Musial, Kazimierz. Pettis integrability of multifunctions with values in arbitrary Banach spaces. *J. Convex Anal.* 18 (2011), no. 3, 769810, 28B20 (28B05 46G10 54C60 54C65)

The paper under review is devoted to characterizations of Pettis integrable multifunctions that take as their values non-empty closed convex subsets of a Banach space. There is an abundant literature concerning such an integrability in case the values are subsets of a separable Banach space but achievements concerning non-separable case almost do not exist. In fact, there are only two (but essential) papers of B. Cascales, B. Kadets and J. Rodriguez [see J. Funct. Anal. 256 (2009) no. 3, 673699, MR2484932 and J. Convex Anal. 17 (2010) no. 1, 229240, MR2642727]. Thus, the main goal of Musial's paper is to build a theory in a non-separable situation.

Denote by $c(X), cb(X), cwk(X), ck(X)$ respectively the collection of all non-empty convex closed, non-empty bounded convex closed, non-empty convex weakly compact, non-empty convex compact subsets of X. If $C \in c(X)$ and $x^* \in X^*$, then we set $s(x^*, C) := \sup\{x^*(x) : x \in C\}$.

If (Ω, Σ, μ) is a complete probability space, X is a Banach space and $\mathit{\Gamma} : \Omega \to c(X)$, then $\mathit{\Gamma}$ is said to be Pettis integrable in $cb(X)$ if each function $s(x^*, \mathit{\Gamma})$ is integrable and for each $E \in \Sigma$ there exists $M_{\mathit{\Gamma}}(E) \in cb(X)$ such that

$$s(x^*, M_{\mathit{\Gamma}}(E)) = \int_E s(x^*, \mathit{\Gamma})\, d\mu.$$

Similarly the Pettis integrability in $cwk(X)$ and $ck(X)$ is defined.

The classical approach to Bochner and Pettis type integrals in case of separable Banach spaces is of Aumann type, that is via selections and approximation by simple functions. Unfortunately that approach totally fails in the general case because selections are in general not strongly measurable and cannot be approximated by simple functions. Also the weak topology restricted to weakly compact sets may be now not metrizable and this immediately eliminates some methods of proofs that used to be applied in case of separable Banach spaces. As a consequence, the methods of proofs applied here are, in several points, completely different from those used in case of separable Banach spaces. The techniques applied are closer to the theory of Pettis integration of functions with values in non-separable Banach spaces.

Here are the most essential results of the paper.

1. Two complete characterizations of scalarly integrable multifunctions with convex weakly compact values that are Pettis integrable in $cwk(X)$ (Theorems 2.5 and 4.6). The proof of Theorem 2.5, when restricted to functions, gives a new proof of the corresponding result of Talagrand for Pettis integrable functions. The characterizations are new also in case of separable Banach spaces. The proofs do not invoke to selections.

2. There is a well known result of Diestel [see J. Diestel and J. J., Jr. Uhl, Vector measures. Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977, MR0453964 and D. B. Dimitrov, Funkcional. Anal. i Priloen 5 (1971) no. 2, 8485, MR0282199] that if a separable Banach space X does not contain any isomorphic copy of c_0, then each X-valued scalarly integrable function is Pettis integrable. It is also known that the result fails for non-separable spaces. The author presents a non-separable version of the above result for multifunctions with convex weakly compact values (Theorem 2.13). Also here no selection theorem is applied. The result is new also for functions (see Theorem 2.14).

3. Each multifunction $\mathit{\Gamma}$ with weakly compact convex values and Pettis integrable in $cwk(X)$ has a representation $\mathit{\Gamma} = G + g$, where g is a Pettis integrable function and G is Pettis integrable in $ck(X)$. In particular, all Pettis integrable selections of G have norm relatively compact ranges of their integrals (Theorems 3.3 and 3.6). This is a significant simplification of the theory.

4. The above results are then applied to obtain a few convergence theorems and a Fatou type theorem for multifunctions that generalize known facts about functions and separable valued multifunctions.

In my opinion the paper under review is a fundamental work that will be very useful for all future integrability investigations.

Reviewer's remarks.

1) In Theorem 1.4 the expression "on $B(X^*)$" repeated twice, should be delated.

2) In Remark 4.9 $l_\infty(\mathit{\Omega})$ should be replaced by l_∞.

Reviewed by (*L. Di Piazza*)