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On the variety $\mathrm{var}\left(E^{*}_{(N-m,1^m)}\right)$. The $\mathbb{Z}$-grading of $M_2(E)$.

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Contents

Introduction

Let A be an algebra over a field F of characteristic zero. We say that a polynomial $f = (x_1, x_2, \cdots, x_n) \neq 0$ in the non-commuting variables $x_1, x_2, \cdots, x_n$ of the free associative algebra $F\langle X\rangle = F\langle x_1, x_2, \cdots\rangle$ over a field F is a *polynomial identity* for A if and only if $f(a_1, a_2, \cdots, a_n) = 0$, for any $a_1, a_2, \cdots, a_n \in A$, and we say that A is a PI-algebra. The set of all polynomial identities of A, $\mathrm{Id}(A)$, is a T-ideal of the free associative algebra $F\langle X\rangle$.

It is possible to see that any algebra A determines a T-ideal of $F\langle X\rangle$. On the other hand, many algebras correspond to the same T-ideal hence the class of all algebras A such that f is a polynomial identity of A for all f lying in a non-empty set $S \subset F\langle X\rangle$ is called the variety $\mathcal{V} = \mathcal{V}(\mathcal{S})$ determined by S. Moreover if $\mathcal{V}$ is a variety and A is an F-algebra such that $\mathrm{Id}(A) = \mathrm{Id}(\mathcal{V})$, then we say that $\mathcal{V}$ is the variety generated by A and we write $\mathcal{V} = \mathrm{var}\,(A)$. Also, Kemer in [26] proved that if the infinite dimensional Grassmann algebra E is not an algebra of the variety $\mathcal{V}$, then $\mathcal{V}$ is generated by a finite dimensional algebra A.

A famous theorem of Kemer says that if A is a PI-algebra over F, then its T-ideal is finitely generated. The explicit set of generators for the T-ideal is well know only for a small number of algebras like the 2×2 matrices over F, $M_2(F)$, due to Razmylov in [31] and Drensky in [17], the infinite dimensional Grassmann algebra E, due to Krakowsky and Regev in [27] and the upper triangular matrices of order n, due to Malcev in [29].

It is well know that if F is a field of characteristic zero, all the polynomial identities of an algebra A come from the multilinear ones. If we set P_n to be the set of polynomials that are linear in the variables $x_1, x_2, \cdots, x_n$, we can consider for any $n \in \mathbb{N}$, the factor space

$$P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}(A)}.$$

We call the n-th codimension of A, the dimension of $P_n(A)$ and we denote it with $c_n(A)$. In general $c_n(A)$ is bounded from above by $n!$, but Regev in [32] proved that if A is PI-algebra, its sequence of codimensions is exponentially bounded. Later Kemer proved in [25] that the sequence of codimensions of any PI-algebra is either polynomially or exponentially bounded. Moreover Giambruno and Zaicev proved in [18] and [19] that the limit

$$\lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists and is a non-negative integer called the PI-*exponent* of A, $\exp(A)$.

The permutation action of S_n on the space P_n of multilinear polynomials in the first n variables induces a structure of S_n-module on $P_n(A)$, and let

$\chi_n(A)$ be its cocharacter. By complete reducibility we can write

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the irreducible S_n-character associated to the partition λ of n and $m_\lambda \geq 0$ is the corresponding multiplicity.

In order to understand better the T-ideals, a useful tool consists in the study of some weaker polynomial identities for associative algebras. In fact let G be a group and A an algebra over F, a G-grading of A is a decomposition of A, as a vector space, into the direct sum of subspaces

$$A = \bigoplus_{g \in G} A_g$$

such that $A_g A_h \subseteq A_{gh}$. Let X be the disjoint union $\bigcup_{g \in G} X_g$, where X_g is a countable set of indeterminates, then we consider the free algebra $F\langle X \rangle^{(gr)}$ generated by X. We call the elements of $F\langle X \rangle^{(gr)}$ *graded polynomials.* A graded polynomial $f(x_1^{(g_1)}, x_2^{(g_2)}, \cdots, x_n^{(g_n)}) \in F\langle X \rangle^{(gr)}$ is a *G-graded polynomial identity* for the algebra A if and only if $f(a_1^{(g_1)}, a_2^{(g_2)}, \cdots, a_n^{(g_n)}) = 0$ for all $a_1^{(g_1)} \in A_{g_1}, a_2^{(g_2)} \in A_{g_2}, \cdots, a_n^{(g_n)} \in A_{g_n}$. Similarly to the ordinary case, we can define the T_G-ideal, the G-graded codimensions and cocharacter sequences, the G-PI-exponent of a G-graded PI-algebra A (see also [5], [12]).

The first chapter of the thesis is introductory. We introduce the algebras with polynomial identity by giving their basic definitions and properties. We only deal with associative algebras over a field F of characteristic zero. We give a brief introduction to the classical representation theory of the symmetric group and of the general linear group via the theory of Young diagrams which is our main tool in the study of the T-ideals of the free algebra. Then we introduce the sequences of codimensions, cocharacters and colengths. We also introduce the graded algebras and the graded polynomial identities and finally we focus our attention on the alternating polynomials such as the standard polynomials, the Capelli polynomials and the Amitsur Capelli-type polynomials.

In the second chapter of this thesis we study the asymptotic behaviour of the variety satisfying Amitsur Capelli-type polynomials associated to a hook-shaped Young diagram.

Let λ be a partition of n, $\lambda \vdash n$, and let χ_λ be the corresponding irreducible S_n-cocharacter, Amitsur and Regev in [1] defined the Amitsur Capelli-type polynomials as follows:

$$e^*_\lambda [x; y] = \sum_{\sigma \in S_n} \chi_\lambda (\sigma) \, x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \ldots x_{\sigma(n)} y_n.$$

The importance of studying the Amistur Capelli-type polynomials is that, as the Capelli polynomials characterize the algebras having the cocharacter lying in a strip-shaped Young diagram [33], they characterize all the algebras having the cocharacter contained in a hook ([1]).

More precisely, given any integers $d, l \geq 0$ we denote by $H(d,l) = \cup_{n\geq 1}\{\lambda = (\lambda_1, \lambda_2, \dots) \vdash n \mid \lambda_{d+1} \leq l\}$ the infinite hook of arm d and leg l. We say that a partition λ lies in $H(d,l)$ if the corresponding Young diagram D_λ is contained in the $H(d,l)$ hook. If $l = 0$, $H(d,0)$ is the set of all partitions with diagram contained in the strip of height d. Regev proved in [33] that, if A is a PI-algebras, then A satisfies the Capelli identities $c_d \equiv 0$ if and only if $\chi_n(A) = \sum_{\lambda\in H(d-1,0)} m_\lambda \chi_\lambda$ with λ partition of n. Successively Amitsur and Regev proved that if A is a PI-algebra, then A satisfies the Capelli-type polynomials $e^*_{M,L} \equiv 0$ if and only if $\chi_n(A) = \sum_{\lambda\in H(M,L)} m_\lambda \chi_\lambda$, whit λ partition of n and $e^*_{M,L}$ are the Capelli-type polynomials associated to the partition $\lambda = (L+1)^{M+1}$.

We denote by E^*_λ the set of the polynomials obtained from e^*_λ by evaluating the variables y_i to 1 in all the possible ways, by Γ_λ the T-ideal generated by E^*_λ and also we say that $\mathcal{V}_\lambda = \mathrm{var}(E^*_\lambda) = \mathrm{var}(\Gamma_\lambda)$, $c_n(E^*_\lambda) = c_n(\Gamma_\lambda)$ and $E_\lambda = \exp(E^*_\lambda) = \exp(\Gamma_\lambda)$.

Giambruno-Zaicev and Benanti-Sviridova proved in [21] and [4] respectively that $c_n\left(E_{k^2,0}\right) \simeq c_n(M_k(F))$, $c_n\left(E_{k^2,k^2}\right) \simeq c_n(M_k(E))$ and $c_n\left(E_{k^2+l^2,2kl}\right) \simeq c_n(M_{k,l}(E))$ where E is the Grassmann algebra.

We study the variety E^*_λ where $\lambda = (N-m, 1^m)$, with $m = k^2$ or $m-1 = k^2$ or $m-1 = k_1^2 + k_2^2$ and we prove that the variety E^*_λ is asymptotically equal to the $k \times k$ matrices, $M_k(F)$, over a field F of characteristic zero, $M_{r\times 2r}(F) \oplus M_{2r\times r}(F)$, and $\bigoplus_{s^2+t^2=m-1} UT(s,t)$ respectively.

In the third chapter we study the algebra of 2×2 matrices over the infinite dimensional Grassmann algebra $A = M_2(E)$ with a $\mathbb{Z}$-grading. In particular we determine a subset of generators for the $\mathbb{Z}$-graded identities of A and finally we compute the nth cocharacter of the homogeneous components of degree 0,1 and -1 and the $(0,r,n-r)$-graded cocharacter $\chi_{(0,r,n-r)}$ of the $\mathbb{Z}$-graded algebra A, i.e. the $S_r \times S_{n-r}$-character of the quotient space of multilinear graded identities $P_{0,r,n-r}/(P_{0,r,n-r} \cap Id^{\mathbb{Z}}(A))$ via the representation theory of the general linear group $GL \times GL \times GL$ (see also [2], [3], [10], [11], [13]).

Chapter 1

Preliminaries

In this chapter we introduce the main objects and the main results of the theory of the algebras with polynomial identities. We also introduce the notion of variety of algebras which is one of the most important in PI-theory.

We also give the basic definition of the representation theory of finite groups over an algebraically closed field of characteristic zero and we describe the classical representation theory of the symmetric group S_n and of the general linear group trough the theory Young tableaux and we also introduce a relation between S_n-characters and GL_n-characters.

We introduce the sequences, ordinary and graded version, of codimensions, cocharacters and colengths.

Finally we introduce the alternating polynomials, Capelli polynomials and Amitsur Capelly-type polynomials and we present some results about the algebras satisfying these particular alternating polynomials.

1.1 Polynomial identities, PI-algebras, T-ideals and varieties of algebras

Throughout this thesis we shell denote by F a field of characteristic zero and by A an associative algebra over F.

We start introducing the definition of a free algbera. Let F be a field and X a set. The free associative algebra on X over F is the algebra $F\langle X\rangle$ of polynomials in the non-commuting indeteminates $x \in X$. The free algebra $F\langle X\rangle$ is defined by the following universal property: given an associative F-algebra A, any map $\varphi : X \to A$ can be uniquely expressed to a homomorphism of algebras $\bar{\varphi} : F\langle X\rangle :\to A$. the cardinality of the set X is called the *rank* of $F\langle X\rangle$.

Definition 1.1.1 Let A be an associative F-algebra and $f = f(x_1, x_2, \cdots, x_n) \in F\langle X\rangle$. We say that $f \equiv 0$ is a *polynomial identity* for A if

$$f(a_1, a_2, \cdots, a_n) = 0 \qquad \text{for all } a_1, a_2, \cdots, a_n \in A.$$

We also say that A satisfies $f \equiv 0$ or, sometimes, that f itself is an identity of A.

Since the trivial polynomial identity $f = 0$ is an identity for any algebra A, we make the following:

Definition 1.1.2 If the associative algebra A satisfies a non-trivial polynomial identity $f \equiv 0$, we call A a *PI-algebra.*

We denote by $\mathrm{Id}(A) = \{f \in F\langle X\rangle | f \equiv 0 \quad \text{on } A\}$ the set of the polynomial identities for A. It is easy to check that $\mathrm{Id}(A)$ is a two-sided ideal of $F\langle X\rangle$. Moreover, $\mathrm{Id}(A)$ is stable under all endomorphisms of $F\langle X\rangle$ i.e. $\varphi\left(\mathrm{Id}(A)\right) \subseteq \mathrm{Id}(A)$, for all the endomorphisms of $F\langle X\rangle$.

Definition 1.1.3 An ideal I of $F\langle X\rangle$ is called a T-ideal if $\varphi(I) \subseteq I$ for all endomorphisms φ of $F\langle X\rangle$.

It follows that $\mathrm{Id}(A)$ is a T-ideals of $F\langle X\rangle$. Actaully it can be easily shown that all T-ideal of $F\langle X\rangle$ are of this type i.e., ideals of polynomial identities for a suitable algbera A. In fact, if I is a T-ideal of $F\langle X\rangle$, then the ideal of polynomial identities of the factor algebra $F\langle X\rangle/I$ is just I.

Let S be a set of polynomials identites of $F\langle X\rangle$. the T-ideal generated by the set S, denotes by $\langle S\rangle_T$, is the smallest T-ideal containing S.

We also say that f follows from S or that f is a consequence of S if $f \in \langle S\rangle_T$.

Two set of polynomials are equivalent if they generate the same T-ideal.

We now give some examples of PI-algebras.

Examples 1.1.4

1. If A is a commutative algebra then $[x_1, x_2] := x_1x_2 - x_2x_1 \equiv 0$ is a polynomial identities of A.

2. If A ia nilpotent algebra of index $n \geq 1$, then A is a PI-algebra. In fact, it satisfies the polynomial identity $x_1x_2 \ldots x_n \equiv 0$ since $A^n = 0$ and $A^{n-1} \neq 0$.

3. If A is a finite dimensional algebra and $\dim A = n$ then A satisfies the polynomial

$$\mathrm{St}_{n+1}(x_1, x_2, \cdots, x_{n+1}) = \sum_{\sigma \in S_{n+1}} (-1)^\sigma x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n+1)}$$

called the *standard identity* of degree $n+1$, where $(-1)^\sigma = \mathrm{sgn}\,\sigma$.

4. Let V a countable dimensional vector space over the field F of characteristic non 2 and let $\varepsilon = \{e_1, e_2, \cdots\}$ be an ordered basis for V. The *Grassmann algebra* E of V over F is the algebra generated by the set $\{e_1, e_2, \cdots\}$ satisfying the relation

$$e_i e_j = -e_j e_i \quad i, j = 1, 2, \cdots .$$

As a vector space over F, E is generated by the elements $e_{i_1} e_{i_2} \ldots e_{i_n}$ where $i_1 < i_2 < \cdots < i_n$ and $n > 0$. By [27] $\mathrm{Id}(E) = \langle [[x_1, x_2], x_3] \rangle_T$.

5. Let $A = M_2(F)$ be the 2×2 matrix algebra over F, then A satisfies the standard polynomial of degree 4, $\mathrm{St}_4(x_1, x_2, x_3, x_4)$ an the Hall identity $[[x_1, x_2]^2, x_3]$.

Since several different algebras might have the same set of polynomial identities it is natural to introduce the notion of variety of algebras.

Definition 1.1.5 Let $S \subseteq F\langle X \rangle$ be a non-empty set of polynomials of $F\langle X \rangle$. The class of all algebras A for which S is a set of polynomial identities, i.e. $f \equiv 0$ on A for all $f \in S$, is called the *variety* $\mathcal{V} = \mathrm{var}(S)$, determined (or generated) by S.

We remark that :

- if $S = \{[x_1, x_2]\}$, then $\mathrm{var}(S)$ is the variety of all commutative algebras;
- if $S = \{f = 0\}$, then $\mathrm{var}(S)$ id the class of all associative algebras.

Notice that any variety $\mathcal{V}$ is closed under taking homomorphic images, subalgebras and direct products. Actually, a theorem of Birkhoff shows that these properties characterize the varieties.

T-ideals and varieties are related by the followig

Theorem 1.1.6 *There is a one-to-one correspondence between T-ideals of $F\langle X \rangle$ and varieties of algberas $\mathcal{V}$. In this correspondence a variety $\mathcal{V}$ corresponds to the T-ideal of identities* $\mathrm{Id}(\mathcal{V})$ *and a T-ideal I corresponds to the variety of algebras determined by I.*

A variety $\mathcal{U}$ is called a *subvariety* of $\mathcal{V}$ if $\mathcal{U} \subseteq \mathcal{V}$. It is clear that $\mathcal{U} \subseteq \mathcal{V}$ if and only if $\mathrm{Id}(\mathcal{V}) \subseteq \mathrm{Id}(\mathcal{U})$.

1.2 Multihomogeneous and multilinear polynomials

In this section we introduce the multihomogeneous and the multilinear polynomials that play an important role in the study of the identities of a given algebra when F is a infinite field of characteristic zero. In fact, in this case, the study of the polynomial identities of an algebra A is equivalent to the study of the corresponding multihomogeneous or multilinear polynomials.

Definition 1.2.1 A polynomial $f = f(x_1, x_2, \cdots, x_n) \in F\langle X\rangle$ is called *homogeneous* of degree k, for some $k \geq 1$, if it is a linear combination of monomials of degree k. we also say that a polynomial f is *homogeneus in the variable* x_i of degree k_i, if x_i appears whit the same degree k_i in every monomial of f.

Moreover a polynomial f i called *multihomogeneous* of multidegree $(k_1, k_2, \cdots, k_n)$ if f is homogeneous in each variables $x_1, x_2, \cdots, x_n$ of degree $k_1, k_2, \cdots, k_n$, respecively.

If $f(x_1, x_2, \cdots, x_n) \in F\langle X\rangle$ is a polynomial, then f can be always decomposed into a sum of multihomogeneous polynomials. In fact, it can be written as:

$$f = \sum_{k_1 \geq 0, k_2 \geq 0, \cdots, k_n \geq 0} f^{k_1, k_2, \cdots, k_n}$$

where $f^{k_1, k_2, \cdots, k_n}$ is the sum of all the monomials in f where $x_1, x_2, \cdots, x_n$ appear at degree $k_1, k_2, \cdots, k_n$, respectively. The polynomials $f^{k_1, k_2, \cdots, k_n}$ which are non-zero are called the multihomogeneous components of f.

An interesting, feature of T-ideals is that if F is an infinite field, they are multihomogeneous i.e., they are generated by multihomogeneous polynomials. In fact, we have the following([15] ,Proposition 4.2.3):

Theorem 1.2.2 *Let F be an infinite field. If $f \equiv 0$ is a polynomial identities for the algebra A, then every multihomogeneous component of f is still a polynomial identity for A.*

A prominent role in characteristic zero is played by the multihomogeneous polynomials of multidegree $(1, 1, \cdots, 1)$.

Definition 1.2.3 A polynomial f is called *linear* in the variable x_i if x_i occurs with degree 1 in every monomial of f (equivalently, if f is homogeneous in the variable x_i of degree $k_i = 1$).

A polynomial is called *multilinear* if f is linear in each of its variables (equivalently, if f is multihomogeneous of multidegree $(1, 1, \cdots, 1)$).

It is obvious that any multilinear polynomial is always of the form

$$f(x_1, x_2, \cdots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)},$$

where $\alpha_\sigma \in F$ and S_n is the symmetric group on $\{1, 2, \cdots, n\}$.

Observe that if $f(x_1, x_2, \cdots, x_n)$ is a polynomial multihomogeneous in the variable x_i of degree k_i, then

$$f(x_1, x_2, \cdots, \alpha x_i, \cdots, x_n) = \alpha^{k_i} f(x_1, x_2 \cdots, x_i, \cdots, x_n),$$

for every $\alpha \in F$. In particular, if f is linear in x_i, then

$$f\left(x_1, x_2, \cdots, \sum_{i=1}^{m} \alpha_i y_i, \cdots, x_n\right) = \sum_{i=1}^{m} \alpha_i f(x_1, x_2, \cdots, y_i, \cdots, x_n),$$

for every $\alpha_i \in F$, $y_i \in F\langle X\rangle$.

This property is useful to prove the following remark.

Remark 1.2.4 Let A be an F-algebra. If a multilinear polynomial f vanishes on a basis of A, then f is a polynomial identity of A.

We shall show now how to reduce an arbitrary polynomial identity to a multilinear one. By using the so-called process of multilinearization we shall prove the following theorem.

Theorem 1.2.5 *If* $\operatorname{char} F = 0$, *every non-zero polynomial* $f \in F\langle X\rangle$ *is equivalent to a finite set of multilinear polynomials.*

Proof. By Theorem 1.2.2 f is equivalent to the set of its multihomogeneous components (i.e. they generate the same T-ideal). Hence we may assume that $f = f(x_1, x_2, \cdots, x_n)$ is multihomogeneous of multidegree $(k_1, k_2, \cdots, k_n)$ with, for instance, $k_1 = \deg_{x_1} f > 1$

Compute the polynomial:

$$\begin{aligned} h(y_1, y_2, x_2, \cdots, x_n) =& f(y_1 + y_2, x_2, \cdots, x_n) - f(y_1, x_2, \cdots, x_n) \\ & - f(y_2, x_2, \cdots, x_n). \end{aligned}$$

Notice that h is a non-zero polynomial. Write now h in the following form:

$$h(y_1, y_2, x_2, \cdots, x_n) = \sum_{i=1}^{k_1-1} h_i(y_1, y_2, x_2, \cdots, x_n)$$

where h_i is the homogeneous component of degree i in y_i. Then, still by Theorem 1.2.2, all the polynomials $h_i = h_i(y_1, y_2, x_2, \cdots, x_n)$, for $i = 1, 2, \cdots, k_1 - 1$, are consequence of h and so, consequences of f. Since $\deg_{y_1} h_i = i < k_1 = \deg_{x_1} f$, for $i = 1, 2, \cdots, k_1 - 1$, by an induction argument we obtain a set of multiliner polynomials which are consequence of f.

We show now that these multilinear polynomials are equivalent to f. Observe first that, for every i, $h_i(y_1, y_1, x_2, \cdots, x_n) = \binom{k_1}{i} f(y_1, x_2, \cdots, x_n)$. Since $\operatorname{char} F = 0$ and $\binom{k_1}{i} \neq 0$, thus f is a consequence of any h_i, for $i = 1, 2, \cdots, k_1 - 1$. By still applying induction the proof is complete □

We observe that the hypothesis of characteristic zero can be changed into $\operatorname{char} F > \deg f$.

In the language of T-ideals the previous theorem takes the following form.

Definition 1.2.6 for every $n \geq 1$ we denote by

$$P_n = \operatorname{span}\left\{x_{\sigma(1)} \cdots x_{\sigma(n)} | \sigma \in S_n\right\}$$

the vector space of multilinear polynomials in $x_1, x_2, \cdots, x_n$.

Corollary 1.2.7 *If $\operatorname{char} F = 0$, every T-ideal is generated, as a T-ideal, by the multilinear polynomials it contains. This means that if I is a T-ideal, then is uniquelly determined by its multilinear parts $I \cap P_n$, for $n = 1, 2, \cdots$.*

1.3 Representations of groups

Throughout this section we denote by G a group, by V or W a vector space over F and $GL(V) \simeq GL_n(F)$, $n = \dim_F V$ the general linear group, i.e., the group of all invertible endomorphism of V.

Definition 1.3.1 A *representation* of a group G on a vector space V is a homomorphism of groups $\rho : G \to GL(V)$.

The *degree* (or *dimension*) of the representation ρ is the dimension of the vector space V. The representation is *faithful* if the kernel of ρ is trivial; ρ is trivial if its kernel coincides with G.

Remark 1.3.2 There is a one-to-one correspondence between the representation of a group G on a finite dimensional vector space and finite dimensional FG-modules (or G-modules)

Proof. Let $\rho : G \to GL(V)$ a representation of G. Then V is a (left) G-module via the action defined by

$$gv := \rho(g)(v), \quad \text{for all } g \in G, v \in V.$$

Let M be a G-module which is finite dimensional as a vector space over F. Then $\rho = G \to GL(M)$ such that :

$$\rho(g)(m) := gm \quad \text{for all } g \in G, m \in M,$$

defines a representation of G on M. □

Since modules and representations are equivalent notions, we shall use the language of G-modules.

Definition 1.3.3 If $\rho : G \to GL(V)$ and $\rho' : G \to W$ are two representations of a group G, we say that ρ and ρ' are *equivalent* if V and W are isomorphic as G-modules.

Definition 1.3.4 A representation $\rho : G \to GL(V)$ is *irreducible* if V is an irreducible G-module.

ρ is *completely reducible* if V is the direct sum of its irreducible submodules.

A prominent role in the representation theory of finite groups is played by Maschke's theorem

Theorem 1.3.5 (Maschke's) *Let G be a finite group and let F be a field of characteristic zero or $p > 0$ and $p \nmid |G|$. Then every G-module V is completely reducible, i.e. V is direct sum of a finite number of irreducible G-modules equals to the number of simple components in the Wedderburn decomposition of the group algebra FG.*

An important tool for studying the representation of a finite group is the theory of characters.

Definition 1.3.6 Let $\rho : G \to GL(V)$ be a representation of G. The function $\chi_\rho : G \to F$ defined by $\chi_\rho(g) = \operatorname{tr} \rho(g)$ is called the *character* of the representation ρ and $\dim V = \deg \chi_\rho$ is called the *degree* of the character χ_ρ.

We say that the character is *irreducible* (or *completely reducible*) if ρ is an irreducible representation (completely reducible, respectively). We notice that $\chi_\rho(1) = \deg \chi_\rho$.

The following theorem shows that the knowledge of the character gives a lot of information for the representation and the number of the irreducible representations (G- modules) is determined by a purely group property of the group.

Theorem 1.3.7 *Let G be a finite group and let F be an algebraically closed field.*

i) Every finite dimensional representation of G is determined, up to isomorphism, by its character.

ii) the number of the non isomorphic irreducible representations (G-modules) is equal to the number of conjugacy classes of G.

1.4 Representations of the symmetric group

In this section we deal with the representation theory of the symmetric group S_n.

Definition 1.4.1 Let $n \geq 1$ be an integer. A *partition* λ of n, and we write $\lambda \vdash n$ or $|\lambda| = n$, is a finite sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ of integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ and $\sum_{i=1}^{r} \lambda_i = n$.

We shall also use the following notation:

$$(\lambda_1^{k_1}, \lambda_2^{k_2}, \cdots, \lambda_t^{k_t}) = (\underbrace{\lambda_1, \cdots, \lambda_1}_{k_1}, \underbrace{\lambda_2, \cdots, \lambda_2}_{k_2}, \cdots, \underbrace{\lambda_t, \cdots, \lambda_t}_{k_t}).$$

Definition 1.4.2 Let $n < m$ and let $\lambda = (\lambda_1, \lambda_2, \cdots) \vdash n$ and $\mu = (\mu_1, \mu_2, \cdots) \vdash m$, we will say that $\mu \geq \lambda$ if $\mu_i \geq \lambda_i$ for all $i = 1, 2, \cdots$.

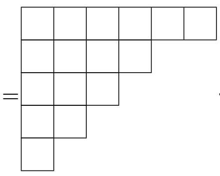
It is well known that there exists a one-to-one correspondence between partitions of n and conjugacy classes of S_n. Hence by Theorem 1.3.7, *all the irreducible non-equivalent S_n-module are indexed by the partitions of n.* Therefore let us denote by χ_λ the S_n-character corresponding to $\lambda \vdash n$.

It is always possible to associate $\lambda \vdash n$ a diagram.

Definition 1.4.3 Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \vdash n$. The *Young diagram* associated to λ, denoted by D_λ, is the finite subset of $\mathbb{Z} \times \mathbb{Z}$ defined as:

$$D_\lambda = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} |\ i = 1, 2, \cdots r,\ j = 1, 2, \cdots, \lambda_i\}.$$

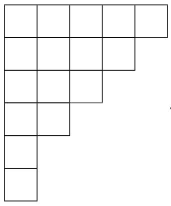
Graphically, a Young diagram is denoted as an array of boxes with the convention that the first coordinate i (the row index) increases from top to bottom and the second coordinate j (the column index) increases from left to right. For example, for $\lambda = (6, 4, 3, 2, 1)$, the corresponding Young diagram is :

$D_{(6,4,3,2,1)}=$  .

Definition 1.4.4 Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \vdash n$. the *conjugate partition* of λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_r)$ such that $\lambda'_1, \lambda'_2, \cdots, \lambda'_r$ are the lengths of the columns of D_λ

We shall always write $h_i(\lambda)$ instead of λ'_i and we shall denote with $h(\lambda)$ the height of D_λ (i.e. the length of the first column).

For instance, if $\lambda = (6,4,3,2,1)$ as above, then its conjugate partition is $\lambda' = (5,4,3,2,1,1)$ and the corresponding Young diagram is:

$D_{(5,4,3,2,1,1)} =$  .

Definition 1.4.5 Let $\lambda \vdash n$ and let D_λ be the corresponding Young diagram. A *Young tableau* T_λ of shape λ is one of the $n!$ arrays obtained by filling the boxes of D_λ with the integer $1, 2, \cdots, n$. We shall also say that T_λ is a λ-tableau and denote the integer (a_{ij}) in the (i,j) box, for $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, \lambda_i$.

Among these Young tableaux a prominent role is played by the so-called standard tableaux.

Definition 1.4.6 A tableau T_λ of shape λ is *standard* if the integers in each row and in each column of T_λ increase from top to bottom and from left to right.

For instance, the following is a standard tableau

$$T_{(4,3,2)} = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 6 & 9 \\ \hline 2 & 5 & 7 \\ \cline{1-3} 3 & 8 \\ \cline{1-2} \end{array} .$$

It is possible to calculate the number d_λ of standard λ-tableaux. First we need to define the hook number.

Definition 1.4.7 For any box $(i,j) \in D_\lambda$, we define the *hook number* of (i,j) as:

$$h_{ij} = \lambda_i + \lambda'_j - i - j + 1$$

where λ' is the conjugate partition of λ.

Note that h_{ij} counts the number of boxes in the hook with edge in (i,j), i.e. the boxes to the right and below (i,j).

In the following we have written inside each boxes its hook number

$$\begin{array}{|c|c|c|c|} \hline 6 & 5 & 3 & 1 \\ \hline 4 & 3 & 1 \\ \cline{1-3} 2 & 1 \\ \cline{1-2} \end{array} .$$

Next we give the *Hook Formula* (see [36]) :

Proposition 1.4.8 *Let $\lambda \vdash n$. The number d_λ of standard λ-tableaux is:*

$$d_\lambda = \frac{n!}{\prod_{i,j} h_{i,j}}$$

where the product runs over all boxes of D_λ.

1.5 The left ideals $FS_n e_{T_\lambda}$ and the two-sided ideals I_λ

Given $\lambda \vdash n$, a tableau $T_\lambda = D_\lambda(a_{ij})$ of shape λ, the symmetric group S_n acts on T_λ as follows: if $\sigma \in S_n$, we define $\sigma T_\lambda := D_{\sigma(a_{ij})}$.

For example, if $\sigma = (134)(25)$ would change the following tableau T_λ:

$$T_\lambda = \begin{array}{|c|c|c|}\hline 1 & 5 & 4 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$$

into:

$$\sigma T_\lambda = \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 \\ \cline{1-2}\end{array} \quad .$$

Definition 1.5.1 The *row-stabilizer* of T_λ is the subgroup R_{T_λ} of S_n which stabilizes the row of T_λ i.e,

$$R_{T_\lambda} = S_{\lambda_1}(a_{11}, a_{12}, \cdots, a_{1\lambda_1}) \times \cdots \times S_{\lambda_r}(a_{r1}, a_{r2}, \cdots, a_{r\lambda_1})$$

where $S_{\lambda_i}(a_{i1}, a_{i2}, \cdots, a_{i\lambda_i})$ denotes the symmetric group acting on the integers $a_{i1}, a_{i2}, \cdots, a_{i\lambda_i}$.

Similarly, the *column-stabilizer* of T_λ is the subgroup C_{T_λ} of S_n which stabilizes the columns of T_λ i.e.,

$$C_{T_\lambda} = S_{\lambda'_1}(a_{11}, a_{12}, \cdots, a_{1\lambda'_1}) \times \cdots \times S_{\lambda'_s}(a_{s1}, a_{s2}, \cdots, a_{s\lambda'_s}).$$

Definition 1.5.2 Given a tableau T_λ, we define

$$e_{T_\lambda} = \sum_{\substack{\sigma \in R_{T_\lambda} \\ \tau \in C_{T_\lambda}}} (-1)^\tau \sigma\tau$$

where $(-1)^\tau$ is the sign of the permutation τ.

Now the left ideals $FS_n e_{T_\lambda}$ are minimal and for different partitions are non-isomorphic (see [23]).

Theorem 1.5.3 *Let λ, μ be partitions of n with $\lambda \neq \mu$.*

1. *Let $T_{1,\lambda}, T_{2,\lambda}$ be two tableaux of the same shape λ. Then, as left FS_n-modules,*

$$FS_n e_{T_{1,\lambda}} \cong FS_n e_{T_{2,\lambda}}.$$

2. *Let T_λ and T_μ be tableaux of shape λ and μ respectively. Then, as left FS_n-modules,*
$$FS_n e_{T_\lambda} \not\cong FS_n e_{T_\mu}.$$

Moreover the following theorem (see [15]) describes the irreducible representation of the symmetric group.

Theorem 1.5.4 *For any $\lambda \vdash n$, let T_λ be a fixed tableau. Then:*

- *$\{FS_n e_{T_\lambda} \mid \lambda \vdash n\}$ is a complete set of irreducible non-equivalent representations of S_n.*
- *The degree of the irreducible representation corresponding to the partition λ is equal to the number d_λ of standard tableaux.*

Definition 1.5.5 Let λ be a partition of n and define $I_\lambda = \sum_{T_\lambda} FS_n e_{T_\lambda}$, where the sum runs all over all $n!$ tableaux T_λ of shape λ.

If we identify FS_n with P_n, we will say that a polynomial $f(x_1, x_2, \cdots, x_n)$ is a λ-polynomial if $f(x_1, x_2, \cdots, x_n) \in I_\lambda$. If, also, $f(x_1, x_2, \cdots, x_n)$ is a polynomial identity and $w_1, w_2, \cdots, w_{m+1} \in I_\lambda$ we will say that $f(x_1, x_2, \cdots, x_n)$ is λ-identity.

Theorem 1.5.6 *Let λ be a partition of n, Then:*

1. *The above defined I_λ is a two-sided ideal in FS_n.*
2. *The decomposition of FS_n into a direct sum of simple algebras is given by*
$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda$$

The standard tableaux come into play if one wants to find, among the $n!$ essential idempotent arising from tableaux of shape λ, some orthogonal ones. In fact, we have

Proposition 1.5.7 *If $T_1, \cdots, T_{d_\lambda}$ are all the standard tableaux of shape λ, then I_λ, the minimal two-sided ideal of FS_n corresponding to λ, has the decomposition*
$$I_\lambda = \bigoplus_{i_1}^{d_\lambda} FS_n e_{T_{\lambda_i}}$$

1.6 Representations of the general linear group

In this section we give some results on the representation theory of the general linear group by restriciting our attention to the case when $GL_m = GL_m(F)$ acts on the free associative algebra of rank m. We refer to ([15], Chapter 12) for the results of this section.

Definition 1.6.1 The *representation* of the general linear group GL_m:

$$\phi : GL_m \to GL_s$$

is called polynomial if the entries $\phi(g)_{pq}$ are polynomials of the entries a_{ij} of $g$for $g \in GL_m(F)$, $i, j = 1, 2, \cdots, m$ and $p, q = 1, 2, \cdots, s$.

When all the entries of $\phi(a_{ij})$ are homogeneous polynomials of degree k, then ϕ is a *homogeneous representation* of degree k.

Let $F_m\langle X\rangle = F\langle x_1, x_2, \cdots, x_m\rangle$ denote the free associative algebra in m variables and let $U = \text{span}_F \{x_1, x_2, \cdots, x_m\}$.

The action of the group $GL_m \cong GL(U)$ on $F_m\langle X\rangle$ can be obtained extending diagonally the natural left action of GL_m on the space U by defining:

$$g(x_{i_1}, x_{i_2}, \cdots, x_{i_k}) = g(x_{i_1})g(x_{i_2}) \cdots g(x_{i_k}),\ g \in GL_m,\ x_{i_1}, x_{i_2}, \cdots, x_{i_m} \in F_m\langle X\rangle.$$

Actually, $F_m\langle X\rangle$ is a polynomial GL_m-module (i.e. the corresponding representation is polynomial).

Let F_m^n be the space of homogeneous polynomials of degree n in the variables $x_1, x_2, \cdots, x_m$, then F_m^n is a (homogeneous polynomials) submodule of $F_m\langle X\rangle$. We observe that:

$$F_m^n = \bigoplus_{i_1+i_2+\cdots+i_m=n} F_m^{(i_1,i_2,\cdots,i_m)}$$

where $F_m^{(i_1,i_2,\cdots,i_m)}$ is the homogeneous subspace spanned by all monomials of degree i_1 in x_1, i_2 in x_2 and so on.

The following theorem states a result similar to Maschke's Theorem about the complete reducibility of GL_m-modules, valid for the polynomial representation of GL_m.

Theorem 1.6.2 ([15], Theorem 12.4.3) *Every polynomial GL_m-module is a direct sum of irreducible homogeneous polynomial subspaces.*

The irreducible homogeneous polynomial GL_m-modules are described by partition of n in not more than m parts and Young diagrams.

Theorem 1.6.3 ([15], Theorem 12.4.4) *Let $P_m(n)$ denote the set of all partitions of n with at most m parts (i.e. whose diagrams have height at most m).*

1. *the pairwise isomorphic irreducible homogeneous polynomial GL_m-modules of degree $n \geq 1$ are in one-to-one correspondence with the patitions $\lambda \in P_m(n)$.*

   We denote by W^λ an irreducible GL_m-module related to λ.

2. *Let $\lambda \in P_m(n)$. Then the GL_m-module W^λ is isomorphic to a submodule of F_m^n. Moreover, the GL_m-module F_m^n has a decomposition:*

$$F_m^n \cong \sum_{\lambda \in P_m(n)} d_\lambda W^\lambda,$$

   where d_λ is the dimension of the irreducible S_n-module corresponding to the partitio λ.

3. *As a subspace of F_m^n, the vector space W^λ is multihomogeneous, i.e.*

$$W^\lambda = \bigoplus_{i_1+i_2+\cdots+i_m=n} W^{\lambda,(i_1,i_2,\cdots,i_m)}$$

   where $W^{\lambda,(i_1,i_2,\cdots,i_m)} = W^\lambda \cap F_m^{i_1,i_2,\cdots,i_m}$.

We want to show now that if $W^\lambda \subseteq F_m^n$, then W^λ is cyclic and generated by a polynomial multihomogeneous of multidegree $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ with $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \in P_m(n)$.

We observe first that the symmetric group S_n acts from the right on F_m^n by permuting the places in which the variables occurs. i.e $\forall x_{i_1}, x_{i_2}, \cdots x_{i_n} \in F_m^n$ and $\forall \sigma \in S_n$

$$x_{i_1} x_{i_2} \ldots x_{i_n} \sigma = x_{i_{\sigma(1)}} x_{i_{\sigma(2)}} \ldots x_{i_{\sigma(n)}}.$$

Let now $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \in P_m(n)$. We denote by s_λ the following polynomial of F_m^n:

$$s_\lambda = s_\lambda(x_1, x_2, \cdots, x_k) = \prod_{i=1}^{\lambda_1} \mathrm{St}_{h_i(\lambda)}(x_1, x_2, \cdots, x_{h_i(\lambda)}),$$

where $h_i(\lambda)$ is the height of the ith column of the diagram of λ and $\mathrm{St}_r(x_1, x_2, \cdots, x_r)$ is the standard polynomial of degree r.

Note that by definition s_λ is multihomogeneous of multidegree $(\lambda_1, \lambda_2, \cdots, \lambda_k)$.

Theorem 1.6.4 ([15], Theorem 12.4.12) *Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$.*

1. *The element $s_\lambda(x_1, x_2, \cdots, x_k)$, defined above, generates an irreducible GL_m-submodule W of F_m^n isomorphic to W^λ.*

2. *Every submodule* $W^\lambda \subseteq F_m^n$ *is generated by a non-zero polynomial, called the* highest weight vector *of* W^λ*, of the type:*

$$f_\lambda = s_\lambda \sum_{\sigma \in S_n} \alpha_\sigma \sigma, \ \alpha_\sigma \in F. \tag{1.1}$$

the highest weight vectors f_λ *is unique up to a multiplicative constant and it is contained in the one-dimensional vector space* $W^{(\lambda_1, \lambda_2, \cdots, \lambda_k)}$.

3. *Let* $\sum_{\sigma \in S_n} \alpha_\sigma \sigma \in FS_n$*. If* $s_\lambda \sum_{\sigma \in S_n} \alpha_\sigma \sigma \neq 0$*, then it generates an irreducible submodule* $W \cong W^\lambda$*,* $W \subseteq F_m^n$*.*

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \in P_m(n)$ and let T_λ be a Young tableau. We denote by f_{T_λ} the highest weight vector obtained from (1.1) by considering the only permutation $\sigma \in S_n$ such that the first column of T_λ is filled in from top to bottom with integers $\sigma(1), \sigma(2), \cdots, \sigma(h_1(\lambda))$, in this order, the second column is filled in with $\sigma(h_1(\lambda)+1), \sigma(h_1(\lambda)+2), \cdots, \sigma(h_1(\lambda)+h_2(\lambda))$, etc.

Proposition 1.6.5 ([15], Proposition 12.4.14) *Let* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \in P_m(n)$ *and let* $W^\lambda \subseteq F_m^n$*. The highest weight vector* f_λ *of* W^λ *can be expressed uniquely as a linear combination of the polynomials* f_{T_λ} *with* T_λ *standard tableau.*

1.7 Relation between S_n-characters and GL-characters

In this section we shall see that the representation theory of the general linear group is related to that of the symmetric group. Let A be an associative algebra.

Definition 1.7.1 For every $n \geq 1$

$$P_n = \mathrm{span}_F \left\{ x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)} | \ \sigma \in S_n \right\}$$

is the vector space of multilinear polynomials in the free algebra $F\langle X \rangle$.

The left action of the symmetric group S_n on P_n is

$$\sigma f(x_1, x_2 \cdots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)}).$$

Since the subspace $P_n \cap \mathrm{Id}(A)$ is invariant under this action,

$$P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

has the S_n-module structure.

Definition 1.7.2 For $n \geq 1$, the S_n-character of $P_n(A)$, denoted by $\chi_n(A)$, is called the *nth cocharacter* of A and the sequence $\{\chi_n(A)\}_{n\geq 1}$ is the *cocharacter sequence* of A.

if we decompose the nth cocharacter into irreducibles, then we obtain

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{1.2}$$

where χ_λ is the irreducible S_n-character associated to the partition $\lambda \vdash n$ and $m_\lambda \geq 0$ is the corresponding multiplicity.

In the previous section we have observed that $F_m\langle X\rangle$ is a GL_m-module. Under the same action, the space $F_m\langle X\rangle \cap \mathrm{Id}(A)$ is invariant, hence

$$F_m(A) = \frac{F_m\langle X\rangle}{F_m\langle X\rangle \cap \mathrm{Id}(A)}$$

inherits a structure of left GL_m-module. We denote by $F_m^n(A)$ the space

$$F_m^n(A) = \frac{F_m^n}{F_m^n \cap \mathrm{Id}(A)}.$$

Clearly $F_m^n(A)$ is a GL_m-submodule of $F_m(A)$ and we denote its character by $\psi_n(A)$. Write

$$\psi_n(A) = \sum_{\substack{\lambda \vdash n \\ \lambda \in P_m(n)}} \bar{m}_\lambda \psi_\lambda$$

where ϕ_λ is the irreducible GL_m-character associated to the partition λ and $\bar{m}_\lambda$ is the corresponding multiplicity. It was proved in [6] and [17] that if the nth cocharacter of A has the decomposition given in (1.2) then $m_\lambda = \bar{m}_\lambda$, for all $\lambda \vdash n$ whose corresponding diagram has height at most m.

We also have the following.

Remark 1.7.3 If

$$\psi_n(A) = \sum_{\substack{\lambda \vdash n \\ \lambda \in P_m(n)}} \bar{m}_\lambda \psi_\lambda$$

is the GL_m-character of $F_m^n(A)$, then $\bar{m}_\lambda \neq 0$ if and only if there exists a tableau T_λ such that the corresponding highest weight vector f_{T_λ} is not a polynomial identity for A. Moreover $\bar{m}_\lambda$ is equal to the maximal number of linearly independent highest weight vectors f_{T_λ} in $F_m^n(A)$.

1.8 Codimensions, colenghts and PI-exponent

In this section we introduce the sequences of codimensions, the sequence of colenghts and the PI-exponent of a PI-algebra A. We recall that, if A is an algebra over a field F of characteristic zero the T-ideal of its identities is determined by its multilinear parts $P_n \cap \mathrm{Id}(A)$, $n \geq 1$. Clearly, the bigger are their dimension, the bigger is the T-ideal. So, these dimension, give us, in some sense, a kind of measure of the T-ideal.

Definition 1.8.1 The non-negative integer

$$c_n(A) = \dim_f \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

is the so called *nth codimension* of A. The sequence $\{c_n(A)\}_{n\geq 1}$ is the *sequence of codimensions* of A.

In general the computation of such sequence for a given algebra is very difficult. However, there exist some algebras whose sequence of codimensions is well known.

Examples 1.8.2

1. If A is not a PI-algebra then $c_n(A) = n!$, for all $n \geq 1$.
2. If A is a nilpotent algebra of index $k \geq 1$, then $c_n(A) = 0$ for all $n \geq k$.
3. If A is commutative, but not nilpotent, then $c_n(A) = 1$, for all $n \geq 1$.
4. Let UT_2 be the algebra of 2×2 upper triangular matrices over F. Then $c_n(UT_2) = 2^{n-1}(n-2) + 2$ (see [28]).
5. Let E be the Grassmann algebra. Then $c_n(E) = 2^{n-1}$, for all $n \geq 1$. (see [27])

Definition 1.8.3 Let A, B be two algebras, or two varieties. The sequence of codimensions are asymptotic equals if

$$\lim_{n\to\infty} \frac{c_n(A)}{c_n(B)} = 1,$$

and we write $c_n(A) \simeq c_n(B)$.

In general, the sequence of codimensions is bounded from above $n!$. In case A is a PI-algebra, i.e. it satisfies a non trivial polynomial identity, the following theorem, proved by Regev in [32], show that its sequence of codimensions is exponentially bounded, i.e. exist a constants t such that $c_n(A) \leq t^n$, for any $n \geq 1$.

Theorem 1.8.4 ([22], Theorem 4.2.4) *Let A be a PI-algebra satisfying a polynomial identity of degree $d \geq 1$. Then the sequence of codimensions satisfies*

$$c_n(A) \leq (d-1)^{2n}, \ n \geq 1.$$

We now define the exponent of a PI-algebra.

Definition 1.8.5 Let A be a PI-algebra. Then the exponent (or PI-exponent) of A is

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}.$$

In case $\mathcal{V} = \mathrm{var}(A)$ is a variety of algebras, we write $\exp(\mathcal{V}) = \exp(A)$ and we call $\exp(A)$ the exponent of the variety $\mathcal{V}$.

Recently Giambruno and Zaicev, in [18] and in [19] proved the following.

Theorem 1.8.6 *Let A be a PI-algebra over any field F of characteristic zero. Then the* $\exp(A)$ *exists and is an integer.*

We now give the exponent for some algebras:

Examples 1.8.7

1. If $A = M_k(F)$ is the algebra of $k \times k$ matrices over a field F, then $\exp(A) = k^2$ (see [34]).
2. If E is the Grassmann algebra and $A = M_k(E)$ is the algebra of $k \times k$ matrices over E, then $\exp(A) = 2k^2$ (see [22], Corollary 6.6.3.).
3. Let $(d_1, d_2, \cdots, d_m)$ a set of positive integer. If we denote by $UT(d_1, d_2, \cdots, d_m)$ the subalgebra of the matrix algebra $M_{d_1+d_2+\cdots+d_m}(F)$ consisting of all matrices of the type

$$\begin{pmatrix} A_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_m \end{pmatrix},$$

where $A_i = M_{d_i}(F)$ for $i = 1, 2, \cdots, m$, then $\exp(UT(d_1, d_2, \cdots, d_m)) = d_1^2 + d_2^2 + \cdots + d_m^2$ (see [22], Corollary 6.6.2.)

1.9 Graded algebras

Definition 1.9.1 Let F be a field and A an associative algebra over F. Let also G be a group. A *G-grading* on A is a decomposition of A, as a vector space, into the direct sum of subspace

$$A = \bigoplus_{g\in G} A_g$$

such that

$$A_g A_h \subseteq A_{gh}.$$

The grading is called *finite* if the set $\{g \in G|\ A_g \neq 0\}$ is finite. Any element $x \in A_g$ is called *homogeneous* of degree g, $\deg x = g$. A subspace $V \subseteq A$ is said to be *graded* or homogeneous if $V = \bigoplus_{g \in G} V \cap A_g$. If e is the identity element of G then A_e is called the *identity* or *neutral component.* The grading is called *trivial* if $A_g = 0$ for any $g \neq e$.

The *support* of a graded algebra is defined as

$$\mathrm{Supp}_G A = \mathrm{Supp}\, A = \{g \in G|\ A_g \neq 0\}.$$

Definition 1.9.2 A map $\varphi : A = \bigoplus_{g \in G} A_g \to B = \bigoplus_{g \in G}$ is called a *homomorphism (isomorphism) of graded algebras* if φ is an ordinary homomorphism (isomorphism) and $\varphi(A_g) \subseteq B_g$, $g \in G$.

It is easy to observe that $\ker \varphi$ is a graded ideal of A.

Definition 1.9.3 An algebra A is called $\mathbb{Z}_2$-*graded* or *superalgebra* with grading $(A^{(0)}, A^{(1)})$ if A has the vector space decomposition $A = A^{(0)} \oplus A^{(1)}$ such that

$$A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)} \text{ and } A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}.$$

Examples 1.9.4

1. Any algebra A can be viewed as a $\mathbb{Z}_2$-graded with *trivial grading*, i.e. $A = A^{(0)} \bigoplus A^{(1)}$ with $A^{(0)} = A$ and $A^{(1)} = 0$.
2. Let E be the Grassmann algebra of countable rank over F. Recall that E is generated by the set $\{e_1, e_2, \cdots\}$ satisfying the relation $e_ie_j = -e_je_i$, $i, j = 1, 2, \cdots$. Let $E^{(0)}$ be the subspace of E generated by the monomials in the e_i's of even length and $E^{(1)}$ the subspace generated by the monomials in the e_i's of odd length. Then $E = E^{(0)} \oplus E^{(1)}$ is a $\mathbb{Z}_2$ graded algebra.

Given a superalbegra A one can obtain a new superalgebra with the help of the Grassmann algebra E.

Definition 1.9.5 Let $A = A^{(0)} \oplus A^{(1)}$ be a superalgebra. The algebra

$$G(A) = (A^{(0)} \otimes E^{(0)}) \oplus (A^{(1)} \otimes E^{(1)})$$

is called the *Grassmann envelope of A*

1.10 Identities of graded algebras

In this section we discuss the finite basis property for graded identities and relations between graded and ordinary identities.

Let G be a group and let $F\langle X\rangle$ be the free associative algebra over F on a countable set $X = \bigcup_{g\in G} X_g$ where $X_g = \left\{x_1^{(g)}, x_2^{(g)}, \cdots\right\}$ are disjoint sets. The indeterminates from X_g are said to be homogeneous degree g. The homogeneous degree of a monomial $x_{i_1}^{(g_1)} x_{i_2}^{(g_2)} \cdots x_{i_t}^{(g_t)} \in F\langle X\rangle$ is defined to be $g_1 g_2 \ldots g_t$, as opposed its total degree, which is defined to be t. Denote by $F\langle X\rangle^{(g)}$ the subspace of the algebra $F\langle X\rangle$ genrated by all the monomials having homogeneous degree g. Since $F\langle X\rangle^{(g)} F\langle X\rangle^{(h)} \subseteq F\langle X\rangle^{(gh)}$, for every $g, h \in G$, it follows that $F\langle X\rangle$ can be naturally endowed with a G-grading if one set:

$$F\langle X\rangle = \bigoplus_{g\in G} F\langle X\rangle^{(g)}.$$

We denote by $F\langle X\rangle^{(gr)}$ the algebra $F\langle X\rangle$ with this grading.

Definition 1.10.1 $F\langle X\rangle^{(gr)}$ is called the *free G-graded algebra of countable rank over F.*

It is easy to prove that the following universal property holds: for any G-graded algebra $A = \bigoplus_{g\in G} A_g$ and for any set-theoretical map $\psi : X \to A$ such that $\psi(X_g) \subseteq A_g$, there exists a homomorphism of G-graded algebra $\overline{\psi} : F\langle X\rangle \to A$ such that $\overline{\psi}_{|X} = \psi$. Let $\overline{\Psi}$ be the set of all such homomorphism, then $\mathrm{Id}^{gr}(A) = \bigcap_{\overline{\psi}\in\overline{\Psi}} \ker \overline{\psi}$ is called the ideal of G-graded polynomial identities of A. This means that a graded polynomial $f(x_1^{(g_1)}, x_2^{(g_2)}, \cdots, x_n^{(g_n)}) \in F\langle X\rangle^{(gr)}$ is a *graded identity* for the algebra A, and we write $f \equiv 0$ in A, if $f(a_1^{(g_1)}, a_2^{(g_2)}, \cdots, a_n^{(g_n)}) = 0$ for all $a_1^{(g_1)} \in A^{(g_1)}, a_2^{(g_2)} \in A^{(g_2)}, \cdots, a_n^{(g_n)} \in A^{(g_n)}$.

Definition 1.10.2 $\mathrm{Id}^{gr}(A) = \left\{f \in F\langle X\rangle^{(gr)} \mid f \equiv 0 \text{ on } A\right\}$ is the *ideal of the graded identities of A.*

Clearly $\mathrm{Id}^{gr}(A)$ is stable under all graded endomorphisms of $F\langle X\rangle$, i.e. $\mathrm{Id}^{gr}(A)$ is a T^{gr}-ideal.

It can be easily proved that any non-trivial graded identity has non-trivial graded multilinear consequence and, in particular, if $\mathrm{char}\, F = 0$ the $\mathrm{Id}^{gr}(A)$ is uniquely determined by all the multilinear polynomials it contains.

There is an obvious way of relating ordinary identities and graded identities of the algebra A. Recalling that the indeterminates from $X^{(g)}$ are denoted $x_i^{(g)}$, then any multilinear graded polynomial can be written as:

$$f = f(x_1^{(g_1)}, x_2^{(g_2)}, \cdots, x_n^{(g_n)}) = \sum_{\sigma\in S_n} \alpha_\sigma x_{\sigma(1)}^{(g_{\sigma(1)})} x_{\sigma(2)}^{(g_{\sigma(2)})} \ldots x_{\sigma(n)}^{(g_{\sigma(n)})}.$$

For a fixed n-tuple $(g_1, g_2, \cdots, g_n) \in G^n$, all the multilinear polynomials in the variable $x_1^{(g_1)}, x_2^{(g_2)}, \cdots, x_n^{(g_n)}$ form the subspace $n!$-dimensional:

$$P_n^{g_1,g_2,\cdots,g_n} = \text{span}\left\{ x_{\sigma(1)}^{(g_{\sigma(1)})} x_{\sigma(2)}^{(g_{\sigma(2)})} \ldots x_{\sigma(n)}^{(g_{\sigma(n)})} \mid \sigma \in S_n \right\}.$$

The intersection

$$P_n^{g_1,g_1,\cdots,g_n} \cap \text{Id}^{gr}(A)$$

consists of all multilinear graded identities of A in the variables $x_1^{(g_1)}, x_1^{(g_2)}, \cdots, x_n^{(g_n)}$. Define

$$\bar{x}_i = \sum_{g \in G} x_i^g$$

for every $i = 1, 2, \cdots$. Then it is clear that the set $\{\bar{x}_1, \bar{x}_2, \cdots\}$ generates the free associative algebra of countable rank. Moreover, given a polynomial $f(x_1, x_2, \cdots, x_n) \in F\langle X\rangle$, f is an ordinary polynomial identity of A if and only if

$$f(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \text{Id}^{gr}(A).$$

This basic observation implies the following technical result.
For $(g_1, g_2, \cdots, g_n) \in G^n$,

$$c_n^{(g_1,g_2,\cdots,g_n)} = \dim \frac{P_n^{(g_1,g_2,\cdots,g_n)}}{P_n^{(g_1,g_2,\cdots,g_n)} \cap \text{Id}^{gr}(A)}$$

is called the *homogeneous nth codimension* associated to $(g_1, g_2, \cdots, g_n)$. Obviously, existence of a graded identity on a graded algebra is much weaker condition than ordinary polynomial identity. For example, if B is an arbitrary algebra with the trivial G-grading, that is $B_g = 0$ for all non-unitary $g \in G$, then B satisfies any graded identities of the type $x \equiv 0$ with $x \in X^g$, $g \neq e$.

1.11 Capelli polynomials and Amitsur's Capelli-type polynomials

In this section we introduce the notion of alternating polynomials and we focus our attention in particular on the *Capelli polynomials* and on the *Amitsur's Capelli-type polynomials.*

Definition 1.11.1 Let $f(x_1, x_2, \cdots, x_i, \cdots, x_j, \cdots, x_n, y_1, y_2, \cdots, y_t)$ be a polynomial linear in each variables x_i's. We say that f is alternating in the variables $x_1, x_2, \cdots, x_n$ if for any $1 \leq i < j \leq n$, the polynomial become zero when we substitute x_i instead of x_j.

Remark 1.11.2 If $f(x_1, x_2, \cdots, x_i, \cdots x_j, \cdots x_n, y_1, y_2, \cdots, y_t)$ is an alternating polynomial in $x_1, x_2, \cdots, x_n$ then for any $1 \le i < j \le n$ we have

$$f(x_1, x_2, \cdots, x_i, \cdots x_j, \cdots, x_n, y_1, y_2, \cdots, y_t) =$$

$$-f(x_1, x_2, \cdots, x_j, \cdots x_i, \cdots x_n, y_1, y_2, \cdots, y_t),$$

Moreover, if σ is a permutation of S_n, and by writing σ as a product of transposition, it follows that

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)}, y_1, y_2, \cdots, y_t) = (-1)^\sigma f(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_t).$$

Definition 1.11.3 The polynomial

$$Cap_m = Cap_m(x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_{m+1}) =$$

$$\sum_{\sigma \in S_n} (-1)^\sigma y_1 x_{\sigma(1)} y_2 x_{\sigma(2)} \cdots y_m x_{\sigma(m)} y_{m+1}$$

is called the Capelli polynomial of rank m or the mth Capelli polynomial.

The Capelli polynomials are multilinear and alternating in $x_1, x_2, \cdots, x_m$. It plays a central role among alternating polynomials since every polynomial wich is alternating in $x_1, x_2, \cdots, x_m$ can be written as a linear combination of Capelli polynomials obtained by specializing the y_i's, in fact we have the follow proposition

Proposition 1.11.4 (Poposition 1.5.4. [22]) *If $f \in F\langle X\rangle$ is a polynomial alternating in $x_1, x_2, \cdots, x_m$, then*

$$\sum_{w_1, w_2, \cdots, w_{m+1}} \alpha_{w_1, w_2, \cdots, w_{m+1}} Cap_m(x_1, x_2, \cdots x_m; w_1, w_2, \cdots, w_{m+1})$$

is a linear combination of Capelli polynomials where $w_1, w_2, \cdots, w_{m+1}$ are suitable (eventually trivial) monomials in $F\langle X\rangle$.

We observe that, if we specialize all the variables y_i's to 1 the Capelli polynomial is equal to standard polynomial of degree m, hence

$$Cap_m(x_1, x_2, \cdots, x_m; 1, 1, \cdots, 1) = \mathrm{St}(x_1, x_2, \cdots, x_m).$$

We now give the definition of the Capelli-type polynomials.

Definition 1.11.5 Let λ be a partition of n, $\lambda \vdash n$, and let χ_λ the corresponding irreducible S_n-cocharacter, the Amitsur's Capelli type polynomials associated to a partition λ is

$$e^*_\lambda = e^*_\lambda[\bar{x}; \bar{y}] = \sum_{\sigma \in S_n} \chi_\lambda(\sigma) x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \ldots x_{\sigma(n)} y_n.$$

The Amitsur's Capelli polynomials are strictly linked with the λ-polynomial, in fact we have the following

Theorem 1.11.6 ([1], Theorem B) *Let A be an algebra. A satisfies the identity $e^*_\lambda[\bar{x};\bar{y}] \equiv 0$ if and only if A satisfies all μ-identities for all $\mu \geq \lambda$ on any degree.*

The Capelli polynomial and the Amitsur's Capelli type polynomial are very important to the study of the cocharacters of the PI-algebra in fact Regev in [33], first, Amitsur and Regev in [1], later, proved the followings:

Theorem 1.11.7 *Let A be a PI-algebra and let λ be a partition of n, then A satisfies the Capelli polynomial of rank d, i.e. $c_d(x_1, x_2, ..., x_d; y_1, y_2, \cdots, y_{d+1}) \equiv 0$ if and only if $\chi_n(A) = \sum_{\lambda \in H(d-1,0)} m_\lambda \chi_\lambda$, where*

$$H(d-1,0) = \bigcup_{n\geq 1} \{\lambda = (\lambda_1, \lambda_2, \cdots)|\ \lambda_d = 0\}.$$

Theorem 1.11.8 *Let A be a PI-algebra and let λ be a partition of n,then A satisfies the Amitsur's Capelli type polynomials associated to the partition $\mu = (L+1)^{M+1}$, i.e. $e^*_{M,L}(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n) \equiv 0$, if and only if $\chi_n(A) = \sum_{\lambda \in H(M,L)} m_\lambda \chi_\lambda$, where*

$$H(M,L) = \bigcup_{n\geq 1} \{\lambda = (\lambda_1, \lambda_2, \cdots)|\ \lambda_{M+1} \leq L\}.$$

Chapter 2

On the variety $\mathrm{var}(E^*_{(N-m,1^m)})$.

In this chapter we study the sequence of codimensions of the variety $\mathcal{V}_\lambda$ of algebras satisfying $E^*_\lambda[\bar{x};\bar{y}]$, with $\lambda \vdash n$ a partition of n such that its associated Young tableau is a hook with the height of the arm and the width of the leg equal to one.

In the first section we give some important results about the Amitsur's Capelli type polynomials and λ-polynomials defined in the previous chapter, we prove that if the $\lambda = (n-m, 1^m)$, such that $m = k^2$, then the sequence of codimensions of the variety $\mathrm{var}(E^*_\lambda[\bar{x};\bar{y}])$ and of the algebra $M_k(F)$ of the $k \times k$ matrices over a field F of characteristic zero are asymptotically equals.

We prove a similar result when $m-1$ is a square or sum of two squares.

2.1 Amitsur's Capelli type polynomials

Let P_n be the set of all multilinear polynomials and let S_n the symmetric group, we can define a right and a left action of S_n on P_n as follows

Definition 2.1.1 Let $f(x_1, x_2, \cdots, x_n)$ be a multilinear polynomial in $x_1, x_2, \cdots, x_n$ and let τ be a permutation of S_n, then

$$\tau \cdot f(x_1, x_2, \cdots, x_n) = f(x_{\tau(1)}, x_{\tau(2)}, \cdots, x_{\tau(n)}). \tag{2.1}$$

Definition 2.1.2 Let $f(x_1, x_2, \cdots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$ be a multilinear polynomial in $x_1, x_2, \cdots, x_n$ and let ρ be a permutation of S_n, then

$$f(x_1, x_2, \cdots, x_n) \cdot \rho = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma\rho(1)} x_{\sigma\rho(2)} \ldots x_{\sigma\rho(n)}. \tag{2.2}$$

The interpretation of this action on the polynomials is that the places in each monomial $x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}$ are changed according to the permutation ρ^{-1} and independent of σ. For example, the first element $x_{\sigma(1)}$ will be placed in the $\rho^{-1}(1)$ place of the new monomial, the second term in the $\rho^{-1}(2)$ place etc.

There is also a natural embedding of $S_n \subset S_m$ for $m \geq n$, by letting S_n act on the first n variables and leaving the other $m - n$ variables invariant. The corresponding effect on the polynomials of P_n is multiplying $f \in P_n$ by $x_{n+1}x_{n+2}\cdots x_m$, i.e. $f(x_1, x_2, \cdots, x_n)x_{n+1}x_{n+2}\cdots x_m \in P_m$. We now obtain the following lemma

Lemma 2.1.3 (Lemma 1, [1]) *Let*

$$f(x_1, x_2, \cdots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)} \in P_n,$$

$p_0 = x_{i_1} x_{i_2} \dots x_{i_k}$, $p_1 = x_{j_1} x_{j_2} \dots x_{j_h}$, $\cdots$, $p_n = x_{t_1} x_{t_2} \cdots x_{t_s}$ *be* $n + 1$ *multilinear monomials in* $x_{n+1}, x_{n+2}, \cdots, x_m$ *of total degree* $m - n$, *and we let some* $p_j = 1$. *There exists a permutation* $\rho \in S_m$ *such that*

$$(f(x_1, x_2, \cdots, x_n) x_{n+1} x_{n+2} \dots x_m) \rho = \sum_{\sigma \in S_n} \alpha_\sigma p_0 x_{\sigma(1)} p_1 x_{\sigma(2)} \dots p_{n-1} x_{\sigma(n)} p_n.$$

As defined in the previous character, if λ be a partition of n and let χ_λ be the corresponding irreducible character, the polynomial e^*_λ is the Amitsur's Capelli type polynomial associated to the partition λ, we set with $E^*_\lambda = E^*_\lambda[\bar{x}; \bar{y}]$ the set of the polynomials obtained from e^*_λ evaluating to 1 the variables $y's$ in all the possible way.

Examples 2.1.4

if $\lambda = (m)$, the corresponding character is $\chi_{(m)} = 1$ (see for instance [24]), hence $e^*_{(m)} = \sum_{\sigma \in S_n} x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \dots y_{n-1} x_{\sigma(n)}$

if $\lambda = (1^m)$, the corresponding character is $\chi_{(1^m)} = (-1)^\sigma$ (see for instance [24]), hence $e^*_{(1^m)} = Cap_m = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \dots y_{n-1} x_{\sigma(n)}$.

Throughout this chapter we set by $\mathcal{V}_\lambda = \mathrm{var}(E^*_\lambda)$ the variety of algebras satisfying the set of polynomials E^*_λ, by Γ_λ the T-ideal generated by E^*_λ, by $c_n(E^*_\lambda)$ and by E_λ the codimension and the exponent, respectively.

For the sequence of codimensions of particular permutation λ we have the following result.

Theorem 2.1.5 (Theorem 3, [21]) *Let* $m = k^2$. *Then* $\mathrm{var}(Cap_{m+1}) = \mathrm{var}(M_k(F) \oplus B)$ *for some finite dimensional algebra* B *such that* $\exp(B) < k^2$. *In particular* $c_n(C_{k^2+1}) \simeq c_n(M_k(F))$.

Theorem 2.1.6 (Theorem 5, [4]) *Let $k, l \in \mathbb{N}$ and let E the Grassmann algebra. Then* $\mathrm{var}(E^*_{(2kl+1)^{k^2+l^2+1}}) = \mathrm{var}(M_{k,l}(E) \oplus G(D))$*, where D is a finite-dimensional superalgebra such that* $\exp(D) < (k+l)^2$*, and*

$$M_{k,l}(E) = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

$0 < l \leq k$, where $P = M_k(E^{(0)})$, $Q = M_{k\times l}(E^{(1)})$, $R = M_{l\times k}(E^{(1)})$ and $S = M_l(E^{(0)}$. In particular,

$$c_n\left(E^*_{(2kl+1)^{k^2+l^2+1}}\right) \simeq c_n(M_{k,l}(G)).$$

Theorem 2.1.7 (Theorem 10, [4]) *Let $s \in \mathbb{N}$, $s > 0$ and let E the Grassmann algebra. Then* $\mathrm{var}(E^*_{(s^2+1)^{s^2+1}}) = \mathrm{var}(M_s(E) \oplus G(D))$*, where D is a finite-dimensional superalgebra such that $exp(D) < 2s^2$. In particular,*

$$c_n\left(E^*_{(s^2+1)^{s^2+1}}\right) \simeq c_n(M_s(E)).$$

We now focus our attention on the variety satisfying the Amitsur's Capelli type polynomial in the case $\lambda = (n-m, 1^m)$, i.e. the Young tableau associated to λ is an hook which arm and leg have height and width equal to one.

2.2 The case $m = k^2$

Let $\lambda = (n-m, 1^m)$, in this section we study the asymptotic behaviour of the variety of algebras satisfying the polynomials $E^*_{(n-m,1^m)}$ where $m = k^2$, with $n, k \in \mathbb{N}$.

We start introducing an important tool.

Definition 2.2.1 Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_r + J$ be a finite dimensional superalgebra where $A_1, A_2, \cdots, A_r$ are simple superalgebras and $J = J(A)$ is the Jacobson radical of A. We say that A is reduced if $A_1 J A_2 J \ldots J A_r \neq 0$.

This reduced algebras are very important because they can be used as building blocks of any proper variety, in fact, if we recall that we may regard an algebra as a superalgebra with trivial grading, we have the following

Theorem 2.2.2 (Corollary 1, [21]) *Let A be finite dimensional algebra. Then there exists a finite number of reduced algebras $B_1, B_2, \ldots, B_t$ and a finite-dimensional algebra D such that*

$$\mathrm{var}\,(A) = \mathrm{var}\,(B_1 \oplus B_2, \oplus \cdots \oplus B_t \oplus D)$$

where $\exp(A) = \exp(B_1) = \exp(B_2) = \cdots = \exp(B_t)$, $\exp(D) < \exp(A)$, *and*

$$c_n(\mathrm{var}(A))) \simeq c_n(B_1 \oplus \cdots \oplus B_t)$$

We start the study of the variety E^*_λ by the following

Lemma 2.2.3 *Let* $\lambda = (n-m, 1^m) \vdash n$ *be a partition of* n *and let* E *be the infinite dimensional Grassmann algebra, then* $E \notin \mathcal{V}_\lambda$.

Proof. We first build a polynomial which is consequence of e^*_λ. We consider the following Young tableaux T_λ associated to the diagram D_λ

$T_\lambda =$

<table>
<tr><td>1</td><td>2</td><td>3</td><td>⋯</td><td>$n-m$</td></tr>
<tr><td>$n-m+1$</td></tr>
<tr><td>$n-m+2$</td></tr>
<tr><td>⋮</td></tr>
<tr><td>n</td></tr>
</table>

It is possible to associate to T_λ two subgroups of S_n: the row-stabilizer R_{T_λ} and the column-stabilizer C_{T_λ}, in this case:

$$R_{T_\lambda} = S_m(1, 2, \cdots, n-m),$$

and

$$C_{T_\lambda} = S_{n-m}(n-m+1, n-m+2, \cdots, n),$$

where $S_t(a_1, a_2, \cdots, a_t)$ is the symmetric group of degree t on the elements $a_1, a_2, \cdots, a_t$. Hence the polynomial corresponding to T_λ will be

$$f_\lambda(\bar{x}) = \sum_{\rho \in R_{T_\lambda}} \rho \cdot \sum_{\sigma \in C_{T_\lambda}} (-1)^\sigma x_{\sigma(1)} x_2 \dots x_{n-m}$$
$$x_{\sigma(n-m+1)} x_{\sigma(n-m+2)} \cdots x_{\sigma(n)}.$$

Now by lemma 2.1.3 we obtain the following polynomial:

$$f^*_\lambda(\bar{x}; \bar{y}) = \sum_{\rho \in R_{T_\lambda}} \rho \cdot \sum_{\sigma \in C_{T_\lambda}} (-1)^\sigma y_1 x_{\sigma(1)} y_2 x_2 \dots y_{n-m} x_{n-m}$$
$$y_{n-m+1} x_{\sigma(n-m+1)} y_{n-m+2} x_{\sigma(n-m+2)} \cdots y_n x_{\sigma(n)}.$$

For Theorem (1.11.6) $f^*_\lambda(\bar{x}, \bar{y})$ is a consequence of $e^*_\lambda[\bar{x}, \bar{y}]$. Now let $E = E_0 \oplus E_1$ be the natural $\mathbb{Z}_2$-grading of E, and consider the following substitution

$$x_i := (h_0 + g_0) \qquad i = 1, \dots, n-m,$$

$$x_{n-m+j} := h_j \qquad j = 1, \ldots, m,$$
$$y_i := g_i \qquad i = 1, \ldots, n,$$

with $g_i \in E_0$ and $h_i \in E_1$ distinct, and $g_0^m \neq 0$.
Under this substitution f_λ^* take the value

$$2(n-m)!(n-m)mh_0h_1 \ldots h_m g_0^{n-m-1} + (m-1)h_1h_2 \ldots h_m g_0^m \neq 0$$

and the lemma is proved. □

Throughout this section we set $R = A + J$, where $A = M_k(F)$ is the algebra of $k \times k$ matrices over a field F and $J = J(A)$ is the Jacobson radical, now we have the following.

Lemma 2.2.4 (Lemma 2 [21]) *Let be $R = A + J$, where $A = M_k(F)$ is the algebra of $k \times k$ matrices over the field F and $J = J(R)$ the Jacobson radical, then J can be decomposed into the direct sum of four A-bimodules*

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$$

where, for $p, q \in \{0, 1\}$, J_{pq} is a left faithful module or a 0-left module according as $p = 1$ or $p = 0$, respectively. Similarly, J_{pq} is a right faithful module or a 0-right module according as $q = 1$ or $q = 0$, respectively. Moreover, for $p, q, i, l \in \{0, 1\}$, $J_{pq}J_{ql} \subseteq J_{pl}$, $J_{pq}J_{il} = 0$ for $q \neq i$ and there exists a finite dimensional nilpotent subalgebra N of J such that $J_{11} = AN \cong A \otimes_F N$ (isomorphism of A-bimodules and of algebras).

We start studying the properties of the different subspaces J_{ij}.

Lemma 2.2.5 *Let $\lambda = \left(n - k^2, 1^{k^2}\right) \vdash n$ be a partition of n with $n, k \in \mathbb{N}$ and let $R = A + J$ be defined as in lemma 2.2.4. If $E_\lambda^* \subseteq Id(R)$ then $J_{01} = J_{10} = 0$*

Proof. We construct the following Young tableaux T_λ associated to the diagram D_λ,

$T_\lambda=$

$n-k^2$	1	2	$\cdots$	$n-k^2-1$
$n-k^2+1$				
$\vdots$				
n				

As in lemma 2.2.3, we obtain that

$$f_\lambda^*(\bar{x}, \bar{y}) = \sum_{\rho_{R_{T_\lambda}}} \rho \cdot \sum_{\sigma \in C_{T_\lambda}} (-1)^\sigma y_1x_1y_2x_2 \cdots y_{n-k^2}x_{\sigma(n-k^2)}$$
$$y_{n-k^2+1}x_{\sigma(n-k^2+1)} \cdots y_nx_{\sigma(n)}.$$

By Theorem (1.11.6) $f_\lambda^*(\bar{x},\bar{y})$ is a consequence of $e_\lambda^*(\bar{x},\bar{y})$. But by hypothesis $E_\lambda^*[\bar{x},\bar{y}] \subseteq Id(R)$, hence $f_\lambda^*(\bar{x},\bar{y}) \in \mathrm{Id}(R)$.

Since Cap_{k^2} is not a polynomial identities for $M_k(F)$ then there exist elements $a_1, a_2, \ldots, a_{k^2}, b_1, b_2, \ldots, b_{k^2}, b_{k^2+1}$ such that

$$Cap_{k^2}(a_1, a_2, \ldots, a_{k^2}, b_1, b_2, \ldots, b_{k^2}, b_{k^2+1}) = e_{kk} \tag{2.3}$$

(see, for instance, Proposition 1.4.7 in [35]). Then we make the following substitution:

$$x_1 = x_2 = \cdots = x_{n-k^2} = a_1,$$

$$x_{n-k^2+i} = a_{i+1} \quad i = 1, 2, \cdots, k^2 - 1,$$

$$x_n = \sum_{t=1}^{k} e_{tt} d_{10} = e d_{10} \in J_{10},$$

$$y_j = b_j \quad j = n - k^2, n - k^2 + 1, \cdots, y_n,$$

and $y_j = e_{rs}$ for $j = 1, 2, \cdots, n - k^2 - 1$ are opportune matrix units.

By the properties of f_λ^* and recalling that $d_{10} \in J_{10}$ implies that $d_{10}a = 0\ \forall a \in A$, we have that, under this evaluation we have that f_λ^* takes the value

$$e_{kk} e d_{10} = 0.$$

Hence, we can say that $d_{10} = 0, \quad \forall d_{10} \in J_{10}$, and finally the conclusion of the lemma, $J_{10} = 0$ holds. To prove that $J_{01} = 0$ it is enough consider the following Young tableau,

$$T_\lambda = \begin{array}{|c|c|c|c|}\hline k^2+1 & k^2+2 & \cdots & n \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1} \vdots \\ \cline{1-1} k^2 \\ \cline{1-1} \end{array}$$

and we have the result. $\square$

Lemma 2.2.6 *Let* $\lambda = \left(n - k^2, 1^{k^2}\right) \vdash n$ *be a partition of* n *with* $n, k \in \mathbb{N}$, $R = A + J$ *be defined as in lemma 2.2.4, and let* $J_{11} \simeq A \otimes_F N$ *as in Lemma 2.2.4. If* $E_\lambda^* \subseteq \mathrm{Id}(R)$ *then* N *is commutative.*

Proof. Let $\mu = \left(n - k^2, 1^{k^2+1}\right) \vdash (n+1)$, we remark that by Theorem 1.11.6 we have that R satisfies all the μ-identities with $\mu \geq \lambda$, so we consider the following tableaux

$$T_\mu = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & n-k^2 \\ \hline n-k^2+1 \\ \cline{1-1} \vdots \\ \cline{1-1} n \\ \cline{1-1} n+1 \\ \cline{1-1} \end{array}$$

in the same way of lemma 2.2.3 we can associate the following polynomial

$$f_\mu^* = \sum_{\rho \in R_{T_\mu}} \rho \cdot \sum_{\sigma \in C_{T_\mu}} (-1)^\sigma x_{\sigma(1)} y_1 x_2 y_2 \dots y_{n-k^2-1} x_{n-k^2} y_{n-k^2} x_{n-k^2+1} \\ y_{n-k^2+1} \cdots y_n x_{\sigma(n+1)}. \tag{2.4}$$

We consider an ordered basis consisting of all matrix units e_{ij} of A, and let $v_1, \dots, v_{k^2}$ such that $v_1 = e_{11}$. Consider now k^2 elements of A, let a_i for $i = 0, \dots, k^2 - 1$, be such that

$$v_1 a_0 v_2 a_1 \dots v_{k^2} a_{k^2-1} = e_{11}$$

and for any non-trivial permutation $\sigma \in S_{k^2}$

$$v_{\sigma(1)} a_0 v_{\sigma(2)} a_1 \dots v_{\sigma(k^2)} a_{k^2-1} = 0.$$

We remark that if we pick two elements of $J_{11} \cong A \otimes_F N$, let be $e \otimes d_1, e \otimes d_2 \in J_{11}$, this elements commutes with A (see the proof of lemma 2.2.4). Then we make the following substitution:

$$\begin{aligned} x_i &= v_1 \otimes 1 \;\; i = 1, 2, \cdots, n - k^2, \\ x_j &= v_j \otimes 1 \;\; j = 2, 3, \cdots, k^2 \\ x_{k^2+1} &= e \otimes d_1, \\ x_{k^2+2} &= e \otimes d_2, \\ y_j &= e_{11} \otimes 1 \quad j = 1, 2, \cdots, n - k^2, \\ y_{n-k^2+j} &= a_j \otimes 1 \quad j = 0, 1, \cdots, k^2 - 1. \end{aligned}$$

Using this substitution and recalling that f_μ^* it is a polynomial identities for R, we have, up to a non-zero scalar

$$e_{11} \otimes [d_1, d_2] = 0$$

This implies that $[d_1, d_2] = 0$, and so the lemma is proved.

□

Lemma 2.2.7 *Let $d_1^2 + \cdots + d_t^2 = k^2$ and let $\lambda' = \left(n - k^2, 1^{k^2+t-1}\right)$, then $UT(d_1, \dots, d_t) \notin \mathcal{V}_{\lambda'}$, where $UT(d_1, \dots, d_t)$ is the upper block triangular matrices algebra.*

Proof. To prove the lemma we build a polynomial that is a consequence of $e^*_{\lambda'}$, and such that is not a polynomial identities for $UT(d_1, \ldots, d_t)$.

Let $\lambda' = \left(n - k^2, 1^{k^2+t-1}\right)$ and let $T_{\lambda'}$ be the following Young tableau associated to λ'

$$T_{\lambda'} = \begin{array}{|c|c|c|c|c|}\hline 1 & k^2+t & k^2+t+1 & \cdots & n \\ \hline 2 \\ \cline{1-1} \vdots \\ \cline{1-1} k^2+t-1 \\ \cline{1-1} \end{array}$$

From it we can construct the following polynomial

$$f^*_{\lambda'} = \sum_{\rho \in R_{T_{\lambda'}}} \rho \cdot \sum_{\sigma \in C_{T_{\lambda'}}} (-1)^\sigma y_1 x_{\sigma(1)} y_2 x_{\sigma(2)} \cdots y_{k^2+t-1} x_{\sigma(k^2+t-1)} y_{k^2+t} x_{k^2+t} \cdots y_n x_n.$$

which, by Theorem (1.11.6) is a consequence of $e^*_{\lambda'}$.

Now consider the upper triangular block matrices algebra

$$UT(d_1, \ldots, d_t) = \begin{pmatrix} A_1 & & & * \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & A_t \end{pmatrix}$$

where $A_i = M_{d_i}(F)$, and we recall that $J = J(UT(d_1, \ldots, d_t))$, the Jacobson radical of $UT(d_1, \ldots, d_t)$, consists of all strictly upper block triangular matrices.

Since $A_1 J A_2 J \ldots J A_t \neq 0$ we can take $c_1, \ldots, c_{t-1} \in J$, $e_i \in A_i$ for $i = 1, \ldots, t$ such that

$$e_1 c_1 e_2 c_2 \ldots c_{t-1} e_t \neq 0, \tag{2.5}$$

where e_i is some matrix unit from A_i, $c_i \in J$ such that $A_i c_i A_{i+1} \neq 0$ and $A_j c_i A_k = 0$ for $j \neq i$ or $k \neq i+1$,also

$$1_i c_i = c_i 1_{i+1} = c_i \tag{2.6}$$

where $1_i \in A_i$ is an identity for A_i.

Now, in a similar way of Lemma 2.2.6 we consider a basis $u^i_1, \ldots, u^i_{d_i}$ of matrix units of A_i for $i = 1, \ldots, t$. We can choose $a^i_1, \ldots, a^i_{d_i+1} \in A_i$ such that

$$a^i_1 u^i_1 a^i_2 \ldots a^i_{d_i} u^i_{d_i} a^i_{d_i+1} = e_i \neq 0 \tag{2.7}$$

and

$$a_1^i u_{\sigma(1)}^i a_2^i \dots a_{d_i}^i u_{\sigma(d_i)}^i a_{d_i+1}^i = 0 \tag{2.8}$$

for any non trivial permutation σ of S_{d_i}.
Let $D_i = d_i^2 + 1$ and consider the following substitution:

$$y_{i+j} = a_j^i \qquad j = 1, 2, \cdots d_1^2 + 1, \qquad i = 1, 2, \cdots, t,$$
$$x_{i+j} = u_j^i \qquad j = 1, 2, \cdots, d_i^2, \qquad i = 1, 2, \cdots, t,$$
$$x_{D_i} = c_i \qquad 1 = 1, 2, \cdots t,$$

and all the other variables take the value of the identity matrix of A_t. We remark, that if $x_i = e_k$ for $i = k^2 + t, \dots, n$ then all the summands of $f_{\lambda'}^*$ are equal to zero except when ρ is trivial. under this substitution the polynomial $f_{\lambda'}^*$ is equal, up to non zero scalar, to

$$e_1 c_1 e_2 c_2 \dots c_{t-1} e_t \neq 0,$$

and so the lemma holds.

□

Theorem 2.2.8 *Let* $\lambda = \left(n - k^2, 1^{k^2}\right) \vdash n$ *be a partition of* n*, then* $\mathrm{var}\,(E_\lambda^*) = \mathrm{var}\,(M_k\,(F) + D)$ *for some finite dimensional algebra* D *with* $\exp\,(D) < k^2$. *In particular*

$$c_n\,(E_\lambda^*) \simeq c_n\,(M_k\,(F))\,.$$

Proof.
By lemma 2.2.3 the infinite dimensional Grassmann algebra G does not satisfy e_λ^* and by [[26], Theorem 2.3] $\mathcal{V}_\lambda$ is generated by a finite dimensional algebra. By theorem 2.2.2, there exist finite-dimensional reduced algebras $B_1, B_2, \dots, B_t$ and a finite-dimensional algebra D such that

$$\mathrm{var}\,(\mathcal{V}_\lambda) = \mathrm{var}\,(B_1 \oplus B_2, \oplus \cdots \oplus B_t \oplus D) \tag{2.9}$$

where $\exp\,(\mathcal{V}_\lambda) = \exp\,(B_1) = \exp\,(B_2) = \cdots = \exp\,(B_t) = k^2$ and $\exp\,(D) < \exp\,(\mathcal{V}_\lambda)$.

Now, we analyze the structure of a finite-dimensional reduced algebra B which satysfies E_λ^*.

Let B be a finite dimensional algebra such that $E_\lambda^* \in \mathrm{Id}(B)$ and $\exp(B) = k^2$.

Hence by [20], B contains a subalgebra isomorphic to the upper block triangular matrix algebra,

$$UT(d_1,\dots,d_t)=\begin{pmatrix} M_{d_1}(F) & & & * \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & M_{d_t} \end{pmatrix}.$$

where $\exp(B)=\exp(UT(d_1,\dots,d_t))=d_1^2+\cdots+d_t^2$

Since $UT(d_1,\dots,d_t)\subseteq B$ it follows that $\mathrm{Id}(B)\subseteq\mathrm{Id}(UT(d_1,\dots,d_t))$, but by Lemma 2.2.7, $UT(d_1,\dots,d_t)$ does not satysfy $E^*_{\left(n-k^2,1^{k^2+t-1}\right)}$, hence $t=1$ and $d_1=k$. So $B=A+J$, where $A=M_k(F)$, $J=J(B)$. Now for Lemma 2.2.5, we have $J_{01}=J_{10}=0$ and J_{00} is a nilpotent ideal, so we write $B=A+J=(A+J_{11})\oplus J_{00}$.

By Lemma 2.2.6, $A+J_{11}=A+AN\cong A\otimes_F N^*$, where N^* is the algebra obtained from N by adjoining a unit element. Since N^* is commutative, it follows that $A+J_{11}$ and A satisfy the same identities and $\mathrm{var}(B)=\mathrm{var}(A\oplus J_{00})$.

By the decomposition in (2.9), it follows that $\mathcal{V}_\lambda$ is generated by $M_k(F)\oplus D$ where D is some finite dimensional algebra and $\exp(D)<k^2$. □

2.3 The case $m-1=k^2$ and $m-1=k_1^2+k_2^2$

In this section we study the varieties $\mathcal{V}_\lambda$, where $\lambda=(n-m,1^m)$ and $m-1=k^2$ or $m-1=k_1^2+k_2^2$. It plays an important role the algebra $M_{k\times l}(F)$, the algebra of $(k+l)\times(k+l)$ matrices over F having the last l rows and the last k columns equal to zero. We start with the case $m-1=k^2$.

Lemma 2.3.1 *Let $m-1=k^2$ and $\lambda=(n-m,1^m)$, $R=A+J$ where $A=M_k(F)$, $J=J(R)$, and let $R\in\mathrm{var}(E^*_\lambda)$. If $J_{01}J_{10}=J_{10}J_{01}=J_{10}J_{00}=J_{00}J_{01}=0$, then $\mathrm{var}(R)=\mathrm{var}(A_1\oplus A_2\oplus J_{00})$ where $A_1=A+J_{10}$ and $A_2=A+J_{01}$.*

Proof. Clearly $\mathrm{Id}(R)\subseteq\mathrm{Id}(A_1\oplus A_2\oplus J_{00})$. Let now $f=f(x_1,\dots,x_n)$ be a multilinear polynomial such that $f\notin\mathrm{Id}(R)$.

Suppose first that

$$f\in\mathrm{Id}(A+J_{11}+J_{10})\cap\mathrm{Id}(A+J_{11}+J_{01})\cap\mathrm{Id}(J_{00})$$

and let $b_1,\dots,b_n\in R$ be such that $f(b_1,\dots,b_n)\neq 0$. We may assume by linearity that $b_1,\dots,b_n$ belong to $A\cup J_{10}\cup J_{10}\cup J_{11}\cup J_{00}$. By the assumption, $b_1,\dots,b_n$ do not belong, at the same time, to $A\cup J_{11}\cup J_{10}$ or to $A\cup J_{11}\cup J_{01}$ or to J_{00}. Thus there exist b_i,b_j, $i\neq j$, such that one of the following three possibilities occurs:

$$b_i\in J_{10}\text{ and }b_j\in J_{01}\tag{2.10}$$

and

$$b_i \in J_{10} \text{ and } b_j \in J_{00} \tag{2.11}$$

and

$$b_i \in J_{01} \text{ and } b_j \in J_{00}. \tag{2.12}$$

Since the J_{kl}'s are A-bimodules, $J_{01}J_{10} = J_{10}J_{01} = J_{10}J_{00} = J_{00}J_{01} = 0$ and, by Lemma 2.2.4 $J_{01}J_{00} = J_{00}J_{10} = J_{00}J_{11} = J_{11}J_{00} = J_{01}J_{01} = J_{10}J_{10} = 0$; we have that each of the above three cases leads to $b_{\sigma(1)} \cdots b_{\sigma(n)} = 0$ for all $\sigma \in S_n$. Thus $f \in \mathrm{Id}(R)$, contrary to the assumption.

We have proved that $\mathrm{Id}(R) \supseteq \mathrm{Id}(A + J_{11} + J_{10}) \cap \mathrm{Id}(A + J_{11} + J_{01}) \cap \mathrm{Id}(J_{00})$. If we prove that $\mathrm{Id}(A + J_{11} + J_{10}) = \mathrm{Id}(A + J_{10})$ and $Id(A + J_{11} + J_{01}) = \mathrm{Id}(A + J_{01})$, we would get that $\mathrm{Id}(R) \supseteq \mathrm{Id}(A_1) \cap \mathrm{Id}(A_2) \cap \mathrm{Id}(J_{00})$ and the proof would be complete.

In order to prove that $\mathrm{Id}(A + J_{11} + J_{10}) = \mathrm{Id}(A + J_{10})$, suppose that there exists $f(x_1, \ldots, x_n) \notin \mathrm{Id}(A + J_{11} + J_{10})$ and let f be multilinear. Since $J_{11} = AN$, A commutes with N and N is commutative by Lemma 2.2.6, we have that for all $b_1, \ldots, b_m \in A + J_{11} + J_{10}$, $a \in A$, $d \in N$

$$b_1 \ldots b_k a d b_{k+1} \ldots b_m = d b_1 \ldots b_k a b_{k+1} \ldots b_m.$$

It follows that if $b_1, \ldots, b_n \in A \cup J_{11} \cup J_{10}$ are such that $f(b_1, \ldots, b_n) \neq 0$, then we can write

$$f(b_1, \ldots, b_n) = d' f\left(b'_1, \ldots, b'_n\right)$$

for some $d' \in N$, $b'_1, \ldots, b'_n \in A \cup J_{10}$. Thus $f \notin \mathrm{Id}(A + J_{10})$ and

$$\mathrm{Id}(A + J_{11} + J_{10}) = \mathrm{Id}(A + J_{01})$$

follows. Similarly, one can show that $\mathrm{Id}(A + J_{11} + J_{01}) = \mathrm{Id}(A + J_{01})$. This completes the proof of the lemma. $\square$

Lemma 2.3.2 *Let* $m - 1 = k^2$ *and* $\lambda = (n - m, 1^m)$, *and let be* $R = A + J$ *where* $A = M_k(F)$ *and* $J = J(R)$. *If* $R \in \mathrm{var}(E^*_\lambda)$, *then*

$$\mathrm{var}(R) = \mathrm{var}(M_{k \times 2k}(F) \oplus M_{2k \times k}(F) \oplus D),$$

with D *a finite dimensional algebra such that* $\exp(D) < k^2$.

Proof. We first claim that if $R \in \mathrm{var}(E^*_\lambda)$ then

$$J_{10}J_{01} = J_{01}J_{10} = J_{10}J_{00} = J_{00}J_{01} = 0. \tag{2.13}$$

We prove this for the case $J_{10}J_{01}$, and the other can be proved in a similar way. Suppose $J_{10}J_{01} \neq 0$ and let $u \in J_{10}$ and $v \in J_{01}$ be such that $uv \neq 0$.

As in the lemmas proved before, we consider the following tableau

$$T_\lambda = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & n-k^2+1 \\ \hline n-k^2+2 \\ \cline{1-1} n-k^2+3 \\ \cline{1-1} \vdots \\ \cline{1-1} n \\ \cline{1-1} \end{array}$$

and we associate the polynomial

$$f_\lambda^* = \sum_{\rho \in R_{T_\lambda}} \rho \cdot \sum_{\sigma \in C_{T_\lambda}} (-1)^\sigma x_{\sigma(1)} y_1 x_2 y_2 \ldots x_{n-k^2+1} y_{n-k^2+1} x_{\sigma(n-k^2+2)}$$
$$y_{n-k^2+2} \cdots y_{n-1} x_{\sigma(n)} y_n.$$

Now, let $b_1, \ldots, b_{k^2}$ be an ordered basis of A consisting of all matrix units e_{ij} such that $b_1 = e_{11}$, and $a_1, \ldots, a_{k^2} \in A$ such that

$$b_1 a_1 b_2 \cdots a_{k^2-1} b_{k^2} a_{k^2} = e_{11}$$

and

$$b_{\sigma(1)} a_1 b_{\sigma(2)} \cdots a_{k^2-1} b_{\sigma(k^2)} a_{k^2} = 0$$

for any non trivial permutation σ of S_{k^2}.

Using the following substitution

$$x_1 = u \in J_{10},$$
$$y_1 = v \in J_{01}$$
$$x_{n-k^2+i} = b_i \quad i = 2, \ldots, k^2,$$
$$y_{n-k^2+j} = a_j \quad j = 1, \ldots, k^2,$$

and all the other variables equal to b_1; recalling the property of Capelli polynomials used in lemma 2.2.6 we have that, under this substitution, f_λ^* take the value $uve_{11} \neq 0$, a contradiction.

Now by Lemma 2.2.3 and Theorem 2.2.2, we have that there exist a finite number of reduced algebras $B_1, \ldots, B_t$ and a finite-dimensional algebra D such that

$$\mathrm{var}\,(\mathcal{V}_\lambda) = \mathrm{var}\,(B_1 \oplus \cdots \oplus B_t \oplus D) \tag{2.14}$$

As in the Theorem 3.3.7 it is possible to prove that $\mathrm{var}(\mathcal{V}_\lambda) = \mathrm{var}\,(B \oplus D)$, where B is a reduced algebra satisfying E_λ^* and $B = A + J$ with $A = M_k\,(F)$. By Lemma 2.3.1 $\mathrm{var}\,(B) = \mathrm{var}\,(A_1 \oplus A_2 \oplus J_{00})$ where $A_1 = A + J_{10}$, $A_2 = A + J_{01}$ and J_{00} is a nilpotent algebra. We prove now that $A + J_{10}$ has the same identities of $M_{k \times 2k}\,(F)$, and then, similarly, it is possible to prove that $A + J_{01}$ has the same identities of $M_{2k \times k}\,(F)$. Now the left A module J_{10} is isomorphic to a distinct sum of irreducible modules, hence it is isomorphic to say $t > 0$ copies of an irreducible $M_k\,(F)$-module. It follows that

$A = M_k(F) + J_{10} \simeq M_{k,k+t}$ and, so, A has the same identities of $M_{k,2k}(F)$. Since $J_{10}A = J_{10}J_{10} = 0$ and $A = M_{k\times k}(F)$, then $A + J_{10}$ has the same identities as $M_{k\times 2k}(F)$.

Hence we have proved that if B is a reduced algebra satisfying E^*_λ then

$$\mathrm{var}(B) = \mathrm{var}(M_{k\times 2k}(F) \oplus M_{2k\times k}(F) \oplus J_{00}).$$

Finally, by the decomposition given in (2.14), we get that

$$\mathrm{var}(\mathcal{V}_\lambda) = (M_{k\times 2k}(F) \oplus M_{2k\times k}(F) \oplus D)$$

where $\exp(D) < k^2$.

□

Lemma 2.3.3 *Let $m - 1 = k_1^2 + k_2^2$, $\lambda = (n - m, 1^m) \vdash n$ and let $R = A \oplus B + J$ where $A = M_{k_1}(F)$, $B = M_{k_2}(F)$ and $J = J(R)$. If $R \in \mathrm{var}(E^*_\lambda)$, and $AJB \neq 0$, then $\mathrm{var}(R) = \mathrm{var}(A_1 \oplus A_2 \oplus D)$ where $A_1 = A + B + AJA + BJB + AJB$, $A_2 = A + B + AJA + BJB + BJA$ and $\exp(D) < m - 1$.*

Proof. As in a previous lemma we consider a partition $\mu \vdash (n + 2)$ and we consider the following tableau

$$T_\mu = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & n-m \\ \hline n-m+1 \\ \cline{1-1} \vdots \\ \cline{1-1} n+1 \\ \cline{1-1} n+2 \\ \cline{1-1} \end{array}$$

and, as before, we obtain the following polynomial

$$f^*_\mu = \sum_{\rho \in R_{T_\mu}} \rho \cdot \sum_{\sigma \in C_{T_\mu}} (-1)^\sigma x_{\sigma(1)} y_1 x_2 y_2 \ldots x_{n-m} y_{n-m} \qquad (2.15)$$
$$x_{\sigma(n-m+1)} \cdots y_{n+1} x_{\sigma(n+2)}.$$

And for theorem B of [1] $f^*_\mu \in \mathrm{Id}(R)$.

First of all, we claim that

$$BJAJA = 0 \text{ and } AJBJA = 0. \qquad (2.16)$$

Suppose, by contradiction, that there exist $x, y \in J$ such that $BxAyB \neq 0$. If 1_A and 1_B denote the unit elements of A and B respectively, then we may assume that $1_B x = x 1_a = x$ and $1_A y = y 1_B = y$. Hence

$$Ax = xB = yA = By = 0. \qquad (2.17)$$

As in the lemma 2.2.5 we can consider the matrices

$$u_1, \ldots, u_{k_1^2}, a_0, a_1, \ldots, a_{k_1^2-1} \in A \quad v_1, \ldots, v_{k_2^2}, b_0, b_1, \ldots, b_{k_1^2-1} \in B$$

and

$$xa_0u_1a_1u_2\dots a_{k_1^2-1}u_{k_1^2}1_Ayb_0v_1b_1v_2\dots b_{k_2^2-1}v_{k_2^2}\neq 0$$

is the only combination of $u_1,\dots,u_{k_1^2},v_1,\dots,v_{k_2^2}$ such that the above product is not zero.

Now, if we consider the following substitution

$$x_1 = x,$$

$$y_i = a_{i-1} \qquad i = 1, 2, \cdots, k_1^2,$$

$$x_j = u_{j-1} \qquad j = 2, 3, \cdots, k_1^2 + 1,$$

$$x_{k_1^2+2} = y,$$

$$y_{k_1^2+1+i} = b_{i-1} \qquad 1 = i, 2, \cdots, k_2^2,$$

$$x_{k_1^2+1+j} = v_{j-1} \qquad j = 2, 3, \cdots, k_2^2 + 1,$$

and all the other variables take the value 1_A, we have that under this sustitution

$$f_\mu^* \neq 0$$

The second equality in (2.16) is proved similarly.

We show now that if $xAyB \neq 0$, with $x, y \in J$ then we may take $x \in AJA$. In fact, since $1_B \in B$, we have that $xB \neq 0$ and also $BxAyB \neq 0$, and this contradicts (2.16). Hence $Bx = 0$. If also $Ax = 0$, then we can say that (2.15) is not a polynomial identity for R. Therefore $Ax \neq 0$ and $x \in AJA$. In the same way it is possible prove that if $AxBy \neq 0$, for some $x, y \in J$, then we may take $y \in BJB$.

Consider now a non-zero product of the type

$$a_1d_1\cdots a_{m-1}d_{m-1}a_m$$

where $a_1,\dots,a_m \in A\cup B$ and $d_1,\cdots,d_{m-1}\in J$. Then, by (2.16), either $a_1,\cdots,a_m\in A$ or $a_1,\cdots,a_m\in B$ or there exists $1\leq k\leq m$ such that $a_1,\cdots,a_{k-1}\in A$, $a_k,\cdots,a_m\in B$ or $a_1,\cdots,a_{k-1}\in B$, $a_k,\cdots,a_m\in A$.

Similarly, if

$$d_0a_1d_1\cdots a_{m-1}d_{m-1}a_m\neq 0$$

where $a_1,\dots,a_{k-1}\in A$, $a_k,\dots,a_m\in B$ and $d_0,d_1,\dots,d_{m-1}\in J$, then, by what we proved above, we may take $d_0\in AJA$.

We next prove that

$$\mathrm{Id}\,(R) = \mathrm{Id}\,(A_1)\cap \mathrm{Id}\,(A_2)\cap \mathrm{Id}\,(A_3)\cap \mathrm{Id}\,(A_4)$$

where $A_3 = A + J$ and $A_4 = B + J$. In this way we can affirm that

$$\mathrm{var}\,(R) = \mathrm{var}\,(A_1\oplus A_2\oplus A_3\oplus A_4)$$

and then, since $\exp(A_3 \oplus A_4) < \max\{k_1^2, k_2^2\} < k_1^2 + k_2^2$, the lemma will be proved. The inclusion

$$Id(R) \subseteq Id(A_1) \cap Id(A_2) \cap Id(A_3) \cap Id(A_4)$$

is obvious, then Let $f \notin Id(R)$, a multilinear polynomial, and suppose that $f \in \mathrm{Id}(A_3) \cap \mathrm{Id}(A_4)$.

Since $AB = BA = 0$, in order to obtain a non-zero evaluation of f we must substitute at least one $y \in J$ such that $AyB \neq 0$ or $ByA \neq 0$. As we remarked in the proof of (2.17), the element y can be taken in AJB or BJA, respectively. Taking into account the relation (2.16) and the above discussion, it follows that all the other variables must be evaluated in $A \cup B \cup AJA \cup BJB$.

Thus either $f \notin \mathrm{Id}(A_2)$ or $f \notin \mathrm{Id}(A_2)$. This prove that

$$\mathrm{Id}(R) \supseteq \mathrm{Id}(A_1) \cap \mathrm{Id}(A_2) \cap \mathrm{Id}(A_3) \cap \mathrm{Id}(A_4).$$

And so the lemma is proved. □

Lemma 2.3.4 *Let* $m-1 = k_1^2 + k_2^2$ *and let* $\lambda = \left(n-m-1, 1^{k_1^2+k_2^2+1}\right) \vdash n$, *and let* $A_1, A_2 \in \mathrm{var}(E_\lambda^*)$ *be the algebras defined in Lemma 2.2.4. Then there exist finite dimensional algebras* D_1 *and* D_2 *such that*

$$\mathrm{var}(A_1) = \mathrm{var}(UT(k_1,k_2) \oplus D_1), \qquad \mathrm{var}(A_2) = \mathrm{var}(UT(k_1,k_2) \oplus D_2)$$

and $\exp(D_1), \exp(D_2) < m-1$.

Proof. Consider the algebra $A' = A + AJA$. By Lemma 2.2.4 we have that $AJA = (AJA)_{11} = AN \cong A \otimes N$ for some nilpotent algebra N. Suppose first that N is non-commutative. Then, by lemma 2.2.6, we can affirm that (2.4) is not a polynomial identities for A'. If $AJAJB \neq 0$, as in the proof of lemma 2.3.3, we can prove that (2.4) is not a polynomial identities for A_1, a contradiction.

Hence $AJAJB = 0$. Since $AB = BA = 0$, we obtain that

$$\mathrm{var}(A_1) = \mathrm{var}((A + B + AJB + BJB) \oplus (A + AJA))$$

and $\exp(A + AJA) < k_1^2 + k_2^2$. In the case N in commutative, $Id(AJA) \subseteq Id(A_1)$ implies that $\mathrm{var}(A_1) = \mathrm{var}((A + B + AJB + BJB) \oplus (A + AJA))$.

Now, considering the summand BJB and using the same arguments we obtain that $\mathrm{var}(A_1) = \mathrm{var}((A + B + AJB) \oplus D)$ for some finite dimensional algebra D such that $\exp(D) < k_1^2 + k_2^2$.

The algebra $A+B+AJB$ contains a subalgebra isomorphic to $UT(k_1,k_2)$, (see [20]), and so $\mathrm{Id}(A + B + AJB) \subseteq \mathrm{Id}(UT(k_1,k_2))$. On the other hand, $\mathrm{Id}(UT(k_1,k_2)) = \mathrm{Id}(M_{k_1}(F))\,\mathrm{Id}(M_{k_2}(F))$. Since $(AJB)^2 = 0$, it is easy to see that if $f_1 \in \mathrm{Id}(M_{k_1}(F))$ and $f_2 \in \mathrm{Id}(M_{k_2}(F))$ then

$f_1f_2 \in \mathrm{Id}\,(A+B+AJB)$. Hence $\mathrm{var}\,(A+B+AJB) = \mathrm{var}\,(UT\,(k_1,k_2))$ and the conclusion of the lemma follows for the algebra A_1. In the same way we can prove the second part of the lemma. □

Theorem 2.3.5 *Let $\lambda = (n-m, 1^m) \vdash n$, let $m \neq k^2$ and suppose $m-1$ is a square or the sum of two squares. Then*

$$\mathrm{var}\,(E^*_\lambda) = \mathrm{var}\,(A \oplus B \oplus D)$$

where D is a finite dimensional algebra with $\exp(D) < m$,

$$A = \begin{cases} M_{r\times 2r}\,(F) \oplus M_{2r\times r}\,(F) & \text{if } m-1 = r^2 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B = \begin{cases} 0 & \text{if } m-1 \text{ is not the sum of two squares,} \\ \bigoplus_{s^2+t^2=m-1} UT\,(s,t) & \text{otherwise.} \end{cases}$$

Proof. By lemma 2.2.3 and theorem 2.3 of [26], $\mathrm{var}\,(E^*_\lambda)$ is generated by a finite dimensional algebra. As in the proof of theorem 3.3.7, and by the 2.2.2, we need only examine finite dimensional reduced algebras in $\mathrm{var}\,(E^*_\lambda)$. We then apply lemmas 2.3.2, 2.3.3 and 2.3.4 to complete the proof of the theorem. □

Chapter 3

On the $\mathbb{Z}$-grading of $M_2(E)$

Throughout this chapter we study the polynomial identities of the $\mathbb{Z}$-grading of $A = M_2(E)$, and we find a subset of generators of the corresponding T-ideal.

We also compute, through the representation theory of $GL_2 \times GL_2 \times GL_2$, the nth cocharacter of the homogeneous component of degree -1, 0 and 1 of the $\mathbb{Z}$-graded algebra A and the graded cocharacter $\chi_{(0,r,n-r)}$ of A.

3.1 The polynomial identities of $M_2^{\mathbb{Z}}(E)$

Let $A = M_2(E)$ be the algebra of 2×2 matrices over the infinite dimensional Grassmann algebra E and let $\mathbb{Z}(+)$ be the additive group of integer.

Now we define a $\mathbb{Z}$-grading of A as follows:

$$M_2^{\mathbb{Z}}(E) = \bigoplus_{n \in \mathbb{Z}} M_2^{(n)}(E),$$

such that

$$M_2^{(0)}(E) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$$

$$M_2^{(1)}(E) = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix},$$

$$M_2^{(-1)}(E) = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix},$$

and

$$M_2^{(n)}(E) = 0 \qquad n \notin \{-1, 0, 1\}.$$

We remark that, since F is a field of characteristic zero, to study of the polynomial identities of $M_2(E)^{\mathbb{Z}}$ we may restrict ourselves to consider only multilinear polynomials.

We adopt the following convention: the symbol $\tilde{}, \bar{}$ indicate alternation on a given set of variables, in particular $\tilde{x_1}\tilde{x_2} = [x_1, x_2] = x_1x_2 - x_2x_1$. We also adopt with $[x_1, x_2, \cdots, x_r] = [[[x_1, x_2]\, x_3], \cdots, x_r]$ the left normed commutator.

Hence, if we denote by x_i the variables of homogeneous degree zero, by y_i the variables of homogeneous degree 1 and by z_i the variables of homogeneous degree -1, we have the following result.

Lemma 3.1.1 *The followings are $\mathbb{Z}$-graded polynomial identities of A:*

$$[x_1, x_2, x_3] \equiv 0,$$
$$y_1y_2 \equiv 0,$$
$$z_1z_2 \equiv 0.$$

Proof. The first is an obvious consequence of [30], [27]. Since the homogeneous component of degree 1 and -1 are nilpotents we have $y_1y_2 \equiv 0$, $z_1z_2 \equiv 0$. □

We also have

Lemma 3.1.2 *The followings are $\mathbb{Z}$-graded polynomial identities of A:*

$$[y_1z_1, y_2z_2]\, y_3 - y_3\, [z_1y_1, z_2y_2] \equiv 0, \tag{3.1}$$
$$[z_1y_1, z_2y_2]\, z_3 - z_3\, [y_1z_1, y_2z_2] \equiv 0. \tag{3.2}$$

Proof. We prove the first one, and the other can be proved in a similar way. First we observe that two elements of the component of degree 1 and -1 satisfy the following

$$ae_{ij}be_{ji} = abe_{ii} = (-1)^{\deg a \deg b} bae_{ii} \tag{3.3}$$

where e_{ij} is a matrix unit of $M_2(F)$, $a, b \in E$ and where $\deg a = 0$ if $a \in E_0$ and $\deg a = 1$ if $a \in E_1$. We remark also that (3.1) is a multilinear polynomial, hence is sufficient prove that (3.1) is equal to zero for elements ae_{ij} such that $a \in E_l$, $l = 0, 1$. Now we use the convention $(-1)^{\deg a} = (-1)^a$ and we consider the following substitution

$$y_1 = ae_{12}, \quad z_1 = be_{21}, \quad y_2 = ce_{12}, \quad z_2 = de_{21}, \quad y_3 = fe_{12}.$$

It is easy to check that the polynomial (3.1) takes the value $\gamma(abcdf)e_{12}$, where

$$\gamma = \left[1 - (-1)^{(a+b)(c+d)}\right] \left[1 - (-1)^{f(a+b+c+d)+ab+cd}\right].$$

Now $\gamma = 0$ for all the 2^5 possibility of choice of $a, b, c, d, f \in E_l$, $l = 0, 1$. □

We now give a polynomial identity which is a consequence of (3.1)

Lemma 3.1.3 *For any $i, j \in \mathbb{N}$*

$$\bar{y}_1\tilde{z}_i y_1 \tilde{z}_j \bar{y}_2 \equiv 0$$

$$\tilde{z}_1\bar{y}_i z_1 \bar{y}_j \tilde{z}_2 \equiv 0.$$

are polynomial identities for $M_2(E)^{\mathbb{Z}}$.

Proof.

$$\bar{y}_1\tilde{z}_i y_1 \tilde{z}_j \bar{y}_2 = y_1\tilde{z}_i y_1 \tilde{z}_j y_2 - y_2\tilde{z}_i y_1 \tilde{z}_j y_1.$$

We remark that $\tilde{z}_i y_1 \tilde{z}_j y_1 = [z_i y_1, z_j y_1]$, hence by lemma 3.1.2, $\bar{y}_1\tilde{z}_i y_1 \tilde{z}_j \bar{y}_2$ is a polynomial identity. The second one can be proved in the same way.

□

3.2 On the $\mathbb{Z}$-graded cocharacter of the matrix algebra $M_2(E)$

In order to describe the decomposition of the nth cocharacter of the homogeneous component of degree 0,1,−1 of the $\mathbb{Z}$-graded algebra, denoted by $A^{(0)}$, $A^{(1)}$ and $A^{(-1)}$ respectively, we shall use the representation theory of $GL_2 \times GL_2 \times GL_2$.

Let λ be a partition of the integer r, $\lambda \vdash r$, μ a partition of the integer s, $\mu \vdash s$ and ν a partition of the integer $n-r-s$.

Let also F_m^n be the space of all homogeneous polynomials of degree n in the variables $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_m, z_1, z_2, \cdots, z_m$. Then

$$F_m^n(A) = \frac{F_m^n}{(F_m^n \cap \mathrm{Id}^{gr}(A))}$$

is a $GL_m \times GL_m \times GL_m$-submodule of $F_m(A)$.

It is well know (see, for instance [15], Theorem 12.4.12) that any irreducible submodule of $F_m^n(A)$ corresponding to the 3-tuple (λ, μ, ν) is cyclic and is generated by a non-zero polynomial $f_{\lambda,\mu,\nu}$, called highest weight vector, of the form:

$$f_{\lambda,\mu,\nu}(x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_m, z_1, z_2, \cdots, z_m) = \prod_{i=1}^{\lambda_1} \mathrm{St}_{h_i(\lambda)}(x_1, x_2, \cdots, x_{h_i(\lambda)})$$

$$\prod_{i=1}^{\mu_1} \mathrm{St}_{h_i(\mu)}(y_1, y_2, \cdots, y_{h_i(\mu)}) \prod_{i=1}^{\nu_1} \mathrm{St}_{h_i(\nu)}(z_1, z_2, \cdots, z_{h_i(\nu)}) \sum_{\sigma \in S_n} \alpha_\sigma \sigma \qquad (3.4)$$

where $\alpha_\sigma \in F$, the right action of S_n on $F_m^n(A)$ is defined by place permutation, $h_i(\lambda)$(respectively $h_i(\mu)$, $h_i(\nu)$) is the height of the ith column of the diagram D_λ (respectively D_μ, D_ν) and $\mathrm{St}_r(x_1, x_2, \cdots, x_r)$ is the standard polynomial of degree r.

If $\mu = \nu = \emptyset$ then the highest weight vector corresponding to the 3-tuple $(\lambda, \emptyset, \emptyset)$, denoted by f_λ, is the polynomial

$$f_\lambda = f_{\lambda,\emptyset,\emptyset} = \prod_{i=1}^{\lambda_1} \mathrm{St}_{h_i(\lambda)}(x_1, x_2, \cdots, x_{x_{h_i(\lambda)}}) \sum_{\sigma \in S_n} \alpha_\sigma \sigma.$$

Similarly we define f_μ and f_ν, the highest weight vectors corresponding to the 3-tuple $(\emptyset, \mu, \emptyset)$ and $(\emptyset, \emptyset, \nu)$ respectively.

Let T_λ, T_μ and T_ν be three Young tableaux. We denote by $f_{T_\lambda,T_\mu,T_\nu}$ the highest weight vector obtained from (3.4) by considering the only permutation $\tau \in S_n$ such that the integers $\tau(1), \tau(2), \cdots, \tau(h_1(\lambda))$, in this order, fill in from top to bottom the first column of T_λ, $\tau(h_1(\lambda)+1), \tau(h_1(\lambda)+2), \cdots, \tau(h_1(\lambda)+h_2(\lambda))$ the second column of T_λ, $\cdots$, $\tau(h_1(\lambda)+\cdots+h_{\lambda_1-1}(\lambda)+1), \tau(h_1(\lambda)+\cdots+h_{\lambda_1-1}(\lambda)+2), \cdots, \tau(r)$ the last column of T_λ; $\tau(r+1), \tau(r+2), \cdots, \tau(r+h_1(\mu))$, in this order, fill in from top to bottom the first column of T_μ,$\cdots$, $\tau(r+h_1(\mu)+\cdots+h_{\mu_1-1}(\mu)+1), \tau(r+h_1(\mu)+\cdots+h_{\mu_1-1}(\mu)+2), \cdots, \tau(s)$ the last column of T_μ; finally $\tau(s+1), \tau(s+2), \cdots, \tau(s+h_1(\nu))$, in this order, fill in from top to bottom the first column of T_ν,$\cdots$, $\tau(s+h_1(\nu)+\cdots+h_{\nu_1-1}(\nu)+1), \tau(s+h_1(\nu)+\cdots+h_{\nu_1-1}(\mu)+2), \cdots, \tau(n)$ the last column of T_ν. As above we also define f_{T_λ}, f_{T_μ} and f_{T_ν}.

We denote by $T(\lambda, \mu, \nu)$ the set of all 3-tuple $(T_\lambda, T_\mu, T_\nu)$ of standard Young tableaux and by $d_{\lambda,\mu,\nu}$ its cardinality. If d_λ (respectively d_μ, d_ν) denotes the number of standard λ-tableaux given by the hook formula [36] (respectively μ and ν standard tableaux) then

$$d_{\lambda,\mu,\nu} = d_\lambda d_\mu d_\nu.$$

We have the following result

Proposition 3.2.1 ([16], Proposition 1) *Let $\lambda \vdash r$, $\mu \vdash s$, $\nu \vdash n-r-s$. Any highest weight vector $f_{\lambda,\mu,\nu}$ can be expressed uniquely as a linear combination of the polynomials $f_{T_\lambda,T_\mu,T_\nu}$ with T_λ,T_μ and T_ν standard tableaux.*

We start now computing the nth cocharacter of the homogeneous component of degree 0, 1 and -1 of the $\mathbb{Z}$-graded algebra $A = M_2(E)$ denoted by $A^{(0)}$,$A^{(1)}$ and $A^{(-1)}$ respectively.

Let the following be the decomposition into irreducibles of the nth cocharacter of $A^{(0)}$:

$$\chi_{(n,0,0)} = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$$

where χ_λ is the irreducible S_n-cocharacter associated to the partition $\lambda \vdash n$ and $m_\lambda \geq 0$ is the corresponding multiplicity. By lemma 3.1.1 and by theorem 4.1.8 of [22] the following theorem holds.

Theorem 3.2.2 *The cocharacter sequence of $A^{(0)}$ is*

$$\chi_{(n,0,0)}(M_2^{\mathbb{Z}}(E)) = \sum_{\substack{\lambda \vdash n \\ \lambda \subset H(1,1)}} \chi_\lambda.$$

Consider now $\chi_{(0,n,0)} = \chi_{S_n}\left(\dfrac{P_{0,n,0}}{P_{0,n,0} \cap \mathrm{Id}^{\mathbb{Z}}(M_2(E))}\right)$ and let

$$\chi_{(0,n,0)}(M_2^{\mathbb{Z}}(E)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$$

its decomposition into irreducibles.

Since lemma 3.1.1 all multilinear polynomials $f \in P_{0,n,0} = \text{span}\left\{y_{\sigma(1)}y_{\sigma(2)}\cdots y_{\sigma(n)} | \sigma \in S_n\right\}$ of degree greater or equals 2 vanish in $A^{(1)}$, we have the following result:

Theorem 3.2.3

$$\chi_{(0,n,0)}(M_2^{\mathbb{Z}}(E)) = \begin{cases} \chi_{(1)} & \textit{if} \quad n = 1 \\ 0 & \textit{if} \quad n \geq 2. \end{cases}$$

Applying the same arguments, we have the following result for $\chi_{(0,0,n)}(M_2^{\mathbb{Z}}(E)) = \chi_{S_n}\left(\dfrac{P_{0,0,n}}{P_{0,0,n} \cap \mathrm{Id}^{\mathbb{Z}}(M_2(E))}\right)$

Theorem 3.2.4

$$\chi_{(0,0,n)}(M_2^{\mathbb{Z}}(E)) = \begin{cases} \chi_{(1)} & \textit{if} \quad n = 1 \\ 0 & \textit{if} \quad n \geq 2. \end{cases}$$

Now, given any $r, s \in \{0, 1, \cdots n\}$ such that $r + s = n$, consider the space $P_{0,r,s}$ of multilinear polynomials in r variables of homogeneous degree 1 and s variables of homogenous degree -1.

The group $S_r \times S_s$ acts on $P_{0,r,s}$ by permuting the variables of homogeneous degree 1 and -1 separately. This action preserves $P_{0,r,s} \cap \mathrm{Id}^{\mathbb{Z}}(M_2(E))$, the $\mathbb{Z}$-graded identities of $M_2(E)$ lying in $P_{0,r,s}$. Then the $S_r \times S_s$-character of the quotient space $\dfrac{P_{0,r,s}}{P_{0,r,s} \cap \mathrm{Id}^{\mathbb{Z}}(M_2(E))}$ is called the *graded cocharacter* of $M_2(E)$.

Recall that there is a one-to-one correspondence between irreducible $S_r \times S_s$-characaters and the set of pair of partitions $\{(\lambda, \mu) | \lambda \vdash r, \mu \vdash s\}$. We denote by $\chi_\lambda \otimes \chi_\mu$ the $S_r \times S_s$-character corresponding to (λ, μ) . We also set $m_{\lambda,\mu} = m_{\emptyset,\lambda,\mu}$

We start the computing of the $\mathbb{Z}$-graded character by the following lemma.

Lemma 3.2.5 *Let*

$$\chi_{(0,r,s)}(M_2^{\mathbb{Z}}(E)) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash s}} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu$$

be the decomposition into irreducibles of the nth graded cocharacter of $M_2(E)$. If $r \notin \{s, s-1, s+1\}$ then $m_{\lambda,\mu} = 0$.

Proof. Let

$$\chi_{(0,r,s)}(M_2^{\mathbb{Z}}(E)) = \chi_{S_r \times S_s} \left(\frac{P_{0,r,s}}{P_{0,r,s} \cap \mathrm{Id}^{\mathbb{Z}}(M_2(E))} \right)$$

$$= \sum_{\substack{\lambda \vdash r \\ \mu \vdash s}} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu.$$

Since $y_1 y_2 \equiv 0$ and $z_1 z_2 \equiv 0$ are $\mathbb{Z}$-graded identities of $M_2(E)$, it is clear that, if $r \neq s$ or $r \neq s-1$ or $r \neq s+1$, all polynomials $f \in P_{0,r,s}$ vanish in $M_2^{\mathbb{Z}}(E)$. Then $m_{\lambda,\mu} = 0$ for all $\lambda \vdash r$, $\mu \vdash s$. Hence it follows that $\chi_{0,r,s}(M_2^{\mathbb{Z}}(E)) = 0$ if $r \neq s$ or $r \neq s-1$ or $r \neq s+1$. Therefore the proof is complete. □

The previous lemma gives a restriction on the number of box of the diagrams D_λ and D_μ, instead the following, which is a consequences of Theorem 4.1.8 [22], gives a restriction on the shape of the two diagrams.

Lemma 3.2.6 *Let*

$$\chi_{(0,r,s)}(M_2^{\mathbb{Z}}(E)) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash s}} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu$$

be the decomposition into irreducibles of the nth graded cocharacter of $M_2(E)$. If $\lambda, \mu \not\subset H(1,1)$ then $m_{\lambda,\mu} = 0$.

At the light of the previous lemmas, we have to examine the following cases:

- $\lambda = (r-t, 1^{t-1}), \mu = (s)$ or $\lambda = (r), \mu = (s-t', 1^{t'-1})$, with $1 \leq t \leq r$ and $1 \leq t' \leq s$,
- $\lambda = (r-t, 1^{t-1}), \quad \mu = (s-t', 1^{t'-1})$, with $1 \leq t \leq r$, $1 \leq t' \leq s$ and $t = t'$,
- $\lambda = (r-t, 1^{t-1}), \quad \mu = (s-t', 1^{t'-1})$, with $1 \leq t \leq r$, $1 \leq t' \leq s$ and $t \neq t'$

3.3 $h_1(\lambda) = 1$ or $h_1(\mu) = 1$

In this section we focus our attention on the pair of Young tableaux such that $h_1(\lambda) = 1$ or $h_1(\mu) = 1$. Given a pair of Young diagrams, we can obtain a pair of standard Young tableaux by filling D_λ with odd numbers and D_μ with even numbers or by filling D_λ with even numbers and D_μ with odd numbers. Throughout this chapter we consider, without loss of generality, the first case, moreover we set $h_1(\lambda) = n$ and $h_(\mu) = m$. We also set

$$f_{T_\lambda, T_\mu} = f(y,z) = \bar{y}_1 z_1 \bar{y}_2 z_1 \ldots z_1 \bar{y}_n z_1 y_1 z_1 \cdots$$

that is the highest weight vector corresponding to the pair of Young tableau obtained from D_λ and D_μ, filling the first column of D_λ with the odd numbers $1, 3, \cdots, 2h_1(\lambda) - 1$. We start with the following

Lemma 3.3.1 *Let $k \geq 1$ and let*

$$f_k = f_k(y,z) = \bar{y}_1 z_1 (y_1 z_1)^k \bar{y}_2 z_1 y_1 \cdots = \bar{y}_1 z_1 \underbrace{y_1 z_1 \ldots y_1 z_1}_{k} \bar{y}_2 z_1 y_1 \cdots,$$

then

$$f_k \equiv (k+1) f_{T_\lambda, T_\mu}$$

Proof. We prove the lemma by induction on k. If $k = 1$ we have

$$f_1 = \bar{y}_1 z_1 y_1 z_1 \bar{y}_2 z_1 y_1 \ldots.$$

Now, by (3.1)

$$[y_1 z_1, y_2 z_1]\, y_1 - y_1\, [z_1 y_1, z_1 y_2] \equiv 0$$

follows that

$$y_1 z_1 y_1 z_1 y_2 \equiv 2 y_1 z_1 y_2 z_1 y_1 - y_2 z_1 y_1 z_1 y_1. \tag{3.5}$$

Hence

$$\begin{aligned}\bar{y}_1 z_1 y_1 z_1 \bar{y}_2 \ldots &= y_1 z_1 y_1 z_1 y_2 \cdots - y_2 z_1 y_1 z_1 y_1 \ldots \\ &\equiv 2 y_2 z_1 y_1 z_1 y_1 \cdots - 2 y_1 z_1 y_2 z_1 y_1 \ldots \\ &\equiv 2 \bar{y}_1 z_1 \bar{y}_2 z_1 y_1 \ldots .\end{aligned}$$

Now suppose the lemma true for $i < k$ and we prove it for k. By (3.5) we have

$$f_k \equiv 2f_{k-1} - f_{k-2} \equiv [2k - (k-1)] f_{T_\lambda, T_\mu} \equiv (k+1) f_{T_\lambda, T_\mu}.$$

□

Lemma 3.3.2

$$\bar{y}_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_{2k-1} z_1 \bar{y}_{2k} z_1 y_1 \cdots$$
$$\equiv \sum_{\sigma \in S_{2k}} \alpha_{\sigma,k} \bar{y}_{\sigma(1)} z_1 \bar{y}_{\sigma(2)} z_1 \tilde{y}_{\sigma(3)} z_1 \tilde{y}_{\sigma(4)} z_1 \cdots \dot{y}_{\sigma(2k-1)} z_1 \dot{y}_{\sigma(2k)} z_1 y_1 \cdots$$

where σ is a permutation of S_{2k} such that $\sigma(1) = 1$, $\sigma(2i-1) < \sigma(2i)$, t, s are odd numbers such that $\sigma(2i+t) < \sigma(2i+s)$ where $1 \le i \le k-1$, $0 < t < s$ and

$$\alpha_{\sigma,k} = (-1)^\sigma \alpha_k$$

with

$$\alpha_k = \begin{cases} 1 & \textit{if } k = 1 \\ k\alpha_{k-1} & \textit{if } k > 1. \end{cases}$$

Proof. We prove the lemma by induction on k. If $k = 1$ the statement is true. Suppose now the lemma true for k and we prove the thesis for $k+1$.

We can write

$$\bar{y}_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_{2k-1} z_1 \bar{y}_{2k} z_1 \bar{y}_{2k+1} z_1 \bar{y}_{2k+2} z_1 y_1 \cdots$$
$$= \sum_{\substack{i,j=1 \\ i \neq j}}^{2k+2} (-1)^\sigma y_{\sigma(i)} z_1 y_{\sigma(j)} z_1 \bar{y}_{\sigma(l_1)} z_1 \bar{y}_{\sigma(l_1)} z_1 \cdots \bar{y}_{\sigma(l_{2k-1})} z_1 \bar{y}_{\sigma(l_{2k})} z_1 y_1 \cdots \quad (3.6)$$

where $\{l_1, l_2, \cdots l_{2k}\} = \{1, 2, \cdots, i-1, i+1, \cdots, j-1, j+1, \cdots, 2k+2\}$ and such that $\sigma(l_s) < \sigma(l_{s+1})$ for all $1 \le s \le 2k-1$. We note that for all pair i, j we find the polynomials

$$y_{\sigma(i)} z_1 y_{\sigma(j)} z_1 \bar{y}_{\sigma(l_1)} z_1 \bar{y}_{\sigma(l_1)} z_1 \cdots \bar{y}_{\sigma(l_{2k-1})} z_1 \bar{y}_{\sigma(l_{2k})} z_1 y_1 \cdots$$

and

$$y_{\sigma(j)} z_1 y_{\sigma(i)} z_1 \bar{y}_{\sigma(l_1)} z_1 \bar{y}_{\sigma(l_1)} z_1 \cdots \bar{y}_{\sigma(l_{2k-1})} z_1 \bar{y}_{\sigma(l_{2k})} z_1 y_1 \cdots$$

with the opposite sign, then we can write (3.6) as

$$\sum_{\substack{i,j=1\\ \sigma(i)<\sigma(j)\\ \sigma\in S_{2k}}}^{2k+2} (-1)^{\sigma}\tilde{y}_{\sigma(i)}z_1\tilde{y}_{\sigma(j)}z_1\bar{y}_{\sigma(l_1)}z_1\bar{y}_{\sigma(l_1)}z_1\cdots\bar{y}_{\sigma(l_{2k-1})}z_1\bar{y}_{\sigma(l_{2k})}z_1y_1\cdots. \tag{3.7}$$

By induction on k we have that, modulo the identities, (3.7) is equivalent to

$$\sum_{\substack{i,j=1\\ \sigma(i)<\sigma(j)\\ \sigma\in S_{2k+2}}}^{2k+2} \sum_{\substack{\tau\in S_{2k+2}\\ \tau(\sigma(i))=\sigma(i)\\ \tau(\sigma(j))=\sigma(j)}} (-1)^{\sigma}\alpha_{\tau,k}\tilde{y}_{\tau(\sigma(i))}z_1\tilde{y}_{\tau(\sigma(j))}z_1\bar{y}_{\tau(l_1)}z_1\bar{y}_{\tau(l_2)}z_1 \\ \dot{y}_{\tau(l_3)}z_1\dot{y}_{\tau(l_4)}z_1\cdots\acute{y}_{\sigma(l_{2k-1})}z_1\acute{y}_{\sigma(l_{2k})}z_1y_1\cdots, \tag{3.8}$$

where τ is a permutation of S_{2k+2} such that $\tau(l_1)=l_1$, $\tau(l_{2r-1})<\tau(l_{2r})$, $\tau(l_{2r+t})<\tau(l_{2r+s})$ where $k\geq 2$, t,s are odd numbers such that $0<t<s$ and $\alpha_{\tau,k}=(-1)^{\tau}\alpha_k$ with

$$\alpha_k=\begin{cases}1 & \text{if } k=1\\ \\ k\alpha_{k-1} & \text{if } k>1.\end{cases}$$

By

$$\bar{y}_iz_1\bar{y}_jz_1\tilde{y}_kz_1\tilde{y}_l\equiv\tilde{y}_kz_1\tilde{y}_lz_1\bar{y}_iz_1\bar{y}_j,$$

for all the pairs $\sigma(i)<\sigma(j)$ we find in (3.8) $k+1$ equivalent polynomials to the following

$$\bar{y}_1z_1\bar{y}_pz_1\tilde{y}_{q_1}z_1\tilde{y}_{q_2}z_1\cdots\dot{y}_{q_r}z_1\dot{y}_{q_{r+1}}z_1\acute{y}_{\sigma(i)}z_1\acute{y}_{\sigma(j)}z_1$$

$$\check{y}_{q_{r+2}}z_1\check{y}q_{r+3}z_1\cdots\bar{\bar{y}}_{q_{2k-3}}z_1\bar{\bar{y}}_{q_{2k-2}}z_1y_1\cdots,$$

hence (3.8) is equivalent to

$$\sum_{\rho\in S_{2k+2}}\alpha_{\rho,k+1}\bar{y}_{\rho(1)}z_1\bar{y}_{\rho(2)}z_1\tilde{y}_{\rho(3)}z_1\tilde{y}_{\rho(4)}z_1\cdots\dot{y}_{\rho(2k+1)}z_1\dot{y}_{\rho(2k+2)}z_1y_1\cdots$$

where ρ is a permutation S_{2k+2} such that $\rho(1)=1$, $\rho(2i-1)<\rho(2i)$, $\rho(2i+t)<\rho(2i+s)$ where $1\leq i\leq k$, t,s are odd numbers such that $0<t<s$ and $\alpha_{\rho,k+1}=(-1)^{\rho}(k+1)\alpha_k$. □

Corollary 3.3.3

$$\bar{y_1}z_1\bar{y_2}z_1\cdots\bar{y}_{2k-1}z_1\bar{y}_{2k+1}z_1y_1\cdots$$

$$\equiv \sum_{\substack{j=1 \\ \sigma\in S_{2k+1}}}^{2k+1} \alpha_{\sigma,k} y_{\sigma(j)} z_1 \bar{y}_{\sigma(l_1)} z_1 \bar{y}_{\sigma(l_2)} \tilde{y}_{\sigma(l_3)} z_1 \tilde{y}_{\sigma(l_4)} z_1 \cdots \dot{y}_{\sigma(l_{2k-1})} z_1 \dot{y}_{\sigma(l_{2k})} z_1 y_1 \cdots$$

where $\{l_1\, l_2, \cdots, l_{2k}\} = \{1, 2, \cdots, j-1, j+1, \cdots, 2k+1\}$ *and* $\sigma(2i-1) < \sigma(2i)$, $\sigma(l_{2i+t}) < \sigma(l_{2i+s})$ *where* $1 \le i \le k$, t, s *are odd numbers such that* $0 < t < s$ *and*

$$\alpha_{\sigma,k} = (-1)^{\sigma} \alpha_k$$

with

$$\alpha_k = \begin{cases} 1 & \text{if } k = 1 \\ \\ k\alpha_{k-1} & \text{if } k > 1. \end{cases}$$

Proof. It is sufficient explicit the first alternating variable and by the previous lemma, the corollary holds. □

Lemma 3.3.4

$$y_1 z_1 \bar{y}_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_n \equiv \bar{y}_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_n z_1 y_1 z_1 y_1 \cdots$$

Proof. We first consider the case $n = 2k$. Then by the previous lemma it is possible write the polynomial

$$y_1 z_1 \bar{y}_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_{2k-1} z_1 \bar{y}_{2k} z_1 y_1 \cdots$$

$$\equiv \sum_{\sigma\in S_{2k}} \alpha_{\sigma,k} y_1 z_1 \bar{y}_{\sigma(1)} z_1 \bar{y}_{\sigma(2)} z_1 \tilde{y}_{\sigma(3)} z_1 \tilde{y}_{\sigma(4)} z_1 \cdots \dot{y}_{\sigma(2k-1)} z_1 \dot{y}_{\sigma(2k)} z_1 y_1 \cdots \tag{3.9}$$

where σ is a permutation of S_{2k} such that $\sigma(1) = 1$, $\sigma(2i-1) < \sigma(2i)$, $\sigma(2i+t) < \sigma(2i+s)$ where $1 \le i \le k-1$,and t, s are odd numbers such that $0 < t < s$ and

$$\alpha_{\sigma,k} = (-1)^{\sigma} \alpha_k$$

with

$$\alpha_k = \begin{cases} 1 & \text{if } k = 1 \\ \\ k\alpha_{k-1} & \text{if } k > 1. \end{cases}$$

According to

$$\bar{y}_i z_1 \bar{y}_j z_1 y_k \equiv y_k z_1 \bar{y}_i z_1 \bar{y}_j \tag{3.10}$$

(3.9) is equivalent to

$$\sum_{\sigma\in S_{2k}} \alpha_{\sigma,k}\bar{y}_{\sigma(1)}z_1\bar{y}_{\sigma(2)}z_1\tilde{y}_{\sigma(3)}z_1\tilde{y}_{\sigma(4)}z_1\cdots\dot{y}_{\sigma(2k-1)}z_1\dot{y}_{\sigma 2k}z_1y_1z_1y_1\cdots$$

$$\equiv \bar{y_1}z_1\bar{y_2}z_1\cdots\bar{y_{2k}}z_1y_1z_1y_1\cdots .$$

Now let be $n = 2k+1$, if we explicit the first alternating variable we have

$$y_1z_1\bar{y_1}z_1\bar{y_2}z_1\cdots\bar{y_{2k+1}}z_1y_1\cdots$$

$$= \sum_{\substack{i=1\\ \sigma\in S_{2k+1}}}^{2k+1} (-1)^{\sigma}y_1z_1y_{\sigma(i)}z_1\bar{y}_{\sigma(l_1)}z_1\bar{y}_{\sigma(l_2)}z_1\cdots\bar{y}_{\sigma(l_{2k-1})}z_1\bar{y}_{\sigma(l_{2k})}z_1y_1\cdots \quad (3.11)$$

where $\{l_1, l_2, \cdots, l_{2k}\} = \{1, 2, \cdots, i-1, i+1, \cdots, 2k\}$ and such that $\sigma(l_s) < \sigma(l_{s+1})$ for $1 \le s \le 2k-1$ By the previous lemma, (3.11) is equivalent to

$$\sum_{\substack{\sigma\in S_{2k+1}\\ \tau\in S_{2k}}}\sum_{i=1}^{2k+1}(-1)^{\sigma}\alpha_{\tau,k}y_1z_1y_{\sigma(i)}z_1\bar{y}_{\tau(\sigma(l_1))}z_1\bar{y}_{\tau(\sigma(l_2))}$$
$$z_1\tilde{y}_{\tau(\sigma(l_3))}z_1\tilde{y}_{\tau(\sigma(l_4))}z_1\cdots\dot{y}_{\tau(\sigma(l_{2k-1}))}z_1\dot{y}_{\tau(\sigma(l_{2k}))}z_1y_1\cdots$$

$$= \sum_{\substack{\tau\in S_{2k}\\ \sigma\in S_{2k+1}}} (-1)^{\sigma}\alpha_{\tau,k}y_1z_1y_1z_1\bar{y}_{\tau(\sigma(l_1))}z_1\bar{y}_{\tau(\sigma(l_2))}$$
$$z_1\tilde{y}_{\tau(\sigma(l_3))}z_1\tilde{y}_{\tau(\sigma(l_4))}z_1\cdots\dot{y}_{\tau(\sigma(l_{2k-1}))}z_1\dot{y}_{\tau(\sigma(l_{2k}))}z_1y_1\cdots$$
$$+\sum_{\substack{\sigma\in S_{2k+1}\\ \tau\in S_{2k}}}\sum_{i=2}^{2k+1}(-1)^{\sigma}\alpha_{\tau,k}y_1z_1y_{\sigma(i)}z_1\bar{y}_{\tau(\sigma(l_1))}z_1\bar{y}_{\tau(\sigma(l_2))}$$
$$z_1\tilde{y}_{\tau(\sigma(l_3))}z_1\tilde{y}_{\tau(\sigma(l_4))}z_1\cdots\dot{y}_{\tau(\sigma(l_{2k-1}))}z_1\dot{y}_{\tau(\sigma(l_{2k}))}z_1y_1\cdots \quad (3.12)$$

where $\sigma(l_{2i-1}) < \sigma(l_{2i})$, $\sigma(2i+t) < \sigma(2i+s)$ where $1 \le i \le k-1$,and t, s are odd numbers such that $0 < t < s$.

As in the previous case the first sumand of (3.12) is equivalent to

$$\sum_{\substack{\tau\in S_{2k}\\ \sigma\in S_{2k+1}}} (-1)^{\sigma}y_1z_1\bar{y}_{\tau(\sigma(l_1))}z_1\bar{y}_{\tau(\sigma(l_2))}z_1\tilde{y}_{\tau(\sigma(l_3))}$$
$$z_1\tilde{y}_{\tau(\sigma(l_4))}z_1\cdots\dot{y}_{\tau(\sigma(l_{2k-1}))}z_1\dot{y}_{\tau(\sigma(l_{2k}))}y_1z_1z_1y_1\cdots . \quad (3.13)$$

We note that the second summand of (3.12) by

$$y_i z_1 y_j z_1 = \bar{y}_i z_1 \bar{y}_j z_1 + y_j z_1 y_i z_1$$

is equivalent to

$$\begin{aligned}
&\sum_{\substack{\tau\in S_{2k}\\ \sigma\in S_{2k+1}}} \sum_{\sigma(i)=2}^{2k+1} (-1)^{\sigma} \alpha_{\tau,k} \acute{y}_1 z_1 \acute{y}_{\sigma(i)} z_1 \bar{y}_{\tau(\sigma(l_1))} z_1 \bar{y}_{\tau(\sigma(l_2))} z_1 \\
&\qquad \tilde{y}_{\tau(\sigma(l_3))} z_1 \tilde{y}_{\tau(\sigma(l_4))} z_1 \cdots \dot{y}_{\tau(\sigma(l_{2k-1}))} z_1 \dot{y}_{\tau(\sigma(l_{2k}))} z_1 y_1 \cdots \\
&+ \sum_{\substack{\tau\in S_{2k}\\ \sigma\in S_{2k+1}}} \sum_{\sigma(i)=2}^{2k+1} (-1)^{\sigma} \alpha_{\tau,k} y_{\sigma(i)} z_1 y_1 z_1 \bar{y}_{\tau(\sigma(l_1))} z_1 \bar{y}_{\tau(\sigma(l_2))} z_1 \\
&\qquad \tilde{y}_{\tau(\sigma(l_3))} z_1 \tilde{y}_{\tau(\sigma(l_4))} z_1 \cdots \dot{y}_{\tau(\sigma(l_{2k-1}))} z_1 \dot{y}_{\tau(\sigma(l_{2k}))} z_1 y_1 \cdots ,
\end{aligned} \tag{3.14}$$

it is easy to check that the first summand of (3.14) is equivalent to zero and, according to (3.10), (3.14) is equivalent to

$$\begin{aligned}
&\sum_{\substack{\tau\in S_{2k}\\ \sigma\in S_{2k+1}}} \sum_{\sigma(i)=2}^{2k+1} (-1)^{\sigma} \alpha_{\tau,k} y_{\sigma(i)} z_1 \bar{y}_{\tau(\sigma(l_1))} z_1 \bar{y}_{\tau(\sigma(l_2))} z_1 \\
&\qquad \tilde{y}_{\tau(\sigma(l_3))} z_1 \tilde{y}_{\tau(\sigma(l_4))} z_1 \cdots \dot{y}_{\tau(\sigma(l_{2k-1}))} z_1 \dot{y}_{\tau(\sigma(l_{2k}))} z_1 y_1 z_1 y_1 \cdots .
\end{aligned} \tag{3.15}$$

Finally the sum of (3.13) and (3.15) is equal to

$$\sum_{\substack{\tau\in S_{2k}\\ \sigma\in S_{2k+1}}} (-1)^{\sigma} \alpha_{\tau,k} y_{\sigma(i)} z_1 \bar{y}_{\tau(\sigma(l_1))} z_1 \bar{y}_{\tau(\sigma(l_2))} z_1 \cdots \dot{y}_{\tau(\sigma(l_{2k-1}))} z_1 \dot{y}_{\tau(\sigma(l_{2k}))} z_1 y_1 z_1 y_1 \cdots ,$$

and by the previous lemma it is equivalent to

$$\bar{y}_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_{2k} z_1 \bar{y}_{2k+1} z_1 y_1 \cdots .$$

□

Lemma 3.3.5

$$\bar{y}_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_{n-1} z_1 y_1 z_1 \bar{y}_n z_1 y_1 \cdots \equiv \beta \bar{y}_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_{n-1} z_1 \bar{y}_n z_1 y_1 z_1 y_1 \cdots$$

where

$$\beta = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ \dfrac{\alpha_{\sigma,k} + \alpha_{\tau,k-1}}{\alpha_{\sigma,k}} & \text{if } k \text{ is even} \end{cases}$$

with $\sigma \in S_{2k}$, $\tau \in S_{2k-1}$ *and* $\alpha_{\sigma,k}$ *is defined as in the Lemma 3.3.2.*

Proof. If $n = 2k+1$ and if we explicit the last alternating variable and we apply lemma 3.3.2 and lemma 3.3.4, the lemma holds.

Now let be $n = 2k$, then if we explicit the last alternate variable we have

$$\bar{y}_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_{2k-1} z_1 y_1 z_1 \bar{y}_{2k} z_1 y_1 \cdots$$

$$\equiv \sum_{\sigma \in S_{2k}} (-1)^\sigma \bar{y}_{\sigma(l_1)} z_1 \bar{y}_{\sigma(l_2)} z_1 \ldots \bar{y}_{\sigma(l_{2k-1})} z_1 y_1 z_1 y_{\sigma(i)} z_1 y_1 \cdots$$

where $\{l_1, l_2 \cdots l_{2k-1}\} = \{1, 2, \cdots i-1, i+1, \cdots, 2k\}$ and $\sigma(l_s) < \sigma(l_{s+1})$ with $1 \le s \le 2k-1$.

Similarly to lemma 3.3.4 we can write the previous as

$$-\bar{y}_2 z_1 \bar{y}_3 z_1 \cdots \bar{y}_{2k-1} z_1 y_1 z_1 y_1 z_1 y_1 \cdots \tag{3.16}$$

$$+ \sum_{\sigma(i)=2}^{2k} \bar{y}_{\sigma(l_1)} z_1 \bar{y}_{\sigma(l_2)} z_1 \cdots \bar{y}_{\sigma(l_{2k-1})} z_1 y_1 z_1 y_{\sigma(i)} z_1 y_1 \cdots .$$

For the previous lemma, the second summand of (3.16) is equivalent to

$$\sum_{\substack{\sigma(i)=2 \\ \sigma \in S_{2k}}}^{2k} (-1)^\sigma y_1 z_1 \bar{y}_{\sigma(l_1)} z_1 \bar{y}_{\sigma(l_2)} z_1 \cdots \bar{y}_{\sigma(l_{2k-1})} z_1 y_{\sigma(i)} z_1 y_1 \cdots . \tag{3.17}$$

We remark that

$$\bar{y}_2 z_1 \bar{y}_3 z_1 \cdots \bar{y}_{2k} z_1 y_1 z_1 y_1 z_1 y_1 \cdots \equiv y_1 z_1 \bar{y}_2 z_1 \bar{y}_3 z_1 \cdots \bar{y}_{2k} z_1 y_1 z_1 y_1 \cdots \tag{3.18}$$

$$+\gamma y_1 z_1 \bar{y}_1 z_1 \bar{y}_2 z_1 \bar{y}_3 z_1 \cdots \bar{y}_{2k} z_1 y_1 \cdots$$

for some scalar γ, in fact by lemma 3.3.2 we have

$$\bar{y}_2 z_1 \bar{y}_3 z_1 \cdots \bar{y}_{2k-1} z_1 y_1 z_1 y_1 z_1 y_1 \cdots$$

$$\equiv \sum_{\substack{j=2 \\ \tau \in S_{2k-1}}}^{2k} \alpha_{\tau,k-1} y_{\tau(j)} z_1 \bar{y}_{\tau(l_1)} z_1 \bar{y}_{\tau(l_2)} z_1 \tilde{y}_{\tau(l_3)} z_1 \tilde{y}_{\tau(l_4)} \cdots \dot{y}_{\tau(l_{2k-2})}$$

$$z_1 \dot{y}_{\tau(l_{2k-1})} z_1 y_1 z_1 y_1 z_1 y_1 \cdots$$

where $\tau(l_1) = 2$, $\tau(l_{2i-1}) < \tau(l_{2i})$, $\tau(l_{2i+t}) < \tau(l_{2i+s})$ where $1 \le i \le k-1$ and t, s are odd numbers such that $0 < t < s$. According to (3.10), the previous is equivalent to

$$\sum_{\substack{j=2 \\ \tau \in S_{2k-1}}}^{2k} \alpha_{\tau,k-1} y_{\tau(j)} z_1 y_1 z_1 \bar{y}_{\tau(l_1)} z_1 \bar{y}_{\tau(l_2)} z_1 \tilde{y}_{\tau(l_3)} z_1 \tilde{y}_{\tau(l_4)} \tag{3.19}$$

$$z_1 \cdots \dot{y}_{\sigma(l_{2k-1})} z_1 \dot{y}_{\sigma(l_{2k})} z_1 y_1 z_1 y_1 \cdots$$

according to $y_i z_1 y_j z_1 \equiv \bar{y}_i z_1 \bar{y}_j z_1 + y_j z_1 y_i z_1$, (3.19) is equivalent to

$$\sum_{\substack{j=2 \\ \tau \in S_{2k-1}}}^{2k} \alpha_{\tau,k-1} y_1 z_1 y_{\tau(j)} z_1 \bar{y}_{\tau(l_1)} z_1 \bar{y}_{\tau(l_2)} z_1 \tilde{y}_{\tau(l_3)} z_1 \tilde{y}_{\tau(l_4)} \tag{3.20}$$

$$z_1 \cdots \dot{y}_{\sigma(l_{2k-1})} z_1 \dot{y}_{\sigma(l_{2k})} z_1 y_1 z_1 y_1 \cdots$$

$$- \sum_{\substack{j=2 \\ \tau \in S_{2k-1}}}^{2k} \alpha_{\tau,k-1} \acute{y}_1 z_1 \acute{y}_{\tau(j)} z_1 \bar{y}_{\tau(l_1)} z_1 \bar{y}_{\tau(l_2)} z_1 \tilde{y}_{\tau(l_3)} z_1 \tilde{y}_{\tau(l_4)}$$

$$z_1 \cdots \dot{y}_{\sigma(l_{2k-1})} z_1 \dot{y}_{\sigma(l_{2k})} z_1 y_1 z_1 y_1 \cdots .$$

Now, the first summand of (3.19) is equivalent to

$$y_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_{2k} z_1 y_1 z_1 y_1 \cdots ,$$

the second one, by (3.10) is equivalent to

$$-\frac{\alpha_{\tau,k-1}}{\alpha_{\sigma,k}} \; y_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_{2k} z_1 y_1 z_1 y_1 \cdots ,$$

if $\gamma = -\dfrac{\alpha_{\tau,k-1}}{\alpha_{\sigma,k}}$ we obtain (3.18). Hence, according to (3.16) and (3.17), (3.18) is equivalent to

$$\sum_{\substack{\sigma(i)=2 \\ \sigma \in S_{2k}}}^{2k} (-1)^{\sigma} y_1 z_1 \bar{y}_{\sigma(l_1)} z_1 \bar{y}_{\sigma(l_2)} z_1 \cdots \bar{y}_{\sigma(l_{2k-1})} z_1 \bar{y}_{\sigma(i)} z_1 y_1 \cdots$$

$$- y_1 z_1 \bar{y}_2 z_1 \bar{y}_3 z_1 \cdots \bar{y}_{2k} z_1 y_1 - \gamma y_1 z_1 \bar{y}_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_{2k} z_1 y_1 \cdots$$

$$\equiv (1-\gamma) y_1 z_1 \bar{y}_1 z_1 \bar{y}_2 z_1 \cdots \bar{y}_{2k} z_1 y_1 \cdots$$

if we set $\beta = 1 - \gamma$. Finally applying Lemma 3.3.4 we have the thesis. □

Theorem 3.3.6 *Let $r, s, m, n \in \mathbb{N}$, let*

$$\chi_{(0,r,s)} = \sum_{\substack{\lambda \vdash n \\ \mu \vdash m}} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu .$$

If $\lambda = (r-n+1, 1^{n-1}) \vdash r$ and $\mu = (s) \vdash s$ then

$$m_{\lambda,\mu} = \begin{cases} 2 & \text{if} \quad r = s \\ 1 & \text{if} \quad |r-s| = 1 \end{cases} .$$

Proof. It is obvious that if $h_1(\lambda) = h_1(\mu) = 1$ then $m_{\lambda,\mu} = 1$.

If $h_1(\lambda) = 2$, by the lemma 3.3.1, we can say that the corresponding highest weight vector f_{T_λ,T_μ} is up to a non zero scalar equivalent to

$$f_{T_\lambda,T_\mu} \equiv \bar{y}_1 z_1 \bar{y}_2 z_1 y_1 z_1 \dots,$$

Then if we consider the substitution $y_1 = (h_1 + g_1)e_{12}$, $z_1 = (h'_1 + g'_1)e_{21}$ and $y_2 = h_2 e_{12}$, where $h_1, h_2 \in E_1$, $g_1, g'_1 \in E_0$ and $g_1^{r+s}, {g'_1}^{r+s} \neq 0$ we have that $f_{T_\lambda,T_\mu} \not\equiv 0$.

Suppose now $h_1(\lambda) > 2$. We prove that any highest weight vector can be written, $\mod \mathrm{Id}^{\mathbb{Z}}(A)$, as a linear combination of polynomials

$$\bar{y}_1 z_1 \bar{y}_2 z_1 \dots z_1 \bar{y}_{r-1} z_1 \bar{y}_r z_1 y_1 z_1 \dots, \tag{3.21}$$

and

$$\bar{y}_1 z_1 \bar{y}_2 z_1 \dots z_1 \bar{y}_{r-1} z_1 y_1 z_1 \bar{y}_r z_1 y_1 z_1 \dots. \tag{3.22}$$

In fact we consider for any $i, j, k \in \mathbb{N}$ a polynomial

$$f = f(y, z) = \dots \bar{y}_i z_1 y_1 \dots z_1 \bar{y}_j z_1 y_1 \dots z_1 \bar{y}_k \dots. \tag{3.23}$$

We recall $\bar{y}_i z_1 \bar{y}_j z_1 = [y_1 z_1, y_j z_1]$, and by (3.1) we have $\bar{y}_i z_1 \bar{y}_j z_1 y_1 \equiv y_1 z_1 \bar{y}_i z_1 \bar{y}_j$.

By induction it is also possible to check that

$$(y_1 z_1)^t y_i \equiv t y_i (z_1 y_1)^t - (t-1)(y_1 z_1)^{t-1} y_i z_1 y_1 \qquad t > 1. \tag{3.24}$$

Now by (3.24) and (3.1) we can write (3.23) as a linear combination of

$$\bar{y}_i z_1 \bar{y}_j z_1 \bar{y}_k z_1 \dots$$

$$\bar{y}_i z_1 \bar{y}_j z_1 y_1 z_1 \bar{y}_k z_1 \dots.$$

Iterating this process we have that any highest weight vector can be written as a linear combination of (3.21) and (3.22), but, by lemma 3.3.5 the polynomials are linearly dependent, hence up to a scalar

$$f \equiv \bar{y}_1 z_1 \bar{y}_2 z_1 \dots z_1 \bar{y}_{n-1} z_1 \bar{y}_n z_1 y_1 z_1 \dots = f_{T_\lambda,T_\mu}.$$

If we consider the substitution $y_1 = (h_1 + g_1)e_{12}$, $z_1 = (h'_1 + g'_1)e_{21}$ and $y_i = h_i$ $i = 2, 3, \cdots, r$ where $h_i, h'_1 \in E_1$, $g_1, g'_1 \in E_0$ and $g_1^n \neq 0$, we have that $f_{T_\lambda,T_\mu} \not\equiv 0$. It is now obvious that if $r = s$ then there are two independent highest weight vectors, $m_{\lambda,\mu} = 2$, instead if the $|r - s| = 1$ then $m_{\lambda,\mu} = 1$.

□

Theorem 3.3.7 *Let $r, s, m, n \in \mathbb{N}$, let*

$$\chi_{(0,r,s)} = \sum_{\substack{\lambda \vdash n \\ \mu \vdash m}} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu .$$

If $\lambda = (r) \vdash r$ and $\mu = (s - m + 1, 1^{m-1}) \vdash s$ then

$$m_{\lambda,\mu} = \begin{cases} 2 & \textit{if} \quad r = s \\ 1 & \textit{if} \quad |r - s| = 1 \end{cases} .$$

3.4 $h_1(\lambda) = h_1(\mu)$

In this section we examine the pairs of Young tableaux which have the same height, $h_1(\lambda) = h_1(\mu) = n$. We set

$$f_{T_\lambda, T_\mu} = f_{T_\lambda, T_\mu}(y, z) = \bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \ldots \bar{y}_n \tilde{z}_n y_1 z_1 y_1 z_1 \ldots .$$

Lemma 3.4.1 *Let be $n \in \mathbb{N}$ and let*

$$\begin{aligned} p_i = p_i(y, z) &= (-1)^{i-1} \bar{y}_2 \tilde{z}_1 \bar{y}_3 \tilde{z}_2 \cdots \bar{y}_i \tilde{z}_{i-1} y_1 \tilde{z}_i \bar{y}_{i+1} \tilde{z}_{i+1} \cdots \bar{y}_n \tilde{z}_n y_1 z_1 \cdots \\ &= \sum_{\substack{\sigma, \tau \in S_n \\ \sigma(i) = 1}} (-1)^\sigma (-1)^\tau y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots \\ &\quad \cdots y_{\sigma(i-1)} z_{\tau(i-1)} y_{\sigma(i)} z_{\tau(i)} y_{\sigma(i+1)} z_{\tau(i+1)} \cdots y_{\sigma(n)} z_{\tau(n)} y_1 z_1 \cdots , \end{aligned}$$

then

$$f_{T_\lambda, T_\mu} = \bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \cdots \bar{y}_n \tilde{z}_n y_1 z_1 \cdots \equiv \begin{cases} n p_{k+1} & \textit{if } n = 2k+1 \\ k \left[p_k + p_{k+1} \right] & \textit{if } n = 2k \end{cases}$$

Proof. It is obvious that

$$f_{T_\lambda, T_\mu} = \sum_{i=1}^{n} p_i, \tag{3.25}$$

and we prove that

$$2p_i \equiv p_{i-t} + p_{i+t} \qquad 1 < i < n, \quad 1 < t < n - i \tag{3.26}$$

in fact, by induction, if we consider a polynomial p_i we have

$$
\begin{aligned}
p_i = &\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(i)=1}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}\\
&\qquad y_{\sigma(i)}z_{\tau(i)}y_{\sigma(i+1)}z_{\tau(i+1)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots\\
= &\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(i)=1}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i)}\\
&\qquad y_{\sigma(i+1)}z_{\tau(i-1)}y_{\sigma(i)}z_{\tau(i+1)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots\\
+ &\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(i)=1}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}\left[z_{\tau(i-1)}y_{\sigma(i)},\right.\\
&\qquad \left. z_{\tau(i)}y_{\sigma(i+1)}\right] z_{\tau(i+1)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots\\
\equiv &\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(i)=1}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i)}\\
&\qquad y_{\sigma(i+1)}z_{\tau(i-1)}y_{\sigma(i)}z_{\tau(i+1)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots\\
+ &\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(i)=1}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i+1)}\left[y_{\sigma(i)}z_{\tau(i-1)},\right.\\
&\qquad \left. y_{\sigma(i+1)}z_{\tau(i)}\right]\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots
\end{aligned}
\tag{3.27}
$$

if we set

$$
\rho_1 = \rho_{1,\sigma} = (\sigma(i)\sigma(i+1)), \qquad \rho_1' = \rho_{1,\tau} = (\tau(i-1)\tau(i)),
$$
$$
\rho_2 = \rho_{2,\sigma} = (1), \qquad \rho_2' = \rho_{2,\tau} = (\tau(i-1)\tau(i)\tau(i+1))^{-1},
$$
$$
\rho_3 = \rho_1\rho_2, \qquad \rho_3' = \rho_1'\rho_2',
$$

we can write (3.27) as

$$\sum_{\substack{\rho_1\sigma\in S_n\\ \rho_1'\tau\in S_n\\ \sigma(i)=1}} (-1)^{\rho_1}(-1)^{\rho_1'}(-1)^{\rho_1\sigma}(-1)^{\rho_1'\tau} y_{\rho_1\sigma(1)} z_{\rho_1'\tau(1)} y_{\rho_1\sigma(2)}$$

$$z_{\rho_1'\tau(2)} \cdots y_{\rho_1\sigma(n)} z_{\rho_1'\tau(n)} y_1 z_1 \cdots$$

$$\sum_{\substack{\rho_2\sigma\in S_n\\ \rho_2'\tau\in S_n\\ \sigma(i)=1}} (-1)^{\rho_2}(-1)^{\rho_2'}(-1)^{\rho_2\sigma}(-1)^{\rho_2'\tau} y_{\rho_2\sigma(1)} z_{\rho_2'\tau(1)}$$

$$y_{\rho_2\sigma(2)} z_{\rho_2'\tau(2)} \cdots y_{\rho_2\sigma(n)} z_{\rho_2'\tau(n)} y_1 z_1 \cdots$$

$$-\sum_{\substack{\rho_3\sigma\in S_n\\ \rho_3'\tau\in S_n\\ \sigma(i)=1}} (-1)^{\rho_3}(-1)^{\rho_3'}(-1)^{\rho_3\sigma}(-1)^{\rho_3'\tau} y_{\rho_3\sigma(1)} z_{\rho_3'\tau(1)} y_{\rho_3\sigma(2)}$$

$$z_{\rho_3'\tau(2)} \cdots y_{\rho_3\sigma(n)} z_{\rho_3'\tau(n)} y_1 z_1 \cdots$$

hence

$$p_i \equiv p_{i+1} + p_{i-1} - p_i, \tag{3.28}$$

and so

$$2p_i \equiv p_{i-1} + p_{i+1}.$$

Now suppose the observation true for $s < t$ and we prove it for t, in fact

$$2p_{i-(t-1)} \equiv p_{i-t} + p_i - (t-2)$$

and

$$2p_{i+(t-1)} \equiv p_{i+t} + p_{i+(t-2)},$$

hence

$$\begin{aligned} p_{i-t} + p_{i+t} &\equiv 2\left(p_{i-(t-1)} + p_{i+(t-1)}\right) - \left(p_{i-(t-2)} + p_{i+(t-2)}\right) \\ &\equiv 4p_i - 2p_i = 2p_i. \end{aligned}$$

Now if $n = 2k+1$, we write (3.25) in the following way

$$f_{T_\lambda,T_\mu} = \sum_{i=1}^{n} p_i = p_{k+1} + \sum_{t=1}^{k} (p_{k-t+1} + p_{k+t+1})$$

and by (3.26) we have

$$f_{T_\lambda,T_\mu} \equiv p_{k+1} + 2kp_{k+1} = (2k+1)p_{k+1} = np_{k+1}.$$

If $n = 2k$, we first observe that

$$p_{k-t}(\bar{y},\tilde{z}) + p_{k+t+1}(\bar{y},\tilde{z}) \equiv p_k(\bar{y},\tilde{z}) + p_{k+1}(\bar{y},\tilde{z}), \tag{3.29}$$

In fact when $t = 1$ we have, by (3.28)

$$p_k + p_{k+1} \equiv p_{k-1} + p_{(k+1)+1}.$$

Then, by induction,

$$\begin{aligned} p_k + p_{k+1} &\equiv p_{k-t+1} + p_{k+t} \\ &\equiv -(p_{k-t+1} + p_{k+t}) + (p_{k-t+2} + p_{k+t-1}) + (p_{k-t} + p_{k+t+1}) \\ &\equiv p_{k-t} + p_{k+t+1}. \end{aligned}$$

Now we write (3.25) in the following way

$$f_{T_\lambda,T_\mu} = \sum_{t=0}^{k-1}(p_{k-t} + p_{k+t+1}),$$

then by (3.29) we have

$$f_{T_\lambda,T_\mu} \equiv k(p_{k-t} + p_{k+t+1}).$$

□

Lemma 3.4.2 *let be $n \in \mathbb{N}$, then*

$$\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_n\tilde{z}_n y_1 \equiv (-1)^{n+1} y_1\tilde{z}_1\bar{y}_1\tilde{z}_2\bar{y}_2\cdots\tilde{z}_n\bar{y}_n$$

Proof. We first consider the case $n = 2k + 1$. By the previous lemma we have that

$$\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_n\tilde{z}_n \equiv np_{k+1},$$

than

$$
\begin{aligned}
&\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_n\tilde{z}_n y_1 \\
\equiv (2k+1)\cdot &\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(k+1)=1}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(k+1)}z_{\tau(k+1)} \\
&y_{\sigma(k+2)}z_{\tau(k+2)}\cdots y_{\sigma(2k+1)}z_{\tau(2k+1)}y_1 \\
= (2k+1)\cdot &\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(k+1)=1}} (-1)^\sigma(-1)^\tau y_{\sigma(k+2)}z_{\tau(k+2)}\cdots y_{\sigma(2k+1)}z_{\tau(2k+1)} \\
&y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(k+1)}z_{\tau(k+1)}y_1 \\
+(2k+1)\cdot &\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(k+1)=1}} (-1)^\sigma(-1)^\tau \left[\left(y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(k+1)}\right)z_{\tau(k+1)},\right. \\
&\left.\left(y_{\sigma(k+2)}z_{\tau(k+2)}\cdots y_{\sigma(2k+1)}\right)z_{\tau(2k+1)}\right]y_1 \\
\equiv (2k+1)\cdot &\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(k+1)=1}} (-1)^\sigma(-1)^\tau y_{\sigma(k+2)}z_{\tau(k+2)}\cdots y_{\sigma(2k+1)}z_{\tau(2k+1)} \\
&y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(k+1)}z_{\tau(k+1)}y_1 \\
+(2k+1)\cdot &\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(k+1)=1}} (-1)^\sigma(-1)^\tau y_1\left[z_{\tau(k+1)}\left(y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(k+1)}\right),\right. \\
&\left.z_{\tau(2k+1)}\left(y_{\sigma(k+2)}z_{\tau(k+2)}\cdots y_{\sigma(2k+1)}\right)\right]
\end{aligned}
\tag{3.30}
$$

If we assume that

$$\rho_1=\rho_{1,\sigma}=(\sigma(1)\,\sigma(2)\,\cdots\,\sigma(2k{+}1))^{k+2},\qquad \rho_1'=\rho_{1,\tau}=(\tau(1)\tau(2)\cdots\tau(2k{+}1))^{k+2},$$

$$\rho_2=\rho_{2,\sigma}=(1),\qquad \rho_2'=\rho_{2,\tau}=(\tau(1)\tau(2)\cdots\tau(k{+}1))^{-1}(\tau(k{+}2)\tau(k{+}3)\,\cdots\tau(2k{+}1))^{-1},$$

$$\rho_3=\rho_1\rho_2\qquad \rho_3'=\rho_1'\rho_2',$$

(3.30) is equivalent to

$$(2k+1)\sum_{\substack{\rho_1\sigma\in S_{2k+1}\\ \rho_1'\tau\in S_{2k+1}\\ \sigma(k+1)=1}}(-1)^{\rho_1}(-1)^{\rho_1'}(-1)^{\rho_1\sigma}(-1)^{\rho_1'\tau}y_1z_{\rho_1'\tau(1)}y_{\rho_1\sigma(1)}z_{\rho_1'\tau(2)}$$
$$y_{\rho_1\sigma(2)}\cdots z_{\rho_1'\tau(2k+1)}y_{\rho_1\sigma(2k+1)}$$
$$+(2k+1)\sum_{\substack{\rho_2\sigma\in S_{2k+1}\\ \rho_2'\tau\in S_{2k+1}\\ \sigma(k+1)=1}}(-1)^{\rho_2}(-1)^{\rho_2'}(-1)^{\rho_2\sigma}(-1)^{\rho_2'\tau}y_1z_{\rho_2'\tau(1)}y_{\rho_2\sigma(1)}z_{\rho_2'\tau(2)}$$
$$y_{\rho_2\sigma(2)}\cdots z_{\rho_2'\tau(2k+1)}y_{\rho_2\sigma(2k+1)}$$
$$-(2k+1)\sum_{\substack{\rho_3\sigma\in S_{2k+1}\\ \rho_3'\tau\in S_{2k+1}\\ \sigma(k+1)=1}}(-1)^{\rho_3}(-1)^{\rho_3'}(-1)^{\rho_3\sigma}(-1)^{\rho_3'\tau}y_1z_{\rho_3'\tau(1)}y_{\rho_3\sigma(1)}z_{\rho_3'\tau(2)}$$
$$y_{\rho_3\sigma(2)}\cdots z_{\rho_3'\tau(2k+1)}y_{\rho_3\sigma(2k+1)}.$$

Hence

$$(2k+1)p_{k+1}y_1\equiv(2k+1)y_1p'_{2k+1}+(2k+1)y_1p'_1-(2k+1)y_1p'_{k+1}\quad(3.31)$$

where

$$p'_i(\bar{y},\tilde{z})=\sum_{\substack{\sigma,\tau\in S_n\\ \sigma(i)=1}}(-1)^\sigma(-1)^\tau\tilde{z}_{\tau(1)}\bar{y}_{\sigma 1}\tilde{z}_{\tau(2)}\bar{y}_{\sigma 2}\cdots\tilde{z}_{\tau(n)}\bar{y}_{\sigma(n)}.$$

By (3.26) we have

$$\begin{aligned}(2k+1)p_{k+1}y_1&\equiv 2(2k+1)p'_{k+1}-(2k+1)p'_{k+1}=(2k+1)p'_{k+1}\\ &\equiv y_1\tilde{z}_1\bar{y}_1\tilde{z}_2\bar{y}_2\cdots\tilde{z}_{2k+1}\bar{y}_{2k+1}.\end{aligned}$$

Now let be $n=2k$, hence by lemma 3.4.1 we have

$$\begin{aligned}&\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{2k+1}\tilde{z}_{2k+1}\\ &\equiv k\left[p_ky_1+p_{k+1}y_1\right],\end{aligned}\quad(3.32)$$

as in the previous case we have

$$k \cdot \sum_{\substack{\sigma,\tau \in S_{2k} \\ \sigma(k)=1}} (-1)^\sigma (-1)^\tau y_{\sigma(k+1)} z_{\tau(k+1)} \cdots y_{\sigma(2k)} z_{\tau(2k)} y_{\sigma(1)} z_{\tau(1)} \cdots y_{\sigma(k)} z_{\sigma(k)} y_1$$

$$+k \cdot \sum_{\substack{\sigma,\tau \in S_{2k} \\ \sigma(k)=1}} (-1)^\sigma (-1)^\tau \left[\left(y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(k)} \right) z_{\tau(k)} \; , \right.$$
$$\left. \left(y_{\sigma(k+1)} z_{\tau(k+1)} \cdots y_{\sigma(2k)} \right) z_{2k} \right] y_1$$

$$+k \cdot \sum_{\substack{\sigma,\tau \in S_{2k} \\ \sigma(k+1)=1}} (-1)^\sigma (-1)^\tau y_{\sigma(k+1)} z_{\tau(k+1)} \cdots y_{\sigma(2k)} z_{\tau(2k)} y_{\sigma(1)} z_{\tau(1)} \cdots y_{\sigma(k)} z_{\sigma(k)} y_1$$

$$+k \cdot \sum_{\substack{\sigma,\tau \in S_{2k} \\ \sigma(k+1)=1}} (-1)^\sigma (-1)^\tau \left[\left(y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(k)} \right) z_{\tau(k)} \; , \right.$$
$$\left. \left(y_{\sigma(k+1)} z_{\tau(k+1)} \cdots y_{\sigma(2k)} \right) z_{2k} \right] y_1 .$$

If we assume

$$\rho_1 = \rho_{1,\sigma} = (\sigma(1)\ \sigma(2)\ \cdots\ \sigma(2k))^k, \qquad \rho'_1 = \rho_{1,\tau} = (\tau(1)\tau(2)\cdots\tau(2k))^k,$$
$$\rho_2 = \rho_{2,\sigma} = (1), \qquad \rho'_2 = \rho_{2,\tau} = (\tau(1)\tau(2)\cdots\tau(k))^{-1}(\tau(k+1)\tau(k+2)\cdots\tau(2k))^{-1},$$
$$\rho_3 = \rho_1\rho_2, \qquad \rho'_3 = \rho'_1\rho'_2,$$

then (3.32) is equivalent to

$$k \cdot \sum_{\substack{\rho_1\sigma \in S_{2k} \\ \rho'_1\tau \in S_{2k} \\ \sigma(k)=1}} (-1)^{\rho_1} (-1)^{\rho'_1} (-1)^{\rho_1\sigma} (-1)^{\rho'_1\tau} y_1 z_{\rho'_1\tau(1)} y_{\rho_1\sigma(1)} z_{\rho'_1\tau(2)}$$
$$y_{\rho_1\sigma(2)} \cdots z_{\rho'_1\tau(2k+1)} y_{\rho_1\sigma(2k+1)}$$

$$+k \cdot \sum_{\substack{\rho_2\sigma \in S_{2k} \\ \rho'_2\tau \in S_{2k} \\ \sigma(k)=1}} (-1)^{\rho_2} (-1)^{\rho'_2} (-1)^{\rho_2\sigma} (-1)^{\rho'_2\tau} y_1 z_{\rho'_2\tau(1)} y_{\rho_2\sigma(1)} z_{\rho'_2\tau(2)}$$
$$y_{\rho_2\sigma(2)} \cdots z_{\rho'_2\tau(2k+1)} y_{\rho_2\sigma(2k+1)}$$

$$-k \cdot \sum_{\substack{\rho_3\sigma \in S_{2k} \\ \rho'_3\tau \in S_{2k} \\ \sigma(k)=1}} (-1)^{\rho_3} (-1)^{\rho'_3} (-1)^{\rho_3\sigma} (-1)^{\rho'_3\tau} y_1 z_{\rho'_3\tau(1)} y_{\rho_3\sigma(1)} z_{\rho'_3\tau(2)}$$
$$y_{\rho_3\sigma(2)} \cdots z_{\rho'_3\tau(2k+1)} y_{\rho_3\sigma(2k+1)}$$

$$+k \cdot \sum_{\substack{\rho_1\sigma\in S_{2k}\\ \rho_1'\tau\in S_{2k}\\ \sigma(k+1)=1}} (-1)^{\rho_1}(-1)^{\rho_1'}(-1)^{\rho_1\sigma}(-1)^{\rho_1'\tau} y_1 z_{\rho_1'\tau(1)} y_{\rho_1\sigma(1)} z_{\rho_1'\tau(2)}$$

$$y_{\rho_1\sigma(2)} \cdots z_{\rho_1'\tau(2k+1)} y_{\rho_1\sigma(2k+1)}$$

$$+k \cdot \sum_{\substack{\rho_2\sigma\in S_{2k}\\ \rho_2'\tau\in S_{2k}\\ \sigma(k+1)=1}} (-1)^{\rho_2}(-1)^{\rho_2'}(-1)^{\rho_2\sigma}(-1)^{\rho_2'\tau} y_1 z_{\rho_2'\tau(1)} y_{\rho_2\sigma(1)} z_{\rho_2'\tau(2)}$$

$$y_{\rho_2\sigma(2)} \cdots z_{\rho_2'\tau(2k+1)} y_{\rho_2\sigma(2k+1)}$$

$$-k \cdot \sum_{\substack{\rho_3\sigma\in S_{2k}\\ \rho_3'\tau\in S_{2k}\\ \sigma(k+1)=1}} (-1)^{\rho_3}(-1)^{\rho_3'}(-1)^{\rho_3\sigma}(-1)^{\rho_3'\tau} y_1 z_{\rho_3'\tau(1)} y_{\rho_3\sigma(1)} z_{\rho_3'\tau(2)}$$

$$y_{\rho_3\sigma(2)} \cdots z_{\rho_3'\tau(2k+1)} y_{\rho_3\sigma(2k+1)},$$

hence

$$kp_k y_1 \equiv ky_1p_k' - ky_1p_{2k}' - ky_1p_1'$$

and

$$kp_{k+1}y_1 \equiv ky_1p_{k+1}' - ky_1p_{2k}' - ky_1p_1',$$

Now

$$k\left[p_ky_1 + p_{k+1}y_1\right] \equiv k\left[y_1p_k' + y_1p_{k+1}'\right] - 2k\left[y_1p_{2k}' - y_1p_1'\right]$$

and by (3.26) we have

$$\begin{aligned} k\left[p_ky_1 + p_{k+1}y_1\right] &\equiv -k\left[y_1p_k' - y_1p_{k+1}'\right] \\ &\equiv -y_1\tilde{z}_1\bar{y}_1\tilde{z}_2\bar{y}_2\cdots\tilde{z}_{2k}\bar{y}_{2k}. \end{aligned}$$

□

Lemma 3.4.3 *Let be $n \in \mathbb{N}$, then for any $1 < i < n-1$*

$$\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_i\tilde{z}_iy_1z_1\bar{y}_{i+1}\tilde{z}_{i+1}\cdots\bar{y}_n\tilde{z}_ny_1z_1\cdots \equiv \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_i\tilde{z}_i\bar{y}_{i+1}\tilde{z}_{i+1}\cdots\bar{y}_n\tilde{z}_ny_1z_1y_1z_1\cdots$$

Proof. We set

$$f_i = f_i(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{i-1}\tilde{z}_{i-1}y_1z_1\bar{y}_i\tilde{z}_i\cdots\bar{y}_n\tilde{z}_ny_1z_1\cdots,$$

and we observe that

$$f_i \equiv 2f_{i+1} - f_{i+2}. \tag{3.33}$$

In fact, if we consider

$$\begin{aligned} &f_{i+1} - f_i \\ = &\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}\left[y_{\sigma(i)}z_{\tau(i)}, y_1z_1\right] \\ &y_{\sigma(i+1)}z_{\tau(i+1)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots \end{aligned}$$

$$\equiv \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i+1)}z_{\tau(i+1)}$$
$$\left[y_{\sigma(i)}z_{\tau(i)}, y_1z_1\right] y_{\sigma(i+2)}z_{\tau(i+2)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots . \tag{3.34}$$

Now if we set

$$\rho_1 = \rho_{1,\sigma} = (\sigma(i)\sigma(i+1)), \qquad \rho_1' = \rho_{1,\tau} = (\tau(i)\tau(i+1)),$$
$$\rho_2 = \rho_{2,\sigma} = (\sigma(i)\sigma(i+1)), \qquad \rho_2' = \rho_{2,\tau} = (\tau(i)\tau(i+1)),$$

then we can write (3.34) as

$$\sum_{\substack{\rho_2\sigma\in S_n\\ \rho_2'\tau\in S_n}} (-1)^{\rho_2}(-1)^{\rho_2'}(-1)^{\rho_2\sigma}(-1)^{\rho_2'\tau}y_{\rho_2\sigma(1)}z_{\rho_2'\tau(1)}y_{\rho_2\sigma(2)}z_{\rho_2'\tau(2)}\cdots y_{\rho_2\sigma(i+1)}z_{\rho_2'\tau(i+1)}$$
$$y_1z_1y_{\rho_2\sigma(i+2)}z_{\rho_2'\tau(i+2)}\cdots z_{\rho_2'\tau(n)}y_{\rho_2\sigma(n)}y_1z_1\cdots$$
$$-\sum_{\substack{\rho_1\sigma\in S_n\\ \rho_1'\tau\in S_n}} (-1)^{\rho_1}(-1)^{\rho_1'}(-1)^{\rho_1\sigma}(-1)^{\rho_1'\tau}y_{\rho_1\sigma(1)}z_{\rho_1'\tau(1)}y_{\rho_1\sigma(2)}z_{\rho_1'\tau(2)}\cdots y_{\rho_1\sigma(i)}z_{\rho_1'\tau(i)}$$
$$y_1z_1y_{\rho_1\sigma(i+1)}z_{\rho_1'\tau(i+1)}\cdots z_{\rho_1'\tau(n)}y_{\rho_1\sigma(n)}y_1z_1\cdots$$
$$= f_{i+2} - f_{i+1}$$

hence

$$f_i \equiv 2f_{i+1} - f_{i+2}.$$

By induction it is easy to check that for any $1 < i < n-2$

$$f_1 \equiv if_i - (i-1)f_{i+1}. \tag{3.35}$$

Moreover

$$f_n \equiv 2f_{T_\lambda,T_\mu} - f_2 \tag{3.36}$$

in fact

$$f_n = \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(n-1)}z_{\tau(n-1)}y_1z_1y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots$$
$$= \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_1z_1y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(n-1)}z_{\tau(n-1)}y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots$$
$$+ \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(n)}z_{\tau(n)}\left[\left(y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(n-1)}\right)z_{\tau(n-1)}, y_1z_1\right]y_1z_1\cdots$$

$$= \sum_{\sigma,\tau\in S_n} (-1)^{\sigma}(-1)^{\tau} y_1 z_1 y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(n-1)} z_{\tau(n-1)} y_{\sigma(n)} z_{\tau(n)} y_1 z_1 \cdots$$
$$+ \sum_{\sigma,\tau\in S_n} (-1)^{\sigma}(-1)^{\tau} \left[\left(y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(n-1)}\right) z_{\tau(n-1)}, y_1 z_1\right]$$
$$y_{\sigma(n)} z_{\tau(n)} y_1 z_1 \cdots . \tag{3.37}$$

Now if we set

$$\rho_1 = \rho_{1,\sigma} = (1), \qquad \rho'_1 = \rho_{1,\tau} = (1)$$
$$\rho_2 = \rho_{2,\sigma} = (\sigma(1)\sigma(2)\cdots\sigma(n)), \qquad \rho'_2 = \rho_{2,\tau} = (\tau(1)\tau(2)\cdots\tau(n)),$$
$$\rho_3 = \rho_1\rho_2, \qquad \rho'_3 = \rho'_1\rho'_2,$$

we can write (3.37) as

$$\sum_{\substack{\rho_1\sigma\in S_n\\ \rho'_1\tau\in S_n}} (-1)^{\rho_1}(-1)^{\rho'_1}(-1)^{\rho_1\sigma}(-1)^{\rho'_1\tau} y_{\rho\sigma(1)} z_{\rho'_1\tau(1)} y_{\rho_!\sigma(2)} z_{\rho'_1\tau(2)} \cdots$$
$$y_{\rho_1\sigma(n-1)} z_{\rho'_1\tau(n-1)} y_{\rho_!\sigma(n)} z_{\rho'_!\tau(n)} y_1 z_1 y_1 z_1 \cdots$$

$$- \sum_{\substack{\rho_2\sigma\in S_n\\ \rho'_2\tau\in S_n}} (-1)^{\rho_2}(-1)^{\rho'_2}(-1)^{\rho_2\sigma}(-1)^{\rho'_2\tau} y_{\rho_2\sigma(1)} z_{\rho'_2\tau(1)} y_1 z_1 y_{\rho_2\sigma(2)} z_{\rho'_2\tau(2)} \cdots$$
$$y_{\rho_2\sigma(n-1)} z_{\rho'_2\tau(n-1)} y_{\rho_2\sigma(n)} z_{\rho'_2\tau(n)} y_1 z_1 \cdots$$
$$+ \sum_{\substack{\rho_3\sigma\in S_n\\ \rho'_3\tau\in S_n}} (-1)^{\rho_3}(-1)^{\rho'_3}(-1)^{\rho_3\sigma}(-1)^{\rho'_3\tau} y_{\rho_3\sigma(1)} z_{\rho'_3\tau(1)} y_{\rho_3\sigma(2)} z_{\rho'_3\tau(2)} \cdots$$
$$y_{\rho_3\sigma(n-1)} z_{\rho'_3\tau(n-1)} y_{\rho_3\sigma(n)} z_{\rho'_3\tau(n)} y_1 z_1 y_1 z_1 \cdots$$
$$= 2 f_{T_\lambda,T_\mu} - f_2.$$

Hence by (3.35) and (3.36) we have that

$$f_n \equiv f_{T_\lambda,T_\mu}.$$

Finally, by induction we have that for any $1 < i < n$

$$f_i \equiv f_{T_\lambda,T_\mu}.$$

□

Remark 3.4.4 If we set

$$f^i = f^i(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_i z_1 y_1 \tilde{z}_i \bar{y}_{i+1}\tilde{z}_{i+1}\cdots\bar{y}_n\tilde{z}_n y_1 z_1\cdots .$$

In a similar way of the previous lemma it is possible to prove that

$$f^i \equiv 2f^{i+1} - f^{i+2}.$$

Lemma 3.4.5 *Let be $n \in \mathbb{N}$, then for any $1 < i \leq n$*

$$\begin{aligned} f^i &= \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_i z_1 y_1 \tilde{z}_i \bar{y}_{i+1}\tilde{z}_{i+1}\cdots\bar{y}_n\tilde{z}_n y_1 z_1\cdots \\ &\equiv \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{n-1}\tilde{z}_{n-1}\bar{y}_n z_1 y_1 \tilde{z}_n y_1 z_1\cdots \end{aligned}$$

Proof. We consider the polynomial f^n, and we have

$$\begin{aligned}
&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(n-1)}z_{\tau(n-1)}y_{\sigma(n)}z_1y_1z_{\sigma(n)}y_1z_1\cdots \\
=&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i)} \\
&\qquad z_1y_1z_{\tau(i)}\cdots y_{\sigma(n-1)}z_{\tau(n-1)}y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots \\
&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i)} \\
&\qquad \left[\left(z_{\tau(i)}\cdots y_{\sigma(n-1)}z_{\tau(n-1)}\right)y_{\sigma(n)}, z_1y_1\right]z_{\tau(n)}y_1z_1\cdots \\
\equiv&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i)} \\
&\qquad z_1y_1z_{\tau(i)}\cdots y_{\sigma(n-1)}z_{\tau(n-1)}y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots \\
&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i)} \\
&\qquad z_{\tau(n)}\left[y_{\sigma(n)}\left(z_{\tau(i)}\cdots y_{\sigma(n-1)}z_{\tau(n-1)}\right), y_1z_1\right]y_1z_1\cdots .
\end{aligned}$$

Now if we set

$$\rho_1 = \rho_{1,\sigma} = (1), \qquad \rho_1' = \rho_{1,\tau} = (1),$$
$$\rho_2 = \rho_{2,\sigma} = (\sigma(i+1)\sigma(i+2)\cdots\sigma(n))^{-1}, \qquad \rho_2' = \rho_{2,\tau} = (\tau(i+1)\tau(i+2)\cdots\tau(n))^{-1},$$
$$\rho_3 = \rho_1\rho_2, \qquad \rho_3' = \rho_1'\rho_2',$$

we can write the polynomial in the following way

$$\sum_{\substack{\rho_1\sigma\in S_n\\ \rho_1'\tau\in S_n}} (-1)^{\rho_1}(-1)^{\rho_1'}(-1)^{\rho_1\sigma}(-1)^{\rho_1'\tau} y_{\rho_1\sigma(1)}z_{\rho_1'\tau(1)}y_{\rho_1\sigma(2)}z_{\rho_1'\tau(2)}\cdots y_{\rho_1\sigma(i)}$$

$$z_1y_1z_{\rho_1'\tau(i)}\cdots y_{\rho_1\sigma(n-1)}z_{\rho_1'\tau(n-1)}y_{\rho_1\sigma(n)}z_{\rho_1'\tau(n)}y_1z_1\cdots$$

$$+\sum_{\substack{\rho_2\sigma\in S_n\\ \rho_2'\tau\in S_n}} (-1)^{\rho_2}(-1)^{\rho_2'}(-1)^{\rho_2\sigma}(-1)^{\rho_2'\tau} y_{\rho_2\sigma(1)}z_{\rho_2'\tau(1)}y_{\rho_2\sigma(2)}$$

$$z_{\rho_2'\tau(2)}\cdots y_{\rho_2\sigma(n)}z_{\rho_2'\tau(n)}y_1z_1y_1z_1\cdots$$

$$-\sum_{\substack{\rho_3\sigma\in S_n\\ \rho_3'\tau\in S_n}} (-1)^{\rho_3}(-1)^{\rho_3'}(-1)^{\rho_3\sigma}(-1)^{\rho_3'\tau} y_{\rho_3\sigma(1)}z_{\rho_3'\tau(1)}y_{\rho_3\sigma(2)}z_{\rho_3'\tau(2)}\cdots y_{\rho_3\sigma(i)}z_{\rho_3'\tau(i)}$$

$$y_1z_1y_{\rho_3\sigma(i+1)}z_{\rho_3'\tau(i+1)}\cdots y_{\rho_3\sigma(n)}z_{\rho_3'\tau(n)}y_1z_1\cdots$$

$$=\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_iz_1y_1\tilde{z}_i\bar{y}_{i+1}\tilde{z}_{i+1}\cdots\bar{y}_n\tilde{z}_ny_1z_1\cdots$$

$$+\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{n-1}\tilde{z}_{n-1}\bar{y}_n\tilde{z}_nz_1y_1y_1z_1\cdots$$

$$-\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_i\tilde{z}_iy_1z_1\bar{y}_{i+1}\tilde{z}_{i+1}\cdots\bar{y}_n\tilde{z}_ny_1z_1\cdots$$

Since $(\operatorname{sgn}\rho_2)(\operatorname{sgn}\rho_2') = (\operatorname{sgn}\rho_3)(\operatorname{sgn}\rho_3')$ and by the lemma 3.4.3 the lemma holds. □

Lemma 3.4.6 *let be $n\in\mathbb{N}$ and $k=\left[\frac{n}{2}\right]$ then*

$$\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_k\tilde{z}_ky_1\tilde{z}_{k+1}\bar{y}_{k+1}\cdots\tilde{z}_n\bar{y}_nz_1y_1\cdots\equiv \tag{3.38}$$

$$\begin{cases} 0 & \text{if} \quad n=2k,\\ \bar{y}_2\tilde{z}_1\bar{y}_3\tilde{z}_2\cdots\bar{y}_{k+1}\tilde{z}_ky_1\tilde{z}_{k+1}\bar{y}_{k+2}\cdots\bar{y}_{n-1}\tilde{z}_ny_1y_1z_1\cdots & \text{otherwise}\end{cases}.$$

Proof. If we explicit all the alternating variables y's , z's except the k-th and the $(k+1)$-th y and z, by lemma 3.1.3, it is easy to check that all the polynomials whit y_1 is in the k-th and the $(k+1)$-th position are equivalent to zero. Then, if we explicit all the alternating variables y's , z's except the $(k-t)$-th and the $(k+1+t)$-th y_i and z_j, for $t\le 1$, we can write the polynomial as

$$\sum_{\substack{i<j\\ l_1\neq l_2\neq 1}}\cdots\bar{y}_i(\widetilde{z_{i_1}y_{l_1}z_{i_2}})y_1(\widetilde{z_{j_1}y_{l_2}z_{j_2}})\bar{y}_j\cdots$$

$$=\sum_{\substack{i<j\\ l_1\neq l_2\neq 1}}\cdots\bar{y}_i(z_{i_1}y_{l_1}z_{i_2})y_1(z_{j_1}y_{l_2}z_{j_2})\bar{y}_j\cdots$$

$$-\sum_{\substack{i<j\\ l_1\neq l_2\neq 1}}\cdots\bar{y}_i(z_{j_1}y_{l_2}z_{j_2})y_1(z_{i_1}y_{l_1}z_{i_2})\bar{y}_j\cdots,$$

and for the same reason of the previous case y_i and y_j can not be y_1.

If we iterate the process for any $t \leq k$ we can affirm that, if $n = 2k+1$ then only the last y can be y_1, otherwise if $n = 2k$, (3.38) is a polynomial identity. □

Lemma 3.4.7 *Let be $n \in \mathbb{N}$ and let*

$$g_i = g_i(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2 \ldots \tilde{z}_{i-1}y_1\tilde{z}_i\bar{y}_i \ldots \bar{y}_{n-1}\tilde{z}_n\bar{y}_n z_1 y_1 \cdots .$$

then

if $n = 2k+1$

$$g_{k+t+1} \equiv -g_{k-t} \equiv \begin{cases} \dfrac{2k-1}{2k+1} g_{n+1} & \text{if } t = k \\ \\ \dfrac{(-1)^{k+t}}{2k+1}(2t+1)\ g_{n+1} & \text{if } 0 \leq t < k, \end{cases}$$

if $n = 2k$

$$g_{k+1+t} \equiv g_{k+1-t} \equiv (-1)^t \frac{t}{k}\ g_{n+1} \qquad \text{if } 0 \leq t \leq k.$$

Proof. In order to prove the lemma we first check the following

$$g_i \equiv -2g_{i+1} - g_{i+2},$$

in fact, if we consider

$$g_{i+1} = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2 \cdots \bar{y}_i\tilde{z}_i y_1\tilde{z}_{i+1}\bar{y}_{i+1} \cdots \tilde{z}_n\bar{y}_n z_1 y_1 \cdots$$

$$\begin{aligned}
&= \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)} \cdots y_{\sigma(i-1)}z_{\tau(i-1)} \\
&\qquad y_1 z_{\tau_(i)}y_{\sigma(i)}z_{\tau(i+1)}y_{\sigma(i)}z_{\tau(i+2)} \cdots y_{\sigma(n-1)}z_{\tau(n)}y_{\sigma(n)}z_1 y_1 \cdots \\
&+ \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)} \cdots y_{\sigma(i-1)}z_{\tau(i-1)} \\
&\qquad \left[y_{\sigma(i)}z_{\tau(i)}, y_1 z_{\tau(i+1)}\right] y_{\sigma(i+1)}z_{\tau(i+2)} \cdots y_{\sigma(n-1)}z_{\tau(n)}y_{\sigma(n)}z_1 y_1 \cdots \\
&\equiv \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)} \cdots y_{\sigma(i-1)}z_{\tau(i-1)} \\
&\qquad y_1 z_{\tau(i)}y_{\sigma(i)}z_{\tau(i)}y_i z_{i+1} \cdots y_{n-1}z_n y_n z_1 y_1 \cdots \\
&+ \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)} \cdots y_{\sigma(i-1)}z_{\tau(i-1)} \\
&\qquad y_{\sigma(i+1)}z_{\tau(i+2)} \left[y_{\sigma(i)}z_{\tau(i)}, y_1 z_{\tau(i+1)}\right] \cdots y_{n-1}z_{\sigma(n)}y_{\tau(n)}z_1 y_1 \cdots .
\end{aligned}$$

If we assume that

$$\rho_1 = \rho_{1,\sigma} = (1), \qquad \rho'_1 = \rho_{1,\tau} = (\tau(1)\tau(i+1)),$$
$$\rho_2 = \rho_{2,\sigma} = (\sigma(i)\sigma(i+1)), \qquad \rho'_2 = \rho_{2,\tau} = (\tau(i)\tau(i+1)\tau(i+2)),$$
$$\rho_3 = \rho_1\rho_2, \qquad \rho'_3 = \rho'_1\rho'_2,$$

we can write the polynomial in the following way

$$\sum_{\substack{\rho_1\sigma\in S_n\\ \rho'_1\sigma\in S_n}} (-1)^{\rho_1}(-1)^{\rho'_1}(-1)^{\rho_1\sigma}(-1)^{\rho'_1\tau} y_{\rho_1\sigma(1)} z_{\rho'_1\tau(1)} y_{\rho_1\sigma(2)} z_{\rho'_1\tau(2)} \cdots y_{\rho_1\sigma(i-1)} z_{\rho'_1\tau(i-1)}$$
$$y_1 z_{\rho'_1\tau(i)} y_{\rho_1\sigma(i)} z_{\rho'_1\tau(i+1)} \cdots y_{\rho_1\sigma(n-1)} z_{\rho'_1\tau(n)} y_{\rho_1\sigma(n)} z_1 y_1 \cdots$$
$$+ \sum_{\substack{\rho_2\sigma\in S_n\\ \rho'_2\sigma\in S_n}} (-1)^{\rho_2}(-1)^{\rho'_2}(-1)^{\rho_2\sigma}(-1)^{\rho'_2\tau} y_{\rho_2\sigma(1)} z_{\rho'_2\tau(1)} y_{\rho_3\sigma(2)} z_{\rho'_2\tau(2)} \cdots y_{\rho_2\sigma(i+1)} z_{\rho'_2\tau(i+1)}$$
$$y_1 z_{\rho'_2\tau(i+2)} y_{\rho_2\sigma(i+2)} z_{\rho'_2\tau(i+3)} \cdots y_{\rho_2\sigma(n-1)} z_{\rho'_2\tau(n)} y_{\rho_2\sigma(n)} z_1 y_1 \cdots$$
$$- \sum_{\substack{\rho_3\sigma\in S_n\\ \rho'_3\sigma\in S_n}} (-1)^{\rho_3}(-1)^{\rho'_3}(-1)^{\rho_3\sigma}(-1)^{\rho'_3\tau} y_{\rho_3\sigma(1)} z_{\rho'_3\tau(1)} y_{\rho_3\sigma(2)} z_{\rho'_3\tau(2)} \cdots y_{\rho_3\sigma(i)} z_{\rho'_3\tau(i)}$$
$$y_1 z_{\rho'_3\tau(i+1)} y_{\rho_3\sigma(i+1)} z_{\rho'_3\tau(i+2)} \cdots y_{\rho_3\sigma(n-1)} z_{\rho'_3\tau(n)} y_{\rho_3\sigma(n)} z_1 y_1 \cdots$$
$$= -g_{i+2} - g_{i+1} - g_i,$$

hence

$$g_i \equiv -2g_{i+1} - g_{i+2}.$$

Moreover by lemma 3.4.6 and 3.1.2 we have

$$g_{k+1} \equiv \frac{(-1)^k}{2k+1} g_{2k+2},$$

and, by an easy computation, we have

$$g_{k+t+1} \equiv g_{k-t} \equiv \begin{cases} -\dfrac{2k-1}{2k+1} g_{n+1} & \text{if } t = k \\[2ex] \dfrac{(-1)^{k+t}}{2k+1}(2t+1)\ g_{n+1} & \text{if } 0 \le t < k. \end{cases}$$

Now let be $n = 2k$ then by lemma 3.4.6 and lemma 3.1.3 we have

$$g_{k+1} \equiv 0$$

and, by induction, it is easy to check that

$$g_{k+1-t} \equiv -g_{k+1+t},$$

it follows that

$$g_{k+1+t} \equiv -g_{k+1-t} \equiv (-1)^t \frac{t}{k}\ g_{2k+1} \qquad \text{if } 0 \le t \le k.$$

□

Lemma 3.4.8 *Let be $n \in \mathbb{N}$ and $0 < i < j < n$, then*

$$\begin{aligned} &\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{i-1}\tilde{z}_{i-1}y_1\tilde{z}_i\bar{y}_i\cdots\tilde{z}_{j-1}\bar{y}_{j-1}z_1\bar{y}_j\tilde{z}_j\cdots\bar{y}_n\tilde{z}_ny_1z_1\cdots \\ \equiv&\alpha_{i,j}\ \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_n\tilde{z}_ny_1z_1y_1z_1\cdots, \end{aligned}$$

where

$$\alpha_{i,j} = \begin{cases} \dfrac{(-1)^{j-i}}{2k+1}\left[(2k+1) - 2\,(j-i)\right] & \text{if} \quad n = 2k+1, \\ \\ \dfrac{(-1)^{k+j-i-1}}{k}\,(k - (j-i+1)) & \text{if} \quad n = 2k. \end{cases}$$

Proof. if we consider

$$\begin{aligned} &\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{i-1}\tilde{z}_{i-1}y_1\tilde{z}_i\bar{y}_i\cdots\tilde{z}_{j-1}\bar{y}_{j-1}z_1\bar{y}_j\tilde{z}_j\cdots\bar{y}_n\tilde{z}_ny_1z_1\cdots \\ =&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_1 \\ &z_{\tau(i)}y_{\sigma(i)}\cdots z_{\tau(j-1)}y_{\sigma(j-1)}z_1y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots \\ =&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i)}\cdots \\ &z_{\tau(j-1)}y_{\sigma(j-1)}z_1y_1z_{\tau(i)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots \\ +&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}\big[y_1z_{\tau(i)}, \\ &y_{\sigma(i)}\cdots z_{\tau(j-1)}y_{\sigma(j-1)}z_1\big]\,y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots \\ \equiv&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i)}\cdots \\ &z_{\tau(j-1)}y_{\sigma(j-1)}z_1y_1z_{\tau(i)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots \\ +&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)} \\ &\big[y_1z_{\tau(i)}, y_{\sigma(i)}\cdots z_{\tau(j-1)}y_{\sigma(j-1)}z_1\big]\,y_1z_1\cdots. \end{aligned}$$

if we assume that

$$\rho_1 = \rho_{1,\sigma} = (1), \qquad \rho_1' = \rho_{1,\tau} = (\tau(i)\tau(i+1)\cdots\tau(j-i)),$$

$$\rho_2 = \rho_{2,\sigma} = (\sigma(i)\sigma(i{+}1)\cdots\sigma(n))^{j-i}, \qquad \rho_2' = \rho_{2,\tau} = (\tau(i)\tau(i{+}1)\cdots\tau(n))^{j-i}$$

$$\rho_3 = \rho_1\rho_2, \qquad \rho_3' = \rho_1'\rho_2',$$

we can write the polynomial in the following way

$$\sum_{\substack{\rho_1\sigma\in S_n\\ \rho_1'\tau\in S_n}} (-1)^{\rho_1}(-1)^{\rho_1'}(-1)^{\rho_1\sigma}(-1)^{\rho_1'\tau} y_{\rho_1\sigma(1)}z_{\rho_1'\tau(1)}y_{\rho_1\sigma(2)}z_{\rho_1'\tau(2)}\cdots y_{\rho_1\sigma(j-2)}$$
$$z_{\rho_1'\tau(j-2)}y_{\rho_1\sigma(j-1)}z_1y_1z_{\rho_1'\tau(j-1)}y_{\rho_1\sigma(j)}z_{\rho_1'\tau(j)}\cdots y_{\rho_1\sigma(n)}z_{\rho_1'\tau(n)}y_1z_1\cdots$$
$$+\sum_{\substack{\rho_2\sigma\in S_n\\ \rho_2'\tau\in S_n}} (-1)^{\rho_2}(-1)^{\rho_2'}(-1)^{\rho_2\sigma}(-1)^{\rho_2'\tau} y_{\rho_2\sigma(1)}z_{\rho_2'\tau(1)}y_{\rho_2\sigma(2)}z_{\rho_2'\tau(2)}\cdots y_{\rho_2\sigma(n+i-j)}$$
$$z_{\rho_2'\tau(n+i-j)}y_1z_{\rho_2'\tau(n+i-j+1)}y_{\rho_2\sigma(n+i-j+1)}\cdots z_{\rho_2'\tau(n)}y_{\rho_2\sigma(n)}z_1y_1z_1\cdots$$
$$-\sum_{\substack{\rho_3\sigma\in S_n\\ \rho_3'\tau\in S_n}} (-1)^{\rho_3}(-1)^{\rho_3'}(-1)^{\rho_3\sigma}(-1)^{\rho_3'\tau} y_{\rho_3\sigma(1)}z_{\rho_3'\tau(1)}y_{\rho_3\sigma(2)}z_{\rho_3'\tau(2)}\cdots y_{\rho_3\sigma(n-1)}$$
$$z_{\rho_3'\tau(n)}z_1y_1z_{\rho_3'\tau(n)}y_1z_1\cdots,$$

and by the lemma 3.4.5 the polynomial is equivalent to

$$\sum_{\substack{\rho_2\sigma\in S_n\\ \rho_2'\tau\in S_n}} (-1)^{\rho_2}(-1)^{\rho_2'}(-1)^{\rho_2\sigma}(-1)^{\rho_2'\tau} y_{\rho_2\sigma(1)}z_{\rho_2'\tau(1)}y_{\rho_2\sigma(2)}z_{\rho_2'\tau(2)}\cdots$$
$$y_{\rho_2\sigma(j-i-1)}z_{\rho_2'\tau(j-i-1)}y_1z_1y_{\rho_2\sigma(j-i)}z_{\rho_2'\tau(j-i)}\cdots y_{\rho_2\sigma(n)}z_{\rho_2'\tau(n)}y_1z_1\cdots$$
$$=\bar y_1\tilde z_1\bar y_2\tilde z_2\cdots\bar y_{j-i-1}\tilde z_{j-i-1}y_1\tilde z_{j-i}\bar y_{j-i}\cdots\tilde z_n\bar y_nz_1,$$

and by lemma 3.4.7, the lemma holds. □

Lemma 3.4.9 *Let be* $n\in\mathbb{N}$ *then*

$$\bar y_1\tilde z_1\bar y_2\tilde z_2\cdots\bar y_{n-1}\tilde z_{n-1}\bar y_nz_1y_1\tilde z_ny_1z_1\cdots$$
$$\equiv\alpha_n\ \bar y_1\tilde z_1\bar y_2\tilde z_2\cdots\bar y_{n-1}\tilde z_{n-1}\bar y_n\tilde z_ny_1z_1y_1z_1\cdots,$$

where

$$\alpha_n = \begin{cases} 1 \quad if \quad n=2k+1 \\ \dfrac{(-1)^{k-1}}{k}+1 \quad if \quad n=2k. \end{cases}$$

Proof. If we consider $k = \left[\frac{n}{2}\right]$, then

$$
\begin{aligned}
&\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{n-1}\tilde{z}_{n-1}\bar{y}_n z_1 y_1\tilde{z}_n y_1 z_1\cdots\\
=&\sum_{\sigma,\tau\in S_n}(-1)^{\sigma}(-1)^{\tau}y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(n-1)}z_{\tau(n-1)}y_{\sigma(n)}z_1y_1z_{\tau(n)}y_1z_1\cdots\\
=&\sum_{\sigma,\tau\in S_n}(-1)^{\sigma}(-1)^{\tau}y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(k)}z_{\tau(k)}\\
&y_{\sigma(n)}z_1y_{\sigma(k+1)}z_{\tau(k+1)}y_{\sigma(k+2)}z_{\tau(k+2)}\cdots y_{\sigma(n-1)}z_{\tau(n-1)}y_1z_{\tau(n)}y_1z_1\cdots\\
+&\sum_{\sigma,\tau\in S_n}(-1)^{\sigma}(-1)^{\tau}y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(k)}z_{\tau(k)}\\
&\left[\left(y_{\sigma(k+1)}z_{\tau(k+1)}y_{\sigma(k+2)}z_{\tau(k+2)}\cdots y_{\sigma(n-1)}\right)z_{\tau(n-1)},y_{\sigma(n)}z_1\right]y_1z_{\tau(n)}y_1z_1\cdots\\
\equiv&\sum_{\sigma,\tau\in S_n}(-1)^{\sigma}(-1)^{\tau}y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(k)}z_{\tau(k)}\\
&y_{\sigma(n)}z_1y_{\sigma(k+1)}z_{\tau(k+1)}y_{\sigma(k+2)}z_{\tau(k+2)}\cdots y_{\sigma(n-1)}z_{\tau(n-1)}y_1z_{\tau(n)}y_1z_1\cdots\\
+&\sum_{\sigma,\tau\in S_n}(-1)^{\sigma}(-1)^{\tau}y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(k)}z_{\tau(k)}\\
&y_1\left[z_{\tau(n-1)}\left(y_{\sigma(k+1)}z_{\tau(k+1)}y_{\sigma(k+2)}z_{\tau(k+2)}\cdots y_{\sigma(n-1)}\right),z_1y_{\sigma(n)}\right]z_{\tau(n)}y_1z_1\cdots.
\end{aligned}
$$

If we assume that

$$
\begin{gathered}
\rho_1=\rho_{1,\sigma}=(\sigma(k+1)\sigma(k+2)\cdots\sigma(n))^{-1},\qquad \rho_1'\rho_{1,\tau}=(1),\\
\rho_2=\rho_{2,\sigma}=(1),\qquad \rho_2'=\rho_{2,\tau}=(\tau(1)\tau(2)\cdots\tau(n-1))^{-1},\\
\rho_3=\rho_1\rho_2,\qquad \rho_3'=\rho_1'\rho_2',
\end{gathered}
$$

we can write the polynomial in the following way

$$
\begin{aligned}
&\sum_{\substack{\rho_1\sigma\in S_n\\ \rho_1'\tau\in S_n}}(-1)^{\rho_1}(-1)^{\rho_1'}(-1)^{\rho_1\sigma}(-1)^{\rho_1'\tau}y_{\rho_1\sigma(1)}z_{\rho_1'\tau(1)}y_{\rho_1\sigma(2)}z_{\rho_1'\tau(2)}\cdots y_{\rho_1\sigma(k)}z_{\rho_1'\tau(k)}y_{\rho_1\sigma(k+1)}\\
&z_1y_{\rho_1\sigma(k+2)}z_{\rho_1'\tau(k+1)}y_{\rho_1\sigma(k+3)}z_{\rho_1'\tau(k+2)}\cdots y_{\rho_1\sigma(n)}z_{\rho_1'\tau(n-1)}y_1z_{\rho_1'\tau(n)}y_1z_1\cdots\\
+&\sum_{\substack{\rho_2\sigma\in S_n\\ \rho_2'\tau\in S_n}}(-1)^{\rho_2}(-1)^{\rho_2'}(-1)^{\rho_2\sigma}(-1)^{\rho_2'\tau}y_{\rho_2\sigma(1)}z_{\rho_2'\tau(1)}y_{\rho_2\sigma(2)}z_{\rho_2'\tau(2)}\cdots y_{\rho_2\sigma(k)}z_{\rho_2'\tau(k)}y_1\\
&z_{\rho_2'\tau(k+1)}y_{\rho_2\sigma(k+1)}z_{\rho_2'\tau(k+2)}y_{\rho_2\sigma(k+2)}\cdots z_{\rho_2'\tau(n-1)}y_{\rho_2\sigma(n-1)}z_1y_{\rho_2\sigma(n)}z_{\rho_2'\tau(n)}y_1z_1\cdots\\
-&\sum_{\substack{\rho_3\sigma\in S_n\\ \rho_3'\tau\in S_n}}(-1)^{\rho_3}(-1)^{\rho_3'}(-1)^{\rho_3\sigma}(-1)^{\rho_3'\tau}y_{\rho_3\sigma(1)}z_{\rho_3'\tau(1)}y_{\rho_3\sigma(2)}z_{\rho_3'\tau(2)}\cdots y_{\rho_3\sigma(k)}\\
&z_{\rho_3'\tau(k)}y_1z_1y_{\rho_3\sigma(k+1)}z_{\rho_3'\tau(k+1)}y_{\rho_3\sigma(k+2)}z_{\rho_3'\tau(k+2)}\cdots y_{\rho_3\sigma(n)}z_{\rho_3'\tau(n)}y_1z_1\cdots.
\end{aligned}
$$

Now by lemma 3.4.3, 3.4.6 and 3.4.8 it is easy to check that the polynomial is equivalent to

$$\alpha_n\ \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{n-1}\tilde{z}_{n-1}\bar{y}_n\tilde{z}_n y_1 z_1 y_1 z_1\cdots,$$

where

$$\alpha_n = \begin{cases} 1 \text{ if } & n = 2k+1 \\ \\ \dfrac{(-1)^{k-1}}{k} + 1 \text{ if } & n = 2k. \end{cases}$$

□

Lemma 3.4.10 *Let be $n \in \mathbb{N}$ and $0 < i < j < n$, then*

$$\begin{aligned} &\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{i-1}\tilde{z}_{i-1}\bar{y}_i z_1\bar{y}_{i+1}\tilde{z}_i\cdots\bar{y}_{j-1}\tilde{z}_{j-2}y_1\tilde{z}_{j-1}\bar{y}_j\tilde{z}_j\cdots\bar{y}_n\tilde{z}_n y_1 z_1\cdots \\ \equiv&\alpha_{i,j}\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_n\tilde{z}_n y_1 z_1\cdots, \end{aligned}$$

where, if $n = 2k+1$

$$\alpha_{i,j} = \begin{cases} \dfrac{(-1)^{j-i}}{n}(n - 2(j-i-2)) & \text{if} \quad i \neq j-2 \\ \\ -\dfrac{2k-1}{2k+1} \quad \text{if} \quad i = j-2, \end{cases}$$

if $n = 2k$

$$\alpha_{i,j} = \begin{cases} \dfrac{(-1)^k + j - i + 1}{k}(k - j + i) & \text{if} \quad i \neq j-k+1 \\ \\ -\dfrac{1}{k} \quad \text{if} \quad i = j-k+1. \end{cases}$$

Proof. If we consider

$$\begin{aligned} &\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{i-1}\tilde{z}_{i-1}\bar{y}_i z_1\bar{y}_{i+1}\tilde{z}_i\cdots\bar{y}_{j-1}\tilde{z}_{j-2}y_1\tilde{z}_{j-1}\bar{y}_j\tilde{z}_j\cdots\bar{y}_n\tilde{z}_n y_1 z_1\cdots \\ =&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i)} \\ &z_1 y_{\sigma(i+1)}z_{\tau(i)}\cdots y_{\sigma(j-1)}z_{\tau(j-2)}y_1 z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1 z_1\cdots \\ =&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i)} \\ &z_{\tau(i)}\cdots y_{\sigma(j-1)}z_{\tau(j-2)}y_1 z_1 y_{\sigma(i+1)}z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1 z_1\cdots \\ +&\sum_{\sigma,\tau\in S_n}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i)} \\ &\left[z_1 y_{\sigma(i+1)}, \left(z_{\tau(i)}\cdots y_{\sigma(j-1)}z_{\tau(j-2)}\right)y_1\right] z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1 z_1\cdots \end{aligned}$$

$$
\begin{aligned}
\equiv & \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i)} \\
& \quad z_{\tau(i)}\cdots y_{\sigma(j-1)}z_{\tau(j-2)}y_1z_1y_{\sigma(i+1)}z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots \\
+ & \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i)} \\
& \quad z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}\left[y_{\sigma(i+1)}z_1, y_1\left(z_{\tau(i)}\cdots y_{\sigma(j-1)}z_{\tau(j-2)}\right)\right]y_1z_1\cdots \\
= & \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i)} \\
& \quad z_{\tau(i)}\cdots y_{\sigma(j-1)}z_{\tau(j-2)}y_1z_1y_{\sigma(i+1)}z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_1\cdots \\
+ & \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i)} \\
& \quad z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_{\sigma(i+1)}z_1y_1z_{\tau(i)}\cdots y_{\sigma(j-1)}z_{\tau(j-2)}y_1z_1\cdots \\
- & \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i)} \\
& \quad z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_{\tau(i)}\cdots y_{\sigma(j-1)}z_{\tau(j-2)}y_{\sigma(i+1)}z_1y_1z_1\cdots,
\end{aligned}
$$

If we assume that

$$
\begin{gathered}
\rho_1 = \rho_{1,\sigma} = (\sigma(i)\sigma(i+1)\cdots\sigma(j-1)), \qquad \rho'_1 = \rho_{1,\tau} = (1), \\
\rho_2 = \rho_{2,\sigma} = (\sigma(i+1)\sigma(i+2)\cdots\sigma(n))^{j-i-1}, \qquad \rho'_2 = \rho_{2,\tau} = (\tau(i)\tau(i+1)\cdots\tau(n))^{j-i-1}, \\
\rho_3 = \rho_1\rho_2, \qquad \rho'_3 = \rho'_1\rho'_2,
\end{gathered}
$$

we can write the polynomial in the following way

$$
\begin{aligned}
& \sum_{\substack{\rho_1\sigma\in S_n \\ \rho'_1\tau\in S_n}} (-1)^{\rho_1}(-1)^{\rho'_1}(-1)^{\rho_1\sigma}(-1)^{\rho'_1\tau} y_{\rho_1\sigma(1)}z_{\rho'_1\tau(1)}y_{\rho_1\sigma(2)}z_{\rho'_1\tau(2)}\cdots y_{\rho_1\sigma(j-2)} \\
& \quad z_{\rho'_1\tau(j-2)}y_1z_1y_{\rho_1\sigma(j-1)}z_{\rho'_1\tau(j-1)}y_{\rho_1\sigma(j)}z_{\rho'_1\tau(j)}\cdots y_{\rho_1\sigma(n)}z_{\rho'_1\tau(n)}y_1z_1\cdots \\
+ & \sum_{\substack{\rho_2\sigma\in S_n \\ \rho'_2\tau\in S_n}} (-1)^{\rho_2}(-1)^{\rho'_2}(-1)^{\rho_2\sigma}(-1)^{\rho'_2\tau} y_{\rho_2\sigma(1)}z_{\rho'_1\tau(1)}y_{\rho_2\sigma(2)}z_{\rho'_1\tau(2)}\cdots y_{\rho_2\sigma(n+i-j-1)} \\
& \quad z_{\rho'_1\tau(n+i-j-1)}y_{\rho_2\sigma(n+i-j)}z_1y_1z_{\rho'_1\tau(n+i-j)}y_{\rho_2\sigma(n+i-j+1)}z_{\rho'_1\tau(n+i-j+1)}\cdots y_{\rho_2\sigma(n)} \\
& \quad z_{\rho'_1\tau(n)}y_1z_1\cdots \\
- & \sum_{\substack{\rho_3\sigma\in S_n \\ \rho'_3\tau\in S_n}} (-1)^{\rho_3}(-1)^{\rho'_3}(-1)^{\rho_3\sigma}(-1)^{\rho'_3\tau} y_{\rho_3\sigma(1)}z_{\rho'_3\tau(1)}y_{\rho_3\sigma(2)}z_{\rho'_3\tau(2)}\cdots y_{\rho_3\sigma(n+i-j+2)} \\
& \quad z_{\rho'_3\tau(n+i-j+2)}y_1z_{\rho'_3\tau(n+i-j+3)}y_{\rho_3\sigma(n+i-j+3)}\cdots z_{\rho'_3\tau(n)}y_{\rho_3\sigma(n)}z_1y_1z_1\cdots.
\end{aligned}
$$

Now by lemma 3.4.3, 3.4.7 and 3.4.9 the lemma holds. □

Theorem 3.4.11 *Let $n, r, s \in \mathbb{N}$, such that $r, s \geq n \geq 1$, let*

$$\chi_{(0,r,s)} = \sum_{\substack{\lambda \vdash r \\ \mu \vdash s}} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu.$$

If $\lambda = (r-n+1, 1^n) \vdash r$ and $\mu = (s-n+1, 1^n) \vdash s$ then

$$m_{\lambda,\mu} = \begin{cases} 2 & \text{if} \quad r = s \\ \\ 1 & \text{if} \quad |r-s| = 1 \end{cases}.$$

Proof. Without loss of generality we suppose that the highest weight vector f starts with a y, hence $r = s$ or $r = s+1$. If $r + s = 2k$ it is obvious that $m_{\lambda,\mu} = 2$. By lemma 3.4.7 we have that if $r + s = 2k+1$ then any highest weight vector is linearly dependent, $\mod \mathrm{Id}^{\mathbb{Z}}(A)$, to f_{T_λ,T_μ}. If $r + s = 2k+2$, lemma 3.4.3, 3.4.5, 3.4.8, 3.4.9 and 3.4.10 show that any highest weight vector is linearly dependent to f_{T_λ,T_μ}. Let $r+s > 2k+2$, we prove that any highest weight vector is linearly dependent, $\mod \mathrm{Id}^{\mathbb{Z}}(A)$, to the set S of the following polynomials:

- $\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2 \dots \tilde{z}_{i-1}y_1\tilde{z}_i\bar{y}_i \dots \bar{y}_{n-1}\tilde{z}_n\bar{y}_n y_1 z_1 \dots,$
- $\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2 \dots \bar{y}_{i-1}\tilde{z}_{i-1}y_1 z_1\bar{y}_i\tilde{z}_i \dots \bar{y}_n\tilde{z}_n y_1 z_1 \dots,$
- $\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2 \dots \bar{y}_i z_1 y_1\tilde{z}_i\bar{y}_{i+1}\tilde{z}_{i+1} \dots \bar{y}_n\tilde{z}_n y_1 z_1 \dots,$
- $\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2 \dots \bar{y}_{i-1}\tilde{z}_{i-1}y_1\tilde{z}_i\bar{y}_i \dots \tilde{z}_{j-1}\bar{y}_{j-1}z_1\bar{y}_j\tilde{z}_j \cdots \bar{y}_n\tilde{z}_n y_1 z_1 \dots,$
- $\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2 \dots \bar{y}_{i-1}\tilde{z}_{i-1}\bar{y}_i z_1\bar{y}_{i+1}\tilde{z}_i \dots \bar{y}_{j-1}\tilde{z}_{j-2}y_1\tilde{z}_{j-1}\bar{y}_j\tilde{z}_j \dots \bar{y}_n\tilde{z}_n y_1 z_1 \dots$

wich are polynomials linearly dependent to $f_{T_\lambda,T_\mu} \mod {}^{\mathbb{Z}}(A)$. Suppose that $r = s+1$, it follows that any highest weight vector $f = f(y,z)$ have at least two non-alternating y, let y_1', y_1''. Suppose that f is not a polynomials of the previous, hence let

$$\begin{aligned} & f = \bar{y}_1\tilde{z}_1 \dots y_1' \dots y_1'' \dots \tilde{z}_n \\ = & \sum_{\sigma,\tau \in S_n} (-1)^\sigma (-1)^\tau y_{\sigma(1)} z_{\tau(1)} \dots y_1' p_1^{\sigma,\tau}(y,z) y_1'' p_2^{\sigma,\tau}(y,z) y_{\sigma(n)} \\ = & \sum_{\sigma,\tau \in S_n} (-1)^\sigma (-1)^\tau y_{\sigma(1)} z_{\tau(1)} \dots y_1' p_2^{\sigma,\tau}(y,z) y_{\sigma(n)} p_1^{\sigma,\tau}(y,z) y_1'' \\ + & \sum_{\sigma,\tau \in S_n} (-1)^\sigma (-1)^\tau y_{\sigma(1)} z_{\tau(1)} \dots y_1' \left[p_1^{\sigma,\tau}(y,z) y_1'', p_2^{\sigma,\tau}(y,z) y_{\sigma(n)}\right] \end{aligned}$$

$$
\begin{aligned}
\equiv & \sum_{\sigma,\tau\in S_n} (-1)^{\sigma}(-1)^{\tau} y_{\sigma(1)} z_{\tau(1)} \dots y_1' p_2^{\sigma,\tau}(y,z) y_{\sigma(n)} p_1^{\sigma,\tau}(y,z) y_1'' \\
+ & \sum_{\sigma,\tau\in S_n} (-1)^{\sigma}(-1)^{\tau} y_{\sigma(1)} z_{\tau(1)} \dots y_1'' p_1^{\sigma,\tau}(y,z) y_{\sigma(n)} p_2^{\sigma,\tau}(y,z) y_1' \\
- & \sum_{\sigma,\tau\in S_n} (-1)^{\sigma}(-1)^{\tau} y_{\sigma(1)} z_{\tau(1)} \dots y_{\sigma(n)} p_2^{\sigma,\tau}(y,z) y_1'' p_1^{\sigma,\tau}(y,z) y_1' \\
& = \alpha \bar{y}_1 \tilde{z}_1 \dots y_1' \dots \bar{y}_n \dots y_1'' + \beta \bar{y}_1 \tilde{z}_1 \dots y_1'' \dots \bar{y}_n \dots y_1' \\
& + \gamma \bar{y}_1 \tilde{z}_1 \dots \bar{y}_n \dots y_1' \dots y_1''.
\end{aligned}
$$

Hence f is, $\mod \mathrm{Id}^{\mathbb{Z}}(A)$, linearly dependent to a set of polynomials whose end with a non-alternating y. Now, in a similar way, it is possible prove that every polynomial obtained before, are linearly dependent, $\mod \mathrm{Id}^{\mathbb{Z}}(A)$, to a set of polynomials whose end with $z_1 y_1$, and we continue this process till every polynomials are in S. If $r = s$ we can obtain the same result. Hence, up to a scalar, we have

$$f \equiv f_{T_\lambda, T_\mu},$$

and if we consider the substitution $y_1 = (h_1 + g_1)e_{12}$, $z_1 = (h_1' + g_1')e_{21}$, $y_i = h_i e_{12}$, $z_j = h_j' e_{21}$ for $i, j = 2, \cdots, n$ where e_{ij} are matrix units, $h_k \neq h_l' \in E_1$, $g_k \neq g_l' \in E_0$ and such that $g_k^{r+s}, g_l'^{r+s} \neq 0$, we have $f_{T_\lambda, T_\mu} \not\equiv 0$, and the theorem holds.

□

3.5 $h_1(\lambda) > h_1(\mu)$

Throughout this section we consider a pairs of Young tableaux whose heights are $h_1(\lambda) = n$ and $h_1(\mu) = m$, such that $n > m$, and we set

$$f_{T_\lambda, T_\mu} = f_{T_\lambda, T_\mu}(y, z) = \bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \dots \bar{y}_m \tilde{z}_m y_{m+1} z_1 \bar{y}_{m+2} z_1 \dots z_1 \bar{y}_n z_1 y_1 \cdots .$$

Lemma 3.5.1 *Let $n, m \in \mathbb{N}$ $n > m$, and let*

$$f'_{T_\lambda, T_\mu} = f'_{T_\lambda, T_\mu}(y, z) = \bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \dots \bar{y}_m \tilde{z}_m \bar{y}_{m+1} z_1 y_1 z_1 \bar{y}_{m+2} z_1 \dots z_1 \bar{y}_n z_1 y_1 \cdots .$$

Then

$$f'_{T_\lambda, T_\mu} \equiv \begin{cases} f_{T_\lambda, T_\mu} & \text{if} \quad n - m = 2k + 1 \\ 2 f_{T_\lambda, T_\mu} & \text{if} \quad n - m = 2k \end{cases}$$

Proof. Let

$$f'_{T_\lambda,T_\mu} = f'_{T_\lambda,T_\mu}(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\dots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1y_1z_1\bar{y}_{m+2}z_1\dots z_1\bar{y}_nz_1y_1\cdots$$
$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} y_{\sigma(1)}z_{\tau(1)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_1z_1y_{\sigma(m+2)}z_1\dots z_1y_{\sigma(n)}z_1y_1\cdots .$$

Now, if we set

$$I_l = \{(a_1,a_2,\cdots,a_l)\,|a_i\in\{1,2,\cdots,n\}\ \ a_i<a_j\ \ 1\le i,l\le n\},$$

we can write f'_{T_λ,T_μ} as

$$\sum_{\substack{(i_1,i_2,\cdots,i_m)\in I_m\\ (j_1,j_2,\cdots,j_{n-m})\in I_{n-m}}}\ \sum_{\substack{\xi,\tau\in S_m\\ \rho\in S_{n-m}}} (-1)^{\sigma'}(-1)^{\xi}(-1)^{\rho}(-1)^{\tau}y_{\xi(i_1)}z_{\tau(1)}y_{\xi(i_2)}z_{\tau(2)}\cdots y_{\xi(i_m)}z_{\tau(m)}$$
$$y_{\rho(j_1)}z_1y_1z_1y_{\rho(j_2)}z_1\dots z_1y_{\rho(j_{n-m})}z_1y_1\cdots,$$

where $\{i_1,i_2,\cdots,i_m,j_1,j_2,\cdots,j_{n-m}\}=\{1,2,\cdots,n\}$, and σ' is the permutation of S_n such that $\sigma'(l)=i_l$ for $l=1,2,\cdots,m$ and $\sigma'(l)=j_{l-n}$ for $j=n+1,n+2,\cdots,m$.

$$\sum_{\substack{(i_1,i_2,\cdots,i_m)\in I_m\\ (j_1,j_2,\cdots,j_{n-m})\in I_{n-m}}} (-1)^{\sigma'}\dot{y}_{i_1}\tilde{z}_1\dot{y}_{i_2}\tilde{z}_2\dots\dot{y}_{i_m}\tilde{z}_m\ddot{y}_{j_1}z_1y_1z_1\ddot{y}_{j_2}z_1\dots z_1\ddot{y}_{j_m}z_1y_1\cdots .$$

Now if $n-m=2k+1$, by lemma 3.3.2 and by (3.3), we have $f'_{T_\lambda,T_\mu}\equiv f_{T_\lambda,T_\mu}$.

If $n-m=2k$, we first observe that for any $i,j\neq 1$

$$\bar{y}_iz_1y_1z_1\bar{y}_j \equiv 2\bar{y}_iz_1\bar{y}_jz_1y_1-\bar{y}_1z_1\bar{y}_iz_1\bar{y}_j, \tag{3.39}$$

in fact

$$\bar{y}_1z_1\bar{y}_iz_1\bar{y}_j = y_1z_1\bar{y}_iz_1\bar{y}_j-\bar{y}_iz_1y_1z_1\bar{y}_j+\bar{y}_iz_1\bar{y}_jz_1y_1$$

and by (3.3)

$$\equiv 2\bar{y}_iz_1\bar{y}_jz_1y_1-\bar{y}_iz_1y_1z_1\bar{y}_j$$

hence

$$\bar{y}_iz_1y_1z_1\bar{y}_j \equiv 2\bar{y}_iz_1\bar{y}_jz_1y_1-\bar{y}_1z_1\bar{y}_iz_1\bar{y}_j.$$

Then by lemma 3.3.2, by (3.3) and by (3.39) we have that f'_{T_λ,T_μ} is equivalent to

$$\sum_{\substack{(i_1,i_2,\cdots,i_m)\in I_m\\ (j_1,j_2,\cdots,j_{n-m})\in I_{m-n}}} (-1)^{\sigma'}\alpha_{\rho,k}\dot{y}_{i_1}\tilde{z}_1\dot{y}_{i_2}\tilde{z}_2\dots\dot{y}_{i_m}\tilde{z}_m\hat{y}_{j_1}z_1\hat{y}_{j_2}z_1$$
$$\breve{y}_{j_3}z_1\breve{y}_{j_4}z_1\dots\breve{y}_1z_1\breve{y}_{j_{n-m-1}}z_1\breve{y}_{j_{n-m}}z_1y_1\cdots+2f_{T_\lambda,T_\mu}.$$

Since the first summand is equivalent, up to a scalar, to

$$\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1\ldots z_1\bar{y}_n z_1\bar{y}_1 z_1 y_1\cdots\equiv 0$$

The lemma holds.

□

Similarly to the Lemma 3.4.7, we can prove the following.

Lemma 3.5.2 *Let be $n, m \in \mathbb{N}$, and let*

$$g'_i = g'_i(y,z) = \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i)}z_1 y_{\sigma(i+1)}z_{\tau(i)}\cdots$$
$$\cdots y_{\sigma(m+1)}z_{\tau(m)}y_{\sigma(m+2)}z_1 y_{\sigma(m+3)}\cdots z_1 y_{\sigma(n)}z_1 y_1\cdots,$$

and

$$g'_{m+1} = f_{T_\lambda,T_\mu},$$

then if $m = 2k$ then

$$\begin{cases} g'_{k+t+1} \equiv -g'_{k-t+1} \\ \\ g'_{k+t+1} \equiv (-1)^t \dfrac{t}{k} g'_{2k+1} \qquad t = 0, 1, \cdots, 2k-1, \end{cases}$$

and, if $m = 2k+1$

$$\begin{cases} g'_{2k+1}(\bar{y},\tilde{z}) \equiv -\dfrac{2k-1}{2k+1} g'_{2k+2}(\bar{y},\tilde{z}) \\ \\ g'_{k-t}(\bar{y},\tilde{z}) \equiv g'_{k+t+1}(\bar{y},\tilde{z}) \equiv \dfrac{(-1)^{k+t}}{2k+1}(2t+1)g'_{2k+2}(\bar{y},\tilde{z}). \end{cases}$$

In a similar way it is possible to prove the following

Lemma 3.5.3 *Let*

$$g^t_i = g^t_i(y,z) = y_1\tilde{z}_1\bar{y}_1\tilde{z}_2\bar{y}_2\ldots\tilde{z}_{m-i+1}\bar{y}_{m-i+1}z_1\bar{y}_{m_i+2}\tilde{z}_{m-i+2}$$
$$\ldots\tilde{z}_{m-1}\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1\ldots y_1\bar{y}_n z_1 y_1\cdots,$$

then

$$g^t_{k+1} \equiv \begin{cases} 0 & \text{if} \quad m = 2k \\ \\ \frac{(-1)^k}{2k+1} g^t_{2k+2} & \text{if} \quad m = 2k+1. \end{cases}$$

Moreover

$$g^t_1 \equiv (-1)^{m+1} f^t_{T_\lambda,T_\mu}$$

where

$$f^t_{T_\lambda,T_\mu} = f^t_{T_\lambda,T_\mu}(y,z) = y_1 z_1 \bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \dots \bar{y}_m \tilde{z}_m \bar{y}_{m+1} z_1 \dots z_1 \bar{y}_n z_1 y_1 \cdots .$$

Lemma 3.5.4 *The polynomials f'_{T_λ,T_μ} and $f^t_{T_\lambda,T_\mu}$ are equivalent,* $\mod \mathbb{Z}(A)$.

Proof.

$$f'_{T_\lambda,T_\mu} = \bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \dots \bar{y}_m \tilde{z}_m \bar{y}_{m+1} z_1 y_1 z_1 \bar{y}_{m+2} z_1 \bar{y}_{m+2} z_1 \dots z_1 \bar{y}_n z_1 y_1 \cdots$$

$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma (-1)^\tau y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(m)} z_{\tau(m)} y_{\sigma(m+1)} z_1 y_1 z_1$$
$$y_{\sigma(m+2)} z_1 \dots z_1 y_{\sigma(n)} z_1 y_1 \cdots$$
$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma (-1)^\tau y_{\sigma(m+1)} z_1 y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(m)} z_{\tau(m)} y_1 z_1$$
$$y_{\sigma(m+2)} z_1 \dots z_1 y_{\sigma(n)} z_1 y_1 \cdots$$
$$\sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma (-1)^\tau \left[y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(m)} z_{\tau(m)}, y_{\sigma(m+1)} z_1 \right] y_1 z_1$$
$$y_{\sigma(m+2)} z_1 \dots z_1 y_{\sigma(n)} z_1 y_1 \cdots$$

$$\equiv \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma (-1)^\tau y_{\sigma(m+1)} z_1 y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(m)} z_{\tau(m)} y_1 z_1$$
$$y_{\sigma(m+2)} z_1 \dots z_1 y_{\sigma(n)} z_1 y_1 \cdots$$
$$+ \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma (-1)^\tau y_1 z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(m)} z_{\tau(m)} y_{\sigma(1)} z_1 y_{\sigma(m+1)} z_1$$
$$y_{\sigma(m+2)} z_1 \dots z_1 y_{\sigma(n)} z_1 y_1 \cdots$$
$$- \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma (-1)^\tau y_1 z_1 y_{\sigma(m+1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(m)} z_{\tau(m)} y_{\sigma(1)} z_1$$
$$y_{\sigma(m+2)} z_1 \dots z_1 y_{\sigma(n)} z_1 y_1 \cdots .$$

If we assume that

$$\rho'_j = \rho_{\tau,j} = (1) \quad j = 1,2,3$$

$$\rho_1 = \rho_{\sigma,1} = (\sigma(1)\sigma(2)\cdots\sigma(m+1))^{-1}, \qquad \rho_2 = \rho_{\sigma,2} = (\sigma(1)\sigma(2)\cdots\sigma(m)),$$

$$\rho_3 = \rho_1\rho_2, \qquad \rho'_3 = \rho'_1\rho'_1,$$

we can write the polynomial in the following way □

$$\sum_{\substack{\rho_1\sigma\in S_n\\ \rho'_1\tau\in S_m}} (-1)^{\rho_1}(-1)^{\rho'_1}(-1)^{\rho_1\sigma}(-1)^{\rho'_1\tau} y_{\rho_1\sigma(1)}z_1y_{\rho_1\sigma(2)}z_{\rho'_1\tau(1)}y_{\rho_1\sigma(3)}z_{\rho'_1\tau(2)}\cdots y_{\rho_1\sigma(m+1)}$$

$$z_{\rho'_1\tau(m)}y_1z_1y_{\rho_1\sigma(m+2)}z_1\ldots z_1y_{\rho'_1\sigma(n)}z_1y_1\cdots$$

$$+\sum_{\substack{\rho_2\sigma\in S_n\\ \rho'_2\tau\in S_m}} (-1)^{\rho_2}(-1)^{\rho'_2}(-1)^{\rho_2\sigma}(-1)^{\rho'_2\tau} y_1z_{\rho'_2\tau(1)}y_{\rho_2\sigma(1)}z_{\rho'_2\tau(2)}y_{\rho_2\sigma(2)}\cdots z_{\rho'_2\tau(m)}y_{\rho_2\sigma(m)}$$

$$z_1y_{\rho_2\sigma(m+1)}z_1y_{\rho_2\sigma(m+2)}z_1\ldots z_1y_{\rho'_2\sigma(n)}z_1y_1\cdots$$

$$-\sum_{\substack{\rho_3\sigma\in S_n\\ \rho'_3\tau\in S_m}} (-1)^{\rho_3}(-1)^{\rho'_3}(-1)^{\rho_3\sigma}(-1)^{\rho'_3\tau} y_1z_1y_{\rho_3\sigma(1)}z_{\rho'_3\tau(1)}y_{\rho_3\sigma(2)}z_{\rho'_3\tau(2)}\cdots y_{\rho_3\sigma(m)}$$

$$z_{\rho'_3\tau(m)}y_{\rho_3\sigma(m+1)}z_1y_{\rho_3\sigma(m+2)}z_1\ldots z_1y_{\rho'_3\sigma(n)}z_1y_1\cdots$$

$$= (-1)^m\bar{y}_1z_1\bar{y}_2\tilde{z}_1\bar{y}_3\tilde{z}_2\ldots\bar{y}_{m+1}\tilde{z}_my_1z_1\bar{y}_{m+2}z_1\ldots z_1\bar{y}_nz_1y_1\cdots$$

$$(-1)m-1y_1\tilde{z}_1\bar{y}_1\tilde{z}_2\ldots\bar{y}_{m-1}\tilde{z}_m\bar{y}_mz_1\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1\ldots z_1\bar{y}_nz_1y_1\cdots$$

$$+y_1z_1\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots y_m\tilde{z}_m\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1\ldots z_1\bar{y}_n.$$

$$f'_{T_\lambda,T_\mu} \equiv (-1)^m g'_1 + (-1)^{m-1}g^t_1 + f^t_{T_\lambda,T_\mu}$$

and by lemma 3.5.3 we have $f'_{T_\lambda,T_\mu} \equiv f^t_{T_\lambda,T_\mu}$.

Lemma 3.5.5 *Let*

$$f'_i = f'_i(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_{i-1}\tilde{z}_{i-1}\bar{y}_iz_1y_1\tilde{z}_i\ldots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1\bar{y}_{m+2}\ldots z_1\bar{y}_nz_1y_1\cdots$$

and

$$f^t_i = f^t_i(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_{m-i+1}\tilde{z}_{m-i+1}y_1z_1$$

$$\bar{y}_{m-i+2}\tilde{z}_{m-i+2}\ldots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1\bar{y}_{m+2}\ldots z_1\bar{y}_nz_1y_1\cdots,$$

then

$$f'_i \equiv f^t_i.$$

Proof.

$$f'_i = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_{i-1}\tilde{z}_{i-1}\bar{y}_iz_1y_1\tilde{z}_i\ldots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1\ldots z_1\bar{y}_n$$

$$\sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i)}z_1y_1z_{\tau(i)}\cdots$$

$$y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(m+2)}\cdots z_1y_{\sigma(n)}z_1y_1\cdots$$

$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_1 z_{\tau(i)} \dots y_{\sigma(m)} z_{\tau(m)} y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(i-1)} z_{\tau(i-1)} y_{\sigma(i)} z_1 y_{\sigma(m+1)} z_1$$

$$y_{\sigma(m+2)} z_1 \dots z_1 y_{\sigma(n)} z_1 y_1 \cdots$$

$$+ \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau \left[\left(y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(i-1)} z_{\tau(i-1)} y_{\sigma(i)}\right) z_1, y_1 \left(z_{\tau(i)} \cdots\right.\right.$$

$$\left.\left. y_{\sigma(m)} z_{\tau(m)}\right)\right] y_{\sigma(m+1)} z_1 y_{\sigma(m+2)} z_1 \dots z_1 y_{\sigma(n)} z_1 y_1 \cdots$$

$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_1 z_{\tau(i)} \dots y_{\sigma(m)} z_{\tau(m)} y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(i-1)} z_{\tau(i-1)} y_{\sigma(i)} z_1 y_{\sigma(m+1)} z_1$$

$$y_{\sigma(m+2)} z_1 \dots z_1 y_{\sigma(n)} z_1 y_1 \cdots$$

$$+ \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(m+1)} z_1 y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(i-1)} z_{\tau(i-1)} y_{\sigma(i)} z_{\tau(i)} \cdots y_{\sigma(m)} z_{\tau(m)}$$

$$y_1 z_1 y_{\sigma(m+2)} z_1 \dots z_1 y_{\sigma(n)} z_1 y_1 \cdots$$

$$- \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(m+1)} z_{\tau(i)} \cdots y_{\sigma(m)} z_{\tau(m)} y_1 z_1 y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(i-1)} z_{\tau(i-1)} y_{\sigma(i)}$$

$$z_1 y_{\sigma(m+2)} z_1 \dots z_1 y_{\sigma(n)} z_1 y_1 \cdots .$$

If we assume that

$$\rho_1 = \rho_{\sigma,1} = (\sigma(1)\sigma(2)\dots\sigma(m))^i, \qquad \rho'_1 = \rho_{\tau,1} = (\sigma(1)\sigma(2)\dots\sigma(m))^{i-1},$$
$$\rho_2 = \rho_{\sigma,2} = (\sigma(1)\sigma(2)\dots\sigma(m+1))^{-1}, \qquad \rho'_2 = \rho_{\tau,2} = (1),$$
$$\rho_3 = \rho_1\rho_2, \qquad \rho'_3 = \rho'_1\rho'_2,$$

we can write the polynomial as □

$$\sum_{\substack{\rho_1\sigma\in S_n\\ \rho'_1\tau\in S_m}} (-1)^{\rho_1}(-1)^{\rho'_1}(-1)^{\rho_1\sigma}(-1)^{\rho'_1\tau} y_1 z_{\rho'_1\tau(1)} y_{\rho_1\sigma(1)} \cdots z_{\rho'_1\tau(m)} y_{\rho_1\sigma(m)} z_1 y_{\rho_1\sigma(m+1)}$$

$$z_1 y_{\rho_1\sigma(m+2)} z_1 \dots z_1 y_{\rho_1\sigma(n)} z_1 y_1 \cdots$$

$$+ \sum_{\substack{\rho_2\sigma\in S_n\\ \rho'_2\tau\in S_m}} (-1)^{\rho_2}(-1)^{\rho_2}(-1)^{\rho_2\sigma}(-1)^{\rho'_2\tau} y_{\rho_2\sigma(1)} z_1 y_{\rho_2\sigma(2)} z_{\rho'_2\tau(1)} \cdots y_{\rho_2\sigma(m+1)} z_{\rho'_2\tau(m)} y_1$$

$$z_1 y_{\rho_2\sigma(m+2)} z_1 \dots z_1 y_{\rho_2\sigma(n)} z_1 y_1 \cdots$$

$$- \sum_{\substack{\rho_3\sigma\in S_n\\ \rho'_3\tau\in S_m}} (-1)^{\rho_3}(-1)^{\rho_3}(-1)^{\rho_3\sigma}(-1)^{\rho'_3\tau} y_{\rho_3\sigma(1)} z_{\rho'_3\tau(1)} y_{\rho_3\sigma(2)} z_{\rho'_3\tau(2)} \cdots y_{\rho_3\sigma(m-i+1)} z_{\rho'_3\tau(m-i+1)}$$

$$y_1z_1y_{\rho_3\sigma(m-i+2)}z_{\rho_3'\tau(m-i+2)}\cdots y_{\rho_3\sigma(m)}z_{\rho_3'\tau(m)}y_{\rho_3\sigma(m+1)}z_1y_{\rho_3\sigma(m+2)}z_1\cdots y_{\rho_3\sigma(n)}z_1y_1\cdots$$

$$=(-1)^{m-1}y_1\tilde{z}_1\bar{y}_1\tilde{z}_2\bar{y}_2\ldots\tilde{z}_m\bar{y}_my_1\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1\ldots z_1\bar{y}_nz_1y_1\cdots$$
$$(-1)^m\bar{y}_1z_1\bar{y}_2\tilde{z}_1\ldots\bar{y}_{m+1}\tilde{z}_my_1z_1\bar{y}_{m+2}z_1\ldots z_1\bar{y}_nz_1y_1\cdots$$
$$-\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_{m-i+1}\tilde{z}_{m-i+1}y_1z_1\bar{y}_{m-i+2}\tilde{z}_{m-i+2}\ldots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1\bar{y}_{m+2}\ldots z_1\bar{y}_nz_1y_1\cdots.$$

Hence
$$f_i'\equiv(-1)^{m-1}g_1^t+(-1)^mg_1'+f_i^t$$
and by lemma 3.5.3 we have $f_i'\equiv f_i^t$.

Remark 3.5.6 Let

$$f_{i,m+1}=f_{i,m+1}(y,z)=\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_{i-1}\tilde{z}_{i-1}y_1$$
$$\tilde{z}_i\bar{y}_i\ldots\bar{y}_{m-1}\tilde{z}_m\bar{y}_mz_1\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1\ldots z_1\bar{y}_nz_1y_1\cdots$$

then $f_{i,m+1}\equiv-2f_{i+1,m+1}-f_{i+2,m+1}$ for $i=1,2,\cdots,m-1$.

Moreover since $f_{1,m+1}=g_1^t\equiv f'_{T_\lambda,T_\mu}$ and $f_{m+1,m+1}=f_0^t\equiv f'_{T_\lambda,T_\mu}$ we have that

$$f_{2,m+1}\equiv\frac{(-1)^{m+1}f_{m+1,m+1}-mf_{1,m+1}}{m}\equiv\frac{(-1)^{m+1}-m}{m}f'_{T_\lambda,T_\mu}$$

and

$$f_{m,m+1}\equiv\frac{(-1)^{m+1}f_{1,m+1}-mf_{m+1,m+1}}{m}\equiv\frac{(-1)^{m+1}-m}{m}f'_{T_\lambda,T_\mu}.$$

Lemma 3.5.7 *The polynomials f_i^t and f'_{T_λ,T_μ} are, up to a non zero scalar, equivalent.*

Proof. We first consider the case $m=2k$. We consider first the polynomial f_k^t, then

$$f_k^t=\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_{m-k+1}\tilde{z}_{m-k+1}y_1z_1\bar{y}_{m-k+2}$$
$$\tilde{z}_{m-k+2}\ldots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1\ldots z_1\bar{y}_nz_1y_1\cdots$$

$$
\begin{aligned}
&= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^{\sigma}(-1)^{\tau} y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(m-k+1)}z_{\tau(m-k+1)}y_1z_1\\
&\qquad y_{\sigma(m-k+2)}z_{\tau(m-k+2)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(m+2)}z_1\cdots y_{\sigma(n)}z_1y_1\cdots\\
&= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^{\sigma}(-1)^{\tau} y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(m-k+1)}z_1y_{\sigma(m-k+2)}\\
&\qquad z_{\tau(m-k+2)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_{\tau(m-k+1)}y_1z_1y_{\sigma(m+2)}z_1\cdots y_{\sigma(n)}z_1y_1\cdots\\
&+ \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^{\sigma}(-1)^{\tau} y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(m-k+1)}\left[z_{\tau(m-k+1)}y_1, z_1\right.\\
&\qquad \left.\left(y_{\sigma(m-k+2)}z_{\tau(m-k+2)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}\right)\right] z_1y_{\sigma(m+2)}z_1\cdots y_{\sigma(n)}z_1y_1\cdots\\
&\equiv \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^{\sigma}(-1)^{\tau} y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(m-k+1)}z_1y_{\sigma(m-k+2)}z_{\tau(m-k+2)}\cdots y_{\sigma(m)}\\
&\qquad z_{\tau(m)}y_{\sigma(m+1)}z_{\tau(m-k+1)}y_1z_1y_{\sigma(m+2)}z_1\cdots y_{\sigma(n)}z_1y_1\cdots\\
&+ \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^{\sigma}(-1)^{\tau} y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(m-k)}z_{\tau(m-k)}y_1z_{\tau(m-k+1)}y_{\sigma(m-k+2)}\\
&\qquad z_{\tau(m-k+2)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{m-k+1}z_1y_{\sigma(m+2)}z_1\cdots y_{\sigma(n)}z_1y_1\cdots\\
&- \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^{\sigma}(-1)^{\tau} y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(m-k)}z_{\tau(m-k)}y_{\sigma(m-k+2)}z_{\tau(m-k+2)}\cdots y_{\sigma(m)}\\
&\qquad z_{\tau(m)}y_{\sigma(m+1)}z_1y_1z_{\tau(m-k+1)}y_{\sigma(m-k+1)}z_1y_{\sigma(m+2)}z_1\cdots y_{\sigma(n)}z_1y_1\cdots.
\end{aligned}
$$

If we assume that

$$\rho_1 = \rho_{\sigma,1} = (1), \quad \rho'_1 = \rho'_{\tau,1} = (\tau(k+1)\tau(k+2)\cdots\tau(2k)),$$

$$\rho_2 = \rho_{\sigma,2} = (\sigma(k+1)\sigma(k+2)\cdots\sigma(2k+1)), \quad \rho'_2 = \rho'_{\tau,2} = (1),$$

$$\rho_3 = \rho_1\rho_2, \qquad \rho'_3 = \rho'_1\rho'_2.$$

we can write the polynomial in the following way

$$\sum_{\substack{\rho_1\sigma\in S_n\\ \rho_1'\tau\in S_m}} (-1)^{\rho_1}(-1)^{\rho_1'}(-1)^{\rho_1\sigma}(-1)^{\rho_1'\tau} y_{\rho_1\sigma(1)}z_{\rho_1'\tau(1)}y_{\rho_1\sigma(2)}z_{\rho_1'\tau(2)}\cdots y_{\rho_1\sigma(k+1)}z_1$$

$$y_{\rho_1\sigma(k+2)}z_{\rho_1'\tau k+1}\cdots y_{\rho_1\sigma(m+1)}z_1\cdots z_1 y_{\rho_1\sigma(n)}z_1 y_1\cdots$$

$$+\sum_{\substack{\rho_2\sigma\in S_n\\ \rho_2'\tau\in S_m}} (-1)^{\rho_2}(-1)^{\rho_2'}(-1)^{\rho_2\sigma}(-1)^{\rho_2'\tau} y_{\rho_2\sigma(1)}z_{\rho_2'\tau(1)}y_{\rho_2\sigma(2)}z_{\rho_2'\tau(2)}\cdots y_{\rho_2\sigma(k)}z_{\rho_2'\tau(k)}y_1$$

$$z_{\rho_2'\tau(k+1)}y_{\rho_2\sigma(k+1)}\cdots y_{\rho_2\sigma(m)}z_{\rho_2'\tau(m)}y_{\rho_2\sigma(m+1)}z_1 y_{\rho_2\sigma(m+2)}z_1\cdots z_1 y_{\rho_2\sigma(n)}z_1 y_1\cdots$$

$$-\sum_{\substack{\rho_3\sigma\in S_n\\ \rho_3'\tau\in S_m}} (-1)^{\rho_3}(-1)^{\rho_3'}(-1)^{\rho_3\sigma}(-1)^{\rho_3'\tau} y_{\rho_3\sigma(1)}z_{\rho_3'\tau(1)}y_{\rho_3\sigma(2)}z_{\rho_3'\tau(2)}\cdots z_{\rho_3'\tau(m-1)}$$

$$y_{\rho_3\sigma(m)}z_1 y_1 z_{\rho_3'\tau(m)}y_{\rho_3\sigma(m+1)}z_1 y_{\rho_3\sigma(m+2)}z_1\cdots z_1 y_{\rho_3\sigma(n)}z_1 y_1\cdots$$

$$= (-1)^k \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_{k+1}z_1\bar{y}_{k+2}\tilde{z}_{k+i}\ldots\bar{y}_{m+1}\tilde{z}_m\bar{y}_{m+2}z_1\ldots z_1\bar{y}_n z_1 y_1\cdots$$

$$(-1)^{k+1}\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_k\tilde{z}_k y_1\tilde{z}_{k+1}\bar{y}_{k+1}\ldots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1\ldots z_1\bar{y}_n z_1 y_1\cdots$$

$$+\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_{m-1}\tilde{z}_{m-1}\bar{y}_m z_1 y_1\tilde{z}_m\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1\ldots z_1\bar{y}_n z_1 y_1\cdots.$$

By lemma 3.4.6 and by lemma 3.5.5 we have

$$f_k^t \equiv -\frac{1}{2k+1}f'_{T_\lambda,T_\mu} + f_{2k}^t.$$

Moreover

$$f_i^t \equiv 2f_{i+1}^t - f_{i+2}^t,$$

hence it is easy to check that, up to a scalar,

$$f_i^t \equiv f'_{T_\lambda,T_\mu}$$

for $i = 1, 2, \ldots, m-1$.

Suppose now $m = 2k+1$, and consider the polynomial f_{k+1}^t, and consider the polynomial f_{k+1}^t, then we have

$$f_{k+1}^t = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_{m-(k+1)+1}\tilde{z}_{m-(k+1)+1}y_1 z_1\bar{y}_{m-(k+1)+2}\tilde{z}_{m-(k+1)+2}\cdots\bar{y}_m\tilde{z}_m$$

$$\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1\ldots z_1\bar{y}_n z_1 y_1\cdots$$

$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(m-(k+1)+1)}z_{\tau(m-(k+1)+1)}y_1z_1y_{\sigma(m-(k+1)+2)}$$

$$z_{\tau(m-(k+1)+2)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(m+2)}z_1\cdots z_1y_{\sigma(n)}z_1y_1\cdots$$

$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_1y_{\sigma(m-(k+1)+2)}z_{\tau(m-(k+1)+2)}\cdots y_{\sigma(m)}z_{\tau(m)}$$

$$y_{\sigma(m+1)}z_{\tau(2)}\cdots y_{\sigma(m-(k+1)+1)}z_{\tau(m-(k+1)+1)}y_1z_1y_{\sigma(m+2)}z_1\cdots z_1y_{\sigma(n)}z_1y_1\cdots$$

$$+ \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}\left[\left(z_{\tau(2)}\cdots y_{\sigma(m-(k+1)+1)}z_{\tau(m-(k+1)+1)}\right)y_1, z_1\right.$$

$$\left.\left(y_{\sigma(m-(k+1)+2)}z_{\tau(m-(k+1)+2)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}\right)\right]z_1y_{\sigma(m+2)}z_1\cdots z_1y_{\sigma(n)}z_1y_1\cdots$$

$$\equiv \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_1y_{\sigma(m-(k+1)+2)}z_{\tau(m-(k+1)+2)}\cdots y_{\sigma(m)}z_{\tau(m)}$$

$$y_{\sigma(m+1)}z_{\tau(2)}\cdots y_{\sigma(m-(k+1)+1)}z_{\tau(m-(k+1)+1)}y_1z_1y_{\sigma(m+2)}z_1\cdots z_1y_{\sigma(n)}z_1y_1\cdots$$

$$+ \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_1z_{\tau(2)}\cdots y_{\sigma(m-(k+1)+1)}z_{\tau(m-(k+1)+1)}y_{\sigma(m-(k+1)+2)}$$

$$z_{\tau(m-(k+1)+2)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(2)}z_1y_{\sigma(m+2)}z_1\cdots z_1y_{\sigma(n)}z_1y_1\cdots$$

$$- \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(m-(k+1)+2)}z_{\tau(m-(k+1)+2)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_1$$

$$z_{\tau(2)}\cdots y_{\sigma(m-(k+1)+1)}z_{\tau(m-(k+1)+1)}y_{\sigma(2)}z_1y_{\sigma(m+2)}z_1\cdots z_1y_{\sigma(n)}z_1y_1\cdots$$

If we assume that

$$\rho_1 = \rho_{1,\sigma} = (1), \qquad \rho'_1 = \rho'_{1,\tau} = (\tau(k+1)\tau(k+2)\ldots\tau(2k)),$$

$$\rho_2 = \rho_{2,\sigma} = (\sigma(k+1)\sigma(k+2)\cdots\sigma(2k+1)), \qquad \rho_2 = \rho_{2,\tau} = (1),$$

$$\rho_3 = \rho_1\rho_2, \qquad \rho'_3 = \rho'_1\rho'_2$$

we can write the polynomial in the following way

$$\sum_{\substack{\rho_1\sigma\in S_n\\ \rho'_1\tau\in S_m}} (-1)^{\rho_1}(-1)^{\rho'_1}(-1)^{\rho_1\sigma}(-1)^{\rho'_1\tau} y_{\rho_1\sigma(1)} z_{\rho'_1\tau(1)} y_{\rho_1\sigma(2)} z_1 y_{\rho_1\sigma(3)} z_{\rho'_1\tau(2)} \cdots y_{\rho_1\sigma(m+1)}$$

$$z_{\rho'_1\tau(m)} y_{\rho_1\sigma(m+2)} z_1 \ldots z_1 y_{\rho_1\sigma(n)} z_1 y_1 \cdots$$

$$+ \sum_{\substack{\rho_2\sigma\in S_n\\ \rho'_2\tau\in S_m}} (-1)^{\rho_2}(-1)^{\rho'_2}(-1)^{\rho_2\sigma}(-1)^{\rho'_2\tau} y_{\rho_1\sigma(1)} z_{\rho'_2\tau(1)} y_1 z_{\rho'_2\tau(2)} y_{\rho_1\sigma(2)} \cdots y_{\rho_1\sigma(m)} z_{\rho'_2\tau(m)}$$

$$y_{\rho_1\sigma(m+1)} z_1 y_{\rho_1\sigma(m+2)} z_1 \ldots z_1 y_{\rho_1\sigma(n)} z_1 y_1 \cdots$$

$$- \sum_{\substack{\rho_3\sigma\in S_n\\ \rho'_3\tau\in S_m}} (-1)^{\rho_3}(-1)^{\rho'_3}(-1)^{\rho_3\sigma}(-1)^{\rho'_3\tau} y_{\rho_3\sigma(1)} z_{\rho'_3\tau(1)} y_{\rho_3\sigma(2)} z_{\rho'_3\tau(2)} \cdots y_{\rho_3\sigma(k+1)} z_{\rho'_3\tau(k+1)}$$

$$y_{\rho_3\sigma(k+2)} z_1 y_1 z_{\rho'_3\tau(k+1)} \cdots y_{\rho_3\sigma(m)} z_{\rho'_3\tau(m)} y_{\rho_3\sigma(m+1)} z_1 y_{\rho_3\sigma(m+2)} z_1 \ldots z_1 y_{\rho_3\sigma(n)} z_1 y_1 \cdots$$

$$- \bar{y}_1 \tilde{z}_1 \bar{y}_2 z_1 \bar{y}_3 \tilde{z}_2 \ldots \bar{y}_{m+1} \tilde{z}_m \bar{y}_{m+2} z_1 \ldots z_1 \bar{y}_n z_1 y_1 \cdots$$

$$+ \bar{y}_1 \tilde{z}_1 y_1 \tilde{z}_2 \bar{y}_2 \ldots \bar{y}_{m-1} \tilde{z}_m \bar{y}_m z_1 \bar{y}_{m+1} z_1 \bar{y}_{m+2} z_1 \ldots z_1 \bar{y}_n z_1 y_1 \cdots$$

$$+ \bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \ldots \bar{y}_{k+1} \tilde{z}_{k+1} \bar{y}_{k+2} z_1 y_1 \tilde{z}_{k+1} \ldots \bar{y}_m \tilde{z}_m \bar{y}_{m+1} z_1 \bar{y}_{m+2} z_1 \ldots z_1 \bar{y}_n z_1 y_1 \cdots ,$$

hence

$$f^t_{k+1} \equiv -g'_2 + h_2 + f'_{k+2}.$$

By remark 3.5.6 and by lemma 3.5.2, 3.5.5 we have that

$$f^t_{k+1} \equiv -\frac{4(k-1)}{2k+1} f'_{T_\lambda,T_\mu} + f^t_{k+2}.$$

Moreover

$$f^t_i \equiv 2f^t_{i+1} - f^t_{i+2}, \qquad i = 0, 1, \cdots, m-2.$$

Hence, up to a scalar, we have that $f^t_i \equiv f'_{T_\lambda,T_\mu}$. □

Lemma 3.5.8 *If $j < i < m$, let*

$$f_{i,j} = f_{i,j}(y,z) = \bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \ldots \bar{y}_{j-1} z_1 \bar{y}_j \tilde{z}_{j-1} \ldots \bar{y}_{i-1} \tilde{z}_{i-2} y_1 \tilde{z}_{i-1} \bar{y}_i \tilde{z}_i \ldots \bar{y}_m \tilde{z}_m$$
$$\bar{y}_{m+1} z_1 \bar{y}_{m+2} z_1 \ldots z_1 \bar{y}_n z_1 y_1 \cdots ,$$

if $i < j < m$ let

$$f_{i,j} = f_{i,j}(y,z) = \bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \ldots \tilde{z}_{i-1} y_1 \tilde{z}_i \bar{y}_i \ldots \tilde{z}_{j-1} \bar{y}_{j-1} z_1 \bar{y}_j \tilde{z}_j \ldots \bar{y}_m \tilde{z}_m$$
$$\bar{y}_{m+1} z_1 \bar{y}_{m+2} z_1 \ldots z_1 \bar{y}_n z_1 y_1 \cdots .$$

Then, up to a scalar,

$$f_{i,j} \equiv f'_{T_\lambda,T_\mu}.$$

Proof. Let $j < i < m$ then

$$f_{i,j} = f_{i,j}(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2 \dots \bar{y}_{j-1}z_1\bar{y}_j\tilde{z}_{j-1} \dots \bar{y}_{i-1}\tilde{z}_{i-2}y_1\tilde{z}_{i-1}\bar{y}_i\tilde{z}_i \dots \bar{y}_m\tilde{z}_m$$
$$\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1 \dots z_1\bar{y}_n z_1 y_1 \cdots$$

$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)} \cdots y_{\sigma(j-1)}z_1y_{\sigma(j)}z_{\tau(j-1)} \cdots y_{\sigma(i-1)}z_{\tau(i-2)}y_1$$
$$z_{\tau(i-1)}y_{\sigma(i)}z_{\tau(i)} \cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(m+2)}z_1 \cdots z_1y_{\sigma(n)}z_1y_1 \cdots$$
$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)} \cdots y_{\sigma(j-1)}z_{\tau(i-2)}y_1z_1y_{\sigma(j)}z_{\tau(j-1)} \cdots y_{\sigma(i-1)}$$
$$z_{\tau(i-1)y_{\sigma(i)}z_{\tau(i)}} \cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(m+2)}z_1 \cdots z_1y_{\sigma(n)}z_1y_1 \cdots$$
$$+ \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)} \cdots y_{\sigma(j-1)} \left[z_1\left(y_{\sigma(j)}z_{\tau(j-1)} \cdots y_{\sigma(i-1)}\right), z_{\tau(i-2)}y_1\right]$$
$$z_{\tau(i-1)}y_{\sigma(i)}z_{\tau(i)} \cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(m+2)}z_1 \cdots z_1y_{\sigma(n)}z_1y_1 \cdots$$
$$\equiv \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)} \cdots y_{\sigma(j-1)}z_{\tau(i-2)}y_1z_1y_{\sigma(j)}z_{\tau(j-1)} \cdots y_{\sigma(i-1)}$$
$$z_{\tau(i-1)y_{\sigma(i)}z_{\tau(i)}} \cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(m+2)}z_1 \cdots z_1y_{\sigma(n)}z_1y_1 \cdots$$
$$+ \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)} \cdots y_{\sigma(j-1)}z_{\tau(i-1)}y_{\sigma(i)}z_{\tau(i)} \cdots y_{\sigma(m)}$$
$$z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(j)}z_{\tau(j-1)} \cdots y_{\sigma(i-1)}z_{\tau(i-2)}y_1z_1y_{\sigma(m+2)}z_1 \cdots z_1y_{\sigma(n)}z_1y_1 \cdots$$
$$- \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)} \cdots y_{\sigma(j-1)}z_{\tau(i-1)}y_{\sigma(i)}z_{\tau(i)} \cdots y_{\sigma(m)}$$
$$z_{\tau(m)}y_{\sigma(m+1)}z_{\tau(i-2)}y_1z_1y_{\sigma(j)}z_{\tau(j-1)} \cdots y_{\sigma(i-1)}z_1y_{\sigma(m+2)}z_1 \cdots z_1y_{\sigma(n)}z_1y_1 \cdots .$$

If we assume that

$$\rho_1 = \rho_{1,\sigma} = (1), \qquad \rho_1' = \rho_{1,\tau}' = (\tau(j-1)\tau(j)\dots\tau(i-2)),$$
$$\rho_2 = \rho_{2,\sigma} = (\sigma(j)\sigma(j+1)\dots\sigma(m+1))^{i-j}, \qquad \rho_2' = \rho_{2,\tau}' = (\tau(j-1)\tau(j)\dots\tau(m))^{i-j}$$
$$\rho_3 = \rho_1\rho_2, \qquad \rho_3' = \rho_1'\rho_2',$$

we can write the polynomial in the following way

$$\sum_{\substack{\rho_1\sigma\in S_n\\ \rho_1'\tau\in S_m}} (-1)^{\rho_1}(-1)^{\rho_1'}(-1)^{\rho_1\sigma}(-1)^{\rho_1'\tau} y_{\rho_1\sigma(1)}z_{\rho_1'\tau(1)}y_{\rho_1\sigma(2)}z_{\rho_1'\tau(2)}\cdots y_{\rho_1\sigma(j-1)}z_{\rho_1'\tau(j-1)}$$

$$y_1 z_1 y_{\rho_1\sigma(j)}z_{\rho_1'\tau(j)}\cdots y_{\rho_1\sigma(m)}z_{\rho_1'\tau(m)}y_{\rho_1\sigma(m+1)}z_1 y_{\rho_1\sigma(m+2)}z_1\cdots z_1 y_{\rho_1\sigma(n)}z_1 y_1\cdots$$

$$+\sum_{\substack{\rho_2\sigma\in S_n\\ \rho_2'\tau\in S_m}} (-1)^{\rho_2}(-1)^{\rho_2'}(-1)^{\rho_2\sigma}(-1)^{\rho_2'\tau} y_{\rho_2\sigma(1)}z_{\rho_2'\tau(1)}y_{\rho_2\sigma(2)}z_{\rho_2'\tau(2)}\cdots y_{\rho_2\sigma(m+i-j+1)}z_1$$

$$y_{\rho_2\sigma(m+i-j+2)}z_{\rho_2'\tau(m+i-j+1)}\cdots y_{\rho_2\sigma(m+1)}z_{\rho_2'\tau(m)}y_{\rho_2\sigma(m+2)}z_1\cdots z_1 y_{\rho_2\sigma(n)}z_1 y_1\cdots$$

$$\sum_{\substack{\rho_3\sigma\in S_n\\ \rho_3'\tau\in S_m}} (-1)^{\rho_3}(-1)^{\rho_3'}(-1)^{\rho_3\sigma}(-1)^{\rho_3'\tau} y_{\rho_3\sigma(1)}z_{\rho_3'\tau(1)}y_{\rho_3\sigma(2)}z_{\rho_3'\tau(2)}\cdots y_{\rho_3\sigma(m+i-j+1)}$$

$$z_{\rho_3'\tau(m+i-j+1)}y_1 z_1 y_{\rho_3\sigma(m+i-j+2)}z_{\rho_3'\tau(m+i-j+2)}\cdots y_{\rho_3\sigma(m)}z_{\rho_3'\tau(m)}y_{\rho_3\sigma(m+1)}$$

$$z_1 y_{\rho_3\sigma(m+2)}z_1\cdots z_1 y_{\rho_3\sigma(n)}z_1 y_1\cdots$$

$$=(-1)^{i-j+1}\bar y_1\tilde z_1\bar y_2\tilde z_2\ldots\bar y_{j-1}\tilde z_{j-1}y_1 z_1\bar y_j\tilde z_j\ldots\bar y_m\tilde z_m\bar y_{m+1}z_1\bar y_{m+2}$$

$$z_1\ldots z_1\bar y_n z_1 y_1\cdots$$

$$(-1)^{i-j}\bar y_1\tilde z_1\bar y_2\tilde z_2\ldots\bar y_{m+i-j+1}z_1\bar y_{m+i-j+2}\tilde z_{m+i-j+1}\ldots\bar y_{m+1}\tilde z_m\bar y_{m+2}$$

$$z_1\ldots z_1\bar y_n z_1 y_1\cdots$$

$$-\bar y_1\tilde z_1\bar y_2\tilde z_2\ldots\bar y_{m+i-j+1}\tilde z_{m+i-j+1}y_1 z_1\bar y_{m+i-j+2}\tilde z_{m+i-j+2}\ldots\bar y_{m+1}$$

$$\tilde z_m\bar y_{m+2}z_1\ldots z_1\bar y_n z_1 y_1\cdots.$$

Hence

$$f_{i,j}\equiv(-1)^{i-j+1}f'_{j-1}+(-1)^{i-j}g'_{m+i-j+1}-f'_{m+i-j+1},$$

by the lemma 3.5.5, 3.5.7 and 3.5.2 we have that $f_{i,j}$ is, up to a scalar, equivalent to f'_{T_λ,T_μ}.

Now let $i<j<m$ then

$$f_{i,j}=\bar y_1\tilde z_1\bar y_2\tilde z_2\ldots\tilde z_{i-1}y_1\tilde z_i\bar y_i\ldots\tilde z_{j-1}\bar y_{j-1}z_1\bar y_j\tilde z_j\ldots\bar y_m\tilde z_m\bar y_{m+1}z_1\bar y_{m+2}z_1\ldots z_1\bar y_n z_1 y_1\cdots$$

$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots z_{\tau(i-1)}y_1z_{\tau(i)}y_{\sigma(i)}\cdots z_{\tau(j-1)}$$

$$y_{\sigma(j-1)}z_1y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(m+2)}z_1\cdots z_1y_{\sigma(n)}z_1y_1\cdots$$

$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots z_{\tau(i-1)}y_{\sigma(j-1)}z_1y_1z_{\tau(i)}y_{\sigma(i)}\cdots z_{\tau(j-1)}$$

$$y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(m+2)}z_1\cdots z_1y_{\sigma(n)}z_1y_1\cdots$$

$$+ \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots z_{\tau(i-1)}\left[y_1\left(z_{\tau(i)}y_{\sigma(i)}\cdots z_{\tau(j-1)}\right),\right.$$

$$\left.y_{\sigma(j-1)}z_1\right]y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(m+2)}z_1\cdots z_1y_{\sigma(n)}z_1y_1\cdots$$

$$\equiv \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots z_{\tau(i-1)}y_{\sigma(j-1)}z_1y_1z_{\tau(i)}y_{\sigma(i)}\cdots z_{\tau(j-1)}$$

$$y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}z_1y_{\sigma(m+2)}z_1\cdots z_1y_{\sigma(n)}z_1y_1\cdots$$

$$+ \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots z_{\tau(i-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(m)}$$

$$z_{\tau(m)}y_{\sigma(m+1)}y_{\sigma(i)}z_{\tau(i)}\cdots z_{\tau(j-1)}y_1z_1y_{\sigma(j)}z_1y_{\sigma(m+2)}z_1\cdots z_1y_{\sigma(n)}z_1y_1\cdots$$

$$- \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots z_{\tau(i-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(m)}z_{\tau(m)}y_{\sigma(m+1)}$$

$$z_1y_{\sigma(j-1)}z_{\tau(i)}y_{\sigma(i)}\cdots z_{\tau(j-1)}y_1z_1y_{\sigma(m+2)}z_1\cdots z_1y_{\sigma(n)}z_1y_1\cdots.$$

If we assume that

$$\rho_1 = \rho_{1,\sigma} = (\sigma(i)\sigma(i+1)\ldots\sigma(j-1)), \qquad \rho_1' = \rho_{1,\tau}' = (1),$$

$$\rho_2 = \rho_{2,\sigma} = (\sigma(i)\sigma i+1\ldots\sigma(m{+}1))^{j-i}, \qquad \rho_2' = \rho_{2,\tau}' = (\tau(i)\tau(i{+}1)\ldots\tau(m)),$$

$$\rho_3 = \rho_1\rho_2, \qquad \rho_3' = \rho_1'\rho_2',$$

we can write the polynomial in the following way

$$
\begin{aligned}
&\sum_{\substack{\rho_1\sigma\in S_n\\ \rho_1'\tau\in S_m}}(-1)^{\rho_1}(-1)^{\rho_1'}(-1)^{\rho_1\sigma}(-1)^{\rho_1'\tau}y_{\rho_1\sigma(1)}z_{\rho_1'\tau(1)}y_{\rho_1\sigma(2)}z_{\rho_1'\tau(2)}\cdots z_{\rho_1'\tau(i-1)}y_{\rho_1\sigma(i)}z_1y_1\\
&\qquad z_{\rho_1'\tau(i)}y_{\rho_1\sigma(i+1)}\cdots z_{\rho_1'\tau(i-1)}y_{\rho_1\sigma(i)}\cdots y_{\rho_1\sigma(m)}z_{\rho_1'\tau(m)}y_{\rho_1\sigma(m+1)}z_1y_{\rho_1\sigma(m+2)}\\
&\qquad z_1\dots z_1y_{\rho_1\sigma(n)}z_1y_1\cdots\\
+&\sum_{\substack{\rho_2\sigma\in S_n\\ \rho_2'\tau\in S_m}}(-1)^{\rho_2}(-1)^{\rho_2'}(-1)^{\rho_2\sigma}(-1)^{\rho_2'\tau}y_{\rho_2\sigma(1)}z_{\rho_2'\tau(1)}y_{\rho_2\sigma(2)}z_{\rho_2'\tau(2)}\cdots y_{\rho_2\sigma(m)}\\
&\qquad z_{\rho_2'\tau(m)}y_1z_1y_{\rho_2\sigma(m+2)}z_1\dots z_1y_{\rho_2\sigma(n)}z_1y_1\cdots\\
-&\sum_{\substack{\rho_3\sigma\in S_n\\ \rho_3'\tau\in S_m}}(-1)^{\rho_3}(-1)^{\rho_3'}(-1)^{\rho_3\sigma}(-1)^{\rho_3'\tau}y_{\rho_3\sigma(1)}z_{\rho_3'\tau(1)}y_{\rho_3\sigma(2)}z_{\rho_3'\tau(2)}\cdots y_{\rho_3\sigma(m+i-j)}z_1\\
&\qquad y_{\rho_3\sigma(m+i-j+1)}z_{\rho_3'\tau(m+i-j)}\cdots y_{\rho_3\sigma(m+1)}z_{\rho_3'\tau(m)}y_1z_1y_{\rho_3\sigma(m+2)}\\
&\qquad z_1\dots z_1y_{\rho_3\sigma(n)}z_1y_1\cdots\\
&\qquad (-1)^{j-i-1}\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\dots\bar{y}_iz_1y_1\tilde{z}_i\dots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1\dots z_1\bar{y}_nz_1y_1\cdots\\
&\qquad -\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\dots\bar{y}_{m-i+1}\tilde{z}_{m-i+1}y_1z_1\bar{y}_{m-i+2}\tilde{z}_{m-i+2}\dots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1\bar{y}_{m+2}\\
&\qquad z_1\dots z_1\bar{y}_nz_1y_1\cdots\\
&\qquad (-1)^{j-i}\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\dots\bar{y}_{m+i-j}z_1\bar{y}_{m+i-j+1}\tilde{z}_{m+i-j}\dots\bar{y}_{m+1}\tilde{z}_m\bar{y}_{m+2}\\
&\qquad z_1\dots z_1\bar{y}_nz_1y_1\cdots.
\end{aligned}
$$

Hence

$$f_{i,j}\equiv(-1)^{j-i-1}f_i'-f_i^t+(-1)^{j-i}g_{m+i-j}',$$

by the lemma 3.5.5, 3.5.7 and 3.5.2 we have that $f_{i,j}$ is , up to a scalar, equivalent to f'_{T_λ,T_μ}. □

Theorem 3.5.9 *Let $n,m,r,s\in\mathbb{N}$ such that $1\le m<n$, let $\lambda=(r-n+1,1^{n-1})$, $\mu=(s-m+1,1^{m-1})$ and let T_λ, T_μ a pair of Young tableaux. If f is the highest weight vector corresponding to T_λ, T_μ, then, up to a scalar,*

$$f\equiv f_{T_\lambda,T_\mu}$$

Proof. We can consider without loss of generality that f starts with a y, hence $r=s$ or $r=s+1$. By lemma 3.5.3 we have that if $r+s=n+m+1$ then any highest weight vector is linearly dependent, $\mod \mathrm{Id}^{\mathbb{Z}}(A)$, to f_{T_λ,T_μ}. If $r+s=n+m+2$, lemma 3.5.1, 3.5.5, 3.5.7 and 3.5.8 show that any highest weight vector is linearly dependent to f_{T_λ,T_μ}. Let $r+s>n+m+2$, we prove that any highest weight vector is linearly dependent, $\mod \mathrm{Id}^{\mathbb{Z}}(A)$, to the set S of the following polynomials:

- $\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\dots\bar{y}_i z_1\bar{y}_{i+1}\tilde{z}_i\dots\bar{y}_{m+1}\tilde{z}_m\bar{y}_{m+2}z_1\dots z_1\bar{y}_n z_1 y_1\dots,$

- $\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\dots\bar{y}_{i-1}\tilde{z}_{i-1}y_1 z_1\bar{y}_i\tilde{z}_i\dots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1\dots z_1\bar{y}_n z_1 y_1\dots,$

- $\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\dots\bar{y}_i z_1 y_1\tilde{z}_i\bar{y}_{i+1}\tilde{z}_{i+1}\dots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1\bar{y}_{m+2}z_1\dots z_1\bar{y}_n z_1 y_1\dots,$

- $\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\dots\bar{y}_{i-1}\tilde{z}_{i-1}y_1\tilde{z}_i\bar{y}_i\dots\tilde{z}_{j-1}\bar{y}_{j-1}z_1\bar{y}_j\tilde{z}_j\dots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}z_1$
  $\bar{y}_{m+2}z_1\dots z_1\bar{y}_n z_1 y_1\dots,$

- $\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\dots\bar{y}_{j-1}\tilde{z}_{j-1}\bar{y}_j z_1\bar{y}_{j+1}\tilde{z}_j\dots\bar{y}_{i-1}\tilde{z}_{i-2}y_1\tilde{z}_{i-1}\bar{y}_i\tilde{z}_i\dots\bar{y}_m\tilde{z}_m\bar{y}_{m+1}$
  $z_1\bar{y}_{m+2}z_1\dots z_1\bar{y}_n z_1 y_1\dots$

Wich are, up to a non zero scalar, equivalent to f_{T_λ,T_μ} Suppose $r = s+1$, it follows that any highest weight vector $f = f(y,z)$ have at least two non-alternating y, let y_1', y_1'', and suppose also

$$f = \bar{y}_1\tilde{z}_1\dots y_1'\dots y_1''\dots\tilde{z}_n$$

hence

$$\begin{aligned} f &= \sum_{\sigma,\tau\in S_n} y_{\sigma(1)}z_{\tau(1)}\dots y_1' p_1^{\sigma,\tau}(y,z)y_1'' p_2^{\sigma,\tau}(y,z)y_{\sigma(m+1)} \\ &= \sum_{\sigma,\tau\in S_n} y_{\sigma(1)}z_{\tau(1)}\dots y_1' p_2^{\sigma,\tau}(y,z)y_{\sigma(m+1)}p_1^{\sigma,\tau}(y,z)y_1'' \\ &+ \sum_{\sigma,\tau\in S_n} y_{\sigma(1)}z_{\tau(1)}\dots y_1'\left[p_1^{\sigma,\tau}(y,z)y_1'', p_2^{\sigma,\tau}(y,z)y_{\sigma(m+1)}\right] \\ &\equiv \sum_{\sigma,\tau\in S_n} y_{\sigma(1)}z_{\tau(1)}\dots y_1' p_2^{\sigma,\tau}(y,z)y_{\sigma(m+1)}p_1^{\sigma,\tau}(y,z)y_1'' \\ &+ \sum_{\sigma,\tau\in S_n} y_{\sigma(1)}z_{\tau(1)}\dots y_1'' p_1^{\sigma,\tau}(y,z)y_{\sigma(m+1)}p_2^{\sigma,\tau}(y,z)y_1' \\ &- \sum_{\sigma,\tau\in S_n} y_{\sigma(1)}z_{\tau(1)}\dots y_{\sigma(m+1)}p_2^{\sigma,\tau}(y,z)y_1'' p_1^{\sigma,\tau}(y,z)y_1'. \end{aligned}$$

$$= \alpha\bar{y}_1\tilde{z}_1\dots y_1'\dots\bar{y}_{m+1}\dots y_1'' + \beta\bar{y}_1\tilde{z}_1\dots y_1''\dots\bar{y}_{m+1}\dots y_1' + \gamma\bar{y}_1\tilde{z}_1\dots\bar{y}_{m+1}\dots y_1'\dots y_1''$$

Hence f is, mod $\mathrm{Id}^{\mathbb{Z}}(A)$, linearly dependent to a set of polynomials such that the non-alternating y, that we can find before the $\bar{y}_{m+1}$ is in a previous position respect to f. Now in a similar way it is possible to prove that every polynomial obtained before are linearly dependent to a set of polynomials such that the number of non-alternating z that we can find before the $\tilde{z}_m$is in a previous position respect to f, and we can continue this process till every polynomials are in S. Obviously if $r = s$ we can obtain the same result. Now, by (3.5) and (3.39), we have that

$$f \equiv f_{T_\lambda,T_\mu}.$$

□

3.6 $h_1(\lambda) < h_1(\mu)$

Throughout this section we consider the pairs of Young tableaux whose heights are $h_1(\lambda) = n$ and $h_1(\mu) = m$, such that $n < m$, and we set

$$f_{T_\lambda,T_\mu} = f_{T_\lambda,T_\mu}(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\dots\bar{y}_n\tilde{z}_n y_1\tilde{z}_{n+1}y_1\dots y_1\tilde{z}_m.$$

Similarly to 3.5.1 it is possible to prove the following lemma

Lemma 3.6.1 *Let $n, m \in \mathbb{N}$ $n < m$, and let*

$$f'_{T_\lambda,T_\mu} = f'_{T_\lambda,T_\mu}(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\dots\bar{y}_n\tilde{z}_n y_1 z_1 y_1\tilde{z}_{n+1}y_1\tilde{z}_{n+2}y_1\dots y_1\tilde{z}_m y_1 z_1\cdots.$$

Then

$$f'_{T_\lambda,T_\mu} \equiv \begin{cases} f_{T_\lambda,T_\mu} & \textit{if} \quad m-n=2k \\ \\ 2f_{T_\lambda,T_\mu} & \textit{if} \quad m-n=2k+1 \end{cases}$$

We can prove, similarly to lemma 3.4.7 the following result.

Lemma 3.6.2 *Let be $n, m \in \mathbb{N}$ and let*

$$g_i = g_i(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\dots\tilde{z}_{i-1}y_1\tilde{z}_i\bar{y}_i\dots\bar{y}_{n-1}\tilde{z}_n\bar{y}_n\tilde{z}_{n+1}y_1\tilde{z}_{n+2}y_1\dots y_1\tilde{z}_m y_1 z_1\cdots.$$

then

if $n = 2k+1$

$$g_{k+t+1} \equiv -g_{k-t} \equiv \begin{cases} \dfrac{2k-1}{2k+1} f'_{T_\lambda,T_\mu} & \textit{if} \quad t=k \\ \\ \dfrac{(-1)^{k+t}}{2k+1}(2t+1)\ f'_{T_\lambda,T_\mu} & \textit{if} \quad 0 \le t < k, \end{cases}$$

if $n = 2k$

$$g_{k+1+t} \equiv g_{k+1-t} \equiv (-1)^t \frac{t}{k}\ f'_{T_\lambda,T_\mu} \qquad \textit{if } 0 \le t \le k.$$

Lemma 3.6.3 *Let $n, m \in \mathbb{N}$, and let*

$f_i = f_i(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_{i-1}\tilde{z}_{i-1}y_1z_1\bar{y}_i\tilde{z}_i\ldots\bar{y}_n\tilde{z}_ny_1\tilde{z}_{n+1}y_1\ldots y_1\tilde{z}_my_1z_1\cdots,$

then f_i and f'_{T_λ,T_μ} are equivalent, $\mod \mathrm{Id}^{\mathbb{Z}}(A)$.

Proof.

We consider the polynomial f_{n-1}, then

$$\begin{aligned} f_{n-1} =& \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_{n-1}\tilde{z}_{n-1}y_1z_1\bar{y}_n\tilde{z}_ny_1\tilde{z}_{n+1}y_1\ldots y_1\tilde{z}_my_1z_1\cdots \\ =& \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}}(-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(n-1)}z_{\tau(n-1)}y_1z_1y_{\sigma(n)} \\ & z_{\tau(n)}y_1z_{\tau(n+1)}y_1\ldots y_1z_{\tau(m)}y_1z_1\cdots \end{aligned}$$

$$\begin{aligned} \sum_{\substack{(i_1,i_2,\cdots,i_{n-1})\in I_{n-1}\\ (j_1,j_2,\cdots,j_{m-n+1})\in I_{m-n+1}}}\ \sum_{\substack{\sigma\in S_n\\ \xi\in S_{n+1}\\ \rho\in S_{m-n+1}}} & (-1)^{\tau'}(-1)^\xi(-1)^\rho(-1)^\sigma y_{\sigma(1)}z_{\xi(i_1)}y_{\sigma(2)} \\ & z_{\xi(i_2)}\cdots y_{\sigma(n-1)}z_{\xi(i_{n-1})}y_1z_1y_{\sigma(n)}z_{\rho(j_1)}y_1z_{\rho(j_2)} \\ & y_1\ldots y_1z_{\rho(j_{m-n+1})}y_1z_1\cdots \end{aligned}$$

$$\begin{aligned} = \sum_{\substack{(i_1,i_2,\cdots,i_{n-1})\in I_{n-1}\\ (j_1,j_2,\cdots,j_{m-n+1})\in I_{m-n+1}}} & (-1)^{\tau'}\bar{y}_1\dot{z}_{i_1}\bar{y}_2\dot{z}_{i_2}\ldots\bar{y}_{n-1}\dot{z}_{i_{n-1}}y_1z_1\bar{y}_n \\ & \ddot{z}_{j_1}y_1\ddot{z}_{j_2}y_1\ldots y_1\ddot{z}_{j_{m-n+1}}y_1z_1\cdots, \end{aligned}$$

where I_l is defined as in the lemma 3.5.1, $I_{n-1}\cup I_{m-n+1} = \{1,2,\cdots,m\}$ and τ' is a permutation of S_m such that $\tau'(l) = i_l$ for $l = 1,2,\cdots,n-1$ and $\tau'(l) = l-n+1$ for $l = n, n+1,\cdots,m$.

Now by lemma 3.5.4, 3.5.5 and 3.5.5, we have that, up to a scalar,

$$\sum_{\substack{(i_1,i_2,\cdots,i_{n-1})\in I_m\\ (j_1,j_2,\cdots,j_{m-n})\in I_{m-n+1}}}(-1)^{\tau'}\bar{y}_1\dot{z}_{i_1}\bar{y}_2\dot{z}_{i_2}\ldots\bar{y}_{n-1}\dot{z}_{i_{n_1}}\bar{y}_n\ddot{z}_{j_1}y_1z_1y_1\ddot{z}_{j_2}y_1\ldots y_1\ddot{z}_{j_{m-n+1}}y_1z_1\cdots$$

$$\begin{aligned} = \sum_{\substack{(i_1,i_2,\cdots,i_{n-1})\in I_m\\ (j_1,j_2,\cdots,j_{m-n})\in I_{m-n+1}}}\ \sum_{\substack{\sigma\in S_n\\ \xi\in S_{n+1}\\ \rho\in S_{m-n+1}}} & (-1)^{\tau'}(-1)^\xi(-1)^\rho(-1)^\sigma y_{\sigma(1)}z_{\xi(i_1)}y_{\sigma(2)}z_{\xi(i_2)}\cdots y_{\sigma(n-1)} \\ & z_{\xi(i_{n-1})}y_{\sigma(n)}y_1z_1z_{\rho(j_1)}y_1z_{\rho(j_2)}y_1\ldots y_1z_{\rho(j_{m-n+1})}y_1z_1\cdots \end{aligned}$$

$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^{\sigma}(-1)^{\tau} y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(n-1)}z_{\tau(n-1)}y_{\sigma(n)}y_1z_1$$

$$z_{\tau(n)}y_1z_{\tau(n+1)}y_1\cdots y_1z_{\tau(m)}y_1z_1\cdots$$

$$= \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_n\tilde{z}_ny_1z_1y_1\tilde{z}_{n+1}y_1\ldots y_1\tilde{z}_my_1z_1\cdots = f'_{T_\lambda,T_\mu}$$

Moreover it can be easy to check, as for (3.33), that for $i = 2,\cdots,n-1$

$$f_i \equiv 2f_{i+1} - f_{i+2}. \tag{3.40}$$

Hence, up to a scalar, $f_i \equiv f'_{T_\lambda,T_\mu}$. □

Lemma 3.6.4 *Let $n, m \in \mathbb{N}$, and let*

$$f^i = f^i(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_iz_1y_1\tilde{z}_i\bar{y}_{i+1}\ldots\bar{y}_n\tilde{z}_ny_1\tilde{z}_{n+1}y_1\ldots y_1\tilde{z}_my_1z_1\cdots,$$

then f_i and f'_{T_λ,T_μ} are equivalent, $\mod \mathrm{Id}^{\mathbb{Z}}(A)$.

Proof. We first observe that, as in the lemma 3.6.1,

$$f^n \equiv \begin{cases} 2f'_{T_\lambda,T_\mu} & \text{if} \quad m-n=2k \\ f'_{T_\lambda,T_\mu} & \text{if} \quad m-n=2k+1 \end{cases}$$

We consider the polynomial f^{n-1}, then

$$f^n = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\ldots\bar{y}_{n-1}z_1y_1\tilde{z}_{n-1}\bar{y}_n\tilde{z}_ny_1\tilde{z}_{n+1}y_1\ldots y_1\tilde{z}_my_1z_1\cdots$$

$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^{\sigma}(-1)^{\tau} y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(n-1)}z_1y_1z_{\tau(n-1)}$$

$$y_{\sigma(n)}z_{\tau(n)}y_1z_{\tau(n+1)}y_1\cdots y_1z_{\tau(m)}y_1z_1\cdots$$

$$= \sum_{\substack{(i_1,i_2,\cdots,i_{n-1})\in I_{n-1}\\ (j_1,j_2,\cdots,j_{m-n+1})\in I_{m-n+1}}} \sum_{\substack{\sigma\in S_n\\ \xi\in S_{n+1}\\ \rho\in S_{m-n+1}}} (-1)^{\tau'}(-1)^{\xi}(-1)^{\rho}(-1)^{\sigma} y_{\sigma(1)}z_{\xi(i_1)}y_{\sigma(2)}$$

$$z_{\xi(i_2)}\cdots y_{\sigma(n-1)}z_1y_1z_{\xi(i_{n-1})}y_{\sigma(n)}z_{\rho(j_1)}y_1z_{\rho(j_2)}$$

$$y_1\cdots y_1z_{\rho(j_{m-n+1})}y_1z_1\cdots$$

$$= \sum_{\substack{(i_1,i_2,\cdots,i_{n-1})\in I_{n-1}\\(j_1,j_2,\cdots,j_{m-n+1})\in I_{m-n+1}}} (-1)^{\tau'} \bar{y}_1 \dot{z}_{i_1} \bar{y}_2 \dot{z}_{i_2} \ldots \bar{y}_{n-1} z_1 y_1 \dot{z}_{i_{n-1}} \bar{y}_n$$

$$\ddot{z}_{j_1} y_1 \ddot{z}_{j_2} y_1 \ldots y_1 \ddot{z}_{j_{m-n+1}} y_1 z_1 \cdots ,$$

where I_l is defined as in the previous lemma, and $I_{n-1} \cup I_{m-n+1} = \{1, 2, \cdots, m\}$ and τ' is a permutation of S_m such that $\tau'(l) = i_l$ for $l = 1, 2, \cdots, n-1$ and $\tau'(l) = l - n + 1$ for $l = n, n+1, \cdots, m$.

Now by lemma 3.5.4, 3.5.5 and 3.5.5, we have that, up to a scalar,

$$\sum_{\substack{(i_1,i_2,\cdots,i_{n-1})\in I_m\\(j_1,j_2,\cdots,j_{m-n})\in I_{m-n+1}}} (-1)^{\tau'} \bar{y}_1 \dot{z}_{i_1} \bar{y}_2 \dot{z}_{i_2} \ldots \bar{y}_{n-1} \dot{z}_{i_{n_1}} \bar{y}_n \ddot{z}_{j_1} y_1 z_1 y_1$$

$$\ddot{z}_{j_2} y_1 \ldots y_1 \ddot{z}_{j_{m-n+1}} y_1 z_1 \cdots$$

$$= \sum_{\substack{(i_1,i_2,\cdots,i_{n-1})\in I_m\\(j_1,j_2,\cdots,j_{m-n})\in I_{m-n+1}}} \sum_{\substack{\sigma\in S_n\\ \xi\in S_{n+1}\\ \rho\in S_{m-n+1}}} (-1)^{\tau'} (-1)^{\xi} (-1)^{\rho} (-1)^{\sigma} y_{\sigma(1)} z_{\xi(i_1)} y_{\sigma(2)}$$

$$z_{\xi(i_2)} \cdots y_{\sigma(n-1)} z_{\xi(i_{n-1})} y_{\sigma(n)} z_{\rho(j_1)} y_1 z_1 y_1 z_{\rho(j_2)}$$
$$y_1 \ldots y_1 z_{\rho(j_{m-n+1})} y_1 z_1 \cdots$$

$$= \sum_{\substack{\sigma\in S_n\\ \tau\in S_m}} (-1)^{\sigma} (-1)^{\tau} y_{\sigma(1)} z_{\tau(1)} y_{\sigma(2)} z_{\tau(2)} \cdots y_{\sigma(n-1)} z_{\tau(n-1)} y_{\sigma(n)} z_{\tau(n)}$$

$$y_1 z_1 y_1 z_{\tau(n+1)} y_1 \ldots y_1 z_{\tau(m)} y_1 z_1 \cdots$$

$$\bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \ldots \bar{y}_n \tilde{z}_n y_1 z_1 y_1 \tilde{z}_{n+1} y_1 \ldots y_1 \tilde{z}_m y_1 z_1 \cdots = f'_{T_\lambda, T_\mu}.$$

Moreover it is easy to check that for $i = 2, \cdots n-1$, as for the remark 3.4.4,

$$f^i \equiv 2f^{i+1} - f^{i+2}$$

hence, up to a scalar, $f_i \equiv f'_{T_\lambda, T_\mu}$. □

Lemma 3.6.5 *Let $i < j \in \mathbb{N}$ and let*

$$f_{i,j} = f_{i,j}(y,z) = \bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \cdots \bar{y}_{i-1} \tilde{z}_{i-1} y_1 \tilde{z}_i \bar{y}_i \cdots \tilde{z}_{j-1} \bar{y}_{j-1}$$

$$z_1 \bar{y}_j \tilde{z}_j \cdots \bar{y}_n \tilde{z}_n y_1 \tilde{z}_{n+1} \ldots y_1 \tilde{z}_m y_1 z_1 \cdots$$

then, up to a scalar,

$$f_{i,j} \equiv f'_{T_\lambda, T_\mu}.$$

Proof. We consider the polynomial $f_{i,j}$, then

$$\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{i-1}\tilde{z}_{i-1}y_1\tilde{z}_i\bar{y}_i\cdots\tilde{z}_{j-1}\bar{y}_{j-1}z_1\bar{y}_j\tilde{z}_j\cdots\bar{y}_n\tilde{z}_n y_1\tilde{z}_{n+1}\ldots u_1\tilde{z}_m y_1 z_1\cdots$$

$$\begin{aligned}
&= \sum_{\sigma,\tau\in S_n}(-1)^{\sigma}(-1)^{\tau}y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_1z_{\tau(i)}\\
&\qquad y_{\sigma(i)}\cdots z_{\tau(j-1)}y_{\sigma(j-1)}z_1y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_{\tau(n+1)}\\
&\qquad y_1\ldots y_1z_{\tau(m)}y_1z_1\cdots\\
&+ \sum_{\sigma,\tau\in S_n}(-1)^{\sigma}(-1)^{\tau}y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}\left[y_1z_{\tau(i)},y_{\sigma(i)}\right.\\
&\qquad \left.z_{\tau(i+1)}\cdots z_{\tau(j-1)}y_{\sigma(j-1)}z_1\right]y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_{\tau(n+1)}\\
&\qquad y_1\ldots y_1z_{\tau(m)}y_1z_1\cdots\\
&\equiv \sum_{\sigma,\tau\in S_n}(-1)^{\sigma}(-1)^{\tau}y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(i)}\cdots\\
&\qquad z_{\tau(j-1)}y_{\sigma(j-1)}z_1y_1z_{\tau(i)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_{\tau(n+1)}\\
&\qquad y_1\ldots y_1z_{\tau(m)}y_1z_1\cdots\\
&+ \sum_{\sigma,\tau\in S_n}(-1)^{\sigma}(-1)^{\tau}y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(i-1)}z_{\tau(i-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(n)}z_{\tau(n)}\\
&\qquad \left[y_1z_{\tau(i)},y_{\sigma(i)}\cdots z_{\tau(j-1)}y_{\sigma(j-1)}z_1\right]y_1z_{\tau(n+1)}y_1\ldots y_1z_{\tau(m)}y_1z_1\cdots.
\end{aligned}$$

if we assume that

$$\rho_1=\rho_{1,\sigma}=(1),\qquad \rho_1'=\rho_{1,\tau}=(\tau(i)\tau(i+1)\cdots\tau(j-i)),$$
$$\rho_2=\rho_{2,\sigma}=(\sigma(i)\sigma(i{+}1)\cdots\sigma(n))^{j-i},\qquad \rho_2'=\rho_{2,\tau}=(\tau(i)\tau(i{+}1)\cdots\tau(n))^{j-i}$$
$$\rho_3=\rho_1\rho_2,\qquad \rho_3'=\rho_1'\rho_2',$$

we can write the polynomial in the following way

$$\sum_{\substack{\rho_1\sigma\in S_n\\ \rho_1'\tau\in S_n}} (-1)^{\rho_1}(-1)^{\rho_1'}(-1)^{\rho_1\sigma}(-1)^{\rho_1'\tau} y_{\rho_1\sigma(1)} z_{\rho_1'\tau(1)} y_{\rho_1\sigma(2)} z_{\rho_1'\tau(2)} \cdots y_{\rho_1\sigma(j-2)}$$

$$z_{\rho_1'\tau(j-2)} y_{\rho_1\sigma(j-1)} z_1 y_1 z_{\rho_1'\tau(j-1)} y_{\rho_1\sigma(j)} z_{\rho_1'\tau(j)} \cdots y_{\rho_1\sigma(n)} z_{\rho_1'\tau(n)} y_1 z_{\rho_1'\tau(n+1)}$$

$$y_1 \cdots y_1 z_{\rho_1'\tau(m)} y_1 z_1 \cdots$$

$$+ \sum_{\substack{\rho_2\sigma\in S_n\\ \rho_2'\tau\in S_n}} (-1)^{\rho_2}(-1)^{\rho_2'}(-1)^{\rho_2\sigma}(-1)^{\rho_2'\tau} y_{\rho_2\sigma(1)} z_{\rho_2'\tau(1)} y_{\rho_2\sigma(2)} z_{\rho_2'\tau(2)} \cdots y_{\rho_2\sigma(n+i-j)}$$

$$z_{\rho_2'\tau(n+i-j)} y_1 z_{\rho_2'\tau(n+i-j+1)} y_{\rho_2\sigma(n+i-j+1)} \cdots z_{\rho_2'\tau(n)} y_{\rho_2\sigma(n)} z_1 y_1 z_{\rho_2'\tau(n+1)}$$

$$y_1 \cdots y_1 z_{\rho_2'\tau(m)} y_1 z_1 \cdots$$

$$- \sum_{\substack{\rho_3\sigma\in S_n\\ \rho_3'\tau\in S_n}} (-1)^{\rho_3}(-1)^{\rho_3'}(-1)^{\rho_3\sigma}(-1)^{\rho_3'\tau} y_{\rho_3\sigma(1)} z_{\rho_3'\tau(1)} y_{\rho_3\sigma(2)} z_{\rho_3'\tau(2)} \cdots y_{\rho_3\sigma(n-1)}$$

$$z_{\rho_3'\tau(n)} z_1 y_1 z_{\rho_3'\tau(n)} y_1 z_{\rho_3'\tau(n+1)} y_1 \cdots y_1 z_{\rho_3'\tau(m)} y_1 z_1 \cdots ,$$

and by lemma 3.6.2, 3.6.3 and 3.6.4 the lemma holds. □

Lemma 3.6.6 *Let $j < i \in \mathbb{N}$ and let*

$$f_{i,j} = f_{i,j}(y,z) = \bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{j-1}\tilde{z}_{j-1}\bar{y}_j z_1\bar{y}_{j+1}\tilde{z}_j\cdots\bar{y}_{i-1}\tilde{z}_{i-2}y_1\tilde{z}_{i-1}\bar{y}_i\tilde{z}_i\cdots\bar{y}_n\tilde{z}_n$$
$$y_1\tilde{z}_{n+1}y_1\cdots y_1\tilde{z}_m y_1 z_1\cdots ,$$

then, up to a scalar,

$$f_{i,j} \equiv f'_{T_\lambda,T_\mu}.$$

Proof. If we consider the polynomial $f_{i,j}$ then

$$\bar{y}_1\tilde{z}_1\bar{y}_2\tilde{z}_2\cdots\bar{y}_{j-1}\tilde{z}_{j-1}\bar{y}_j z_1\bar{y}_{j+1}\tilde{z}_j\cdots\bar{y}_{i-1}\tilde{z}_{i-2}y_1\tilde{z}_{i-1}\bar{y}_i\tilde{z}_i\cdots\bar{y}_n\tilde{z}_n y_1\tilde{z}_{n+1}y_1\cdots y_1\tilde{z}_m y_1 z_1\cdots$$

$$= \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(j-1)}z_{\tau(j-1)}y_{\sigma(j)}z_1y_{\sigma(j+1)}z_{\tau(j)}\cdots y_{\sigma(i-1)}$$

$$z_{\tau(i-2)}y_1z_{\tau(i-1)}y_{\sigma(i)}z_{\tau(i)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_{\tau(n+1)}y_1\dots y_1z_{\tau(m)}y_1z_1\cdots$$

$$= \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(j-1)}z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(i-1)}z_{\tau(i-2)}$$

$$y_1z_1y_{\sigma(j+1)}z_{\tau(i-1)}y_{\sigma(i)}z_{\tau(i)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_{\tau(n+1)}y_1\dots y_1z_{\tau(m)}y_1z_1\cdots$$

$$+ \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(j-1)}z_{\tau(j-1)}y_{\sigma(j)}\left[z_1y_{\sigma(j+1)},\right.$$

$$\left.\left(z_{\tau(j)}\cdots y_{\sigma(i-1)}z_{\tau(i-2)}\right)y_1\right]z_{\tau(i-1)}y_{\sigma(i)}z_{\tau(i)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_{\tau(n+1)}$$

$$y_1\dots y_1z_{\tau(m)}y_1z_1\cdots$$

$$\equiv \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(j-1)}z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(i-1)}z_{\tau(i-2)}$$

$$y_1z_1y_{\sigma(j+1)}z_{\tau(i-1)}y_{\sigma(i)}z_{\tau(i)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_{\tau(n+1)}y_1\dots y_1z_{\tau(m)}y_1z_1\cdots$$

$$+ \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(j-1)}z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(i-1)}y_{\sigma(i)}z_{\tau(i)}\cdots y_{\sigma(n)}$$

$$z_{\tau(n)}\left[y_{\sigma(j+1)}z_1, y_1\left(z_{\tau(j)}\cdots y_{\sigma(i-1)}z_{\tau(i-2)}\right)\right]y_1z_{\tau(n+1)}y_1\dots y_1z_{\tau(m)}y_1z_1\cdots$$

$$= \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(j-1)}z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(j)}\cdots y_{\sigma(i-1)}z_{\tau(i-2)}$$

$$y_1z_1y_{\sigma(j+1)}z_{\tau(i-1)}y_{\sigma(i)}z_{\tau(i)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_{\tau(n+1)}y_1\dots y_1z_{\tau(m)}y_1z_1\cdots$$

$$+ \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(j-1)}z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(i-1)}y_{\sigma(i)}$$

$$z_{\tau(i)}\cdots y_{\sigma(n)}z_{\tau(n)}y_{\sigma(j+1)}z_1y_1z_{\tau(j)}\cdots y_{\sigma(i-1)}z_{\tau(i-2)}y_1z_{\tau(n+1)}y_1\dots y_1z_{\tau(m)}y_1z_1\cdots$$

$$- \sum_{\sigma,\tau\in S_n} (-1)^\sigma(-1)^\tau y_{\sigma(1)}z_{\tau(1)}y_{\sigma(2)}z_{\tau(2)}\cdots y_{\sigma(j-1)}z_{\tau(j-1)}y_{\sigma(j)}z_{\tau(i-1)}y_{\sigma(i)}$$

$$z_{\tau(i)}\cdots y_{\sigma(n)}z_{\tau(n)}y_1z_{\tau(j)}\cdots y_{\sigma(i-1)}z_{\tau(i-2)}y_{\sigma(j+1)}z_1y_1z_{\tau(n+1)}y_1\dots y_1z_{\tau(m)}y_1z_1\cdots,$$

If we assume that

$$\rho_1 = \rho_{1,\sigma} = (\sigma(j)\sigma(j+1)\cdots\sigma(i-1)), \qquad \rho_1' = \rho_{1,\tau} = (1),$$

$$\rho_2 = \rho_{2,\sigma} = (\sigma(j{+}1)\sigma(j{+}2)\cdots\sigma(n))^{i-j-1}, \qquad \rho_2' = \rho_{2,\tau} = (\tau(j)\tau(j{+}1)\cdots\tau(n))^{i-j-1},$$

$$\rho_3 = \rho_1\rho_2, \qquad \rho_3' = \rho_1'\rho_2',$$

we can write the polynomial in the following way

$$\sum_{\substack{\rho_1\sigma\in S_n\\ \rho'_1\tau\in S_n}} (-1)^{\rho_1}(-1)^{\rho'_1}(-1)^{\rho_1\sigma}(-1)^{\rho'_1\tau} y_{\rho_1\sigma(1)}z_{\rho'_1\tau(1)}y_{\rho_1\sigma(2)}z_{\rho'_1\tau(2)}\cdots y_{\rho_1\sigma(j-2)}z_{\rho'_1\tau(j-2)}$$

$$y_1z_1y_{\rho_1\sigma(j-1)}z_{\rho'_1\tau(j-1)}y_{\rho_1\sigma(j)}z_{\rho'_1\tau(j)}\cdots y_{\rho_1\sigma(n)}z_{\rho'_1\tau(n)}y_1z_{\rho'_1\tau(n+1)}$$

$$y_1\cdots y_1z_{\rho'_1\tau(m)}y_1z_1\cdots$$

$$+\sum_{\substack{\rho_2\sigma\in S_n\\ \rho'_2\tau\in S_n}} (-1)^{\rho_2}(-1)^{\rho'_2}(-1)^{\rho_2\sigma}(-1)^{\rho'_2\tau} y_{\rho_2\sigma(1)}z_{\rho'_2\tau(1)}y_{\rho_2\sigma(2)}z_{\rho'_2\tau(2)}\cdots y_{\rho_2\sigma(n+i-j-1)}$$

$$z_{\rho'_2\tau(n+i-j-1)}y_{\rho_2\sigma(n+i-j)}z_1y_1z_{\rho'_2\tau(n+i-j)}y_{\rho_2\sigma(n+i-j+1)}$$

$$z_{\rho'_2\tau(n+i-j+1)}\cdots y_{\rho_2\sigma(n)}z_{\rho'_2\tau(n)}y_1z_{\rho'_2\tau(n+1)}y_1\cdots y_1z_{\rho'_2\tau(m)}y_1z_1\cdots$$

$$-\sum_{\substack{\rho_3\sigma\in S_n\\ \rho'_3\tau\in S_n}} (-1)^{\rho_3}(-1)^{\rho'_3}(-1)^{\rho_3\sigma}(-1)^{\rho'_3\tau} y_{\rho_3\sigma(1)}z_{\rho'_3\tau(1)}y_{\rho_3\sigma(2)}z_{\rho'_3\tau(2)}\cdots y_{\rho_3\sigma(n+i-j+2)}$$

$$z_{\rho'_3\tau(n+i-j+2)}y_1z_{\rho'_3\tau(n+i-j+3)}y_{\rho_3\sigma(n+i-j+3)}\cdots z_{\rho'_3\tau(n)}y_{\rho_3\sigma(n)}z_1y_1z_{\rho'_3\tau(n+1)}$$

$$y_1\cdots y_1z_{\rho'_3\tau(m)}y_1z_1\cdots.$$

Now by lemma 3.6.2, 3.6.3 and 3.6.4 the lemma holds. □

Similarly to 3.5.9 it is possible to prove the following.

Theorem 3.6.7 *Let $n, m, r, s \in \mathbb{N}$ such that $1 \le n < m$, let $\lambda = (r-n+1, 1^{n-1})$, $\mu = (s-m+1, 1^{m-1})$ and let T_λ, T_μ a pair of Young tableaux, if f is the highest weight vector corresponding to T_λ and T_μ then, up to a scalar,,*

$$f \equiv f_{T_\lambda,T_\mu}.$$

Theorem 3.6.8 *Let $n, m, r, s \in \mathbb{N}$ such that $m \ne n$ and $r \ge n \ge 1$ $s \ge m \ge 1$, let*

$$\chi_{(0,r,s)} = \sum_{\substack{\lambda\vdash r\\ \mu\vdash s}} m_{\lambda,\mu}\chi_\lambda \otimes \chi_\mu.$$

If $\lambda = (r-n+1, 1^{n-1}) \vdash r$ and $\mu = (s-m+1, 1^{m-1}) \vdash s$ then

$$m_{\lambda,\mu} = \begin{cases} 2 & \text{if} \quad r = s \\ 1 & \text{if} \quad |r-s| = 1 \end{cases}$$

Proof. If $r+s = n+m$ it is obvious that $m_{\lambda,\mu} = 2$. Now let $r+s \ne n+m$, then by theorem 3.6.7 and 3.6.8, any highest weight vector is, $\mod \mathrm{Id}^{\mathbb{Z}}(A)$, equivalent to f_{T_λ,T_μ}.

If $m < n$ and $r = s$ we have two different highest weight vectors linearly independent,

$$\bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \cdots \bar{y}_m \tilde{z}_m \bar{y}_{m+1} z_1 \bar{y}_{m+2} z_1 \cdots z_1 \bar{y}_n z_1 y_1 \cdots y_1 z_1,$$

and

$$\tilde{z}_1 \bar{y}_1 \tilde{z}_2 \bar{y}_2 \cdots \tilde{z}_m \bar{y}_m z_1 \bar{y}_{m+1} z_1 \tilde{z}_{m+2} z_1 \ldots z_1 \bar{y}_n z_1 y_1.$$

If $r = s + 1$, or $r = s - 1$ respectively, every highest weight vector is equivalent to

$$\bar{y}_1 \tilde{z}_1 \bar{y}_2 \tilde{z}_2 \cdots \bar{y}_m \tilde{z}_m \bar{y}_{m+1} z_1 \bar{y}_{m+2} z_1 \cdots z_1 \bar{y}_n z_1 y_1 \cdots y_1 z_1 y_1,$$

and, respectively

$$\tilde{z}_1 \bar{y}_1 \tilde{z}_2 \bar{y}_2 \cdots \tilde{z}_m \bar{y}_m z_1 \bar{y}_{m+1} z_1 \tilde{z}_{m+2} z_1 \ldots z_1 \bar{y}_n z_1 y_1 z_1.$$

Similarly if $n < m$.

Hence we have

$$\chi_{(0,r,s)} = \sum_{\substack{\lambda \vdash r \\ \mu \vdash s}} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu,$$

where

$$m_{\lambda,\mu} = \begin{cases} 2 & \text{if} \quad r = s \\ 1 & \text{if} \quad |r - s| = 1 \end{cases}$$

□

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