fy$ implies $Tx \preccurlyeq Ty$ for all $x, y \in X$. If f is the identity mapping on X, then T is nondecreasing.

In the sequel, we consider the following properties of regularity. Let (X, d, s) be a quasi-b-metric space and let $\alpha : X \times X \to [0,+\infty)$ be a function. Then

(H) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geqslant 1$ for all $k \in \mathbb{N}$.
(H′) if $\{x_n\}$ is a sequence in X such that $\alpha(x_{n+1}, x_n) \geqslant 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geqslant 1$ for all $k \in \mathbb{N}$.

(r) If $\preccurlyeq$ is a partial order on X, then X is regular if for each sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$, and x_{n-1} and x_n are comparable for all $n \in \mathbb{N}$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)}$ and x are comparable for all $k \in \mathbb{N}$.

2 Main results

Our first result is the following theorem that generalizes the main result of [4].

Theorem 1. *Let (X, d, s) be a right-complete quasi-b-metric space and $T : X \to X$ be a right α-implicit contractive mapping. Suppose that*:

(i) *T is an α-admissible mapping,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$,*
(iii) *T is continuous.*

Then there exists a fixed point z of T, that is, $z = Tz$.

Proof. By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. We define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. First, suppose that $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$; in this case, the proof is completed since $z = x_n = x_{n+1} = Tx_n = Tz$. Then, throughout the proof, we assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Now, since the mapping T is α-admissible and $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geqslant 1$, we deduce that $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geqslant 1$. By iterating the process above, we get

$$\alpha(x_{n-1}, x_n) \geqslant 1 \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

By an application of the contractive condition (1) with $x = x_{n-1}$ and $y = x_n$, we have

$$\begin{aligned} &F\big(\alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n), d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), \\ &\quad d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\big) \leqslant 0, \end{aligned}$$

that is,

$$\begin{aligned} &F\big(\alpha(x_{n-1}, x_n) d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n), \\ &\quad d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\big) \leqslant 0. \end{aligned}$$

By using the conditions (2), (d2) and (F1) we get

$$\begin{aligned} &F\big(d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad s\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big], 0\big) \leqslant 0. \end{aligned}$$

Since the function F satisfies also the condition (F2), then there exists $\psi \in \Psi_s$ such that

$$d(x_n, x_{n+1}) \leqslant \psi\big(d(x_{n-1}, x_n)\big) \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

From (3), it is easy to derive that

$$d(x_n, x_{n+1}) \leqslant \psi^n\bigl(d(x_0, x_1)\bigr) \quad \text{for all } n \in \mathbb{N}.$$

Now, we shall prove that $\{x_n\}$ is a right-Cauchy sequence. Take $m > n$; by using the condition (d2), we write

$$\begin{aligned} d(x_n, x_m) &\leqslant sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-n-1} d(x_{m-1}, x_m) \\ &\leqslant \sum_{k=n}^{m-1} s^k \psi^k\bigl(d(x_0, x_1)\bigr). \end{aligned}$$

By condition ($\psi 2$) the series $\sum_{k=1}^{+\infty} s^k \psi^k(d(x_0, x_1))$ is convergent and so $\{x_n\}$ is a right-Cauchy sequence.

Since (X, d, s) is a right-complete quasi-b-metric space, then there exists a point $z \in X$ such that $x_n \to z$ as $n \to +\infty$, that is,

$$\lim_{n\to+\infty} d(x_n, z) = \lim_{n\to+\infty} d(z, x_n) = 0.$$

Next, we prove that z is a fixed point of T. Indeed, we write

$$\begin{aligned} d(z, Tz) &\leqslant s\bigl[d(z, x_{n+1}) + d(x_{n+1}, Tz)\bigr] \\ &= sd(z, x_{n+1}) + sd(Tx_n, Tz). \end{aligned}$$

Finally, by using the continuity of T, on letting $n \to +\infty$, we obtain $d(z, Tz) = 0$, that is, $Tz = z$ and hence z is a fixed point of T. □

Our next result is analogous to Theorem 1, but we do not require the continuity of mapping T. Precisely, we obtain this result by assuming the continuity of the function F and the condition (H) stated above.

Theorem 2. *Let (X, d, s) be a right-complete quasi-b-metric space and $T : X \to X$ be a right α-implicit contractive mapping with respect to a continuous function $F \in \Gamma$. Suppose that*:

(i) *T is an α-admissible mapping,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$,*
(iii) *condition (H) holds true,*
(iv) *if $x_n \to x$ as $n \to +\infty$, then $\limsup_{n\to+\infty} d(x_n, y) \geqslant d(x, y)$ for all $y \in X$.*

Then there exists a fixed point z of T.

Proof. Following the same lines in the proof of Theorem 1, we get that the sequence $\{x_n\}$, defined by the schema $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, is a right-Cauchy sequence, with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$, which converges to some $z \in X$. Next, from condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \geqslant 1$ for all $k \in \mathbb{N}$. We need to show that $Tz = z$.

Since $\alpha(x_{n(k)}, z) \geqslant 1$, by an application of the contractive condition (1) with $x = x_{n(k)}$ and $y = z$, and condition (F1), we obtain

$$\begin{aligned} &F\big(d(x_{n(k)+1}, Tz), d(x_{n(k)}, z), d(x_{n(k)}, x_{n(k)+1}), d(z, Tz), \\ &\quad s\big[d(x_{n(k)}, z) + d(z, Tz)\big], d(z, x_{n(k)+1})\big) \leqslant 0. \end{aligned}$$

Letting $k \to +\infty$ and by using the continuity of F and (iv), we get

$$\begin{aligned} &F\big(d(z, Tz), 0, 0, d(z, Tz), sd(z, Tz), 0\big) \\ &\quad \leqslant F\Big(\limsup_{n\to+\infty} d(x_{n(k)+1}, Tz), 0, 0, d(z, Tz), sd(z, Tz), 0\Big) \leqslant 0. \end{aligned}$$

Finally, by condition (F2), we deduce that $d(z, Tz) \leqslant 0$ and hence $z = Tz$. □

For an improvement of above results, we consider sufficient conditions to establishing the uniqueness of fixed point. Precisely, we need the following additional conditions for the functions $\alpha : X \times X \to [0, +\infty)$ and $F : [0, +\infty)^6 \to \mathbb{R}$:

(U) For all $x, y \in \mathrm{Fix}(T)$, we have $\alpha(x, y) \geqslant 1$, where $\mathrm{Fix}(T)$ denotes the set of all fixed points of T.

(F3) There exists $\varphi \in \Phi$ such that for all $u, v > 0$, the condition $F(u, u, 0, 0, u, v) \leqslant 0$ implies $u \leqslant \varphi(v)$.

Theorem 3. *Adding conditions* (U) *and* (F3) *to the hypotheses of Theorem* 1 (*resp. Theorem* 2)*, we obtain that z is a unique fixed point of T.*

Proof. The proof is obtained by contradiction. Assume that there exist $z, w \in \mathrm{Fix}(T)$ with $z \neq w$. By an application of the contractive condition (1), we get

$$F\big(\alpha(z, w)d(Tz, Tw), d(z, w), d(z, Tz), d(w, Tw), d(z, Tw), d(w, z)\big) \leqslant 0.$$

Then, by condition (U), we write

$$F\big(d(z, w), d(z, w), 0, 0, d(z, w), d(w, z)\big) \leqslant 0.$$

Since F satisfies also condition (F3), then

$$d(z, w) \leqslant \varphi\big(d(w, z)\big).$$

By a similar argument, we obtain

$$d(w, z) \leqslant \varphi\big(d(z, w)\big).$$

Then, by combining the last two inequalities, we write

$$d(z, w) \leqslant \varphi\big(d(w, z)\big) \leqslant \varphi^2\big(d(z, w)\big) < d(z, w),$$

which is a contradiction and hence $z = w$. □

Next, to underline the unifying power of implicit relations, we deduce some corollaries from our theorems.

Corollary 1. *Let* (X, d, s) *be a right-complete quasi-b-metric space and* $T : X \to X$ *be a given mapping. Suppose that there exists a function* $\alpha : X \times X \to [0, +\infty)$ *such that*

$$\begin{aligned}&\alpha(x, y)d(Tx, Ty)\\&\quad \leqslant ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(x, Ty) + Ld(y, Tx)\end{aligned}$$

for all $x, y \in X$*, where* $a, b, c, e, L \geqslant 0$ *and* $sa + sb + c + (s + s^2)e < 1$*. Assume also that*:

(i) T *is an* α*-admissible mapping,*
(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$*,*
(iii) T *is continuous or conditions* (H) *and* (iv) *of Theorem* 2 *hold true.*

Then there exists a fixed point z *of* T*. Moreover, if* $a + e + L < 1$ *and condition* (U) *holds true, then* z *is a unique fixed point of* T*.*

Proof. The proof of existence of a fixed point for mapping T follows by an application of Theorem 1 (resp. Theorem 2), by assuming

$$F(t_1, \dots, t_6) = t_1 - at_2 - bt_3 - ct_4 - et_5 - Lt_6,$$

where $a, b, c, e, L \geqslant 0$ and $sa + sb + c + (s + s^2)e < 1$. In fact, in view of Example 2, the function F defined above satisfies conditions (F1) and (F2), and this concludes the proof. Also, F satisfies condition (F3) if $a + e + L < 1$. Thus, by an application of Theorem 3, we get uniqueness of the fixed point. □

Corollary 2. *Let* (X, d, s) *be a right-complete quasi-b-metric space and* $T : X \to X$ *be a given mapping. Suppose that there exists a function* $\alpha : X \times X \to [0, +\infty)$ *such that*:

$$\alpha(x, y)d(Tx, Ty) \leqslant k \max\bigl\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\bigr\}$$

for all $x, y \in X$*, where* $k \in [0, (s + s^2)^{-1})$*. Assume also that*:

(i) T *is an* α*-admissible mapping,*
(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$*,*
(iii) T *is continuous or conditions* (H) *and* (iv) *of Theorem* 2 *hold true.*

Then there exists a fixed point z *of* T*. Moreover, if condition (U) holds true, then* z *is a unique fixed point of* T*.*

Proof. Consider the function $F : [0, +\infty)^6 \to \mathbb{R}$ defined by

$$F(t_1, \dots, t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\},$$

where $k \in [0, (s + s^2)^{-1})$, see Example 2. Also F satisfies condition (F3). Thus, by Theorem 1 (resp. Theorem 2) we obtain existence of a fixed point z of T, and by Theorem 3 we deduce uniqueness of the fixed point z. □

Corollary 3. *Let (X, d, s) be a right-complete quasi-b-metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\alpha : X \times X \to [0, +\infty)$ such that*

$$\alpha(x, y)d(Tx, Ty) \leqslant kd(x, y) \tag{4}$$

for all $x, y \in X$, where $k \in [0, s^{-1})$. Assume also that:

(i) *T is an α-admissible mapping,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$,*
(iii) *T is continuous or conditions* (H) *and* (iv) *of Theorem* 2 *hold true.*

Then there exists a fixed point z of T. Moreover, if condition (U) holds true, then z is a unique fixed point of T.

Next, we give an illustrative example.

Example 3. Let $X = [0, +\infty)$, $d(x, y) = (x - y)^2 + x$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$, then it is clear that $(X, d, 2)$ is a complete quasi-b-metric space. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} x^2 - 2x + 2 & \text{if } x > 2, \\ x/7 & \text{if } x \in [0, 2] \end{cases}$$

and the function $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, for all $x, y \in [0, 1]$ with $Tx \neq Ty$, we write

$$d(Tx, Ty) = (Tx - Ty)^2 + Tx \leqslant \frac{1}{7}\left[(x - y)^2 + x\right] = \frac{1}{7}d(x, y),$$

which ensures that condition (4) of Corollary 3 is satisfied. Also, condition (4) holds true if $Tx = Ty$ or $\{x, y\} \cap (1, +\infty) \neq \emptyset$, since the left hand side of inequality (4) reduces to zero in both the cases. Next, it is easy to show that T is an α-admissible mapping and that condition (H) holds true. Finally, we have

$$\alpha(1, T1) = \alpha\left(1, \frac{1}{7}\right) = 1.$$

Notice that, for all $\{x_n\} \subset X$ with $x_n \to x$ as $n \to +\infty$, we have $d(x_n, y) \to d(x, y)$. Thus, condition (iv) of Theorem 2 holds true. Since all the hypotheses of Corollary 3 are satisfied, then mapping T has a fixed point in X. Moreover, the fixed point is unique since condition (U) holds true.

We conclude this section by establishing the counterparts of above results in the case where we consider a left-complete quasi-b-metric space. At first, Definition 8 is changed as follows.

Definition 9. Let (X, d, s) be a quasi-b-metric space and $T : X \to X$ be a given mapping. We say that T is a left α-implicit contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $F \in \mathit{\Gamma}$ such that

$$F\bigl(\alpha(x,y)d(Tx,Ty), d(x,y), d(Ty,y), d(Tx,x), d(Tx,y), d(Ty,x)\bigr) \leqslant 0$$

for all $x, y \in X$.

Theorem 4. *Let (X, d, s) be a left-complete quasi-b-metric space and $T : X \to X$ be a left α-implicit contractive mapping. Suppose that*:

(i) *T is an α-admissible mapping,*
(ii) *there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geqslant 1$.*

If one of the following conditions holds:

(iv) *T is continuous,*
(v) *conditions (H') holds and if $x_n \to x$ as $n \to +\infty$, then $\limsup_{n\to+\infty} d(y, x_n) \geqslant d(y, x)$ for all $y \in X$,*

then there exists a fixed point z of T, that is, $z = Tz$. Moreover, if conditions (U) *and* (F3) *hold true, then z is a unique fixed point of T.*

Corollary 4. *Let (X, d, s) be a left-complete quasi-b-metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\alpha : X \times X \to [0, +\infty)$ such that*:

$$\begin{aligned}&\alpha(x,y)d(Tx,Ty)\\&\quad \leqslant ad(x,y) + bd(Ty,y) + cd(Tx,x) + ed(Tx,y) + Ld(Ty,x)\end{aligned}$$

for all $x, y \in X$, where $a, b, c, e, L \geqslant 0$ and $sa + sb + c + (s + s^2)e < 1$. Assume also that:

(i) *T is an α-admissible mapping,*
(ii) *there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geqslant 1$,*
(iii) *T is continuous or condition (iv) of Theorem 4 holds true.*

Then there exists a fixed point z of T. Moreover, if $a + e + L < 1$ and condition (U) *holds true, then z is a unique fixed point of T.*

Corollary 5. *Let (X, d, s) be a left-complete quasi-b-metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\alpha : X \times X \to [0, +\infty)$ such that*

$$\begin{aligned}&\alpha(x,y)d(Tx,Ty)\\&\quad \leqslant k\max\bigl\{d(x,y), d(Ty,y), d(Tx,x), d(Tx,y), d(Ty,x)\bigr\}\end{aligned}$$

for all $x, y \in X$, where $k \in [0, (s + s^2)^{-1})$. Assume also that:

(i) *T is an α-admissible mapping,*
(ii) *there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geqslant 1$,*
(iii) *T is continuous or condition (iv) of Theorem 4 holds true.*

Then there exists a fixed point z of T. Moreover, if condition (U) *holds true, then z is a unique fixed point of T.*

3 Results in ordered quasi-*b*-metric spaces

In this section, we prove some fixed and common fixed point results in ordered quasi-*b*-metric spaces. In the following theorems, we always assume, for convenience, that $F \in \Gamma$ is such that $F(0, t_2, t_3, t_4, t_5, t_6) \leqslant 0$ for all $t_2, t_3, t_4, t_5, t_6 \geqslant 0$. However, the theorems can be established without this hypothesis, by giving a direct proof.

Theorem 5. *Let $(X, d, s, \preccurlyeq)$ be a right-complete ordered quasi-b-metric space and $T : X \to X$ be a nondecreasing mapping with respect to $\preccurlyeq$. Suppose that there exists a function $F \in \Gamma$ such that*

$$F\big(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big) \leqslant 0$$

for all comparable $x, y \in X$. Assume also that the following hypotheses are satisfied:

(i) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$,*
(ii) *T is continuous or* (r) *and* (iv) *of Theorem* 2 *hold true and F is continuous.*

Then there exists a fixed point z of T. Moreover, if the function F satisfies condition (F3)*, then* $\mathrm{Fix}(T)$ *is well-ordered if and only if T has a unique fixed point.*

Proof. Consider the function $\alpha : X \times X \to [0, +\infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preccurlyeq y \text{ or } y \preccurlyeq x, \\ 0 & \text{otherwise.} \end{cases}$$

Since $F \in \Gamma$, then condition (F1) holds true and so mapping T is α-implicit contractive, that is,

$$F\big(\alpha(x, y)d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big) \leqslant 0$$

for all $x, y \in X$. In view of hypothesis (i), we deduce that $\alpha(x_0, Tx_0) \geqslant 1$. Next, since mapping T is nondecreasing for all $x, y \in X$, we get immediately that $\alpha(x, y) \geqslant 1$ implies $\alpha(Tx, Ty) \geqslant 1$, and hence T is α-admissible. Finally, we use hypothesis (ii). Precisely, in the case that T is a continuous mapping, the existence of a fixed point z of T is an immediate consequence of our Theorem 1.

On the other hand, in the case that condition (r) holds true, define a sequence $\{x_n\} \subset X$ such that $\alpha(x_{n-1}, x_n) \geqslant 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$. Now, in view of the definition of function α, we deduce that x_{n-1} and x_n are comparable for all $n \in \mathbb{N}$. Next, by condition (r), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)}$ and x are comparable for all $k \in \mathbb{N}$. It follows from the definition of function α that $\alpha(x_{n(k)}, x) \geqslant 1$ for all $k \in \mathbb{N}$, that is, condition (H) holds true. Thus, the existence of a fixed point z of T is a consequence of our Theorem 2.

Notice that if $\mathrm{Fix}(T)$ is well-ordered, then $\alpha(x, y) \geqslant 1$ for all $x, y \in \mathrm{Fix}(T)$, that is, condition (U) holds true and hence the uniqueness of the fixed point is an immediate consequence of our Theorem 3. Clearly, if T has a unique fixed point, then $\mathrm{Fix}(T)$ is well-ordered. □

We continue our study by establishing some common fixed point results. Precisely, we use the following lemma, that is, a consequence of the axiom of choice, to prove a common fixed point theorem for two self-mappings defined on an ordered quasi-b-metric space.

Lemma 1. (*See* [13, *Lemma* 2.1].) *Let X be a nonempty set and $f : X \to X$ a function. Then there exists a subset $E \subset X$ such that $f(E) = f(X)$ and $f : E \to X$ is one-to-one.*

Let $T, f : X \to X$ be two mappings. Then, a point $u \in X$ is called coincidence point of T and f if $Tu = fu$. Also, a point v such that $v = Tu = fu$ is called point of coincidence of T and f. Finally, T and f are said to be weakly compatible if they commute at their coincidence points. We denote by $PC(T, f)$ the set of all points of coincidence of T and f.

Definition 10. Let (X, d, s) be a quasi-b-metric space and $T, f : X \to X$ be two mappings. T is f-continuous if $\lim_{n\to+\infty} d(Tx_n, Tx_0) = 0 = \lim_{n\to+\infty} d(Tx_0, Tx_n)$ whenever $\lim_{n\to+\infty} d(fx_n, fx_0) = 0 = \lim_{n\to+\infty} d(fx_0, fx_n)$ for all $x_0 \in X$.

Theorem 6. *Let $(X, d, s, \preccurlyeq)$ be an ordered quasi-b-metric space and $T, f : X \to X$ be such that T is an f-nondecreasing mapping, $T(X) \subset f(X)$ and $f(X)$ is right-complete. Suppose that there exists a function $F \in \varGamma$ such that*

$$F\big(d(Tx, Ty), d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\big) \leqslant 0 \qquad (5)$$

for all $x, y \in X$ with fx and fy comparable. Assume also that the following hypotheses are satisfied:

(i) *there exists $x_0 \in X$ such that $fx_0 \preccurlyeq Tx_0$,*
(ii) *T is f-continuous.*

Then there exists a coincidence point z of T and f. Moreover, if the function F satisfies condition (F3) *and T and f are weakly compatible, then $PC(T, f)$ is well-ordered if and only if T and f have a unique common fixed point.*

Proof. By Lemma 1, there exists $E \subset X$ such that $f(E) = f(X)$ and $f : E \to X$ is one-to-one. Define

$$S : f(E) \to f(E) \quad by \quad Sfx = Tx \quad \text{for all } fx \in f(E).$$

Since f is one-to-one on E, S is well-defined. Also S is continuous and nondecreasing since T is f-continuous and f-nondecreasing. Note that, by (5), for all $x, y \in X$ with fx and fy comparable, we have

$$F\big(d(Sfx, Sfy), d(fx, fy), d(fx, Sfx), d(fy, Sfy), d(fx, Sfy), d(fy, Sfx)\big) \leqslant 0.$$

Now, hypothesis (i) ensures that $fx_0 \preccurlyeq Sfx_0$. Thus all the hypotheses of Theorem 5 are satisfied, since $f(E)$ is right-complete, and hence S has a fixed point on $f(E)$, say fz. Then, $fz \in f(E)$ is a point of coincidence of T and f, that is, $Tz = S(fz) = fz$. Thus z is a coincidence point of T and f.

Now, we prove that T and f have a unique point of coincidence if $PC(T,f)$ is well-ordered. Let $fw \in PC(T,f)$ with $fw \neq fz$, clearly fz and fw are comparable since $PC(T,f)$ is well-ordered. From (5), we get

$$F\big(d(Tz,Tw), d(fz,fw), d(fz,Tz), d(fw,Tw), d(fz,Tw), d(fw,Tz)\big) \leqslant 0,$$

that is,

$$F\big(d(fz,fw), d(fz,fw), d(fz,fz), d(fw,fw), d(fz,fw), d(fw,fz)\big) \leqslant 0.$$

By condition (F3) there exists $\varphi \in \Phi$ such that $d(fz,fw) \leqslant \varphi(d(fw,fz))$. Similarly, we deduce that $d(fw,fz) \leqslant \varphi(d(fz,fw))$. Then

$$d(fz,fw) \leqslant \varphi\big(d(fw,fz)\big) \leqslant \varphi^2\big(d(fz,fw)\big) < d(fz,fw),$$

which is a contradiction; it follows that $fw = fz$. Since T and f are weakly compatible, we have

$$ffz = fTz = Tfz = TTz.$$

Since T and f have a unique point of coincidence, we deduce that fz is a unique common fixed point of T and f. Now, if T and f have a unique common fixed point, clearly $PC(T,f)$ is well-ordered. □

Obviously, we have the following theorems.

Theorem 7. *Let $(X,d,s,\preccurlyeq)$ be a left-complete ordered quasi-b-metric space and $T : X \to X$ be a nondecreasing mapping with respect to $\preccurlyeq$. Suppose that there exists a function $F \in \Gamma$ such that*

$$F\big(d(Tx,Ty), d(x,y), d(Ty,y), d(Tx,x), d(Tx,y), d(Ty,x)\big) \leqslant 0$$

for all comparable $x, y \in X$. Assume also that the following hypotheses are satisfied:

(i) *there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq x_0$,*
(ii) *T is continuous.*

Then there exists a fixed point z of T. Moreover, if the function F satisfies condition (F3)*, then $\mathrm{Fix}(T)$ is well-ordered if and only if T has a unique fixed point.*

Theorem 8. *Let $(X,d,s,\preccurlyeq)$ be an ordered quasi-b-metric space and $T, f : X \to X$ be such that T is an f-nondecreasing mapping, $T(X) \subset f(X)$ and $f(X)$ is left-complete. Suppose that there exists a function $F \in \Gamma$ such that*

$$F\big(d(Tx,Ty), d(fx,fy), d(Ty,fy), d(Tx,fx), d(Tx,fy), d(Ty,fx)\big) \leqslant 0$$

for all $x, y \in X$ with fx and fy comparable. Assume also that the following hypotheses are satisfied:

(i) *there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq fx_0$,*
(ii) *T is f-continuous.*

Then there exists a coincidence point z of T and f. Moreover, if the function F satisfies condition (F3) *and T and f are weakly compatible, then $PC(T,f)$ is well-ordered if and only if T and f have a unique common fixed point.*

4 Application to integral equation

Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [1, 14, 16, 26, 27] and references therein). Inspired by Aydi et al. [4] and many others, we develop an approach for solving a nonlinear problem represented by the following integral equation:

$$x(t) = \int_0^t K\big(t, s, x(s)\big)\,\mathrm{d}s, \tag{6}$$

where $K : [0, I] \times [0, I] \times \mathbb{R} \to \mathbb{R}$ is continuous, with $I > 0$. Let $X = C([0, I], \mathbb{R})$ be the space of all continuous functions defined on $[0, I]$. It is well known that this space endowed with the metric

$$d(x, y) = \begin{cases} \|(x-y)^2\|_\infty + \|x\|_\infty & \text{for all } x, y \in X \text{ with } x \neq y, \\ 0 & \text{otherwise} \end{cases}$$

is a right-complete quasi-b-metric space with $s = 2$, where

$$\|x\|_\infty = \max_{t\in[0,I]} \big|x(t)\big|$$

is the usual supremum norm. Also, notice that $(X, d, 2, \preccurlyeq)$ is a right-complete ordered quasi-b-metric space, where $\preccurlyeq$ denotes the usual order, that is, $x \preccurlyeq y$ if $x(t) \leqslant y(t)$ for all $t \in [0, I]$.

Theorem 9. *Assume that the following hypotheses are satisfied*:

(i) *there exists a function* $p : [0, I] \times [0, I] \to [0, +\infty)$ *with* $p(t, \cdot) \in L^1([0, I])$ *for all* $t \in [0, I]$*, such that for all* $t, s \in [0, I]$*,*

$$0 \leqslant K(t, s, y(s)) - K\big(t, s, x(s)\big) \leqslant p(t, s)\big(y(s) - x(s)\big) \quad x, y \in X,\ x \preccurlyeq y,$$

and

$$\big|K\big(t, s, x(s)\big)\big| \leqslant p(t, s)\big|x(s)\big| \quad \textit{for all } x \in X,$$

(ii) $\sup_{t\in[0,I]} \int_0^t p(t, s)\,\mathrm{d}s = \lambda < 1/2$,

(iii) *there exists* $x_0 \in X$ *such that* $x_0(t) \leqslant \int_0^t K(t, s, x_0(s))\,\mathrm{d}s$ *for all* $t \in [0, I]$.

Then (6) *has a solution in* X.

Proof. Define the integral operator $T : X \to X$ by

$$Tx(t) = \int_0^t K\big(t, s, x(s)\big)\,\mathrm{d}s$$

for all $x, y \in X$. Notice that T is well-defined and (6) has a solution if and only if the operator T has a fixed point. Precisely, we have to show that our Theorem 5 is applicable to the operator T. Then, for all comparable $x, y \in X$, we write

$$\left|Tx(t)\right| \leqslant \int_0^t \left|K\big(t, s, x(s)\big)\right| \mathrm{d}s \leqslant \int_0^t p(t, s)\left|x(s)\right| \mathrm{d}s = \lambda\|x\|_\infty$$

and

$$\begin{aligned}
&\left|Tx(t) - Ty(t)\right| \\
&\quad\leqslant \int_0^t \left|K\big(t, s, x(s)\big) - K\big(t, s, y(s)\big)\right| \mathrm{d}s \leqslant \int_0^t p(t, s)\left|x(s) - y(s)\right| \mathrm{d}s \\
&\quad= \int_0^t p(t, s)\sqrt{\big(x(s) - y(s)\big)^2}\, \mathrm{d}s \leqslant \lambda\sqrt{\left\|(x - y)^2\right\|_\infty}.
\end{aligned}$$

Therefore, we get

$$\|Tx\|_\infty \leqslant \lambda\|x\|_\infty \quad \text{and} \quad \left\|(Tx - Ty)^2\right\|_\infty \leqslant \lambda^2\left\|(x - y)^2\right\|_\infty,$$

and hence

$$d(Tx, Ty) \leqslant \lambda d(x, y)$$

for all comparable $x, y \in X$ such that $Tx \neq Ty$. Also, it is an obvious fact that the above inequality holds true if $Tx = Ty$. Thus, the integral operator T satisfies all the hypotheses of Theorem 5, where $F : [0, +\infty)^6 \to \mathbb{R}$ is given by $F(t_1, \dots, t_6) = t_1 - at_2 - bt_3 - ct_4 - et_5 - Lt_6$, with $b = c = e = L = 0$ and $2a < 1$ (see Example 2). Thus, T has a fixed point, that is, (6) has a solution in X. $\square$

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