The Henstock-Kurzweil-Stieltjes type integral for real functions on a fractal subset of the real line

D. Bongiorno, G. Corrao

Dipartimento di Ingegneria Elettrica, Elettronica e delle Telecomunicazioni,
di Tecnologie Chimiche, Automatica e Modelli Matematici (DIEETCAM)
Università degli Studi di Palermo, Viale delle Scienze, 90128 Palermo, Italy
E-mail addresses: donatella.bongiorno@unipa.it, giusi.corrao@unipa.it

Abstract

The aim of this paper is to introduce an Henstock-Kurzweil-Stieltjes type integration process for real functions on a fractal subset E of tha real line.

Key words: s-set; s-derivatives; s-integral

Contents

1 Introduction

During last years, some mathematicians have been obliged to define a new concept of derivation (see [1] and [2]) and a new concept of integration (see, [6] and [9]) to solve some physical and engineering problems. In fact even if the geometry of fractal sets is a well explored subject (see, [3], [4], [5], [7]), in this sets there are so many irregularities to render inapplicable the standard methods and the technique of the ordinary calculus to give reasonable solutions of practical problems. Just think of the fact that the usual derivative of the classical Lebesgue-Cantor staircase function is zero almost everywhere (see [1] and [6]) and of the fact that the Riemann's integral of functions defined on a fractal sets is undefined. So, in 1991, de Guzmann, Martin and Reyes (see [1]) in order to study the problem of the existence and the uniqueness of the solutions of differential equations of the type: $\frac{d\,x(t)}{dt} = f(t, x(t))$ in which t or $x(t)$ takes values in a non-continuous set (i.e. a fractal set) introduced, for functions defined on a fractal set E, a new concept of derivative, called the s-derivative. Later on, Jiang and Su (see [6]) and more recently Parvate and Gangal (see [9]) introduced, for function defined in a fractal set E, a new concept of Riemann-Stieltjes type integral, called the s-integral and the F^s-integral, respectively. Both authors defined such integration process in an analogous fashion as the classical Riemann integral but with the Hausdorff measure and the mass function taking over the role of the distance, respectively. Since the Hausdorff measure and the mass function are proportional (see [9], section 4), we can say that Jiang and Su in [6] and Parvate and Gangal in [9] defined indipendently the same integral. Both authors developed in a way analogous to the standard calculus the rest of the integration theory. Therefore properties like linearity and additivity with respect to integration domain are valid. About the Fundamental theorem of calculus, both authors use an extra hypothesis to prove it.

In this paper we will compare the hypothesis of the Fundamental theorem of Calculus used by Jiang and Su with the one used by Parvate and Gangal and we will notice that without such additional hypothesis the Fundamental theorem of Calculus lose its classical formulation. Therefore in in the fractal case, it is not possible to prove a theorem like this:

If $E \in \mathbb{R}$ is a closed fractal set and if $F : E \to \mathbb{R}$ is s-differentiable on E with a continuous derivative therefore

$$\int_{E\cap[a,b]} F'_s\, d\mathcal{H}^s \;=\; F(b) - F(a).$$

Moreover, in this paper we introduce an Henstok-Kurzweil-Stieltjes type integration process defined for real functions on a fractal subset E of the real line of Hausdorff dimension s with $0 < s \le 1$, since it is well known that numerical algorithms are based very rarely on Lebesgue integrals but are based more often on Riemann sums and in the classical literature the best formulation of the Fundamental theorem of calculus is done by the Henstock-Kurzweil integral that it is based on Riemann sums too (see [?].

Finally, we will show that the Henstok-Kurzweil-Stieltjes integral on the fractal set E contains properly the Riemann-Stieltjes integral defined by Jiang and Su and by Parvate and Gangal.

2 Preliminaries

In all the paper we denote by $\mathbb{R}$ the set of all real numbers and by E a nonempty closed subset of the interval $[a, b] \subset \mathbb{R}$, with $a = \min E$ and $b = \max E$. Moreover, we denote, by $\mathcal{L}(\cdot)$ the Lebesgue measure on $\mathbb{R}$.

Now, given s, with $0 \le s \le 1$, we recall that the *s-dimensional Hausdorff measure* of E is defined as:

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} \mathcal{L}(A_i)^s : E \subset \bigcup_{i=1}^{\infty} A_i,\ \mathcal{L}(A_i) \le \delta \right\}.$$

$\mathcal{H}^s(\cdot)$ is a Borel regular measure (see [7]).

Moreover, we recall that, the unique number s for which $\mathcal{H}^t(E) = 0$ if $t > s$ and $\mathcal{H}^t(E) = +\infty$ if $t < s$ is called the Hausdorff dimension of E.

Whenever E is $\mathcal{H}^s$-measurable with $0 < \mathcal{H}^s(E) < \infty$, therefore E is said to be an *s-set*. Without loss of generality we can assume in all the paper, that $\mathcal{H}^s(E) = 1$.

Definition 2.1. (see [1], [5], and [8])
Let $f : E \to \mathbb{R}$ be a function and let $x_0 \in E$. The *s-derivative* of f, on the right and on the left, at the point x_0 is:

$$f_s'^{+}(x_0) = \lim_{\substack{x\to x_0^+ \\ x\in E}} \frac{f(x) - f(x_0)}{\mathcal{H}^s(\widetilde{[x_0, x]})} \quad \text{if} \quad \mathcal{H}^s(\widetilde{[x_0, x]}) > 0 \quad \text{for all} \quad x > x_0$$

$$f_s'^{-}(x_0) = \lim_{\substack{x\to x_0^- \\ x\in E}} \frac{f(x_0) - f(x)}{\mathcal{H}^s(\widetilde{[x, x_0]})} \quad \text{if} \quad \mathcal{H}^s(\widetilde{[x, x_0]}) > 0 \quad \text{for all} \quad x < x_0$$

where these limits exist.

It is said that the *s-derivative* of f at x_0 exists if $f_s'^{+}(x_0) = f_s'^{-}(x_0)$, we be denote by $f_s'(x_0)$ such common value.

Remark. Of course, if f is s-derivable at the point x_0 then f is continuous at x_0, according to the topology induced on E for the usual topology of $\mathbb{R}$.

2.1 The s-integral

Definition 2.2. Let $A \subset [a, b]$ be an interval. The set $\widetilde{A} = A \cap E$ is called an *interval of E*.

Definition 2.3. *A partition of E* is any collection $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^p$ of pairwise disjoint intervals $\widetilde{A}_i$ and points $x_i \in \widetilde{A}_i$ such that $E = \bigcup_i \widetilde{A}_i$.

Definition 2.4. (see [5]) Let $f : E \to \mathbb{R}$ be a function. A number σ is said to be the *s-integral of f* on E, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|\sum_{i=1}^{n} f(x_i)\ \mathcal{H}^s(\widetilde{A}_i) - \sigma| < \varepsilon$$

for each partition $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^p$ of E with $\mathcal{H}^s(\widetilde{A}_i) < \delta$, for $i = 1, 2, ..., n$.

By

$$\sigma = (R) \int_E f(t) d\mathcal{H}^s(t)$$

we will denote the s-integral and by s-$R(E)$ we will denote the collection of all functions that are s-integrable on E.

Theorem 2.1. *(see [5]) Let $f : E \to \mathbb{R}$ be a continuous function at all points of E. If $F : E \to \mathbb{R}$ is a function $\mathcal{H}^s$-absolutely continuous on E and $F_s^{'}(x) = f(x)$ at $\mathcal{H}^s$-a.e. point $x \in E$, then*

$$(R) \int_E f(t)\, d\mathcal{H}^s(t) = F(b) - F(a).$$

Definition 2.5. (see [5]) Let $f : E \to \mathbb{R}$ be a function. We say that f is $\mathcal{H}^s$-absolutely continuous on E, if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

whenever $\sum_{k=1}^n \mathcal{H}^s \left[\widetilde{(a_k, b_k)} \right] < \delta$, $a_k, b_k \in E$, $k = 1, \ldots, n$, and $a_1 < b_1 \leq a_2 < b_2 \leq .... \leq a_n < b_n$.

Lemma 2.2. *Let $f : E \to \mathbb{R}$ be $\mathcal{H}^s$-absolutely continuous on E. If $[\alpha, \beta]$ is a contiguous interval of E therefore*

$$f(\alpha) = f(\beta).$$

Proof. Since $[\alpha, \beta]$ is a contiguous interval of E it follows that $\mathcal{H}^s([\alpha, \beta]) = 0$. Moreover, since f is $\mathcal{H}^s$-absolutely continuous on E,

$\forall \varepsilon > 0$ we have that

$$|f(\beta) - f(\alpha)| < \varepsilon.$$

The assertion follows by the arbitrariness of ε. □

Definition 2.6. (see, [8]) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. A point $x \in \mathbb{R}$ is said a *point of change of* f if f is not constant over any open interval (c, d) containing x. The set of all points of change of f is called the set of change of f and it is denoted by $Sch(f)$.

Theorem 2.3. *(see [8]) If $F : \mathbb{R} \to \mathbb{R}$ is s-differentiable on E, F_s' is a continuous function on E and* $\mathrm{Sch}(F) \subseteq E$ *then*

$$(R) \int_E F_s'(x) d\mathcal{H}^s(x) = F(b) - F(a).$$

Example 2.1. Let $E \subset [0,1]$ be the classical Cantor set. Let F be a function on $[0,1]$ defined by

$$F(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{3} \\ 3x - 1, & \frac{1}{3} < x < \frac{2}{3} \\ 1, & \frac{2}{3} \leq x \leq 1. \end{cases}$$

and let f be the restriction of F on the set E.

Note that:

- f is not $\mathcal{H}^s$-absolutely continuous on E.
- $Sch(f) \not\subseteq E$.
- f is s-differentiable at each point of E and $f'_s(x) = 0$ for all points $x \in E$.
- f'_s is s-integrable on E and

$$(R)\int_E f'_s \, d\mathcal{H}^s = 0 \neq f(1) - f(0) = 1.$$

(i) f is not $\mathcal{H}^s$-absolutely continuous on E.

(ii) $Sch(f) \not\subseteq E$.

(iii) f is s-differentiable at each point of E and $f'_s(x) = 0$ for all points $x \in E$.

(iv) f'_s is s-integrable on E and

$$(R)\int_E f'_s \, d\mathcal{H}^s = 0 \neq f(1) - f(0) = 1.$$

3 The s-Henstock-Kurzweil-Stieltjes integral

Definition 3.1. A *gauge on* E is any positive real function δ defined on E.

Definition 3.2. *A partition of* E is any collection $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^p$ of a pairwise disjoint intervals of E $\widetilde{A}_i$ and points $x_i \in \widetilde{A}_i$ such that $E = \bigcup_i \widetilde{A}_i$.

Definition 3.3. Let $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^p$ be a partition of E. If δ is a gauge on E, then we say that P is a δ-*fine partition of* E whenever $\widetilde{A}_i \subseteq]x_i - \delta(x_i), x_i + \delta(x_i)[$ for all $i = 1, 2, ..., p$.

The following Cousin's lemma, addresses the existence of δ-fine partitions.

Lemma 3.1. *If δ is a gauge on E, then there exists a δ-fine partition of E.*

Proof. If $[\alpha\,,\,\beta] \subset [a\,,\,b]$ and if $\widetilde{[\alpha\,,\,\beta]} = \emptyset$, we said that $\widetilde{[\alpha\,,\,\beta]}$ has a δ-*fine* partition of E.

Let c be the midpoint of $[a,b]$ and let us observe that if P_1 and P_2 are δ-fine partitions of $\widetilde{[a,c]}$ and $\widetilde{[c,b]}$, respectively, then $P = P_1 \cup P_2$ is a δ-fine partition of E. Using this observation, we proceed by contradiction.
Let us suppose that E does not have a δ-fine partition then at least one of the intervals $\widetilde{[a,c]}$ or $\widetilde{[c,b]}$ does not have a δ-fine partition, let us say $\widetilde{[a,c]}$. Therefore $\widetilde{[a,c]}$ is not empty. Let us relabel the interval $[a,c]$ with $[a_1,b_1]$ and let us repeat indefinitely this bisection method. So, we obtain a sequence of nested intervals:

$$[a,b] \supset [a_1,b_1] \supset ... \supset [a_n,b_n] \supset ...$$

Since the length of the interval $[a_n,b_n]$ is $(b-a)/2^n$ therefore, for the Nested Intervals Property, there is a unique number $\xi \in [a,b]$ such that:

$$\bigcap_{n=0}^{\infty} [a_n,b_n] = \{\xi\}.$$

It is trivial to notice that the interval $\widetilde{[a_n,b_n]}$ is not empty.

Let $\xi_n \in \widetilde{[a_n,b_n]}$, therefore $|\xi_n - \xi| < |b_n - a_n| = (b-a)/2^n$. So $\lim_{n\to\infty} \xi_n = \xi$. Now since E is a closed set, $\xi \in E$.

Since $\delta(\xi) > 0$, we can find $k \in \mathbb{N}$ such that $\widetilde{[a_k,b_k]} \subset [\xi - \delta(\xi), \xi + \delta(\xi)]$. Therefore $\{\widetilde{[a_k,b_k]}, \xi\}$ is a δ-fine partition of $\widetilde{[a_k,b_k]}$, contrarily to our assumption. □

Let $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^p$ be a partition of E, let $f : E \to \mathbb{R}$ be a function and let us consider the following sum

$$S(f,P) = \sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i).$$

Definition 3.4. We say that the function f is *s-HK-integrable* on E, if there exists $I \in \mathbb{R}$ such that for all $\varepsilon > 0$, there is a gauge δ on E with:

$$|S(f,P) - I| < \varepsilon$$

for each δ-fine partition $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^p$ of E.

The number I is called the *s-HK-integral* of f on E and we write

$$I = (HK) \int_E f \; d\mathcal{H}^s.$$

The collection of all functions that are s-HK integrable on E will be denoted by s-$HK(E)$.

Remark. The number I from Definition 5 is determinate uniquely by the s-HK-integrable function f. In fact, let us suppose that I and J satisfies Definition 5, let us assume that $J \neq I$ and let us take $\varepsilon = |I - J|/2$. Then we can find two gauges δ_1 and δ_2 on E so that

$$\left| \sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i) - I \right| < \varepsilon$$

for each δ_1-fine partition $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^{p}$ of E and

$$\left| \sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i) - J \right| < \varepsilon$$

for each δ_2-fine partition $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^{p}$ of E.

Now let $\delta = \min\{\delta_1, \delta_2\}$. Then, for each δ-fine partition of E we have

$$|I - J| \leq \left| I - \sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i) \right| + \left| \sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i) - J \right| < 2\varepsilon = |I - J|,$$

that is a contradiction.

3.1 Linearity properties

Theorem 3.2. *(a) If f and g are s-HK-integrable on E, then $f + g$ is also s-HK-integrable on E and*

$$(HK) \int_E (f + g)\ d\mathcal{H}^s = (HK) \int_E f\ d\mathcal{H}^s + (HK) \int_E g\ d\mathcal{H}^s.$$

(b) If f is s-HK-integrable on E and $k \in \mathbb{R}$, then kf is s-HK-integrable on E and

$$(HK) \int_E kf\ d\mathcal{H}^s = k \cdot (HK) \int_E f\ d\mathcal{H}^s.$$

Proof. (a): Let I and J be the s-HK-integrals of f and g, respectively. Given $\varepsilon > 0$, let δ' and δ'' be two gauges on E such that:

$$\left| \sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i) - I \right| < \frac{\varepsilon}{2},$$

if the partition $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^{p}$ is δ'-fine and

$$\left| \sum_{i=1}^{p} g(x_i)\mathcal{H}^s(\widetilde{A}_i) - J \right| < \frac{\varepsilon}{2}$$

if the partition $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^p$ is δ''-fine.
Now let $\delta = \min\{\delta', \delta''\}$ therefore if P is a δ-fine partition, then it is both δ'-fine and δ''-fine. So

$$\left|\sum_{i=1}^{p}(f+g)(x_i)\mathcal{H}^s(\widetilde{A}_i) - (I+J)\right|$$
$$\leq \left|\sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i) - I\right| + \left|\sum_{i=1}^{p} g(x_i)\mathcal{H}^s(\widetilde{A}_i) - J\right| < \varepsilon.$$

for each δ-fine partition of E.

The proof follows by arbitrariness of ε.

(b): Let I be the s-HK-integral of f. Then, $\forall\, \varepsilon > 0$ there is a gauge δ on E with:

$$\left|\sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i) - I\right| < \varepsilon$$

for each δ-fine partition $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^p$ of E. Therefore, $\forall k \in \mathbb{R}$ we have:

$$\left|\sum_{i=1}^{p} kf(x_i)\mathcal{H}^s(\widetilde{A}_i) - kI\right| = |k| \left|\sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i) - I\right| < |k|\,\varepsilon.$$

The proof follows by arbitrariness of ε.

□

Theorem 3.3. *Let f be s-HK-integrable on E and $f(x) \geq 0$ for all $x \in E$, then*

$$(HK)\int_E f\; d\mathcal{H}^s \geq 0.$$

Proof. Let δ be a gauge on E such that

$$\left|\sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i) - {\scriptstyle (HK)}\int_E f\; d\mathcal{H}^s\right| < \varepsilon$$

if the partition $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^p$ is δ-fine.
Since $f(x) \geq 0$ for all $x \in E$, then

$$\sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i) \geq 0.$$

Therefore

$$-\,\varepsilon \leq \sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i) -\; \varepsilon < (HK)\int_E f\; d\mathcal{H}^s < \sum_{i=1}^{p} f(x_i)\mathcal{H}^s(\widetilde{A}_i) \;+\; \varepsilon,$$

by the arbitrariness of ε it follows that

$$(HK)\int_E f\ d\mathcal{H}^s \geq 0.$$

$\square$

Corollary 3.4. *Let f and g be s-HK-integrable on E. If $f \leq g$ for all $x \in E$, then*

$$(HK)\int_E f\ d\mathcal{H}^s \leq (HK)\int_E g\ d\mathcal{H}^s.$$

Proof. Let $h := g - f$. By theorem 3.2 (b) h is s-HK-integrable and

$$(HK)\int_E h\ d\mathcal{H}^s = (HK)\int_E g\ d\mathcal{H}^s - (HK)\int_E f\ d\mathcal{H}^s,$$

and since $f \leq g$ then $h(x) \geq 0$ for all $x \in E$ and $(HK)\int_E h\ d\mathcal{H}^s \geq 0$.
Therefore the conclusion is immediate. $\square$

3.2 Cauchy Criterion and Cauchy extension

Theorem 3.5. *A function $f : E \to \mathbb{R}$ is s-HK integrable on E if and only if for each $\varepsilon > 0$ there exists a gauge δ on E such that*

$$|S(f, P_1) - S(f, P_2)| < \varepsilon$$

for each pair δ-fine partitions P_1 and P_2 of E.

Proof. ($\Rightarrow$) Let $\varepsilon > 0$ be given. Since $f \in s\text{-}HK(E)$, there exists a gauge δ on E such that

$$\left|S(f, P) - (HK)\int_E f\ d\mathcal{H}^s\right| < \frac{\varepsilon}{2}$$

for each δ-fine partition P of E. If P_1 and P_2 are two δ-fine partitions of E, we have that

$$\begin{aligned}
&|S(f, P_1) - S(f, P_2)| \\
&\leq \left|S(f, P_1) - (HK)\int_E f\ d\mathcal{H}^s\right| + \left|S(f, P_2) - (HK)\int_E f\ d\mathcal{H}^s\right| \\
&< \varepsilon.
\end{aligned}$$

($\Leftarrow$) For each $n \in \mathbb{N}$, let δ_n be a gauge on E such that

$$|S(f, Q_n) - S(f, R_n)| < \frac{1}{n}$$

for each pair δ_n-fine partitions Q_n and R_n of E.

Let $\Delta_n(x) = min\{\delta_1(x), ..., \delta_n(x)\}$ be a gauge on E. For each $n \in \mathbb{N}$, let P_n be a Δ_n-fine partition on E.

Clearly, if $m > n$ then P_m and P_n are Δ_n-fine partitions on E; hence

$$|S(f, P_n) - S(f, P_m)| < \frac{1}{n}$$

for $m > n$.

Conseguently, $\{S(f, P_n)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers; therefore this sequence converges to some real numer $A = \lim_{n\to\infty} S(f, P_n)$.

Then

$$|S(f, P_n) - A| < \frac{1}{n}$$

for each n.

Let $\varepsilon > 0$ be given and choose N such that $\frac{1}{N} < \frac{\varepsilon}{2}$. If P is an arbitrary Δ_N-fine partitions on E, then

$$|S(f, P) - A| \le |S(f, P) - S(f, P_N)| + |S(f, P_N) - A| < \frac{1}{N} + \frac{1}{N} < \varepsilon.$$

Thus $f \in s\text{-}HK(E)$ and $A = (HK) \int_E f \; d\mathcal{H}^s$. □

Theorem 3.6. *If $f \in s\text{-}HK(E)$, then $f \in s\text{-}HK(\widetilde{A})$ for each closed interval $\widetilde{A} \subset E$.*

Proof. Let $\widetilde{A}$ be a closed interval of E. Since $f \in s\text{-}HK(E)$, it follows that by Theorem 3.4, for each $\varepsilon > 0$ there exists a gauge δ on E such that

$$|S(f, P) - S(f, Q)| < \varepsilon$$

for each pair δ-fine partitions P and Q of E.

Let $\{\widetilde{A_1}, \widetilde{A_2}, ..., \widetilde{A_N}\}$ be a finite collection of pairwise non-overlapping intervals of E, such that $\widetilde{A} \notin \{\widetilde{A_1}, \widetilde{A_2}, ..., \widetilde{A_N}\}$ and such that

$$E = \widetilde{A} \cup \bigcup_{k=1}^{N} \widetilde{A_k}.$$

For each $k \in \{1, ..., N\}$ we fix a δ-fine partition P_k of $\widetilde{A_k}$. If $P_{\widetilde{A}}$ and $Q_{\widetilde{A}}$ are δ-fine partitions of $\widetilde{A}$, then it is clear that $P_{\widetilde{A}} \cup \bigcup_{k=1}^{N} P_k$ and $Q_{\widetilde{A}} \cup \bigcup_{k=1}^{N} P_k$ are δ-fine partitions of E.

Thus

$$
\begin{aligned}
&\left|S(f,P_{\widetilde{A}})-S(f,Q_{\widetilde{A}})\right| \\
&=\left|S(f,P_{\widetilde{A}})+\sum_{k=1}^{N}S(f,P_k)-S(f,Q_{\widetilde{A}})-\sum_{k=1}^{N}S(f,P_k)\right| \\
&=\left|S\Big(f,P_{\widetilde{A}}\cup\bigcup_{k=1}^{N}P_k\Big)-S\Big(f,Q_{\widetilde{A}}\cup\bigcup_{k=1}^{N}P_k\Big)\right| \\
&<\varepsilon.
\end{aligned}
$$

Therefore, by Theorem 3.4, $f \in s\text{-}HK(\widetilde{A})$. □

3.2.1 Saks-Henstock Lemma

Definition 3.5. *A subpartition of* E is a finite collection $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^{p}$ of a pairwise disjoint intervals $\widetilde{A}_i \subset E$ and points $x_i \in \widetilde{A}_i$ for $i = 1, 2, ..., p$.

Definition 3.6. Let $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^{p}$ be a subartition of E. If δ is a gauge on E, then we say that P is a δ-fine *subpartition of* E whenever $\widetilde{A}_i \subset]x_i - \delta(x_i), x_i + \delta(x_i)[$ for all $i = 1, 2, ..., p$.

Lemma 3.7 (The Saks-Henstock Lemma)**.** *Let $f \in s\text{-}HK(E)$ and $\varepsilon > 0$. If δ is a gauge on E such that*

$$\left|S(f,P)-(HK)\int_E f\ d\mathcal{H}^s\right| < \varepsilon$$

for each δ-fine partition P of E, then

$$\left|\sum_{i=1}^{p}\left\{f(x_i)\mathcal{H}^s(\widetilde{A}_i)-(HK)\int_{\widetilde{A}_i} f\ d\mathcal{H}^s\right\}\right| \le \varepsilon$$

and

$$\sum_{i=1}^{p}\left|f(x_i)\mathcal{H}^s(\widetilde{A}_i)-(HK)\int_{\widetilde{A}_i} f\ d\mathcal{H}^s\right| \le 2\varepsilon$$

for each δ-fine subpartition $P_0 = \{(\widetilde{A}_i, x_i)\}_{i=1}^{p}$ of E.

Proof. Let $\varepsilon > 0$. The set $E \setminus \bigcup_{i=1}^{p} \widetilde{A}_i$ is a finite union of disjoint intervals. Let $\widetilde{K}_1, \widetilde{K}_2, ..., \widetilde{K}_m$ be the closure of these intervals.

From theorem 3.5 we have that $f \in s\text{-}HK(\widetilde{K}_j)$ for each $j = 1, 2, ..., m$. Hence for each $\eta > 0$ and for each j there exists a gauge δ_j on $\widetilde{K}_j$ such that

$$\left| S(f, P_j) - (HK) \int_{\widetilde{K}_j} f \; d\mathcal{H}^s \right| < \frac{\eta}{m}$$

for each δ_j-fine partition P_j of $\widetilde{K}_j$.

Let $P = P_0 \cup \bigcup_{j=1}^m \widetilde{P}_j$. Then P is a δ-fine partition of E and since $S(f, P) = S(f, P_0) + \sum_{j=1}^m S(f, P_j)$ we have that

$$(HK) \int_E f \; d\mathcal{H}^s = \sum_{i=1}^p (HK) \int_{\widetilde{A}_i} f \; d\mathcal{H}^s + \sum_{j=1}^m (HK) \int_{\widetilde{K}_j} f \; d\mathcal{H}^s.$$

Therefore

$$\left| \sum_{i=1}^p \left\{ f(x_i)\mathcal{H}^s(\widetilde{A}_i) - (HK) \int_{\widetilde{A}_i} f \; d\mathcal{H}^s \right\} \right| = \left| S(f, P_0) - \sum_{i=1}^p (HK) \int_{\widetilde{A}_i} f \; d\mathcal{H}^s \right|$$
$$= \left| \left\{ S(f, P) - \sum_{j=1}^m S(f, P_j) \right\} - \left\{ (HK) \int_E f \; d\mathcal{H}^s - \sum_{j=1}^m (HK) \int_{\widetilde{K}_j} f \; d\mathcal{H}^s \right\} \right|$$
$$\leq \left| S(f, P) - (HK) \int_E f \; d\mathcal{H}^s \right| + \sum_{j=1}^m \left| S(f, P_j) - (HK) \int_{\widetilde{k}_j} f \; d\mathcal{H}^s \right|$$
$$< \varepsilon + m\frac{\eta}{m} = \varepsilon + \eta.$$

Since $\eta > 0$ is arbitrary, the first inequality is proved.

To prove the second part, let

$$P_0^+ = \left\{ (\widetilde{A}_i, x_i) \in P_0 : \quad f(x_i)\mathcal{H}^s(\widetilde{A}_i) - (HK) \int_{\widetilde{A}_i} f \; d\mathcal{H}^s \geq 0 \right\}$$

and let $P_0^- = P_0 \setminus P_0^+$. Let us note that both P_0^+ and P_0^- are δ-fine subpartition of E, so they satisfy the first part of Lemma.

Thus,

$$\begin{aligned}&\sum_{i=1}^{p}\left|f(x_i)\mathcal{H}^s(\widetilde{A}_i)-(HK)\int_{\widetilde{A}_i} f\ d\mathcal{H}^s\right|\\&=\sum_{(\widetilde{A}_i,x_i)\in P_0^+}\left\{f(x_i)\mathcal{H}^s(\widetilde{A}_i)-(HK)\int_{\widetilde{A}_i} f\ d\mathcal{H}^s\right\}\\&-\sum_{(\widetilde{A}_i,x_i)\in P_0^-}\left\{f(x_i)\mathcal{H}^s(\widetilde{A}_i)-(HK)\int_{\widetilde{A}_i} f\ d\mathcal{H}^s\right\}\\&\le\varepsilon+\varepsilon=2\varepsilon.\end{aligned}$$

□

Theorem 3.8 (Cauchy extension). *Let $f : E \to \mathbb{R}$ be a function s-HK integrable on $\widetilde{[c,b]}$ for each $c \in E \cap (a,b)$. If $\lim_{c\to a^+}(HK)\int_{\widetilde{[c,b]}} f\ d\mathcal{H}^s$ exists, then f is s-HK integrable on E and*

$$(HK)\int_E f\ d\mathcal{H}^s = \lim_{c\to a^+}(HK)\int_{\widetilde{[c,b]}} f\ d\mathcal{H}^s.$$

Proof. Let $\varepsilon > 0$ and let $\{c_n\}_{n=0}^{\infty}$ be a strictly decreasing seguence of real numbers of E such that $c_0 = b$ and $\inf_{n\in\mathbb{N}} c_n = a$.

Let $A = \lim_{c\to a^+}(HK)\int_{\widetilde{[c,b]}} f\ d\mathcal{H}^s$. Choose N such that

$$\left|(HK)\int_{\widetilde{[x,b]}} f\ d\mathcal{H}^s - A\right| < \frac{\varepsilon}{4}$$

for $x \in \widetilde{(a,c_N]}$ and $|f(a)|\mathcal{H}^s\left(\widetilde{[a,c_N]}\right) < \frac{\varepsilon}{4}$.

From Theorem 3.5, since f is s-HK integrable on $\widetilde{[c,b]}$ for each $c \in E\cap(a,b)$, then $f \in s\text{-}HK(\widetilde{[c_k,c_{k-1}]})$. Therefore, for $\varepsilon > 0$ and for each positive integer k, we let δ_k be a gauge on $\widetilde{[c_k,c_{k-1}]}$ such that

$$\left|S(f,Q_k) - (HK)\int_{\widetilde{[c_k,c_{k-1}]}} f\ d\mathcal{H}^s\right| < \frac{\varepsilon}{4(2^k)}$$

for each δ_k-fine partition Q_k of $\widetilde{[c_k,c_{k-1}]}$.

Define a gauge δ on E by setting

$$\delta(x) = \begin{cases} c_N - a & \text{if } x = a \\ \min\{\delta_k(c_k), \delta_{k+1}(c_k), \frac{1}{2}(c_{k-1} - c_k), \frac{1}{2}(c_k - c_{k+1})\} & \\ & \text{if } x = c_k \text{ and k=1,2,..} \\ \min\{\delta_k(c_k), \frac{1}{2}(x - c_k), \frac{1}{2}(c_{k-1} - x)\} & \\ & \text{if } x \in (\widetilde{c_k, c_{k-1}}) \text{ and k=1,2,..} \\ \frac{1}{2}(b - c_1) & \text{if } x = b \end{cases}$$

and let $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^p$ be a δ-fine partition of E. Let $\widetilde{A}_i = [\widetilde{y_{i-1}, y_i}]$ for $i = 1, 2, ..., p$ and $a = y_0 < y_1 < ... < y_p = b$. Since $a \notin \bigcup_{k=1}^{\infty} [\widetilde{c_k, c_{k-1}}]$, our choice of δ implies that $x_1 = a$, $y_1 < c_N$ and $y_1 \in (\widetilde{c_{r+1}, c_r}]$ for some unique positive integer r. We also observe that for each $k \in \{1, 2, ..., r\}$, our choice of δ implies that

$$\{(\widetilde{A}, x) \in P : \widetilde{A} \subseteq [\widetilde{c_k, c_{k-1}}]\}$$

is a δ_k-fine partition of $[\widetilde{c_k, c_{k-1}}]$.

Since

$$S(f, P) = f(a)\mathcal{H}^s([\widetilde{a, y_1}]) + \sum_{\substack{(\widetilde{A},x)\in P \\ \widetilde{A}\subseteq[\widetilde{y_1,c_r}]}} f(x)\mathcal{H}^s(\widetilde{A}) + \sum_{k=1}^{r} \left(\sum_{\substack{(\widetilde{A},x)\in P \\ \widetilde{A}\subseteq[\widetilde{c_k,c_{k-1}}]}} f(x)\mathcal{H}^s(\widetilde{A}) \right)$$

and

$$(HK)\int_{[\widetilde{y_1,b}]} f \, d\mathcal{H}^s = (HK)\int_{[\widetilde{y_1,c_r}]} f \, d\mathcal{H}^s + \sum_{k=1}^{r} \left((HK)\int_{[\widetilde{c_k,c_{k-1}}]} f \, d\mathcal{H}^s \right)$$

thus

$$\begin{aligned}
&\left| S(f,P) - \lim_{c\to a^+} (HK)\int_{\widetilde{[c,b]}} f\, d\mathcal{H}^s \right| \\
&\le |f(a)|\, \mathcal{H}^s(\widetilde{[a,y_1]}) + \left| \sum_{\substack{(\widetilde{A},x)\in P\\ \widetilde{A}\subseteq \widetilde{[y_1,c_r]}}} f(x)\mathcal{H}^s(\widetilde{A}) - (HK)\int_{\widetilde{[y_1,c_r]}} f\, d\mathcal{H}^s \right| \\
&+ \left| \sum_{k=1}^{r} \left(\sum_{\substack{(\widetilde{A},x)\in P\\ \widetilde{A}\subseteq \widetilde{[c_k,c_{k-1}]}}} f(x)\mathcal{H}^s(\widetilde{A}) - (HK)\int_{\widetilde{[c_k,c_{k-1}]}} f\, d\mathcal{H}^s \right) \right| \\
&+ \left| (HK)\int_{\widetilde{[y_1,b]}} f\, d\mathcal{H}^s - \lim_{c\to a^+} (HK)\int_{\widetilde{[c,b]}} f\, d\mathcal{H}^s \right| \\
&< \varepsilon.
\end{aligned}$$

□

4 Relationship between the s-integral and the s-Henstock-Kurzweil-Stieltjes integral

4.1 The Vitali-Carathéodory Theorem

Theorem 4.1 (The Vitali-Carathéodory Theorem). *Let E be an s-set, let f be $\mathcal{L}$-integrable on E and let $\varepsilon > 0$. Then there exist functions u and v on E such that $u \le f \le v$, u is upper semicontinuous and bounded above, v is lower semicontinuous and bounded below, and*

$$(L)\int_E (v-u)\, d\mathcal{H}^s < \varepsilon. \tag{1}$$

Proof. Assume first that $f \ge 0$ and that f is not identically 0. Since f is the pointwise limit of an increasing sequence of simple functions s_n (see [9]), f is the sum of the simple functions $t_n = s_n - s_{n-1}$ (taking $s_0 = 0$), and since t_n is a linear combination of characteristic functions, we see that there are measurable sets M_i (not necessarily disjoint) and constants $c_i > 0$ such that

$$f(x) = \sum_{i=1}^{\infty} c_i \chi_{M_i}(x) \qquad (x \in E).$$

Since

$$(L)\int_E f\ d\mathcal{H}^s = \sum_{i=1}^{\infty} c_i \mathcal{H}^s(M_i) \tag{2}$$

the series in (3) converges. Since $\mathcal{H}^s$ is Borel regular (see [7], pag. 57) and E is an s-set, therefore $\mathcal{H}^s \llcorner E$ is a Radon measure. So, there are compacts sets K_i and open sets V_i such that $K_i \subset M_i \subset V_i$ and

$$c_i \mathcal{H}^s(V_i - K_i) < 2^{-i-1}\varepsilon \qquad (i = 1, 2, 3, ...). \tag{3}$$

Put

$$v = \sum_{i=1}^{\infty} c_i \chi_{V_i}, \qquad u = \sum_{i=1}^{N} c_i \chi_{K_i},$$

where N is chosen so that

$$\sum_{N+1}^{\infty} c_i \mathcal{H}^s(M_i) < \frac{\varepsilon}{2}. \tag{4}$$

Then v is lower semicontinuous, u is upper semicontinuous, $u \leq f \leq v$, and

$$v - u = \sum_{i=1}^{N} c_i(\chi_{V_i} - \chi_{K_i}) + \sum_{N+1}^{\infty} c_i \chi_{V_i} \leq \sum_{i=1}^{\infty} c_i(\chi_{V_i} - \chi_{K_i}) + \sum_{N+1}^{\infty} c_i \chi_{V_i}$$

so that (4) and (5) imply (2).

In the general case, write $f = f^+ - f^-$, attach u_1 and v_1 to f^+, attach u_2 and v_2 to f^-, as above, and put $u = u_1 - u_2,\ v = v_1 - v_2$. Since $-v_2$ is upper semicontinuous and since the sum of two upper semicontinuous functions is upper semicontinuous, u and v have the desired properties. □

We now show that the s-HK integral is more general than the Lebesgue integral.

Theorem 4.2. *If E is an s-set, then $\mathcal{L}(E) \subseteq$ s-HK(E) and*

$$(L)\int_E f\ d\mathcal{H}^s = (HK)\int_E f\ d\mathcal{H}^s$$

for each $f \in \mathcal{L}(E)$.

Proof. Let $f \in \mathcal{L}(E)$ and $\varepsilon > 0$. For the Vitali-Carathéodory Theorem, there are upper and lower semicontinuos functions g and h, respectively, such that $-\infty \leq g \leq f \leq h \leq +\infty$ and $(L)\int_E (h - g)\ d\mathcal{H}^s < \varepsilon$. Find a gauge δ on E so that

$$g(t) \leq f(t) + \varepsilon \qquad \text{and} \qquad h(t) \geq f(t) - \varepsilon$$

for each $x, t \in E$ with $|x - t| < \delta(x)$. Now if $P = \{(\widetilde{A}_i, x_i)\}_{i=1}^{p}$ is a δ-fine partition of E, then

$$(L)\int_{\widetilde{A}_i} g\ d\mathcal{H}^s \leq (L)\int_{\widetilde{A}_i} f\ d\mathcal{H}^s \leq (L)\int_{\widetilde{A}_i} h\ d\mathcal{H}^s. \tag{5}$$

Since $g(t) \leq f(x_i) + \varepsilon$, then $g(t) - \varepsilon \leq f(x_i)$, for each $t \in \widetilde{A_i}$ we have that

$$(L)\int_{\widetilde{A}_i} (g-\varepsilon)\; d\mathcal{H}^s \leq (L)\int_{\widetilde{A}_i} f(x_i)\; d\mathcal{H}^s$$

and therefore

$$(L)\int_{\widetilde{A}_i} g\; d\mathcal{H}^s - \varepsilon\,\mathcal{H}^s(\widetilde{A}_i) \leq f(x_i)\,\mathcal{H}^s(\widetilde{A}_i).$$

Moreover, since $h(t) \geq f(x_i) + \varepsilon$, then $f(x_i) \leq h(t) + \varepsilon$, for each $t \in \widetilde{A_i}$ and therefore

$$f(x_i)\,\mathcal{H}^s(\widetilde{A}_i) \leq (L)\int_{\widetilde{A}_i} h\; d\mathcal{H}^s + \varepsilon\,\mathcal{H}^s(\widetilde{A}_i).$$

The last two results lead to

$$(L)\int_{\widetilde{A}_i} g\; d\mathcal{H}^s - \varepsilon\,\mathcal{H}^s(\widetilde{A}_i) \leq f(x_i)\,\mathcal{H}^s(\widetilde{A}_i) \leq (L)\int_{\widetilde{A}_i} h\; d\mathcal{H}^s + \varepsilon\,\mathcal{H}^s(\widetilde{A}_i), \tag{6}$$

$i = 1, ..., p$. It follows that:

$$\left|S(f,P) - (L)\int_E f\; d\mathcal{H}^s\right| \leq \sum_{i=1}^{p}\left|f(x_i)\mathcal{H}^s(\widetilde{A}_i) - (L)\int_{\widetilde{A}_i} f\; d\mathcal{H}^s\right|$$

For the (7), we have that

$$\begin{aligned}\left|f(x_i)\mathcal{H}^s(\widetilde{A}_i) - (L)\int_{\widetilde{A}_i} f\; d\mathcal{H}^s\right| &\leq \left|(L)\int_{\widetilde{A}_i} h\; d\mathcal{H}^s + \varepsilon\,\mathcal{H}^s(\widetilde{A}_i) - (L)\int_{\widetilde{A}_i} f\; d\mathcal{H}^s\right| \\ &= \left|\varepsilon\,\mathcal{H}^s(\widetilde{A}_i) + (L)\int_{\widetilde{A}_i} (h-f)\; d\mathcal{H}^s\right|.\end{aligned}$$

Since

$$-(L)\int_{\widetilde{A}_i} h\; d\mathcal{H}^s \leq -(L)\int_{\widetilde{A}_i} f\; d\mathcal{H}^s \leq -(L)\int_{\widetilde{A}_i} g\; d\mathcal{H}^s$$

and therefore

$$0 \leq (L)\int_{\widetilde{A}_i} (h-f)\; d\mathcal{H}^s \leq (L)\int_{\widetilde{A}_i} (h-g)\; d\mathcal{H}^s$$

we have that

$$\begin{aligned}\left|S(f,P) - (L)\int_E f\; d\mathcal{H}^s\right| &\leq \sum_{i=1}^{p}\left|f(x_i)\mathcal{H}^s(\widetilde{A}_i) - (L)\int_{\widetilde{A}_i} f\; d\mathcal{H}^s\right| \\ &\leq \sum_{i=1}^{p}\left[\varepsilon\,\mathcal{H}^s(\widetilde{A}_i) + (L)\int_{\widetilde{A}_i} (h-g)\; d\mathcal{H}^s\right] \\ &< \varepsilon\,(\mathcal{H}^s(E) + 1)\end{aligned}$$

and then the theorem is proved. □

Remark. The converse of previus theorem is not true. The following example explains this assertion.

Example 4.1. Let E be the classical Cantor set and let $F : E \to \mathbb{R}$ be the function defined as follow

$$F(x) = \frac{(-1)^{n+1}2^n}{n} \qquad \text{if } x \in \widetilde{\left[\frac{2}{3^n}, \frac{1}{3^{n-1}}\right]}, \quad \forall\, n = 1, 2, 3, ...$$

We will prove that F is s-HK-integrable on E, but it is not s-L-integrable on E.

$$(HK)\int_{\widetilde{[\frac{2}{3^k},1]}} F\, d\mathcal{H}^s = \sum_{n=1}^{k} (HK)\int_{\widetilde{[\frac{2}{3^n},\frac{1}{3^{n-1}}]}} F\, d\mathcal{H}^s =$$

$$= \sum_{n=1}^{k} \frac{(-1)^{n+1}\; 2^n}{n}\; \mathcal{H}^s\left(\widetilde{\left[\frac{2}{3^n}, \frac{1}{3^{n-1}}\right]}\right) =$$

$$= \sum_{n=1}^{k} \frac{(-1)^{n+1}\; 2^n}{n} \cdot \frac{1}{2^n} = \sum_{n=1}^{k} \frac{(-1)^{n+1}}{n}$$

For the Cauchy extension, we have:

$$(HK)\int_{E\cap[0,1]} F\, d\mathcal{H}^s = \lim_{k\to\infty} (HK)\int_{\widetilde{[\frac{2}{3^k},1]}} F\, d\mathcal{H}^s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$$

F is not s-L-integrable on E. In fact, if F were s-L-integrable on E, therefore $|F|$ would be s-L-integrable on E.

Following the previous sequnce of steps, it follows easily that

$$(HK)\int_{E\cap[0,1]} |F|\, d\mathcal{H}^s = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

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