Quantum Reynolds number for superfluid counterflow turbulence

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Abstract

In this paper we define a quantum analogous of the Reynolds number for the transition to superfluid turbulence. The kinematic viscosity in the classical Reynolds number is replaced by the vorticity quantum, which has the same physical dimensions. We also explore several possible definitions of such a quantum Reynolds number, according to the velocity used (the mutual velocity of the normal and superfluid component, related to the heat flux, or the total barycentric velocity, related to the mass flow). We discuss three critical values of the Reynolds number, characterizing the appearance of the first vortex line, of the transition from laminar to type I turbulence, and the transition from type I to type II turbulence. In particular, we explore the dependence of the critical Reynolds number on the system temperature.

Key words: Reynolds number; superfluid turbulence; quantized vortex line.
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Contents

1 Introduction

The characterization of the transition from laminar flows to turbulent flows is a central topic in the theory of turbulence [1, 2]. In classical fluids, this transition is often characterized in terms of the dimensionless Reynolds number [3, 4]. It would be interesting to have a similar characterization of the transition to superfluid turbulence in an analogous way, by defining a suitable dimensionless number. The aim of this short paper is to provide a very simple estimation of the order of magnitude of a quantum Reynolds number for this transition in superfluid counterflow. The interest of this analysis is its simplicity and intuitive character. More rigorous analyses should start from the Hall-Vinen-Bekarevich-Khalatnikov equations of motion for both components, but we take advantage of the peculiar features of quantum turbulence, namely, the quantized character of the vorticity, which must be equal to the vorticity quantum $\kappa = h/m = 9.97 \times 10^{-4} \approx 10^{-3}$ cm^2/s, with h Planck's constant and m the mass of helium atom.

The dimensionless Reynolds number plays an important role in describing the transition from classical laminar to turbulent flows. It is defined as

$$Re = \frac{\overline{V} D}{\nu}, \tag{1}$$

with D the typical size of the duct or of the object, $\overline{V}$ the average fluid velocity, and ν the kinematic viscosity, which has dimensions of $length^2\ time^{-1}$. It comes from Navier-Stokes equations and it reflects the interplay between viscous forces and inertial forces. More precisely a classical fluid is turbulent when the inertial forces are stronger than the viscous ones, and the laminar behavior is expected in the opposite situation [3, 4].

In superfluids there is not a typical viscosity; in fact, according to the Landau's theory, superfluid is composed by two unscindible components: the normal component and the superfluid component. The former is a quasi-classical fluid composed by excitations carrying the whole entropy of the superfluid, whereas the latter is a viscousless fluid which does not carry entropy nor heat. According to this theory we can define a Reynolds number for the normal component, but it deals with the turbulence of the normal component, and not with quantum turbulence.

Turbulence in a quantum fluid is instead related to the presence of quantum vortices, which are vortex lines reflecting the local vorticity of the superfluid component whose core size is fixed in about one Amgstrong and their circulation is quantized as multiple of κ (and this explains why they are usually called quantized vortices). Because of the thinness of the vortex line, the amount of the quantized vortices and their distribution all over the vessel depend strongly on the experiments; indeed vortex arrays are usually characterized by the average vortex line length per unit volume, L, the polarization vector $\mathbf{p} =< \mathbf{s}' >$ and the anisotropy tensor $\Pi =< U - \mathbf{s}'\mathbf{s}' >$ of the vortex lines in a small volume Λ, where $\mathbf{s}'$ is the unit vector tangent to the vortex line, U is identity matrix and $\mathbf{s}'\mathbf{s}'$ is the dyadic product [1, 2]. For instance, for a fast enough rotation of the vessel, vortex lines appear distributed parallel to the axis of rotation, while in counterflow experiments a randomly distributed tangle of vortices is created when the applied heat flux is higher than a critical value.

In this short paper we define a quantum Reynolds number which characterizes the appearance of these different regimes in counterflow experiments: the appearance of the first vortex, turbulent TI regime and turbulent TII regime. The value of this number is expected to be useful to express the frontiers between a flow without vortices and flows with vortices, and to summarize several results for quantum turbulence arising in counterflow situations and in coflow situations (barycentric velocity different from zero), as for instance Poiseuille flows. In spite of its physical appel, this number is not as widely used as it could be expected.

2 Definition of quantum Reynolds number

Since the viscosity of the superfluid component is zero, and quantum vortices are related to the quantum of vorticity, which has the same dimensions as kinematic viscosity, one may define a quantum Reynolds number analogous to (1) where $\overline{V}$ is the velocity characterizing the experiment: for instance the counterflow velocity $V_{ns} = V_n - V_s$ (V_n being the velocity of the normal component) in counterflow experiments, the rotation velocity or the superfluid velocity V_s in a rotating vessel (note that the presence of an array of vortices parallel to the rotation axis keeps the superfluid velocity, the normal velocity and the velocity of rotation locked), the barycentric velocity $\rho V = \rho_s V_s + \rho_n V_n$ in a Poiseuille flow (ρ, ρ_s and ρ_n being the density of the superfluid, the superfluid component and the normal component, respectively). In this view we could introduce several Reynolds numbers according to the velocity used such as

$$Re_q = \frac{V_{ns} D}{\kappa}, \qquad Re_b = \frac{VD}{\kappa}. \tag{2}$$

However, because the vortex formation is due to the presence of the counterflow, amongst the several possibilities mentioned in (2), we are led to prefer a quantum Reynolds number dependent to the counterflow velocity, as:

$$Re_q = \frac{V_{ns} D}{\kappa}. \tag{3}$$

Indeed, all Reynolds numbers introduced in (2) must be linked to the quantum Reynolds number defined in (3). For example, in counterflow situations one has $\rho_s V_s + \rho_n V_n = 0$, and therefore $V_s = -(\rho_n/\rho_s) V_n$. This leads to the relations

$$V_{ns} = V_n - V_s = (\rho/\rho_s) V_n = -(\rho/\rho_n) V_s, \tag{4}$$

so that in this case it is:

$$Re_q = \frac{\rho}{\rho_s} \frac{V_n D}{\kappa} = \frac{\rho}{\rho_n} \frac{V_s D}{\kappa}. \tag{5}$$

Though (3) is the possibility most directly related to the transition to turbulence, the relative velocity V_{ns} is not easy to measure; however, it may be linked to V_n or

V_s or, in counterflow situations, to the heat flux $q = \rho_s T s V_{ns}$, namely

$$Re_q = \frac{qD}{\kappa \rho_s T s}, \tag{6}$$

where T is the temperature and s is the entropy density.

3 The appearance of the first vortex

The first step is to characterize the appearance of the first vortex. The clue comes from the well-established arguments for the appearance of the first vortex in a rotating vessel [1]. Since the vorticity is quantized, vortices will not appear until there is the possibility of reaching the minimum vorticity required by the quantum κ. An estimation of the possible vorticity of superfluid velocity in a channel of diameter D is $2\pi(D/2)V_s \approx \pi V_s D$. Therefore, the minimum value of V_s required to set a vortex will be $\pi V_s D \approx \kappa$. In fact, a detailed analysis [1] indicates that instead of κ one must take $\kappa \ln(D/(2a_0))$, where a_0 is the radius of the core of the vortex line. Thus, we will have

$$\left(\frac{V_s D}{\kappa}\right)_{crit} \approx \frac{1}{\pi} \ln\left(\frac{D}{2a_0}\right). \tag{7}$$

Thus, the critical value of the quantum Reynolds number (3) for the formation of the first vortex in counterflow is expected to be of the order of

$$\left(\frac{V_{ns} D}{\kappa}\right)_{crit} \approx \frac{1}{\pi} \frac{\rho}{\rho_n} \ln\left(\frac{D}{2a_0}\right). \tag{8}$$

Expression (8) depends on the ratio (ρ/ρ_n), which is temperature-dependent. This formula shows a different critical value in terms of the temperature for the appearance of the first quantum vortex. Unfortunately, we don't know experiments which show this critical value, apart from an estimation made in a rotating vessel which carries to the definition of (7) and then to (8) in counterflow situation. Indeed, the fact that a vortex line may appear does not imply that it will be necessarily stable. Thus, this regime does not necessarily imply the existence of a stable vortex line, but rather the repeated and random appearance and destruction of a single vortex line.

Now we will give an estimation of the critical value (8) for the appearance of the first vortex in the Martin and Tough's counterflow experiments [5]. There $D \approx 10^{-3}$ m and $a_0 \approx 10^{-10}$ m; and hence $\ln(D/(2a_0)) \approx 15.4$. Thus, keeping in mind the values $\rho_n/\rho = 0.231$ at $T = 1.7$ K; $\rho_n/\rho = 0.1131$ at $T = 1.5$ K, our naive estimation of the critical value of the quantum Reynolds number for the formation of the first vortex will be

$$\left(\frac{V_{ns} D}{\kappa}\right)_{crit} = 43 \qquad \text{at} \qquad \mathrm{T} = 1.5\,\mathrm{K}, \tag{9}$$

$$\left(\frac{V_{ns}D}{\kappa}\right)_{crit} = 21 \qquad \text{at} \qquad \mathrm{T} = 1.7\,\mathrm{K}. \tag{10}$$

Both estimations (9) and (10) give hints to understand the temperature dependence of the critical quantum number. Indeed, since in $V_{ns}D/\kappa$ neither D nor κ depend on temperature, our estimation shows that this dependence goes as ρ/ρ_n. Since at lower temperature the relative proportion of ρ_n decreases, the ratio ρ/ρ_n increases accordingly.

In general, the Reynolds number (8) applied to the Martin and Tough's experiments becomes $Re_q = 4.85\rho/\rho_n$. A plot of this Reynolds number as function of the temperature T is therefore the same of ρ/ρ_n vs T.

Finally, it is convenient to recall that the critical Reynolds numbers in quantum turbulence cannot be compared to the classical Reynolds number in classical turbulence, because the nature of turbulence and the reasons for its appearance are strongly different, even if both kinds of turbulence show similarities.

4 Transition to quantum turbulence

4.1 Transition to the turbulence TI

The condition (8) states the threshold for the appearance of the first vortex. In pure counterflow the regime without vortices and the regime with a few of pinned vortex line, are both called *laminar regimes*; instead the *turbulent regime* is characterized by the formation of a disordered tangle of quantized vortices. Three different kinds of regimes are detected: the laminar regime, and two different turbulent regimes: TI turbulence, with a tangle with a low vortex line density, and TII turbulence, with a greater vortex line density in the tangle.

To distinguish these two kinds of turbulence, two critical values for the applied heat flux are experimentally detected: one for the transition from the laminar regime to the first kind of turbulence (TI regime) and one for the transition to the second kind of turbulence (TII regime) where a higher amount of vortices is revealed. The two critical values for the Reynolds numbers for the transition, first to the TI turbulence and then to the TII turbulence are the arguments of the current section.

In the counterflow experiments the amount of vortices L is related to the heat flux by $L \sim q^2$, where q is the modulus of the applied heat flux, that in the two-fluid model is proportional to the counterflow velocity V_{ns}, and the proportionality coefficient depends strongly on the turbulence regime (TI or TII).

Quantum Reynolds number (3) characterizes also the transition from the laminar regime to the first type of turbulence. By using the experimental data from Martin and Tough, at $T = 1.5$ K the TI regime is for $V_{ns} = 1.3$ cm/s and at $T = 1.7$ K for $V_{ns} = 0.95$ cm/s. Thus the quantum Reynolds numbers for the transitions from pinned vortices to the TI turbulent regime are

$$Re_q = \frac{V_{ns}D}{\kappa} = 130 \quad (1.5\ K); \qquad\qquad Re_q = \frac{V_{ns}D}{\kappa} = 95 \quad (1.7\ K). \tag{11}$$

Also the transition to the TI regime is characterized by the dependence on the temperature.

In order to understand the conceptual difference between (9) and (10) and (11), it is convenient to write the generalized Vinen equation describing L in terms of V_{ns} and the wall effects. This is [6]

$$\frac{dL}{dt} = \alpha_0 V_{ns} L^{3/2}\left[1 - \omega\frac{L^{-1/2}}{D}\right] + \alpha_1\frac{V_{ns}^2}{\kappa}L - \beta\kappa L^2, \tag{12}$$

where α_1 and α_0 are constants which depend on the regime of the helium II, ω is a parameter which takes into account of the presence of the walls and β is a parameter which models the destruction of vortices. The dimensionless coefficients appearing in (12) may depend on temperature.

It is easy to show that $L = 0$ is a stable solution for

$$\frac{V_{ns}D}{\kappa} < \frac{\alpha_0\omega}{\alpha_1} \tag{13}$$

whereas it becomes unstable otherwise. The difference between the Reynolds number (9)–(10) and (11) is thus related to the stability of vortex lines. Between (9)–(10) and (11), a single vortex appears and disappears randomly, whereas beyond (11), the vortex lines appear and live enough to yield a macroscopic accumulation of them, leading to a non-vanishing value of L.

Expression (13) shows that the the critical value of the Reynolds number will depend on the temperature, because the coefficients on the left hand side are functions of temperature. However, the dependence on temperature may be different from that obtained in (8).

In the previous arguments nothing has said about the remnant vortices, whose existence was experimentally proved in the past, even if the amount of these vortices is still a mystery. It is believed that these vortices are created when helium is cooled below the critical temperature for the appearance of the superfluidity. The amount of the remnant vortices perhaps depends on the details of the cooling, and hence in principle it might be possible to have a superfluid helium free from remnant vortices. We haven't discussed about this kind of vortices because their presence does not mean turbulence and because we are interested to the lowest critical Reynolds number for the appearance of the first vortex. If one wanted to take into account of these vortices, then one should modify equation (12), as in Ref. [7].

4.2 Transition from turbulence TI to TII

Quantum Reynolds number (3) characterizes also the transition from the first type of turbulence to the second one. By using the experimental data from Martin and

Tough, at $T = 1.5$ K the transition TI–TII is for $V_{ns} = 2.1$ cm/s, and at $T = 1.7$ K for $V_{ns} = 1.8$ cm/s. Thus the quantum Reynolds numbers for the TI–TII transition are

$$\frac{V_{ns}D}{\kappa} = 210, \qquad \frac{V_{ns}D}{\kappa} = 180. \tag{14}$$

There are two different interpretations to explain the transition from TI to TII. The one proposed by Jou and Mongiovì [8] means that vortex lines in the TI regime are polarized and the transition to TII regime is explained by the randomization of these vortices, and vortex tangle becomes isotropic. This process is achieved by means of the appearance of helical Kelvin waves along the vortices. When the amplitude of these helical waves becomes comparable to the average separation of the original vortex lines, the different vortex lines mutually cross each other and recombine, yielding free vortex loops which distribute in the volume of the fluid. The second view is by Barenghi and Melotte [9] , and it supports the two-fluid model: in the TI regime quantum turbulence is achieved but the normal component is instead laminar; the transition to the TII regime refers to the turbulent status of the normal component.

Thus, the idea following the first view would be that when the quantum Reynolds number Re_q is less than the critical values (11), the fluid is laminar, with very few vortices. When the quantum Reynolds number exceeds the critical value (11), the superfluid becomes turbulent but vortices are still polarized and at last up the second critical velocity (14) vortex tangle loses its polarization and it becomes uniformly distributed. The polarization $\mathbf{p} =< \mathbf{s}' >\simeq \nabla \times V_s/(\kappa L)$ is higher in the TI regime and depends on the location, while its value vanishes in the TII regime. In this view the quantum Reynolds number, by itself, is not sufficient to characterize the transition TI–TII, because this transition is due to the vanishing of the polarization.

According to the Barenghi's interpretation, when quantum Reynolds number exceeds the critical value (11), superfluid becomes turbulent but not yet the normal fluid. This one could become turbulent when the classical Reynolds number specialized on him

$$Re_{classic} = \frac{V_n D}{\nu} = \frac{\rho_s}{\rho}\frac{V_{ns}D}{\nu} = \frac{\rho_s}{\rho}\frac{\kappa}{\nu}Re_q, \tag{15}$$

with ν the normal component viscosity coefficient, exceeds the well-known critical values for turbulence. In the Martin and Tough's experiments (15) becomes $Re_{classic} = 9,66Re_q$ at $T = 1.5$ K and $Re_{classic} = 8,38Re_q$ at $T = 1.7$ K.

Equation (14) shows that, being ρ_s/ρ lower when the temperature is higher, the classical Reynolds number for the transition laminar–TI regime appears higher than the classical Reynolds number for the transition TI–TII.

5 Conclusions

In this short paper we dealt with the definition of a quantum Reynolds numbers in counterflow experiments in helium II. Since there the viscosity is very low and related to the normal component, and turbulence is characterized by the formation of quantized vortex lines, namely related to the rotation of the viscousless superfluid component, the classical definition of Reynolds number in terms of the viscosity is useless because the vortex formation is not connected to the kinematic viscosity of the normal component.

The definition of the quantum Reynolds number is made by using the quantum of vorticity κ instead of the viscosity, and the velocity $\overline{V}$ can be any velocity characterizing the experiments. In agreement with the rotating vessel, the appearance of the first vortex is established by the superfluid velocity V_s, which can be related to the counterflow velocity V_{ns} or the heat flux q. Then the transition to the turbulence TI and from the turbulence TI to TII is determined by using the well-known critical counterflow velocities experimentally detected in the Martin and Toungh's experiments.

Note that the critical values of the Reynolds number in (10) and (9), in (11) and in (14) are more or less of the same order of magnitude. They have in common the fact that the values at 1.5 K are higher than at 1.7 K; however, the ratio between the highest and lowest values is lower at 1.5K than at 1.7 K. This indicates that the relation between the three mentioned Reynolds numbers is not a purely numerical factor, but it also depends on temperature.

Furthermore, the explanation of the transition from the turbulence TI to the turbulence TII is still an open question in this paper, because according to the relation (15) and the values obtained in (14) the corresponding critical Reynolds numbers $Re_{classic}$ are of the same order than for the classical turbulence, since it depends on dynamical coefficients dependent on temperature, appearing in (12).

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References

[1] R.J. Donnelly, *Quantized vortices in helium II* (Cambridge University Press, Cambridge, UK, 1991).

[2] C.F. Barenghi, R.J. Donnelly, W.F.Vinen, *Quantized Vortex Dynamics and Superfluid Turbulence* (Springer, Berlin, 2001).

[3] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, UK, 1995).

[4] L.D. Landau, E.M. Lifshitz, *Fluid Mechanics* (Elsevier, Oxford, UK, 1987).

[5] K. P. Martin, J. T. Tough, Evolution of superfluid turbulence in thermal counterflow Phys. Rev. b **27** (1983) 2788.

[6] M. S. Mongiovì, D. Jou, Evolution equations in superfluid turbulence in: M.P. Das (Ed.) *Condensed Matter: New Research* (Nova Science Publishers, New York, 2006).

[7] L. Ardizzone, G. Gaeta, M. S. Mongiovì, Propagation of fourth sound in turbulent superfluids via extended thermodynamics, J. Non-Equilib. Thermodyn., **36** (2011) 179.

[8] M. S. Mongiovì, D. Jou, Generalization of Vinen's equation including transition to superfluid turbulence, J Phys. Condens. Matter **17** (2005) 4423.

[9] D. J. Melotte, C.F. Barenghi, Transition to Normal Fluid Turbulence in Helium II, Phys. Rev. Lett. **80** (1998) 4181.