

# Fractal dimension of superfluid turbulence: a random-walk toy model

Michele Sciacca<sup>1,2</sup>, David Jou<sup>2,3</sup>, Maria Stella Mongiovi<sup>4</sup>

<sup>1</sup>*Dipartimento di Scienze Agrarie e Forestali  
Università degli Studi di Palermo, Italy  
michele.sciacca@unipa.it*

<sup>2</sup>*Departament de Física  
Universitat Autònoma de Barcelona, Bellaterra, Catalonia, Spain*

<sup>3</sup>*Institut d'Estudis Catalans, Barcelona, Catalonia, Spain  
david.jou@uab.es*

<sup>4</sup>*Dipartimento di Ingegneria Chimica, Gestionale, Informatica, Meccanica  
Università degli Studi di Palermo, Italy  
m.stella.mongiovi@unipa.it*

Communicated by Nicola Bellomo

## Abstract

This paper deals with the fractal dimension of a superfluid vortex tangle. It extends a previous model [1] (which was proposed for very low temperature), and it proposes an alternative random walk toy model, which is valid also for finite temperature. This random walk model combines a recent Nemirovskii's proposal, and a simple modelization of a self-similar structure of vortex loops (mimicking the geometry of the loops of several sizes which compose the tangle). The fractal dimension of the vortex tangle is then related to the exponents describing how the vortex energy per unit length changes with the length scales, for which we take recent proposals in the bibliography. The range between 1.35 and 1.75 seems the most consistent one.

*Keywords:* superfluid turbulence, quantum vortices, random walks, fractal dimension.

*AMS subject classification:* 76F05, 82D50, 82B99.

## 1. Introduction.

Superfluid turbulence in helium II is described as a chaotic tangle of quantized vortex lines, whose core has a typical size of the order of the atomic radius of helium. The quantum of circulation is  $\kappa = h/m$ , where  $h$  is the Planck's constant and  $m$  the mass of helium atom. Quantum turbulence is present in many experimental situations: rotating helium, counterflow turbulence (heat flux without mass flux), turbulence caused by oscillating

grid and so on. In contrast with classical fluids, an increase of forces acting on liquid helium II does not imply a growth in the circulation of the vortices, which is fixed, but of the total length of vortex lines. Thus, it is clear why the growth of turbulence is usually connected to the vortex line length per unit volume,  $L$ , briefly called *vortex line density* [2–6]. However, current descriptions aim to go beyond the use of  $L$  by incorporating additional information, as for instance some statistical features of the vortex length distribution, or scaling laws in vortex size, and relating them with the energy provided to the tangle per unit time and volume [7]. Our analysis of the fractal dimension of the tangle goes along this way of research.

According to the two-fluid model of Tisza [8] and Landau [9], helium II is composed of a normal component (a viscous classical fluid carrying the whole entropy of helium II and having velocity  $\mathbf{v}_n$  and density  $\rho_n$ ) and a superfluid component (an inviscid fluid having velocity  $\mathbf{v}_s$  and density  $\rho_s$ ), such that the total density is  $\rho = \rho_n + \rho_s$  and the barycentric velocity is  $\rho\mathbf{v} = \rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n$ . In counterflow experiments, heat flux is carried away from the heater to the helium bath by the normal component  $\mathbf{q} = \rho_s T \mathbf{v}_n$ , and the superfluid component counterflows in the opposite direction to conserve the total mass. This mechanism implies a counterflow velocity defined by  $\mathbf{V}_{ns} = \mathbf{v}_n - \mathbf{v}_s = \mathbf{q}/(sT\rho_s)$ ,  $s$  being the entropy density and  $T$  temperature. In counterflow experiments the total energy supplied to helium II is dissipated by viscous forces, or furnished to the vortex tangle (as contribution to the vortex line formation and destruction) [10]. The viscous dissipation is due essentially to the interaction between the quasi-particles (phonons and rotons) that compose the normal component and the vortex tangle.

Previous studies [1,11,12] argued that the vortex tangle has a fractal structure because the net breaking process of bigger vortices into smaller vortices does not depend on the size along the energy cascade, between the extreme big and small scales [13]. In a previous paper [1] a geometrical model of superfluid turbulence at very low temperature was proposed, with a hierarchy of self-similar vortex loops whose behaviour mimics the features of a cascade of Kelvin helical waves (Kelvin wave model), so relating the fractal dimension of the vortex tangle to the amplitude and wavelength of the Kelvin waves.

In this paper, following a Nemirovskii’s proposal [14], a model analogous to the Kelvin wave model is presented: the *random walk model*. It describes the geometry of the vortex tangle with a hierarchy of self-similar vortex loops, where reconnections are faster than Kelvin wave propagation rate. The random walk model is more realistic for higher values of  $L$  ( $L > 10^5$  cm<sup>-2</sup> see Table 1), and it shares the essential main results obtained in [1].

Both models are then applied to the recent results on Kelvin-wave cascade, and fractal dimension is related to the exponent of the potential law in the wavenumbers  $K$ -space. Both models are simplified sketches (toy models) of a much more complex reality, and they are aimed to explore an intuitive grasp of this topic, rather than an exact description of it.

Furthermore, we use for the energy spectrum a recent proposal by Sonin [15] from which previous results are recovered and clarified. We discuss the fractal dimension of the vortex tangle in terms of the exponents describing how the vortex energy per unit length changes with the length scales and the structure of the random walk loops. At the end of the paper the applicability of both models at finite temperature is analyzed.

This paper is organized as follow: in Section 2 the random walk model and the scaling laws are proposed and the results discussed, Section 3 is devoted to energetic consideration on superfluid turbulence; and in Section 4 the main conclusions are summarized.

## 2. A self-similar random walk model.

Superfluid turbulence is characterized by a relevant amount of thin vortex lines, which interact and connect one to each other. The spatial distribution and the dynamical evolution of these vortices strongly depend on the type of experiment. The counterflow experiments (heat flow without mass flow) have been widely studied and their most appealing feature is the presence of an almost homogeneous and isotropic vortex tangle. The dynamics of vortices inside the tangle is governed by the external heat flow and by the presence of other vortices and by other parts of the same vortices [16].

The quantum vortex tangle can be described as an ensemble of chaotic lines, whose evolution depends on two main ingredients: the motion of the line elements, and the reconnections between them. The vortex motion obeys the Biot-Savart law, supplemented by the mutual friction force, due to the external counterflow [3–6,16].

An important feature in vortex dynamics is the possibility of vortex reconnections: when two vortices approach closely they break and reconnect, so modifying the topology of the vortex tangle [2,5,6,16]. Vortex reconnections randomize the vortex tangle and initiate the physical mechanisms of the decay of the tangle in smaller and smaller loops. Another way to transfer energy from larger scales to smaller ones is the Kelvin waves cascade.

How the supplied energy is dissipated depends also on the temperature. When temperature is approximately zero, normal component is almost absent, as well as the viscous forces and the subsequent loss of energy due to the interaction between normal component and vortices. The energy sup-

plied to the largest scales (biggest loops) is then transferred to smaller scales by reconnections and Kelvin waves (without kinetic energy loss) till the ultimate length scale, where sound emission appears, which according to the Vinen's analysis, is  $l_{min} \simeq (\kappa^3/\epsilon)^{1/4}$ , where  $\epsilon$  is the energy communicated to the system per unit volume and time, which is proportional to  $L^2$  [17]. In this regime, one could expect the vortex tangle to exhibit fractal features, if the mentioned processes act in a self-similar way on several orders of spatial lengths.

As stated, the energy transfer from the largest scales to the smallest scales is caused by reconnections among vortices and Kelvin waves, which in some cases evolve in a wave cascade, a process in which the nonlinear interaction between Kelvin waves triggers waves of shorter and shorter wavelength. A similar phenomena is seen in fiber optics for the propagation of optical pulses [18,19]. The model proposed in [1] assumes the vortex tangle as made by helical vortex loops wrapped by coils (Kelvin waves) and mimics the transfer of energy from larger to smaller loops (daughter loops) by discrete steps. This model is valid when the time a Kelvin wave needs to propagate (for some wavelengths) along the vortex is less than the time between two reconnections. Thus it is worth to compare (see Section 2.2) the time of evolution of Kelvin waves (which depends on the wavenumber) and the time between two consecutive reconnections. When reconnections are faster than wave propagation, the model proposed in [1] is no longer valid, and an alternative model has to be proposed: the random walk model.

Thus, we adopt a random walk model inspired by Nemirovskii [14]. We assume that the average vortex loop consists of many arches, smoothly connected to each other. Its structure is determined by many previous reconnections. These arches are uncorrelated, because Kelvin waves do not have the time to propagate far enough. Thus each loop has a random-walk structure.

### 2.1. *Description of the model: geometrical scaling laws and fractal dimension.*

We assume that, due to the frequent breaking and reconnection processes, the tangle may be described as an ensemble of self-similar objects [1]. We take as reference configuration a collection of  $N_0$  vortex loops of length  $l_0$ ; each loop is formed by  $N'_0$  small arches of length  $b_0$ , connected randomly but smoothly.

The next generation of smaller vortex loops is assumed to be composed of  $N_0 r$  vortex loops, of length  $l_0/\beta$ . Thus, the  $n$ -th generation is composed of  $N_n = r^n N_0$  vortex loops of length  $l_0/\beta^n$ . Here,  $r$  and  $\beta$  must be posi-

tive numbers higher than 1, whose value will depend on the details of the dynamics.

At each step, the random walk structure of each loop is formed by  $N'_n$  small arches of length  $b_n$ , a process which mimics the results of successive random reconnections. Besides a rule for the number  $N_n$  of loops in the  $n$ -th generation and the total length  $l_n$  of the loop, it is necessary to give scaling rules for  $N'_n$  and  $b_n$ , the number of small arches and their respective lengths  $b_n$ .

We assume then that each loop, at the  $n$ -th generation, has  $N'_n = N'_0(r')^n$  arches, and that the length of these arches is scaled as  $b_n = b_0/(\beta')^n$ , i.e. they might scale in a different proportion than the total length of the loop. We introduce finally two scaling coefficients  $\alpha$  and  $\gamma$ , putting  $r' = r^\alpha$  and  $\beta' = \beta^\gamma$ .

In Figure 1 a sketch of the vortex cascade is shown. The largest loop in figure is one of the  $N_n$  loops at the  $n$ -th generation, and the smallest ones refer to the daughters at  $n+1$ -th generation. According to our assumptions, the largest loop has a mean radius  $R_n = l_0/\beta^n$ , and it has  $N'_n$  arches of length  $b_n$ . Each arch represents a step of the random walk, and dots between arches refer to reconnections just or previously happened.

Because we have assumed a self-similar structure in the vortex tangle, it is possible that this structure leads to a fractal dimension of superfluid turbulence, as it happens in classical turbulence [20–22]. Our aim is to model the fractal dimension of the tangle, by separating the scaling behaviour of vortex length and of vortex energy.

The main assumption of the model is that in the breaking and reconnection processes the energy at each loop generation remains unchanged because at these length scales there is no friction neither sound radiation. We will see in the following subsection the conditions of validity of this hypothesis.

We will call  $E^{in}$  this amount of energy. Thus, we impose

$$(1) \quad E_n^{in} = E_{n+1}^{in} \quad \Rightarrow \quad N_n E'_n = N_{n+1} E'_{n+1}$$

with  $E'_n$  the energy of each loop at the  $n$ -th generation, whereas the total length  $L_n = N_n N'_n b_n$  may change with  $n$ . The underlying physical idea is that the energetic contributions of very close parts of the vortex lines may interfere with each other, thus leading to a nonlinear relation between the energy and the length.

In [1] it was outlined an interesting relation between energy and fractal dimension  $D_F$  of vortices, which is over any particular model. It states that: If  $E'_n \propto l_n^{\alpha'}$  ( $\alpha'$  being a constant) and if the energy of the successive generations of vortex loops of the tangle is constant, then  $\alpha' = D_F$ .

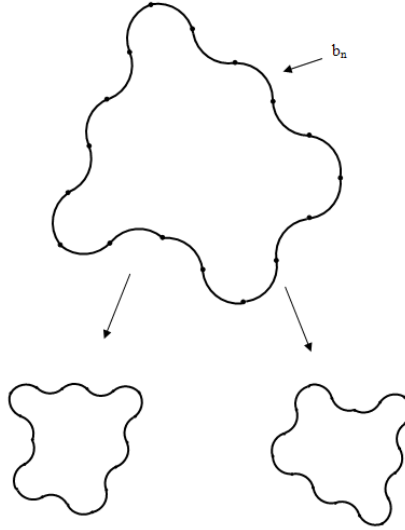


Figure 1. The self-similar vortex loop cascade: each loop of the  $n$ -th generation breaks into several loops of the  $(n+1)$ -th generation, which are assumed similar to the respective parent loop.

As a concrete illustration we consider an expression recently proposed by Sonin [15] for the energy density distribution in terms of the wavevector  $K$  in Kelvin-wave cascade, which is

$$(2) \quad E_{KV} \approx \Lambda \frac{\rho\kappa^2}{\delta_0^2} \left( \frac{\epsilon L_0^2}{\rho\kappa^3} \right)^{\frac{1}{2p-1}} K^{-\frac{2p+1}{2p-1}},$$

where  $\Lambda$  is a logarithm factor,  $\delta_0$  the intervortex spacing in the vortex tangle,  $\epsilon$  the supplied energy flux in the  $K$ -space, i.e. flowing from large vortices to small vortices. In (2),  $p$  is a natural number expressing the number of “Kelvons” — i.e. quasiparticles associated to Kelvin waves — participating in the collision. Expression (2) comes from

$$(3) \quad E_{KV} \approx \Lambda \rho \kappa^2 L K^2 m(K),$$

with  $m(K) = |u(K)|^2$  the intensity of the Kelvin mode, where  $\mathbf{u}(K)$  is the Fourier component of the displacement expansion for the vortex line of length  $L$ . Sonin proposed [15]

$$(4) \quad m(K) \approx \left( \frac{\epsilon L_0^2}{\rho\kappa^3} \right)^{\frac{1}{2p-1}} K^{-\frac{6p-1}{2p-1}}.$$

For  $p = 3$  one has Kozik and Svistunov proposal [23]  $m(K) \simeq K^{-17/5}$ , and for  $p = 2$  one obtains that of L'vov *et al.*,  $m(K) \simeq K^{-11/3}$  [24]. Lively debate has aroused about these expressions, because of the important conceptual differences behind their derivations, despite the close numerical values of the corresponding exponents in both expressions.

However, here we are interested in (2) as providing an explicit expression for  $E(K)$  for the sake of a plausible illustration. Since  $K \sim 1/l$ , and using  $l \equiv b_n$  we may rewrite (2) in the form

$$(5) \quad E'_n = \Lambda \frac{\rho\kappa^2}{\delta_0^2} \left( \frac{\epsilon L_0^2}{\rho\kappa^3} \right)^{\frac{1}{2p-1}} b_n^{\frac{2p+1}{2p-1}} N'_n.$$

Since the total number of vortex loops at the  $n$ -th generation is  $N_n$ , the total energy stored on the vortex loops of the  $n$ -th generation will be

$$(6) \quad E_n^{in} = N_n E'_n = \Lambda \frac{\rho\kappa^2}{\delta_0^2} \left( \frac{\epsilon L_0^2}{\rho\kappa^3} \right)^{\frac{1}{2p-1}} b_n^{\frac{2p+1}{2p-1}} N'_n N_n.$$

In our scaling assumptions this may be rewritten as

$$(7) \quad E_n^{in} = N_0 N'_0 b_0^{\frac{2p+1}{2p-1}} \Lambda \frac{\rho\kappa^2}{\delta_0^2} \left( \frac{\epsilon L_0^2}{\rho\kappa^3} \right)^{\frac{1}{2p-1}} \frac{r^{n(\alpha+1)}}{\beta^{n\gamma(2p+1)/(2p-1)}}.$$

Imposing the condition  $E_n^{in} = E_{n+1}^{in}$  we obtain in the limit of high  $n$ , the relation  $r^{\alpha+1} = \beta^{\gamma(2p+1)/(2p-1)}$ .

The total length of vortex loops in the  $n$ -th generation will be

$$(8) \quad L_n = N_n N'_n b'_n = N_0 N'_0 b'_0 \frac{r^{n(\alpha+1)}}{\beta^{n\gamma}}.$$

We are now in conditions to obtain the fractal dimension  $D_F$  of the vortex tangle. The usual definition of the fractal dimension  $D_F$  is [13,20]

$$(9) \quad D_F = - \lim_{n \rightarrow \infty} \frac{\log(N_n/N_0)}{\log(l_n/l_0)},$$

where  $N_n$  is the number of loops and  $l_n = L_n/N_n$  the length of a loop. In our case,  $N_n/N_0 = r^n$  and  $l_n/l_0 = N'_0 r^{\alpha n} / \beta^{\gamma n}$ . Substituting these values in (9), and using the relation between  $r$  and  $\beta$  obtained from expression (7), in the limit of high values of  $n$  it follows that

$$(10) \quad D_F = \frac{2p+1}{2p-1-2\alpha},$$

which is higher than 1 for  $\alpha < (2p - 1)/2$ . Then, if  $p = 3$ , it turns out that  $D_F = 7/(5 - 2\alpha)$  and  $\alpha < 2.5$ , and if  $p = 2$ ,  $D_F = 5/(3 - 2\alpha)$  and  $\alpha < 1.5$ . Note that a plausible value for the fractal dimension of the tangle is  $1 \leq D_F \leq 2$ . In this view we find out that  $0 \leq \alpha \leq \frac{2p-3}{4}$ , which requires that  $p \geq \frac{3}{2}$ . More in details,  $1 \leq D_F \leq 2$  for  $0 \leq \alpha \leq 0.75$  if  $p = 3$ , and  $0 \leq \alpha \leq 0.25$  if  $p = 2$ .

We note that  $p$  by itself does not specify the geometrical fractal dimension of the tangle, but also the exponent  $\alpha$  is needed. Recall (Section 2) that our scaling with respect to the features of one vortex is based on exponents  $\beta$  and  $\alpha$  that are both positive coefficients; the former one is related to the reduction in size of the small arches of the loop; the second one is related to the increment of the number of arches in a loop. For instance, if  $\alpha = 1$  the number of arches in a loop will increase as  $r^n$ , i.e. at the same rate that the number of loops in the tangle is assumed to increase with the index  $n$  of “generation”, if  $\alpha > 1$  the number of arches increases faster, if  $\alpha < 1$  the number of arches increases slower, and if  $\alpha = 0$  the number of arches in each loop is always the same. The most plausible values of  $\alpha$  indeed are in the range  $(0, 1)$ . This indicates that smaller loops (belonging to higher values of  $n$ ) are expected to have a higher number of arches, and their number grows slower than the number of loops.

Now we can compare these results with the ones obtained by several authors in [12,25,26]. In these papers the values of the fractal dimension were found at different temperature and by means of different equations (taking or not account of the friction through the normal component). The range of values obtained numerically by these authors shows that the fractal dimension is in the range  $[1.35, 1.75]$ , which according to our results brings to  $0 \leq \alpha \leq 0.5$  if  $p = 3$ , and  $0 \leq \alpha \leq 0.07$  if  $p = 2$ .

Assuming that the smaller loops (belonging to higher values of  $n$ ) will have a higher number of arches, (and their number grows slower than the number of loops), the model with  $p = 3$  seems closer to our results. However, we cannot get definitive conclusions and exclude other possibilities.

## 2.2. Range of validity of the model.

The rate of reconnections depends strongly on the amount of vortices in the tangle: it is expected that the number of reconnections per unit volume is higher for high values of  $L$ . As noted in [14,27], in a fully developed vortex tangle, because of very frequent breaking and reconnections, the vortex loops (as a whole) do not live long enough to perform a true evolution due to this deterministic motion. We also assume that turbulence is widely developed in such a way that vortices pinned to the wall of the container



can be neglected.

The rate of reconnections (at any temperature) is estimated as [14,27]:

$$(11) \quad \frac{dN}{dt} \simeq \kappa L^{5/2},$$

where  $N$  is the number of reconnections per unit volume. It means that the time interval between two consecutive reconnections per unit volume is  $\Delta t \simeq \kappa^{-1} L^{-5/2}$ . After a reconnection, two waves run along each vortex line (in opposite direction one to each other, as in Figure 3 in [14]), but Kelvin waves can be also triggered in the waves cascade, if it starts. The propagation speed of Kelvin waves along a vortex line can be approximated by  $v_g \propto K$  ( $K$  being the wave number) quite well for  $K < 500 \text{ cm}^{-1}$  as shown in Figure 3 of [28]. The time a Kelvin wave needs to run for a length  $a\lambda$  ( $a$  being a constant of the order of unity and the wavelength  $\lambda = K^{-1}$ ) is  $\Delta t_1 \simeq \kappa^{-1} K^{-2}$ .

According to these results, for low temperature the model proposed in [1] is valid when  $\Delta t_1 < \Delta t$ , namely for  $L^{5/2} < K^2$  or for  $K > \bar{K} = L^{5/4}$ . This means that the model proposed in [1] can be applied in the whole or to a confined range of wavelengths, settled by the threshold wavelength  $\bar{K}$ .

According to the Vinen's result [17], the ultimate length scale, below which sound emission appears, is  $l_{min} \propto (\kappa^3/\epsilon)^{1/4}$ , where  $\epsilon$  is the energy communicated to the system per unit volume and time, which is proportional to  $L^2$ ; therefore one can assume  $l_{min} \simeq L^{-1/2}$ . Therefore, the model in [1] is valid for  $\bar{K} < K < l_{min}^{-1} \simeq L^{1/2}$ . The random walk model should instead be valid for  $K < \bar{K}$  down to the smallest scales, where the model loose to be valid. To check whether there are situations at  $T = 0$  in which the random walk model can be applied, we have to require that the energy cascade can reach the smallest scales which are bounded by the Vinen's value  $l_{min}^{-1}$ . This means that  $\bar{K} > l_{min}^{-1}$ . For the relation (11) the answer is affirmative and it happens for  $L > 10^3 \text{ cm}^{-2}$ .

### 3. Energetic analysis of superfluid turbulence at finite temperature.

The above arguments refer to a vortex tangle at low temperatures, in such a way that the normal component (and thus friction) can be neglected, and the only dissipation occurs because of sound emission. Now we try to extend our remarks to finite temperature, where the viscous effects cannot be ignored.

At higher temperature the viscous component of helium II cannot be underestimated, at least for small Reynolds numbers: part of the energy is

lost because of viscous forces. The other part of the energy is consumed in the vortex tangle: in the formation and destruction of vortices and in the friction between vortices and quasi-particles of helium II. In the latter case, friction deviates the trajectories of the vortex elements and causes the decay of the amplitude of Kelvin waves, which are created by reconnections (these waves propagate through the vortex lines transferring energy from higher scales to smaller scales).

It is worth noting that in [10] it was found that the energy supplied to the system (tangle) is split in pure friction and in vortex destruction and formation. In more details, let the total power per unit volume  $P/V$  delivered to the vortex tangle in a time  $\Delta t$  in the superfluid reference frame be

$$(12) \quad E_{tan} = (P/V)_{tangle} \Delta t = \langle \mathbf{v}_{ns} \cdot \mathbf{F}_{MF} \rangle \Delta t,$$

where  $\mathbf{v}_{ns}$  is the microscopic counterflow velocity (the relative velocity of normal component with respect to the superfluid component),  $\mathbf{F}_{MF}$  is the microscopic mutual friction force, and  $\langle \cdot \rangle$  denotes the average in a mesoscopic volume  $\Lambda$ . The total power is split in (see [10] for more details)

$$(13) \quad E_1 = (P/V)_1 \Delta t = \langle \mathbf{v}_L \cdot \mathbf{F}_{MF} \rangle \Delta t = \epsilon_V \frac{dL}{dt} \Delta t,$$

and

$$(14) \quad E_2 = (P/V)_2 \Delta t = \langle (\mathbf{v}_{ns} - \mathbf{v}_L) \cdot \mathbf{F}_{MF} \rangle \Delta t = \alpha \rho_s \kappa L \langle |(\mathbf{v}_{ns} - \mathbf{v}_i)_\perp|^2 \rangle \Delta t,$$

where  $\mathbf{v}_L$  is the vortex line velocity,  $\epsilon_V$  is the energy per unit length,  $\alpha$  is a friction coefficient,  $\mathbf{v}_i$  is the induced velocity and  $\perp$  stands for the orthogonal component to the unit vector  $\mathbf{s}'$  ( $\mathbf{s}'$  being the derivative of the generic position  $\mathbf{s}$  with respect to the arc-length  $\xi$ ).

Therefore, a part of the energy, (13), is used in formation and destruction of vortex line to self-maintain the vortex line length in steady state, and the second part, (14), refers to an effective dissipation of the energy. Therefore, the presence of the normal component might modify the conclusions of the previous paragraphs, mainly because now energy cannot keep constant at each step of the self-similar model. But, we claim that the self-similar random walk model can be applied also at finite temperature after the following assumptions:

- For high Reynolds numbers viscous forces can be neglected.

- In the vortex tangle, energy is supplied to the largest scales and transferred to the smaller scales by reconnections and Kelvin waves. At finite temperature, part of the energy is lost in the attenuation of the amplitude of Kelvin waves (because of normal fluid), which is more evident at higher wave numbers [29]. To avoid this trouble, it is sufficient assuming that Kelvin waves do not evolve for enough time before a new reconnection happens. According to the estimations of the above paragraphs, it would occur for  $K < \bar{K}$ , but since we are interested to consider also the small length scales, then  $\bar{K} > l_{min}^{-1}$  (which means  $L > 10^3 \text{cm}^{-2}$ ).
- Moreover, according to equations (13) and (14), the energy supplied to the system (free of any viscosity) is employed to self-maintain vortex tangle because of formation and destruction ( $E_1$ ), and is lost in the interaction between vortex and quasi particles (energy  $E_2$ ). In steady state the exchange of energy  $E_1$  is null: indeed, the amount delivered to the largest scales,  $E^{in}$ , is then radiated at smallest scales,  $E^{out}$ , so we have  $E_1 = E^{in} + E^{out} = 0$ .

At finite temperature, the expression of the energy  $E^{in}$  to use in our models to calculate the fractal dimension of the ensemble of vortices is the same used at low temperature in equation (1) and in the following ones.

Note that these conclusions are still valid after removing the first restriction of those we have just commented. The conclusions of this section are summarized in Table 1.

Table 1. In this table an overview on the Kelvin wave model and the random walk model is given in terms of the temperature  $T$  (see also Section 3 for finite temperature), the vortex line density  $L$ , and the wavenumber of the Kelvin wave  $K$ . The quantities  $\bar{K}$  and  $l_{min}^{-1}$  are defined by  $\bar{K} = L^{5/4}$   $l_{min}^{-1} = L^{1/2}$ .

	Kelvin wave model	Random-walk model
Low temperature	$\bar{K} < K < l_{min}^{-1}$	$\bar{K} > l_{min}^{-1}$ or $L > 10^3 \text{cm}^{-2}$
High temperature	No because normal fluid damps out the Kelvin wave amplitude	$\bar{K} > l_{min}^{-1}$ or $L > 10^3 \text{cm}^{-2}$

#### 4. Conclusions.

In summary, we have proposed a simple toy model which allows us to interpret the fractal dimension of a vortex tangle in energetic and geometrical terms.

At very small temperature and at not too small length scales, we claim

that no dissipation occurs. In that limit we have found that  $E'_n/l_n \propto l_n^{D_F-1}$  (if smaller length scales contribute less to the energy than the large length scales, then  $D_F > 1$ ). In a previous paper a toy model was proposed to confirm this thesis [1].

According to the results of the last section, the toy model proposed in [1] is applicable when vortex line density  $L$  is not too high, because Kelvin waves must have the time to propagate. Otherwise a new model is required. In this paper we have proposed simplified random walk models of vortex tangle, which are more suitable and realistic for higher vortex line densities. Our models can be also applied to finite temperature, after some important restrictions: high Reynolds number, steady state and high value of  $L$  (see the end of Section 3). The main difference between situations at finite temperature and situations at  $T = 0$  is the loss of energy: in the latter almost all the energy supplied to the system is kept constant, from the largest scales to the smallest scales, till the critical length is reached; in the former it is not properly true.

In equation (10) we have expressed the fractal dimension  $D_F$  in terms of a characteristic dynamical exponent  $p$  — related to the number of “Kelvons” participating in a reconnection process —, and a geometrical exponent  $\alpha$ . In fact, both exponents appear in the energy expression (6):  $p$  appears in an explicit way and  $\alpha$  appears implicitly through  $N'_n$ , which refers to the number of arches of length  $b_n$  which compose a single loop. Thus, the size of the loops depends not only on  $b_n$  but also on  $N'_n$ . Small values of the  $N'_n$  will mean that there are many small loops of length  $b_n$ ; in contrast, high values of  $N'_n$  mean that the characteristic elementary lengths (or “arches”) are assembled in a small number of big loops of length  $N'_n b_n$ . We have shown that the fractal dimension in our model depends on  $\alpha$ , and the comparison with the results of the numerical simulations suggested that the preferred model is that with  $p = 3$  and  $0 \leq \alpha \leq 0.5$ .

In [1] we proposed the Kelvin wave model which, according to the results of Section 2 and Section 4 (see also Table 1), is applicable at different temperature and for  $L < 10^3 \text{ cm}^{-2}$ . In that paper we used different expressions for the energy in “Large amplitude limit” and “Long wavelength limit” for the Kelvin wave wrapping the vortex loop. By using the energy distribution (2) recently proposed by Sonin, and assuming to consider Kelvin wave in the “Large amplitude limit”, the expression (10) of the fractal dimension is the same in the second model in [1].

Since the geometrical form for the loops is here rather different than in [1], the fact that the relation between the fractal dimension and the energy/length relation is the same shows that it is a considerably robust feature of the tangle, as it is independent of the particular form assumed

for the loops.

Moreover, results from numerical simulations and experiments confirm and could give hints about the results achieved in this paper. In [12] numerical experiments performed for low  $L$  and null normal component show an increasing value of  $L$  over the time and a fractal dimension higher than 1. According to the results of [1] and the present paper (even if  $L$  is too small), the two results are correlated: a growth of  $L$  reflects  $D_F > 1$ . Also, in [29] the dynamics of the high amount of vortex lines is influenced by the presence of the normal component. A constant value of  $L$  means that the fractal dimension should be 1, even because Kelvin waves are damped by the normal component.

In the future, it would be interesting to relate this fractal dimension to transport properties of the tangle; another topic for research would be how this fractal dimension is modified in very narrow cylinders or pores, where the interaction of the loops with the walls is frequent, or in rotating tangles, where rotational effects could introduce a marked anisotropy between the rotation direction and the radial directions.

### Acknowledgements.

The authors acknowledge the support of the Università di Palermo (under Progetto CoRI 2012, Azione d) and the collaboration agreement between Università di Palermo and Universitat Autònoma de Barcelona.

M.S. acknowledges the hospitality of the “Group of Fisica Estadística of the Universitat Autònoma de Barcelona”, the financial support of the Istituto Nazionale di Alta Matematica (GNFM–Gruppo Nazionale della Fisica Matematica).

M.S. and M.S.M. acknowledge the support of the Università di Palermo (under Grant Nos. Fondi 60% 2012). D.J. acknowledges the financial support from the Dirección General de Investigación of the Spanish Ministry of Economy and Competitiveness under grant FIS2012-13370-C02-01 and of the Direcció General de Recerca of the Generalitat of Catalonia, under grant 2009 SGR-00164 and to Consolider Programm Nanotherm.

### REFERENCES

1. D. Jou, M. S. Mongiovì, M. Sciacca, and C. F. Barenghi, Vortex length, vortex energy and fractal dimension of superfluid turbulence at very low temperature, *Journal of Physics A: Mathematical and Theoretical*, vol. 43, p. 205501 (11 pages), 2010.
2. R. J. Donnelly, *Quantized vortices in helium II*. Cambridge University Press, 1991.

3. C. F. Barenghi, R. J. Donnelly, and W. F. Vinen, *Quantized Vortex Dynamics and Superfluid Turbulence*. Springer, 2001.
4. S. K. Nemirovskii and W. Fiszdon, Chaotic quantized vortices and hydrodynamic processes in superfluid helium, *Reviews of Modern Physics*, vol. 67, pp. 37–84, 1995.
5. M. Tsubota, M. Kobayashi, and H. Takeuchi, Quantum hydrodynamics, *Physics Report*, vol. 522, pp. 191–238, 2013.
6. S. K. Nemirovskii, Quantum turbulence: theoretical and numerical problems, *Physics Report*, vol. 524, pp. 85–202, 2013.
7. D. Jou, M. S. Mongiovì, and M. Sciacca, Statistical mechanics and thermodynamics of turbulent quantum vortex tangles, *Communications in Applied and Industrial Mathematics*, vol. 1, pp. 216–229, 2010.
8. L. Tisza, Transport phenomena in helium II, *Nature*, vol. 141, p. 913, 1938.
9. L. D. Landau, The theory of superfluidity of helium II, *Journal of Physics*, vol. 5, pp. 71–90, 1941.
10. M. S. Mongiovì, D. Jou, and M. Sciacca, Energy and temperature of superfluid turbulent vortex tangles, *Physical Review B*, vol. 75, p. 214514 (10 pages), 2007.
11. D. Kivotides, J. C. Vassilicos, D. C. Samuels, and C. F. Barenghi, Kelvin waves cascade in superfluid turbulence, *Physical Review Letters*, vol. 86, pp. 3080–3083, 2001.
12. A. W. Baggaley and C. F. Barenghi, Spectrum of turbulent Kelvin-waves cascade in superfluid helium, *Physical Review B*, vol. 83, p. 134509 (14 pages), 2011.
13. B. Mandelbrot, *Les objets fractales: forme, hasard et dimension*. Flammarion, 1984.
14. S. K. Nemirovskii, Kinetics of a network of vortex loops in He II and a theory of superfluid turbulence, *Physical Review B*, vol. 77, p. 214509 (14 pages), 2008.
15. E. B. Sonin, Symmetry of Kelvin-wave dynamics and the Kelvin-wave cascade in the  $t = 0$  superfluid turbulence, *Physical Review B*, vol. 85, p. 104516 (6 pages), 2012.
16. K. W. Schwarz, Three-dimensional vortex dynamics in superfluid  $^4\text{He}$ : homogeneous superfluid turbulence, *Physical Review B*, vol. 38, pp. 2398–2417, 1988.
17. W. F. Vinen, Decay of superfluid turbulence at a very low temperature: the radiation of sound from a Kelvin wave on a quantized vortex, *Physical Review B*, vol. 64, p. 134520 (4 pages), 2001.
18. G. P. Agrawal, *Nonlinear Fiber Optics*. Academic Press, 2001.

19. T. Brugarino and M. Sciacca, Singularity analysis and integrability for a HNLS equation governing pulse propagation in a generic fiber optics, *Optics Communication*, vol. 262, pp. 250–256, 2006.
20. T. Boder, M. H. Jensen, G. Paladin, and A. Vulpiani, *Dynamical systems approach to turbulence*. Cambridge University Press, 1998.
21. R. Benzi, G. Paladin, and A. Vulpiani, Characterisation of intermittency in chaotic systems, *Journal of Physics A: Mathematical and General*, vol. 18, pp. 2157–2165, 1985.
22. D. Jou, Intermittent turbulence: a short introduction, *Scientia Marina*, vol. 61, pp. 57–62, 1997.
23. E. Kozik and B. Svistunov, Kelvin-wave cascade and decay of superfluid turbulence, *Physical Review Letters*, vol. 92, p. 035301 (4 pages), 2004.
24. J. Laurie, V. S. L'vov, S. Nararenko, and O. Rudenko, Interaction of Kelvin waves and nonlocality of energy transfer in superfluids, *Physical Review B*, vol. 81, p. 104526 (14 pages), 2010.
25. D. Kivotides, C. F. Barenghi, and D. C. Samuels, Fractal dimension of superfluid turbulence, *Physical Review Letters*, vol. 87, p. 155301 (3 pages), 2001.
26. D. Kivotides and A. Leonard, Geometrical physics of the many vortex filament problem, *Europhysics Letters*, vol. 66, p. 69, 2004.
27. S. K. Nemirovskii and M. W. Nedoiko, Applications of a Gaussian model of vortex tangle in the superfluid turbulent He II, in *Quantized Vortex Dynamics and Superfluid Turbulence* (C. F. Barenghi, R. J. Donnelly, and W. F. Vinen, eds.), pp. 205–211, Springer, 2001.
28. T. Brugarino, M. S. Mongiovì, and M. Sciacca, Waves on a vortex filament: exact solutions of dynamical equations, (*submitted*), 2013.
29. A. W. Baggaley and C. F. Barenghi, Vortex-density fluctuations in quantum turbulence, *Physical Review B*, vol. 84, p. 020504 (4 pages), 2011.