A Route to Agnosticism in Mathematics

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Introduction

The ontological problem is characterized by Quine as the question on what there is. This question can be split into smaller issues according to the way we describe the world: common sense, chemistry, computer science, physics, and so on. Mathematics plays a crucial role in our web of beliefs, because it seems indispensable to the scientific picture of the world. Because this is so, part of the ontological problem takes the form of what mathematics is about: that is, whether mathematical objects exist or not. If there are such objects, like numbers, it seems we ought to regard them as *abstracta*: causally inert objects that are outside of space-time and mind-language independent. This is because no one can build a telescope to detect where mathematical objects are, no one can break a Turing machine, and no one can causally interact with numbers. The view that mathematical *abstracta* exist is called Platonism, whereas the view according to which there is no commitment to such objects is called anti-Platonism.

Anti-Platonism is a broad name for a large number of different positions that are divided into two classes: anti-Platonist realism and anti-Platonist anti-realism. According to the former, mathematical objects exist but are not *abstracta*, whereas, according to the latter, mathematical objects do not exist at all. Both views are generally driven by the weak epistemology of abstract objects, that is, by the fact that it is hard to account for how the correlation between belief states and abstract objects occurs. Epistemological arguments against Platonism are based on manifold theories of knowledge (e.g. the causal theory of knowledge, reliabilism, etc.) but they all point to the same problem: no epistemic story can bridge the gap between belief states and abstract objects. And if we lack such an epistemic story, it might be questionable whether or not to include *abstracta* in our ontology.

The Platonist has a straightforward reply to such epistemological worries: the indispensability arguments. According to one of the most important versions of the indispensability argument, if mathematical objects are indispensable to our best scientific theories, we ought to have ontological commitment to such objects independently of any epistemological concern. Because the indispensability argument aims to overcome any epistemological objection, it is perhaps the best argument for Platonism. In this regard, there have been many attempts at disarming the indispensability argument without committing to any sort of revisionism of actual scientific practice.

I should like to present the best examples of such attempts by advocating the distinction between the 'hard road' and the 'easy road' to anti-Platonism: hard-road strategies paraphrase physical, or mathematical, statements in order to dispense with mathematical objects; easy-road strategies also reject abstract objects but do not require that physics and mathematics be paraphrase-bound.

I will examine three 'hard roads': Field's nominalization of classical mechanics and Newtonian gravitational theory, Chihara's constructibility theory, and Hellman's modal reconstruction of arithmetic. I intend to consider the best-developed of all nominalist reconstructions: the best anti-realist hard roads available to us.

According to Field, it is possible to show that the mathematical objects we refer to in classical mechanics and Newtonian gravitational theory are dispensable. Roughly speaking, Field's argument proceeds as follows: if the physical consequences we can derive by using mathematics are the same as those we can derive from a body of pure physical assertions, then mathematical objects are dispensable. As a result, the statements that existentially quantify over mathematical objects, such as 'there are infinitely many prime numbers', turn out to be false. Under the assumption that mathematical objects do not exist, a large number of mathematical assertions turn out to be false. But according to Field, mathematics does not need to be true to be useful: it merely needs to be a device for making calculations easier. And if it is possible to extend Field's approach to all our best scientific theories, the indispensability argument will be disarmed.

Chihara provides an extensive reconstruction of mathematics without ontological commitment to mathematical *abstracta*. In this regard, Chihara settles on a nominalistic interpretation of mathematics through the constructibility theory, namely a first-order language equipped with constructibility quantifiers. Constructibility quantifiers are concrete tokens (marks on paper, or on screen, etc.) that explain the rules for building sequences of concrete tokens. Such sequences are in turn employed to replace mathematical objects, formal relations, mathematical properties and so on. To put it another way, Chihara's goal is to find a surrogate of mathematical objects that is nominalistically acceptable. Moreover, Chihara makes use of the modal operator 'it is possible to construct X', where X is conceived as a token located in space and time.

Hellman develops a modal strategy that offers some analogies to Chihara's. According to Hellman, mathematics is about possible structures. For instance, Peano Arithmetic is the study of the possible structures (i.e. progressions or ω-sequences) that satisfy Peano axioms. Hellman's modal structuralism assumes that mathematical structures are possible, but does not force any commitment to actual structures. In addition, there is no ontological commitment to the objects that occur in structures. Hellman also presents a truth-translation scheme according to which all ordinary mathematical statements can be translated into statements about possible structures. The translation scheme allows Hellman to recover arithmetic, real analysis, and set theory partially, given the hypothetical existence of suitable mathematical structures.

For a long time, 'hard-road' strategies attracted the attention of anti-

Platonists. However, there have been many objections to such approaches. For example, some philosophers point out how Field's nominalization involves mathematical *abstracta* (i.e. models) surreptitiously; others stress technical difficulties in extending Field's approach to other branches of contemporary physics. Turning to Chihara, it is open to question whether or not constructibility quantifiers require possible worlds semantics, and Hellman, for his part, needs to clarify how we can get knowledge of possible mathematical structures. These and other problems have led anti-Platonists to develop new strategies that do not require any paraphrasing of mathematical, or physical statements. In this regard, I will consider the following 'easy roads' to anti-Platonism: Azzouni's neutralism, Yablo's presuppositionalism, and Maddy-Sober's objection to confirmation holism.

Azzouni seeks to reject Quine's criterion for ontological commitment, where commitment is tracked down in formulas of the form $\exists x P(x)$. In Quine's view, if existential quantifiers range over a domain of discourse, and the truth of an existential formula such as $\exists x P(x)$ is given by the satisfaction of an object x, existential quantifiers are always ontologically marked. By applying Quine's criterion to a domain of discourse, we get a straightforward way of identifying what that discourse is about. However, according to Azzouni, existential quantifiers are neutral in the sense that quantification is not sufficient to carry any ontological commitment: we also need a story that shows how the objects in question are independent of psychological processes and language. Azzouni notices how a sentence such as 'there are numbers' leads to a commitment to numbers so long as 'there is' carries ontological weight; but Azzouni distinguishes the ontological commitment of a sentence from what a sentence is about: there are sentences about something even if they do not force ontological commitment to what they refer to. In the end, Azzouni's strategy does not need any paraphrasing of mathematical statements insofar as quantification is released from ontological commitment and if, in addition, mathematical objects are mind– and language– dependent.

According to Yablo, the physical content of a theory can be represented as what is left after subtracting the proposition that mathematical objects exist from that theory. In this regard, Yablo develops a general approach called 'logical subtraction' in order to strip away unwanted propositions that presuppose the existence of mathematical objects. Here the challenge is to show that even though a proposition such as 'there are no numbers' is false, its failure makes no difference to how matters stand in relation to the physical world. Although complex, the entire procedure does not require any sort of paraphrasing of mathematical, or physical, sentences. To make logical subtraction work, Yablo invokes powerful philosophical mechanisms: propositions, possible worlds and subject matters.

Consider this argument for Platonism: it could be argued that the empirical success of a theory confirms both the existence of physical objects and the mathematical entities involved. If mathematics is true because it is empirically confirmed, we ought to commit ourselves to the existence of mathematical entities as well as we do to the existence of physical objects. In this regard, confirmation holism states that scientific theories are confirmed as a whole. I intend to present two objections to confirmation holism: on the one hand, Maddy argues that confirmation holism does not corroborate the existence of mathematical objects that are employed in physical theories; on the other hand, Sober argues that mathematics does not need empirical confirmation to be true, nor is it affected by the falsification of empirical theories. In point of fact, both Maddy and Sober dismiss the indispensability argument that is based on confirmation holism.

The 'easy roads' to anti-Platonism I mentioned earlier lead to either realism or anti-realism. But other 'easy roads' can be considered: Balaguer's anti-metaphysical view and Bueno's agnostic nominalism. These positions neither endorse nor reject the existence of abstract objects, but lead to the conclusion that we should suspend our judgment on whether mathematical *abstracta* exist or not. The issue is not that the metaphysical debate is meaningless or faulty: instead, according to Balaguer, there is no fact of the matter as to whether or not abstract objects exist whereas, according to Bueno, we cannot ever know whether or not such objects exist: perhaps they exist, perhaps they do not.

For Balaguer, we do not have any good arguments for and against the existence of abstract objects. Philosophers have not come up with any good arguments because there is no fact of the matter as to whether abstract objects exist. If this is so, Balaguer argues that we do not have any reasons for choosing between Platonism and anti-Platonism. Under the assumption that full-blooded Platonism and fictionalism are respectively the best forms of Platonism and anti-Platonism, Balaguer argues that the metaphysical debate is factually empty. This is a valid argument inasmuch as the sentence 'there are abstract objects' does not have any (possible-worlds-style) truth-conditions. Balaguer aims to show that if our usage does not determine truth-conditions for 'there are abstract objects', we can imply that there is no fact of the matter.

Bueno presents a view where the commitment to mathematical *abstracta* is avoided without denying their existence. If *abstracta* are truly mind–language independent, we cannot rule out their existence. Bueno argues that agnosticism can account for the application of mathematics to the physical world and, in addition, agnosticism can take mathematical statements at face value as the Platonist does. With regard to the former point, Bueno claims that mathematicians deal with mathematical artifacts whereas, with regard to the latter, the agnostic can maintain that mathematical objects would be *abstracta* if they happened to exist.

By way of a conclusion, I intend to advance my own agnostic view. I would argue that even if there is no epistemic role for mathematical *abstracta*, that claim, on its own, does not say anything about the non-existence of mathematical *abstracta*. In other words, that claim is compatible with the existence, or non-existence, of mathematical *abstracta*. I would also argue that abstract objects have good reasons for lacking any epistemic role. The fact that abstract objects have no epistemic role just implies that nothing epistemically relevant to mathematical practice follows from the existence, or non existence, of mathematical *abstracta*. If this is so, it does not matter for mathematical practice whether or not *abstracta* exist. My own agnostic position leads to the distinction between agnosticism about abstract objects and global agnosticism; that is to say, my own agnostic view is not an old-fashioned agnosticism. Finally, I will show how my own agnostic view could open an easy-road to anti-Platonism that is compatible with the indispensability argument. Additionally, I will demonstrate how it overcomes Field's challenge, and how it partially recovers confirmation holism.

Chapter 1

Arguments for and against Platonism

People take for granted the existence of the external world before they start philosophizing. Some of them will reconsider their position later on, whereas others will find compelling arguments for supporting their pre-philosophical belief. It is not clear when exactly philosophy comes into the picture, but philosophy is certainly involved once we ask ourselves what the external world is. Imagine being a novice at philosophizing but, nonetheless, you want to figure out what the external world is. The first thing you may notice is that we are surrounded by medium-sized objects perceived through the senses, and that the existence of such objects is somehow independent of us. Later on, you will find that philosophers qualify these objects as mind– and language– independent. Perhaps you are on the right track, but many important items have been left out: objects that we do not perceive directly, such as elementary particles, molecules, exoplanets. And such objects merit an important role in the ontological picture no less than medium-sized objects. This is because perceived objects are not sufficient to account for most empirical phenomena. For this reason, scientists postulate the existence of objects that can be detected only by extending the senses through more and more sophisticated instruments. Ultimately, determining what exists involves not just perception but also a considerable number of scientific concepts.

Here lies the first question: what kind of scientific concepts should we take into account? It seems as if we should look to our current best scientific theories, because there is no reason to believe in the existence of something that scientists dismissed a long time ago, e.g. the phlogiston. Here lies another issue: if scientific theories are indispensable to addressing the ontological problem, what is the nature of the entities involved? These entities, say, are generally physical ones. However, scientific theories *do* quantify over objects that do not seem to be physical at all: numbers, for instance. And because our best scientific theories allow the quantification over abstract objects, we ought to commit ourselves to the existence of mathematical *abstracta*.[1] That is, we ought to commit ourselves to the existence of objects that are not physical on the basis of the way we describe the physical world.

Abstract objects seem to play an indispensable role not just in physical theories but also in our common talk: propositions, concepts, and meanings do not seem to refer to concrete objects, or to the mereological sum of concrete objects.[2] Although people may have a sort of fundamental intuition[3] that the world is made up of physical objects, abstract entities abound in the scientific enterprise as well as in ordinary speech. This is because many sentences, taken literally, seem to refer to *abstracta*. Hence, some philosophers have come to the conclusion that abstract objects *do* exist, and unless

[1]We should at least commit ourselves to the existence of mathematical objects that are indispensable to our best scientific theories. I will examine this statement in sec. n. 1.3 when I consider the indispensability argument.

[2]Mereology studies the relationships between an entity and its constituted parts. Unlike sets, mereological sums can be conceived as concrete entities. At least in the light of Leonard and Goodman's analysis (1940), mereology is a formal theory that can be used to develop a nominalistic program, i.e. calculus of individuals.

[3]Goodman and Quine (1947) claimed that whatever exists is concrete, and they regard such a claim as a *fundamental intuition*. Goodman and Quine aim to build a nominalistic syntax in order to avoid referring to abstract objects. Moreover, Goodman and Quine reject every predicate, definition, and sentence that is supposed to refer to abstract objects. It is well known how Quine distanced himself from his early position.

we can dispense with that concept, the fundamental intuition is seriously under attack.

Addressing the ontological problem is not straightforward. The answer to the question 'what is there?' cannot simply be "Everything", because 'everything' does not take into account the fact that people have different ontological positions on the same matter.[4] And even if the disagreement in question was faulty, that is, people think they disagree whereas they do not, philosophers should explain in what sense there *seem* to be different ontological positions. In any case, 'everything' does not specify what we are ontologically committed to, and thus it does not address the ontological problem.

Quine recognizes the existence of plural ontologies and, in addition, advances a way to settle the ontological disagreement: Quine's criterion of ontological commitment. The idea is to look to our best scientific theories, regiment them into an interpreted first-order language, and track down the ontological commitment in any formula of the form $\exists x P(x)$.[5] I will examine Quine's criterion in detail later on, but notice this:[6] if existential quantifiers range over a domain of discourse, and the truth of the formula $\exists x P(x)$ is given by an object that satisfies x, existential quantifiers force ontological commitment to that object *independently* of its nature. The criterion applies irrespective of whether x is a physical or an abstract object. If after regimenting our best scientific theories we find a statement such as 'there are numbers', and numbers are considered abstract objects, Quine's criterion forces ontological commitment to mathematical *abstracta*.

But do abstract objects *really* exist? Philosophers disagree widely on this matter. Concrete objects can be defined as things that have causal powers in space and time: from things we perceive through the senses, like a tennis ball, to objects we discover by using sophisticated instruments,

[4] See Quine (1948).

[5] Where x occurs free in P. See Quine (1948), (1951) and (1951b).

[6] When I consider Azzouni's position on Quine's criterion, I will explain that matter. See sec. n. 3.1.2.

like MRI.[7] By contrast, abstract objects are not in space-time, nor have causal powers: no one can build a telescope to detect the portion of space-time where numbers are, no one can break Turing machines, no one can causally interact with numbers, and so on.[8] Because abstract objects have such odd properties, philosophers have, for a long time, questioned their existence. I shall call 'anti-Platonists' those philosophers who reject the existence of abstract objects, and 'Platonists' those who believe that mind– and language– independent abstract objects exist.[9]

Anti-Platonism is a term that encompasses many different positions: nominalists, fictionalists, idealists, non-Platonist realists, and so on. It is extremely difficult to take account of all these perspectives, so I need a philosophical excuse to focus on some of them. I believe that many anti-Platonists criticize Platonism mainly on the basis of epistemological arguments. Because anti-Platonists are generally skeptical about finding a good epistemology for abstract objects, I will start with the epistemological objections to abstract objects. More specifically, I will first examine the objections based on the causal theory of knowledge.

1.1 Between a rock and a hard place: Benacerraf's argument

The causal theory of knowledge was not initially used as an argument against Platonism. Goldman indeed employed the causal theory of knowledge only to strengthen the classical concept of knowledge.[10] According to this concept, an epistemic agent knows that P iff P is both a true and justified belief that is held by an epistemic agent. The definition highlights three notions (truth, justification, belief) that play an important role in what we

[7] Magnetic resonance imaging (MRI). To avoid terminological ambiguities, I should point out that 'concrete' is synonymous with 'physical'.

[8] We causally interact with *concreta*: inscriptions, numerals, diagrams, and so on.

[9] I borrow the term 'anti-Platonist' from Balaguer's *Platonism and Anti-Platonism in Mathematics* (1998).

[10] See Goldman (1967).

regard intuitively as knowledge. But there are at least two problems in such a definition: first, how epistemic agents form their belief is missing and, secondly, the definition is open to Gettier's counter-examples.[11] Goldman aims at overcoming both problems by supplying the concept of knowledge with causality. On this view, an agent knows that P iff P is both a true and justified belief, and P is caused by its truthmaker.[12] In other words, Goldman's causal theory of knowledge requires the causal connection between belief and its truthmaker in addition to the notions of truth, justification, and belief.[13]

It is important to ask which kind of knowledge Goldman is referring to. In Goldman's view, the causal theory of knowledge concerns *only* empirical knowledge:

> My concern will be with knowledge of empirical propositions only, since I think that the traditional analysis is adequate for knowledge of non-empirical truths.[14]

Whereas Goldman applies the causal theory of knowledge only to empirical contexts, Benacerraf runs his argument by using the causal theory of knowledge against non-empirical truths.[15] Of course, this is an important

[11]Suppose Tyler believes that 'Lance owns a Chrysler'. Tyler can infer from that sentence that 1) either Lance owns a Chrysler or Lance is in Boston; 2) either Lance owns a Chrysler or Lance is in Miami; 3) either Lance owns a Chrysler or Lance is in Rome; and so on. Now, suppose that Lance is really in Rome, but he has not kept Tyler posted. Does Tyler know that Lance is in Rome? Intuitively, the answer is "no". However, according to Gettier, if the classical concept of knowledge was true, Tyler would know that Lance is in Rome. This is because Tyler's belief is true (Lance is in Rome) and justified (Tyler had used logic to make that inference). Gettier's counter-example suggests that there is something wrong with the classical concept of knowledge. See Gettier (1963).

[12]P's truthmaker is the fact that makes P true.

[13]There are several problems with Goldman's suggestion that I will not elaborate on, because I would like to develop a discussion on Benacerraf's use of the causal theory of knowledge. A good, concise reconstruction of those problems can be found in Dancy (1986), Ch. 2.

[14]Goldman (1967, p. 357).

[15]Benacerraf does not seem to assume that mathematical truths are empirical. See

shift for those who believe that mathematical knowledge is *not* empirical.

Benacerraf's argument aims to challenge mathematical Platonism, but it can be applied to any object that is supposed to be causally inert. Let us call 'broad Platonism' the claim that abstract objects exist, and see how the argument goes:

(P_1) If broad Platonism is true, we should have knowledge of abstract objects.

(P_2) If P_1 is true, we should be causally related to abstract objects.

(P_3) But we are not causally related to abstract objects.

(C) Broad Platonism is false.

Since mathematical Platonism is an instance of broad Platonism, the argument can be applied to mathematical entities as well. Benacerraf's epistemological argument against mathematical Platonism is as follows: to have mathematical knowledge, we should be causally related to mathematical entities. But because we cannot causally interact with mathematical entities, i.e. they are *abstracta*, mathematical knowledge turns out to be impossible. As a result, mathematical Platonism must be rejected if we want to account for mathematical knowledge.[16]

Along the same lines as Benacerraf's epistemological argument, the causal theory of reference can be used against Platonism.[17] If mathematical objects are not stipulated, as the Platonist claims, we should refer to them through

Benacerraf (1973).

[16]When I talk about 'mathematical Platonism' from now on, I will refer to it as 'Platonism'.

[17]See Kripke (1980). Roughly speaking, according to the causal theory of reference, there are two ways to fix reference: by dubbing or by description. In the former case, reference is fixed by perceiving objects, e.g. this dog is called 'Fido', whereas in the latter, reference is fixed via stipulation, e.g. 'Italy' is the area that borders France, Switzerland, Austria and Slovenia. In the case of proper names, reference occurs by a causal chain that stretches back to the dubbing of the object with that name.

sense organs. But given the inertness of mathematical objects, it is impossible to explain how we can refer to them successfully. As a consequence, Platonism makes the way we refer to mathematical objects inexplicable.

Both versions of Benacerraf's argument are considered problematic nowadays. However, Benacerraf's argument has the benefit of highlighting the epistemological weakness of abstract objects, although within a restricted linguistic and epistemological background. I will elaborate on the causal theory of knowledge, but let me first note that Benacerraf's argument is not necessarily a step toward anti-realism. In fact, it is an argument for anti-Platonism. Roughly speaking, a realist might accept that mathematical entities exist in space and time; in this case, the interaction with mathematical entities would not be mysterious.[18] I am not, of course, saying that Benacerraf's argument is useless for anti-realists. I merely think that in order to serve as an argument for anti-realism, Benacerraf's argument should be supported by additional assumptions, for example by the claim that mathematical entities do not exist in space and time, nor in our mind.

Let us examine Benacerraf's argument starting with the least controversial premise: the fact that we do not causally interact with abstract objects. Some philosophers, such as Gödel, argue that abstract objects are causally related to us.[19] According to Gödel, epistemic agents interact with mathematical *abstracta* via mathematical intuition, which makes possible mathematical knowledge by connecting the epistemic agents to abstract objects, such as perception connects epistemic agents to physical objects.

Gödel's analogy is rather obscure. We can explain how perception works through chemical stimulation of the sense organs, whereas we do not have an account of intuition of *abstracta* that is compatible with cognitive sciences. In fact, it is not clear how to provide such an account since abstract objects are outside of space and time.

To overcome Gödel's problem, Penelope Maddy attempted to explain how we can get knowledge of mathematical objects via perception. Maddy's

[18] The most famous advocate of this view is John Stuart Mill.
[19] Gödel (1947).

position is called 'set-theoretical monism', where mathematical intuition is nothing but perception:

> for the monist, all sets have physical grounding and spatio-temporal location, and all physical objects are sets. These manoeuvres produce a radical 'one-worldism' – a reality at once mathematical and physical – that should appeal to philosophers of this stripe.[20]

Notice that Maddy is not supporting Platonism. If mathematical objects are outside of space and time, there is no way they can be perceived. As Balaguer highlights: "Maddy hasn't naturalized platonism at all — she's *abandoned* it.[21]

I have already mentioned how the causal theory of knowledge requires a causal connection between beliefs and their truthmakers. This requirement is captured by the principle of causal knowledge: if an epistemic agent knows that P, then he or she stands in a causal relation to P.[22] It may sound trivial, but standing in a causal relation to P is not a sufficient condition to know that P. If there is a causal connection between an undiscovered particle and us, this connection does not imply that we know the properties of the particle in question. Nevertheless, according to the causal theory of knowledge, the principle is a *necessary* condition to know that P. Hence rejecting such a principle is the easiest way to defend Platonism from Benacerraf's attack.

There are many strategies for dismissing the principle of causal knowledge; but let me focus on the most prominent. I have previously mentioned how Goldman applies the causal theory of knowledge only to empirical contexts. If one adheres to Goldman's original idea, one needs to argue that mathematical knowledge is empirical before turning the principle of causal

[20] Maddy (1990, p. 180).

[21] Balaguer (1998, p. 29). Since Maddy does not support Platonism but monism, I do not need to discuss her approach.

[22] Philosophers refer to the principle of causal knowledge using different terms. Colyvan, for example, calls it 'the eleatic principle'. See Colyvan (2001), Ch. 3.

knowledge against Platonism. Of course, such a response is not sufficient to dismiss every objection to Platonism, because it is addressed to those who believe that mathematical knowledge is empirical. Certainly, mathematics includes some empirical elements, such as the fact that mathematicians make use of pens, paper and computer software in order to do their job. More importantly, empirical factors can influence mathematicians during their research. However, what is at issue here is not that mathematical research involves some empirical elements, but whether mathematical knowledge is empirical or not. Here I will not address this problem, because the Platonist does not need to argue that mathematical knowledge is empirical in order to undermine Benacerraf's argument, and the solution does not involve any epistemological terms.

Platonists can avoid the problem concerning the nature of mathematical knowledge by advocating the indispensability argument: we ought to commit ourselves to *all* and only those objects that are indispensable to the best scientific theories that we use.[23] If mathematical objects must exist because they are indispensable to our best theories that we use to describe the world, the Platonist does not even need to reject the principle of causal knowledge altogether. He, or she, can allow that the principle works in the empirical sciences, but deny that mathematical knowledge needs causality. And if the principle of causal knowledge cannot be applied to mathematics, then Benacerraf's argument is flawed.

Colyvan supports the indispensability argument and offers some naturalistic reasons for invalidating the principle of causal knowledge.[24] I call them 'naturalistic' because they aim at undermining the principle of causal knowledge through the interpretation of scientific practice, instead of advocating a pure philosophical argument. In a nutshell, Colyvan argues that scientific practice constantly violates the principle of causal knowledge, and

[23] I present the indispensability argument in sec. n. 1.3. I am going to argue that the indispensability argument rises the most serious problem for those who want to distinguish between concrete and abstract objects.

[24] Colyvan (2001).

mentions some case studies where that happened. In this regard, Colyvan presupposes that anti-Platonists would endorse a weak form of naturalism: if a philosophical doctrine is inconsistent with science, we must drop our philosophical misconception and adopt a new one that is consistent with it. If the principle of causal knowledge turns out to be inconsistent with scientific practice, Colyvan argues, the principle must be rejected.

Colyvan quotes several examples from the history of science to show how the principle of causal knowledge does not seem likely to be true. Among them, a nice example comes from the discovery of germanium.[25] Although there had been no causal contact with germanium before 1886, Mendeleeff had noted that there was a gap in the Periodic Table of the Elements, corresponding to the position of germanium. So Mendeleeff had good reasons for believing in the existence of germanium before 1886, because his belief had been founded on a solid background theory, i.e. the Periodic Table of the Elements. If Mendeleeff had endorsed the principle of causal knowledge, the discovery of germanium would have had a temporary setback. As a consequence, the principle of causal knowledge turns out to be inconsistent with scientific practice and must be rejected.

I think Colyvan is right to emphasize the fact that scientists do not always need a causal confirmation before committing themselves to the truth of an empirical hypothesis. Nevertheless, causal interactions play an important role for scientists in confirming, or falsifying, empirical hypotheses. Here are a few examples: the existence of germanium was only confirmed in 1886 by Winkler, when he was able to isolate germanium in a sulfide mineral called argyrodite; the existence of the planet Neptune was proved *only* after Galle detected the planet at the Berlin Observatory; physicists and engineers build accelerators that produce collisions between particles to track down the existence of smaller and smaller objects; and so on. Even if the causal theory of knowledge is wrong, causal interactions remain an essential test-bed for empirical hypothesis.

[25]Colyvan (2001, pp. 44-45).

1.2 Benacerraf revisited: the reliability claim

Benacerraf's epistemological argument against the existence of abstract objects is based on the causal theory of knowledge. I will now consider how to recast Benacerraf's argument without referring to the causal theory of knowledge. In this regard, I will examine Field's version of Benacerraf's argument within the reliabilist theory of justification.

Reliabilism is a doctrine aimed at making precise our common understanding of the term 'justification'.[26] More precisely, reliabilism aims to provide an account of 'justified belief' that is close enough to what people ordinarily mean by that. The goal is to highlight the process of beliefs-formation without using epistemic terms. The idea behind reliabilism is that people are intuitively capable of distinguishing reliable processes of beliefs-formation from unreliable ones even without a theory of knowledge in the background.

According to reliabilism, an epistemic agent is reliable on a certain matter M if he, or she, has mostly true beliefs about M. In this regard, reliabilism highlights the *tendency* for an expert epistemic agent to produce true beliefs. Even an expert epistemic agent can be wrong about his field, of course, but what matters for him, or her, to be considered reliable is the tendency of having mostly true beliefs about M. Expert mathematicians, for example, can be wrong about the truth-value of a mathematical proposition; nonetheless, we consider a mathematician reliable if he, or she, has mostly true beliefs about mathematics.

Reliabilism is a theory of justification. If a belief arises out of a reliable process, then it is justified. This does not require that the belief in question be true — under the assumption that justification and truth are two different matters. As I said, reliabilism grasps the fact that we regard someone as an expert on a particular matter if he, or she, has mostly true beliefs about that matter. Nonetheless, reliabilism may raise some worries for those who

[26]There are many versions of reliabilism but I will refer mainly to Goldman's (1979) view.

look for a stronger theory of justification, since it is not clear how many times an agent must have true beliefs in order to be considered reliable.[27]

Let us examine how Field's argument against Platonism based on reliabilism goes. Field argues that the Platonist cannot explain why expert mathematicians have mostly reliable beliefs about mathematics. For mathematicians to be reliable, a correlation between belief states and mathematical facts must hold.[28] As previously observed, expert mathematicians can be wrong about mathematical facts, but this is not what is at issue. The point is that *if* there is some kind of correlation between mathematicians' beliefs and mathematical facts, philosophers must explain why this correlation does not occur by accident. To accept Field's challenge, Platonists do not even need to commit themselves to the existence of mathematical facts, because the correlation between beliefs and facts can be stated in terms of sentences: expert mathematicians tend to accept true mathematical sentences. The process that leads mathematicians to true beliefs is captured by this principle: most of the time,

The reliability claim: If expert mathematicians accept a mathematical sentence S, then S is true.

Field argues that the Platonist should accept the reliability claim, since it just expresses a non-accidental correlation between mathematical facts, or mathematical sentences, and expert mathematicians' beliefs. If an ordinary person, by some fluke of luck, guesses the truth-value of a mathematical sentence, we do not need to account for how he, or she, got it right — the correlation simply occurred by accident. But expert mathematicians *do not* guess the truth-values of mathematical sentences. This is why we need to explain how they generally get them right.

[27] I will address the problem when I come to consider in what sense Field's argument is a challenge for the Platonist.

[28] Field's argument is presented in several papers. See for example Field (1989, pp. 230–239).

I would like to examine what the minimal requirements for the reliability claim are. To do this, I am going to semi-formalize the reliability claim. *Prima facie*, notice how the reliability claim includes an 'if-then' conditional which, I think, should be interpreted as a material implication in order to avoid some unwanted consequences that I will present in the next paragraph.[29] If this is so, the reliability claim can be rephrased as follows: most of the time,

RC1: $\forall S$[(expert mathematician accepts S) $\rightarrow$ S is true]

where S is any mathematical sentence. Now, consider this case: it might occur that expert mathematicians reject a mathematical sentence that turns out to be true afterwards. An interesting example is perhaps the case of the well-ordering theorem. At the time when Zermelo proved the well-ordering theorem, there were a few mathematicians (e.g. Lebesgue) who rejected the well-ordering theorem on the basis of the fact that Zermelo had used the axiom of choice in his proof.[30] Does this case falsify the reliability claim? It does not, if the conditional is material — i.e. if the antecedent of RC1 is false, then RC1 is trivially true. A way of rejecting the reliability claim is to argue that expert mathematicians have mostly false beliefs about mathematics, but this move is rather implausible. Another possible strategy is to argue that 'mostly' is vague but, as showed previously, even this condition can be dropped: the only requirement is that there must be a *non-accidental* correspondence between expert mathematicians' beliefs and mathematical facts.

The reliability claim involves the truth predicate. What does 'truth' mean in this context? According to Field, the reliability claim only requires the deflationistic conception of truth.[31] There is a large amount of literature

[29] Linnebo (2006) also argued that the material conditional is the minimal requirement to run Field's argument.

[30] According to the well-ordering theorem, every set can be well-ordered; and a set x is well-ordered iff every non-empty subset of x has a least element.

[31] "I argue that in doing this we need not rely on any notion of fact, nor even on any notion of truth beyond a thoroughly disquotational one." See Field (1989, p. 26).

on what 'deflationism' is supposed to mean, but, generally speaking, every deflationist regards 'true' as a predicate that occurs in any instance of the biconditional schema: "S" is true if and only if S, where S is a sentence. Roughly speaking, deflationists argue that the concept of truth is fully captured by every instance of the biconditional schema, and thus the concept of truth cannot be defined by appealing to another property. So the concept of truth is primitive.[32]

In Field's view, deflationism should take into account the truth predicate purely disquotationally: that is, deflationism should not depend on (1) any non-disquotational concept of truth; (2) it should not involve any verificationist account;[33] (3) it should not need truth-conditions.[34] In short, the concept of truth is just a useful simplification device to express infinitely many conjunctions and disjunctions that could not be formulated otherwise. If there were only a finite number of sentences, Field argues, we could build a theory of meaning without referring to truth at all.[35]

According to Field, the concept of truth does not require truth-conditions. For this reason the reliability claim need not the correspondence theory of truth to be formulated. Moreover, because mental states are supposed to be finite, the reliability claim does not require the concept of truth either:

> I have denied that a 'mathematical realist' need be committed to a correspondence theory of truth for mathematical sentences. Nonetheless, I believe that even on this weak construal of realism, we should not be realists about mathematics. [...] The problem

[32]Deflationism raises several important issues about the role of truth, if any, in language: if 'S' is true means S, is truth simply redundant? Is true a property of sentences as man and red are properties of certain objects? And so on.

[33]Verificationism is defined by Field broadly: 'the verification conditions of a type of utterance might be given by the class of sensory stimulations that would or should lead to the acceptance of an utterance in that class'. See Field (1994b, p. 249).

[34]Field regards truth-conditions as correlations between belief states, or mental states, and sentences. See Field (1994a, p. 408).

[35]See Field (1994a, p. 406). Field's deflationism aims at capturing any legitimate concept of truth.

> can be put without use of the term of art 'know', and also without talk of truth (though talk of disquotational truth enables us to give a more snappy formulation of it).[36]

If Field is right, the reliability claim can be finally reformulated as follows: most of the time

RC2: $\forall S[(\text{expert mathematician accepts } S) \rightarrow S]$

1.2.1 Field's argument against Platonism

Let us see Field's argument against Platonism. If Platonists believe that some kind of correlation between expert mathematicians' beliefs and mathematical facts must occur, then:

(P_1) Platonists should explain how expert mathematicians are reliable.

(P_2) Any explanation of reliability can be either casual or non-casual.

(P_3) Platonists are not able to provide such explanations.

(C) Platonism is not justified.

Field does not seek to advance a conclusive argument against Platonism. In point of fact, Field does not argue that Platonism is false but, more likely, that it is not justified unless Platonists are able to provide a plausible account of reliability. In this regard, Field draws an analogy between the challenge faced by Platonists and an hypothetical situation in a remote Nepalese village:

> It is rather as if someone claimed that his or her belief states about the daily happenings in a remote village in Nepal were nearly all disquotationally true, despite the absence of any mechanism to explain the correlation between those belief states and the happenings in the village.[37]

[36] Field (1989, p. 60).
[37] Field (1989, pp. 26-27).

Suppose that, for instance, I believe that the sun sets because there is an entity that moves it every day. Even in the absence of a better explanation, you should be suspicious about my explanation unless I can account for how the correlation between that mysterious entity and the sun is supposed to occur. If I am not able to provide such an explanation, my belief is simply not justified.[38] This is not to say that my belief is false, since justification and truth are different from one another. Perhaps what I believe is true, perhaps it is not.

In contrast to Benacerraf's argument, Field does not appeal to the causal theory of knowledge but employs the concept of explanation. The problem, of course, is to figure out what kind of explanation Field refers to. When Field mentions 'explanation', it seems that he actually means 'correlation'. If this is so, mathematical *abstracta* cannot correlate belief states to mathematical facts, since abstract objects are a-causal.[39] But what about non-causal explanations? Are they possible? Here is Field's answer:

> Perhaps then some sort of non-causal explanation of the correlation is possible? Perhaps, but it is very hard to see what this supposed non-causal explanation could be.[40]

As far as I know, Field does not spell out what he means by 'non-causal explanation'; that is, what sort of non-explanation the Platonist should provide. So what is a non-causal correlation between belief states and mathematical facts supposed to be? In other words, the problem we are facing is as follows:

Field's challenge: how can Platonists account for the fact that the reliability claim usually holds?

[38]Even if I can provide an explanation, you may be still unconvinced. Explaining the correlation is a necessary but not sufficient condition for a belief to be justified.

[39]Platonists may refuse to explain the correlation in terms of causality for the same reasons as they reject Benacerraf's argument. Kasa (2010), for example, argues that Field's argument is similar to Benacerraf's argument.

[40]Field (1989, p. 231).

Here is a trivial solution: mathematicians become reliable by working hard, that is, by learning over time how to manipulate mathematical statements correctly. For example, mathematicians become expert on set theory after many trials and errors by learning basic algebraic operations, transfinite induction, forcing, and so on. After years of studying, they finally become expert, and when they accept a statement of set theory P, then P usually follows from the axioms of set theory.

The trivial solution to Field's challenge seems plausible. In addition, notice how the trivial solution has the advantage of distinguishing between expert mathematicians and someone who simply guesses the truth-value of mathematical statements. Expert mathematicians know which statements follow from which mathematical theories, whereas the lucky ordinary person does not. But imagine a supercomputer that proves a sentence S only if S follows from the axioms of a mathematical theory that has been accepted by expert mathematicians.[41] Because the supercomputer can prove that S follows from a mathematical theory, it is absolutely reliable. But here lies a problem that the trivial solution does not address: why do expert mathematicians accept certain mathematical theories over others? The reliability claim can be presented in that new form, and the trivial solution does not address it.

To sum up, the matter of providing for a non-causal explanation of the reliability claim is not a clear-cut one. But this is how I intend to interpret Field's challenge: the Platonist should tell us some kind of story that must bridge the gap between belief states and abstract objects. Otherwise, the existence of abstract objects is not justified. The story does not need to be based on the causal theory of knowledge but needs to correlate belief states with *abstracta*. So long as the Platonists is unable to provide such a story, the existence of abstract objects is not justified.

[41] This example is quoted from Linnebo (2006).

1.3 The indispensability argument(s)

If mathematical objects exist and are outside of space and time, it is clearly hard to provide a good epistemology for them. Does this mean that the existence of abstract objects cannot be justified *at all*? The indispensability argument aims at overcoming the problem as follows: if mathematical *abstracta* are indispensable to our best scientific theories, then the existence of such *abstracta* is justified independently of any epistemological concern. And because the main problem for Platonism is the lack of a good epistemology, the indispensability argument is the first and foremost argument for Platonism.

The indispensability argument is mainly due to Quine and Putnam, but it would be more accurate to refer to it as a class of (fairly) homogeneous arguments. Indeed, the indispensability argument is often used to force ontological commitment to mathematical abstract objects, but it can be also used to sustain semantic realism.[42] In addition, some philosophers argue that nominalism is compatible with some versions of the indispensability argument.[43] However, in this section I will regard the indispensability argument only as an argument to endorse Platonism.

The indispensability argument forces us to commit ourselves to the existence of mathematical entities. Here is how the indispensability argument for Platonism goes:[44]

(P_1) We ought to have ontological commitment to all and only those objects that are indispensable to our best scientific theories.

(P_2) Mathematical objects are indispensable in that regard.

(C) We ought to have ontological commitment to mathematical objects. Therefore, Platonism is true.

[42]According to semantic realism, the truth-value of mathematical statements transcend knowability. Mathematics can be objective even without ontological commitment to mathematical objects. Putnam (1967, pp. 69-70), for example, endorses semantic realism.

[43]See for example Azzouni (2009).

[44]Colyvan (2001) presents a thorough analysis of the indispensability argument.

Contemporary scientific theories, like physics, provide the best description of the world. According to Quine, our best scientific theories are the reference point to settle the ontological dispute. In other words, in order to figure out what exists, we ought to look to what our best scientific theories quantify over. For example, if in contemporary physics we quantify over elementary particles, then such objects exist — until otherwise proven. Yet our best scientific theories do not quantify just over physical objects, but also over mathematical *abstracta* such as numbers, groups, topological spaces, and so on. Unless abstract objects are dispensable, so it is claimed, we ought to commit ourselves to *abstracta* as we do for physical objects that are indispensable to our best scientific theories.

The indispensability argument is very powerful, because it forces ontological commitment to objects that are outside of space and time on the basis of empirical considerations. Although mathematical *abstracta* are epistemically , we ought to include them in our ontology to formulate our best scientific theories of the physical world: pragmatic reasons overcome epistemological demands altogether.

The force of the indispensability argument depends on the fact that empirical theories are confirmed by experience. If mathematics were employed within theories that are not confirmed by experience, the indispensability argument would be seriously undermined. Nonetheless, it is important to realize that, strictly speaking, empirical confirmation is unnecessary in order to run an indispensability argument. In this regard, Resnik developed a variant of the indispensability argument without referring to empirical confirmation.[45] That is, even if empirical theories are not confirmed by experience, or even false, the indispensability argument can still be used to argue for the existence of mathematical entities. Here is Resnik's argument:

(P_1) Scientists assume the existence of mathematical objects and the truth of much mathematics.

[45] See Resnik (1995, pp. 169-170).

(P_2) These assumptions are indispensable for scientists, because scientific laws could not be derived without taking mathematical sentences to be true.

(C) We are justified in drawing scientific conclusions only if we take the mathematics used in science to be true.

Resnik's argument does not refer to empirical confirmation, and perhaps it is less persuasive than the one I presented previously. However, it highlights the basic premise of the indispensability argument: scientists presume that the mathematics employed in their deductions is *true*. This is why Azzouni argues that the kernel of the indispensability argument is as follows:[46]

(P_1) Certain statements that quantify over mathematical entities are indispensable to science.

(C) Those statements are true.

On this view, truth is an unavoidable premise of the indispensability argument, and it is even more important than empirical confirmation. This is why I am going to argue later on that the strategies based on the rejection of confirmation holism cannot disarm the indispensability argument but, at best, can only make it less effective. Roughly speaking, according to confirmation holism, the empirical success of that theory confirms not only the existence of physical entities involved, but also mathematics, because mathematics is part of the corporate body.[47]

Here is an important issue that I would like to introduce. Every version of the indispensability argument that I mentioned refers to the truth of mathematical statements, or to the existence of mathematical entities. However, none of these arguments is explicit on what the nature of mathematical entities is supposed to be. At best, the indispensability arguments

[46] Azzouni (2009, p. 139). Azzouni calls it an enthymemic blueprint version of the indispensability argument.

[47] I examine confirmation holism in sec. n. 3.3.

tell us that we ought to assume the existence of *some kind of* mathematical entities in order to formulate our best scientific theories. And to be fair, Azzouni does not even refer to the *existence* of mathematical entities because, according to Azzouni, only quantification and truth are necessary to run the indispensability argument.[48]

It is notable that the indispensability arguments do not tell us about the nature of mathematical entities. In other words, they are not explicit on whether mathematical entities are *abstracta*, *concreta*, fictions, artifacts, or what. In addition, the indispensability arguments are also indeterminate on what kind of mathematical objects we ought to be committed to: do we need to have ontological commitment to numbers? Sets? Or maybe, categories? At this stage, it is only necessary to observe that we need more assumptions to settle these matters.

1.3.1 Three forms of naturalism

I would now like to examine the first premise (P_1) of the former indispensability argument that I presented earlier: we ought to have ontological commitment to *all* and *only* those objects that are indispensable to our best scientific theories. P_1 contains a couple of adjectives that highlight two important assumptions: 'all' refers to confirmation holism, and 'only' involves naturalism. I will discuss both confirmation holism and naturalism during the remainder of this section, but let me first introduce what I mean by 'naturalism'.[49]

Naturalism rejects first philosophy. Roughly speaking, first philosophy is the idea that philosophy comes before, and is prior to, science. According to naturalism, philosophy cannot challenge the methods of scientific inquiry and, more specifically, it cannot address the ontological problem by itself. There are several types of naturalism, but many can be sorted according

[48]This is because Azzouni distinguishes between quantification and ontological commitment. See sec. n. 3.1.2.

[49]As previously stated, I will examine confirmation holism in sec. n. 3.3.

what role ontology plays in our picture of the world. I will refer to three different forms of naturalism exemplified by Carnap, Maddy, and Quine.

Carnap is skeptical about the possibility of addressing the ontological problem according to both internal and external standards of justification.[50] Given a linguistic framework, i.e. a collection of inference rules and assumptions, an existential question such as 'are there x?' is internally justified if it is possible to answer that question according to the inference rules within the framework. For instance, 'are there prime numbers?' is justified within the framework of arithmetic because, given certain inference rules, we can determine whether any natural number is prime or not. On the other hand, Carnap argues that the ontological problems that take the form of 'does x exist?' are supposed to be addressed independently of any given framework, and thus they cannot be justified according to any internal standards of justification. In addition, Carnap argues that internal existential questions do not force any ontological commitment: 'there are prime numbers' is just a fruitful convention that we adopt in order to achieve practical goals. In point of fact, whether or not a framework is useful for practical reasons is the only external question that Carnap is willing to admit. Thus, ontological problems are generally regarded by Carnap as pseudo-scientific questions. In other words, Carnap's naturalism implies the abandonment of the ontological problem.

Maddy's naturalism is more moderate than Carnap's. According to Maddy, ontology is continuous to science, although mathematics and empirical sciences have their own standard of justification. From this viewpoint, naturalism is

> 'the conviction that a successful enterprise, be it science or mathematics, should be understood and evaluated on its own terms, that such an enterprise should not be subject to criticism from, and does not stand in need of support from, some external, sup-

[50] See Carnap (1956).

posedly higher point of view'.[51]

In contrast to Carnap, Maddy argues that the ontological problems can be addressed by internal standards of justification. Mathematicians, for example, have their own internal standards of justification, and according to them the question of whether or not mathematical objects exist is straightforward: look at the axioms of whatever mathematical theory and ask yourselves what kind of objects the theory assumes. Because ZFC assumes the existence of certain sets, then it follows that we are ontologically committed to mathematical objects, i.e. sets. In Maddy's view, internal standards of justification are sufficient to settle the ontological disagreement. As a consequence, mathematicians do not need *any* external justification: neither from philosophy, nor from empirical sciences. This is because the fact that mathematics is indispensable to our best scientific theories is not necessary in order to have ontological commitment to mathematical objects. Therefore, Maddy's naturalism implies the rejection of the indispensability argument, but the ontological problem is not considered pseudo-scientific.[52]

According to Quine, the ontological problem cannot be solved by employing internal standards of justification, in contrast to Maddy. Quine argues that if a theoretical hypothesis is adopted on the basis of practical grounds, we have strong evidence for its truth and, as a consequence, for the existence of the entities employed to formulate that hypothesis. I should like to point out that Quine does not endorse certain naive forms of scientism: the fact that an hypothesis is confirmed by experience is the best we can ever hope for in determining what exists. In contrast to what Maddy claims, internal standards of justification are not sufficient for Quine: the existence of mathematical objects can be justified by looking at our best *empirical* scientific theories.

[51]Maddy (1997, p. 161).

[52]I will examine Maddy's objection to the indispensability argument in sec. n. 3.3.1. In addition, I reconsider Maddy's rejection of first philosophy in sec. n. 3.1.4.

1.3.2 Dodging the indispensability argument: an outline

We saw that the indispensability argument forces us in committing ourselves to the existence of mathematical objects, and that it is the most powerful argument to which the Platonist can resort. In the chapters that follow I will examine the most important strategies opposing Platonism that are based on the rejection of the indispensability argument. I will split such strategies into two classes: hard roads and easy roads. This distinction is made by Colyvan, although I distance myself in part from his view.[53] Whereas Colyvan refers to hard roads and easy roads to *nominalism*, I will talk about hard roads and easy roads to *anti-Platonism*. This is because I think that Colyvan's distinction faces two problems: first, it regards Field's program as the only hard-road, and secondly some easy-roaders do not consider themselves nominalists. Despite this, I think that Colyvan has an interesting point, because there are basically two strategies of denying the existence of abstract objects: the first approach requires the paraphrase of mathematical or scientific sentences, whereas the second approach does not demand any paraphrases. Hence I will recast Colyvan's original distinction by calling the first approach 'the hard road to anti-Platonism', and the second one 'the easy road to anti-Platonism'.

Before I examine hard and easy roads, let me sum up the strategies I am going to consider. In the next chapter I will analyse the following hard roads to anti-Platonism:

- Showing how we can dispense with mathematical objects from physics (Field).
- Reformulating mathematics within a constructivist account of quantifiers (Chihara).
- Reconstructing mathematics without referring to mathematical objects but possible structures (Hellman).

[53] See Colyvan (2010).

Secondly, I will consider the following easy roads to anti-Platonism:

- Rejecting Quine's criterion of ontological commitment (Azzouni).
- Showing that it makes no difference to how matters stand in relation to the physical world if mathematical propositions are false (Yablo).
- Rejecting confirmation holism (Maddy/Sober).

Lastly, I will examine a less debated option in contemporary metaphysical literature: agnosticism. Indeed, my final goal is to support a specific agnostic view on the metaphysical debate between Platonism and anti-Platonism. As will become clear later on, my own position is an easy road because it does not require any paraphrasing of mathematical statements, nor does it demand to endorse nominalism. My own agnosticism is a genuine anti-Platonist position in the sense that I do not commit myself to the existence of abstract objects.

1.4 Burgess and Rosen's challenge to nominalism

Before concluding my discussion of Field's nominalization program, I would like to present some general objections to nominalism. Nominalists in the philosophy of mathematics argue that mathematical *abstracta* do not exist. In this regard, Burgess and Rosen present a few arguments called 'scientific', because they arise from scientific practice, in order to undermine nominalism.[54] Burgess and Rosen argue that the literal meaning of mathematical existential statements such as 'there are infinitely many natural numbers' is expressed by Platonism, and this is commonly accepted by the scientific community. The reason is as follows: 'there are infinitely many natural numbers' is true only if natural numbers exist, and because scientists know that such objects are not *concreta*, then natural numbers must be *abstracta*. In Burgess and Rosen's view, either nominalists are able to show how scientists

[54] See Burgess and Rosen (2005) and Burgess (2004).

could dispense with mathematical objects from science, or they can change the literal meaning of mathematical sentences commonly accepted by the scientific community. But either way, nominalists interfere with the scientific community and, as a result, nominalism turns out to be inconsistent with scientific practice.

According to Burgess and Rosen there are two varieties of nominalism: revolutionary and hermeneutic. This distinction is further split into two more types: revolutionaries can be either 'naturalized' or 'alienated'; hermeneuticists are either on 'content' or on 'attitude'.

Let us first examine revolutionary nominalism. Revolutionary nominalists claim that mathematicians are wrong when they refer to mathematical objects, because mathematical objects simply do not exist. On the one hand, naturalized nominalists aim at showing how mathematics is dispensable from our best scientific theories. Field's reconstruction of Newtonian physics is the typical example of naturalized nominalism.[55] However, because contemporary scientists employ mathematical objects to make prediction about the physical world, Burgess and Rosen argue that naturalized nominalism is incompatible with the scientific practice. On the other hand, "alienated" nominalists appeal to philosophical arguments in order to refute the existence of mathematical objects. These philosophers are "alienated" because they favor philosophy over science. But scientists can simply regard Field's argument based on the reliability claim or Benacerraf's argument as skeptical challenges. It is as if someone demanded an argument for perception to be reliable before they would accept that physical objects exist. Thus, when nominalists argue that mathematical objects do not exist because we lack epistemic access to them, Burgess and Rosen reject such a demand by tagging it as skeptical.[56]

[55] I examine Field's program in sec. n. 2.1.

[56] In addition, Burgess and Rosen endorse Lewis' position (1991, p. 53), according to which epistemologists cannot argue against mathematicians, because what mathematics has accomplished over time is definitely more impressive than what philosophers have achieved. This long quote from Lewis (1991, p. 59) is self-explanatory: "How would you

In Burgess and Rosen's view, revolutionary nominalism is overall inconsistent with scientific practice. Are they right? In point of fact, Chihara replies to Burgess and Rosen by pointing out that nominalism aims at building the big picture of the world, in which all aspects of knowledge are coherent with one another:

> In this search for the big picture, *coherence* is an essential ingredient [...] Take the philosophy of language, for example. Here, we seek an understanding of the nature of language and our mastery of language that is consistent with our general scientific, epistemological, and metaphysics views [...] In general one would not expect a contemporary philosopher's account of language to contradict any of our prevailing views of science and scientific knowledge without very compelling reasons.[57]

Abstract objects are out of the picture because they lack good epistemology. However, Burgess and Rosen may reply that, despite the big picture of the world, nominalistic reconstructions are more complicated than the theories where quantification over mathematical *abstracta* is allowed. In other words, nominalistic reconstructions are fruitless in the eyes of the scientific community. But why should nominalistic reconstructions be useful in that regard? It is clear, I think, that nominalists do not wish to compete against mathematicians or physicists. A nominalist like Field, for example, recognizes that mathematics is useful to our best scientific theories even though there are no abstract objects. Generally speaking, nominalists do not claim that empirical sciences plus mathematics provide a wrong picture of the world, but that Platonism, as *philosophical* doctrine, is false. Burgess and Rosen's argument arises from two questionable premises: existential

like the job of telling the mathematicians that they must change their ways, and abjure countless errors, now that philosophy has discovered that there are no classes? Can you tell them, with a straight face, to follow philosophical argument wherever it may lead? If they challenge your credentials, will you boast of philosophy?".

[57]Chihara (2005, p. 490).

mathematical statements can be true only if there are abstract objects and, moreover, that such a conception is part of our scientific world-view.

According to Burgess and Rosen, there are two types of hermeneutic nominalists: those who focus on the content of mathematical statements, and those who focus on mathematicians' attitudes towards mathematical statements. According to the former, nominalism is compatible with empirical sciences and mathematics, because it is possible to interpret the content of mathematical statements without postulating abstract objects. Burgess and Rosen argue that because the existence of abstract objects is implied by the literal interpretation of existential mathematical statements, content hermeneuticists change the literal meaning of such statements. As a result, content hermeneuticists are in opposition to the mathematical community, which endorses the literal interpretation of existential mathematical statements. On the other hand, those hermeneuticists who focus on attitude, claim that mathematicians can pretend that mathematical objects exist — whereas they do not. However, Burgess and Rosen argue that no expert mathematicians warn people not to believe in what existence mathematical sentences literally say, i.e. certain abstract objects exist. Expert mathematicians *do* recognize the literal meaning of mathematical sentences, and *do* take it for granted.

I will argue that mathematical statements, taken literally, do not force any commitment to abstract objects. The mathematical community does not believe that the literal meaning of mathematical existential statements is what the Platonist adheres to. Moreover, I intend to achieve my goal without endorsing nominalism: either revolutionary or hermeneutic.

Chapter 2

Removing Plato's Beard: Hard Roads to anti-Platonism

2.1 Physics without numbers

Nominalism can be defined in two different ways, according to what commitment they take on what exists: on the one hand, nominalism is the negative view that there are no abstract objects; on the other hand, it is the positive view that only concrete, or physical, objects exist. The early nominalistic strategies in the philosophy of mathematics were based on the conviction that mathematics ultimately refers to physical objects, such as linguistic tokens or mental states.[1] In contrast to those strategies, Field's approach aims at undermining the indispensability argument by showing that the mathematics we use in our best scientific theories is actually dispensable.

Field's claim may be rather controversial at first glance. Is it really possible to dispense with mathematics whilst contemporary scientists employ it in order to make empirical deductions? Regarding this, I would like to emphasize that Field does not dispense with mathematics, but with mathematical

[1]For example, Leonard and Goodman (1940) or Goodman and Quine (1947).

abstracta. In fact, it is a common misconception about Field's program to regard his nominalism as a sort of revisionism of scientific practice because mathematicians would assume that quantifiers range over abstract objects.[2]

Field does intend to revise mathematical, or scientific, practice: he wants to prove that each statement of classical mechanics that quantifies over mathematical objects can in principle be nominalized. Field's goal is merely philosophical in the sense that he wants to explain the success of mathematics without presuming the existence of mathematical objects. That is, how mathematics can be applied to the physical world if nominalism is true.

It is important to understand what Field means by 'nominalism'. According to Field, a theory is nominalistically statable if it does not overlap in non-logical vocabulary with a mathematical theory.[3] This definition is rather technical, so let me spell it out as follows: a nominalistic sentence is an assertion that does not refer to mathematical objects but only to the physical world. In other words, a nominalistic sentence is a sentence that talks about only the physical world. However, because our best scientific theories involve mathematical *abstracta*, such theories are not nominalistic in Field's sense. This is because physical theories contain mixed assertions that involve both mathematical and physical objects. Consider for example Newton's law of universal gravitation $F = G_0 \frac{M_1 M_2}{r^2}$. Even though the formula describes physical events, numbers are employed in order to represent both mass and force. And since that mixed formula involves numbers, it seems that we ought to commit ourselves to the existence of numbers for that formula to be true.

Perhaps a nominalist could argue that physical theories do not need to contain true mixed assertions in order to describe the physical world. What physical theories *do* need is consistency: given an empirical phenomenon O, a physical theory cannot predict both O and $\neg O$. However, as Field himself notes, consistency cannot be used instead of truth in the context

[2]As showed in the previous chapter, Burgess and Rosen misconceive Field's program, i.e. they regard it as revolutionary nominalism.

[3]See Field (1980, p. 108), note n. 8.

of the empirical theories because, if this was so, false predictions about the physical world would be possible.[4] For example, it would be easy to imagine a consistent physical theory that entails the existence of twelve planets in the Solar System. In other words, the consistency of a successful physical theory is not sufficient by itself to account for its predictive power. Our best scientific theories, such as contemporary physics, must be true in order to describe the world. But according to Field, scientific theories can be true despite the fact that existential mathematical statements are false. Mathematics turns out to be just a useful tool to simplify calculations.

2.1.1 Truth and conservativeness

Here lies a problem that I should like to consider. If existential mathematical statements are false, how can mathematics be useful and make reliable predictions about the physical world? After all, mathematics cannot be successful by accident. Even if Field is right to claim that mathematics does not need to be true to be useful, we should nonetheless explain why scientists are willing to adopt a false theory and still able to account for physical phenomena. Field's answer to this problem is called 'fictionalism': although there are no abstract objects, if we regard 'truth' as 'truth in the of story mathematics', existential mathematical statements are still considered true. For example, a mathematical sentence such as 'there are natural numbers' is true iff it is true in the story of mathematics.

Since mathematics is just a story for the fictionalist, some philosophers could raise a faulty analogy between mathematics and stories like novels. Consider the sentence 'Robin Hood steals money from the rich'. This sentence is literally false, since Robin Hood does not exist, but it is nonetheless true in Robin Hood's legends. Here is the analogy with mathematics: 'there are natural numbers' is literally false, because there are no numbers; but it is still true in the story of mathematics.[5] But even if mathematics is just a

[4] See Field (1982).

[5] It seems to me that an analogy between fictionalism and formalism can be drawn. The formalist argues that mathematics is a game. So what is the difference between games

story, mathematics is still different from novels and legends. Novels do not have predictive power: mathematics does.

I would like to highlight that Field does not need to stress the difference between mathematics and novels (or legends). In fact, Field should explain how mathematics can be applied to the physical world if mathematical existential statements are false.[6] Roughly speaking, this is Field's answer: if a physical theory, i.e. mathematics plus physical assertions, is nothing but a conservative extension over a nominalistic body of assertions about the physical world, it is possible to get a purely nominalistic physics that does not refer to mathematical objects. In this regard, Field states the following conservativeness principle:

The conservativeness principle: A mathematical theory M is conservative if and only if for any assertion A about the physical world and any body N of such assertions, A doesn't follow from $N + M$ unless it follows from N alone.[7]

Roughly speaking, the conservativeness principle states that we can dispense with mathematical objects *if* the physical consequences that we derive by using mathematics are the same as those we can derive from a body of physical assertions without mathematics. Of course, it is one thing to state the conservativeness principle, another is to prove that the principle is true. But Field provides an ingenious nominalization of Newtonian mechanics and theory of gravitation where mathematics is just a useful device for simplifying calculation and, as a result, mathematics need not carry any ontological weight in classical physics. If Field's approach really works, it provides an hard road to nominalism for classical mechanics.

I would now like to emphasize that conservativeness is stronger than both consistency and truth. According to Field:

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and stories? The fictionalist can argue that mathematics is false whereas, according to formalism, mathematical statements lack truth-values.

[6] Platonists, for their part, must explain how mathematics can be applied to the physical world if mathematical objects are outside of space and time.

[7] Field (1989, p. 58).

> Unlike consistency, conservativeness does not follow from truth; our anti-realist, then, is not really substituting a weaker goal in place of the realist's goal of truth, he or she is substituting a different goal. [...] Conservativeness might loosely be thought of as 'necessary truth without the truth'.[8]

Despite the fact that truth does not entail conservativeness, consistency is similar to conservativeness.[9] Consider an analogy between the role of conservativeness in Field's nominalism and the one of consistency in Hilbert's program. According to Hilbert, the finitary fragment of elementary number theory is certain, whereas the ideal fragment might be considered problematic. After formalizing a mathematical theory into an axiomatic system, Hilbert intends to provide a finitary (metamathematical) proof of the axiomatized mathematics. Similarly, the 'certain' part of Field's nominalization is the nominalistic body of assertions, whereas the 'ideal' part is represented by mathematics.[10] Nevertheless, it is important to stress a difference between Hilbert's program and Field's: whereas Hilbert justifies the ideal part of mathematics by finitary methods, Field shows that mathematical objects are dispensable by proving that mathematics is conservative. Again, mathematics is not useless for Field: it is still indispensable to making calculations easier to handle. More precisely, Field's aim is to show that mathematics does not need to be true to be employed in physics: it needs to be conservative.

Field argues that good (i.e. applied) mathematics is always conservative.[11] But what about physics? Does physics need to be conservative to be

[8]Field (1989, p. 59).

[9]Indeed a theory is consistent iff it has a model, i.e. there exists an interpretation under which all theorems are *true*. For this reason, consistency follows from truth.

[10]Urquhart (1990) points out how such a comparison is rather inaccurate, because that it is not clear what the metaphysical and epistemic status of the finitary fragment of Hilbert's program is. For example, according to Hilbert, the objects of the finitary fragment of number theory are numerals, but they are neither physical nor mental constructions. See Hilbert (1926). In contrast to Hilbert's view, Field distances himself from finitism.

[11]"Good mathematics is conservative; a discovery that accepted mathematics isn't con-

good? A conservativeness principle for physics may be as follows:

The conservativeness principle (in physics): if M is a physical theory, and N a collection of assertions about observables, one does not get any more conclusions from N+M then one does from N alone.

Physical theories do not need to be conservative in that sense. In point of fact, conservativeness is generally a *bad* requirement for physics. Suppose that one believes that unobservables exist, like Field does. Because unobservables *do* have causal power, they are nominalistically acceptable. In other words, Field does not need to dispense with unobservable.[12]

2.1.2 Dispensing with mathematical objects in physics

In *Science without Numbers*, Field proves that mathematics is conservative.[13] The conservativeness principle requires a nominalistic body of assertions to operate on, whilst the representation theorem provides the link between mathematical objects and their nominalistic counterparts.[14] According to the representation theorem, there is a function (isomorphism) that preserves the structure from the points of space-time to the set of quadruples of real numbers. In other words, the representation theorem states that we can translate every nominalistic statement into a statement about abstract objects, such that every statement about space-time turns out to be equivalent to its abstract counterpart. Roughly speaking, we can continue talking about space-time without referring to real numbers. In

servative would be a discovery that it isn't good." Field (1980, p. 13).

[12]There are non-mathematical abstract entities that are postulated by physics. A perfect gas is an example in that regard. However, the nominalist does not need to dispense with perfect gases, because they are explicitly *postulated* to make calculations easier to handle. In other words, scientists are aware that perfect gases do not exist.

[13]Field has two separate procedures for showing that mathematics is conservative: he employs set theory plus inaccessible cardinals or, alternatively, Field proves that if standard set theory is consistent, then standard set theory is conservative. See Field (1980, pp. 16-19).

[14]See Field (1980, p. 27).

addition, because Hilbert's representation theorem requires an isomorphism between physical space and $\mathbb{R}^4$, physical space must have the same structure of $\mathbb{R}^4$. In other words, Field ought to commit himself to the existence of many uncountable physical objects.[15] This would be problematic for nominalists who endorse finitism, such as Goodman and Quine, but not for those who reject finitism, like Field does.

Let us examine how Field intends to nominalize Newtonian classical mechanics. Field's approach requires the nominalization of Euclidean geometry; that is, it requires dispensing with real numbers in Euclidean geometry. This is because the scientific treatment of physical space is represented by the set of real numbers. In this regard, Field appeals to Hilbert's axiomatization of geometry in order to dispense with real numbers by replacing the statements that refer to real numbers with synthetic geometrical relations, such as 'congruence' and 'collinearity'. For example, if x, y, z are space-time points, we say that 'y is collinear to x and z' if y is a point on the line-segment whose endpoints are x and z; 'xy is congruent with zw' when the distance from point x to point y is the same as the distance from z to point w. Similarly, it is possible to nominalize every predicate that is necessary to recover plane and solid geometry. Synthetic predicates are employed in the treatment of the four-dimensional hyperspace of kinematics and dynamics. Ultimately, classical mechanics and Newtonian gravitation theory are fully reconstructed by including other predicates such as simultaneity, mass density and gravitational potential.[16]

Hilbert's axiomatization of Euclidean geometry might be problematic for the nominalist, because it requires that first-order variables range over points, lines, and spaces, whereas second-order variables range over sets of points, sets of lines, and sets of spaces.[17] If quantification over sets is allowed, as Hilbert does, it seems that the nominalist should assume the existence of sets. So here lies a question for Field: how can the nominalist eventually

[15] Field addresses this problem in (1980), Ch. 4.
[16] See Field (1980), Ch. 8.
[17] See Field (1980, p. 37).

dispense with sets? Field's solution is to replace standard second-order logic with a suitable nominalistic surrogate: Goodman's second-order logic. By this way Field can replace sets with mereological sums, which are supposed to have causal power — whereas sets do not. As a result, Field can operate on (concrete) mereological sums of regions of space-time, instead of referring to sets of regions of space-time. In the end, Field's reconstruction does not give up second-order logic but

> it involves what might be called the complete logic of the part-whole relation, or the complete logic of Goodmanian sums, and this is not a recursively axiomatizable logic. To clarify this, note that the theory as I've suggested it be written is still a second-order theory, that is, it still involves second-order logic [...] and because also we haven't invoked variables for functions or for predicates of more than one place, no nominalistically dubious entities need be invoked to serve in the range of the second-order quantifiers.[18]

I would now like to present substantivalism, which is the concept of space-time endorsed by Field.[19] According to substantivalism, the universe is made up of space-time and its parts whereas, according to relationalism, the universe consists of physical objects that are related with one another. More precisely, according to the former, a physical object is nothing but a part of space-time whereas, according to the latter, the existence of space-time is unnecessary. Field argues that substantivalism is more appropriate than relationalism because the latter is ultimately reducible to the former. In fact, suppose that the best candidate for replacing space-time is the notion of field, which assigns causal properties to space-time regions. For example, an electromagnetic field assigns to n-ple of space-time's points a relation of electromagnetic intensity. Relationalists could argue that space-time is dispensable if, for every sentence about space-time regions, it is possible

[18] Field (1980, p. 38).

[19] See Field (1984).

to build a corresponding sentence about field's properties. However, Field replies, substantialists can dismiss this objection by claiming that fields are ultimately properties of space-time. As a result, the minimal requirement is merely the existence of space-time.

2.1.3 Shapiro's objections

Suppose that Field's nominalization works. Is it legitimate to employ so much mathematics in order to show that mathematical objects are dispensable? As Shapiro points out

> There is an interesting irony in Field's development of fictionalism [...] By assuming the mathematics, Field shows that, in a sense, mathematics is not necessary for science.[20]

In point of fact, there have been many objections to Field's approach, and I intend to split them into two classes: those who reject the conservativeness of mathematics, and those who criticize the possibility of extending Field's nominalization towards other branches of physics. Because there are a considerable number of objections, I will elaborate on the most important critiques for each class of objections.[21]

Consider Field's claim that good mathematics is conservative. It is not clear what kind of conservativeness Field invokes:[22] on the one hand, syntactic conservativeness states that a nominalistic statement can be derived from $N + M$ if it can be derived from N alone; on the other hand, semantic conservativeness states that if a nominalistic statement is true in every model of $M + N$, then it is true in every model of N alone. We know that Gödel's completeness theorem establishes a correspondence between semantic truth and syntactic provability in first-order logic, but since Field chooses second-order logic for Euclidean geometry, the two distinct notions of conservativeness remain into play.

[20] Shapiro (1997, p. 227).
[21] Most objections are collected in Chihara (1990).
[22] See Shapiro (1983).

Let us suppose that mathematics is just semantically conservative: if $M + N$ entails a nominalistic statement n, then n is true in every model N, but it does imply that n is derived from N alone. As a consequence, if it is possible to build a n that is true in every model N, but n cannot be derived from N alone, then mathematics cannot useful in Field's sense: that is, mathematics does not merely simplify deductions.[23] One might reply to this objection by adopting first-order logic. However, Field himself notes that since the representation theorem maps space-time onto the set of real numbers, the theorem implies that *any* model of N is uncountable.[24] If N was formulated within first-order logic, Löwenheim-Skolem's theorem would hold, and thus N would have a countable model.[25] For this reason, Field adopts Goodman's second-order logic instead of first-order logic.[26]

Consider a further objection. If nominalists dispense with mathematical objects, they should still make use of the notion of model. This is because the notion of logical consequence is required in order to formulate semantic conservativeness, which is stated in terms of models. However, models are abstract objects that are not available to the nominalist. In other words, even if mathematics was conservative, the nominalist could make no use of that result. If mathematics were dispensable in classical mechanics, Field would still need the notion of logical consequence in order to prove the conservativeness theorem. In the next section I will present how Field intends to show that conservativeness is nominalistically acceptable.

2.1.4 Field's modal deflationism

Nominalists have two strategies for stating conservativeness: either (1) they could dispense with the notion of logical consequence, or (2) they could

[23] In point of fact, Shapiro builds a Gödel-style sentence G that belongs to the nominalistic body of assertion N such that $ZFC + N \vdash G$ but $N \nvdash G$. See Shapiro (1983). In this regard, Field (1985, p. 255) replies that G is not an assertion about space-time.

[24] See Field (1985).

[25] The theorem states that consistent first-order theories have a model with a countable domain.

[26] Field does not seem to be satisfied with his choice, though. See Field (1980, p. 115).

employ modal logic. Because the first option is too hard, Field chooses the second one reinforcing classical logic through modality without possible worlds semantics. The idea is to express the notions of logical consequence and consistency in modal terms by avoiding ontological commitment to possible worlds.

Let us see how Field's strategy works. In the first instance, Field introduces a primitive notion of logical consistency, where a mathematical theory T is consistent if $\Diamond T$ (i.e. T is possible). Let A_T be the (finite) conjunction of the axioms of T, B a mathematical assertion of T, and B^* the result of restricting B to non-mathematical entities. According to Field, T's conservativeness can be modally expressed in this way: if $\Diamond B$, then $\Diamond(B^* \wedge A_T)$.[27] Moreover, Field reinterprets the notion of logical consequence in terms of modal operators without referring to models and truth. Instead of 'P' in A_T is true iff 'P' is true in a model of A_T, we say $\Box(A_T \rightarrow P)$. This is a nominalistic surrogate of the notion of truth in a model, which aims to overcome the objection according to which both conservativeness and logical consequence involve abstract objects, i.e. models.

A problem with Field's formulation is that standard set theory, such as ZFC, is not finitely axiomatized (i.e. there are infinitely many axioms). And standard set theory is essential to prove that mathematics is conservative. So how can we express the conjunction of all the axioms of set theory? Field's idea is to appeal to a finitely axiomatizable theory: von Neumann-Bernays-Gödel set theory (NBG).

To sum up, Field recasts consistency, conservativeness, and logical consequence in modal terms. But here lies another problem: since both conservativeness and logical consequence involve modality, one might tend to interpret such operators in terms possible worlds semantics. Why should a commitment to possible worlds be better than mathematical *abstracta*? Field's answer is that modal operators must be taken as primitive. This response might be considered rather hasty, but the point is that Field does

[27] See Field (1989, p. 120).

not actually regard modality as a surrogate of ontology.[28] What Field does is to offer an epistemology for his modal concepts: that is to say, a deflationistic view of mathematical knowledge. On this view, an epistemic agent has mathematical knowledge if he, or she, knows that (1) certain mathematical assertions follow from certain mathematical assumptions, and that (2) those assumptions are consistent. In other words, an epistemic agent has mathematical knowledge of A_T if he, or she, knows that (1') $\Box(A_T \rightarrow B)$, and that (2') $\Diamond A_T$.[29]

2.1.5 Extending Field's program

One of the main problems of Field's approach concerns its extension. It is questionable whether or not Field's program can be extended towards other branches of physics aside from classical mechanics. Suppose that classical mechanics is nominalizable in a way that one can dispense with mathematical *abstracta* and nominalize the basic elements of vector calculus. Nonetheless, Malament argues, quantum mechanics cannot be still nominalized because we lack a sort of the representation theorem for it.[30] In classical mechanics the nominalistic surrogates of abstract objects are the physical regions of space-time. But in quantum mechanics we deal with Hilbert's spaces in order to represent quantum events, or propositions, that are not dispensable in the absence of a representation theorem. This is because quantum mechanics is a probabilistic theory in which quantum states are functions from quantum events to probabilities. Malament's point is that probabilities in quantum mechanics are represented by real numbers that are not dispensable:

[28] See Field (1989, p. 256).

[29] Here is a problem: how can we know that $\Diamond A_T$? For example, we cannot prove the consistency of NBG within NBG itself: we should know that the existence of a weakly inaccessible cardinal is possible. Alternatively, we might appeal to an inductive argument: no one has found yet a contradiction in NBG, hence we can reasonably suppose that NBG is consistent. However, this inductive argument is rather controversial, as Resnik points out (1984).

[30] See Malament (1982).

> I do not see how Field can get started at all. I suppose one can think of the theory as determining a set of models — each a Hilbert space. But what form would the recovery (i.e. representation) theorem take? The only possibility that comes to my mind is a theorem of the sort sought by Jauch, Piron, *et al.* They start with "propositions" (or "eventualities") and lattice-theoretic relations as primitive, and then seek to prove that the lattice of propositions is necessarily isomorphic to the lattice of subspaces of some Hilbert space. But of course no theorem of the sort would be of any use to Field. What could be worse than *propositions* (or *eventualities*)?[31]
>
> Even a generous nominalist like Field cannot feel entitled to quantify over possible dynamical states.[32]

In contrast to Malament's view, Balaguer argues that Field's program can be extended to quantum mechanics.[33] According to Balaguer, all we need is to find the nominalistic counterparts of the statements that assign probabilities to quantum events. To do this, Balaguer argues that we could regard probabilities as physical properties of physical systems, i.e. propensities. In a second step, Balaguer outlines a representation theorem for quantum mechanics in this way: for each Hilbert space we employ in quantum mechanics, the set of closed subspaces of an Hilbert space can be represented by using propensities. Nonetheless, Balaguer's solution remains a highly controversial topic.[34]

[31] Malament (1982, p. 534).

[32] Malament (1982, p. 533).

[33] See Balaguer (1996).

[34] According to Bueno, Balaguer's solution is incompatible with many interpretations of quantum mechanics and, moreover, it is unclear whether or not propensities are available to the nominalist. See Bueno (2003).

2.2 Mathematics and constructibility quantifiers

In the previous section, I presented the way Field employs modal terms in order to dispense with models. But nominalists have other reasons to advocate modality. If the universe does not contain as many *concreta* as those that are necessary to represent mathematical objects, nominalists could make use of modal concepts. In this regard, it is first important to consider this objection: if modal operators are employed in nominalistic reconstructions in order to dispense with mathematical entities, nominalists might be ontologically committed to *possibilia*. And ontological commitment to *possibilia*, or possible worlds, is as much problematic as the commitment to abstract objects. So here lies a first worry for modal nominalism: philosophers who employ modal operators should clarify what they mean by 'possible' and 'necessary' without committing themselves to *possibilia* or possible worlds.

Modal nominalists should handle a further problem: ontology is reduced only by increasing ideology. As will see, the modal nominalist must increase the complexity of the theory in order to dispense with *abstracta*. Chihara, for example, extends classical logic to the system of modal logic S5 in order to dispense with mathematical *abstracta*. However, the semantics of sentences in S5 is presented in terms of possible worlds: 'it is possible that P' means that there is a possible world w such that P is true in w; 'it is necessary that P' means that P is true in any possible world w. Again, ontological commitment to possible worlds is as much questionable as the one to mathematical *abstracta*.

2.2.1 The constructibility theory

Chihara's constructibility theory aims at reinterpreting mathematics without ontological commitment to possible worlds. Before I examine Chihara's theory, I would like to mention that Chihara neither believes that good mathematics is conservative, as Field does, nor he endorses fictionalism. According to Chihara, mathematical sentences do not literally refer to abstract objects. The semantics of a mathematical statement such as 'there are numbers' is not captured by a Platonist interpretation, i.e. 'there are numbers' is true if numbers exist and are *abstracta*. Because Platonism does not express the literal interpretation of mathematical statements, Chihara does not argues that mathematics is false, as Field does. However, Chihara does not seem to endorse any position about the literal meaning of such statements. He merely claims that mathematics does not need to be true to be useful:

> My general position has been that theorems of mathematics, whatever their literal meaning may be (assuming that they have a "literal meaning"), do not have to be true to be justifiably used by scientists to draw the inferences they do in their scientific work.[35]
>
> It can be seen that, no matter how one may analyse the literal meaning of mathematical sentences, one can make good sense of mathematical practice and the applications of mathematics in science without requiring mathematical theorems, literally construed, to be true. It is enough that sentences expressing the structural content of the theorems be true.[36]

For Chihara, it does not matter what the literal interpretation of mathematical statements is. In other words, Chihara seems to endorse an agnostic viewpoint on the literal meaning of mathematical statements. Nominalists should just provide a nominalistic account of mathematical structuralism without committing themselves to the existence of abstract objects. This is sufficient for the modal nominalist. In Chihara's view, what it matters is that mathematics can recovered in the constructible setting.

It is important to understand what Chihara means by 'constructible'. Bur let me first elaborate on how Chihara distances himself from the Pla-

[35] Chihara (2004, p. 252).
[36] Chihara (2004, p. 295).

tonistic view on mathematical structures, or *ante rem* structuralism. According to *ante rem* structuralism, mathematical structures exist and are *abstracta*.[37] On this view, mathematics also involves objects that are positions in the domain of a structure. In contrast to *ante rem* structuralism, Chihara's approach aims at holding together nominalism and structuralism. The idea is to provide a nominalistic interpretation of mathematics based on the constructibility theory, which makes use of special quantifiers, i.e. constructibility quantifiers, in order to avoid quantification over mathematical objects:

> Thus, the point of showing these philosophers how mathematics can be done in terms of constructibility quantifiers was not to convert scientists to using a new system of mathematics, but rather to show that the undeniable usefulness of mathematics in science did not require that one believe in the [things apparently] talked about in mathematics.[38]

As will show soon, the constructibility theory aims at dispensing with mathematical objects by using modality.[39] The constructibility theory is basically a first-order theory plus constructibility quantifiers that do not carry any ontological commitment.[40] Constructibility quantifiers tell us which concrete tokens we can construct. For instance, consider a sentence such as 'it is possible to construct houses made entirely of ice'. By uttering this sentence, I am not committing myself to the *actual* existence of houses made entirely of ice; I am just saying that houses made entirely of ice are *possible*. Constructibility quantifiers occur in ordinary language, although they are not pre-theoretic terms of ordinary language, because they occur in a formal

[37] See Shapiro (1997).

[38] Chihara (1990, p. 188).

[39] I will take into account the latest versions of the constructibility theory developed in Chihara (1990) and (2004).

[40] For those who accept Quine's criterion of ontological commitment, existential quantifiers carry ontological commitment whereas, according to Chihara, constructible quantifies do not.

system, i.e. the constructibility theory.

Constructibility quantifiers are sequences of concrete tokens. Chihara represents the assertion 'it is possible to construct an open-sentence $-$' by the quantifier ($\mathbf{C}-$), and the assertion 'every open-sentence $-$' by the quantifier ($\mathbf{A}-$). Sentences are all open in the constructibility theory. Open-sentences are concrete marks on paper, on screen, and so on: that is to say, they are spatially and temporally located sentences.

The constructibility theory is a formal system that tells us how to build open-sentences without assuming the existence of any open-sentence. In point of fact, the constructibility theory does not aim at providing information about how to detect open-sentences, or which objects satisfy a given open-sentence. Still, it may be helpful to give an example of what an open-sentence could be. I am typing right now the sentences 'x is an American actor' and 'x is my favorite fiction novel'. These are open-sentences that you are reading on screen, or on printed paper, that can be satisfied by John Wayne and The Lord of the Rings respectively. In other words, typing 'The Lord of the Rings is my favorite fiction novel', I wrote a concrete sentence token that can be expressed by saying that 'The Lord of the Rings' satisfies the open-sentence 'x is my favorite fiction novel'.[41]

Satisfaction is the basic relation between objects and open-sentences where standard quantifiers are involved, whereas it is the relation among open-sentences if constructibility quantifiers occur. In the language of constructibility theory, a sequence of symbols '$(\mathbf{C})xy$' could be read in English such as 'it is possible to construct an open-sentence y, such that y satisfies x'; on the other hand, '$(\mathbf{A})xy$' could be read in English such as 'every open-sentence y that it is possible to construct is such that y satisfies x'. Again, it is important to emphasize that if an open-sentence is constructible, we do not commit ourselves to an actual open-sentence token, or to a possible world where that open-sentence exists. The constructibility theory does not

[41] In intuitionistic mathematics, proofs are mental constructions that are in principle constructible whereas, on Chihara's account, mathematics involves concrete open-sentences: tokens that are said to be constructible.

tell us which objects satisfy a given open-sentence, such as Euclid's geometry does not tell us how to recognize straight lines, points and circles:

> Euclid's geometry does not tell us how to recognize straight lines, how to tell if a line is really straight, or if a line really intersects a point. It does not tell us how to construct points, straight lines, or arcs. The important point is this: it doesn't matter that Euclid's geometry does not tell us these things. The usefulness of that kind of modal theory does not depend on its giving us that kind of information. That's not the way we use that geometry. Similarly, my Constructibility Theory is not designed to give us information about how to tell what is an open-sentence or what things satisfy any given open-sentence.[42]

Open-sentences are all monadic, that is, sentences such as 'x is human'. But in order to recover mathematics we also need to express relations, functions, and many other mathematical concepts. How could we express for instance that a set has the same cardinality of another one without functions? Or that every natural number has a unique successor? Because binary relations can be defined as sets of ordered pairs $< x, y >$, and because ordered pairs $< x, y >$ can be represented by the set $\{\{x\}, \{x, y\}\}$, Chihara needs to reformulate set-theoretic ordered pairs in terms of open-sentences.

In Chihara's view, an ordered pair is an open-sentence that it is satisfied by other open-sentences; that is, it is satisfied by all and only couples $\{x, x\}$ and $\{x, y\}$ that could be constructed.[43] Starting with ordered pairs, Chihara aims at constructing relations, equinumerosity,[44] and natural numbers.[45] In

[42]Chihara (2004, p. 172).

[43]For example, an ordered pair $\{Field, Yablo\}$ is a couple $\{Field, Field\}$ or a couple $\{Field, Yablo\}$. A Couple $\{x, y\}$ is an open-sentence that is satisfied by only open-sentences x and y.

[44]In point of fact, Chihara proves a constructibility version of Hume's principle. The definition is complex, and it requires to spell out several technical concepts. For a summary see Chihara (2004, pp. 178-179).

[45]To prove the theorems Peano's Arithmetic, Chihara makes use of the hypothesis of infinity. It is called 'hypothesis' instead of 'axiom' because, as Chihara (1990, p. 71)

the end, Chihara carries out an extensive reconstruction of the relations that are indispensable to recovering a large amount of mathematical results into the language of the constructibility theory.[46]

I do not need to examine the constructibility theory in detail. But let me highlight how Chihara's theory is nothing but a theory of types for open-sentences. Indeed, open-sentences are stratified into different levels as follows: we start with concrete objects at level 0; at the next level we quantify over objects of level 0; at the next level we quantify over open-sentences of level 1; and so on.

Consider an open-sentence such as 'x is an Afro-American President'. This concrete token is an open-sentence of level 1, and Barack Obama is the object that satisfies the open-sentence. I am now writing an open-sentence of level 2 'there is at least an object that satisfies F', which is satisfied by the open-sentence 'x is an Afro-American President' that I wrote earlier. In principle, I could iterate the process building a hierarchy of open-sentences that includes different variables for each level: objects (level 0), properties, (level 1), attributes (level 2), qualities (level 3), and so on.

Let us see how stratification works in mathematics. At level zero, we have objects such as x, y, z. At level one, we find the property (couple) $\{x, y\}$ that is satisfied by the objects x and y.[47] At level two, we have the attribute (ordered pair) $< x, y >$ that is satisfied by all and only properties $\{x, x\}$ and $\{x, y\}$ that could be constructed at the first level. By going on

points out, 'I cannot suppose, as did Russell, that the question of whether the domain of objects is finite or not is simply a matter of fact, to be settled, if at all, by the appropriate scientific investigation. Nor is there any reason to maintain that the hypothesis will hold no matter what domain of objects we may select, since I want to allow interpretations in which the domain of objects is finite [...] we can regard the number theorist as implicitly adopting the Hypothesis of Infinity as an axiom. But in the present system, the hypothesis will function as merely a hypothesis: certain theorems will presuppose the hypothesis and others will not''. For further details see Chihara (1990, pp. 68-73).

[46]For a detailed analysis see Chihara (1990), Ch. 3-5.

[47]There are obviously several ways to satisfy such a property, for example $\{Field, Yablo\}$, $\{Field, Field\}$, and so on.

to the next levels we can construct relations, numbers, and so on.[48]

The constructibility theory supports nominalism about structures or, in other words, *in re* structuralism. The idea behind *in re* structuralism is that mathematics is a science of structures, but structures are not considered abstract entities. However, the problem is that it is standard to regard structures as models of a first-order theory, i.e. set-theoretical entities. As Chihara points out,

> The goal is to find such "things" which are also nominalistically acceptable, so that they can be used as the "realizations" of mathematical theories without requiring the background theory to carry a commitment to the sorts of metaphysical entities that led to so much trouble for the [*ante rem*] structuralists.[49]

Thus, we need a nominalistically acceptable open-sentence that describes structures and, moreover, we need an appropriate open-sentences that represent the relations between objects. According to Chihara, the notion of realization aims at substituting the standard (Platonistic) notion of truth in a model. More precisely, a realization is an ordered pair consisting of an open-sentence that is satisfied by the elements of the domain (i.e. constructible open-sentences), and of an open-sentence that represents the relations between such elements. Since open-sentences, ordered pairs and relations, can be built in the constructibility theory, such notions turn out to be nominalistically acceptable. As a result, Chihara can dispense with the Platonistic notion of model.

[48]Suppose that there are less than $10^{10^{10}}$ particles in the universe. It should not be possible to satisfy the open-sentence 'x is an open-sentence that contains $10^{10^{10}}$ concrete tokens'. However, as I said previously, the constructibility theory does not aim at telling us what opens-sentences are possible.

[49]Chihara (2004, p. 220).

2.2.2 Modality without possible worlds

Modal concepts are nominalistically acceptable insofar as the constructibility theory does not imply that the entities that satisfy open-sentences exist, nor constructibility quantifiers are ontologically committing. But here lies a question for Chihara: what about constructibility quantifiers semantics? Does it require possible worlds semantics? In point of fact, Chihara presents constructibility quantifiers semantics in terms of possible worlds,[50] but further adds that

> It should be emphasized again that the above appeal to possible worlds was made to relate the constructibility quantifiers to familiar and heavily studied areas of semantical research. I, personally, do not take possible world semantics to be much more than a useful device to facilitate modal reasoning.[51]
>
> For purposes of formal development, however, I have found it simpler to regard the modal universal quantifier as a primitive of the system.[52]

What does Chihara means by 'primitive'? *Prima facie*, two ways of defining 'primitive' are available: on the one hand, a term is primitive if it is not defined by other terms; on the other, if the term is a pre-theoretical notion.[53]

According to Chihara, possibility is taken as primitive in the sense that it is not defined in the system where that notion occurs, as the membership relation is a primitive of set theory.[54] Possibility can occur as a defined notion in another system but, Chihara argues, model theory is not required.

[50]Chihara develops an extension of the semantic of first-order language that includes constructibility quantifiers. See Chihara (1990, pp. 27-37).

[51]Chihara (1990, p. 38).

[52]Chihara (1990, p. 39).

[53]Shapiro argues that 'primitive' means pre-theoretical according to Chihara. See Shapiro (1997, p. 232).

[54]Chihara (2004, p. 204).

Basically, Chihara regards the existence of open-sentences in a possible world as a *façon de parler*. What matters is just *our* world:

> To say that someone has constructed an open-sentence is not to say that an entity of a certain sort has been constructed but only that the person has done something — he has performed the appropriate series of actions. This is one reason why, when the objects we are discussing are open-sentence tokens, I prefer the 'It is possible to construct' reading to the 'It is possible for there to be' reading. Still, it is useful to treat open-sentence tokens as ordinary objects[55]

In short, possible worlds are merely a useful myth.[56] In Chihara's view, when I say something like 'it is possible to construct two different open-sentences', I am not saying that there are two distinct open-sentences in two distinct possible worlds: I am merely stipulating that such open-sentences could exist. In other words, possible worlds seem to me nothing but fictions in this context.

But how can fictions provide genuine explanations? In this regard, Chihara distinguishes two kinds of explanations:[57] we can either have scientific explanations of natural phenomena, or explanations of the meaning and use of expressions. According to Chihara, myths are not involved in scientific explanations but can be used for clarifying the meaning of modal notions. Thus, possible worlds are myths that make constructibility quantifiers easier to understand, as well as Flatland world can be used as a metaphor to explain complex geometric concepts.

2.2.3 Towards modal fictionalism

Since possible worlds are considered myths by Chihara, I think I can label his account of possible worlds as modal fictionalism. As previously stated, Chi-

[55] Chihara (1990, p. 40).
[56] See Chihara (1990, p. 60).
[57] See Chihara (2004, p. 196).

hara does not endorse fictionalism just because he distances himself from Field's program, and because he does not take existential mathematical statements as false. Thus, I do not think I am forcing Chihara's position by presenting Rosen's account of modal fictionalism and applying it to Chihara's.[58]

Modal fictionalism aims at providing an analysis of modality in terms of fictions. Basically, the idea is to express possible worlds semantics in fictions. Consider the following realist interpretation of modal operators:

1. $\Diamond P$ if and only if at some world w, P holds.
2. $\Box P$ if and only if at all worlds w, P holds.

The modal realist can turn a modal statement P into a non-modal one P^* about possible worlds as follows: P if and only if P^*. For instance, a statement such as 'it is possible to construct the number two' becomes 'there is a possible world in which the number two exists'.

Modal realists have two ways of conceiving modality: possible worlds can be either *abstracta* or *concreta*.[59] By contrast, modal fictionalists opt for a deflationistic view in which possible worlds' semantics works in fictional contexts. In addition, modal fictionalists typically introduce the fictional operator 'according to the fiction F' that operates on sentences. This is how modal operators are interpreted:

1. $\Diamond P$ iff, according to the many-worlds fiction, at some world w, P holds.

[58] See Rosen (1990) and (1995).

[59] David Lewis is perhaps the most famous philosopher who conceived possible worlds as concrete objects. Lewis (1986, p. 2) advocates " a thesis of plurality of worlds, or modal realism, which holds that our world is but one world among many. [...] The other worlds are of a kind with this world of ours. To be sure, there are differences of kind between things that are parts of different worlds - one world has electrons and another has none, one has spirits and another has none - but these differences of kind are no more than sometimes arise between things that are parts of one single world, for instance in a world where electrons coexist with spirits. The difference between this and the other worlds is not a categorical difference. Nor does this world differ from the others in its manner of existing."

2. $\Box P$ iff, according to the many-worlds fiction, at all worlds w, P holds.

The fictional operator singles out a collection of sentences that are true in a domain but false in another. It explains how, for instance, the sentence 'Siegfried killed the dragon' is true in Norse mythology but false in the real world (since there are no dragons). Ontological commitment to possible worlds is avoided by invoking the fictional operator. If one has an anti-realist attitude towards fictions, it is also trivial to account for how we have epistemic access to them, in contrast to realism about possible worlds. Modal fictionalism is thus compatible with nominalism.

If the modal fictionalist is right, it is possible to provide an anti-realist semantics of modality without ontological commitment to possible worlds. Whereas the modal realist accepts the schema P if and only if P^*, the modal fictionalist appeals to the schema P if and only if, according to the many-worlds fictions, P^*. I suggest to apply the latter schema to Chihara's account of possible worlds as follows: for instance, 'it is possible to construct the number 2' if and only if, according to the the many-worlds fiction, there is a possible world in which there is the number 2.

It is important to emphasize a difference between modal realism and fictionalism: according to the former, every modal statement, or proposition, has a determinate truth-value even if we do not know which it is; on the other hand, according to the latter, it is pointless to speculate on a sentence's truth-value if the story does not say anything about it. For example, because *The Lord of the Rings* does not say anything about hobbits' blood type, it makes no sense to mull over Bilbo's blood type. As a consequence, the sentence 'Bilbo's blood type is $A+$' lacks either truth-value, or it is false.[60]

I would like to present a problem that arises out of modal fictionalism. Hale notices how the statement (PWF) 'there exist possible fictional

[60]In point of fact, even if a story leaves many questions open, it might be important to speculate on the truth-value of certain sentences to appreciate the story in question. Consider Kafka's incomplete novel *Amerika*. It would be rather dismissive to say: 'Kafka does not tell us anything about what Karl does after he got to Oklahoma City. The problem is meaningless; stop wondering about it!'

worlds' should be false according to modal fictionalism.[61] Here is a dilemma: is PWF false necessarily or contingently? If PWF is false necessarily, the sentence 'according to fictions, P' is trivially true independently of the content of P, as there are no fictional worlds. Suppose that PWF is false contingently. If this is so, there may exist fictional worlds, and thus modal fictionalism turns out to be self-refusing. Either way, Hale argues that modal fictionalism is hard to sustain.

2.2.4 Burgess' objection

Chihara rejects Burgess and Rosen's distinction between hermeneutic nominalism and revolutionary nominalism.[62] Chihara does not consider his program revolutionary, because it does not dictate that scientists have to adopt the constructibility theory, nor hermeneutic, because it does not provide an alternative semantics of mathematical sentences. For Chihara, it does not matter what literal meaning is. But consider the following Burgess' objection.[63] Suppose one utters the sentences:

1. There are numbers.

2. There are numbers greater than 10^{10} that are prime.

To paraphrase those statements, nominalists have two main strategies according to Burgess. The nominalist could argue that either (2) does not imply (1), or (2) implies (1) but (2) is false. The former is the hermeneutic strategy, whereas the latter is revolutionary. Instead, Chihara argues that (1) is false, but he does not say anything about whether or not (2) implies (1), or whether (2) is true or false. In other words, Chihara does not want to endorse hermeneutic or revolutionary nominalism. He is reluctant to adopt any hypothesis on what people mean when they utter sentences like (2). But how could Chihara distinguish between a mathematician and someone

[61] See Hale (1995).
[62] See sec. n. 1.4.
[63] See Burgess (2005).

like Humpty Dumpty who gives any truth-value whatsoever to sentences? Mathematicians, Burgess argues, know what the truth-value of (2) is: (2) is true. Chihara's reply emphasizes how the constructibility theory does not depend on mathematicians' linguistic practices, nor it states the actual meaning of mathematical assertions.[64] Nonetheless, I believe that Burgess and Rosen's distinction between hermeneutic nominalism and revolutionary nominalism points at something important. Taking for granted that Burgess and Rosen's distinction is too narrow in order to encompass every nominalistic program, it is nevertheless helpful to set what the nominalistic goals should be.[65] It would be hard evaluating the success of a nominalistic reconstruction without considering any goals.

[64]See Chihara (2006, pp. 331-336).
[65]See Baker (2006).

2.3 Mathematics without numbers

Hellman's aim is to interpret mathematical statements modally without quantification over abstract objects. According to Hellman, there are four requirements for every philosophical interpretation of mathematics:[66] 1) mathematical statements must be either true or false; 2) philosophers must account for how we can get mathematical knowledge; 3) mathematics must be *a priori*; 4) philosophers must explain how mathematics can be applied to the physical world. Platonists do not have any problems of accounting for the first and the third requirement, but they have troubles with the second one. For Hellman, it is possible to meet all the four *desiderata* by endorsing his modal structuralism. According to Hellman's modal structuralism,

> mathematics is the free exploration of structural possibilities, pursued by (more or less) rigorous deductive means.[67]

On this view, the basic requirement is that mathematical structures are possible, which is called the 'hypothetical component' of modal structuralism.[68] Hellman's modal structuralism does not lead to ontological commitment to mathematical objects, nor to the existence of actual mathematical structures. Modal logic is required in order to avoid representing structures as models or sets, and modal logic is presented in terms of a second-order language that allows quantification over possible structures. Hellman's structuralism is a form of modal nominalism or, in other words, of eliminativism that avoids commitment to both structures and objects. Hellman's nominalism is often called 'structuralism without structures'.

Consider the case of arithmetic in light of structuralism. It does not matter what the identity of each single natural number is: what matter are the relations in arbitrary ω-sequences. More generally, Hellman intends to recover arithmetic, real analysis, and even part of set theory in a structuralistic setting without commitment to *abstracta*. Hellman's strategy must be

[66] See Hellman (1989, pp. 2-6).
[67] Hellman (1989, p. 6).
[68] See Hellman (1989, p. 16).

distinguished from Chihara's, since the latter makes use of constructibility quantifiers, whereas the former does not.

2.3.1 Hellman's program

Imagine an arithmetical sentence (A) such as $2+2=4$. Hellman interprets A as the conditional 'if there were any ω-sequence, A would hold in it'. The first part of that conditional is expressed by $\Diamond X$, where X is an ω-sequence, and arithmetical sentences are sentences that would hold in possible ω-sequences. The modal structuralist is thus committed to possible ω-sequences, which is the so-called 'categorical component' of modal structuralism: $\Diamond X$, where X is an ω-sequence satisfying Peano axioms. In point of fact, Hellman's structuralism requires second-order arithmetic to be formulated, and it is important to point out that second-order Peano Arithmetic is categorical, i.e. there is one intended model.[69]

Hellman's nominalization starts with standard second-order logic plus the comprehension principle $\Box\exists R\forall x_1 \ldots x_n[R(x_1 \ldots x_n) \leftrightarrow A]$.[70] Arithmetical truths are proved within second-order Peano Arithmetic by modal operators in this way: $\Box(\mathrm{PA}_2 \rightarrow A)$, where PA_2 is the conjunction of the second-order Peano axioms, and A is an arithmetical sentence. More precisely, an arithmetical sentence goes over to a conditional of the form $\Box\forall S(\mathrm{PA}_2 \rightarrow A)$, where S is the relation variable that replaces the successor constant. If A is logically implied by PA_2, then A is true; otherwise $\neg A$ in virtue of the categoricity of second-order arithmetic. Any pair of models of PA_2 is isomorphic, and every arithmetical sentence is either true or false in every model of PA_2.

To express that an arithmetical sentence holds in *any* ω-sequence, Hellman says $\Box\forall X\forall f(\mathrm{PA}_2 \rightarrow A)$, where X is an ω-sequence and A is an arith-

[69]By contrast, there are non-standard models in first-order arithmetic. Quantification over both predicates and first-order variables is allowed in second-order logic. Many interpretations of predicates are admitted once quantification over predicates is allowed. This is why there are many possible interpretations of second-order logic.

[70]Where R occurs free in A. Note that universal quantifiers are not boxed.

metical sentence. Hellman also proves that there could be an infinite totality, which indicates how the modal structuralist postulates the existence of potential infinity.[71]

Arithmetic can be fully recovered in such a modal setting. And if ω-sequences are possible, then natural numbers are dispensable. But suppose that ω-sequences were not possible. If this was so, the conjunction of Peano axioms PA_2 would be false, and the conditional $\mathrm{PA}_2 \rightarrow A$ would be true for every A. Therefore, arithmetic would be trivialized. This is why Hellman must assume the above-mentioned 'categorical component' of modal structuralism as fundamental law of arithmetic.

Since Hellman does not postulate actual ω-sequences, he does not commit himself to the existence of abstract ω-sequences. The modal structuralist assumes that ω-sequences could exist, but such sequences are just a dummy names that do not refer to anything. This is because whereas Platonists regard a true mathematical sentence as 'true in a model', the modal structuralist regards that sentence as 'true in a possible model'. In Hellman's view, true mathematical sentences in a model are false. Modal structuralism is thus distinct from formalism, i.e. it is distinct from the view that mathematical sentences have no truth-values. The price is to take modal operators as primitive notions: that is, modal operators are not given in terms of set-theoretical semantics.

According to Hellman, mathematics explores *a priori* truths, in the sense that it studies truths about possible structures. But why should we endorse Hellman's interpretation of standard mathematics? After all, non-modal mathematical sentences are false according to the modal structuralist. To address this problem, Hellman provides a translation scheme that proves the equivalence between the modal interpretation of arithmetic and the standard (Platonistic) one.[72] The equivalence theorem aims at showing that:

[71]However, because the axiom of comprehension allows impredicative sentences, its constructive nature is not guaranteed. See Hellman (1989, p. 33).

[72]See Hellman (1989, pp. 41-44).

1. A is Platonistically true iff PA_2 logically implies A

2. PA_2 logically implies A iff A is a modal structuralist truth.

3. A is Platonistically true iff A is a modal structuralist truth.

2.3.2 Beyond the modal interpretation of arithmetic

Let us now examine how to extend Hellman's program to real analysis and set theory. Real analysis can be developed in second-order logic by employing first-order variables for real numbers plus a continuity principle.[73] With regard to set theory, Hellman recovers a large amount of set-theoretical notions by the elementary theory of finite sets and classes, where the categoricity of second-order arithmetic is fully available.[74] However, that theory is not nominalistic on its own, and thus it requires both plural quantification and mereology.[75]

Hellman employs a further strategy for paraphrasing sets. Suppose that the cumulative hierarchy is the standard model of set theory. Hellman shows that if A is a set-theoretical sentence that is either true or false in the cumulative hierarchy, then A is either true or false in all the possible models of set theory. In this regard, Hellman employs ZF_2's quasi-categoricity theorem: if M_1 and M_2 are two models of ZF_2 (second-order Zermelo-Fraenkel set theory) that have the same cardinality and ordinal height, then M_1 and M_2 are isomorphic. However, the models of ZF_2 are different than standard ZF, because the ordinal height of any model of ZF_2 is a strongly inaccessible cardinal.[76] This point may be problematic from a nominalistic point of view, since it is no obvious matter to establish the possibility of the existence of structures with inaccessibly many objects.

Last but not least, Hellman also presents the general form of applied mathematics for modal structuralism: if there were structures satisfying

[73]See Hellman (1996).

[74]See Feferman and Hellman (1995) and Hellman (1996, pp. 107-110).

[75]See sec. n. 2.3.4

[76]κ is a strongly inaccessible cardinal iff κ is regular and is a strong limit, i.e. κ has a strong limit if $2^\lambda < \kappa$, for every $\lambda < \kappa$.

the conjunction of the axioms of second-order Zermelo-Fraenkel set theory that also includes *Urelemente* (that is, non-mathematical objects), modal set-theoretical statements would hold in such structures. Roughly speaking, mathematics can be applied in a modal setting if we assume an hypothetical statement about structures satisfying the conjunction of the axioms of ZF_2 plus *Urelemente* are possible.

2.3.3 The wolf and the lamb

Before I examine Hellman's use of mereology and plural quantification, I would like to consider Quine's critique of second-order logic. According to Quine, second-order logic is nothing but set theory in disguise,[77] because if we quantify over predicates, we ought to commit ourselves to sets. Here is how the objection goes: there is a difference between the position of predicates and the position of names in a sentence. In first-order logic, quantifiers range over variables that stand for names of entities of some sort, such as $\exists x(x$ is a glass) where x stands for a name of a glass. In second-order logic, on the other hand, we quantify over predicates treating them as they are names. But if predicates are names, then

> The quantifier '$\exists F$' or '$\forall F$' says not that some or all predicates are thus and so, but that some or all entities of the sort named by predicates are thus and so.[78]

And if predicates name sets, then second-order logic is nothing but set theory in disguise. Boolos' reply to Quine is based on the distinction between names and ranges of predicates. Predicates have ranges but they do not necessary name sets. Even though quantifiers range over a domain of discourse, there is no unique interpretation of second-order logic. Logic has no a specific subject matter, because it is neutral about the choice of the domain. If a second-order sentence says something about sets, the domain contains sets; otherwise there is no ontological commitment to sets.

[77] See Quine (1970, pp. 66-67).

[78] Quine (1970, p. 67).

Moreover, second-order logic is weaker than set theory, because the notion of validity can be defined in set theory, but it cannot purely be defined in second-order language.[79] According to Boolos, this point shows that second-order logic is not set theory in disguise.

2.3.4 Ontological innocence

Hellman postulates possible (non-actual) infinitely many *Urelemente* (individuals). These individuals follow the axioms of mereology and the term 'finite set' is interpreted as 'finite sum of individuals'.[80] Infinite sums of individuals are expressed using plural quantification by the following postulate:

> There are (possibly) some individuals one of which is an atom and each one of which fused with a unique atom not overlapping that individual is also one of them.[81]

where an atom is an individual without proper parts.[82] Adding that postulate plus Goodman's axioms of mereology, we get a nominalistic model for the elementary theory of finite sets and classes. First-order variables range over individuals; finite set variables range over finite sum of individuals; and class variables range over arbitrary individuals. If a countable infinity of atoms is possible, then we can show the modal existence of ω-sequences.

It is important to understand that sums of individuals are as much concrete as the individuals themselves. A mereological sum of individuals is a thing that has all such individuals as its parts, and every part overlaps some of those individuals. If a particle is an individual, then a mereological sum of particles is a concrete item that is made up of all particles and nothing else.

[79] See Boolos (1975, pp. 518-519).

[80] See Goodman (1977).

[81] Hellman (1996, p. 108). Things that overlap are things that have a part in common.

[82] x is a proper part of y iff x is a part of y and $x \neq y$. In other words, a proper part of something is a part of it that is distinct from the whole.

I have not explained yet how Hellman employs plural quantification. Plural quantification was originally motivated by the fact that some sentences in natural language cannot be naturally formulated by using first-order logic. There are sentences such as 'some critics admire only one another' or 'there are some pens on the desk' that contain plural quantifiers. Boolos and others suggest that such statements should be literally taken as containing plural quantifiers. In this regard, there two possible options: one could include plural quantifiers, i.e. $\exists xx$ and $\forall xx$, in a first-order language, or one could allow plural quantification over predicates in second-order logic.

It is interesting to examine how plural quantification can be used to eliminate ontological commitment to mathematical *abstracta.* Remember that instead of claiming that there are abstract objects satisfying Peano axioms, Hellman argues that ω-sequences satisfying Peano axioms could exist. Without plural quantification, quantifiers are interpreted as ranging over (monadic) predicates and (polyadic) relations in second-order logic. By contrast, plural quantification combined with mereology enables a reduction of polyadic relations to monadic predicates. Roughly speaking, relations and functions are both dispensable within plural quantification.[83] As a consequence, plural quantification does not depend on mastering mathematical concepts such as functions and relations.

I would now like to emphasize a few aspects of Hellman's approach that might be controversial. In the first place, Hellman neither clarifies what an individual is, nor he spells out which entities are mereologically acceptable. In addition, modal structuralists need to assume modal operators as primitive, in the sense that such operators are not required to be given a set-theoretic semantics. Hellman's structuralism relies on the general assumption that modal existence and structures are both possible, but modal structuralists should account for how we can have epistemic access to modal existence. Hellman himself understands the problem:

> What sort of evidence can we have for the various modal-existence

[83] Hellman (2005, p. 559).

> postulates arising in mathematics, as illustrated above? [...] It seems that we must fall back on indirect evidence pertaining to our successful practice internally and in applications and, perhaps, the intuitive pictures and ideas we have of various structures as supporting the coherence of our concepts of them[84]

This last remark does not undermine the epistemic objection to modal structuralism, because even if Hellman can dispense with mathematical objects, he should address the problem of epistemic access to possible structures.

[84] Hellman (2005, p. 557).

Chapter 3

Trimming Plato's Beard: Easy Roads to anti-Platonism

3.1 Deflating existential consequences

Platonists and anti-Platonists disagree on what there is. It is unclear, however, if the dispute arises from the fact that the opponents do not actually share the same criterion of existence. In that case, the disagreement would be faulty in the following sense: Platonists and anti-Platonists seem to disagree on what there is, whereas the debate is about the right criterion for what exists. Just to mention some well-known examples, one could opt for observability, causally efficacy, being in space and time, and so on. Azzouni suggests the following criterion that, perhaps, both Platonists and anti-Platonists could adopt: anything exists if it is mind- and language- independent. It is not straightforward to make precise what objects fall under that criterion, but some cases can be presented. The monster I dreamed last night while I was sleeping does not exist, since dreams are mind-dependent; a fictional character such as Sherlock Holmes does not exist, since it is made up by Conan Doyle and his readers. More generally, existence does not take place merely because one is thinking about it, or when people depict it. Dreams, fictional characters, and hallucinations, are all in the same boat: they do not exist and do not have properties either.[1]

I would like to make a remark about properties. When I open a book and read that 'Sherlock Holmes has grey eyes', I am not discovering a property of a fictional character. Properties, according to Azzouni, are strictly connected with things that exist: thus, fictional characters have no properties. Nevertheless, we constantly express true or false sentences about fictional characters and other non-existing objects. For example, it is true that 'Sherlock Holmes has green eyes' within Conan Doyle's novels, whereas 'Sherlock Holmes has yellow eyes' is false.[2] But statements about fictional objects are not true or false because of a truth-maker, i.e. because of an existing (fictional) object. In fact, we pretend that such statements are true or false during the fiction (myths, novels, films, etc.),[3] but we do not commit ourselves to the fact that Sherlock Holmes has properties by uttering a statement such as 'Sherlock Holmes has grey eyes'. Things that do not exist cannot have properties or, in other words, there is nothing to discover about non-existing objects. Everyone who is acquainted with the fact that Sherlock Holmes is a fictional character also knows that he does not really have an eye colour.

Here what seems to emerge is a pluralistic concept of truth, in which true in a fiction is different from true in the real world. But looking at our linguistic practices, we do not distinguish a fictional conception of truth from a realist one: truth does not draw any line between real and unreal.

[1]See Azzouni (2004, pp. 83–87).

[2]Although, I think, Conan Doyle never says that 'Sherlock Holmes does not have yellow eyes', we can easily infer it from the fact that he has grey eyes.

[3]See Azzouni (2010a, p. 112). More precisely, Azzouni distinguishes internal statements from external ones during a fiction. External statements are those that we pretend to be true or false, whereas internal statements are those that are true or false *simpliciter*. For example, we pretend that 'Hamlet is a prince' is true, whereas 'Hamlet is portrayed as a prince in Shakespeare's play *Hamlet*' is true *simpliciter*. See Azzouni (2010a, p. 114-123).

If we know that Sherlock Holmes is a fictional character, it is because we read Conan Doyle's novels, or because we watched movies where Sherlock Holmes is depicted as a fictional detective, and so on. In other words, there are usually facts in the world that induce the identification of fictional characters, and such facts play an indispensable role in identifying both fictional characters and real objects. When a term in a sentence does not refer to, the term does not have truth-makers, although there may be some external factors that induce a truth value of a sentence.[4]

Azzouni's picture of truth is a form of deflationism, but an anomalous one, because it is compatible with the correspondence view of truth. Deflationists usually aim at a concept of truth that does not fit with the correspondence view of truth. However, according to Azzouni, the correspondence theory of truth has to be recovered even within a deflationist picture, because correspondence grasps an important intuition that arises from our linguistic practice. The sentence 'snow is white' is true in virtue of how the world is made: the fact that snow is white. But a statement such as 'Sherlock Holmes lives in Baker street' is true even in the absence of truth-makers. More specifically, 'Sherlock Holmes lives in Baker street' is true in virtue of Conan Doyle's novels, despite the fact that there are no fictional characters.

A satisfactory concept of truth should neither be entirely developed from truth-makers, nor imply the existence of the objects under the quotation marks. Azzouni's idea makes sense if we distinguish the ontological commitment of a sentence from what a sentence is about. As a consequence, there are sentences about something even if we do not commit ourselves to the existence of what the sentences are about. If we can express true mathematical sentences without committing to mathematical objects, then it will be possible to provide an easy-road to nominalism.

[4]See Azzouni (2012a).

3.1.1 A criterion for what exists

I will say more on the relation between ontological commitment and existence in the next section.[5] I would now like to point out that Azzouni's mind– and language– independent criterion is not just a necessary condition for what exists: it is also sufficient. This point does not rule out other criteria insofar as some of them may be co-extensive with Azzouni's criterion. Let us pick out the criterion of being in space-time. It may be that, for every object x, if x is mind– and language– independent, then x is in space-time, and vice versa. Thus Azzouni's criterion may coincide with being in space-time. But even in that case, Azzouni's criterion will remain the most adequate criterion, because it can be applied to *abstracta*: to objects that are mind– and language– independent, although neither observable, nor located in space-time, nor causally efficacious, and so forth. This point, I think, is decisive for the debate between Platonists and anti-Platonists. In fact, if they share the same criterion for what exists, the metaphysical disagreement in question will not be a faulty one. Given the same criterion, the debate is *really* about what exists.

Azzouni's criterion has an epistemic value, because it emerges from our epistemic practices.[6] How do epistemic agents discover the properties and the relations of existing objects? To settle this issue, we need a non-trivial explanation of the method by which we discover the properties and relations of mind– and language– independent objects.[7] Consider a concrete object such as an apple: apples are mind– and language– independent we perceive through causal connections. In this case, we can provide an epistemic account of such connections that involves neuro-physiological concepts that tell us how the sense organs work. However, causality is not the only way of meeting such an epistemic requirement. This is because there could be other contexts where causality is not operative – although epistemic requirements

[5]See sec. n. 3.1.2.

[6]Azzouni (2012b, p. 956).

[7]In Azzouni (2004) this method is explicitly presented in terms of reliability.

are empirical.[8]

The epistemic requirement is a test for determining whether or not an object is stipulated. But it neither depends on the metaphysical criterion, nor it can be deduced from that criterion. Whereas Azzouni's criterion tells us what it is the necessary and sufficient condition for what exists, the epistemic requirement is an empirical test to distinguish what exist from what is merely stipulated. Azzouni's criterion does not apply to objects whose existence is stipulated. Hence if numbers do not exist, they must depend on our linguistic practices in the same way as fictional characters do. Here the metaphysical criterion joins the epistemic requirement: mathematical objects are mind– and language– dependent because the methods of satisfying the epistemic requirement cannot be applied: no epistemic story about how we can get knowledge of abstract objects is possible.[9] Platonists can reply by invoking the indispensability argument that forces ontological commitment to mathematical objects independently of any epistemological concern. Azzouni's counter-reply is that a sentence such as 'there are numbers' seems to commit to the existence of numbers so long as one believes that 'there is' *always* carries ontological weight. This is why it is important to stress the difference between quantification and ontological commitment.

3.1.2 Existence and ontological commitment

A metaphysical criterion for what exists can be tied to a criterion for what a discourse is committed to. In *On "on what there is"*, Azzouni claims that:

> We start with a distinction between a 'criterion for what exists' (CWE) and a 'criterion for recognizing what a discourse commits us to' (CRD). A nominalist, for example, claims that only concrete objects (of one sort or another) exist; platonists, notoriously, think otherwise [...] These are all variant of CWEs.

[8]See Azzouni (2004, p. 101).
[9]See Azzouni (2004, pp. 103–107).

> Quine (1948), however, is quite clearly not offering a CWE, but only a CRD.[10]

But even accepting the connection between commitment and existence, many questions about the nature of such a commitment still remain open. One could recognize that a discourse commits us to entities and, nonetheless, leave their properties unspecified. It is possible to fix such properties by combining ontological commitment with a criterion for what exists.[11]

One of the common criteria for ontological commitment is Quine's: regiment a discourse into an interpreted first-order language in order to track ontological commitment in every formula of the form $\exists x P(x)$.[12] By applying Quine's criterion to a domain of discourse, one gets a straightforward way of identifying what that discourse is about.[13] Quine's criterion is normative because it says what we have to commit ourselves to; the criterion is also ontologically parsimonious, because it is supported by Russell's theory of descriptions to cut off those names that do not carry ontological commitment;[14] finally, it avoids quantification over predicates as well as the corresponding ontological commitment to properties.

In Quine's hands, the criterion is used to grasp the commitment within our best scientific theories. But the criterion is not sufficient for the Platonist, because other assumptions are required in order to show that mathematical objects are outside of space-time and causally inert. Neither the criterion for which anything exists if it is mind- and language- independent can fill the gap, because *abstracta* and *concreta* are both considered by the

[10] Azzouni (1998, p. 2).

[11] We may, for example, assume that what exists is concrete.

[12] Where x occurs free in P. Quine's criterion seems plausible within the context of our best scientific theories at least. See Quine (1948).

[13] Quine's criterion should be applied to our best scientific theories, although, in principle, it could also be used in other contexts.

[14] How can we say that an entity does not exist without presupposing its existence? Quine's (1948) makes use of Russell's theory of definite descriptions to avoid the suggestion that the use of the word 'Pegasus' in 'Pegasus does not exist' presupposes that there is a Pegasus.

Platonist mind- and language- independent.

According to Azzouni, Quine's criterion for ontological commitment and the metaphysical criterion are disjointed; that is, Quine's way of recognizing ontological commitment via regimentation does not follow from the metaphysical criterion and vice versa. More generally, it is not the case that metaphysical criteria indicate the way by which we should regiment scientific theories, nor regimented existential formulas cast light on what exists. The truth of an interpreted existential formula must be defined as the satisfaction by something in a domain. Retrieving the distinction between object language and metalanguage, existential quantifiers in the object language have an ontological connotation if the quantifiers in the metalanguage also have an ontological connotation. An argument for ontological commitment in Quine's style may proceed as follows: if existential quantifiers range over a domain of discourse, and the truth of an existential formula like $\exists x P(x)$ is given by an object that satisfies x, existential quantifiers are always ontologically marked. Ontological commitment eventually follows from objectual quantifiers and from the notion of satisfaction defined by a Tarskian theory of denotation. However, Azzouni argues, Tarski's theory of truth can be applied to terms that refer to nothing at all, insofar as it is possible to extend the notion of reference *even* to those statements where vacuous terms occur.

Talking about 'extension', perhaps, is not the best way of characterizing Azzouni's strategy, since it lies in our current linguistic practices (vernacular). In the vernacular we constantly refer to objects that do not exist – a straightforward example is our talking of fictional characters. Endorsing Azzouni's view, we get a notion of reference according to which it is possible to make sense of *both* vacuous and denoting terms: a notion of reference that is labeled as reference*.

Reference is narrower than reference*. In Azzouni's view, if objectual quantifiers are taken as ontologically neutral — they do not always commit to existing objects— there is no need to change Tarski's semantics. Of course, speakers need to be able to distinguish committing uses of 'there is' from non-committing ones.

In *Talking about Nothing*, Azzouni refines the idea from *Deflating Existential Consequence*: reference* presented a singular conception of reference in which both vacuous and denoting terms occurred, whereas the new distinction between reference^r from reference^e introduces a plural conception of reference. More precisely, reference^r is the relation that connects terms (names, demonstatives, quantifiers) and objects, whereas reference^e does not require the existence of an object to which we refer^e to. For example, 'Barack Obama' refers^r to Barack Obama, whereas '2' refers^e to 2. The ordinary way of speaking does not differentiate ^{r} from ^{e} but, mixing both, leads to the disagreement between Platonism and anti-Platonism in the philosophy of mathematics. In other words, the Platonist mistakes refer^r for refer^e.

Reference^r and reference^e may sound rather artificial since ordinary speakers do not use them in the vernacular. After all, here is a bit of wordplay to make the point, what reference does it not refer to? However, words such as 'there is', 'exist', or 'refer to', do not always carry ontological commitment *even* in the vernacular. Consider an ordinary sentence such as 'there are fictional hobbits' (H), and let us examine some ways of interpreting it:[15]

1. One may provide a paraphrase of (H) in order to avoid ontological commitment. 'There are depictions of hobbits', or 'there are no hobbits', are standard ways of paraphrasing (H) without ontological commitment to fictional entities. It is clear we need to distinguish proper interpretations of (H) from incorrect ones stressing what kind of goal a paraphrase must achieve. What it is required of a satisfactory paraphrase is what it keeps the meaning of the original statement. 'There are no hobbits' works insofar as the meaning of (H) is a denial of the existence of hobbits. But the paraphrase is still inadequate because the reference to fictional discourse is missed. Consider the paraphrase

[15] There are other ways of reading (H). See Azzouni (2004, pp. 63-78).

'there are depictions of hobbits', which alludes to a fictional context. This new paraphrase can yield a sentence with an unwanted truth value. Suppose that I appreciate some virtues of hobbits, as courage and heartiness, but dislike Tolkien's way of depicting them. If I substitute 'fictional hobbits' with 'depictions of hobbits', I could not appreciate hobbits' virtues unless I liked the way hobbits are depicted. These counter-examples show how the most common paraphrasing options fail to preserve the meaning that (H) sustains.

2. Another possibility is to consider (H) as a sentence about hobbits, and when I assert (H) I commit myself to the existence of fictional entities. All the uses of 'there is' are ontologically committing under this interpretation. It is appropriate to attribute the adjective 'meinongian' to such fictional entities, since they are 'real' non-existing objects. However, ordinary speakers clearly do not intend to commit themselves to the existence of fictional objects when utter a sentence such as (H). Thus, the meinongian route does not have a grip on our common linguistic practices.

3. I would now like to consider a reading of (H) based on the distinction between literal meaning and metaphorical meaning. According to this interpretation, (H) can be read on two different levels: it is false taken literally, since there are no hobbits, but it is true metaphorically. When (H) is understood metaphorically, the words do not carry ontological commitment any more. This strategy has two main advantages over the others: first, it does not seem to deny our common linguistic intuitions as the Meinongian option does; secondly, it does not require any paraphrases. The couple literal/metaphorical works insofar as the distinction between the two levels is plain, that is, when nobody has troubles to make out whether or not a word operates metaphorically. The notion of pretense, or make-believe, plays an essential role: properties can be attributed to non-existing objects by pretending that certain

statements are true during the fiction. For example, I can say that 'Frodo Baggins is the ring-bearer' is true, despite the fact that Frodo does not exist. However, there are statements about fictional characters that are true *simpliciter* in the vernacular, i.e. independently of any pretense.[16] Some examples are statements such as 'Pegasus does not exist', or 'Bilbo Baggins is a fictional character'. In such contexts no pretense comes into play, but those statements are still true, not just truth-apt, in contrast with a statement such as 'Frodo Baggins is the ring-bearer'.

4. The temptation to consider (H) true, and not merely truth-apt, is quite strong. When (H) is considered literally true, the problem is how to avoid ontological commitment to fictional entities. An interesting suggestion comes up from the word 'fictional' when it is used to draw a line between committing uses of 'there is' and non-committing. But ordinary speakers do not always have such a word in order to distinguish between fiction and non-fiction. Context and background information play a crucial and indispensable role to tell whether or not a term is committing ontologically without invoking fictional operators.

I would like to point out that none of those options constitute a conclusive argument for the right interpretation of (H). Azzouni's neutralism is an alternative reading in which quantifiers, nouns, and demonstratives are taken as ontologically neutral. Such a viewpoint is different and even stronger than claiming that 'there is' sometimes is ontologically committing. Azzouni does not argue that the uses of 'there is' are vague or that shows a pluralistic conception of existence. Ordinary speakers are able to recognize whether or not a term carries ontological commitment. If they are not, they have no access to salient data. Background information is essential in that regard: for example, if I did not know that Sherlock Holmes is a fictional detective, I could believe he was a real detective who lived in London a long

[16] Azzouni (2010a, pp. 114-115).

time ago. This does imply that 'there is' is essentially vague: I was wrong because I lacked relevant information.

Azzouni aims at rejecting Quine's criterion by claiming that existential quantifiers are ontologically neutral. Quantification, on its own, is not sufficient to carry ontological commitment, which is justified only for those objects that are independent of our psychological processes and language. And If quantification is released from ontological commitment, nominalists do not need to reject the indispensability argument. Mathematical statements are indispensable because make it possible to represent empirical phenomena and manipulate scientific theories. Following Azzouni's terminology, we use *assertively* mathematical statements in empirical science.[17]

Despite the indispensable use of quantifiers, mathematical objects are stipulated: they are made up and depend on our psychological processes.[18] Of course, an argument is needed to show that mathematical objects are fictional. We require a test for discovering the properties of a mind- and language- independent object, and Azzouni aims to show that standard methods for showing the existence of abstract objects fail in that attempt.

The Platonist could argue that mathematical objects are known *a priori*, where 'a priori' means independently of experience. But even if we take *a priori* knowledge for granted, we should still account for our *a priori* intuition of mathematical objects. The best candidate would lie in some cognitive mechanism, though it is unclear what kind of cognitive mechanism could justify the intuition of objects that are supposed to be outside of space and time. Another difficulty for Platonism is how to explain the applications of mathematics. According to the indispensability argument, we do not need to explain the correlation between mathematical *abstracta* and the physical world, because such *abstracta* are indispensable to formulating our best empirical theories.[19] But look how tricky the argument is: it does not tell us whether or not mathematical objects are mind– and language– indepen-

[17] Azzouni (2010a, p. 298)
[18] Azzouni (2004, p. 103).
[19] Colyvan (2001, p. 11).

dent. In the absence of an epistemic story that would explain the connection between the world and mathematical *abstracta*, we have no reasons to believe that mathematical *abstracta* are mind– and language– independent. The point is not a conclusive argument against Platonism, but it provides a good reason to believe that mathematical objects are mind– and language–dependent.

Consider the following counter-argument: if nothing satisfies $\exists x P(x)$ then, according to standard semantics, the formula turns out to be false. Thus, Azzouni must either show that scientists regard existential statements as false, or he must provide an alternative semantics for those formulas. Azzouni avoids the dilemma by claiming that a sentence can be either true or false without truthmakers: that is to say, without any object that determines the truth-value of a sentence.[20] There are other factors in the world that can force the truth-value of that sentence: truth-value inducings. These factors are not objects, and thus quantifiers do not range over truth-value inducings. Azzouni prefers to talk about 'truth-value inducings' over 'truth-value inducers' because 'truth-value inducers' seems to refer to existing objects. However, according to Azzouni, quantifiers range over fictions in mathematics. Instead, truth-value inducings are conventions, sociological facts, fruitfulness of applications, and so on.

It is important to notice that Azzouni is not a formalist, where by 'formalism' I mean the fact that mathematical statements do not lack content.[21] On the contrary, mathematicians prove *truths* and their consequences that may turn out to be indispensable to our best scientific theories. This is why Azzouni is not a standard fictionalist, if we mean by 'fictionalism' the position that mathematical statements are false. Of course, the standard fictionalist could change the truth-value of a mathematical statement from false to true via the fictional operator 'according to'. For example, they may

[20]For instance, a sentence such as $2 + 2 = 4$ is true without truthmakers. See Azzouni (2004, p. 57).

[21]See Azzouni (2004, p. 48): 'The use of the truth idiom in science prevents instrumentalist construals of applied mathematical doctrine'.

say that 'Hercules is a demigod' is literally false, since there are no demigods, but it is true according to Greek myths; according to Peano Arithmetic, it is true that there exists a number which is $2 + 2$. More generally, false mathematical statements become true via fictional operator in metaphorical contexts. Mathematics is fiction alike not in the sense that mathematicians are story-tellers, but because mathematical statements semantics is treated as fictional statements semantics.

In contrast to standard fictionalism, Azzouni does not need to stress the distinction between literal meaning and metaphorical meaning — i.e. because quantifiers are ontologically neutral. In addition, Azzouni's strategy avoids all the technical difficulties of hard-road programs providing an easy road to nominalism. The strategy is 'easy' because does not paraphrase mathematical statements, but the price is to get rid of Quine's criterion of ontological commitment. In the next section, I will examine Azzouni's counter-proposal to Quine's criterion.

3.1.3 Grades of ontological commitment

Neutral quantifiers range over both fictions and existing objects. To put it another way, neutral quantifiers range over *posits*. Posits are tracked in a first-order regimented discourse by formulas of the form $\exists x P(x)$, where $P(x)$ is any formula with x free. If quantifiers range over mind– and language–independent posits, the commitment is called 'ontological', whereas if they range over fictions, the commitment does not have existential weight. Azzouni calls the latter 'quantifier commitment' in order to emphasize the lack of ontological commitment.[22] Note that it is possible to disagree with Azzouni about whether or not mathematical objects are fictional, but still accept the distinction between quantifier and ontological commitment.

Many posits are indispensable to building up the scientific image of the world. Azzouni rearranges the posits that are indispensable to scientific theories according to their epistemic burdens. In this regard, Azzouni

[22] Azzouni (2004, p. 127).

subdivides posits into three categories: ultra–thin, thin, and thick posits. Ultra–thin posits have no epistemic burdens at all, because they are merely postulated by writing down definitions or axioms. Both mathematical objects and fictional characters are the most common examples of ultra-thin posits. Mathematical posits cannot change their epistemic status because play no epistemic role in mathematical proofs: that is to say, mathematical objects are always mind– and language– dependent. In *Deflating existential consequences*, Azzouni called ‘thin’ those posits that pay the ‘Quinean rent’ by contributing to our web of beliefs. Thin posits were exemplified by Quinean virtues such as conservativism, modesty, simplicity, testing, and refutability.[23] More recently, Azzouni has revised his position claiming that *every* posit satisfies Quinean virtues:

> What I should have said about this is what I'm saying now: that thin posits are the items we commit ourselves to on the basis of our theories about what the things we thickly access are like.[24]

This quote emphasizes the role of theories in scientific inquiry altogether, but requires some additional clarifications. Let us first see what Azzouni means by ‘thickly access’. Thick epistemic access is determined by causal relations between epistemic agents and posits. Thick epistemic access makes possible the detection of objects directly through sensory organs or technical instruments. There is a wide range of posits to which we are thickly connected: from the birds that we observe with the naked eye, or through binoculars, to a Geiger counter that detects the emission of nuclear radiations. Roughly speaking, thick epistemic access operates under the following conditions:

Robustness: The result of thick epistemic access to an object is independent of epistemic agents' expectations. According to robustness, detecting properties of thick posits is a process epistemically independent

[23] Quine and Ullian (1978).
[24] Azzouni (2012b, p. 963).

of theories. Robustness aims to limit the role of confirmation holism or, in other words, the role of theories in scientific practice. The idea is to provide a picture of scientific practice that takes into account what Azzouni calls 'gross regularities'.[25] Scientists and engineers do their job by practicing with instruments regularly without being necessarily aware of what theories are behind their activities. An engineer can use an oscilloscope correctly without being aware of what happens on the subatomic level, or what chemical mechanism allows his movement of hands during muscles stretching. Even though gross regularities can be presumably incorporated into scientific theories, the point is that scientists become skilled independently of the scientific theories behind their job. Gross regularities stress how scientific practice is epistemically independent of scientific theories as a whole. As a result, confirmation holism is partially limited by scientific practice. Azzouni is not denying that many theories are required to build sophisticated instruments in contemporary science. Of course, it would have been impossible to build a modern microscope without optics and the study of electromagnetic waves, or a particle accelerator without the knowledge of electromagnetic fields and atomic properties of materials as tritium. However, the process whereby scientists and engineers become skilled is epistemically independent of scientific theories as a whole. I can learn to use a digital ammeter to measure the electric current without being an engineer, or even if I do not know what it is going on at the subatomic level.

Refinement: Thick epistemic access can be refined. This process is essential in scientific research as well as our common experience. Think about when we move closer to a butterfly to observe more carefully the color of its wings, or when Torricelli built the first mercury barometer to measure the atmospheric pressure. Because many scientific theories are involved when a new instrument is created, or to refine a current

[25] See Azzouni (2000, pp. 63-70).

one, theories play an important role in the activity of refining. However, because the most important reason to refine our thick epistemic access is primarily the achievement of robustness, refinement seems to be subordinate to robustness.

Monitoring: Epistemic agents repeat an experiment several times to confirm an hypothesis, or to check whether or not an objects has a property. Such activities are part of a general process that is called 'monitoring'. For example, physicians can analyze drug's effects inside an organism by checking it over a period of time. Monitoring is strongly tied to observations in a limited portion of space and time.

Grounding: Thick epistemic access requires an empirical explanation of the connection between the properties of posits and our ability to know them. This connection depends on empirical assumptions about how thick epistemic access operates, and it may involve a complex scientific apparatus that accounts for how perception functions. However, an epistemic story is required to bridge the gap between sensory organs and properties — i.e. 'how we know what we know'.[26]

The above conditions characterize thick epistemic access. However, some conditions can be applied to ultra–thin posits. Consider refinement, for example. Mathematicians, one could argue, do not build new tools in order to refine their knowledge of mathematical posits. What is required to prove theorems, conjectures and results, is merely to acquire a certain mathematical skill. This picture is, however, inaccurate: computers have become important in refining mathematical knowledge. The most popular example is perhaps the proof of the four color theorem by Appel and Haken.[27] The proof of the four color theorem covers almost a billion cases, so every case

[26] Azzouni (2004, p. 136).

[27] The theorem states that the regions of any simple planar map can be colored with only four colors, in such a way that any two adjacent regions have different colors. A planar map is a set of pairwise disjoint subsets of the plane (regions). A simple map is one whose regions are connected open sets.

cannot be taken into account by hand. Leaving aside philosophical worries about the status of this kind of proofs, computer's assistance has changed the picture of contemporary mathematics. We can now build instruments in order to refine our mathematical knowledge. Even monitoring condition can be applied to ultra–thin posits, because mathematicians are able to monitor proofs by computer assistant programs. Someone could argue that computers are only heuristic tools that may be dispensed with if humans could compute every single case. But I do not think this maneuver is relevant, because robustness and grounding are the only indispensable features of thick epistemic access. Robustness is vital to circumnavigate the existence of mathematical objects based exclusively on confirmation holism, whereas grounding requires an epistemic story between mathematical objects and sensory organs. The second point matches with Azzouni's argument against the existence of mathematical objects based on reliabilist epistemology.

To determine what exists, we first look at thick epistemic access, because thick posits are mind– and language– independent. The metaphysical criterion has an epistemological value when we have epistemic access to mind– and language– independent posits. Consider a theoretical object such as Higgs boson. In order to claim that Higgs boson exists one has to show that we have thick epistemic access to it. Scientists discovered the existence of Higgs boson by making use of a very powerful particle accelerator which allows the collision between two beams of accelerated particles and detectors. In this sense, the accelerator is thickly in touch with particles. But considering that Higgs boson decays very quickly and particle collisions are infrequent, scientists had to refine the accelerator to get an high luminosity plus use sophisticated computers to reconstruct the decay process. The entire procedure can be partially monitored but gaps are eventually filled by using statistical analysis. Finally, it is plausible to think about an epistemic story that explains the connection between scientists and measuring instruments. Perhaps a story that involves visual perception through retina encoding and processing of light.

I have already shown how the interplay between robustness and refinement occurs. Robustness limits the role of theories in scientific practice, whereas refinement gets them back partially. But since refinement aims to achieve robustness, holism is still overshadowed. In Azzouni's view, the role played by theories is recovered by objects to which scientists do not have thick epistemic access. This is because scientists can deduce the existence of posits from background theories even without thick epistemic access. The classic example is perhaps the deduction of Neptune through the irregular orbit of Uranus based on Newton's laws of motion and gravitation. Neptune was seen through a telescope in 1846 but its existence had been already argued for by the mathematician Le Verrier. Drawing a moral from this event, it seems we can infer the existence of something even without thick epistemic access: existence and thick posits are not co-extensive.

Before Neptune was observed through a telescope, which provided thick epistemic access to the planet, it had been considered a thin posit. Even though we do not have thick epistemic access to thin posits, it is counter-intuitive to claim that such posits do not exist. After all, Neptune had already been a mind– and language– independent object before it was observed. A few years later, scientists confirmed Le Verrier's hypothesis by using a high-performance telescope. Neptune did not start to exist magically: it has simply changed status becoming a thick posit or, in other words, the role of Neptune within our web of belief has changed.

Nepture's case shows how thin posits can change status into thick posits in order to achieve thick epistemic access. Hence thick and thin are movable categories.[28] Technological limits, calculus mistakes and wrong hypothesis, can prevent or slow down thin/thick shift. Thin posits are derived by thick ones by suing mathematics. Ultimately, thin posits are

> items we commit ourselves to on the basis of our theories about what the things we thickly access are like'.[29]

[28]Azzouni (2012b, p. 958).
[29]Azzouni (2012b, p. 963).

The distinction between ultra–thin, thin, and thick, casts light on some classical arguments against the existence of abstract objects. Benacerraf's problem about how we can get knowledge of mathematical objects raises when one mistakes ultra–thin posits for thick ones. Quine's solution to Benacerraf's problem regards ultra–thin posits as thin posits such that thick epistemic access does not apply to them. If mathematical objects are thin posits, there will be no need to have an epistemic story about them. But it is possible to argue against this maneuver by claiming that whereas thin posits can turn into thick, by contrast ultra–thin posits cannot, because we would require robust and grounded epistemic access to them.

3.1.4 Ontological nihilism and objectivity

Penelope Maddy introduces several types of philosophers that she takes to have played a significant role in determining what people mean by 'philosophy': the first philosopher, the second philosopher, and the second metaphysician.

Maddy's primary opponent is the first philosopher, who attempts to solve the ontological problem from a pure philosophical foundation.[30] Opposing this figure is Maddy's second philosopher, who begins her ontological inquiry by accepting the large amount of work that physicists, linguists, psychologists, and so on, have provided so far. The idea is summarized by the slogan 'science comes first, philosophy as second'.

There is another problematic figure: this is the second metaphysician. The second metaphysicians requires the naturalization of epistemology in order to defend a metaphysical claim. Azzouni's position is considered by Maddy an example of second metaphysics; that is to say, it is a strategy for defending nominalism.[31] According to Maddy, Azzouni's program leads to a form of ontological nihilism where the question about what exists cannot be settled at all. In Azzouni's view, Maddy argues, all the competing criteria

[30]Descartes is portrayed as the classic example of first philosopher. See Maddy (2007, pp. 11-19).

[31]Maddy (2007, pp. 392-411).

for what exists are on par with respect to our scientific theories. Science cannot answer ontological questions on its own without metaphysics. Maddy draws that conclusion from this Azzouni's thought experiment: imagine two communities with opposite beliefs about fictions. The members of the first group do not believe in the existence of fictional entities, whereas the members of the second community do. Azzouni argues that:

> On what grounds can we adjudicate between these views? There is nothing in the second community's practices that fixes what its fictional terms refer to in such a way that evidence can be brought to bear for and against these alternatives to decide among them.[32]

In principle, the issue whether or not fictional entities exist seems to be indeterminate, since the opponents agree on all the relevant facts that are necessary to settle the debate. Scientific evidence cannot help either. Hence Maddy notes that, in Azzouni's view, scientific practice cannot answer ontological questions, because metaphysics cannot be naturalized within our scientific world–view.

With respect to Azzouni's thought experiment I should like to point out two things: first, Azzouni seems to be skeptical about the possibility of settling the ontological debates even by running *pure* philosophical arguments and secondly, Azzouni cannot be considered a naturalist only if by 'naturalist' we mean Maddy's second philosopher. The second philosopher is resolute in determining what exists on the basis of the scientific practice without any external metaphysical criteria. In fact, the second philosopher no longer regards science and ontology as two separate subjects, since science has the resources to answer the ontological question. Any distinction between ontology and scientific practice disappears according to second philosopher.

In Maddy's view, Azzouni is not engaged in figuring out what exists, because there is no objective way of figuring out whether or not fictional

[32]Azzouni (2004, p. 94).

objects *really* exist. At best we can take as existent only what our linguistic community considers as such. But the ontological question, Maddy insists, sounds like 'what is there?' and not what people are inclined to think there is.[33] In other words, the second metaphysician dodges the initial ontological question.

Turning to Azzouni's thought experiment, it is easy for the second philosopher to decide which of the two community is wrong: because there is no scientific evidence that supports the claim that fictional objects exist, then they do not exist. Therefore, the scientific practice has *normative* force in determining what exists (and what not):

> the ordinary science, not the criterion, is what's doing the work: the criterion provides no independent ground for charging the scientist with 'double–think' or 'intellectual dishonesty'.[34]

To sum up, Azzouni is regarded by Maddy as a metaphysical nihilist because, the ontological problem about what *really* exists turns out to be unsolvable. Unlike Azzouni who begins his inquiry with the demand for a general criterion of existence, Maddy's second philosopher looks at what kind of scientific evidence we have in order to settle the ontological dispute.

I believe Maddy's proposal is attractive: the endless metaphysical disputes are solved by looking at science without any criterion for what exists. However, as far as I know, Maddy does not say how we can look to the practice of science. And it is here that Maddy diverges from Quine's attitude. Unlike Quine, Maddy does not seem to be interested in advancing criteria for ontological commitment, nor in challenging them. For Maddy, ontological commitment is not based on a criterion but on what scientists regard as real — with the possibility of scientific revisions, of course. The point, I think, is that Maddy is concerned only with this methodological

[33]Maddy (2007, p. 399).

[34]Maddy (2007, p. 402). 'Double–think' is a Quine's expression, whereas 'intellectual dishonesty' is Putnam's.

shift in ontological inquiries: from the priority of philosophical arguments to the naturalization of ontology in sciences.

Before I consider Azzouni's reply to Maddy, I will present an objection to Azzouni's approach due to Raley. According to Raley, Azzouni's view cannot be regarded as genuinely nominalist if by 'nominalism' we mean the metaphysical position according to which abstract objects do not exist.[35] Azzouni's general strategy, Raley notes, can be summarized saying that we cannot justify a criterion for ontological commitment if we do not justify a criterion for what exists. However, Raley argues, Azzouni undermines his argument in favour of nominalism by adopting this strategy.

To see what Raley has in mind go back to the aforementioned thought experiment on communities with different beliefs about fictional entities. In principle, Azzouni's thought experiment could be applied not only to fictional entities, but to observed entities, or to entities in space and time, and so on. Even though Azzouni has not extended his argument so far, it is not hard to imagine different communities with dissimilar criteria of existence. If Azzouni's thought experiment works even in such possible scenarios, it shows that there is no philosophical argument powerful enough to validate the superiority of any criterion of existence over its competitors. If we agree on all relevant facts that would be applicable to make a decision, then the choice of the correct criterion for what exists turns out to be philosophically indeterminate.

Raley notices that Azzouni's position on the indeterminacy of metaphysical criteria presupposes Azzouni's distinction between criterion–transcendent and criterion–immanent words.[36] Consider as an example the word 'gold'. Criteria for detecting gold have changed over time, and some of them have altered what fell before under the term 'gold', i.e. its extension. When a new criterion is discovered, it changes the extension of the word 'gold'. A long time ago, people had criteria for identifying gold unlike ours such as color, weight, sturdiness, and so on. The discovery of the atomic num-

[35]Raley (2009, p. 74).

[36]Raley (2009, pp. 77-80).

ber of gold has changed what falls before under 'gold'. Certain items that were consider golden before its identification with the atomic number 79 are no longer considered golden nowadays. Gold is 'criterion–transcendent' in the sense that the criteria for identifying what is gold have been changing over time, and have been modifying the extension of the word 'gold'. More generally, criterion–transcendent words are retroactive: when a new criterion for detecting something is changed, previous uses of it are considered mistaken.[37]

Azzouni notices that not every term in language is criterion–transcendent. Think about the word 'hammer'. The hammers crafted by blacksmiths in the Iron age are unlike the hammers we use nowadays. At least shape and materials are different. The word 'hammer' nowadays has a different extension than the Iron Age. Nevertheless, we do not regard the uses of 'hammer' in the Iron Age as mistaken. An Iron Age hammer is still considered a hammer. Crafting a new hammer does not change what fell before under the word 'hammer'. When a term is immune to this kind of revision — i.e. the revision is not applied retroactively — the term is 'criterion–immanent'.

Azzouni argues that words as 'true', 'refer', and 'exist' are all criterion–transcendent.[38] The word 'exist' is criterion–transcendent, and so it is similar to 'gold'. However, Raley notes, this implies that if our linguistic attitude to the word 'exist' changed, the revision would be applied retroactively. But then the problem of what *really* exists independently of any linguistic attitude is simply dismissed in Azzouni's view. In this sense, Azzouni turns out to be a metaphysical nihilist.

I do not think that Azzouni rejects Raley's point of view altogether. He just restricts himself to pointing out that even if 'exist' is a criterion-transcendent word, he does not commit himself to a non-factualistic view — i.e. that there is no fact of the matter about what exists. In the same way, to claim that 'gold' is criterion–transcendent does not imply that there is no

[37] Azzouni (2010b, p. 87).

[38] Azzouni (2010b, p. 89). I will not present Azzouni's argument but just focus on Raley's objection.

fact of the matter about what the term 'gold' refers to.[39] In other words, giving up the idea that terms have a fixed reference is not enough to endorse non-factualism. Despite this, there might be a worry about the objective reference of some words. The extension of 'gold' seems to be objective by virtue of the atoms of gold independently of which meaning linguistic communities attribute to 'gold'. Unlike the case of fictional objects, we cannot stipulate properties of mind– and language– independent objects such as gold. There is a strong normative constraint involved in criterion–transcendent words due to their epistemic role. As we do not stipulate the properties of gold, so we do not stipulate what things exist.

Raley's point resembles Maddy's objection to Azzouni about the ontological problem and its objectivity. Maddy is concerned about the source of Azzouni's ontological criterion: the fact that the criterion arises from our common linguistic practices and not from science. Maddy's concern, I believe, is that Azzouni's criterion functions so long as it is shared in our linguistic practices: it works more by convention instead of being grounded on an objective practice.

Criterion–transcendent words are bonded to their epistemic role, but his maneuver partially answers Maddy's worry. What is at issue is the extent to which Azzouni endorses Maddy's view of naturalized ontology. Bear in mind that Maddy's second philosopher regards ontological inquiry as a subject that can be completely naturalized leaving no room for criteria that come from common linguistic practices. Maddy's objection is basically methodological: ontology begins from science and not by looking at our linguistic practices.

3.1.5 Defeasibility condition and fuzzy posits

I would now like to present and discuss a different objection raised by Colyvan. According to Colyvan, Azzouni's boundary line between what is real and what is not can be boiled down to the distinction between causal and a–

[39] Azzouni (2010b, p. 98).

causal entities.[40] The reason is straightforward: both thick and thin posits are entities with which we can have causal interaction, whereas ultra-thin entities lack it. If Colyvan is right, he can run an argument against the relation between causality and existence, for example by showing, first, how causality fails to establish the existence of stars and planets outside of our own light cone and secondly, how a–causal entities play an indispensable role in the scientific explanations.[41] According to Colyvan, causal idleness of mathematical entities cannot be an excuse against their real role in scientific theorizing, unless one is able to show the dispensability of mathematics in empirical sciences.[42]

Colyvan strikes another objection noticing how thick, thin, and ultra-thin posits can differ from each other in degrees. The point is not just that the distinction between posits is fluid, but seems to imply the existence of fuzzy posits: that is, posits that satisfy only partially Azzouni's epistemic burdens. The example described by Colyvan is Alpha Centauri, which satisfies the conditions of thick epistemic access, but it is less thick than Saturn, because the refinement condition is partially satisfied— Alpha Centauri is much farther away than Saturn. In this sense, Alpha Centauri can be considered a borderline thick posit.[43] In addition, Colyvan suggests that even mathematical entities could be considered borderline posits (very-thin): entities between ultra–thin and thin which enjoy Quine's virtues and have a place in ontology.

Azzouni's answer is based on two different replies: first, Azzouni shows why thin posits are not thick and secondly, he argues against the existence of very–thin posits. According to Azzouni, the reasons why thin posits are not thick come from the scientific practice itself.[44] Bear in mind that thin posits are postulated on the basis of the theories that scientists have developed about thick posits. Scientists postulate the existence of posits to

[40]Colyvan (2010, pp. 289).
[41]See for example Colyvan (2001, pp. 39-57).
[42]Colyvan (2010, p. 233).
[43]Colyvan (2010, pp. 230).
[44]Azzouni (2012b, p. 961).

which we have thick epistemic access: thin posits are postulated but not *ex abrupto*. Scientists are also able to distinguish thin posits from thick ones. Geologists for example are able to distinguish whether a fossil is thick or thin by taking account of trails, other fossils, DNA data, and so on. As Azzouni notices:

> So it's scientific theories themselves that enable us to thinly posit animals and kinds of animals (apart from the animals we have actually interacted with), in particular, to thinly posit the existence of animals that were the sources of the fossils we have discovered.[45]

Qualified scientists are able to distinguish thin from thick posits. But what about very–thin posits? Do they exist? Colyvan is worried about whether mathematical objects have an "excuse" to be considered very–thin or not. According to Colyvan:

> The posits I have in mind, are more than ultra-thin, because they pay their Quinean rent — they are part of a well–confirmed empirical theory exhibiting the Quinean virtues — but they are not thin, because they do not come equipped with an excuse as to why they are not thick posits.[46]

Colyvan characterizes very–thin posits as those posits that would be thin if they satisfied what Colyvan calls 'the defeasibility condition'. The defeasibility condition is an excuse whereby thin posits are not thick but, nonetheless, are listed in our ontology. Defeasibility conditions are basically stories about why posits are thin but not thick. Very–thin and thin posits are alike in the sense that they both play an important role in our best scientific theories, except for the fact that very–thin posits satisfy the defeasibility condition. Colyvan also notes how the defeasibility condition limits

[45] Azzouni (2012b, p. 962).

[46] Colyvan (2005, p. 221).

Azzouni's anti-realism only to ultra–thin posits and allows mathematical entities to be part of our ontology. The defeasibility condition for mathematical entities is this: mathematical entities are not thick simply because they are *abstracta*.[47] Since we can provide a straightforward defeasibility condition for mathematical posits, they deserve ontological commitment as well as other thin posits.

As mentioned earlier, Azzouni has revised his position about ultra–thin posits, according to which even they enjoy Quine's virtues. As a consequence, Colyvan should not have further reasons for claiming that very–thin posits exist. But Colyvan's argument could be reformulated as follows: mathematical objects can be considered thin posit because we have a straightforward excuse why they are not thick — i.e. they are abstract. The problem is whether or not Colyvan's excuse is legitimate and, more generally, what counts as a legitimate excuse. According to Azzouni, Colyvan's excuse misses its target because mathematical objects play no *epistemic* role in mathematical proofs. No epistemic story has a role in the explanation of how we establish mathematical truths. We can tell several epistemic stories for those posits taken into account by the empirical sciences: that is to say, we have scientific reasons that are rooted in our scientific theories for certain posits to be thin, whereas mathematical posits lack such reasons.

It is important to understand what scientific reasons are. The abstractness of mathematical objects, Azzouni argues, does not count as a scientific reason because it is not rooted in our scientific theories. This point requires some additional clarifications. Scientific reasons seem to coincide with epistemic reasons in this context. According to Azzouni's epistemic puzzle, we do not have an epistemic story about the role that mathematical posits play in mathematical practice. So, even if mathematical posits were concrete objects, they would not play an epistemic role in the *practice* of mathematical proofs.[48] Instead, thin posits play an epistemic role in the practice of empirical science. Biologists, for example, can provide an epistemic story

[47] Colyvan (2005, p. 223).
[48] Azzouni (2012b, p. 963).

for thin posits like dinosaurs, even though biologists have never encountered a dinosaur. The existence of dinosaurs is postulated on the basis of fossil records and other empirical data; the stories provided by biologists count as scientific insofar as they are epistemic stories. Instead, in the case of mathematical posits, the indispensability argument does not provide an epistemic story: the argument aims to overcome such a story. Colyvan's argument does not count as an excuse because the story he provides to characterize mathematical posits as thin does not have any epistemic values.

3.1.6 Bueno and Zalta's objection

Bueno and Zalta challenge Azzouni's view by running an argument that is similar to some of Burgess and Rosen's objections to nominalism.[49] Bueno and Zalta start with noticing how many contemporary nominalists reject the existence of mathematical objects by using formal apparatus such as second-order modal logic or fictional operators. These strategies notoriously fail to keep the literal meaning of mathematical statements. The beauty of Azzouni's approach, on the other hand, is to regard mathematical statements as literally true without committing to the existence of the objects these statements are supposed to be about. However, Bueno and Zalta argue that Azzouni's view faces a serious dilemma because of what it implies about the notion of reference. Let us see how the dilemma goes:[50]

(S_1) Either numerals, like '2', refers to numbers, like 2, or they do not.

(S_2) If numerals refer to numbers, nominalism is false.

(S_3) According to Azzouni, nominalism is true and mathematical sentences can be taken at face value. Thus numerals do not refer to numbers.

(C) However, if numerals do not refer to numbers, either mathematical sentences cannot be taken as literally true, or Azzouni has to appeal to a non-standard notion of reference.

[49] See sec. n. 1.4.

[50] Bueno and Zalta (2005, p. 297).

With regard to the former point, if mathematical sentences are not literally true, Azzouni's project runs into the same objections that other nominalistic programs do, because it does not do justice to mathematical practice. Alternatively, Azzouni might substitute the standard notion of reference with something else, but when a non-standard notion of reference is involved, mathematical language cannot be taken literally either. As a result, Bueno and Zalta argue that either Azzouni should give up his nominalism by claiming that numerals refer to numbers, or he should drop the idea that mathematical statements are literally true.

Azzouni replies to Bueno and Zalta in *Talking about Nothing*.[51] According to Azzouni, Bueno and Zalta's dilemma arises from an equivocation in the concept of reference. Bear in mind that, distinguishing between reference^r and reference^e, Azzouni argues that numerals refer^e to numbers but dot not refer^r. Since we can refer to numbers by reference^e, which is neutral from an ontological point of view, the second premise of Bueno and Zalta's dilemma turns out to be false.

I think that Bueno and Zalta can still press their argument against Azzouni. They could argue that mathematical sentences are not taken at face value by using the notion of reference^e. A very straightforward way of doing this is to argue that reference^e is a non–standard notion of reference. Even though numerals can refer to numbers, the price we must pay in order to sustain nominalism is to adopt a non–standard notion of reference. The crucial point, I think, relies on what Bueno and Zalta mean by 'non–standard'. This passage should make their claim clearer:

> Note that we are not saying that the notion of reference requires the existence of an object to which we refer. But it does require that there be something to which we refer. In any case, a non–standard notion of reference is indicative that mathematical language is not being taken literally.[52]

[51] See Azzouni (2010a, pp. 43-45).

[52] Bueno and Zalta (2005, pp. 298-299).

Using Azzouni's terminology, they do not seem to require the identification of reference with reference^r. We can quantify over things that do not exist, like fictions.[53] What I think Bueno and Zalta object to Azzouni can be summarised as follows: if the notion of reference is neutral, then it is non–standard. Therefore mathematical sentences cannot be taken literally, and Azzouni's approach looses its face–value virtues. In short, we can see Bueno and Zalta's argument not as a true dilemma but, instead, as a point against the neutral concept of reference.

It does not seem to me that Bueno and Zalta provide an argument to show how reference^e is non–standard. At this point Azzouni's response could be articulated by showing how reference^e is standard. He could for example show how reference^e can be tracked down in our linguistic practices. The vernacular would indicate some contexts in which the reference is taken as ontologically neutral, and reference^e aims to encode such contexts. If the neutral concept of reference can be found in natural language, this point shows that reference^e is a standard notion of reference. Moreover, Azzouni has another line of defense against Bueno and Zalta by showing how reference^e can be exemplified in a Tarskian theory of truth.[54] Of course, this strategy is effective insofar as one accepts that the Tarkian theory of truth grasps what we intuitively mean by 'reference'. If we can really frame reference^e into a Tarkian theory of truth, this maneuver will significantly undermine Bueno and Zalta's objection.

[53]Fictions, even if not objects, would be something in any case.

[54]See Azzouni (2010a), Ch. 5, for further clarifications.

3.2 How logical subtraction works

Anti-Platonists are sometimes misrepresented as denying the role played by mathematics in scientific practice. On the contrary, what is at issue is if mathematics can continue playing that role without postulating the existence of abstract objects: ontology is challenged, not mathematics. To argue against the existence of abstract objects, anti-Platonists can commit themselves to the nominalistic content of scientific theories. Field is a good example in that regard. According to Field, one can extract the pure nominalistic content from Newton's theory of universal gravitation, that is, what that theory says about the physical world. At a deep level Newton's theory describes the behavior of regions of space–time, and the extraction of its nominalistic content can be seen as a sort of subtraction by which one takes away mathematical objects from physics — the nominalistic content is what is left. In the second chapter, I elaborated on how the success of Field's program requires that any nominalistic consequences of Newtonian gravitation theory can be proved from its nominalistic content.[55] In other words, Newtonian gravitation theory has to be conservative over its nominalistic content.

In contrast to what Colyvan argues,[56] the anti–Platonist does not have to endorse Field's program, nor commit himself to any specific nominalistic content. Azzouni's criterion of existence does not tell us what the nominalistic content of physical theories is, and some alternatives to Field's program have been advanced during the last few years. In this chapter I will present Yablo's approach to stripping away unwanted ontological consequences from hypotheses that presuppose the existence of mathematical objects.

According to Yablo, the nominalistic content of a theory T can be represented as what is left after subtracting the existence of mathematical objects from T itself. Yablo's strategy does not involve the commitment to any nominalistic content, unlike Field's case. Generally, a strategy is called 'easy'

[55] See sec. n. 2.1.1.

[56] See Colyvan (2010).

insofar it does not require any paraphrases of mathematical statements in order to avoid ontological commitment. Azzouni's and Yablo's approaches do not need paraphrases at all: both strategies are easy in that regard. Easy roaders argue that mathematical statements are true and useful even if there are no mathematical objects; hard roaders argue that mathematical statements can be useful even in absence of mathematical objects.

Hard roaders are often committed to nominalistic content. Field, for example, appeals to the representation theorems in order to prove the equivalence between the set of quadruples of real numbers and its nominalistic counterpart — the "structure" of space-time. In short, a representation theorem allows to paraphrase statements that quantify over mathematical objects where space-time is involved, and it shows how Newtonian mechanics is committed to regions of space-time. If Field's strategy truly exemplifies the hard road, as Colyvan claims, it does not lead to a form of ontological nihilism. However, as I showed earlier, other hard roads do not require any nominalistic content, i.e Chihara's constructibility theory and Hellman' modal structuralism.

In contrast to Field, easy roaders deny that mathematics should be paraphrase-bound, and some avoid the commitment to what exists. Yablo's logical subtraction avoids stating the nominalistic content of physical theories: it employs 'a strategy of saying less with more'.[57] On the other hand, it requires being shown how and in what sense a commitment to mathematical objects can be eliminated from empirical theories without substantial changes. Indeed such a subtraction may not be possible. Imagine, for example, that a commitment to hobbits is eliminated from *The Lord of Rings*, and ask yourself what kind of story is left. It does not seem that subtraction is possible in such a case, or at least without changing the story-line in most details. In other words, what would *The Lord of Rings* be without hobbits? Hence, a crucial requirement on Yablo's approach is to explain why subtracting mathematical objects is different from the hobbits case,

[57] Yablo (2014, p. 1009).

and how subtraction can be accomplished without important alterations to the empirical theories. Yablo's strategy can be split into two parts: first, we must find a way to carve out the part of a hypothesis, say **m**, that concerns mathematical objects and secondly, we must strip away **m** from that hypothesis.[58]

3.2.1 Aboutness and possible worlds

It is important to mention that Yablo intends to apply logical subtraction, or simply subtraction, to propositions and not directly to mathematical entities. Propositions are conceived as sets of possible worlds, and subject matters are sets of proposition. I shall first present Yablo's concept of subject matters.

The introduction of subject matters aims to provide a general account of aboutness. Aboutness plays a major role beyond mathematics, as it is also involved in most linguistic practices. Imagine being in front of the painting 'Bélisaire' when, all of a sudden, someone says: 'this picture depicts Belisarius' poverty and misery after he was defeated'. The painting is *about* Belisarius' poverty and misery. Similarly, a sentence such as 'Siegfried killed the dragon' is *about* both Siegfried and the dragon; the American Revolution is *about* both taxes and freedom; red is *about* red things; and so on. Aboutness is a natural and intuitive notion we employ in many contexts, but does not merely involve language. There is a strict correlation between aboutness and truth-conditions in the sense that aboutness is tied up with the ways a sentence is true. 'All ravens are black' is about ravens and their color, and that sentence is true because all ravens are black. If 'all ravens are black' was about different things, let us say mugs and green, the truth-conditions of that sentence would change.[59]

Truth-conditions and aboutness are so tied up together that most philosophers have argued that aboutness can be reduced to the truth-conditions of

[58] Yablo (2014, p. 1012).

[59] Unless mugs and green are synonymous with ravens and black respectively.

statements. ‘All ravens are black’ is true by the virtue of how things stand in the world: the fact that all ravens are black. And the statement is consequently about ravens’ color. However, that picture is partially misleading. ‘All ravens are black’ and ‘all non-black things are non-crows’ are about different things — one is about crows, the other is not — even though the truth-conditions of both statements are the same. For Yablo, aboutness deserves to be investigated independently of the traditional analysis of meaning and semantics.

What is aboutness? I will start with presenting a couple of examples and then will go into details. Basically, Yablo conceives aboutness in terms of possible worlds à la Lewis by distinguishing between, say, the nineteenth century (which is an event of human history) and **the nineteenth century** (which is a subject matter that groups possible worlds, according to what is going on in their respective nineteenth centuries). Here is an intuitive picture of how **the nineteenth century** can group worlds: it associates worlds with the events between 1800 and 1900, in a way that every world corresponds to the course of events between 1800 and 1900. **The nineteenth century** is a subject matter, the nineteenth century is not. Here is another example. Queen Victoria is a figure of the nineteenth century. She is a person and not a subject matter. Queen Victoria got married with Prince Albert but she cannot group worlds. **Queen Victoria** is the subject matter that groups worlds, for example on the basis of what she did in different worlds. In a world she got married to Prince Albert, in another she could have been assassinated, and so on.

The nineteenth century groups worlds on the basis of how matters stand in relation to their specific nineteenth centuries as well as **Queen Victoria** groups worlds on the basis of how Queen Victoria stands in different worlds. Subject matters are ‘patterns of cross-world variation’.[60] When we switch from a world to another one, for example from a fictional world to the actual world, how matters stand may be different: a depiction of Queen

[60]Yablo (2014, p. 27).

Victoria may not the Queen Victoria who ruled the British Empire. People, however, recognize that there is a sense in which we are talking about the same thing. In Yablo's view, we are talking about **Queen Victoria** in both worlds (fictional and actual). **Queen Victoria** is a pattern through different worlds.

The way subject matters group worlds can be defined more precisely. Subject matters group worlds by inducing relations on them. **Queen Victoria**, for instance, induces relations in a way that worlds are partitioned into classes, such that every class is a way of grouping worlds. For example, a class can group all the worlds where Queen's Victoria got married to Prince Albert, another class groups all the worlds in which Queen's Victoria became Empress of India, and so on. More generally, subject matters split worlds into classes, and every class is a partition on worlds. The main difference with the set-theoretic notion is that we are operating on worlds instead of sets.

The identity of subject matters is given in terms of worlds. Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be subject matters; $\mathbf{m}_1$ and $\mathbf{m}_2$ are equivalent if and only if worlds differing where $\mathbf{m}_1$ is concerned differ also with respect to $\mathbf{m}_2$, and vice versa. Consider for instance **Queen Victoria** and **Prince Albert's wife**. These subject matters are equivalent to each other if they group worlds in the same way: intuitively, if Queen Victoria is Prince Albert's wife in all worlds.

Appealing to possible worlds may seem an unnecessary complication. After all we are mainly interested to a specific world: the actual one. But keep in mind that possible worlds also contain all information about how matters stand in relation to a specific world. From a subject matter $\mathbf{m}$ we can build up a function that maps worlds to their $\mathbf{m}$-conditions.[61] These $\mathbf{m}$-conditions tell us how matters stand in relation to a world w, when $\mathbf{m}$ is concerned. Thus, the function $\mathbf{m}(w)$ encodes information about a specific world as far as $\mathbf{m}$ is concerned. For instance, **The number of planets** tells

[61]The domain may be empty. In this case, there would be no worlds to map.

us how many planets exist by mapping worlds to how matters stand in w. If w is the Solar System, then **the number of planets**$(w) = 9$.

Subjects matters are patterns through worlds. More precisely, a subject matter is nothing but a set of propositions, and a proposition is a set of possible worlds. The main advantage of such an approach is to give a way of defining relations over subject matters. An outstanding relation is orthogonality: two subject matters are orthogonal if how matters stand with respect to one puts no constraints on how matters stand where the other is concerned in any world.[62] That is, if one subject matter is somehow independent from another one. For instance, **the number of stars** is orthogonal to **the number of comets** if any number of stars is compatible with any number of comets. Comets are icy bodies that heat up by the effect of Solar radiation, whereas stars are shiny sphere of plasma. The number of the former does not influence the number of the latter (and vice versa), **the number of stars** is orthogonal to **the number of comets**.[63] Another example, perhaps more intuitive, is the relation of orthogonality that holds between **the number of cats** and **the number of dogs**.

Orthogonality can be defined formally if subject matters are understood as dividing worlds into classes. Consider two subject matters **m** and **n** which split worlds into classes. We say that **m** and **n** are orthogonal iff the intersection between classes induced by **m** and classes induced by **n** is non-empty. Turning back to the previous example, let us pick up the relations 'x has as many stars as y' ($=_m$) and 'z has as many comets as y' ($=_n$), where x and y are worlds. These relations induce classes which are represented as sets of worlds. For instance, a class may be the one that contains worlds where there are 10,000,000 stars, or where there are 2,000,000 comets, and so on.

[62] Yablo (2014, p. 1013).

[63] Orthogonality can be even used to formulate some philosophical positions. If **how matters stand physically** is orthogonal to **mathematical objects exist**, then the physical world neither demands nor preclude the existence of mathematical objects (in any world). See Yablo (2012, p. 1013). In this case, physical world is thus compatible with the (non-) existence of mathematical objects.

Thus, **the number of stars** is orthogonal to **the number of comets** iff the intersection between classes induced by the former and classes induced by the latter is non-empty.

The parthood relation is also important. It is somewhat intuitive that there is a sense in which **the nineteenth century** includes **Queen Victoria** as its part, at least in the actual world. Let us call 'cell' a class induced by a subject matter, and remember that such a class represents the way by which a subject matter groups worlds. If **Queen Victoria** is part of **the nineteenth century**, then **the nineteenth century** is intuitively 'larger' than **Queen Victoria** in the sense that the former induces more cells than the latter does.

I have not presented yet how we can evaluate sentences when a specific subject matter is involved. Suppose A is a sentence about $\mathbf{m}$, and we want to establish whether or not A is true about $\mathbf{m}$ relative to a possible world. In this case, we evaluate the proposition that is true if A is true about $\mathbf{m}$ relative to w. Since $\mathbf{m}$ is encoded in terms of possible worlds, i.e. $\mathbf{m}$ is a set of sets of possible worlds, it makes sense to evaluate A's true-value about $\mathbf{m}$ as regards how matters can stand in different worlds. The idea is that A is true about $\mathbf{m}$ in a world w if A is true outright without changing how matters stand in w where $\mathbf{m}$ is concerned. Suppose that 'there are nine planets' is about **the number of planets**, and evaluate whether or not that sentence is true about **the number of planets** in the Solar System. Remember that **the number of planets** is a set of propositions, where each proposition expresses how many planets there are in worlds. In order to evaluate 'there are nine planets', we can single out the proposition that expresses how many planets there are in the Solar System: if the number is 8, then the sentence is true.

Yablo also distinguishes between what A says about $\mathbf{m}$ and the parts of A about $\mathbf{m}$. The point has important consequences in evaluating sentences about $\mathbf{m}$ because, according to Yablo, the part of A about $\mathbf{m}$ can be true even if the whole is mistaken. This is because a sentence can be true about

m in some worlds while being false in others for reasons *unrelated* to **m**. Consider again the sentence 'there are nine planets'. The sentence can be true about **planets**, in a world w, if (1) w is a world where mathematical objects exist, or (2) w is a nominalistic world where that sentence is false but for reasons that do not concern how matters stand regarding **planets**. The point is rather subtle, so let me focus on another example. Suppose 'the number of animals on the Earth on nth day is m' is true about **the animals** in a Platonistic world, but we do not want to commit ourselves to numbers. A way out is to regard that statement as false in a Platonistic world, but true about **the animals** in a world without numbers. The point is that the two worlds are alike with respect to **the animals** although not with respect to **numbers**.

The last consideration suggests how Yablo intends to employ subject matters in order to set out an anti-Platonistic strategy: some propositions may remain about **m** even after taking out what they presuppose (e.g. numbers). More generally, subject matters are indispensable to maintaining what a theory is about after propositions that contain mathematical objects are stripped away. Such a strategy requires to be formulated in terms of possible worlds insofar as worlds give a way of drawing a distinction between Platonistic worlds and nominalistic worlds.

It is important to notice how Yablo's considerations that I have discussed so far do not strike a blow for nominalism. They do not aim at constituting an argument against Platonism because, as far as **physical objects** is concerned, propositions can indifferently be true in a world that has, or does not have, abstract objects.[64] **Physical objects** does not rule out the existence of mathematical objects, and nominalistic worlds are possible because physical objects do not require the existence of mathematical ones. According to Yablo, physical objects do not require the existence of numbers as well as Barack Obama does not presuppose the existence of the set {Barack Obama}. The following passage is meaningful in that regard:

[64] Yablo (2014, p. 81).

> To argue from *we cannot imagine–without–numbers a complex world* to *we cannot imagine a complex world lacking in numbers* is like going from *we cannot imagine a tree non–perceptually* to *we cannot imagine unperceived tree.*[65]

But this is not, of course, an argument against Platonism. Yablo's consideration merely aims to provide a starting point for Platonism and nominalism by establishing that both hypotheses are possible regarding **physical objects**. I will show later on to what extent this point is essential to evaluating whether or not Yablo's strategy can be truly considered an easy road to anti-Platonism.

I have not explained yet how to individuate the subject matter of a specific sentence. At first glance, the problem is that many subject matters can be associated with even a simple sentence. For example, any sentence about **the number of planets** is also about **planets**, and to some extent also about **the composition of planets**, and so on. In other words, the nature of aboutness emerges as intrinsically holistic, and constitutes a serious issue in order to find what a sentence is exactly about. Yablo's solution is to look to the ways a sentence is true by picking out two distinct worlds. Consider the sentence (A) 'my next car will be manufactured in the US', and suppose there are two distinct worlds where that sentence is true — for instance, in w_1 my car is a Chrysler, in w_2 it is a Ford. A is true in both worlds since Chryslers and Fords are both manufactured in the US. But even if how matters stand in w_1 is different with respect to w_2, this does not affect A's subject matter: A is still about **my new car** either way. Indeed, A is true about **my new car** independently of which my car is. What changes in the two worlds is *the way* A is true. If a subject matter is a set of propositions (sets of worlds), how matters stand in relation to worlds can change the way a subject matter is internally structured. As a result, in order to find the exact subject matter of A, one should present the set of A's worlds according to A's changing ways of being true.

[65] Yablo (2012, p. 1014).

3.2.2 Logical subtraction

I presented Yablo's general view of aboutness and mentioned that subject matters can be used to hold what a sentence is about after propositions are subtracted. I have not explained yet how we can subtract propositions from other propositions. This can be accomplished by using Yablo's logical subtraction. The idea is basically to strip a proposition from another in order to carve out the remainder and evaluate its truth-value.

I intend to restrict subtraction to the case in which B is part of A, where A and B are two propositions. The reason is that one of the premises of the indispensability argument can be reformulated by saying that (1) the proposition about the existence of certain mathematical objects, such as numbers, is part of the propositions that are contained in our best scientific theories; (2) and the former proposition cannot be subtracted from the propositions that are contained in our best scientific theories. This is a way of saying that scientific theories indispensably presuppose the existence of mathematical objects. Thus, the anti-Platonist who follows Yablo's route must show that it is possible to subtract that presupposition from propositions in our best scientific theories.

Let us consider how the notion of part works more carefully. Given two propositions A and B, B is part of A if and only if both conditions hold:

1. A implies B.

2. A's subject matter includes that of B.

A's subject matter includes that of B iff how B is true (false) cannot change without changing how A is true (false). The way an hypothesis is true (false) is called truthmaker (falsemaker) which is obtained by presenting the set of worlds where an hypothesis is true (false). For example, if 'the number of planets in the universe is > 5' is true about **the number planets**, the truthmaker is 9 for the Solar System, perhaps 100 billion for the Milky Way, and so on.

Let us designate 'B is part of A' as $B < A$. What is left by subtracting B from A is called 'the remainder $A - B$', such that $A - B < A$ and where $A - B$ is disjoint from B. The remainder should in principle exist if B is part of A. However, the remainder may not be easy to find. Consider this Wittgenstein's quote:

> what is left over if we subtract from the fact that I raise my arm the fact that my arm goes up?[66]

Wittgenstein's question is not unique and can be extended to many other situations. What does *Tom is red* add to *Tom is colored*? What does *Alma believes water is wet* add to *water exists*? How can we subtract *water exists* from *Alma believes water is wet*?[67] Yablo's reply lies on 'extrication'.

Extrication aims to extend a hypothesis A to worlds where its presupposition B is subtracted: in short, $A - B$ is the result of extricating A from worlds where B is false. In Wittgenstein's quote, we must extend the fact that I raise my arm to regions of logical space where my arm stays down, where A's logical space is configured by A's ways of being true or false.

Let us examine how extrication works. We want to know how to extricate A from B's true-conditions — called 'home regions' — in order to project A into worlds where B is not true — called 'away regions'. The most interesting case is to study the remainder's behavior in worlds where B is false and A is true, i.e. where presuppositions do not hold. For example, suppose that I want to extricate 'I raise my arm' from 'my arm goes up'. The problem can be reformulated by saying that we want to understand what 'I raise my arm' adds to 'my arm goes up'. More specifically, we want to figure out what is going on when I raise my arm in the regions of logical space where my arm stays down.

Here is Yablo's solution: the remainder's (away) behavior should be modeled starting with A's (home) behavior. The reason is that the process of extrication is conceived by Yablo as a sort of projection from home to

[66] Wittgenstein (1953, § 62).
[67] See Yablo (2014, p. 136).

B's away behavior when A is true. This consideration leads Yablo to the following guidelines in order to evaluate the remainder's truth-value:

Agreement: $A - B$ is true (false) at home just when A is true (false).

Rectitude: $A - B$ is true (false) at home for the reasons that A is true (false) given B.

Integrity: $A - B$ is true (false) for the reasons away as it is true (false) at home.

Determination: $A - B$ is true (false) away if it has reasons to be true (false) and none to be false (true) at home.

The four guidelines indicate how extrication should be evaluated according to what A adds to B. But it is still hard to understand how extrication works, so let me spell the point out. What does 'I raise my arm' add to 'my arm goes up'? Perhaps someone could suggest that the remainder is the act of will, because to raise my arm I must will my arm to go up. The hypothesis is that 'I will my arm up' is $A - B$. But the hypothesis violates the agreement condition: it does not seem that when my arm goes up, I will raise my arm. My arm might have gone up for other reasons than the act of will.[68] In other words, 'I will my arm up' fails to be the remainder because it does not imply 'my arm goes up' $\to$ 'I raise my arm'.

Yablo suggests a way of summarizing the above-mentioned guidelines. $A-B$ is true in w if A adds only truth to B in w, false if it adds only falsity to B in w.[69] Otherwise $A - B$ is undefined: $A - B$ is undefined if A adds neither truth nor falsity to B in w, or else A adds both truth and falsity to B in w.

I have not presented yet when A adds truth, or falsity, to B. Yablo comes up with the following definitions:

[68] Yablo (2014, p. 146).
[69] Yablo (2014, p. 148).

1. A adds truth to B in w iff $B \to A$ is true in w for a reason compatible with B.

2. A adds falsity to B in w iff $B \to \neg A$ is true in w for a reason compatible with B.

Let us see how the definitions work. First, we must consider B's behavior in home regions. If B is true in w, A adds truth, or falsity, to B according to whether A is true or false in w. Home conditions are easily evaluable. Instead, it is more difficult to evaluate A in away regions: if B is false, A adds truth (falsity) to B if $B \to A$ ($B \to \neg A$) is true for a reason compatible with B. What is that reason? According to Yablo, the reason is a truthmaker for $B \to A$ ($B \to \neg A$) that takes as much advantage as it can from B. Basically, a truthmaker that does not rule out B. To spell out what kind of truthmaker we are looking for, I will cite two cases mentioned by Yablo.

Consider $P \wedge Q$ and examine whether $P \wedge Q$ adds truth, or falsity, to Q. According to the definitions stated previously, either $Q \to P \wedge Q$ has a truthmaker compatible with Q, or $Q \to \neg(P \wedge Q)$ does. In the former case $P \wedge Q$ adds truth, in the latter falsity.[70] A truthmaker compatible with Q does not rule out Q. Now, let us consider a world where P is true. In this case, $Q \to P \wedge Q$ has a truthmaker compatible with Q in the fact that P. On the other hand, in a world where P is false, $Q \to \neg(P \wedge Q)$ has a truthmaker compatible with Q in the fact that $\neg P$.

Let us now examine a case where the remainder is undefined. Imagine that we want to subtract *Both of Herb's dogs have fleas* from *Herb has exactly two dogs*. First, we must consider an away world w where Herb does not have two dogs — where Herb for example has three or four dogs. By definition, *Herb has exactly two dogs* $\to$ *Both of Herb's dogs have fleas* is true in w if it has a reason compatible with *Herb has exactly two dogs*: the fact that Herb has two dogs with fleas. On the other hand, *Herb has exactly two dogs* $\to$ *it is not the case that both of Herb's dogs have fleas* is false in w

[70] It cannot add both in the same worlds if the remainder is evaluable.

if it has a reason compatible with *Herb has exactly two dogs*: the existence of a third dog without fleas.[71] The remainder is undefined in w because *Both of Herb's dogs have fleas* adds both truth and falsity to *Herb has exactly two dogs*.

Undefined remainders are particularly interesting when A presupposes B. And presuppositions are important for Platonism. An occurrence of the indispensability argument can be formulated by saying that a proposition such as *there are nine planets in our Solar System* presupposes the proposition *there are numbers*, because it would not be true if numbers did not exist. In other terms, some propositions in our best scientific theories presuppose that 'there are numbers' is true.

Consider the sentence 'the King of France is bald' and its presupposition 'France has a king'. According to Yablo, 'the King of France is bald' does not seem to have a truth-value because it adds neither truth, nor falsity, to 'France has a king' in a world where France is a Republic. In other words, 'the King of France is bald' is unevaluable because the remainder cannot be extricated. We say that a presupposition fails catastrophically when the question of truth, or falsity, does not longer arise. In this case, 'France has a king' is a presupposition for 'the King of France is bald' that fails catastrophically. By contrast, 'the King of France is in my garage' strikes us as evaluable — it sounds false — because it adds falsity to 'France has a king'. That is, 'France has a king' is a presupposition that fails non–catastrophically. As a general rule, a presupposition B fails non–catastrophically iff $A - B$ is evaluable despite B's falsity. To say that a presupposition fails non-catastrophically is to say that the remainder is evaluable.

3.2.3 Mathematics is extricable

Let us examine how Yablo's approach works for propositions that presuppose the existence of mathematical objects. We must find out whether mathe-

[71] Yablo (2014, p. 149).

matical presuppositions fail non–catastrophically. If this is so, running an argument against Platonism is to provide a non-catastrophic account of presuppositions failure for propositions about mathematical objects. The challenge is to show how presuppositions such as 'there are no numbers' is non–catastrophic in the sense that its failure makes no difference to how matters stand in relation to the physical world. Numbers should be extricated from (the content of) empirical sentences in the same way 'France has a king' can be extricated from 'the King of France is in my garage'. The indispensability argument can be formulated as the thesis that numbers cannot be extricated from (the content of) the propositions that are contained in our best scientific theories.

Given a couple of propositions A and B, where A presupposes B, the remainder is evaluable when B fails non-catastrophically. First, we must show that the remainder exists and secondly, we have to figure out 'how much' we can extricate from B. Although extrication is a matter of degree, running an argument against Platonism implies that propositions that involve mathematical objects should be completely extricated from the propositions in our best scientific theories. Showing that numbers can be partially extricated is not sufficient.

There are many cases where A cannot be extricated from B. 'Tomato is crimson' cannot be extricated from 'tomato is red'. 'Tomato is crimson' could be extricated only if *tomato is red* $\rightarrow$ *tomato is crimson*, or alternatively *tomato is red* $\rightarrow$ *tomato is not crimson*, is true in w for reasons compatible with tomato is red. The most interesting case is to consider the worlds where tomato is not red. But 'tomato is crimson' does not add truth or falsity to 'Tomato is red' because the property of being crimson and redness cannot be separated. Thus, the remainder is unevaluable. More generally, we say that B is *perfectly inextricable* from A iff $A - B$ is evaluable only in B's worlds. In such a case, A does not add truth (falsity) to B except when B is true.[72]

[72] Yablo (2014, p. 153).

In some cases A can be extricated when B is false. Consider Wittgenstein's quote and suppose that we want to subtract 'Lance raised his arm' from 'Lance's arm went up'. In a world where Lance is dead, the remainder is evaluable because 'Lance raised his arm' adds falsity to 'Lance's arm did not go up'. Specifically, the fact that Lance is dead is a truthmaker for *Lance's arm went up* $\rightarrow$ *Lance did not raise his arm.* On the other hand, if Lance is dead, he cannot raise his arm. In other words, there are no truthmakers for *Lance's arm went up* $\rightarrow$ *Lance raised his arm.* Let us now consider a world where Lance is alive and attempts to raise his arm. Trying to raise an arm is not enough for the arm to go up: namely, it is not a truthmaker for *Lance's arm went up* $\rightarrow$ *Lance raised his arm.* Same thing for *Lance's arm went up* $\rightarrow$ *Lance did not raise his arm.* Thus, the remainder is therefore unevaluable. More generally, we say that B is partially extricable from A iff $A - B$ is evaluable in some B's away regions but not in others.

The last option is when B can be perfectly extricated from A, that is, when A adds only truth, or only falsity, to B in *any* away region. The challenge against Platonism is whether or not *there are numbers* (B) can be perfectly extricated from a proposition such as *there are nine planets in our Solar System* (A). For perfect extricability *there are nine planets in our Solar System* has to add only truth, or only falsity, to *there are numbers* in every away region, i.e. in every numberless world. A adds truth iff *there are numbers* $\rightarrow$ *there are nine planets in our Solar System* is true in any numberless world for a reason compatible with B. Let us see how we can proceed. We start with a numberless world, where there are the planets Earth, Mars, Venus, Jupiter, Saturn, Mercury, Uranus and Neptune. The existence of those planets is a truthmaker that is compatible with B — because it does not rule out that numbers exist. On the other hand, A adds falsity iff *there are numbers* $\rightarrow$ *it is not the case that there are nine planets in our Solar System* is false in any numberless world for a reason compatible with B. Our truthmaker must preclude the latter conditional to make the remainder evaluable. Indeed that is so, because it precludes the above-mentioned planets of the Solar System plus Pluto, or only Mars, or Earth plus Venus, and so on. What we get when we subtract *there are numbers* from *there are nine planets in our Solar System* is thus evaluable: the remainder is the proposition that it is true in one kind of numberless world, and false in numberless worlds not of that kind. As a consequence, *there are numbers* is perfectly extricable. It is interesting how that procedure can be applied to other propositions that involve mathematical objects.

3.2.4 Some remarks on Yablo's approach

I intend to analyze Yablo's approach and evaluate its contribution towards anti-Platonism. I am going to raise some objections to the introduction of subject matters, show how the so called 'Yablo's easy road' is not a road to nominalism, and suggest what contribution logical subtraction could bring to the debate between Platonism and anti-Platonism.

Yablo needs that propositions that talk about the existence of mathematical objects have to add only truth, or falsity, to scientific propositions that include numbers in every numberless world for reasons that do not concern **physical objects**. This is why subject matters are essential to carve mathematical content out of a given hypothesis. Without subject matters, we would not have a guarantee that propositions are still about what they were about before subtraction. If a proposition is originally about **physical objects**, the remainder should be still about the same subject matter after that subtraction is applied.

It is questionable whether or not the ontological status of subject matters is acceptable to the anti-Platonist. Which entities are they supposed to be? Let us check on some definitions of subject matters provided by Yablo. A subject matter can be conceived as follows:[73]

1. An equivalence relations.

[73]Yablo (2014, p. 28).

2. A partition.

3. A specification for each world of what is going on there **m**-wise.

4. A set of propositions.

Even if the criterion of identity for subject matters is not given set-theoretically, but in possible worlds terms, all those definitions are set-theoretical: sets of propositions are sets; equivalence relations are defined on sets; for each partition there is an equivalence class (set); and finally a specification of what is going on in a world can be constructed as a function from worlds to certain propositions.[74] Anti-Platonists reject the existence of mathematical objects in the absence of a suitable epistemological access to *abstracta*, and it is not clear why subject matters should be in a better position epistemically, since subject matters are *mathematical abstracta*. The treatment should not be worse than the disease that it is supposed to cure.

Perhaps we may adopt a more pragmatic attitude, and employ any object that is indispensable to solving a philosophical problem without being worried about the nature of such entities. The main advantage of Yablo's perspective is that many formal relations can be defined as regards subject matters such as orthogonality, parts-of, and soon.[75] The informal notion of presupposition, for example, can be made precise by using the parthood relation, which can be extended to non-parts-based subject matter by adopting the concept of partition.[76]

Possible worlds provide information about the actual world in a way that they have a grip on reality. However, subject matters remain problematic entities even including possible worlds into our ontology. Subject matters are even more abstract than possible worlds, since they group worlds by inducing equivalence relations. From an epistemic point of view, it is not clear why the existence of an equivalence class would be less problematic than numbers or other mathematical objects. In fact, they are not. Since

[74] Yablo (2014, p. 28).

[75] Yablo explicitly mentions this point on (2014, p. 29).

[76] Yablo (2014, p. 27).

subject matters are sets of propositions, why should we subtract propositions that include mathematical objects from empirical theories if what we get at the end are sets?

To be clear, I am not arguing against the existence of subject matters. I believe that subject matters are introduced by Yablo to handle a specific philosophical problem: what an hypothesis is about. Subject matters serve that purpose and perhaps have several advantages over other strategies. But I am not sure whether or not subject matters can be employed in favor of anti-Platonism; even possible worlds are problematic within a nominalistic perspective. Hellman is aware that possible worlds are not acceptable from a nominalistic point of view, and so he treats modal operations as primitive notions. In point of fact, numbers are more acceptable than subject matters, because the existence of numbers is introduced on a solid scientific ground: numbers are indispensable to our best scientific theories. Aside from an elegant treatment of aboutness, subject matters are clearly not as indispensable as mathematical objects, and thus Quinean naturalism cannot justify the existence of entities that are not indispensable.

Still, Yablo does not seem interested in figuring out what exists,[77] nor does endorse nominalism. Yablo does not consider himself a nominalist but an anti-Platonist.[78] The nominalist is an anti-Platonist, but the anti-Platonist is not necessarily a nominalist: both argue against the existence of abstract objects, although the nominalist is committed to what is nominalistically acceptable. In Azzouni's case, the mind– and language– independence criterion is the guideline that distinguishes what is nominalistically acceptable from what is not. But an anti-Platonist could argue against the existence of abstract objects without embracing any positive view about

[77]I refer to the skeptical conclusion in an older Yablo's paper: "I conclude that the existence-questions of most interest to philosophers are moot" (1998, p. 260). More recently, (2014, p. 80).

[78]More specifically, Yablo calls himself a quizzicalist (2009). According to the quizzicalist, there is no fact of the matter about the existence of abstract objects if the presupposition that abstract objects exist fails non-catastrophically.

what is nominalistically acceptable. For example, Yablo's position can be stated in this way: there is no fact of the matter about the existence of abstract objects if the presupposition that abstract objects exist fails non-catastrophically.[79] Despite this, the main reason to be an anti-Platonist is tied to epistemic worries about the nature of abstract objects, but it is not clear whether Yablo is supposed to be an anti-Platonist, since subject matters are abstract objects. As showed earlier, subject matters are problematic entities from an epistemological point of view.

To avoid any epistemological problem, an ideal solution would characterize logical subtraction independently of subject matters. However, logical subtraction would present another issue without subject matters. Note that logical subtraction operates on propositions: we subtract the proposition A from the proposition B, and what we get at the end of the process is a proposition $A - B$. But are propositions acceptable to the anti-Platonist? Propositions present the same issue of subject matters: they are abstract objects. The reason is that propositions are *sets* of worlds.[80] Thus, it is not clear why we should use logical subtraction against Platonism. In principle, a way of retrieving logical subtraction as an argument against Platonism might be avoiding both subject matters and propositions. But this goal goes beyond the present work.

Independently of whether logical subtraction can be formulated without subject matters and propositions, let me point out that logical subtraction does not count as an argument against Platonism. At best, logical subtraction can show that numbers are orthogonal to how matters stand physically; it tells us that even though there are no mathematical objects, the remainder can be still evaluated in numberless worlds, according to how matters stand in relation to the physical world. In other words, logical subtraction shows that physical conditions cannot be used to prove, or disprove, the existence

[79] Yablo (2009, p. 522).

[80] "If we assume, again following Lewis, that a proposition too is a set of worlds, then truthmakers are of the same category as at least some of what they make true, namely other propositions". See Yablo (2014, p. 74).

of abstract objects. But this is not a happy result for the nominalist and for many anti-Platonists. Indeed logical subtraction is not an argument against the existence of abstract objects: it merely states that physical objects do not require the existence of abstract objects, because there are truthmakers for propositions about physical objects, in numberless worlds, that are compatible with the existence of abstract objects. That is an important result, but is still far from being an argument for the nominalist or against Platonism — although it undercuts the indispensability argument and, as a result, it may be considered a first step to nominalism.

With or without subject matters, logical subtraction is not an argument for or against Platonism. If I am right, it will make sense to ask ourselves whether Yablo's strategy is an easy road to nominalism or not. Yablo's strategy does not force us in being specific about the nominalistic content of physical theories. In contrast to Field's program, logical subtraction does not tell us that what exists in a Newtonian world are the regions of space–time, or an alternative surrogate. Logical subtraction is neutral about what exists and, more specifically, is neutral about the actual world because nominalistic and Platonistic worlds are both possible. The proposition 'there are nine planets in our Solar System' is true in a Platonistic world but false in nominalistic worlds for reasons that do not concern **planets**. And when we strip numbers away of that proposition, the remainder is still true about **planets** in nominalistic worlds. Despite this, logical subtraction does not aim to tell us which world is the actual world, but assumes that both worlds are possible. Because Yablo's strategy does not rule out the existence of Platonistic worlds, it is not a road to nominalism.

Logical subtraction shows how it is possible for the existence of propositions that include mathematical objects to be extricated from propositions that describe the physical world, but it says nothing about the outcome of such an operation. Consider this analogy: suppose that we know that it is possible to extricate hobbits from *The Lord of the Rings* in a way that what we get at the end of subtraction remains true about the events of

Middle Earth. But what is the remainder supposed to be? *The Lord of the Rings* without hobbits seems to be metaphysically inscrutable. The analogy with the case of mathematics should be clear: even if we could extricate the propositions that include mathematical objects from the propositions in our best scientific theories, what we would get at the end of that process is not clear. Even though the remainder is still true about **physical objects**, what it is remains an enigma.

I already argued that logical subtraction is not a road to nominalism, but I have not said what kind of road it is. It seems to be that Yablo's easy road is a 'no road'. Let me explain what I mean. Yablo assumes that nominalistic and Platonistic worlds are both relatively possible. Mathematical propositions are dispensable to our best scientific theories because the remainder is both evaluable and true in numberless worlds. In this sense mathematics is indispensable from an explanatory point of view, but it does not entail the existence of mathematical objects. Does it mean that mathematics is dispensable from our scientific theories? In Yablo's view, it seems that we can dispense with mathematics in the sense that certain mathematical propositions are false in numberless worlds. But on this view the descriptive role of mathematics remains untouched: Mathematics is explicatorily indispensable and extricable. Because of this, the explanatory role of mathematics in scientific theories seems to me instrumental. To see why, notice first how Yablo distinguishes three levels of mathematical explanation:[81]

- Mathematics is **descriptive** when it assists to provide generalization of a certain phenomenon. For example, we observe that certain combinations of tiles are never seen in rectangular floors, and we use numbers in order to describe that phenomenon (19, 37, 71, and so forth).

- Mathematics occurs **structurally** when one finds certain formal relations, or structures, among objects. For example, the number of tiles is never prime in rectangular floors.

[81] Such grades of mathematical involvement may not occur all together in the description of a single physical event. See Yablo (2012, pp. 1020-1021).

- Mathematics occurs **substantially** if the previous structural relation can be used to provide further generalizations. Given prime numbers that divide integers exactly, i.e. prime factors, the fundamental theorem of arithmetic states that every positive integer has a single unique prime factorization. The theorem gives a way of individuating an important relation between prime numbers and positive integers.

As far as the first two levels occur, mathematics provides the conceptual backdrop to understand and unify physical regularities, but the explanation of such regularities remains physical. The second level is a genuine mathematical explanation. And according to a certain version of the indispensability argument, the existence of mathematical objects can be deduced from the fact that mathematical explanations are indispensable to explaining physical phenomena. But even if the explanation of a physical event is purely mathematical, the scientist is still trying to "carve physical phenomena at the explanatory joints".[82] In Yablo's view, it seems that mathematical explanations are subordinate (i.e. instrumental) to physical explanation.

If Yablo's view is a form of instrumentalism, one could argue that logical subtraction cannot be used to draw ontological conclusions. It cannot, because logical subtraction tells us nothing about the criteria for what exists. However, if mathematics is perfectly extricable from empirical theories, how matters stand physically neither demands nor preclude the existence of mathematical objects in numberless worlds. This conclusion is unsatisfactory for the nominalist but seems to open the possibility of being agnostic. If the physical world is compatible with, or without, the existence of abstract objects, there is a stalemate between Platonism and nominalism.

[82]Yablo (2012, p. 1023).

3.3 On confirmation holism

It has been argued that the indispensability argument is strongly tied to confirmation holism. Following this path, the anti-Platonist can reject the role of confirmation holism within scientific practice in order to run a knock-down argument against the existence of mathematical objects. After all, the connection between the indispensability argument and holism is grounded on a famous passage in Quine:

> The dogma of reductionism survives in the supposition that each statement, taken in isolation from its fellows, can admit of confirmation or information at all. My counter-suggestion, issuing essentially from Carnap's doctrine of the physical world in the *Aufbau*, is that our statements about the external world face the tribunal of sense experience not individually but only as a corporate body.[83]

In this passage Quine states his doctrine of holism. Science can make accurate predictions to the extent that theories are confirmed empirically. To see how confirmation holism comes in the picture, imagine a theory that describes an empirical phenomenon by employing mathematics. According to confirmation holism, the empirical success of that theory confirms not only the existence of the physical entities involved, but also the mathematics, because mathematics is part of the corporate body. Basically, confirmation holism states that theories are always confirmed as a whole and, as a consequence, the distinction between ontology and epistemology cannot be drawn when theories are empirically successful. If mathematics is true because it is empirically confirmed, we ought to commit ourselves to abstract objects as well as particles or quarks.

[83] Quine (1951c, p. 38).

3.3.1 Maddy's non-factualistic view

Given the connection between the indispensability argument and confirmation holism, Maddy notices how the indispensability argument does not corroborate the existence of mathematical objects that do not find any applications within physical theories.[84] If Maddy is right, the indispensability argument cannot be used in order to justify abstract objects which play an important role within pure mathematics.

Maddy is mainly worried about set theory and its ontological status. If confirmation holism were true, set-theoretic objects like large cardinals would be rejected on the basis of non-mathematical standards, in contrast with actual mathematical practice. In fact, mathematicians do not wait for a mathematical theory to be confirmed by physics or other empirical theories, but they proceed independently of empirical confirmation.

Since confirmation holism does not guarantee the existence of objects that are postulated within pure mathematics, Maddy aims to provide an alternative picture of mathematics that does not need to be grounded on empirical sciences. Mathematics has its own ontology that starts out of mathematical practice, and it does not require any empirical confirmation. Eventually, some mathematical theories might become indispensable for empirical sciences, and by that time experiments can provide further grounds to strengthening our mathematical beliefs. But still, empirical confirmation is not required to justify the existence of mathematical objects.

Although Maddy rejects confirmation holism, I do not think that her position can be considered anti-Platonist. Maddy does not seem mainly worried about the existence of abstract objects, but rather she is interested in figuring out whether mathematical theories are true or simply useful to empirical sciences.[85] Of course, the question about the truth of mathematics has also important consequences for the indispensability argument, especially for undecidable problems like the continuum hypothesis. If physi-

[84] Maddy (1992, p. 278).
[85] I am referring to Maddy (1992).

cists were able to show that space-time is continuous, the indispensability argument would provide strong reasons for the existence of $P(\mathbb{N})$. More generally, applied mathematics turns out to be true under the assumption that the indispensability argument is valid, and from the fact that mathematics is true we can easily get to the existence of mathematical objects.

According to Maddy, there is a strict connection between the indispensability argument and the factualistic view about mathematical problems. The factualist claims that there are facts of the matter whether mathematical statements are true or false, and such facts are grounded on our best scientific theories. For instance, despite the independence of the continuum hypothesis from Zermelo-Fraenkel set theory, the factualist argues that there is a fact of the matter whether or not the continuum hypothesis is true. If we have physical grounds to believe the continuum hypothesis, it makes sense to add new axioms to Zermelo-Fraenkel set theory in order to settle that question once and for all. This attitude is firmly based on the indispensability argument.

Maddy seems mainly interested in running an argument against the factualist rather than challenging the indispensability argument directly. To do that, Maddy contrasts factualism with non-factualism, according to which there is no fact of the matter whether mathematical questions are true or false. The continuum hypothesis for example cannot be decided on the basis of achievements in physics. Perhaps we might adopt new axioms in order to prove (disprove) the continuum hypothesis, but such reasons should involve only mathematics, and nothing else. As Maddy points out:

> Set theorists do not regularly keep an eye on developments in fundamental physics. Furthermore, I doubt that the set-theoretic investigation of independent questions would be much affected even if quantum mechanics did end up requiring a new and different account of space-time.[86]

Maddy endorses a non-factualistic view about undecidable mathematical

[86] Maddy (1992, p. 289).

statements, but it is not obvious that her point makes a strong case against the indispensability argument. In fact, in Maddy's view, the indispensability argument is aimed to address the right answer to undecidable mathematical problems and guides mathematical practice. Basically, the factualist is warned about the fact that empirical sciences should not interfere with mathematics: mathematical problems are problems for mathematicians. But is non-factualism sufficient in order to reject the indispensability argument? At best Maddy can argue that the true-value of *certain* mathematical propositions should not be based on the development of empirical sciences, because such an attitude would contrast with scientific practice. But even if the indispensability argument is strictly connected to factualism, to reject factualism is not sufficient. Indeed, a Platonist might agree with Maddy that the continuum hypothesis's truth-value should not be decided on an empirical basis and, nonetheless, he can argue that mathematical objects exist as long as they are indispensable to our best scientific theories. In other words, Platonism is not ruled out by non-factualism.

Maddy is concerned about raising a methodological objection rather than making an ontological claim. Basically, she is interested in what kind of consequences the indispensability argument may have for mathematical practice altogether, and her critique follows from that consideration. A different question is whether or not Maddy's worry about confirmation holism can be used against Platonism.

3.3.2 Sober's objection to the indispensability argument

Sober's critique aims at undermining both Platonism and constructive empiricism. According to constructive empiricism, we cannot draw any ontological conclusions about the existence of unobservables from the fact that unobservable are indispensable to contemporary physics. More specifically, constructive empiricism does not regard the question of truth about unobservable as essential to science but only to empirical adequacy. In short, a theory is empirically adequate if what it says about observables is true.[87] Quantification over unobservables is allowed but their existence cannot be established. As a consequence, constructive empiricism is a form of skepticism about the existence of unobservables in general — without any distinction between mathematical entities and physical ones. Since constructive empiricism invites us to suspend judgment on the existence of any unobservable, such a perspective turns out quite radical from an ontological point of view. In this regard, Sober aims to weaken constructive empiricism by combining it with some elements of scientific realism. The outcome is called 'contrastive empiricism'.

Let us see how contrastive empiricism works. Imagine two different scientific hypothesis that aim to explain the same phenomenon or physical event. How can we establish which hypothesis is more adequate? According to Sober, adequacy is always relative to a set of observations O, and an hypothesis is more adequate than another if it is favored by O. Sober provides a model of what he means by 'favorite' in order to evaluate the adequacy of an hypothesis over another: the likelihood principle.

The likelihood principle O favors H_1 over H_2 iff $P(O/H_1) > P(O/H_2)$

$P(O/H)$ is the probability that the hypothesis confers on the set of observations. The likelihood principle states that H_1 is favored over H_2 if O is more probable given H_1 than given H_2.

What is the relationship between likelihood and indispensability? Scientists may tend to consider H_1 indispensable if the probability that H_1 confers on O is very high with respect to H_2.[88] Because a set of observations cannot favor an hypothesis over *all possible* alternatives, indispensability is a matter of degree. An hypothesis may be regarded as true, but no hypothesis is indispensable in principle. New observations may favor other hypothesis

[87]The scientific Image, p. 12.

[88]What if both H_1 and H_2 confer on the same set of observations the same probability? In this case, it seems that none of them is truly indispensable.

and change what we currently regard as indispensable. Indispensability is strictly connected to observation in Sober's view.

The empirical science makes progress by facing discrimination problems between competitive hypothesis, and such problems are solved by appealing to the likelihood principle.[89] From this perspective there is no longer a distinction between truth and empirical adequacy, because H_1 is true insofar as it is possible to discriminate between 'H_1 is empirically adequate' and 'H_2 is empirically adequate'. And the decision between H_1 and H_2 is made on the basis of the likelihood principle. When an hypothesis that quantifies over mathematical objects is considered empirically adequate, its success may be regarded as an empirical reason for the existence of mathematical objects. However, if the likelihood principle is used to favor one hypothesis over another, for Sober empirical adequacy cannot confirm (dis-confirm) mathematics.

Imagine two mathematical theories, M_1 and M_2, employed in different physical theories. M_1 is indispensable if the probability that M_1 confers on O is very high in contrast to M_2. According to contrastive empiricism, we must use the likelihood principle in order to determine whether or not M_1 is indispensable with respect to M_2. Therefore, M_1 is empirically adequate iff a set of observations O favors M_1 over M_2, that is, when $P(O/M_1) > P(O/M_2)$. Basically, if the probability that M_1 confers on O is greater than what M_2 confers on O.

I would like to make a couple of remarks. First, if the argument is formulated in this way, then the likelihood principle can be used only to discriminate between competing mathematical theories, in order to choose which one is indispensable. Secondly, the likelihood principle can establish not only if a mathematical theory is empirically confirmed, but even if a theory is rejected by a given set of observations. In short, mathematical theories can turn out to be empirically falsifiable.

Can mathematics be confirmed, or refuted, by observation? Sober's

[89] Sober (1993, p. 39).

answer is that mathematical theories cannot confer probability on observations.[90] This is because mathematical theories are not indispensable in the same sense as empirical hypotheses: empirical hypotheses are *a posteriori* indispensable, whereas mathematical theories are *a priori* indispensable. As will see, that distinction is essential in Sober's discussion of the indispensability argument, but even more important is the fact that Sober discusses a version of the indispensability argument within his contrastive empiricism. Indeed Sober's argument can be summarized by this way: if contrastive empiricism is true, then a specific indispensability argument is ruled out because it is incompatible with contrastive empiricism. So before discussing the distinction between *a priori* and *a posteriori* indispensability, let us see first Sober's version of the indispensability argument:

P1: H_1 or H_2

P2: $(H_1 \wedge M)$ entails O and $(H_2 \wedge M)$ entails $\neg O$

P3: H_1 does not entail O (or $\neg O$)

P4: H_2 does not entail $\neg O$ (or O)

P5: O

C: M

Let us see how the argument works. In the first premise we have two empirical hypothesis that we want to evaluate as regards empirical adequacy. In the second premise H_1, together with a mathematical theory M, predicts an observable phenomenon O, whereas H_2 does not. The third and forth premises state that M is indispensable because H_1, or H_2, cannot predict O by their own. And finally M has to be true because we observe O.

Sober notices a deductive flaw in the argument. The observation is that O does not prove that M is true, because O is a redundant premise in the argument (**P5**). Indeed, **P5** does not seem to affect the force of the argument

[90]Sober (1993, p. 46).

and, therefore, that premise can be removed without any particular issues. As a result, the indispensability argument cannot be used in order to deduce the truth of a mathematical theory from a set of observations: observation does not confirm, nor dis-confirm, mathematical theories.

Compare the previous indispensability argument with the following indispensability argument *a posteriori*:

P1: H_1 or H_2

P2: $(H_1 \wedge M)$ entails O and $(H_2 \wedge M)$ entails $\neg O$

P3: $M \wedge O$

C: H_1

In this case, the observation O refutes H_2 and confirms H_1, and H_1 turns out indispensable as regards H_2. Since the assumption that O is true plays an essential role in the argumentation, observation is not a redundant premise and it can be used to prove the indispensability of H_1.

3.3.3 Confirmation holism and Platonism

The last point shows how Sober rejects empirical confirmation only for mathematical theories. This may seem as a decisive maneuver against the existence of mathematical objects, but I do not think that Sober is really interested in challenging Platonism. Indeed Sober is not worried about the problem of ontological commitment, and I am going to argue that the Platonist should not be worried about Sober's objection.

First, notice how Sober does not have anything against mathematical theories that are used with empirical hypotheses, if their truth is stated independently of *any* observations. Indeed Sober recasts the indispensability argument in the following way:[91]

P1: H_1 or H_2 or $\ldots$ or H_n

[91] Sober (1993, p. 47).

P2: For each H_i, H_i entail M

C: M

The last is a valid indispensability argument (*a priori*), which states that mathematical theories are true if they are entailed by empirical hypothesis, independently of any observations. Sober does not seem to have anything against this argument. Indeed, he argues that the indispensability argument is *a priori*, because mathematics does not need empirical confirmation to be true. Mathematics is not even affected by falsification of an empirical theory, of course. To show this, Sober gives the following example in which arithmetic is not dropped in the presence of non-additive quantities. Two gallons of salt plus two gallons of water does not yield four gallons, yet scientists are not inclined to consider $2+2=4$ false. Empirical confirmation may increase our confidence in mathematics, of course, but it cannot be used to claim the truth, or falsity, of mathematical theories.

In Sober's view mathematics is both true and *a priori*.[92] However, the existence of abstract objects is still unquestioned after his critique, because the indispensability argument *a priori* can run even without confirmation — although weakened. Moreover, one of Sober's main goals is to combine some elements of Van Fraassen's empiricism with realism, and thus Platonism is not really an issue for contrastive empiricism:

> I will argue that contrastive empiricism captures what makes sense in standard versions of realism and empiricism, while avoiding the excesses of each.[93]

[92]Mathematics is explicitly regarded as true by Sober, although with a little trace of criticism (1993, p. 53): "Perhaps the indispensability of mathematical statements in empirical science is some sort of reason to regard those statements as true. Nothing I have said here shows that this vague statement is wrong."

[93]Sober (1993, p. 53). Sober means by 'realist' the one who is "persuaded by indispensability arguments to affirm the existence of numbers, genes, and quarks" (1993, p. 53).

Sober's target is rather the Quinean doctrine of epistemological holism: the idea for which our beliefs face the tribunal of experience, or are confirmed, as a whole. Quine's holism is incompatible with contrastive empiricism and, more specifically, it conflicts with the likelihood principle. Keep in mind that for Sober empirical sciences aim at solving discrimination problems between competitive hypotheses through observations. But hypotheses are not confirmed in general on the basis of the empirical adequacy of an hypotheses: "what it is true of the whole may not be true of the parts".[94] Quinean holism is false because for contrastive empiricism, hypothesis are compared one by one.

It is questionable that contrastive empiricism represents an accurate picture of scientific practice. Basically, Sober's idea is that observations cannot falsify, nor confirm, mathematical theories. According to Colyvan, Sober's objection fails because it does not take into account the symmetric character of confirmation holism:[95] a theory is confirmed as a whole, whereas it is usually dis-confirmed only partially.[96] For Colyvan, that is the reason why mathematics is usually not affected by falsification of an empirical hypothesis. The relation between mathematics and an adequate empirical hypothesis is analogous to the relation between a programming language and a successful program. If the program works without compilation errors, the programming language shares credits with the successful program. But if the program does not work:

> the job of the computer programmer (in part) is to seek out the faulty part of the program and correct it. [...] In such a case the programmer seeks to make a small local change in the defective part of the program. Changing the programming language, for

[94]Sober (1993, p. 54).

[95]Colyvan (2001, p. 135).

[96]This is essentially Quine's point: we tend to disturb the whole system as little as possible. Before altering the whole system, we prefer to change a statement very close to the periphery.

instance, is not such a change.[97]

It is the same for mathematics. Mathematics is empirically confirmed when the an empirical hypothesis is empirically confirmed, but it is rarely dis-confirmed.

I believe Colyvan's argument is partially misleading. It is true that computer programmers do not change the programming language, and it is even true that the programming language is one of the reason why the program works. However, it does not seem to me this point confirms the existence of virtual objects. In the same way, mathematical objects should not be taken to exist simply because mathematics share credits with successful empirical hypotheses. Colyvan's argument seems to me rather weak.

Sober's point is that the confirmation of empirical hypotheses cannot be used to show the truth of mathematical theories. He is not claiming that mathematics cannot share credits for the success of empirical hypothesis, as Colyvan seems to think. Indeed if mathematics did not share credit for this, mathematical theories could be easily dispensed with.

I agree with Colyvan about the fact that Sober's objection is stated presupposing contrastive empiricism. Nevertheless, I do not think that Sober's point against the indispensability argument relies on contrastive empiricism. Indeed Sober's concern can be isolated from the specific framework of contrastive empiricism this way: can empirical confirmation be used by the realist to claim mathematical statements are true? This general worry seems interesting even if contrastive empiricism is highly problematic.

Mathematics does not need empirical confirmation to be true. As Maddy noticed, that claim would be in contrast with mathematical practice, because mathematicians in many cases do not wait for a mathematical theory to be empirically confirmed in order to regard it as true. We may even proceed beyond Sober and Maddy, asking ourselves if empirical confirmation favors the existence of abstract objects outside of space and time. I am going to

[97]Colyvan (2001, p. 133).

argue in the next chapter that empirical theories do not confirm, or disconfirm, the existence of abstract objects.

Chapter 4

Agnosticism in the Philosophy of Mathematics

4.1 Balaguer's anti-metaphysical conclusion

For Balaguer, there are no good arguments *for or against* Platonism. As a consequence, Platonism and anti-Platonism are both workable philosophies of mathematics, and it does not matter which of them we choose. The ontological debate cannot be settled not just because philosophers lack good arguments, but because there is no fact of the matter as to whether abstract objects exist or not. Balaguer's position can be split into two different claims:[1] on the one hand, according to the epistemic conclusion, we do not currently have any good arguments for and against the existence of abstract objects; on the other hand, according to the metaphysical conclusion, philosophers could *never* find such arguments because there is no fact of the matter. The metaphysical conclusion is the only non-factualistic claim, whereas the epistemic conclusion sounds more like 'there is much more work to do for philosophers'.

[1] Balaguer himself makes the above distinction. See Balaguer (1998, pp. 151-152).

4.1.1 Full-blooded Platonism and fictionalism

Before I examine how Balaguer's argument works in detail, I would like to mention that Balaguer aims at showing that the challenge between Platonism and anti-Platonism boils down to the opposition between full-blooded Platonism and fictionalism. According to full-blooded Platonism the mathematical universe is plentiful, that is, all possible mathematical objects exist.[2] In this context, 'possibility' means logical possibility or, more precisely, the fact that if a mathematical theory is consistent, then the objects entailed by that theory exist. Now, I would like to point out how Balaguer is not advocating any formal notions of consistency,[3] but his argument depends on a broader concept: the intuitive notion of consistency. Balaguer does not say too much about this notion. Basically, we know that this notion is primitive, in the sense that it can neither be defined in terms of abstract objects, nor it depends on a formal notion of consistency.[4] But it is not clear to me what Balaguer means. He claims that:

> anyone who has taught an introductory logic course can attest that students can be pretty reliable judges of whether a set of sentences is consistent, even if they have no conception whatsoever of syntactic or semantic consistency. Thus, the idea here is that before we developed the notions of syntactic and semantic consistency, our knowledge of intuitive consistency was good enough to give rise to some mathematical knowledge.[5]

[2]Balaguer's claim may be open to some ambiguities about the existence of possible mathematical object versus actual mathematical objects. However, this distinction does not arise for Balaguer because there is no distinction between possible and actual objects. See Balaguer (1998, p. 6).

[3]There are two ways for a mathematical theory to be consistent from a formal point of view: T is syntactically consistent if contractions cannot be derived in it; T is semantically consistent if T has a model.

[4]See Balaguer (1998, pp. 70-71).

[5]Balaguer (1998, p. 72).

I do not argue that we lack an intuitive notion of consistency. Perhaps, Balaguer means by 'intuitive consistency' something like: " I have made many arithmetical operations and I have not found any contradictions so far — unless I have not made mistakes. So I am pretty sure that arithmetical operations do not yield contradictions. Arithmetical operations are thus intuitively consistent". However, I am skeptical of basing full-Platonism on such an intuitive notion. There are many formal systems that seem intuitively consistent whereas, in fact, they turn out to be inconsistent. The most famous example is perhaps the failure of Frege's comprehension axiom.[6]

According to Balaguer, full-blooded Platonism is the only tenable version of Platonism, because it can solve both Benacerraf's problems.[7] In outline, there is no need to account for how we can get knowledge of abstract entities if full-blooded Platonism is true. This is because we only need to establish whether or not a mathematical theory is intuitively consistent, a process that does not require any interaction with abstract objects. In addition, Benacerraf's problem of multiple reduction disappears: given multiple equivalent descriptions of the same mathematical object, there is no unique sequence of *abstracta* that we have to pick out if mathematical universe is plentiful.[8]

Balaguer argues that full-blooded Platonism can solve the epistemological worries that arise from a reliabilist account of knowledge. Remember that according to Field's reliability claim, Platonists must account for the correlation between our mathematical beliefs and the mathematical facts, that is, they must explain why if mathematicians accept a mathematical

[6]It may be possible to argue that a methodology does not have to be fool-proof in order to give us knowledge. That is right. Nevertheless, I am still skeptical that we can appeal to such a weak notion of consistency in the case of mathematics. Mathematicians want to *prove* the consistency of mathematical theories and they do not rely on an intuitive notion.

[7]Benacerraf (1965) and (1973).

[8]Benacerraf focuses his analysis on Zermelo's and von Neumann's set-theoretic reduction of natural numbers.

theory T, then T describes part of the mathematical realm. Balaguer handles Field's challenge by arguing that if mathematicians accept T, then T is intuitively consistent. This does not imply that T is formally consistent, nor that consistency is the only requirement for a theory to be accepted. Balaguer's point is that if full-Blooded Platonism is true, and T is intuitively consistent, then T truly describes part of the mathematical realm. In other words, the correlation between our mathematical beliefs and the mathematical facts holds under the assumption that all possible mathematical objects exist.

It seems to me that Balaguer's solution does not address a basic question: why should we regard full-blooded Platonism as true?[9] Full-blooded Platonism cannot be true just because it handles the epistemological problem. In fact, Balaguer argues that

> To assume (at some level) that full-blooded Platonism is true is just to assume that our mathematical singular terms refer; but it seems plausible to claim that this assumption is inherent (in some sense and at some level) in mathematical practice.[10]

However, whether mathematical singular terms refer to is highly controversial. To play fair, Balaguer claims that his argument does not require that full-blooded Platonism is true: it demands only that we acquire mathematical beliefs through the consistency of mathematical theories.

> My claim is that *people can acquire knowledge of the mathematical realm — even if they do not assume (at any level) that FBP is true* — by simply having a method of mathematical belief acquisition that (as a general rule) leads them to believe purely

[9]Balaguer argues that we do not need to know that full-blooded Platonism is true in order to construct an adequate epistemology. But if it is false, then the methodology Balaguer describes is not reliable. I am not saying that Balaguer's methodology is unreliable, but just that it would be significant to know whether or not full-blooded Platonism is true.

[10]Balaguer (1998, p. 56).

> mathematical sentences and theories only if they are consistent. [...] And, of course, *the reason we can acquire knowledge in these ways is that these methods of belief acquisition are reliable.*[11]

People can acquire mathematical knowledge because our method of mathematical belief acquisition is reliable. However, I do not see how Balaguer can account for how the correlation between belief acquisition and abstract objects would occur if full-blooded Platonism is false. In the case of physical objects, if external-world realism is true then sense perception is reliable. But since Balaguer does not think there is any good reason to believe that full-blooded Platonism is true, it does not seem that there is any good reason to believe that his method of acquiring beliefs (namely full-blooded Platonism) is reliable.

After presenting the advantages of full-blooded Platonism over traditional Platonism, Balaguer attempts to show that the best alternative to full-blooded Platonism is fictionalism. Balaguer's strategy aims at ruling out both anti-realist and realist versions of anti-Platonism. With regard to the former point, Balaguer argues that anti-realists face all the same problems as fictionalism does: they must account for the indispensable applications of mathematics to physics.[12] As a consequence, if anti-realist views have all in common the same problem, Balaguer argues that we can single out one of them as a representative of anti-realism altogether. Fictionalism is the best candidate because, Balaguer continues, it has the advantage of taking mathematical sentences at face value: mathematical sentences are literally about abstract objects.

Balaguer's point is that only Field's fictionalism takes mathematical statements at face value. This claim is rather controversial. On the one hand, it is true that fictionalism regards mathematics as about abstract objects, and so the fictionalist does not need to change standard semantics.

[11] Balaguer (1998, p. 56).

[12] Balaguer takes into account if-thenism, conventionalism, and formalism. See Balaguer (1998, pp. 100-104).

However, since according to Field mathematical statements are false, mathematics is not taken at face value. To overcome this problem, one should regard mathematical statements as true without postulating the existence of abstract objects —- as for instance Azzouni does.

Let see how Balaguer plans on dismissing anti-Platonist realism. According to Balaguer, anti-Platonist realists can be divided into two different classes: those who argue that mathematical entities are mental objects (psychologism), and those who argue that mathematical entities are physical objects (empiricism). In point of fact, Balaguer considers only the latter a form of realism because, for the psychologist, mathematical objects ultimately depend on our psicological process, whereas one of the main features of mathematical realism is that mathematical objects are mind-language independent. In any case, the refutation of both psychologism and Mill's position dates back to Frege, and Balaguer does not add anything new to the well-known Frege's arguments.[13]

I have showed so far how Balaguer attempts to prove that full-blooded Platonism and fictionalism are respectively our best forms of Platonism and anti-Platonism. This reduction is essential for Balaguer's argument, but there are some important issues that are been left out. Balaguer does not consider the easy-road strategies that I have presented earlier and, moreover, he does not take seriously Chihara's and Hellman's nominalism. And unless Balaguer can turn *every* anti-Platonist account into Field's fictionalism, his program will be at best incomplete.

I would like to consider another argument Balaguer gives: his defense of nominalistic scientific realism. In a nutshell, scientific realism is the view that we ought to believe that our best scientific theories, taken at face value, are (approximately) true. In Balaguer's view, nominalists should endorse an alternative version of scientific realism about the empirical sciences. The empirical sciences have two interpretations: on the one hand, the nominalist

[13]Compare Balaguer (1998, pp. 104-107) to Frege (1884). In addition, Balaguer also criticizes Kitcher's position, but I do not need to spell this criticism out, because it is unnecessary to my analysis. See Balaguer (1998, pp. 107-109).

content of an empirical theory is the true part of the theory; on the other hand, the Platonistic content of an empirical theory is the part of the theory that is false. Balaguer aims at separating the two interpretations from each other without nominalization, in contrast with what Field has attempted to do.

Balaguer comes up with a thought experiment to show that the behavior of the physical world does not depend on mathematical objects:

> If all the objects in the mathematical realm suddenly disappeared, nothing would change in the physical world; thus, if empirical science is true right now, then its nominalistic content would remain true, even if the mathematical realm disappeared.[14]

The point is that mathematical objects do not have any causal role in empirical science. Does this mean that they do not have any role at all? Balaguer assumes that mathematical objects do not have any causal role, and then he considers the sentence (A) : 'The physical system S is 40°C'. For Balaguer, A expresses a mixed fact that involves both numbers and temperature, even though A's true-value does not depend upon any causal relations between numbers and temperature. So what does A's truth-value depend on? According to Balaguer, A involves two facts: both a (nominalistic) fact about S's temperature and a (Platonistic) fact about numbers. But if these two facts hold independently of one another, there is no problem of regarding the nominalistic content as true and the Platonistic content as false. Mixed statements contain a nominalistic content that express the complete picture of the physical world: the physical world would be the same even if there were no numbers.

I would like to emphasize that Balaguer does not argue that the empirical sciences can be nominalized because, for him, it does not matter to present the nominalistic content of empirical theories. Balaguer merely

[14] Balaguer (1998, p. 132).

requires that the truth-values of mixed statements depend on both nominalistic and Platonistic facts, and that such facts are independent of one another. For Balaguer, the existence of causally inert mathematical objects is irrelevant to how the physical world is, because if mathematical objects suddenly disappeared nothing would change in the physical world. Note that Balaguer is not claiming that mathematical objects are irrelevant to how we describe the physical world, but that the question of whether or not mathematical objects exist is independent of whether mathematics is (in)dispensable to our best scientific theories. If nominalistic and Platonistic facts are independent of one another, nominalistic scientific realism turns out to be a tenable position. This does not mean that nominalistic scientific realism is true, but only that is it possible to endorse fictionalism without nominalizing the empirical theories.

Since nominalistic scientific realism is the view that fictionalism can be sustained without nominalization, I think that Balaguer's position can be considered an easy road to anti-Platonism. Nonetheless, there are two important premises at stake: first, Balaguer assumes that the existence of Platonistic content is plausible and secondly, Balaguer argues that nominalistic and Platonistic facts hold independently of one another. With regard to the former point, I am going to argue that it is not straightforward that the existence of Platonistic facts is plausible; with regard to the latter point, Balaguer's claim implies the rejection of confirmation holism.

At first it is essential to figure out what Balaguer means by 'Platonistic content'. According to Balaguer, the Platonistic content of an empirical theory is what the theory implies about an abstract mathematical realm.[15] Now, I ask myself in what sense an empirical theory implies something about mathematical objects and, as far as I gather, Balaguer does not address this

[15]See Balaguer (1998, p. 131). In point of fact, Balaguer individuates the Platonistic content not just in the empirical theories but also in sentences as 'the physical system S is 40°C'.

problem openly.[16] He only claims that for an empirical theory to be about a Platonistic content does not imply that abstract objects exist.[17] This seems right, because the fact that 'A Scandal in Bohemia' is about Sherlock Holmes does not imply that Sherlock Holmes exists. But here is the problem: on the one hand, Balaguer does not commit himself to the existence of a Platonistic content; on the other hand, he argues that it is plausible for the empirical theories to have a Platonistic content. So what kind of argument does Balaguer have to show that a Platonistic content is plausible? His argument seems to be based on the claim that full-blooded Platonism is defensible, because it can solve Benacerraf's problems. But as I previously indicated, Balaguer's solution depends on a problematic notion of consistency and does not account for the correlation between our mathematical beliefs and mathematical facts. Thus, I doubt that the existence of a Platonistic content is really plausible.

Now, I would like to examine Balaguer's claim that nominalistic and Platonistic facts are disjointed. To do this, I will pretend that the existence of both Platonistic and nominalistic facts is plausible. So how can Balaguer argue that such facts are independent of one another? Although Balaguer does not refer to confirmation holism explicitly, I think it plays an essential role in his argument. According to Balaguer,

> it seems that empirical science predicts that the behavior of the physical world is not dependent in any way upon the existence of mathematical objects. But this suggests that what empirical science says about the physical world — that is, its complete picture of the physical world — could be true even if there aren't any mathematical objects.[18]

[16]To play fair, Balaguer points out that our empirical theories have full-blown mathematical theories. And this is the best way to make sense of their references to mathematical objects (personal communication, December 18, 2014).

[17]See Balaguer (1998, p. 200), footnote n. 4.

[18]Balaguer (1998, p. 133).

The fact that the empirical sciences are confirmed by experience does not depend on the existence of mathematical objects, because mathematical objects are causally inert. But if a theory can talk about the physical world, this means that the theory in question is confirmed by experience. And thus Balaguer's claim implies that empirical confirmation is true independently of the existence of mathematical objects. In other words, empirical confirmation, what the theory truly says about the physical world, does not imply the existence of mathematical objects. Therefore, Balaguer's claim implies the rejection of confirmation holism. And if confirmation holism is rejected, the indispensability argument is disarmed in this sense: the fact that empirically confirmed mathematical theories are indispensable to our scientific theories cannot be used to show the existence of *abstracta*.

4.1.2 There is no fact of the matter

I presented Balaguer's strategy to reduce Platonism and anti-Platonism to their best representatives, i.e. full-blooded Platonism and fictionalism respectively. Balaguer's next move is to show how the metaphysical debate about the existence of mathematical objects cannot be settled at all. To do this, Balaguer first provides an argument that proves that fictionalism and full-blooded Platonism are both defensible philosophies of mathematics and secondly, that there is no fact of the matter as to whether abstract objects exist.

Because there is a stalemate between full-blooded Platonism and fictionalism, Balaguer argues that we do not have any reasons to choose between Platonism and anti-Platonism. According to Balaguer, there are neither good argument for full-blooded Platonism nor for fictionalism, because both positions can defend themselves from the best arguments that philosophers have provided against them. On the one hand, full-blooded Platonists can explain how we can get mathematical knowledge of abstract objects through the intuitive consistency of mathematical theories; on the other hand, fictionalists can account for the indispensable use of mathematics in the empirical sciences by endorsing nominalistic scientific realism. Bear in mind that if nominalistic scientific realism can be sustained, the fictionalist can maintain that the nominalistic content of the empirical theories is true, whereas the Platonistic content is false. In a nutshell, Balaguer argues that if the best arguments against both full-blooded Platonists and fictionalists can be blocked, the metaphysical debate ends up in a stalemate.

Even though philosophers have not found yet a good argument for or against the existence of abstract objects, perhaps philosophers could settle the metaphysical debate in the long run. Hence I would like to consider Balaguer's metaphysical claim now: philosophers could *never* settle the metaphysical debate. It is important to note that Balaguer does not claim that the dispute over the existence of mathematical objects is meaningless, because he does not commit himself to any sort of verificationism. In fact, the metaphysical debate is *factually empty*.[19]

Balaguer's strategy is to argue that the sentence (P) 'there are objects that exist outside of space-time' does not have truth-conditions. More precisely, even though that sentence has disquotational truth-conditions, i.e. P is true if there are abstract objects, however it does not have possible-worlds-style truth-conditions: there are no possible worlds where P is true. In other words, Balaguer aims at showing how there is no fact of the matter as to which possible worlds count as worlds in which P is true.

Let us see how Balaguer's argument works. At first Balaguer notices that our whole conception of what existence amounts to be is bounded up with space and time. That is, we do not have *any* idea of what the existence outside of space and time is like. If we have no idea of what existence outside of space and time could be like then, Balaguer argues, we cannot even imagine what a possible world where there are *abstracta*. Of course, this argument does not establish that *abstracta* are fictional objects. And in point of fact, Balaguer claims that despite the fact that we cannot imagine what existence outside of space and time could be like, there could exist

[19] Balaguer (1998, pp. 158-159).

abstract objects.

Balaguer's next move aims at showing that if we cannot imagine what existence outside of space and time is like, then there is no fact of the matter as to which possible worlds count as worlds in which P is true. Balaguer argues that the fact that we cannot imagine what existence outside of space and time is like does not depend on our ignorance of the truth-value of P, but on the fact that our usage does not determine P's (possible-worlds-style) truth-conditions. And if our usage does not determine what the (possible-worlds-style) truth-conditions of a sentence are, we have good reasons to imply that there is no fact of the matter.

We should be very careful in regarding Balaguer's anti-metaphysical conclusion as agnostic. Nevertheless, it is true that Balaguer's conclusion resembles agnosticism in a certain sense: it may follow that we should suspend the debate between Platonism and anti-Platonism. But even though agnosticism may follow, it is because of how the world *is*: because there is no fact of the matter about P's truth conditions.

4.1.3 Non-factualism and tertium non datur

We saw how Balaguer distinguishes different kinds of truth-conditions. According to Azzouni and Bueno, two notions of truth are also involved in Balaguer's analysis: one is deflationary, whereas the other is stated in possible-worlds-style.[20] Let us assume that the notion of possible-worlds truth generalises the deflationary conception of truth. Thus, the actual world turns out to be one instance of Balaguer's metaphysical conclusion that 'there is no fact of the matter as to whether abstract objects exist'. If this is so, Azzouni and Bueno notice that if the sentence (P) 'there are abstract objects' has no possible-worlds-style truth-conditions, as Balaguer claims, then P does not also have truth conditions in deflationary sense. However, this claim is in-

[20] See Azzouni and Bueno (2008, p. 760). According to the deflationary theory of truth, to say that a statement is true is just to assert the statement itself. For example, to say that 'snow is white' is true is equivalent to saying that snow is white.

consistent with Balaguer's view that P has disquotational truth-conditions, but it has no possible-worlds-style truth-conditions.

Azzouni and Bueno go further, and argue how Balaguer's claim that 'there is no fact of the matter whether abstract objects exist' turns out to be incompatible with classical logic. In order to sustain non-factualism, Balaguer should be able first to assert that 'either abstract objects exist or they do not' is true. But if 'there are abstract objects' has no disquotational truth-conditions, then 'either abstract objects exist or they do not' does not have disquotational truth-conditions either. This is because

> If one is in the classical setting, one must be able to assert (A or $\neg A$), for any sentence A. If one nevertheless claims there is no fact of the matter whether A or $\neg A$ is true, then one must be able to say this in a way that is compatible with one's commitment to (A or $\neg A$) for every sentence A in one's language.[21]

As a result, Balaguer cannot run his argument unless he rejects the principle of bivalence.[22] Alternatively, Azzouni and Bueno suggest how Balaguer could have argued that 'either abstract objects exist or they do not' is true, but the choice between them is arbitrary because there is no fact of the matter. Unfortunately, Balaguer's anti-metaphysical claim is way stronger than that (i.e. 'there are abstract objects' has no possible-worlds truth-conditions). However, it turns out that even 'there are abstract objects' has no disquotational truth-conditions, and, therefore, 'there are abstract objects or such objects do not exist' has no disquotational truth-conditions either. In Balaguer's view, *tertium non datur* is rejected.

[21] Azzouni and Bueno (2008, p. 756).

[22] Balaguer points out that he rejects the principle of bivalence (personal communication, December 18, 2014).

4.2 Agnostic nominalism

Platonism has a great advantage over nominalistic strategies that paraphrase mathematical statements. The Platonist can take mathematical discourse at face value, since mathematical statements are literally true on this account and, as a consequence, he or she can also provide a unified semantics for both scientific and mathematical statements. On the other hand, the Platonist requires much more work in order to explain how we can get knowledge of abstract objects and, moreover, how we can refer to them. These last issues are by contrast an easier task for the nominalist, because if mathematical *abstracta* are fictional objects, then there is no need to explain how we can get knowledge of a-casual entities, nor one has to explain how reference to *abstracta* is possible. The application of mathematics, however, is problematic for both Platonism and hard-road nominalism. If abstract objects exist, it is unclear how they are connected with the physical world; but if mathematical theories are literally false, as many nominalists claim, it is difficult to understand how such theories can be successfully applied to the physical world.[23]

Since Platonism and hard-road nominalism have both their advantages and disadvantages, it is interesting to ask ourselves if we can get all the benefits of Platonism without ontological commitment to *abstracta*. To address this problem, Bueno advances a view where commitment to abstract objects is avoided without denying their existence.[24] The outcome is called 'agnostic nominalism', which stems from the following consideration: if abstract objects are mind– and language– independent, it is not clear how we can rule out their existence. At best, it is possible to argue that there are no good reasons to believe that abstract objects exist but, still, this is not sufficient to prove that they do not exist. As Bueno emphasizes,

[23]Field and Hellman, for instance, do not take mathematical discourse at face value. Remember that for Field existential mathematical statements are literally false and, according to Hellman, standard arithmetical statements can be reformulated into statements about possible ω–structures.

[24]Bueno (2009, p. 64).

> rather than insisting that mathematical objects do not exist, we could argue that we don't have good reason to believe in their existence. But even if the latter claim were established, it wouldn't settle the issue regarding the non-existence of mathematical entities.[25]

Bueno means by lacking 'good reasons' that mathematical practice does not require the existence of abstract objects. Indeed mathematical practice starts with concrete objects such as diagrams, inscriptions, and so on. From these concrete objects mathematicians develop certain intuitions about mathematical facts, that is, facts about what follows from certain assumptions given a specific domain of objects. Furthermore, mathematicians come up with new intuitions of such facts, and the process can be iterated indefinitely. According to Bueno, mathematical intuitions are basically intuitions of relations between objects that are introduced by comprehension principles, which provide the background assumptions for mathematical inquiry in order to introduce mathematical objects and their properties.[26]

Bueno's point is that mathematical practice presupposes comprehension principles. This point allows Bueno to endorse a specific fictionalist view of mathematical objects, according to which mathematical objects are artifacts created on the basis of comprehension principles.[27] Artifacts are not mind–independent *abstracta*. Mathematical artifacts require a physical basis where they are recorded, such as paper or memory. In addition, mathematical artifacts depend on someone, or a community, who can understand them. Mathematical objects are artifacts because if nobody can understand a specific piece of mathematical work, or every copy of it is lost, these objects simply stop existing. But the fact that, for Bueno, *abstracta* have no role in mathematical practice is not an argument to deny their existence: "perhaps

[25] Bueno (2008, p. 100).

[26] For example, Peano's axioms provide the context to investigate natural numbers. In general, comprehension principles do not need to be presented axiomatically.

[27] Bueno explicitly adopts Thomasson's artifact theory and applies it to mathematics. See Bueno (2009, p. 71).

they exist, perhaps they do not". More precisely, Bueno agrees with the Platonist that mathematical objects would be *abstracta* if they happened to exist. Indeed,

> objects introduced via comprehension principles are introduced as entities that are not located in space-time. Since, typically, there is no specification for time or space in the comprehension principles, the agnostic nominalist can explain why mathematical objects are, thus, introduced as abstract objects.[28]

Platonism is not ruled out by the agnostic nominalist. But what are the advantages of endorsing such a view? According to Bueno, agnostic nominalism has all the advantages of nominalism without its problems. Basically, agnostic nominalism can account for the applications of mathematics to the physical world, and moreover mathematical statements are taken at face value.

Let us see first how the agnostic nominalist can take mathematical statements at face value. Bueno's strategy is basically to employ Azzouni's neutral quantifiers. The idea is that if the standard interpretation of quantifiers is neutral from an ontological point of view, as Azzouni claims, a statement like 'there are infinitely many natural numbers' can be true without committing to either *abstracta* or *concreta*. But because quantifiers are neutral, they cannot dictate what exists and what does not. In addition, the agnostic nominalist notices how abstract objects are even compatible with Azzouni's mind– and language– independence criterion. This is because the Platonist can agree with Azzouni on the criterion for what exists and claims in fact that abstract objects are mind– and language– independent. And when Azzouni argues that mathematical objects are actually ontologically dependent on us, the Platonist can still reply that Azzouni is begging the question.[29]

[28]Bueno (2008, p. 104).

[29]'And if one insists that the only existing things are those that are causally accessible to us, we would simply be begging the question against the Platonist, who has less restrictive ontological constraints' (2008, p. 100).

As a result, if the anti-Platonist cannot claim that mathematical objects are mind– and language– independent without reaching a stalemate, there are only two alternatives: we can either show that mathematics is actually dispensable, like Field does, or we can endorse agnosticism. By the second option, the nominalist maintains the agreement with the Platonist on mathematical statements semantics.

How does the agnostic nominalist explain the application of mathematics? Given that for Bueno mathematical objects are artifacts, or fictions, one may expect that agnostic nominalism would have as many problems as fictionalism does. However, mathematical discourse is not taken to be false within agnostic nominalism. Since quantifiers are ontologically neutral, they do not force us to be committed to anything, and so mathematical discourse can be true even without the existence of objects. In this sense, agnostic nominalism can be regarded as an easy road to nominalism: mathematical practice is naturally agnostic. Mathematical artifacts describe empirical phenomena as well as metaphors can have a grip on reality.

At this point the Platonist can reply by noticing that mathematics does not provide only description of the physical world, but it also provides genuine explanations of empirical phenomena. Thus, the indispensability argument strikes again in a new form. Imagine that mathematical explanations involve the existence of mathematical facts about that physical event. If there are mathematical explanations of physical phenomena, and such explanations are indispensable to our best scientific theories, then there are mathematical facts. But these facts are about mathematical objects, and therefore mathematical objects must exist.

To argue against this new form of the indispensability argument, Bueno runs a counter-reply to show how mathematical explanations do not require the existence of mathematical objects. What we call 'mathematical explanations' are actually just descriptions of empirical phenomena. Bueno's point is that mathematics must be firstly interpreted in order to provide a suitable explanation of physical events.[30] Mathematics by itself cannot explain a physical event unless we properly attribute a physical meaning to constants and variables. Uninterpreted mathematical statements describe only relations among mathematical objects: not physical ones. And even when mathematics is properly interpreted, the event under description is still physical.

Interpretation is also the key to explain the applications of mathematics to the physical world. Let us consider the sentence 'there is the real number $6.02214129 \times 10^{23}$'. This sentence does not have any grip on the physical world unless we interpret that constant physically as the Avogadro constant: that is, as the number of molecules per mole of a physical substance. Only after we have interpreted that real number as the Avogadro constant, can we use it to understand the interactions between molecules. But again, we are describing physical interactions: not mathematical facts. And if no mathematical facts can explain physical events, the indispensability argument based on mathematical explanations is easily blocked.

I would like to make a last remark before I turn to Azzouni's objection to agnostic nominalism. The underdetermination of mathematical theories by the physical world can be used in order to show the indeterminacy of mathematical objects. Roughly speaking, if the same empirical results are obtained by using different mathematical theories, we cannot choose which theory is true on the basis of empirical considerations.[31] So long as underdetermination comes into picture, it does not matter which mathematical theory we pick out and, as a consequence, it does not also matter what kind of mathematical objects we take to exist. Imagine two different mathematical theories M_1 and M_2, where M_1 entails the existence of objects $x_1, x_2, ..., x_n$, and M_2 entails the existence of $y_1, y_2, ..., y_n$. If M_1 and M_2 describe the

[30] See Bueno (2012, p. 972).

[31] The case of mathematics may be similar to the question of different interpretations of quantum mechanics. If, say, the Copenhagen and many-worlds interpretations of quantum mechanics lead to the same empirical results, it seems that the nature of the wave-function will be underdetermined by those interpretations.

same empirical phenomenon, we cannot choose between $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ on the basis of empirical considerations. And, from an empiricist point of view, it does not matter matter what kind of mathematical objects we are committed to.

Establishing the indeterminacy of mathematical objects is not enough for the agnostic. We can be agnostic, say, about the existence of sets and categories under the assumption that category theory and set theory can be used to describe the same empirical phenomena. But this point does not undermine the thesis that mathematical objects exist, because we also need a nominalistic theory that describes the same empirical phenomena as those theories with mathematical objects do.[32]

4.2.1 Azzouni's objection to agnostic nominalism

Azzouni's critique of agnostic nominalism[33] is based on Bueno's acceptance of the epistemic role puzzle, according to which mathematical practice does not require the existence of mathematical objects. Azzouni argues that agnostic nominalism is incompatible with the epistemic role puzzle and, moreover, agnostic nominalism does not provide any further advantages over Azzouni's nominalism. This is because agnostic nominalism either depends on a reformulation of current linguistic practices, or it depends on a broader form of agnosticism. The last point is particularly interesting, because if it is possible to show how we cannot draw a sharp line between agnostic nominalism and global agnosticism, then agnostic nominalism will not be restricted only to mathematics, but it will spread out to every object — i.e. agnostic nominalism is just global agnosticism.

I am going to spell out what global and local agnosticism are. But first, I would like to point out the difference between skepticism and agnosticism, since there are forms of agnosticism that are not skeptical. Roughly speaking, the skeptic argues that knowledge is impossible. Cartesian doubt is

[32] Hellman's reconstruction of real analysis, for instance.

[33] Azzouni (forthcoming a).

perhaps the most common and radical form of skepticism, since it is well known how Descartes casts doubt on any area of knowledge: from common sense to scientific truths. In this sense, Cartesian doubt is what we can call a 'global form of skepticism'. But one does not need to endorse global skepticism to be considered a skeptic. Indeed, local forms of skepticism are still possible, as when a specific area of knowledge, like mathematics, is in doubt. In this way, what would be at issue is only mathematical knowledge.

However, Bueno neither endorses global skepticism nor a local one. He is not a skeptic, at least according to the definition that I provided above. For Bueno, mathematical knowledge is not at issue, since it can be explained by looking at what follows from comprehension principles. More likely, the agnostic argues that if it is unclear how to establish whether a certain thing exists or not, then it is rational to suspend judgment on its existence. In short, the agnostic is agnostic about objects, whereas the skeptical is skeptical about propositions. We can also distinguish between global and local agnosticism: the global agnostic suspends judgment on existential assertions independently of any context, whereas the local agnostic suspends judgment on a targeted domain.

This distinction is useful to understand Azzouni's argument against agnostic nominalism. Although Azzouni does not make this distincion, I think that he aims to show that, given certain assumptions, local agnosticism depends on global agnosticism. To do that, Azzouni starts with noticing how both noun phrases and quantifiers can be used non-referringly in the natural language, where by 'non-referringly' Azzouni means that those expressions do not refer to objects. The most common example is, perhaps, when we are engaged in an activity that involves fictional characters. For example, I can talk about Sherlock Holmes even if the name 'Sherlock Holmes' does not refer to a real detective. Ordinary speakers do not need to specify whether an expression is used non-referringly if they can track the relevant information down from the context.

Azzouni observes that when an expression is used non-referringly, it is pointless being agnostic about the existence of the objects to which that expression is not supposed to refer to. For instance, if it is clear that we are telling a fairy tale about hobbits and dragons, why should one be agnostic about their existence? The point is not that hobbits and dragons are fictional objects because nobody has ever met them, but because they are explicitly made up by speakers. In other words, it is clear from the context that these objects do not exist. One can still be agnostic about their existence, of course, independently of what the speakers mean. But if this is so, why should not agnosticism spread out to any domain?

To see how Azzouni's argument works, let us keep in mind that, according to Bueno, abstract objects do not play any role in mathematical practice. Because of this, Azzouni can argue that Bueno's thesis may be reformulated saying that mathematical terms do not refer to *abstracta*. But if it is true that mathematical terms do not refer to *abstracta*, why should we be agnostic about their existence? Bueno has no reasons to be agnostic about these objects because, Azzouni argues, he admits that they have no role in mathematical practice. The situation is almost identical to the case of fairy tales that I mentioned earlier: if hobbits are made up by the storyteller, why should one be agnostic about their existence? After all, it is clear that this is just a story about hobbits, and if we are still not sure whether or not hobbits exist, something important is missing: the fact that the story is just a fairy tale.

If Azzouni's analogy is accurate, the only reason to be agnostic about the existence of fictional characters relies on the fact that Bueno had already endorsed global agnosticism; that is to say, we are going to be agnostic about what the storyteller says independently of any content. In the same way, if Bueno knows that abstract objects have no role in mathematical practice — the story teller is just telling a *story* — he can be agnostic about their existence only by endorsing global agnosticism. Azzouni is not running a counter-argument to global agnosticism, but points to a serious difficulty for those who want to be local agnostics. If Azzouni's argument works, it seems that agnostic nominalism depends on general sympathy for a global form of agnosticism that is not targeted for the specific context of mathematics.

To sum up, it seems hard to be both agnostic and nominalist unless one adopts global agnosticism. But consider the following response to Azzouni: because quantifiers are ontologically neutral, Bueno is simply not ruling out that that mathematical terms can be used non-referringly. The agnostic does not know whether or not mathematical terms are used referringly: he, or she, suspends judgment on the use of the notion of reference. However, this solution is ruled out because, Azzouni argues, mathematical terms are used non-referringly: if names are introduced into a context where they are deliberately being used non-referringly, it is pointless to be agnostic. The fact that mathematical practice does not require the existence of mathematical objects indicates for Azzouni that mathematical terms are used non-referringly.

Let us assume that the agnostic nominalist agrees with Azzouni that mathematical practice does not refer to abstract objects. In this regard, the agnostic nominalist may be tempted to insist that agnosticism is not about *actual* mathematical expressions, but about the possibility that mathematical terms may refer to abstract objects. But if this is so, Azzouni insists, the worry would not be about actual mathematical practice but about a possible mathematical practice. Again, agnostic nominalism would emerge from global agnosticism: how can we know that there are not other practices where mathematical terms refer to abstract objects?

Azzouni's point is that if the agnostic nominalist accepts that the non-referringly use is the standard one, he ro she does not have any specific reason to be agnostic, aside from general sympathy for global agnosticism. Basically, Azzouni rightly notes a tension between nominalism and agnosticism, and so the agnostics face a challenge: they should either challenge the epistemic role puzzle, leaving behind his nominalism, or they should stretch their agnosticism, but then failing to be locally agnostic about mathematical *abstracta*.

Can the agnostic break that impasse? I will attempt to show how the agnostic should not be worried about the epistemic role puzzle and, in addition, that it is possible to be agnostic about mathematical objects without endorsing global agnosticism. But first, it is important to understand how the agnostic nominalist got in this unpleasant situation. The reason is this: one should be careful to agree with the nominalist that abstract objects have no role in mathematical practice and, at the same time, wink at the Platonist by saying that perhaps *abstracta* exist, perhaps they do not. But does the epistemic role puzzle imply that mathematical terms are used non-referringly? The agnostic should be wise enough to address that question negatively by pointing out that whether or not mathematical terms are used referringly is indeterminate. In other words, the agnostic could argue that mathematicians are neutral about whether or not mathematical terms are used referringly.

The agnostic, unfortunately, cannot simply get away with the last remark. Here is how Azzouni could reply: unless mathematicians intentionally use mathematical terms referringly, the agnostic should accept that these terms function non-referringly. Mathematicians, Azzouni continues, do not try to study objects that are outside of space and time. Therefore, mathematical terms function non-referringly and, as a consequence, the only reason to be agnostic about mathematical objects must be, only, on prior acceptance of global agnosticism.

I will attempt to disarm Azzouni's argument. The agnostic can point out that many mathematicians are Platonists, and for them mathematical terms refer to objects outside of space and time. It is clearly an empirical question whether or not mathematical terms are used referringly, but mathematicians' answer to that problem is not unanimous. In fact, Azzouni seems to think that the answer is agreed by everyone and that mathematical terms are clearly used non-referringly. However, because there is a widespread disagreement among mathematicians on whether or not mathematical terms refer to, the agnostic does not sound preposterous. secondly, the agnostic can try to prove that Azzouni is assuming his nominalism in order to make his argument: he assumes that mathematical objects are fictions. But in point of fact, there is a strong dis-analogy between hobbits and mathematical objects: the former are clearly fictional characters, made up by epistemic agents, whereas the status of the latter is controversial.

Chapter 5

Conclusion

5.1 Four claims

In the previous chapters I examined some easy-roads to anti-Platonism. I would now like to highlight a common idea behind such easy-roads. The point, I think, is that we cannot find evidence to support the existence of *abstracta* as regards the way the physical world behaves. According to Azzouni and Bueno, the evidence we lack is epistemic; according to Balaguer, it is the mechanism of the physical world that does not require abstract objects, because nothing in the world follows from the existence, or non-existence, of *abstracta*. These may be summarized as follows,

Azzouni's epistemic role puzzle: Mathematical objects play no epistemic role whatsoever in mathematical practice. If mathematical objects ceased to exist, mathematical work would go on as usual.

Bueno's agnostic nominalism: Mathematical practice does not require the existence of abstract objects, but this claim still does not settle the issue regarding whether mathematical abstract objects exist. Moreover, even though mathematical explanations do not require the existence of mathematical objects, the claim does not settle the issue regarding whether mathematical abstract objects exist either.

Balaguer's modal anti-factualism: The sentence 'there are objects that exist outside of space-time' does not have truth-conditions. Thus, there is no fact of the matter as to which possible worlds count as worlds where 'there are objects that exist outside of space-time' is true, because our usage does not determine how worlds would be like for 'there are objects that exist outside of space-time' to have (possible-worlds-style) truth-conditions. This argument does not aim to establish that abstract objects do not exist, i.e. it is not an argument for nominalism. In point of fact, Balaguer argues that if our usage does not determine truth-conditions for 'there are abstract objects', then there is no fact of the matter.

Each claim leads to a different conclusion: Azzouni argues for nominalism (no *abstracta*); Bueno's conclusion is agnosticism (we do not know whether or not *abstracta* exist); Balaguer's final claim is non-factualism (no facts can settle the ontological disagreement). But they all have in common the idea that we cannot find evidence to support the existence of *abstracta* as regards the way the physical world behaves, because *abstracta* are supposed to be outside of space and time.

Yablo highlights a further problem. Logical subtraction is basically a philosophical tool to show how physical conditions do not prove (or disprove) propositions about the existence of abstract objects. More precisely, logical subtraction aims to show how the proposition *there are numbers* is perfectly extricable from the propositions that are involved in our best scientific theories. However, logical subtraction is far from being an argument for nominalism: at best it shows that propositions about *physical objects* do not require the existence of numbers, because we can find truthmakers for such propositions in numberless worlds that are compatible with the existence of numbers. More likely, logical subtraction tends to breed a form of agnosticism: even though *there are numbers* can be extricated from empirical sciences, this does not show that numbers do not exist, because both Platonistic and numberless possible worlds are not ruled out. In other words, as stated in

Yablo's modal orthogonality claim: If *how matters stand physically* is orthogonal to *mathematical objects exist*, then the physical world neither demands nor precludes the existence of mathematical objects (in any world). The physical world is thus compatible with the (non-)existence of mathematical objects. This consideration is not an argument against Platonism because, as far as **physical objects** are concerned, they can indifferently be in a world that has or does not have abstract objects.

Yablo's orthogonality claim may resemble Balaguer's anti-factualism, because they both lead to a stand-off between Platonism and anti-Platonism, and because Yablo and Balaguer both employ the concept of possible worlds. But there are possible worlds where abstract objects may exist for Yablo, whereas for Balaguer there is no fact of the matter. Yablo's orthogonality claim, on the other hand, might also resemble agnostic nominalism. But whereas Bueno argues that we cannot know whether or not abstract objects exist, knowledge of *abstracta* is not at issue for Yablo. Besides, it is interesting to note how Balaguer, Bueno, and Yablo all have in common a stand-off between Platonism and anti-Platonism. This does not prove that the ontological debate is meaningless, but that some easy-roads that I presented earlier are either agnostic (Bueno), or imply agnosticism (Balaguer), or are compatible with agnosticism (Yablo).

5.2 My own agnostic view

To some extent, agnosticism is a new view in philosophy of mathematics. As far as I know, there are only two philosophers who have argued that we cannot settle the debate between Platonism and anti-Platonism. Consequently, so it is claimed, we ought to suspend judgment on whether or not mathematical *abstracta* exist. According to Balaguer, the ontological debate cannot be settled because there is no fact of the matter as to whether or not abstract objects exist. On the other hand, according to Bueno, even though mathematical practice does not involve *abstracta*, there is no way to establish whether or not there are such objects.[1]

It is important to distinguish Bueno's agnosticism from Balaguer's anti-factualism. As previously shown, Balaguer's agnostic conclusion follows from his non-factualism, that is, from the idea that there is no fact of the matter. Balaguer's point is that our usage does not determine how worlds would be made for abstract objects to exist: namely, the statement 'there are abstract objects' lacks (possible-worlds-style) truth-conditions. By contrast, Bueno argues that even though abstract objects play no role in mathematical practice, we cannot ever know whether or not such objects exist. Since they are supposed to be outside of space and time, there is no way of (dis-)proving their existence. To support agnosticism, Bueno endorses Azzouni's neutralist quantifiers: existential quantifiers do not force any ontological commitment to *abstracta*.

But here lies a question: if abstract objects have no role in mathematical proofs, or in mathematical practice, why should we suppose that they might exist? After all, if *abstracta* disappeared nothing would change in the way mathematicians prove theorems. Consider the following thought experiment:[2]

> Imagine that mathematical objects ceased to exist sometime in 1968. Mathematical work went on as usual.[3]

Some years later, Balaguer came up with a similar thought experiment, although Balaguer commits himself to the nominalistic content of empirical theories:

[1]Quine's non-factualism about translation is an important antecedent. Quine argues that there is no fact of the matter about how words such as 'gavagai' should be translated. 'Gavagai' means 'rabbit' relative to a translation scheme, but there is no fact of the matter about what translation is the right one.

[2]What follows is *not* Azzouni's objection to agnostic nominalism.

[3]Azzouni (1994, p. 56).

> If all the objects in the mathematical realm suddenly *disappeared*, nothing would change in the physical world; thus, if empirical science is true right now, then its nominalistic content would remain true, even if the mathematical realm disappeared.[4]

If abstract objects disappeared nothing would change in the way mathematicians prove theorems or, more generally, in the physical world. There is no scientific study of the epistemic access to mathematical *abstracta*, whereas in the empirical sciences we attempt to improve our epistemic access to physical entities by building complex instruments and machines. In the empirical sciences a mechanism for refining our epistemic access is involved, in contrast to what happens in mathematics. The way mathematicians prove theorems does not require epistemic access to mathematical objects.

Platonists could easily agree with the considerations I have cited. After all, the indispensability argument is a way of supplying the lack of an epistemic role for abstract objects. In addition, Platonists could agree that no one can dictate that an object exists only by thinking of it or symbolizing it, and they could also agree that anything exists if it is mind– and language–independent. In this regard, the dispute between the Platonist and anti-Platonists is mainly about how we recognize that an object is mind– and language– independent. For example, this is what Azzouni claims:

> if an object has no epistemic role, then it's mind- and language-dependent, and therefore (by our criterion) it doesn't exist. Mathematical abstracta have no epistemic role. Conclusion: there are no mathematical abstracta.[5]

What does Azzouni mean by 'epistemic role'? An object has an epistemic role if the way we discover its properties involves either physical interactions with that object or with theoretically-related objects.[6] According

[4]Balaguer (1998, p. 132).
[5]Azzouni (forthcoming b).
[6]See Azzouni (2012b, pp. 956-957).

to Azzouni's terminology, in the former case the epistemic access is thick whereas in the latter, it is thin.

Azzouni's criterion of existence needs to be distinguished from how we determine what exists. The criterion is a necessary and sufficient condition for existence,[7] but it often requires to be combined with an *empirical* test to know whether or not an object exists. Even though we do not need an empirical test for gold to exist, because the existence of gold is independent of our epistemic access to it, the way to determine whether or not an item is golden is empirical. To detect existing objects we can employ the senses, but also instruments that extend our epistemic access. In addition, we can also deduce the existence of entities from relevant background assumptions, i.e., theory. For example, although we do not have thick access to items that are outside our light cone, scientists can nonetheless commit themselves to their existence. In other words, there are posits that may take to exist because they have an excuse for not being thick.

Azzouni does not draw up a list of what such excuses are, but claims that they arise from scientific practice itself. Scientists are able to decide whether an item is thin on the basis of their internal standards, so that thin posits can be taken to exist even without our having thick epistemic access to them. However, mathematical objects do not benefit from any excuse of that kind (they are neither thick nor thin) because the epistemic role puzzle applies to them: no epistemic story can be told about the role that mathematical objects would play in the way mathematicians prove theorems. If such posits had any role, mathematical practice would change when the epistemic access to *abstracta* was mistaken. But nothing like that is involved in mathematical practice. A mathematical mistake

> can be a matter of our failure to execute a computation correctly [...] but it can also be a matter of conceptualizing a class of objects the wrong way. What it never involves, however, is that the mechanism of our epistemic access to the *abstracta* under

[7] Azzouni (2012b, p. 955).

> study is misleading.[8]

Because of this, mathematical posits cannot be thick.[9] Neither can they be thin, since we would need a reason rooted in our scientific practices:

> this much seems true about thin posits: they are what we commit ourselves to that goes beyond what we have thick epistemic access to [...] on the basis of our current best scientific theories.[10]

For Azzouni, the fact that mathematical posits are acausal does not count as a reason for considering them thin posits, because it does not come directly from mathematical practice, but it is just a philosophical gloss on that practice. The nature of mathematical posits is not involved in mathematical practice.

In Azzouni's view, the epistemic role puzzle explains why mathematical posits are not thin, and he argues that this is inconsistent with agnostic nominalism. According to Azzouni, agnostic nominalists face the following challenge: either they accept that mathematical terms are used non-referringly, or they may argue that they do not know whether or not mathematical terms are employed referringly. The former case is a consequence of the epistemic role puzzle: if mathematical objects have no epistemic role in mathematical practice, why should we suppose that mathematical terms refer to anything? And if they do not refer to anything, it is pointless to be agnostic about the existence of mathematical objects unless one has a broad skeptical attitude that goes beyond mathematics. On the other hand, suppose that the agnostic does not know whether mathematical terms are used referringly or used non-referringly. After all, if quantifiers are ontologically

[8] Azzouni (forthcoming b).

[9] Azzouni's epistemic role puzzle could also be applied to mathematical objects even if they were *concreta*. For instance, even if mathematical objects were perceived, we should nonetheless account for the epistemic role that such *concreta* have in mathematical proofs. Notice that even if mathematical objects are metaphysically necessary (i.e. they cannot disappear), we should still account for their epistemic role in mathematical proofs.

[10] Azzouni (2012b, p. 962).

neutral, one could be agnostic about whether they refer to a domain of existing objects. But this option is also problematic. For neutral quantifiers to work, mathematical terms are used non-referringly unless mathematicians intend to employ them referringly. Again, if the agnostic accepts the epistemic role puzzle, he or she should also agree that mathematical terms are employed non-referringly. And even if some kind of correspondence between mathematical terms and *abtracta* were to occur, it would not happen by accident but would hold because mathematicians intend to refer to *abstracta*. Reference does not often happen by accident.

Notice that even if we supposed that mathematical terms refer to mind– and language- independent *abstracta*, there would be no empirical way to determine whether or not such objects exist. This is because mind– and language- independent *abstracta* are immunized against any empirical test whatsoever. Given the inertness of mind– and language- independent mathematical *abstracta*, the physical interaction between the way mathematicians prove theorems and abstract objects cannot in principle occur. By contrast, galaxies that are far away from our light cone are not causally inert, despite the fact that we cannot physically interact with them. Even if mind– and language– independent *abstracta* existed, they would be irrelevant to mathematical practice. After all, the reason why the indispensability argument is so important for the Platonist is because it shifts the burden of proof from epistemology to the indispensability of quantification over *abstracta* in physics. Since scientists cannot avoid quantification over mathematical *abstracta*, so it is claimed, these objects must exist. But quantification, on its own, is neutral about whether or not mathematical terms are used referringly. In that sense the agnostic should employ neutralist quantifiers.

To sum up, whether or not abstract objects existed, mathematical practice would go as usual, and quantification on its own does not tell us whether or not mathematical terms are used referringly. But do mathematical terms refer to anything? The agnostic should reject the claim that mathematical terms are commonly used non-referringly. After all, many mathematicians think that mathematical terms *do* refer to something, and some of them think that their terms refer to *abstracta*. Perhaps these mathematicians are wrong about how language works, but the matter of whether mathematicians should use their terms non-referringly is not at stake. What I am arguing is that if mathematical terms intend to refer to abstract objects, we cannot know whether or not mind– and language– independent *abstracta* exist. We find out whether an object exists by looking to our (thin or thick) epistemic access to it, but epistemic access cannot be used to determine whether or not such *abstracta* exist, because of their specific nature: they are causally inert objects.

It might seem that for the agnostic the nature of mathematical objects (i.e. being mind– and language– independent *abstracta*) comes before the problem of whether mathematical terms are used referringly. And this seems to reverse our natural way of reasoning: first, we ask ourselves whether or not a term refers to and secondly, we may say something about the nature of the entity that the term refers to. This is because speculating on the nature of non-existing entities is simply pointless; so if mathematical terms are used non-referringly, it does not make sense to wonder about the nature of mathematical objects. Perhaps the agnostic is just begging the question by putting the nature of mathematical objects ahead of whether mathematical terms are used referringly or not. However, this problem does not have a single answer: some mathematicians employ mathematical terms referringly, others do not. To me, the interesting point is how our world would behave if mathematical terms referred to mind– and language– independent *abstracta*.

Let us suppose that someone is using mathematical terms referringly. In this case, we will require a test in order to determine whether or not mathematical *abstracta* are mind– and language– independent. The way to figure it out usually relies on its epistemic role; however, because of Azzouni's epistemic role puzzle, mathematical posits must be mind– and language– dependent. Thus, the agnostic should attempt to disarm the epistemic role puzzle.

Certainly, the epistemic role puzzle on its own neither proves, nor disproves, the existence of abstract objects. In other words, even if there is no epistemic role for *abstracta*, that claim, on its own, is compatible with the (non-) existence of *abstracta*. To become an argument against Platonism, the epistemic role puzzle requires something more: if there is no epistemic role for X, either there is no X, or X has a "good reason" that allows X to be a thin posit. According to Colyvan, the fact that mathematical objects are *abstracta* may count as a good excuse for them to be considered (ultra-) thin posits. But Colyvan's claim is implausible. Physicists offer *physical* explanations to commit themselves to the existence of more items than the matter within our light cone; biologists use fossil constraints and rates of molecular change to deduce the time in geologic history when two species diverged; and so on. Abstractness is not a good excuse to regard mathematical objects as existing posits, since it is not rooted in our scientific practices.[11]

In contrast to Colyvan, I do not think there are any good excuse that allows mathematical *abstracta* to be existing posits. What I am arguing is that even though there is no epistemic role for mind– and language– independent mathematical *abstracta*, we still have a good reason to be agnostic about their existence. Let us call it 'the exclusion condition': *abstracta* are supposed to be mind– and language– independent objects that are causally inert. If mathematical *abstracta* enjoy the exclusion condition, the fact that they have no epistemic role merely implies that nothing epistemically relevant to mathematical practice follows from the existence, or non existence, of such *abstracta*. In other words, the existence of objects that enjoy the exclusion condition does not influence our epistemic practices such as the way mathematicians prove theorems. It does not matter for mathematical practice whether or not mind– and language– independent mathematical *abstracta* exist.

This is how one might reply: the exclusion condition does not come from

[11] See also Azzouni (2012b, p. 963) and sec. n. 3.1.5.

mathematical practice; it is a merely philosophical gloss on that practice, and so it does not count as a good reason.[12] As mentioned earlier, I agree that the abstractness of mathematical objects cannot be an excuse to regard such objects as existing posits. Such an excuse should come from our scientific theories. However, even though the exclusion condition is a pure philosophical condition, I do not see that there is problem with respect to agnosticism is concerned. Agnostics continue looking to science to discover what exists and what does not; they are not trying to engage with science by advocating some philosophical excuse, but just pointing out that certain philosophical questions are indeterminate. Agnosticism is clearly consistent with scientific practice.

The fact that our epistemic practices are restricted within space and time does not imply that the objects that are beyond such practices are mind– and language– dependent. Our concept of existence is tightly bounded by space and time. Nevertheless, even though we cannot imagine what the existence outside of space and time would be like, the point is not relevant when addressing the problem of whether or not *abstracta* exist. This is because the problem cannot be addressed within our epistemic practices. Moreover, since nothing that is epistemically relevant follows from the existence of mind– and language– independent abstract objects, the matter of whether or not such objects exist is irrelevant to mathematical practice. In point of fact, my claim is even stronger: for *any* practice in space and time, the matter of whether mind– and language– independent *abstracta* exist or not is irrelevant.

I have so far applied the exclusion condition to abstract objects that are supposed to be mind– and language– independent, and I have claimed that nothing that is epistemically relevant follows from their existence. In addition, the exclusion condition separates local agnosticism from global agnosticism, because it implies that we should be agnostic *only* about mind– and language– independent *abstracta*. I now wish to clarify how agnosticism

[12]This is how Azzouni replies to Colyvan. See Azzouni (2012b, p. 964).

does not spill over fictional or physical objects. The point is important for those who endorse local agnosticism, that is, for those who do not want to be agnostic about any object whatsoever. What the agnostic needs to be is not to be agnostic about everything.[13]

Consider first the case of objects that have causal power. The exclusion condition does not apply to such objects, and so we can tell whether or not they exist: for a posit to exist, we either have thick access to it, or we are thinly connected with it. In other words, the way we discover the properties of objects that have causal power involves either physical interactions with such objects or with theoretically-related objects.

Consider the case of fictional characters that are supposed to be *abstracta*. There are two possible options: they are either mind– and language– dependent or independent. On the one hand, if fictional characters are considered mind– and language– dependent, the exclusion condition does not apply to them. This is because even if fictional objects were causally inert, they would need to be considered mind– and language– independent for the exclusion condition to apply to them. For example, I am not agnostic about the existence of Sherlock Holmes exists insofar as I consider Sherlock Holmes a mind– and language– dependent object. On the other hand, suppose that fictional objects were mind– and language– independent. In this case, the exclusion condition would hold, and thus we should be agnostic about the existence of such objects. I nevertheless do not think that there are any good reasons to conceive of fictional characters as mind– and language– independent objects. If someone recognizes that Sherlock Holmes is a fictional character, Sherlock Holmes should be considered a mind– and language– dependent object.

Now, consider the indispensabilist objection: even if the agnostic is right, one may claim that certain *abstracta*, i.e. mathematical *abstracta*, must ex-

[13] A caveat: consider the case of mind– and language– independent objects that are causally inert. Because of the exclusion condition, we can never know whether or not they exist or not. The exclusion condition is indeed applied to *any* mind– and language– independent *abstracta* whatsoever.

ist insofar as they are indispensable to our best scientific theories. I would answer in these terms: even if we ought to assume that certain existence of mathematical entities to formulate our best scientific theories, the indispensability argument does not tell us whether such entities are mind– and language– independent. But consider the following counter-replies: one could argue that (1) Platonism is our best view of mathematical objects, because we cannot break Turing machines, nor we can causally interact with numbers, and so on; (2) we also have good reasons to claim that mathematical objects are not *concreta*, because there are not as many concrete objects as we need in the universe to represent all mathematical objects; (3) if mathematical objects are mind– and language dependent, they simply do not exist. Regarding this, I wish to point out I do not need to show that Platonism is false, since I argue that there is no way to prove, or disprove the existence of mind– and language– independent *abstracta*. I just suggested that agnosticism may be compatible with the indispensability argument.

I would now like to suggest how the agnostic does not need to reject confirmation holism altogether. It seems there are at least two ways of rejecting confirmation holism: on the one hand, according to Sober and Maddy, the confirmation of empirical hypotheses cannot be used to show the truth of mathematical theories; on the other hand, according to Balaguer, the confirmation of empirical hypotheses does not show the existence of abstract objects, because the behavior of the physical world is not dependent on the existence of abstract objects. Agnostics can partially endorse confirmation holism: they can commit themselves to posits to which we have epistemic access, but they remain agnostic about the existence of mind– and language– independent *abstracta*. In other words, empirical confirmation cannot be used to prove, or disprove, the existence of mind– and language– independent abstract objects. In addition, agnosticism seems compatible with Field's argument against Platonism. If we are not able to account for the correlation between belief states and mind– and language– independent abstract objects, Field argues that Platonists cannot explain why expert mathematicians are reliable. But if such *abstracta* does not play any role in mathematical practice, there is no need to disarm Field's argument: the agnostic overcomes Field's argument.

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