A theoretical link between gradient and nonlocal elasticity models, including higher order boundary conditions

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SUMMARY.

The paper presents a recently developed [1] rational derivation of the strain gradient elasticity model from the nonlocal (or integral) model. This kind of derivations are generally recovered just by an expansion into a Taylor series of the nonlocal strain field up to a certain order, and then operating the integration (or averaging) over the spatial interaction domain. The latter procedure is fully consistent when the analysis is performed over an unbounded domain, but when a classical bounded domain is analyzed it lacks in reproducing the so-called higher-order boundary conditions. In the present contributions the complete derivation is achieved employing an extended version of the Principle of Virtual Power (PVP), written in a special form in order to comply with the non-simple nature of the nonlocal material. Namely, extra terms are invoked in both internal and external virtual power. The body domain is then decomposed in two disjointed subdomains: a core domain (V_c) and a boundary layer domain (V_b). The latter domain is a boundary layer with a fixed thickness equal to the finite interaction distance, R. The passage from the nonlocal format to the gradient one is achieved inserting the standard strains Taylor expansion for the core domain contribution, whereas for the boundary layer domain, a unidirectional expansion series along the normal direction to the external surface S is adopted. By means of the above procedure, and with the subsequent passage to the limit for $R \to 0$, it is observed that the PVP transforms itself into another PVP which describes a strain gradient model. The inspection of the PVP derived does lead to the governing field equations as well as the higher order boundary equation of a standard strain gradient model.

1 INTRODUCTION

The theoretical link between gradient (or weak nonlocal) and integral type (or strong nonlocal) models has been the subject of many contributions [2]. The usual approach followed to derive gradient theory starts from from the nonlocal integral model and then expanding into a Taylor up to the second order term the nonlocal field, a second order gradient theory is recovered. Alternatively, when the kernel functional of the integral operator is chosen in such a way to result a Green function of some partial differential equation then a mathematical equivalence is obtained between the two approaches. In both cases the link is rather complete when the analysis regards unbounded bodies, whereas when bounded bodies are considered the issue of higher order boundary conditions remains not fully understood.

This paper is devoted to consistently transform a given nonlocal mechanical problem of elasticity into a counterpart gradient elasticity problem, including the proper derivation of the related higher order boundary conditions. The fundamental theoretical equipment require to assess this result is the Principle of the Virtual Power (PVP), written in a specific format related to the non local nature of the problem.

The procedure proposed starts dividing the bounded domain V of the nonlocal elastic body into two subdomains $V = V_c \cup V_b$, where V_c is the core domain, and V_b is a boundary layer domain of thickness R

$$V_b = \{\boldsymbol{x} \in V | \, \|\boldsymbol{x} - \boldsymbol{x}_b\| = \delta \leq R\} \tag{1}$$

where $\boldsymbol{x}_b$ is a generic point on the boundary $S = \partial V$ of the body. The quantity $R \geq 0$ is the radius of the nonlocal influence sphere, i.e. the sphere domain in which a generic point $\boldsymbol{x}'$ gives a nonlocal constitutive contribution to the point $\boldsymbol{x}$ (See Fig. 1)

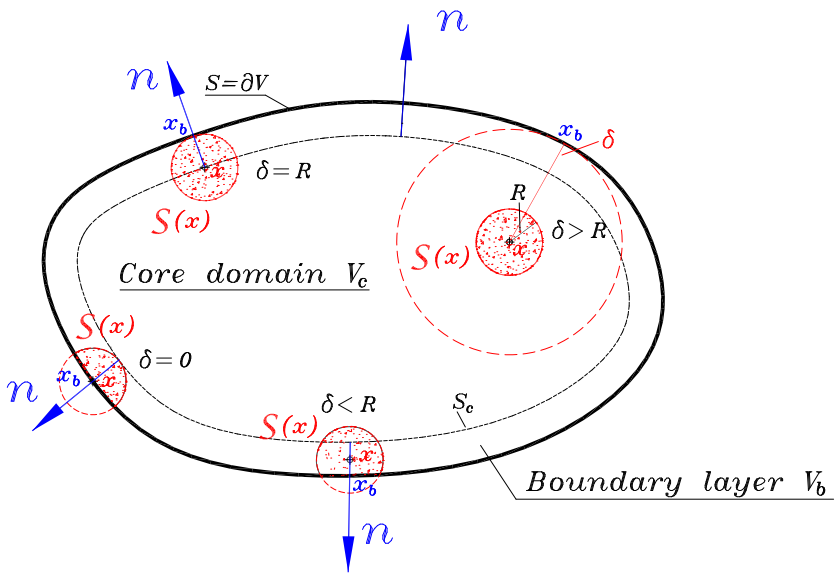

Figure 1: Geometrical sketch showing the domain decomposition into a *core domain* (V_c) and a *boundary layer* ($V_b = V \setminus V_c$), and the influence region $\mathcal{S}(\boldsymbol{x})$ (red dotted area) in different typical locations.

A second point of our procedure consists in the approximation inside the spherical influence region $\mathcal{S}(\boldsymbol{x})$, $\forall \boldsymbol{x} \in V_c$ of the local strains by a Taylor series up to the second order term. Whereas at the points $\boldsymbol{x} \in V_b$ the approximation is achieved by a Taylor series expansion along a direction on the normal $\boldsymbol{n}$ to the surface $S_c = \partial V_c$. The procedure based on the transformation of the Principle of Virtual Power follows three steps, namely:

a) written with reference to a nonlocal elastic problem;

b) the strain field are expanded into a Taylor series if the point is in V_c or in a normal directional expansion if the point is in V_b;

c) finally a limit for $R \to 0$ is performed and we observe that the PVP originally written for a nonlocal problem is consistently transformed into a PVP for a gradient material model.

2 NONLOCAL ELASTICITY MODEL

Let us consider an elastic body occupying a finite domain V of boundary surface $S = \partial V$. $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}$ is the small strain tensor related to the displacement vector $\boldsymbol{u} = \{u_i\}$ by compatibility

relation $\boldsymbol{\varepsilon} = \nabla^s \boldsymbol{u}$. The material is not simple, namely, the stress at a point $\boldsymbol{x} \in V$ is reactive not only to the local strain $\boldsymbol{\varepsilon}(\boldsymbol{x})$, but also to a nonlocal one, say $\boldsymbol{\eta}(\boldsymbol{x})$, expressed as

$$\boldsymbol{\eta}(\boldsymbol{x}) = \mathcal{A}(\mathrm{D}\boldsymbol{\varepsilon}, \boldsymbol{x}) := \int_V \alpha(|\boldsymbol{r}|) \underbrace{[\boldsymbol{\varepsilon}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{\varepsilon}(\boldsymbol{x})]}_{\mathrm{D}\boldsymbol{\varepsilon}} \,\mathrm{d}V(\boldsymbol{r}), \tag{2}$$

where $\boldsymbol{r}$ is the position vector of the integration point $\boldsymbol{x}'$ with respect to the field point $\boldsymbol{x}$, i.e. $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$. Equation (2) defines $\boldsymbol{\eta}$ as the mean weighted value of the strain difference field $\mathrm{D}\boldsymbol{\varepsilon}(\boldsymbol{r}) := \boldsymbol{\varepsilon}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{\varepsilon}(\boldsymbol{x})$ at the point $\boldsymbol{x}$. The material is featured by a stress power as

$$\mathcal{W} := \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{s} : \dot{\boldsymbol{\eta}}, \tag{3}$$

where $\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\eta}}$ are the intrinsic (constitutively independent) deformation modes of the material and $\boldsymbol{\sigma}$ and $\boldsymbol{s}$ are (symmetric) stresses power-conjugate of $\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\eta}}$, respectively. The scalar $\mathcal{W}$ is the power for unit volume expended by the surrounding material through the actual deformation mechanism over a material particle at the generic time t. The deformation of the body is constrained by the boundary conditions

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \qquad \text{on } S_u. \tag{4}$$

The averaging operator $\mathcal{A}$, of eq. (2), is self-adjoint, that is the following Green-type identity holds:

$$\int_V \boldsymbol{s} : \mathcal{A}(\mathrm{D}\boldsymbol{\varepsilon}) \,\mathrm{d}V = \int_V \mathcal{A}(\mathrm{D}\boldsymbol{s}) : \boldsymbol{\varepsilon} \,\mathrm{d}V \tag{5}$$

for any (tensor, or vector, or scalar) fields $\boldsymbol{s}, \boldsymbol{\varepsilon}$. Nonlocal theory features and specific properties of the kernel function α are available in [3, 4].

2.1 *The principle of the virtual power (PVP) for a nonlocal elasticity model*

Let us assume that the body is in an equilibrium state under some body forces $\boldsymbol{b}$ in V and tractions $\boldsymbol{t}$ on S. Also, let $\tilde{\boldsymbol{\varepsilon}}$ and $\tilde{\boldsymbol{\eta}}$, denote any virtual strain field descending from some (continuous) virtual displacement field $\tilde{\boldsymbol{u}}$, such that $\tilde{\boldsymbol{u}} = \boldsymbol{0}$ on $S_u \subseteq S$. Then, the extended PVP for the considered nonlocal material can be cast, recalling the stress power (3), as

$$\underbrace{\int_V \left(\boldsymbol{\sigma} : \tilde{\boldsymbol{\varepsilon}} + \boldsymbol{s} : \tilde{\boldsymbol{\eta}}\right) \mathrm{d}V}_{L_{\mathrm{int}}} = \underbrace{\int_V \boldsymbol{b} \cdot \tilde{\boldsymbol{u}} \,\mathrm{d}v + \int_S \boldsymbol{t} \cdot \tilde{\boldsymbol{u}} \,\mathrm{d}a}_{L_{\mathrm{ext}}} \tag{6}$$

under the kinematic conditions

$$\tilde{\boldsymbol{\eta}} = \mathcal{A}(\mathrm{D}\tilde{\boldsymbol{\varepsilon}}) \quad \text{and} \quad \tilde{\boldsymbol{\varepsilon}} = \nabla^s \tilde{\boldsymbol{u}} \quad \text{in } V \qquad \tilde{\boldsymbol{u}} = \boldsymbol{0}. \quad \text{on } S_u. \tag{7}$$

L_{int}, L_{ext} denote the internal and external virtual powers. Extended forms of the PVP for nonlocal models are not known from the wide literature, except one proposed by the authors [**?**].

By $(7)_1$, the Green identity (5) can be used to rewrite (6) as follows

$$\int_V \underbrace{\left[(\boldsymbol{\sigma} + \mathcal{A}(\mathrm{D}\boldsymbol{s})\right]}_{\boldsymbol{T}} : \tilde{\boldsymbol{\varepsilon}} \,\mathrm{d}V = \int_V \boldsymbol{b} \cdot \tilde{\boldsymbol{u}} \,\mathrm{d}v + \int_S \boldsymbol{t} \cdot \tilde{\boldsymbol{u}} \,\mathrm{d}a \tag{8}$$

where the stress tensor $\boldsymbol{T} = \{T_{ij}\}$, defined as

$$\boldsymbol{T} := \boldsymbol{\sigma} + \mathcal{A}(\mathrm{D}\boldsymbol{s}), \tag{9}$$

denotes the (nonlocal) *total stress*, power-conjugate of the local strain $\boldsymbol{\varepsilon}$. Equation (8) exhibits the format of a standard PVP, $\boldsymbol{T}$ playing the role of Cauchy stress, so that, following standard procedures it is possible to derive the equilibrium equations relating the nonlocal stress $\boldsymbol{T}$, and thus the stresses $(\boldsymbol{\sigma}, \boldsymbol{s})$, to the loads $(\boldsymbol{b}, \boldsymbol{t})$, namely

$$\left.\begin{array}{ll} \nabla \cdot \boldsymbol{T} + \boldsymbol{b} = \boldsymbol{0} & \text{in } V \\ \boldsymbol{n} \cdot \boldsymbol{T} = \boldsymbol{t} & \text{on } S_t := S/S_u \end{array}\right\} \tag{10}$$

In agreement with the absence of extra tractions applied on S, no extra boundary conditions emerge with the present nonlocal model. This is in contrast to a gradient model, which instead contemplates extra boundary conditions.

2.2 *The constitutive equations of nonlocal elasticity*

A general formulation of the constitutive equations of nonlocal elasticity can be achieved by assuming the existence of a free energy as $\psi = \psi(\boldsymbol{\varepsilon}, \boldsymbol{\eta})$, that is, a function of the constitutively independent deformation modes by which the stress power (3) is constructed. Then, under isothermal conditions, the Clausius-Duhem inequality reads, recalling the mentioned stress power, as

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{s} : \dot{\boldsymbol{\eta}} - \dot{\psi}(\boldsymbol{\varepsilon}, \boldsymbol{\eta}) \geq 0 \tag{11}$$

where $\dot{\boldsymbol{\varepsilon}}$, $\dot{\boldsymbol{\eta}}$ denote strain rates occurring in a real deformation mechanism of the material. Assuming that the material can undergo independent deformation modes $\boldsymbol{\varepsilon}$ and $\boldsymbol{\eta}$ and following a classical reasoning, inequality (11) gives the state equations:

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}, \qquad \boldsymbol{s} = \frac{\partial \psi}{\partial \boldsymbol{\eta}}. \tag{12}$$

A simple expression for ψ may be [5]

$$\psi = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{C} : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{C} : \underbrace{\mathcal{A}(\mathrm{D}\boldsymbol{\varepsilon})}_{\boldsymbol{\eta}} \tag{13}$$

from which we obtain

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{C} : \mathcal{A}(\mathrm{D}\boldsymbol{\varepsilon}), \qquad \boldsymbol{s} = \frac{1}{2}\boldsymbol{C} : \boldsymbol{\varepsilon}. \tag{14}$$

Then, substituting (14) into (9) gives the constitutive equation for the total stress, i.e.

$$\boldsymbol{T} = \boldsymbol{C} : (\boldsymbol{\varepsilon} + \mathcal{A}(\mathrm{D}\boldsymbol{\varepsilon}))\,. \tag{15}$$

This expression is quite similar to the elasticity model of Eringen [3]

3 DERIVATION OF THE PVP FOR GRADIENT ELASTICITY

In the previous section we have established the PVP for nonlocal elasticity. In the present section we want to find out a way to derive a gradient elasticity model from the above nonlocal model. For this purpose, let the domain V occupied by the nonlocal material described in Section 2 be decomposed in two subdomains, say $V = V_c \cup V_b$ (see Figure 1). The subdomain V_c (here called *core domain*) collects all point $\boldsymbol{x} \in V$ having a distance $\delta = \delta(\boldsymbol{x})$ from the boundary surface S larger than the influence distance R, i.e. $V_c = \{\boldsymbol{x} \in V : \delta(\boldsymbol{x}) > R\}$. The subdomain V_b denotes the *boundary layer* of constant thickness R, which collects all points $\boldsymbol{x}$ having a distance $\delta(\boldsymbol{x})$ from S smaller than R, i.e. $V_b = \{\boldsymbol{x} \in V : \delta(\boldsymbol{x}) < R\}$ and circumvents the core domain V_c. Obviously, at all point $\boldsymbol{x} \in S_c := \partial V_c$ it is $\delta(\boldsymbol{x}) = R$. A featuring difference between V_c and V_b is that the sphere $\mathcal{V}(\boldsymbol{x})$ of radius R is entirely contained within V whenever it is centered at a point $\boldsymbol{x} \in V_c$, whereas instead it exceeds the boundary surface S whenever it is centered at a point $\boldsymbol{x} \in V_b$, the more the closer is $\boldsymbol{x}$ to S.

After [4] one can state that at any point $\boldsymbol{x} \in V_c$ the distance $\delta(\boldsymbol{x})$ from the boundary surface S can be considered sufficiently large to admit that the material particle at $\boldsymbol{x}$ is subjected to particle interactions from the whole surrounding material up to a maximal distance R, equally distributed in all directions. Instead, for a particle located at a point $\boldsymbol{x} \in V_b$ (where $\delta < R$), the particle interactions originating from all points $\boldsymbol{x}'$ out of S are missing, even if $|\boldsymbol{x}' - \boldsymbol{x}| < R$. The latter physical circumstances are macroscopically accounted by the nonlocal model by means of the specific *interaction region* (= effective integration domain), $\mathcal{B}(\boldsymbol{x})$. In fact, at a point $\boldsymbol{x} \in V_c$, $\mathcal{B}(\boldsymbol{x})$ turns out to coincide with a sphere $\mathcal{V}(\boldsymbol{x})$ of radius R centered at $\boldsymbol{x}$, entirely contained within V; whereas at a point $\boldsymbol{x} \in V_b$, $\mathcal{B}(\boldsymbol{x})$ is a sphere like in the previous case, but has a spherical cup of height $R - \delta$ (protruding from S) cut out, i.e. $\mathcal{B}(\boldsymbol{x}) \subset \mathcal{V}(\boldsymbol{x})$.

At this point, let us reconsider (2) and let us expand the virtual strain field $\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x} + \boldsymbol{r})$ by a Taylor series up to, say, the second-order term, that is

$$\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x} + \boldsymbol{r}) - \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}) = \boldsymbol{r} \cdot \nabla\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + \frac{1}{2}\boldsymbol{r}\,\boldsymbol{r} : \nabla\nabla\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}). \tag{16}$$

We can therefore write, with the aid of a partial indicial notation,

$$\mathcal{A}(\mathrm{D}\tilde{\boldsymbol{\varepsilon}}) = \underbrace{\int_V r_i \alpha(|\boldsymbol{r}|)\,\mathrm{d}v(\boldsymbol{r})}_{W_i}\ \tilde{\boldsymbol{\varepsilon}}_{,i} + \underbrace{\int_V \frac{1}{2} r_i r_j \alpha(|\boldsymbol{r}|)\,\mathrm{d}v(\boldsymbol{r})}_{I_{ij}}\ \tilde{\boldsymbol{\varepsilon}}_{,ij} \tag{17}$$

where the quantities

$$\left.\begin{aligned} W_i(\boldsymbol{x}) &:= \int_V r_i \alpha(|\boldsymbol{r}|)\,\mathrm{d}v(\boldsymbol{r}) = \int_{\mathcal{B}(\boldsymbol{x})} r_i \alpha(|\boldsymbol{r}|)\,\mathrm{d}v(\boldsymbol{r}) \\ I_{ij}(\boldsymbol{x}) &:= \int_V \frac{1}{2} r_i r_j \alpha(|\boldsymbol{r}|)\,\mathrm{d}v(\boldsymbol{r}) = \int_{\mathcal{B}(\boldsymbol{x})} \frac{1}{2} r_i r_j \alpha(|\boldsymbol{r}|)\,\mathrm{d}v(\boldsymbol{r}) \end{aligned}\right\} \tag{18}$$

define, respectively, the first and the second-order weighted moments of the interaction region $\mathcal{B}(\boldsymbol{x})$ with respect to the planes through $\boldsymbol{x}$ parallel to the co-ordinate planes.

For every $\boldsymbol{x} \in V_c$, where $\mathcal{B}(\boldsymbol{x})$ is a sphere (entirely contained within V), by the isotropy of the kernel function α and the radial symmetry of $\mathcal{B}$ it is

$$W_i = 0, \qquad I_{ij} = R^2 I_0 \delta_{ij}, \qquad \forall i, j = (1, 2, 3) \tag{19}$$

where $R^2 I_0$ = second order weighted moment of the sphere of radius R with respect to a plane through the center.

Instead, for any $\boldsymbol{x}$ within the boundary layer V_b —in which the sphere of radius R centered in $\boldsymbol{x}$ exceeds the boundary surface S— assuming local Cartesian orthogonal axeses $\bar{x}_i$ with $\bar{x}_3$ lying upon the normal $\boldsymbol{n}$ to S, we have

$$\left.\begin{array}{ll} W_1 = W_2 = 0, \qquad W_3 = RW_n \qquad (W_n < 0) \\ I_{11} = I_{22} = R^2 I_t, \qquad I_{33} = R^2 I_n, \qquad I_{ij} = 0 \; \forall i \neq j \end{array}\right\} \tag{20}$$

which holds for any $\boldsymbol{x} \in V_b$ whereas W_n, I_t and I_n turns out to be functions of δ. Therefore we can rewrite (17) as

$$\mathcal{A}(\mathrm{D}\tilde{\boldsymbol{\varepsilon}}) := \int_V \alpha(|\boldsymbol{r}|)\,[\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}+\boldsymbol{r}) - \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x})]\,\mathrm{d}v(\boldsymbol{r}) = \int_{\mathcal{B}(\boldsymbol{x})} \alpha(|\boldsymbol{r}|)\,[W_i \tilde{\boldsymbol{\varepsilon}}_{,i} + I_{ij}\tilde{\boldsymbol{\varepsilon}}_{,ij}]\,\mathrm{d}v(\boldsymbol{r}), \tag{21}$$

that is , by (19) and (20),

$$\mathcal{A}(\mathrm{D}\tilde{\boldsymbol{\varepsilon}}) := \left\{\begin{array}{ll} R^2\, I_0\, \tilde{\boldsymbol{\varepsilon}}_{,ii} = R^2\, I_0 \Delta\tilde{\boldsymbol{\varepsilon}} & \forall \boldsymbol{x} \in V_c \\ RW_n \partial_n \tilde{\boldsymbol{\varepsilon}} + R^2 I_n \partial^2_{nn}\tilde{\boldsymbol{\varepsilon}} + R^2 I_t \Delta_{\mathrm{T}}\tilde{\boldsymbol{\varepsilon}} & \forall \boldsymbol{x} \in V_b \end{array}\right\} \tag{22}$$

where Δ = Laplacian, $\Delta_{\mathrm{T}} := \Delta - \partial^2_{nn}$ tangential Laplacian over a surface having a constant distance δ from S.

At this step, it is possible to approximate the internal virtual power L_{int} of (6) making reference to the domain decomposition of Figure 1 and writing

$$\begin{aligned} L_{\mathrm{int}} &= \int_{V_c} [\boldsymbol{\sigma} : \tilde{\boldsymbol{\varepsilon}} + \boldsymbol{s} : \mathcal{A}(\mathrm{D}\tilde{\boldsymbol{\varepsilon}})]\,\mathrm{d}v + \underbrace{\int_{V_b} [\boldsymbol{\sigma} : \tilde{\boldsymbol{\varepsilon}} + \boldsymbol{s} : \mathcal{A}(\mathrm{D}\tilde{\boldsymbol{\varepsilon}})]\,\mathrm{d}v}_{A} \\ &= \int_{V_c} [\boldsymbol{\sigma} : \tilde{\boldsymbol{\varepsilon}} + \underbrace{(-R^2 I_0 \nabla \boldsymbol{s})}_{\boldsymbol{\tau}} \vdots \nabla\tilde{\boldsymbol{\varepsilon}}]\mathrm{d}v + \underbrace{\int_{S_c} R^2 I_0 \boldsymbol{s} : \partial_n \tilde{\boldsymbol{\varepsilon}}\,\mathrm{d}a_c}_{B} + A \\ &= \int_{V_c} [\boldsymbol{\sigma} : \tilde{\boldsymbol{\varepsilon}} + \boldsymbol{\tau} \vdots \nabla\tilde{\boldsymbol{\varepsilon}})]\mathrm{d}v + A + B \end{aligned} \tag{23}$$

where we have introduced the double stress $\boldsymbol{\tau}$ defined as

$$\boldsymbol{\tau} := -R^2 I_0 \nabla \boldsymbol{s}. \tag{24}$$

Then, let the virtual transformation within the boundary layer V_b be featured by a linearly varying displacement along the normal fibers. Hence denoting by l the local abscissa on the generic fiber

($0 \leq l \leq R$), we can write:

$$\left.\begin{array}{l} \tilde{\boldsymbol{u}}(\boldsymbol{x}+l\boldsymbol{n}(\boldsymbol{x})) \approx \tilde{\boldsymbol{u}}(\boldsymbol{x}) + l\partial_n\tilde{\boldsymbol{u}}(\boldsymbol{x}) \\ \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}+l\boldsymbol{n}(\boldsymbol{x})) \approx \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + l\partial_n\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}) \\ \partial_n\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}+l\boldsymbol{n}(\boldsymbol{x})) \approx \partial_n\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}) \\ \partial^2_{nn}\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}+l\boldsymbol{n}(\boldsymbol{x})) \approx 0 \\ \Delta_{\mathrm{T}}\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}+l\boldsymbol{n}(\boldsymbol{x})) \approx \Delta_{\mathrm{T}}\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{x}) \end{array}\right\} \forall l : 0 \leq l \leq R. \tag{25}$$

Next, taking into account (25), we can write the sum $A + B$ of (23) as

$$\begin{aligned} A+B &= \int_{S_c}\int_0^R j\left\{\boldsymbol{\sigma} : (\tilde{\boldsymbol{\varepsilon}} + l\partial_n\tilde{\boldsymbol{\varepsilon}}) + \boldsymbol{s} : (RW_n\partial_n\tilde{\boldsymbol{\varepsilon}} + R^2 I_t\Delta_{\mathrm{T}}\tilde{\boldsymbol{\varepsilon}})\right\}\mathrm{d}l\,\mathrm{d}a_c + \int_{S_c} R^2 I_0\boldsymbol{s} : \partial_n\tilde{\boldsymbol{\varepsilon}}\,\mathrm{d}a_c \\ &= \int_{S_c}\left\{\underbrace{\left(\frac{1}{R}\int_0^R jR\boldsymbol{\sigma}\mathrm{d}l\right)}_{\boldsymbol{s}^{(0)}} : \tilde{\boldsymbol{\varepsilon}} + \underbrace{\left(\frac{1}{R}\int_0^R jR^3 I_t\,\boldsymbol{s})\mathrm{d}l\right)}_{\boldsymbol{s}^{(1)}} : \Delta_{\mathrm{T}}\tilde{\boldsymbol{\varepsilon}} + \right. \\ &\left. + \underbrace{\left(\frac{1}{R}\int_0^R jR(l\boldsymbol{\sigma} + RW_n\boldsymbol{s})\mathrm{d}l + R^2 I_0\boldsymbol{s}\right)}_{\boldsymbol{s}^{(2)}} : \partial_n\tilde{\boldsymbol{\varepsilon}}\right\}\mathrm{d}a_c \end{aligned} \tag{26}$$

where j is the Jacobian to pass from the geometry of the surface S_l located at a constant distance l from S_c ($0 \leq l \leq R$) to the geometry S_c. We admit that for $R \to 0$ there exist the finite limits

$$\left.\begin{array}{l} \boldsymbol{s}^{(0)} := \dfrac{1}{R}\displaystyle\int_0^R jR\boldsymbol{\sigma}\mathrm{d}l \longrightarrow \boldsymbol{\sigma}^{(0)} \\ \boldsymbol{s}^{(1)} := \dfrac{1}{R}\displaystyle\int_0^R jR^3 I_t\boldsymbol{s}\mathrm{d}l \longrightarrow \boldsymbol{\sigma}^{(1)} \\ \boldsymbol{s}^{(2)} := \dfrac{1}{R}\displaystyle\int_0^R jR(l\boldsymbol{\sigma} + RW_n\boldsymbol{s})\mathrm{d}l + R^2 I_0\boldsymbol{s} \longrightarrow \boldsymbol{\sigma}^{(2)} \end{array}\right\} \tag{27}$$

and that as a consequence the quantity $A + B$ of (26) becomes, at the limit for $R \to 0$, as follows:

$$\begin{aligned} A+B &= \int_{S_c}(\boldsymbol{s}^{(0)} : \tilde{\boldsymbol{\varepsilon}} + \boldsymbol{s}^{(1)} : \Delta_{\mathrm{T}}\tilde{\boldsymbol{\varepsilon}} + \boldsymbol{s}^{(2)} : \partial_n\tilde{\boldsymbol{\varepsilon}})\mathrm{d}a_c \quad \longrightarrow \\ &\longrightarrow \quad \underbrace{\int_S(\boldsymbol{\sigma}^{(0)} : \tilde{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^{(1)} : \Delta_{\mathrm{T}}\tilde{\boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^{(2)} : \partial_n\tilde{\boldsymbol{\varepsilon}})\mathrm{d}a.}_{J} \end{aligned} \tag{28}$$

Next, using the notation $\boldsymbol{K} := -\nabla_{\mathrm{T}}\boldsymbol{n}$ (Weingartern tensor) and $H := -\nabla_{\mathrm{T}} \cdot \boldsymbol{n}$ (twice the mean surface curvature) and applying the surface divergence theorem where appropriate, we can write the

following equalities

$$\begin{aligned}\int_S \boldsymbol{\sigma}^{(0)} : \tilde{\boldsymbol{\varepsilon}}\mathrm{d}a &= \int_S \boldsymbol{\sigma}^{(0)} : (\boldsymbol{n}\partial_n\tilde{\boldsymbol{u}} + \nabla_{\mathrm{T}}\tilde{\boldsymbol{u}})\mathrm{d}a \\ &= \int_S \Big[\boldsymbol{n}\cdot\boldsymbol{\sigma}^{(0)}\cdot\partial_n\tilde{\boldsymbol{u}} - (\nabla_{\mathrm{T}} + H\boldsymbol{n})\cdot\boldsymbol{\sigma}^{(0)}\cdot\tilde{\boldsymbol{u}}\Big]\,\mathrm{d}a,\end{aligned} \tag{29}$$

$$\begin{aligned}\int_S \boldsymbol{\sigma}^{(1)} : \Delta_{\mathrm{T}}\tilde{\boldsymbol{\varepsilon}}\mathrm{d}a &= -\int_S (\nabla_{\mathrm{T}} + H\boldsymbol{n})\boldsymbol{\sigma}^{(1)}\vdots\nabla_{\mathrm{T}}\tilde{\boldsymbol{\varepsilon}}\,\mathrm{d}a = \int_S \Delta_{\mathrm{T}}\boldsymbol{\sigma}^{(1)} : \nabla\tilde{\boldsymbol{u}}\,\mathrm{d}a \\ &= \int_S \Delta_{\mathrm{T}}\boldsymbol{\sigma}^{(1)} : (\boldsymbol{n}\partial_n\tilde{\boldsymbol{u}} + \nabla_{\mathrm{T}}\tilde{\boldsymbol{u}})\,\mathrm{d}a \\ &= \int_S \Big\{\boldsymbol{n}\cdot\Delta_{\mathrm{T}}\boldsymbol{\sigma}^{(1)}\cdot\partial_n\tilde{\boldsymbol{u}} - (\nabla_{\mathrm{T}} + H\boldsymbol{n})\cdot\Big[\Delta_{\mathrm{T}}\boldsymbol{\sigma}^{(1)}\Big]\cdot\tilde{\boldsymbol{u}}\Big\}\mathrm{d}a\end{aligned} \tag{30}$$

where we have used the equality $(\nabla_{\mathrm{T}} + H\boldsymbol{n})\cdot(\nabla_{\mathrm{T}} + H\boldsymbol{n}) = \Delta_{\mathrm{T}}$, and finally

$$\begin{aligned}\int_S \boldsymbol{\sigma}^{(2)} : \partial_n\tilde{\boldsymbol{\varepsilon}}\,\mathrm{d}a = \int_S \boldsymbol{\sigma}^{(2)} : \partial_n\nabla\tilde{\boldsymbol{u}}\,\mathrm{d}a &= \int_S \boldsymbol{\sigma}^{(2)} : (\boldsymbol{n}\underbrace{\partial^2_{nn}\tilde{\boldsymbol{u}}}_{\approx 0} + \nabla_{\mathrm{T}}\partial_n\tilde{\boldsymbol{u}} + \boldsymbol{K}\cdot\nabla_{\mathrm{T}}\tilde{\boldsymbol{u}})\mathrm{d}a \\ &= \int_S \big[-(\nabla_{\mathrm{T}} + H\boldsymbol{n})\cdot\boldsymbol{\sigma}^{(2)}\cdot\partial_n\tilde{\boldsymbol{u}} + (\boldsymbol{K}\cdot\boldsymbol{\sigma}^{(2)}) : \nabla_{\mathrm{T}}\tilde{\boldsymbol{u}}\big]\mathrm{d}a \\ &= -\int_S \big[(\nabla_{\mathrm{T}} + H\boldsymbol{n})\cdot\boldsymbol{\sigma}^{(2)}\cdot\partial_n\tilde{\boldsymbol{u}} + (\nabla_{\mathrm{T}} + H\boldsymbol{n})\cdot(\boldsymbol{K}\cdot\boldsymbol{\sigma}^{(2)})\cdot\tilde{\boldsymbol{u}}\big]\mathrm{d}a\end{aligned} \tag{31}$$

where the following equality has been used [6]:

$$\partial_n\,\nabla\tilde{\boldsymbol{u}} = \boldsymbol{n}\partial^2_{nn}\tilde{\boldsymbol{u}} + \nabla_{\mathrm{T}}\partial_n\tilde{\boldsymbol{u}} + \boldsymbol{K}\cdot\nabla_{\mathrm{T}}\tilde{\boldsymbol{u}}. \tag{32}$$

Then the integral J of (28), noting that $\boldsymbol{n}\cdot\boldsymbol{K} = \boldsymbol{0}$, can be cast in the following form, i.e.

$$\begin{aligned}-J &= \int_S \Big\{(\nabla_{\mathrm{T}} + H\boldsymbol{n})\cdot\boldsymbol{\sigma}^{(0)} + (\nabla_{\mathrm{T}} + H\boldsymbol{n})\cdot\Big[\Delta_{\mathrm{T}}\,\boldsymbol{\sigma}^{(1)} + \nabla_{\mathrm{T}}\cdot\boldsymbol{K}\cdot\boldsymbol{\sigma}^{(2)}\Big]\Big\}\cdot\tilde{\boldsymbol{u}}\,\mathrm{d}a \\ &+ \int_S \Big\{-\boldsymbol{n}\cdot\boldsymbol{\sigma}^{(0)} - \boldsymbol{n}\cdot\Big[\Delta_{\mathrm{T}}\,\boldsymbol{\sigma}^{(1)} + (\nabla_{\mathrm{T}} + H\boldsymbol{n})\cdot\boldsymbol{\sigma}^{(2)}\Big]\Big\}\cdot\partial_n\tilde{\boldsymbol{u}}\,\mathrm{d}a \\ &= \int_S (\boldsymbol{q}\cdot\tilde{\boldsymbol{u}} + \boldsymbol{m}\cdot\partial_n\tilde{\boldsymbol{u}})\mathrm{d}a\end{aligned} \tag{33}$$

Here, the symbols $\boldsymbol{q}$ and $\boldsymbol{m}$ denote some extra (ordinary and double) boundary tractions carried in by the approximation process, whereas correspondingly the right hand side of (33) represents a concomitant extra virtual external power. At the limit for $R \to 0$, assuming that (24) continues to hold, we can obtain from (23)

$$L_{\mathrm{int}} \quad \longrightarrow \quad \int_V (\boldsymbol{\sigma} : \tilde{\boldsymbol{\varepsilon}} + \boldsymbol{\tau}\vdots\nabla\tilde{\boldsymbol{\varepsilon}})\mathrm{d}v - \int_S (\boldsymbol{q}\cdot\tilde{\boldsymbol{u}} + \boldsymbol{m}\cdot\partial_n\tilde{\boldsymbol{u}})\mathrm{d}a. \tag{34}$$

This enables us to state that, as a consequence of the previous approximation process within the decomposed domain $V = V_c \cup V_b$ and the subsequent passage to the limit for $R \to 0$, the PVP (6)

featuring the nonlocal elasticity model takes on a limit form as

$$\underbrace{\int_V (\boldsymbol{\sigma} : \tilde{\boldsymbol{\varepsilon}} + \boldsymbol{\tau} \vdots \nabla\tilde{\boldsymbol{\varepsilon}}) \mathrm{d}v}_{L_{\mathrm{int}}} = \underbrace{\int_V \boldsymbol{b} \cdot \tilde{\boldsymbol{u}} \, \mathrm{d}v + \int_S (\boldsymbol{p} \cdot \tilde{\boldsymbol{u}} + \boldsymbol{m} \cdot \partial_n \tilde{\boldsymbol{u}}) \mathrm{d}a.}_{L_{\mathrm{ext}}} \tag{35}$$

where we have set $\boldsymbol{p} := \boldsymbol{t} + \boldsymbol{q}$. We note that the surface integral of (34) originates from a contribution to the internal virtual power pertaining to V_b, but it is transformed into a contribution to the external virtual power through the passage to the limit for $R \to 0$, as shown by (34). Equation (35) can be recognized as the PVP for a first strain gradient elasticity model featured by body forces $\boldsymbol{b}$ within V, as well as by ordinary tractions $\boldsymbol{p}$ and by double tractions $\boldsymbol{m}$ over S. The exploitation of the latter PVP, in which the set $(\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{m})$ is taken as the primitive load parameters, leads to the field and boundary equilibrium equations governing the derived gradient model, including the related higher order boundary conditions. This task is achieved in next Section.

4 THE CORRESPONDING GRADIENT ELASTICITY MODEL

Since the analytical procedure by which the field and boundary equilibrium equations can be derived from (35) is well known from the literature [6, 7, 8], we skip this procedure and limit ourselves to report the final field and boundary equations. These read

$$\nabla \cdot \boldsymbol{T} + \boldsymbol{b} = \boldsymbol{0}, \qquad \boldsymbol{T} := \boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\tau} \qquad \text{in } V \tag{36}$$

$$\left.\begin{aligned} &\boldsymbol{p} = \boldsymbol{n} \cdot \boldsymbol{T} - \left(\nabla_{\mathrm{T}} + H\boldsymbol{n}\right) \cdot (\boldsymbol{n} \cdot \boldsymbol{\tau}) = \boldsymbol{0} \\ &\boldsymbol{m} = \boldsymbol{n}\boldsymbol{n} : \boldsymbol{\tau} \end{aligned}\right\} \text{ on } S_t. \tag{37}$$

The total stress $\boldsymbol{T} = \{T_{ij}\}$ is the gradient counterpart of the analogous total stress $\boldsymbol{T}$ pertaining to the nonlocal model. The boundary equilibrium equations (37) hold on S_t, whereas on S_u the displacement $\boldsymbol{u}$ and its normal derivative $\partial_n \boldsymbol{u}$ have to be specified, namely

$$\boldsymbol{u} = \bar{\boldsymbol{u}}, \qquad \partial_n \boldsymbol{u} = \bar{\boldsymbol{g}} \qquad \text{on } S_u. \tag{38}$$

Equations (36) and (37) have been recently [9] interpreted as the equilibrium equations of a Cauchy continuum circumvented by a membrane-like boundary layer S under the surface stress $\boldsymbol{\Sigma} = \boldsymbol{n} \cdot \boldsymbol{\tau}$ and obeying the principles of surface mechanics [10].

The constitutive equations of the derived gradient model can be obtained starting from the free energy, which —in accord with the stress power of (35)— is taken in the form $\psi = \psi(\boldsymbol{\varepsilon}, \nabla\boldsymbol{\varepsilon})$. Hence, by the Clausius-Duhem inequality, that is,

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\tau} \vdots \nabla\dot{\boldsymbol{\varepsilon}} - \dot{\psi}(\boldsymbol{\varepsilon}, \nabla\boldsymbol{\varepsilon}) \geq 0 \tag{39}$$

we easily obtain

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}, \qquad \boldsymbol{\tau} = \frac{\partial \psi}{\partial (\nabla \boldsymbol{\varepsilon})}, \tag{40}$$

and thus

$$\boldsymbol{T} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} - \nabla \cdot \left(\frac{\partial \psi}{\partial (\nabla \boldsymbol{\varepsilon})} \right). \tag{41}$$

A simple form for ψ may be chosen as

$$\psi = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{C} : \boldsymbol{\varepsilon} + \frac{1}{2}\ell^2 \boldsymbol{C} :: \left[(\nabla\boldsymbol{\varepsilon})^T \cdot (\nabla\boldsymbol{\varepsilon})\right], \tag{42}$$

where $\boldsymbol{C}$ is the classical fourth-order moduli tensor of isotropic elasticity and ℓ is an internal length scale parameter. By (42), equations (40) and (41) take the form

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon}, \qquad \boldsymbol{\tau} = \ell^2 \nabla\boldsymbol{\sigma}, \qquad \boldsymbol{T} = \boldsymbol{C} : \left(\boldsymbol{\varepsilon} - \ell^2 \Delta\boldsymbol{\varepsilon}\right), \tag{43}$$

which conform to the well known Aifantis model of gradient elasticity [11]. Equation $(43)_3$ is the gradient counterpart of (15).

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References

[1] Borino, G. and Polizzotto, C., "A method totransform a nonlocal model Into a gradient one within elasticity and plasticity," *To appear* , (2013).

[2] Mühlhaus, H.-B. and Aifantis, E.C., "A variational principle for gradient plasticity", *Int. J. Solids Strct.*, **28**, 845-858, (1991).

[3] Eringen, E.C., *Nonlocal Continuum Fields Theories*,, Springer-Verlag, New York, NY, (2002)

[4] Bažant, Z. and Jirásek, M., "Nonlocal integral formulations and damage: survey of progress", *J. Eng. Mech. ASCE*, **128**, 1119-1149, (2002).

[5] Polizzotto, C., "Nonlocal elasticity and related variational principles", *Int. J. Solids Struct.*, **113**, 1521-1533, (2001).

[6] Mindlin, R.D., "Second gradient of strain and surface-tension in linear elasticity", *Int. J. Solids Strct.*, **1**, 417-438, (1965).

[7] Germain, P., "The method of virtual power in continuum mechanics. Part 2: Microstructure", *SIAM I. App. Mech.*, **25**, 536-668, (1973).

[8] Fried, E. and Gurtin, M. E., "Tractions, balances, and boundary conditions for nonsimple materials with application to liquid flow at small-length scales", *Arch. Rat. Mech. Anal.*, **182**, 513-554, (2006).

[9] Polizzotto, C., "A Gradient elasticity theory for second-grade materials and higher order inertia", *Int. J. Solids Strct.*, **49**, 2121-3137, (2012).

[10] Gurtin, M.E. and Murdoch, I., "Surface stress — Solids", *Int. J. Solids Struct.*, **14**, 431-440, (1978).

[11] Aifantis, E. C., "On the role of gradients in localizations of deformation and fracture", *Int. J. Eng. Sci.*, **30**, 1279-1299, (1992).