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Further generalization of fixed point theorems in Menger PM-spaces

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Abstract
In this work, we establish some fixed point theorems by revisiting the notion of ψ-contractive mapping in Menger PM-spaces. One of our results (namely, Theorem 2.3) may be viewed as a possible answer to the problem of existence of a fixed point for generalized type contractive mappings in *M*-complete Menger PM-spaces under arbitrary *t*-norm. Some examples are furnished to demonstrate the validity of the obtained results.

MSC: 47H10; 45D05

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1 Introduction and preliminaries

In 1942 Menger [1] initiated the study of probabilistic metric spaces; see also [2–4]. Successively, Sehgal and Bharucha-Reid [5, 6] established fixed point theorems in probabilistic metric spaces (for short, PM-spaces). Indeed, by using the notion of probabilistic q-contraction, they proved a unique fixed point result, which is an extension of the celebrated Banach contraction principle [7]. For the interested reader, a comprehensive study of fixed point theory in the probabilistic metric setting can be found in the book of Hadžić and Pap [8], see also [9] for further discussion on generalizations of metric fixed point theory. Recently, Choudhury and Das [10] gave a generalized unique fixed point theorem by using an altering distance function, which was originally introduced by Khan *et al.* [11]. For other results in this direction, we refer to [12–19]. In particular, Dutta *et al.* [20] defined nonlinear generalized contractive type mappings involving altering distances (say, ψ-contractive mappings) in Menger PM-spaces and proved their theorem for such kind of mappings in the setting of G-complete Menger PM-spaces.

On contributing to this study, we weaken the notion of ψ-contractive mapping and establish some fixed point theorems in G-complete and M-complete Menger PM-spaces, besides discussing some related results and illustrative examples. Indeed, we not only derive the result of Dutta *et al.* [20], Theorem 12, as a particular case of our result, but also we notice that our Theorem 2.3 may be viewed as a possible answer to the problem of existence of a fixed point for generalized type contractive mappings in M-complete Menger PM-spaces under arbitrary t-norm.

Here, we state some allied definitions and results which are needed for the development of the present topic. We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of non-negative real numbers and by $\mathbb{N}$ the set of positive integers.

Definition 1.1 ([8, 21]) A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

We shall denote by D^+ the set of all distribution functions, while $H \in D^+$ will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Definition 1.2 ([21]) A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if the following conditions hold:

(a) T is commutative and associative,
(b) T is continuous,
(c) $T(a,1) = a$ for all $a \in [0,1]$,
(d) $T(a,b) \leq T(c,d)$, whenever $a \leq c$ and $b \leq d$, for $a,b,c,d \in [0,1]$.

The following are three basic continuous t-norms from the literature:

(i) The minimum t-norm, say T_M, defined by $T_M(a,b) = \min\{a,b\}$.
(ii) The product t-norm, say T_p, defined by $T_p(a,b) = a \cdot b$.
(iii) The Lukasiewicz t-norm, say T_L, defined by $T_L(a,b) = \max\{a + b - 1, 0\}$.

These t-norms are related in the following way: $T_L \leq T_p \leq T_M$.

Definition 1.3 A Menger PM-space is a triple (X,F,T) where X is a nonempty set, T is a continuous t-norm and F is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of F at the pair (x,y), the following conditions hold:

(PM1) $F_{x,y}(t) = H(t)$ if and only if $x = y$ for all $t \in \mathbb{R}^+$,
(PM2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x,y \in X$ and $t \in \mathbb{R}^+$,
(PM3) $F_{x,y}(t + s) \geq T(F_{x,z}(t), F_{z,y}(s))$ for all $x,y,z \in X$ and $s,t \in \mathbb{R}^+$.

Definition 1.4 Let (X,F,T) be a Menger PM-space. Then

(i) A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.
(ii) A sequence $\{x_n\}$ in X is called Cauchy sequence if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ whenever $n,m \geq N$.
(iii) A Menger PM-space is said to be M-complete if every Cauchy sequence in X is convergent to a point in X.
(iv) A sequence $\{x_n\}$ is called G-Cauchy if $\lim_{n\to\infty} F_{x_n,x_{n+m}}(t) = 1$ for each $m \in \mathbb{N}$ and $t > 0$.
(v) The space (X,F,T) is called G-complete if every G-Cauchy sequence in X is convergent.

According to [21], the (ϵ,λ)-topology in a Menger PM-space (X,F,T) is introduced by the family of neighborhoods N_x of a point $x \in X$ given by

$$N_x = \{N_x(\epsilon,\lambda) : \epsilon > 0, \lambda \in (0,1)\},$$

where

$$N_x(\epsilon,\lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}.$$

The (ϵ,λ)-topology is a Hausdorff topology. In this topology a function f is continuous in $x_0 \in X$ if and only if $f(x_n) \to f(x_0)$, for every sequence $x_n \to x_0$, as $n \to \infty$.

The following class of functions was introduced in [10] and will be used in proving our results in the next section.

Definition 1.5 ([10]) A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a ϕ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii) $\phi(t)$ is strictly increasing and $\phi(t) \to \infty$ as $t \to \infty$,
(iii) ϕ is left continuous in $(0,\infty)$,
(iv) ϕ is continuous at 0.

Definition 1.6 ([22]) Let (X,F,T) be a Menger PM-space. The probabilistic metric F is triangular if it satisfies the condition

$$\frac{1}{F_{x,y}(t)} - 1 \le \left(\frac{1}{F_{x,z}(t)} - 1\right) + \left(\frac{1}{F_{z,y}(t)} - 1\right)$$

for every $x,y,z \in X$ and each $t > 0$.

In the sequel, the class of all ϕ-functions will be denoted by Φ. Also we denote by Ψ the class of all continuous non-decreasing functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\psi^n(a_n) \to 0$, whenever $a_n \to 0$, as $n \to \infty$.

We conclude this section recalling the following fixed point theorem of Dutta *et al.*, see [20], which is the main inspiration of our paper.

Theorem 1.1 *Let (X,F,T) be a G-complete Menger space and $f : X \to X$ be a mapping satisfying the following inequality*:

$$\frac{1}{F_{fx,fy}(\phi(ct))} - 1 \le \psi\left(\frac{1}{F_{x,y}(\phi(t))} - 1\right), \tag{1.1}$$

where $x,y \in X$, $c \in (0,1)$, $\phi \in \Phi$, $\psi \in \Psi$ and $t > 0$ such that $F_{x,y}(\phi(t)) > 0$. Then f has a unique fixed point.

A mapping $f : X \to X$ satisfying condition (1.1) is usually called ψ-contractive mapping. However, for some discussion on this notion and Theorem 1.1, the reader can refer to the recent paper of Gopal *et al.* [23], where analogous results are proved by using some control functions.

2 Main results

In this section, firstly we weaken the class of functions Ψ by assuming the continuity only at point $t = 0$. Precisely, we denote by Ψ_0 the class of all non-decreasing functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that ψ is continuous at 0, $\psi(0) = 0$ and $\psi^n(a_n) \to 0$ whenever $a_n \to 0$ as

$n \to \infty$; then we utilize this class to prove some fixed point theorems. We start with a revised version of Theorem 1.1 useful to obtaining an affirmative answer to an existence problem of a fixed point in a G-complete Menger space.

Theorem 2.1 *Let (X,F,T) be a G-complete Menger space and $f : X \to X$ be a mapping satisfying the following inequality*:

$$\frac{1}{F_{fx,fy}(\phi(ct))} - 1 \le \psi\left(\frac{1}{F_{x,y}(\phi(t))} - 1\right), \tag{2.1}$$

where $x, y \in X$, $c \in (0,1)$, $\phi \in \Phi$, $\psi \in \Psi_0$ and $t > 0$ such that $F_{x,y}(\phi(t)) > 0$. Then f has a unique fixed point.

Proof Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X so that $x_{n+1} = fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. We suppose $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, otherwise f has trivially a fixed point.

Notice that in view of the fact that $\sup_{t\in\mathbb{R}} F_{x_0,x_1}(t) = 1$ and by (ii) of Definition 1.5, one can find $t > 0$ such that $F_{x_0,x_1}(\phi(t)) > 0$. Since $F_{x_0,x_1}(\phi(t)) > 0$ implies that $F_{x_0,x_1}(\phi(\frac{t}{c})) > 0$, therefore (2.1) gives that

$$\begin{aligned} \frac{1}{F_{x_1,x_2}(\phi(t))} - 1 &= \frac{1}{F_{fx_0,fx_1}(\phi(\frac{ct}{c}))} - 1 \\ &\le \psi\left(\frac{1}{F_{x_0,x_1}(\phi(\frac{t}{c}))} - 1\right). \end{aligned} \tag{2.2}$$

From (2.2) we deduce that $F_{x_1,x_2}(\phi(t)) > 0$ and so $F_{x_1,x_2}(\phi(\frac{t}{c})) > 0$. Again, by applying (2.1), we get

$$\begin{aligned} \frac{1}{F_{x_2,x_3}(\phi(t))} - 1 &= \frac{1}{F_{fx_1,fx_2}(\phi(t))} - 1 \\ &\le \psi\left(\frac{1}{F_{x_1,x_2}(\phi(\frac{t}{c}))} - 1\right), \end{aligned}$$

that is,

$$\frac{1}{F_{x_2,x_3}(\phi(t))} - 1 \le \psi\left(\frac{1}{F_{x_1,x_2}(\phi(\frac{t}{c}))} - 1\right).$$

On using (2.2) and the hypothesis that ψ is non-decreasing, the above expression becomes

$$\frac{1}{F_{x_2,x_3}(\phi(t))} - 1 \le \psi^2\left(\frac{1}{F_{x_0,x_1}(\phi(\frac{t}{c^2}))} - 1\right). \tag{2.3}$$

Repeating the above procedure successively n times, we obtain

$$\frac{1}{F_{x_n,x_{n+1}}(\phi(t))} - 1 \le \psi^n\left(\frac{1}{F_{x_0,x_1}(\phi(\frac{t}{c^n}))} - 1\right).$$

If we change x_0 with x_r in the previous inequalities, then for all $n > r$ we get

$$\frac{1}{F_{x_n,x_{n+1}}(\phi(c^r t))} - 1 \le \psi^{n-r}\left(\frac{1}{F_{x_r,x_{r+1}}(\phi(\frac{c^r t}{c^{n-r}}))} - 1\right).$$

Since $\psi^n(a_n) \to 0$ whenever $a_n \to 0$ as $n \to \infty$, therefore the above inequality implies that

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}\bigl(\phi\bigl(c^r t\bigr)\bigr) = 1. \tag{2.4}$$

Now, let $\epsilon > 0$ be given, then by using the properties (i) and (iv) of a function ϕ we can find $r \in \mathbb{N}$ such that $\phi(c^r t) < \epsilon$. It follows from (2.4) that

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(\epsilon) \geq \lim_{n\to\infty} F_{x_n,x_{n+1}}\bigl(\phi\bigl(c^r t\bigr)\bigr) = 1. \tag{2.5}$$

By using a triangle inequality, we obtain

$$F_{x_n,x_{n+p}}(\epsilon) \geq T\underbrace{\bigl(F_{x_n,x_{n+1}}(\epsilon/p), T\bigl(F_{x_{n+1},x_{n+2}}(\epsilon/p), \ldots, \bigl(F_{x_{n+p-1},x_{n+p}}(\epsilon/p)\bigr)\cdots\bigr)\bigr)}_{p\text{-times}}.$$

Thus, letting $n \to \infty$ and making use of (2.5), for any integer p, we get

$$\lim_{n\to\infty} F_{x_n,x_{n+p}}(\epsilon) = 1 \quad \text{for every } \epsilon > 0. \tag{2.6}$$

Hence $\{x_n\}$ is a G-Cauchy sequence. Since (X, F, T) is G-complete, therefore $x_n \to u$, as $n \to \infty$, for some $u \in X$.

Now we show that u is a fixed point of f.

Since

$$F_{fu,u}(\epsilon) \geq T\bigl(F_{fu,x_{n+1}}(\epsilon/2), F_{x_{n+1},u}(\epsilon/2)\bigr), \tag{2.7}$$

by using the properties (i) and (iv) of a function ϕ, we can find $s > 0$ such that $\phi(s) < \frac{\epsilon}{2}$. Again, since $x_n \to u$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, we have $F_{x_n,u}(\phi(s)) > 0$.

Therefore, for $n > n_0$, we obtain

$$\begin{aligned}\frac{1}{F_{x_{n+1},fu}(\frac{\epsilon}{2})} - 1 &\leq \frac{1}{F_{fx_n,fu}(\phi(s))} - 1 \\ &\leq \psi\left(\frac{1}{F_{x_n,u}(\phi(\frac{s}{c}))} - 1\right).\end{aligned}$$

Since ψ is continuous at 0 and $\psi(0) = 0$, we obtain

$$\lim_{n\to\infty} F_{x_{n+1},fu}(\epsilon/2) = 1. \tag{2.8}$$

From (2.7) and (2.8), we get $F_{fu,u}(\epsilon) = 1$ for every $\epsilon > 0$, which in turn yields that $fu = u$. □

The following example illustrates our Theorem 2.1.

Example 2.1 Let $X = [0, 1]$ and d be the usual metric on X. Define $f : X \to X$ as $fx = \frac{1}{4}\sin x$ and

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{if } t > 0, \\ 0 & \text{if } t = 0 \end{cases}$$

for all $x, y \in X$. Then (X, F, T) is a complete Menger PM-space with $T_p(a,b) = a \cdot b$. Define $\phi \in \Phi$ by $\phi(s) = \frac{s}{c}$ for all $s \in \mathbb{R}^+$, with $c = \frac{1}{2}$, and $\psi \in \Psi_0$ by

$$\psi(s) = \begin{cases} s & \text{if } s \in \mathbb{Q} \cap \mathbb{R}^+, \\ \frac{s}{2} & \text{otherwise.} \end{cases}$$

By the mean valued theorem with function sin, we obtain that

$$\begin{aligned} \frac{1}{F_{fx,fy}(\phi(ct))} - 1 &= \frac{d(fx,fy)}{t} \\ &= \frac{1}{4t} |\sin x - \sin y| \\ &\leq \frac{|x-y|}{4t} \\ &= \frac{d(x,y)}{4t} \leq \psi\left(\frac{1}{F_{x,y}(\phi(t))} - 1\right). \end{aligned}$$

Thus, f satisfies all the hypotheses of Theorem 2.1; here $u = 0$ is a fixed point of f.

Next we consider the uniqueness problem of a fixed point in a G-complete Menger space; to this aim we give the following condition:

$(*)$ $F_{u,v}(0) = 0$ if $u, v \in \operatorname{Fix}(f)$, where $\operatorname{Fix}(f)$ denotes the set of all fixed points of a mapping f, that is, $\operatorname{Fix}(f) := \{x \in X : x = fx\}$.

Theorem 2.2 *Adding condition* $(*)$ *to the hypotheses of Theorem* 2.1, *we obtain uniqueness of the fixed point.*

Proof We prove uniqueness of the fixed point. Let u and v be two fixed points of f, that is, $u = fu$ and $v = fv$. First, we prove that $F_{u,v}(\phi(s)) > 0$ for all $s > 0$. By condition (ii) of Definition 1.5, we have $\phi(s/c^n) \to \infty$ as $n \to \infty$. Since $\sup_{n\in\mathbb{N}} F_{u,v}(\phi(s/c^n)) = 1$, we deduce that there exists $n \in \mathbb{N}$ such that $F_{u,v}(\phi(s/c^n)) > 0$. Now, by using (2.1), we obtain

$$\begin{aligned} \frac{1}{F_{u,v}(\phi(\frac{s}{c^{n-1}}))} - 1 &= \frac{1}{F_{fu,fv}(\phi(\frac{cs}{c^n}))} - 1 \\ &\leq \psi\left(\frac{1}{F_{u,v}(\phi(\frac{s}{c^n}))} - 1\right), \end{aligned}$$

that implies $F_{u,v}(\phi(\frac{s}{c^{n-1}})) > 0$. By repeating a similar reasoning n times, we deduce that $F_{u,v}(\phi(s)) > 0$ for all $s > 0$.

Next, we show that $F_{u,v}(\phi(s)) = 1$. In fact, for every $s > 0$, we have that $F_{u,v}(\phi(\frac{s}{c^i})) > 0$ for all $1 \leq i \leq n$ and for all $n \in \mathbb{N}$. Therefore, by using (2.1), we get

$$\frac{1}{F_{u,v}(\phi(s))} - 1 \leq \psi\left(\frac{1}{F_{u,v}(\phi(\frac{s}{c}))} - 1\right) \leq \cdots \leq \psi^n\left(\frac{1}{F_{u,v}(\phi(\frac{s}{c^n}))} - 1\right).$$

Thus, since $\psi^n(a_n) \to 0$ whenever $a_n \to 0$ as $n \to \infty$, we get $F_{u,v}(\phi(s)) = 1$.

It follows that $F_{u,v}(t) = H(t)$ for all $t > 0$. In fact, if t is not in range of ϕ, since ϕ is continuous at 0, then there exists $s > 0$ such that $\phi(s) < t$. This implies $F_{u,v}(t) \geq F_{u,v}(\phi(s)) = 1$, yielding thereby $u = v$. □

Our next step is to furnish a fixed point theorem in an M-complete Menger PM-space.

Theorem 2.3 *Let (X, F, T) be an M-complete Menger PM-space and $f : X \to X$ be a ψ-contractive mapping, where the function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, continuous at* 0, *$\psi(0) = 0$ and $\sum_{n=1}^{\infty} \psi^n(a_n) < \infty$, whenever $a_n \to 0$ as $n \to \infty$. Then f has a fixed point provided that F is triangular.*

Proof In view of the assumptions on the function ψ, it is clear that $\psi \in \Psi_0$. Then, following similar arguments to those given in Theorem 2.1, one obtains $F_{x_n,x_{n+1}}(\epsilon) \to 1$ as $n \to \infty$. Now, we shall show that $\{x_n\}$ is an M-Cauchy sequence. By the properties of ϕ, given $\epsilon > 0$, we can find $s > 0$ such that $\epsilon > \phi(s) > 0$. Therefore,

$$\frac{1}{F_{x_n,x_{n+p}}(\epsilon)} - 1 \leq \frac{1}{F_{x_n,x_{n+p}}(\phi(s))} - 1.$$

Now, since F is triangular, we get

$$\begin{aligned}
\frac{1}{F_{x_n,x_{n+p}}(\epsilon)} - 1 &\leq \frac{1}{F_{x_n,x_{n+1}}(\epsilon)} - 1 + \frac{1}{F_{x_{n+1},x_{n+2}}(\epsilon)} - 1 + \cdots + \frac{1}{F_{x_{n+p-1},x_{n+p}}(\epsilon)} - 1 \\
&\leq \frac{1}{F_{x_n,x_{n+1}}(\phi(s))} - 1 + \frac{1}{F_{x_{n+1},x_{n+2}}(\phi(s))} - 1 \\
&\quad + \cdots + \frac{1}{F_{x_{n+p-1},x_{n+p}}(\phi(s))} - 1 \\
&\leq \psi^n\left(\frac{1}{F_{x_n,x_{n+1}}(\phi(\frac{s}{c^n}))} - 1\right) + \psi^{n+1}\left(\frac{1}{F_{x_{n+1},x_{n+2}}(\phi(\frac{s}{c^{n+1}}))} - 1\right) \\
&\quad + \cdots + \psi^{n+p}\left(\frac{1}{F_{x_{n+p-1},x_{n+p}}(\phi(\frac{s}{c^{n+p}}))} - 1\right) \\
&\leq \sum_{k=n}^{\infty} \psi^k\left(\frac{1}{F_{x_k,x_{k+1}}(\phi(\frac{s}{c^k}))} - 1\right).
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \psi^n(a_n) < \infty$, where $a_n = (\frac{1}{F_{x_n,x_{n+1}}(\phi(\frac{s}{c^n}))} - 1) \to 0$ as $n \to \infty$, we obtain $F_{x_n,x_{n+p}}(\epsilon) \to 1$ as $n \to \infty$. Thus $\{x_n\}$ is an M-Cauchy sequence in X. The rest of the proof of this theorem can be completed on the lines of Theorem 2.1. This concludes the proof. □

Clearly, on the same lines of Theorem 2.2 one can solve the uniqueness problem of a fixed point in an M-complete Menger space. To avoid repetition, we give the statement of this theorem without the proof.

Theorem 2.4 *Adding condition* $(*)$ *to the hypotheses of Theorem* 2.3, *we obtain uniqueness of the fixed point.*

Remark 2.1 Our Theorem 2.3 is proved in an M-complete Menger PM-space under arbitrary t-norm, therefore Theorem 2.3 can be realized as a possible answer to the problem of existence of a fixed point for generalized type contractive mappings in Menger PM-spaces.

As an application of Theorems 2.1 and 2.2, we prove the following common fixed point theorem for a finite family of mappings which runs as follows.

Theorem 2.5 *Let* (X, F, T) *be a G-complete Menger PM-space,* $\{f_i\}_1^m$ *be a finite family of self-mappings defined on X and denote* $f = f_1 f_2 f_3 \cdots f_m$. *If* $f : X \to X$ *satisfies all the hypotheses of Theorem* 2.2, *then the family* $\{f_i\}_1^m$ *has a unique common fixed point provided that* $f_i f_j = f_j f_i$ *whenever* $i \neq j$, *with* $i, j \in \{1, 2, \dots, m\}$.

Proof Notice that all the hypotheses of Theorems 2.1 and 2.2 are satisfied in respect of the mapping f, therefore there exists a unique $x \in X$ such that $fx = x$. Now

$$\begin{aligned} f(f_i x) &= \big((f_1 f_2 \cdots f_m) f_i\big) x \\ &= (f_1 f_2 \cdots f_{m-1})\big((f_m f_i) x\big) = (f_1 f_2 \cdots f_{m-1})(f_i f_m x) \\ &= \cdots \\ &= f_1 f_i (f_2 f_3 \cdots f_m x) \\ &= f_i f_1 (f_2 f_3 \cdots f_m x) = f_i (fx) = f_i x, \end{aligned}$$

which shows that $f_i x$ is also a fixed point of f. Since x is the unique fixed point of f, therefore $f_i x = x$ and hence x is also a fixed point of all mappings f_i for $i \in \{1, 2, \dots, m\}$. □

By setting $f_1 = f_2 = \cdots = f_m = g$ in Theorem 2.5, we deduce the following fixed point theorem for mth iterates of a mapping g.

Corollary 2.1 *Let* (X, F, T) *be a G-complete Menger PM-space and* $g : X \to X$ *be a mapping such that* g^m *satisfies all the hypotheses of Theorem* 2.2. *Then g has a unique fixed point.*

Remark 2.2 Results similar to Theorem 2.5 and Corollary 2.1 can be outlined in respect of Theorems 2.3 and 2.4.

Finally, by using the following example, we show that Corollary 2.1 can be situationally more useful than Theorems 2.1 and 2.2.

Example 2.2 Let $X = [0, 1]$ be equipped with the usual metric d on X. Define $f : X \to X$ as follows:

$$fx = \begin{cases} 0 & \text{if } x \in \{0, \frac{1}{2}, 1\}, \\ 1 & \text{if } x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1). \end{cases}$$

Also define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t + d(x,y)} & \text{if } t > 0, \\ 0 & \text{if } t = 0 \end{cases}$$

for all $x, y \in X$. Then (X, F, T) is a complete Menger PM-space with $T_p(a,b) = a \cdot b$. Notice that $f^2 x = 0$ for every $x \in X$ and hence the condition

$$\frac{1}{F_{f^2x,f^2y}(\phi(ct))} - 1 \leq \psi\left(\frac{1}{F_{x,y}(\phi(t))} - 1\right)$$

is always satisfied for every choice of functions $\phi \in \Phi$ and ψ as in Theorem 2.5, with any constant $c \in (0,1)$. On the other hand, f does not satisfy condition (2.1). In fact, for instance, putting $x = 0$ and $y \in (\frac{1}{2}, 1)$ the inequality

$$F_{fx,fy}(ct) \geq F_{x,y}(t)$$

does not hold true and consequently condition (2.1) is not satisfied for $\phi(s) = \psi(s) = s$ for all $s \in \mathbb{R}^+$. Moreover, if we choose again $x = 0$ and make $y \to 0$, then f does not satisfy the condition

$$\frac{1}{F_{fx,fy}(\phi(ct))} - 1 \leq \psi\left(\frac{1}{F_{x,y}(\phi(t))} - 1\right)$$

for any choice of $\phi \in \Phi$, $\psi \in \Psi_0$ and $c \in (0,1)$.

Thus we conclude that f does not meet the requirements of Theorem 2.1, whereas the power mapping f^2 satisfies all the conditions of Corollary 2.1 substantiating the utility of Corollary 2.1 (and hence Theorem 2.5) over Theorem 2.1.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References

1. Menger, K: Statistical metric. Proc. Natl. Acad. Sci. USA **28**, 535-537 (1942)
2. Sherwood, H: Complete probabilistic metric spaces. Z. Wahrscheinlichkeitstheor. Verw. Geb. **20**, 117-128 (1971)
3. Shisheng, Z: On the theory of probabilistic metric spaces with applications. Acta Math. Sin. Engl. Ser. **1**, 366-377 (1985)
4. Wald, A: On a statistical generalization of metric spaces. Proc. Natl. Acad. Sci. USA **29**, 196-197 (1943)
5. Sehgal, VM, Bharucha-Reid, AT: Fixed point of contraction mappings in PM-spaces. Math. Syst. Theory **6**, 97-102 (1972)
6. Sehgal, VM: Some fixed point theorems in functional analysis and probability. PhD dissertation, Wayne State University, Michigan (1966)

7. Banach, S: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3**, 133-181 (1922)
8. Hadžić, O, Pap, E: Fixed Point Theory in Probabilistic Metric Spaces. Kluwer Academic, New York (2001)
9. Van An, T, Van Dung, N, Kadelburg, Z, Radenović, S: Various generalizations of metric spaces and fixed point theorems. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. (2014). doi:10.1007/s13398-014-0173-7
10. Choudhury, BS, Das, KP: A new contraction principle in Menger spaces. Acta Math. Sin. Engl. Ser. **24**, 1379-1386 (2008)
11. Khan, MS, Swaleh, M, Sessa, S: Fixed points theorems by altering distances between the points. Bull. Aust. Math. Soc. **30**, 1-9 (1984)
12. Chauhan, S, Bhatnagar, S, Radenović, S: Common fixed point theorems for weakly compatible mappings in fuzzy metric spaces. Matematiche **68**, 87-98 (2013)
13. Chauhan, S, Shatanawi, W, Kumar, S, Radenović, S: Existence and uniqueness of fixed points in modified intuitionistic fuzzy metric spaces. J. Nonlinear Sci. Appl. **7**, 28-41 (2014)
14. Chauhan, S, Radenović, S, Imdad, M, Vetro, C: Some integral type fixed point theorems in non-Archimedean Menger PM-spaces with common property (E.A) and application of functional equations in dynamic programming. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **108**, 795-810 (2014)
15. Choudhury, BS, Dutta, PN, Das, KP: A fixed point result in Menger spaces using a real function. Acta Math. Hung. **122**, 203-216 (2008)
16. Choudhury, BS, Das, KP: A coincidence point result in Menger spaces using a control function. Chaos Solitons Fractals **42**, 3058-3063 (2009)
17. Ćirić, LB: On fixed point of generalized contractions on probabilistic metric spaces. Publ. Inst. Math. (Belgr.) **18**, 71-78 (1975)
18. Gajić, L, Rakočević, V: Pair of non-self-mappings and common fixed points. Appl. Math. Comput. **187**, 999-1006 (2007)
19. Miheţ, D: Altering distances in probabilistic Menger spaces. Nonlinear Anal. **71**, 2734-2738 (2009)
20. Dutta, PN, Choudhury, BS, Das, KP: Some fixed point results in Menger spaces using a control function. Surv. Math. Appl. **4**, 41-52 (2009)
21. Schweizer, B, Sklar, A: Probabilistic Metric Spaces. Elsevier, New York (1983)
22. Vetro, C, Vetro, P: Common fixed points for discontinuous mappings in fuzzy metric space. Rend. Circ. Mat. Palermo **57**, 295-303 (2008)
23. Gopal, D, Abbas, M, Vetro, C: Some new fixed point theorems in Menger PM-spaces with application to Volterra type integral equation. Appl. Math. Comput. **232**, 955-967 (2014)