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The role of virtual work in Levi-Civita's parallel transport

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According to current history of science, Levi-Civita introduced parallel transport solely to give a geometrical interpretation to the covariant derivative of absolute differential calculus. Levi-Civita, however, searched a simple computation of the curvature of a Riemannian manifold, basing on notions of the Italian school of mathematical physics of his time: holonomic constraints, virtual displacements and work, which so have a remarkable, if not dominant, role in the origin of parallel transport.

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1 Introduction

Tullio Levi-Civita (1873-1941)'s teachers gave him a strong training in mathematical physics and mechanics, and developed absolute differential calculus, an inheritance of Beltrami's investigations on Riemannian manifolds [1, 2]. In the mid-1910s, Levi-Civita investigated relativity by absolute differential calculus: he studied the curvature of four-dimensional Riemannian manifolds, modelling space-time, through the parallel transport of vectors over these manifolds.

We usually find that Levi-Civita's parallel transport is motivated by the search for a geometrical interpretation to the covariant derivative of absolute differential calculus. However, in the introduction to [3] Levi-Civita states that he wishes to simplify the computation of the curvature of a Riemannian manifold V_n with dimension $n \geq 2$, and introduces parallelism on it. Going further, the principle of virtual work seems to be one of the basis for parallel transport according to Levi-Civita.

2 Virtual work and parallel transport

Balance exists if and only if the total power (work) on any admissible, or virtual, velocity (first-order displacement) field vanishes. This law is basically the same since Lagrange's formulation, where inertia is a force to be added to the active ones, and dynamics is reduced to statics.[1] The reactions of perfect constraints spend no work on admissible displacements; if $\vec{\delta}P_i$ is the first-order displacement of the point where active forces plus inertia $\vec{F}_i$ are applied, the law of virtual work is ([5], p. 14)

$$\delta L = \sum_i \vec{F}_i \cdot \vec{\delta} P_i = 0, \tag{1}$$

Levi-Civita in [3] embeds V_n in a Euclidean space S_N with dimension $N \leq n(n+1)/2$, and considers two unit vectors $\vec{\alpha}, \vec{\alpha}'$ emerging from nearby points $P, P' \in V_n$. In S_N, $\vec{\alpha}, \vec{\alpha}'$ are parallel if, for any auxiliary direction $\vec{f}$ emerging from P, $\text{angle}(\widehat{\vec{\alpha}, \vec{f}}) = \text{angle}(\widehat{\vec{\alpha}', \vec{f}})$. Parallelism in V_n asks this relation to hold for any direction $\vec{f}$ of the tangent plane $T_P^{S_N}(V_n)$, which is an intrinsic definition, since it depends solely on metrics in V_n, $ds^2 = \sum_{i,k=1}^n a_{ik} dx_i dx_k$.

The manifold V_n may be described by the system ([3], eq. (1), p. 4)

$$y_\nu = y_\nu(x_1, \ldots, x_n), \qquad \nu = 1, 2, \ldots, N \tag{2}$$

with $y_\nu \in S_N$, $x_n \in V_n$. Eq. (2) describes an n d.o.f. system subjected to N smooth holonomic bilateral constraints. The point P varies on a smooth curve $\mathscr{C} \in V_n$, parameterized by the abscissa s, thus the components $\alpha_\nu = \alpha_\nu(s)$, so that

$$\mathscr{C} \equiv y_\nu(s) = y_\nu(x_1(s), \ldots, x_n(s)),\ \nu = 1, \ldots, N \Rightarrow y'_\nu = \sum_{i=1}^n \frac{\partial y_\nu}{\partial x_i} x'_i \qquad \nu = 1, 2, ..., N, \tag{3}$$

In the analog constrained system, s is an evolution parameter (e.g., time), and $\mathscr{C}$ is a trajectory in the manifold of admissible configurations. Derivation with respect to s yields the direction cosines $y'_\nu \in S_N$, while x'_i are the direction cosines of the same unit direction in V_n. Levi-Civita poses the direction cosines of $\vec{\alpha}$ to be $\xi^{(i)}$, $i = 1, 2, \ldots, n$ with respect to V_n, and α_ν, $\nu = 1, \ldots, N$ with respect to S_N. Then, from eq. (3) we have ([3], eq. (7), p. 6)

$$\alpha_\nu = \sum_{l=1}^n \frac{\partial y_\nu}{\partial x_l} \xi^{(l)} \qquad \nu = 1, 2, \ldots, N. \tag{4}$$

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[1] Lagrange gives credit to D'Alembert, but the actual D'Alembert's principle is different [4].

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Levi-Civita considers an arbitrarily fixed direction $\vec{f}$ of S_N, with direction cosines f_ν, thus

$$\cos\left(\widehat{\vec{f},\vec{\alpha}}\right) = \sum_{\nu=1}^{N} \alpha_\nu f_\nu \Rightarrow d\cos\left(\widehat{\vec{f},\vec{\alpha}}\right) = ds\sum_{\nu=1}^{N} \alpha'_\nu(s) f_\nu. \tag{5}$$

Levi-Civita's intrinsic condition of parallelism requires the variation in eq. (5) to vanish only for the directions tangent to V_n at $P \in \mathscr{C}$, which are those compatible with the constraints (2) ([3], p. 7). Thus, by replacing the f_ν with quantities proportional to them, Levi-Civita's intrinsic definition of parallelism implies ([3], eq. (I), p. 7)

$$\sum_{\nu=1}^{N} \alpha'_\nu(s)\delta y_\nu = 0 \tag{6}$$

for any variation δy_ν, that is, for any admissible virtual displacement compatible with the constraints in eq. (2). If the $\alpha'_\nu(s)$ are some mechanical actions in S_N, eq. (6) is a formulation of the virtual work law in S_N related to the smooth bilateral holonomic system defined by eq. (2), hence related to a Riemannian manifold. From eq. (2) it follows that ([3], p. 7)

$$\delta y_\nu = \sum_{k=1}^{n} \frac{\partial y_\nu}{\partial x_k}\delta x_k \quad \nu = 1,2,\ldots,N, \Rightarrow \sum_{\nu=1}^{N} \alpha'_\nu(s)\frac{\partial y_\nu}{\partial x_k} = 0 \qquad (k = 1,2,\ldots,n), \tag{7}$$

for the arbitrariness of the δx_k and (6). Eq. (7) are the parallelism conditions for the directions $\vec{\alpha}$ moving along $\mathscr{C}$. To have an intrinsic expression, Levi-Civita replaces the direction cosines α_ν with their expression eq. (4), to include the intrinsic direction cosines $\xi^{(i)}$, and, if Γ_i^{jl} are Christoffel symbols of second kind, he deduces ([3], eq. (I_a), p. 8)

$$\frac{d\xi^{(i)}}{ds} + \sum_{j,l=1}^{n} \Gamma_i^{jl} x'_j \xi^{(l)} = 0 \qquad (i = 1,2,\ldots,n), \tag{8}$$

Levi-Civita shows that eq. (8) may be expressed by the covariant quantities associated with the $\xi^{(i)}$ [2], thus the intrinsic condition of parallelism has the same form of Lagrange equations of motion on a Riemannian manifold ([3], eq. (I_c), p. 12). It seems apparent that one key guide to Levi-Civita is mathematical physics, not only pure geometry, and, in particular, the principle of virtual work for a mechanical system under smooth holonomic bilateral constraints.

3 Final remarks

We claim that the virtual work principle played a key role in the origin of Levi-Civita's parallel transport in a Riemannian manifold, eq. (6). Levi-Civita's jargon clearly refers to this, e.g., with the locutions 'constraint' and 'displacements compatible with constraint'. Furthermore, he specifies that eq. (6) is obtained "for all the displacements δy_ν compatible with the constraints (2)", highlighting in italic this phrase in [3]. Indeed, for the analog constrained mechanical system, the unit directions emerging from the points of V_n assume the role of admissible displacements, and the dual forms on them are reactions provided by the geometrical links between the elements of the mechanical system ([6], ch. II, sect. 8).

In his monograph on absolute differential calculus [7] (ch. V, (b), sects. 10-15), Levi-Civita highlights the role of analytical mechanics in developing differential geometry notions, and says that his symbolic equation of parallelism formally recalls the principle of virtual work. He uses the same principle in discussing the geodetic principle for the dynamics of a material particle moving in a four-dimensional space-time manifold ([7], ch. XI, sect. 12).

It seems apparent to us that the key notions of the Italian school of mathematical physics were a strong basis for Levi-Civita's non-Euclidean geometrical definitions.

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[2] Levi-Civita adopts the locution 'moment', traditional in the Italian school of mathematical physics of his time, defining a mechanical action dual to a Lagrangean parameter of admissible (virtual) displacements.

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