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On Graded Cocharacters and Corresponding Multiplicities

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Introduction

Let G be a group and A an algebra over a field F. A G-grading on A is a decomposition of A, as a vector space, into the direct sum of subspaces $A = \bigoplus_{g \in G} A_g$ such that $A_g A_h \subseteq A_{gh}$, for any $g, h \in G$.

The description of all possible G-gradings of A is an important problem in the structure theory of graded rings and its applications. Many properties of the ideals of ordinary identities (with trivial grading) have an analogue for graded identities and in this setting are more easily described ([18],[36]). Kemer developed the structure theory of ideals of polynomial identities in the spirit of the ideal theory of commutative algebras (see [36]). In his approach he used $\mathbb{Z}_2$-graded algebras in an essential way showing also the relevance of their graded polynomial identities. In particular, the study of these "weaker" identities was one of his key ingredients for answering positively the famous Specht problem in characteristic zero.

It turned out fairly soon that the study of G-graded polynomial identities of algebras graded by a group G, was a problem of independent interest, with various relations to other objects as, for example, group algebras. For instance, it was proved in [7], [17] that if G is a finite abelian group and A is a G-graded algebra, then A is PI if and only if its neutral component is PI. It was soon discovered that one may consider the graded identities satisfied by an algebra as an "approximation" of the ordinary ones. Several ordinary invariants for T-ideals were transferred to the graded case and have been extensively studied, see for example [8] and its bibliography.

In particular, the description of all gradings on matrix algebras plays an important role in PI-theory (see, for example [13], [47]) and in the theory of Lie superalgebras and colour Lie superalgebras (see [10]).

Let $M_n(F)$ be the $n \times n$ matrix algebra over a field F of characteristic zero. Concerning the ordinary polynomial identities, the picture is completely clear only for 2×2 matrices. The results of Razmyslov [43] and Drensky [22] give a basis of the polynomial identities. The asymptotic behavior of the codimension sequence $c_n(M_2(F))$, $n = 0, 1, 2, \ldots$, was determined by Regev (see [44]) and the explicit formula for $c_n(M_2(F))$ was established by Procesi [42]. The papers [23], [27] contain explicit formulas for the S_n-cocharacters of $M_2(F)$.

If G is an arbitrary group, for the algebra $M_n(F)$ of $n \times n$ matrices, there are two important classes of G-gradings: the elementary gradings and the fine gradings. In fact in [11] it was proved that if F is an algebraically closed field, every G-grading on $M_n(F)$ is a tensor product of an elementary grading and a fine grading. Moreover, if G is a cyclic group then any G-grading on a matrix algebra $M_n(F)$ is an elementary grading.

In case of characteristic zero, an explicit basis of the graded identities for the algebra of 2×2 matrices was exhibited in [21]. In [37] the result of [21] was extended to algebras over an infinitive field of characteristic different from 2. Further significant progress in describing the graded identities satisfied by matrix algebras was made by Vasilovsky in [50], [51]. He described the $\mathbb{Z}_n$ and the $\mathbb{Z}$-graded identities of the matrix algebra $M_n(F)$ over a field of characteristic zero with a particular $\mathbb{Z}_n$-grading. Namely Vasilovsky proved that for such grading the ideal of $\mathbb{Z}_n$-graded identities for $M_n(F)$ is generated by the polynomials $x_{e1}x_{e2} - x_{e2}x_{e1}$ and $x_{i1}x_{(n-i)2}x_{i3} - x_{i3}x_{(n-i)2}x_{i1}$, $0 \leq i \leq n-1$, where x_{ij} is the jth variable of homogeneous degree i. This result was generalized in the paper [6] to any elementary grading of $M_n(F)$ induced by $g = (g_1, \ldots, g_n) \in G^n$, where the group G is finite arbitrary and the elements $g_1, \ldots, g_n$ are pairwise different. Later on both results of Vasilovsky were established over infinite fields, see [4], [5].

This thesis is devoted to the study of the G-graded cocharacter sequence for a PI-algebra, with particular attention to the algebra of upper triangular matrices of order three.

The first chapter of the thesis is introductory. We introduce the algebras with polynomial identity by giving their basic definitions and properties. We only deal with associative algebras over a field F of characteristic zero. The set of all polynomial identities of A, $Id(A)$, is a T-ideal of the free associative algebra $F\langle X\rangle$, where $X = \{x_1, x_2, \ldots\}$ is a countable set. Then we recall the definition of multihomogeneous and multilinear polynomials.

In the second chapter we give a brief introduction to the classical representation theory of the symmetric group and of the general linear group via the theory of Young diagrams which is our main tool in the study of the T-ideals of the free algebra. Then we introduce the sequence of codimensions, cocharacters and colengths, and we restate all results in case A is a G-graded PI-algebra.

In the third chapter we characterize the ideal of graded identities of A in case the multiplicities are bounded by a constant. We shall do this in three different ways. In fact we shall prove that the multiplicities are bounded by a constant if and only if $Id^G(A) \not\supseteq Id^G(UT_2^G)$, where UT_2^G is the algebra of upper triangular matrices of order two with a generic G-grading. Another characterization will be given in terms of S_n-characters: in fact we shall prove that the characters appearing with non-zero multiplicities in $\chi^G_{n_1,\ldots,n_s}(A)$ have corresponding Young diagrams contained in a hook shaped part of the plane.

In the last chapter we study the algebra of upper triangular matrices of order three, $UT_3(F)$, graded by a finite abelian group G. Recall that all G-gradings on $UT_n(F)$, for F algebraically closed of characteristic zero, are elementary (see [53]).

We compute the multiplicities or the proper multiplicities of the graded cocharacter sequences of $UT_3(F)$.

Chapter 1

Polynomial Identities and PI-Algebras

In this chapter we give the basic definitions and results of the theory of polynomial identities of associative algebras.

1.1 Basic Definitions

Throughout this paper we shall be dealing with associative algebras over a field.

We start with the basic definition of free algebra. Let F be a field and X a set. The free associative algebra on X over F, is the algebra $F\langle X\rangle$ of polynomials in the non-commuting indeterminates $x \in X$.

A linear basis of $F\langle X\rangle$ consists of all words in the alphabet X (including the empty word 1). Such words are called monomials and the product of two monomials is given by juxtaposition. The elements of $F\langle X\rangle$ are called polynomials and if $f \in F\langle X\rangle$, we write $f = f(x_1, \ldots, x_n)$ to indicate that $x_1, \ldots, x_n \in X$ are the only indeterminates occurring in f. We shall also assume that X is an infinite set.

we define $\deg u$, the degree of a monomial u, as the length of the word

u. Also $\deg_{x_i} u$, the degree of u in the indeterminate x_i, is the number of occurrence of x_i in u. Accordingly, the degree $\deg f$ of a polynomial $f = f(x_1, \ldots, x_n)$ is the maximum degree of a monomial in f; $\deg_{x_i} f$, the degree of f in x_i, is the maximum of $\deg_{x_i} u$, for u a monomial in f.

The algebra $F\langle X\rangle$ is defined, up to isomorphism, by the following universal property: given an associative F-algebra A, any map $X \to A$ can be uniquely extended to a homomorphism of algebras $F\langle X\rangle \to A$. The cardinality of X is called the rank of $F\langle X\rangle$. We shall consider the free algebra $F\langle X\rangle$ of countable rank on the set $X = \{x_1, x_2, \ldots\}$.

Definition 1.1.1 *Let A be an F-algebra and $f = f(x_1, \ldots, x_n) \in F\langle X\rangle$. We say that $f \equiv 0$ is a polynomial identity of A if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$.*

Let Φ denote the set of all homomorphism $\varphi : F\langle X\rangle \to A$. Then it is clear that $f \equiv 0$ is a polynomial identity for A if and only if $f \in \bigcap_{\varphi\in\Phi} \mathrm{Ker}\varphi$. We shall usually say that $f \equiv 0$ is an identity on A or that A satisfies $f \equiv 0$; sometimes we shall say that f itself is an identity of A.

Since the trivial polynomial $f = 0$ is an identity for any algebra A, we make the following:

Definition 1.1.2 *If A satisfies a non-trivial identity $f \equiv 0$, then we say that A is a PI-algebra.*

For $a, b \in A$, let $[a, b] = ab - ba$ denote the Lie commutator of a and b. We next give some examples of PI-algebras.

Example 1.1.3 *Let $UT_n(F)$ be the algebra of $n \times n$ upper triangular matrices over F. Then $UT_n(F)$ is a PI-algebra since it satisfies the identity*

$$[x_1, x_2] \cdots [x_{2n-1}, x_{2n}] \equiv 0.$$

1.2 T-ideals and varieties of algebras

Given an algebra A, we define

$$Id(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \quad \text{on} \quad A\},$$

the set of polynomial identities of A. Clearly, $Id(A)$ is a two-sided ideal of $F\langle X\rangle$. Moreover, if $f = f(x_1, \ldots, x_n)$ is any polynomial in $Id(A)$, and $g_1, \ldots, g_n$ are arbitrary polynomials in $F\langle X\rangle$, it is clear that $f(g_1, \ldots, g_n) \in Id(A)$. Since any endomorphism of $F\langle X\rangle$ is determined by mapping $x \mapsto g$, $x \in X$, $g \in F\langle X\rangle$, it follows that $Id(A)$ is an ideal invariant under all endomorphism of $F\langle X\rangle$. The ideals with this property are called T-ideals.

Definition 1.2.1 *An ideal I of $F\langle X\rangle$ is a T-ideal if $\varphi(I) \subseteq I$ for all endomorphisms φ of $F\langle X\rangle$.*

Hence $Id(A)$ is a T-ideal of $F\langle X\rangle$. On the other hand, it is easy to check that all T-ideals of $F\langle X\rangle$ are actually of this type.

Now we need the notion of a variety of algebras.

Definition 1.2.2 *Given a non-empty set $S \subseteq F\langle X\rangle$, the class of all algebras A such that $f \equiv 0$ on A for all $f \in S$ is called the variety $\mathcal{V} = \mathcal{V}(S)$ determined by S.*

A variety $\mathcal{V}$ is called non-trivial if $S \neq 0$ and $\mathcal{V}$ is proper if it is non-trivial and contains a non-zero algebra.

The following theorem helps us to decide whether a given class of algebras is a variety.

Theorem 1.2.3 *A non-empty class $\mathcal{V}$ of algebras is a variety if and only if it satisfies the following properties:*

1. *if $A \in \mathcal{V}$, and $B \to A$ is a monomorphism, then $B \in \mathcal{V}$;*
2. *if $A \in \mathcal{V}$, and $A \to B$ is an epimorphism, then $B \in \mathcal{V}$;*

3. *if* $\{A_\gamma\}_{\gamma\in\Gamma}$ *is a family of algebras and* $A_\gamma \in \mathcal{V}$, *for all* $\gamma \in \Gamma$, *then* $\prod_{\gamma\in\Gamma} A_\gamma \in \mathcal{V}$.

There is a close correspondence between T-ideals and varieties of algebras.

Theorem 1.2.4 *There is a one-to-one correspondence between T-ideals of* $F\langle X\rangle$ *and varieties of algebras. In this correspondence a variety* $\mathcal{V}$ *corresponds to the T-ideal of identities* $Id(\mathcal{V})$ *and a T-ideal* I *corresponds to the variety of algebras satisfying all the identities in* I.

1.3 Homogeneous and multilinear polynomials

When the base field F is infinite, the study of the identities of a given algebra can be reduced to the study of homogeneous and multilinear polynomials. In this section we give the basic definitions and results.

Let $F_n = F\langle x_1, \ldots, x_n\rangle$ be the free algebra of rank $n \geq 1$ over F. This algebra can be naturally decomposed as

$$F_n = F_n^{(1)} \oplus F_n^{(2)} \oplus \cdots$$

where, for every $k \geq 1$, $F_n^{(k)}$ is the subspace spanned by all monomials of total degree k. The $F_n^{(i)}$'s are called the homogeneous components of F_n.

This decomposition can be further refined as follows: for every $k \geq 1$ write

$$F_n^{(k)} = \bigoplus_{i_1+\cdots+i_n=k} F_n^{(i_1,\ldots,i_n)}$$

where $F_n^{(i_1,\ldots,i_n)}$ is the subspace spanned by all monomials of degree i_1 in x_1, $\ldots i_n$ in x_n. Such decomposition extend in an obvious way to $F\langle X\rangle$ for X countable.

Definition 1.3.1 *A polynomial* f *belonging to* $F_n^{(k)}$ *for some* $k \geq 1$, *will be called homogeneous of degree* k. *If* f *belongs to some* $F_n^{(i_1,\ldots,i_n)}$, *it will* *be called multihomogeneous of multidegree* $(i_1, \ldots, i_n)$. *We also say that a polynomial* f *is homogeneous in the variable* x_i, *if* x_i *appears with the same degree in every monomial of* f.

A useful property of T-ideals is that if F is an infinite field, they have a corresponding decomposition in multihomogeneous polynomials. If $f(x_1, \ldots, x_n) \in F\langle X \rangle$, we can always write

$$f = \sum_{i_1 \geq 0, \ldots, i_n \geq 0} f^{(i_1, \ldots, i_n)}$$

where $f^{(i_1, \ldots, i_n)} \in F_n^{(i_1, \ldots, i_n)}$ is the sum of all monomials in f where $x_1, \ldots, x_n$ appear at degree $i_1, \ldots, i_n$, respectively. The polynomials $f^{(i_1, \ldots, i_n)}$ which are non-zero are called the multihomogeneous components of f.

Theorem 1.3.2 *Let* F *be an infinite field. If* $f \equiv 0$ *is a polynomial identity for the algebra* A, *then every multihomogeneous component of* f *is still a polynomial identity for* A.

One of the most important consequences of the previous theorem is that over an infinite field every T-ideal is generated by its multihomogeneous polynomials.

Among multihomogeneous polynomials a special role is played by the multilinear ones.

Definition 1.3.3 *A polynomial* f *is linear in the variable* x_i *if* x_i *occurs with degree* 1 *in every monomial of* f. *A polynomial which is linear in each of its variables is called multilinear.*

In the language above we can say that a polynomial $f(x_1, \ldots, x_n) \in F\langle X \rangle$ is multilinear if it is multihomogeneous of multidegree $(1, \ldots, 1)$.

Since in a multilinear polynomial $f(x_1, \ldots, x_n)$ each variable appears in each monomial at degree 1, it is clear that this polynomial is always of the form

$$f(x_1,\dots,x_n)=\sum_{\sigma\in S_n}\alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}$$

where $\alpha_\sigma \in F$ and S_n is the symmetric group on $\{1,\dots,n\}$.

The most important property of multilinear polynomials is given in the following remark

Remark 1.3.4 *Let A be an F-algebra spanned by a set B over F. If a multilinear polynomial f vanishes on B, then f is a polynomial identity of A*

Theorem 1.3.5 *If char$F = 0$, every non-zero polynomial $f \in F\langle X\rangle$ is equivalent to a finite set of multilinear polynomials.*

We can record this results in the language of T-ideals.

Corollary 1.3.6 *If char$F = 0$, every T-ideal is generated, as a T-ideal, by the multilinear polynomials it contains.*

Chapter 2

S_n-Representations

2.1 Finite dimensional representations

Let V be a vector space over a field F nd let $GL(V)$ be the group of invertible endomorphisms of V. Recall the following.

Definition 2.1.1 *A representation of a group G on V is a homomorphism of groups $\rho : G \to GL(V)$.*

Let us denote by $\mathrm{End}(V)$ the algebra of F-endomorphisms of V. If FG is the group algebra of G over F and ρ is a representation of G on V, it is clear that ρ induces a homomorphism of F-algebras $\rho' : FG \to \mathrm{End}(V)$ such that $\rho'(1_G) = 1$.

Throughout we shall be dealing only with the case when $\dim_F V = n < \infty$, i.e., with finite dimensional representations. In this case n is called the dimension or the degree of the representation ρ. Now, a representation of a group G uniquely determines a finite dimensional FG-module (or G-module) in the following way. If $\rho : G \to GL(V)$ is a representation of G, V becomes a (left) G-module by defining $gv = \rho(g)(v)$ for all $g \in G$, $v \in V$. It is also clear that if M is a G-module which is finite dimensional as a vector space over F, then $\rho : G \to GL(M)$, such that $\rho(g)(m) = gm$, for $g \in G$, $m \in M$, defines a representation of G on M.

Definition 2.1.2 *If $\rho : G \to GL(V)$ and $\rho' : G \to GL(W)$ are two representations of a group G, we say that ρ and ρ' are equivalent, and we write $\rho \sim \rho'$, if V and W are isomorphic as G-modules.*

Definition 2.1.3 *A representation $\rho : G \to GL(V)$ is irreducible if V is an irreducible G-module. ρ is completely reducible if V is the direct sum of its irreducible submodules*

The basic tool for studying the representations of a finite group in case $\mathrm{char}F = 0$, is Maschke's theorem. Recall that an algebra A is semisimple if $J(A) = 0$ where $J(A)$ is the Jacobson radical of A.

Theorem 2.1.4 (Maschke) *Let G be a finite group and let $charF = 0$ or $charF = p > 0$ and $p \nmid |G|$. Then the group algebra FG is semisimple.*

As a consequence of Wedderburn's theorem, it follows that, under the hypotheses of Maschke's theorem,

$$FG \cong M_{n_1}(D^{(1)}) \oplus \cdots \oplus M_{n_k}(D^{(k)})$$

where $D^{(1)}, \ldots, D^{(k)}$ are finite dimensional division algebras over F. In light of these results one can classify all the irreducible representations of G: M is an irreducible G-module if and only if M is an irreducible $M_{n_i}(D^{(i)})$-module, for some i. On the other hand, $M_{n_i}(D^{(i)})$ has (up to isomorphisms) only one irreducible module, isomorphic to $\sum_{j=1}^{n_i} D^{(i)} e_{ij}$.

From the above it can also be deduced that every G-module V is completely reducible. Hence if $\dim_F V < \infty$, V is the direct sum of a finite number of irreducible G-modules. We record this fact in the following.

Corollary 2.1.5 *Let G be a finite group and F a field of characteristic zero or $p > 0$ and $p \nmid |G|$. Then every representation of G is completely reducible* *and the number of inequivalent irreducible representations of G equals the number of simple components in the Wedderburn decomposition of the group algebra FG.*

Recall that an element $e \in FG$ is an idempotent if $e^2 = e$. It is well known that since FG is finite dimensional semisimple, every one-sided ideal of FG is generated by an idempotent. Moreover every two-sided ideal is generated by a central idempotent. We say that an idempotent is minimal (central resp.) if it generates a minimal one-sided (two-sided resp.) ideal. We record this in the following.

Proposition 2.1.6 *If M is an irreducible representation of G, then $M \cong J_i$, a minimal left ideal of $M_{n_i}(D^{(i)})$, for some $i \in 1, \dots, k$. Hence there exists a minimal idempotent $e \in FG$ such that $M \cong FGe$.*

When the field F is a splitting field for the group G, e.g., F is algebraically closed, then the following properties hold.

Proposition 2.1.7 *Let F be a splitting field for G. Then the number of non-equivalent irreducible representations of G equals the number of conjugacy classes of G.*

Since by Corollary 2.1.5 this number equals the number of simple components of FG, it follows that when F is a splitting field for G, it equals the dimension of the center of FG over F.

A basic tool in representation theory is provided by the theory of characters. From now on assume that F is a splitting field for G of characteristic zero and let $\mathrm{tr} : End(V) \to F$ be the trace function on $End(V)$.

Definition 2.1.8 *Let $\rho : G \to GL(V)$ be a representation of G. Then the map $\chi_\rho : G \to F$ such that $\chi_\rho(g) = tr(\rho(g))$ is called the character of the representation ρ and $\dim V = deg\chi_\rho$ is called the degree of the character χ_ρ.*

We say that the character χ_ρ is irreducible if ρ is irreducible. Since $\chi_\rho(g) = \chi_\rho(h)$ provided g is conjugate to h in G, χ_ρ is constant on the conjugacy classes of G, i.e., χ_ρ is a class function of G. Notice that $\chi_\rho(1) = \deg\chi_\rho$.

2.2 S_n-representations

In this section we describe the ordinary representation theory of the symmetric group S_n, $n \geq 1$. Since $\mathbb{Q}$, the field of rational numbers, is a splitting field for S_n, for any field F of characteristic zero, the group algebra FS_n has a decomposition into simple components which are algebras of matrices over the field F itself. Moreover, by Proposition 2.1.7, the number of irreducible non-equivalent representations equals the number of conjugacy classes of S_n. Recall the following.

Definition 2.2.1 *Let $n \geq 1$ be an integer. A partition λ of n s a finite sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $\sum_{i=1}^{r} \lambda_i = n$. In this case we write $\lambda \vdash n$.*

If $r = 1$, then $\lambda_1 = n$ and we write $\lambda = (n)$. For the partition λ with $\lambda_1 = \ldots = \lambda_n = 1$ the notation $\lambda = (1^n)$ is usually used. More generally, we write $\lambda = (k^d)$ as soon as $\lambda = (k, \ldots, k)$ and $n = kd$.

It is well known that the conjugacy classes of S_n are indexed by the partitions of n: if $\sigma \in S_n$, we decompose σ into the product of disjoint cycles, including 1-cycles. this decomposition is unique if we require that

$$\sigma = \pi_1 \pi_2 \cdots \pi_r$$

with $\pi_1, \pi_2, \ldots, \pi_r$ cycles of length $\lambda_1 \geq \cdots \geq \lambda_r \geq 1$, respectively. Then the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ uniquely determines the conjugacy class of σ.

Since, as we mentioned above, all the irreducible characters of S_n are indexed by the partitions of n, let us denote by χ_λ the irreducible S_n-character corresponding to $\lambda \vdash n$.

It is standard to use the notation $d_\lambda = \chi_\lambda(1)$ for the degree of χ_λ. It follows that FS_n has the following decomposition

$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda \cong \bigoplus_{\lambda \vdash n} M_{d_\lambda}(F),$$

where $I_\lambda = e_\lambda FS_n \cong M_{d_\lambda}(F)$ is the minimal two-sided ideal of FS_n corresponding to $\lambda \vdash n$, and $e_\lambda = \sum_{\sigma \in S_n} \chi_\lambda(\sigma)\sigma$ is the essential central idempotent deduced from Remark .

Proposition 2.2.2 *Let F be any field of characteristic zero and $n \geq 1$. Then there is a one-to-one correspondence between irreducible S_n-characters and partitions of n. Let $\{\chi_\lambda \mid \lambda \vdash n\}$ be a complete set of irreducible characters of S_n and let $d_\lambda = \chi_\lambda(1)$ be the degree of χ_λ, $\lambda \vdash n$. then*

$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda \cong \bigoplus_{\lambda \vdash n} M_{d_\lambda}(F),$$

where $I_\lambda = e_\lambda FS_n$ and $e_\lambda = \sum_{\sigma \in S_n} \chi_\lambda(\sigma)\sigma$ is up to a scalar, the unit element of I_λ.

Definition 2.2.3 *If $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, the Young diagram associated to λ is the finite subset of $\mathbb{Z} \times \mathbb{Z}$ defined as $D_\lambda = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i = 1, \ldots, r, j = 1, \ldots, \lambda_i\}$.*

There are two standard notations. In one notation, a Young diagram D_λ is denoted as an array of boxes corresponding to the points (i,j). In the other notation, and this is the one we shall adopt, the array of boxes denoting D_λ is such that the first coordinate i (the row index) increases from top to bottom and the second coordinate j (the column index from left to right).

For a partition $\lambda \vdash n$ we shall denote by λ' the conjugate partition of λ; $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ is the partition such that $\lambda'_1, \ldots, \lambda'_s$ are the lengths of the columns of D_λ. Hence D'_λ is obtained from D_λ by flipping D_λ along its main diagonal.

Definition 2.2.4 *Let $\lambda \vdash n$. A Young tableau T_λ of the diagram D_λ is a filling of the boxes of D_λ withe the integers $1, 2, \ldots, n$. We shall also say that T_λ is a tableau of shape λ.*

Of course there are $n!$ distinct tableaux. Among these a prominent role is played by the so called standard tableaux.

Definition 2.2.5 *A tableau T_λ of shape λ is standard if the integers in each row and in each column of T_λ increase from left to right and from top to bottom, respectively.*

There is a strict connection between standard tableaux and degrees of the irreducible S_n-characters.

Theorem 2.2.6 *Given a partition $\lambda \vdash n$, the number of standard tableaux of shape λ equals d_λ, the degree of χ_λ, the irreducible character corresponding to λ.*

Next we give a formula to compute the degree d_λ of the irreducible character χ_λ: the Hook Formula. First we need some further terminology.

Given a diagram D_λ, $\lambda \vdash n$, we identify a box of D_λ with the corresponding point (i, j).

Definition 2.2.7 *For any box $(i, j) \in D_\lambda$, we define the hook number of (i, j) as $h_{ij} = \lambda_i + \lambda'_j - i - j + 1$, where λ' is the conjugate partition of λ.*

Note that h_{ij} counts the number of boxes in the "hook" with edge in (i, j), i.e., the boxes to the right and below (i, j).

Proposition 2.2.8

$$d_\lambda = \frac{n!}{\prod_{i,j} h_{ij}}$$

where the product runs over all boxes of D_λ.

Next we describe a complete set of minimal left ideals of FS_n. Given any tableau T_λ of shape $\lambda \vdash n$, let us denote by $T_\lambda = D_\lambda(a_{ij})$, where a_{ij} is the integer in the (i,j) box. then

Definition 2.2.9 *The row-stabilizer of T_λ is*

$$R_{T_\lambda} = S_{\lambda_1}(a_{11}, a_{12}, \ldots, a_{1\lambda_1}) \times \cdots \times S_{\lambda_r}(a_{r1}, a_{r2}, \ldots, a_{r\lambda_r})$$

where $S_{\lambda_1}(a_{i1}, a_{i2}, \ldots, a_{i\lambda_i})$ denotes the symmetric group acting on the integers $a_{i1}, a_{i2}, \ldots, a_{i\lambda_i}$.

Hence R_{T_λ} is the subgroup of S_n consisting of all permutation stabilizing the rows of T_λ.

Definition 2.2.10 *The column-stabilizer of T_λ is*

$$C_{T_\lambda} = S_{\lambda'_1}(a_{11}, a_{21}, \ldots, a_{\lambda'_1 1}) \times \cdots \times S_{\lambda'_r}(a_{1\lambda_1}, a_{2\lambda_2}, \ldots, a_{\lambda'_s \lambda_1})$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ is the conjugate partition of λ.

Hence C_{T_λ} is the subgroup of S_n consisting of all permutations stabilizing the columns of T_λ.

Definition 2.2.11 *For a given tableau T_λ, we define*

$$e_{T_\lambda} = \sum_{\sigma \in R_{T_\lambda}, \tau \in C_{T_\lambda}} (sgn\tau)\sigma\tau.$$

It can be shown that $e_{T_\lambda}^2 = a e_{T_\lambda}$, where $a = \frac{n!}{d_\lambda}$ is a non-zero integer, i.e., e_{T_λ} is an essential idempotent of FS_n.

Given a partition $\lambda \vdash n$, the symmetric group S_n acts on the set of Young tableaux of shape λ as follows: If $\sigma \in S_n$ and $T_\lambda = D_\lambda(a_{ij})$, then $\sigma T_\lambda = D_\lambda(\sigma(a_{ij}))$. This action has the property that

$$R_{\sigma T_\lambda} = \sigma R_{T_\lambda} \sigma^{-1} \quad \text{and} \quad C_{\sigma T_\lambda} = \sigma C_{T_\lambda} \sigma^{-1}.$$

It follows that $\sigma e_{T_\lambda} \sigma^{-1} = e_{\sigma T_\lambda}$.

We record the most important facts about e_{T_λ} in the following.

Corollary 2.2.12 *For every Young tableau T_λ of shape $\lambda \vdash n$, the element e_{T_λ} is a minimal essential idempotent of FS_n and $FS_n e_{T_\lambda}$ is a minimal left ideal of FS_n with character χ_λ. If T_λ and T^*_λ are Young tableaux of the same shape, then e_{T_λ} and $e_{T^*_\lambda}$ are conjugated in FS_n through some $\sigma \in S_n$; moreover, $\sigma e_{T_\lambda} \sigma^{-1} = e_{\sigma T_\lambda}$.*

The above proposition says that for any two tableaux T_λ and T^*_λ of the same shape λ, $FS_n e_{T_\lambda} \cong FS_n e_{T^*_\lambda}$, as S_n-modules.

2.3 Inducing S_n-representations

In this section we regard the group S_n embedded in S_{n+1} as the subgroup of all permutations fixing the integer $n+1$. The next theorem gives a decomposition into irreducibles of any S_n-module induced up to S_{n+1}.

Let us denote by M_λ an irreducible S_n-module corresponding to the partition $\lambda \vdash n$. We have

Theorem 2.3.1 *Let the group S_n be embedded into S_{n+1} as the subgroup fixing the integer $n+1$. Then*

1. *If $\lambda \vdash n$, then $M_\lambda \uparrow S_{n+1} \cong \sum_{\mu \in \lambda^+} M_\mu$ where λ^+ is the set of all partitions of $n+1$ whose diagram is obtained from D_λ by adding one box;*

2. *If $\mu \vdash n+1$, then $M_\mu \downarrow S_n \cong \sum_{\lambda \in \mu^-} M_\lambda$ where μ^- is the set of all partitions of n whose diagram is obtained from D_μ by deleting one box.*

We go one step further and we state a more general result. First we need some definitions.

We embed the group $S_n \times S_m$ into S_{n+m} by letting S_m act on $\{n+1, \ldots, n+m\}$. Recall that if M is an S_n-module and N is an S_m-module, then $M \otimes_F N$ has a natural structure of $S_n \times S_m$-module.

Definition 2.3.2 *If M is an S_n-module and M is an S_m-module, then the outer tensor product of M and N is defined as*

$$M \widehat{\otimes} N = (M \otimes N) \uparrow S_{n+m}.$$

Recall that (n) denotes a one-row partition $\mu \vdash n$, i.e., $\mu_1 = n$. We have

Theorem 2.3.3 (Young Rule) *Let $\lambda \vdash n$ and $m \geq 1$. Then*

$$M_\lambda \widehat{\otimes} M_{(m)} \cong \sum M_\mu$$

where the sum runs over all partitions μ of $n+m$ such that we have $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \mu_{n+m} \geq \lambda_{n+m}$.

Definition 2.3.4 *An unordered partition of n is a finite sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_t)$ such that $\sum_{i=1}^{t} \alpha_i = n$. In this case we write $\alpha \models n$.*

Definition 2.3.5 *A Young tableau is semistandard if the numbers are non-decreasing along the rows and strictly increasing down the columns.*

We now consider the obvious partial order on the set of partitions. Let $\lambda = (\lambda_1, \ldots, \lambda_p) \vdash n$ and $\mu = (\mu_1, \ldots, \mu_q) \vdash m$, then $\lambda \geq \mu$ if and only if $p \geq q$ and $\lambda_i \geq \mu_i$ for all $i = 1, \ldots, p$. In the language of Young diagrams $\lambda \geq \mu$ means that D_μ is a subdiagram of D_λ.

Let $\lambda \vdash n$, $\mu \vdash m$. We say that $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$ for all $i \geq 1$, i.e.,$D_\lambda \supseteq D_\mu$. if $\lambda \geq \mu$, we define the skew-partition $\lambda \setminus \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots)$; the corresponding diagram $D_{\lambda\setminus\mu}$ is the set of boxes of D_λ which do not belong to D_μ.

Definition 2.3.6 *A skew-tableau $T_{\lambda\setminus\mu}$ is a filling of the boxes of the skew-diagram $D_{\lambda\setminus\mu}$ with distinct natural numbers. if repetitions occur, then we have the notion of (generalized) skew-tableau. We also have the natural notion of standard and semistandard skew-tableau.*

Definition 2.3.7 *Let $\alpha = (\alpha_1, \ldots, \alpha_t) \models n$. We say that α is a lattice permutation if for each j the number of i's which occur among $\alpha_1, \ldots, \alpha_j$ is greater then or equal to the number of $(i+1)$'s for each i.*

We can now formulate

Theorem 2.3.8 (Littlewood-Richardson Rule) *Let $\lambda \vdash n$ and $\mu \vdash m$. Then*

$$M_\lambda \widehat{\otimes} M_\mu \cong \sum_{\nu \vdash n+m} k^\mu_{\nu\setminus\lambda} M_\nu$$

where $k^\mu_{\nu\setminus\lambda}$ is the number of semistandard tableau of shape $\nu \setminus \lambda$ and content μ which yield lattice permutations when we read their entries from right to left and downwards.

2.4 S_n-actions on multilinear polynomials

In this section we introduce an action of the symmetric group S_n on the space of multilinear polynomials in n fixed variables. We assume throughout this section that $\mathrm{char} F = 0$. We start with a remark about arbitrary irreducible S_n-modules.

Lemma 2.4.1 *Let M be an irreducible left S_n-module with character $\chi(M) = \chi_\lambda$, $\lambda \vdash n$. Then M can be generated as an S_n-module by an element of the form $e_{T_\lambda} f$ for some $f \in M$ and some Young tableau T_λ of shape λ. Moreover, for any Young tableau T^*_λ of shape λ there exists $f' \in M$ such that $M = FS_n e_{T^*_\lambda} f'$.*

The previous lemma says that, given a partition $\lambda \vdash n$ and a Young tableau T_λ of shape λ, any irreducible S_n-module M such that $\chi(M) = \chi_\lambda$ can be generated by an element of the form $e_{T_\lambda} f$ for some $f \in M$. By the definition of R_{T_λ}, for any $\sigma \in R_{T_\lambda}$ we have that $\sigma e_{T_\lambda} f = e_{T_\lambda} f$, i.e., $e_{T_\lambda} f$ is stable under the R_{T_λ}-action. The number of R_{T_λ}-stable elements in an arbitrary S_n-module M is closely related to the number of irreducible S_n-submodules of M having character χ_λ.

Lemma 2.4.2 *Let T_λ be a Young tableau corresponding to $\lambda \vdash n$ and let M be an S_n-module such that $M = M_1 \oplus \cdots \oplus M_m$ where $M_1, \ldots, M_m$ are irreducible S_n-submodules with character χ_λ. Then m is equal to the number of linearly independent elements $g \in M$ such that $\sigma g = g$ for all $\sigma \in R_{T_\lambda}$.*

Now let A be a PI-algebra and $Id(A)$ its T-ideal of identities. As we showed in Corollary 1.3.6, in characteristic zero, $Id(A)$ is determined by its multilinear polynomials.

We introduce

$$P_n = \text{span}\{\sigma \in S_n \mid \sigma \in S_n\},$$

the vector space of multilinear polynomials in $x_1, \ldots, x_n$ in the free algebra $F\langle X\rangle$. We define a map

$$\varphi : FS_n \to P_n$$

by setting

$$\varphi : \sum_{\sigma \in S_n} \alpha_\sigma \sigma \mapsto \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

It is clear that φ is a linear isomorphism. This isomorphism turns P_n into an S_n bimodule; if $\sigma, \tau \in S_n$, then

$$\sigma(x_{\tau(1)} \cdots x_{\tau(n)}) = x_{\sigma\tau(1)} \cdots x_{\sigma\tau(n)} = (x_{\sigma(1)} \cdots x_{\sigma(n)})\tau.$$

The interpretation of the left S_n-action on a polynomial $f(x_1, \ldots, x_n) \in P_n$, for $\sigma \in S_n$, is

$$\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

that is , of permuting the variable according to σ.

The right action of τ on $f(x_1, \ldots, x_n)$ is that of changing the places in each monomial $x_{\sigma(1)} \cdots x_{\sigma(n)}$ according to the permutation τ and is independent of σ. This means that the i-th factor $x_{\sigma(i)}$ will be placed in the

$\tau^{-1}(i)$ place of the new monomial, $i = 1, \ldots, n$. Note that T-ideals are not invariant in general under the right action.

Denote $P_n = P_n(x_1, \ldots x_n) = P_n(x)$. If $y_1, \ldots, y_n$ are other variables, one can consider $P_n(y_1, \ldots y_n) = P_n(y)$. If A is a PI-algebra and $\mathrm{char} F = 0$, it suffices to study the multilinear identities of A. Namely one should study $P_n(x) \cap Id(A)$, $P_n(y) \cap Id(A)$, etc. However, the correspondance $x_i \mapsto y_i$ yields the isomorphism $P_n(x) \cong P_n(y) \cong FS_n$ and it suffices to study just $P_n(x) \cap Id(A)$.

Since T-ideals are invariant under permutations of the variables, we obtain that $P_n \cap Id(A)$ is a left S_n-submodule of P_n. Hence

$$P_n(A) = \frac{P_n}{P_n \cap Id(A)}$$

has an induced structure of left S_n-module. If $F\langle X\rangle$ is the free algebra of countable rank on $X = \{x_1, x_2, \ldots\}$, then $P_n(A)$ is the space of multilinear elements in the first n variables of relatively free algebra $F\langle X\rangle \diagup Id(A)$. If $\mathcal{V} = \mathrm{var}(A)$, we also write $P_n(\mathcal{V}) = P_n(A)$.

Definition 2.4.3 *The non-negative integer*

$$c_n(A) = \dim P_n(A) = \frac{P_n}{P_n \cap Id(A)}$$

is called the nth codimension of the algebra A.

Definition 2.4.4 *For $n \geq 1$, the S_n-character of $P_n(A) = \frac{P_n}{P_n \cap Id(A)}$ is called the nth cocharacter of A and is denoted $\chi_n(A)$*

Now, if we decompose the nth cocharacter into irreducibles, we obtain

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the irreducible S_n-character associated to the partition $\lambda \vdash n$ and $m_\lambda \geq 0$ is the corresponding multiplicity. Then we have

Definition 2.4.5

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda$$

is called the nth colength of A.

In other words, $l_n(A)$ counts the number of irreducible S_n-modules appearing in the decomposition of $P_n(A)$. We recall that if $\mathcal{V}$ is a variety of algebras we write $c_n(\mathcal{V}) = c_n(A)$, $\chi_n(\mathcal{V}) = \chi_n(A)$ and $l_n(\mathcal{V}) = l_n(A)$ where A is an algebra generating $\mathcal{V}$.

We recall these important properties for codimensions and colengths.

Theorem 2.4.6 *If the algebra A satisfies an identity of degree $d \geq 1$, then $c_n(A) \leq (d-1)^{2n}$.*

Theorem 2.4.7 *If $\mathcal{V}$ is a non-trivial variety, the sequence of colengths of $\mathcal{V}$ is polynomially bounded, i.e., there exist constant C and k such that*

$$l_n(\mathcal{V}) \leq Cn^k$$

for all $n \geq 1$.

2.5 Representation of the general linear group

In this section we survey the information on representation theory of the general linear group over an algebraically closed field of characteristic zero in a form which we need for our study of PI-algebras. We restrict our attention to the case when $GL_m = GL_m(F)$ acts on the free associative algebra of rank m and consider the so-called polynomial representations of GL_m which have many properties similar to those of the representations of finite groups. We refer to ([24], Chapter 12) for the results of this section.

Definition 2.5.1 *The representation of the general linear group GL_m:*

$$\phi : GL_m \to GL_s$$

is called polynomial if the entries of the $s \times s$ matrix $\phi(a_{ij})$ are polynomial functions of the entries of the $m \times m$ matrix a_{ij}, for all $a_{ij} \in GL_m$. When all the entries of $\phi(a_{ij})$ are homogeneous polynomials of degree k, then ϕ is a homogeneous representation of degree k.

Let $F_m\langle X\rangle = F\langle x_1, \dots, x_m\rangle$ denote the free associative algebra in m variables and let $U = \mathrm{span}_F\{x_1, \dots, x_m\}$.

The action of the group $GL_m \cong GL(U)$ on $F_m\langle X\rangle$ can be obtained extending diagonally the natural left action of GL_m on the space U by defining:

$$g(x_{i_1}, \dots, x_{i_k}) = g(x_{i_1}) \cdots g(x_{i_k}), \quad g \in GL_m \quad x_{i_1}, \dots, x_{i_k} \in F_m\langle X\rangle.$$

Actually, $F_m\langle X\rangle$ is a polynomial GL_m-module (i.e. the corresponding representation is polynomial).

Let F_m^n be the space of homogeneous polynomials of degree n in the variables $x_1, \dots, x_m$, then F_m^n is a (homogeneous polynomial) submodule of $F_m\langle X\rangle$. We observe that:

$$F_m^n = \bigoplus_{i_1+\cdots+i_m=n} F_m^{(i_1,\dots,i_m)}$$

where $F_m^{(i_1,\dots,i_m)}$ is the multihomogeneous subspace spanned by all monomials of degree i_1 in $x_1, \dots, i_m$ in x_m.

The following theorem states a result similar to Maschke's Theorem about the complete reducibility of GL_m-modules, valid for the polynomial representations of GL_m.

Theorem 2.5.2 *Every polynomial GL_m-module is a direct sum of irreducible homogeneous polynomial submodules.*

The irreducible homogeneous polynomial GL_m-modules are described by partition of n in not more than m parts and Young diagrams.

Theorem 2.5.3 *Let $P_m(n)$ denote the set of all partitions of n with at most m parts (i.e. whose diagrams have height at most m).*

1. *The pairwise non isomorphic irreducible homogeneous polynomial GL_m-modules of degree $n \geq 1$ are in one-to-one correspondence with the partitions $\lambda \in P_m(n)$. We denote by W^λ an irreducible GL_m-module related to λ.*

2. *Let $\lambda \in P_m(n)$. Then the GL_m-module W^λ is isomorphic to a submodule of F_m^n. Moreover, the GL_m-module F_m^n has a decomposition:*

$$F_m^n \cong \sum_{\lambda \in P_m(n)} d_\lambda W^\lambda,$$

   where d_λ is the dimension of the irreducible S_n-module corresponding to the partition λ.

3. *As a subspace of F_m^n, the vector space W^λ is multihomogeneous, i.e.*

$$W^\lambda = \bigoplus_{i_1+\cdots+i_m=n} W^{\lambda,(i_1,\ldots,i_m)}$$

   where $W^{\lambda,(i_1,\ldots,i_m)} = W^\lambda \cap F_m^{\lambda,(i_1,\ldots,i_m)}$.

We want to show that if $W^\lambda \subseteq F_m^n$, then W^λ is cyclic and generated by a multihomogeneous polynomial of multidegree $(\lambda_1, \ldots, \lambda_k)$ with $\lambda = (\lambda_1, \ldots, \lambda_k) \in P_m^n$.

We observe first that the symmetric group S_n acts from the right on F_m^n by permuting the places in which the variables occur, i.e. for all $x_{i_1}, \ldots, x_{i_n} \in F_m^n$ and for all $\sigma \in S_n$

$$x_{i_1} \cdots x_{i_n} \sigma = x_{i_{\sigma(1)}}, \ldots, x_{i_{\sigma(n)}}.$$

Let now $\lambda = (\lambda_1, \ldots, \lambda_k) \in P_m(n)$. We denote by s_λ the following polynomial of F_m^n:

$$s_\lambda = s_\lambda(x_1, \ldots, x_k) = \prod_{i=1}^{\lambda_1} St_{h_i(\lambda)}(x_1, \ldots, x_{h_i(\lambda)}),$$

where $h_i(\lambda)$ is the height of the ith column of the diagram of λ and

$$St_r(x_1, \ldots, x_r) = \sum_{\tau \in S_r} (\mathrm{sgn}\tau) x_{\tau(1)} \cdots x_{\tau(r)}$$

is the standard polynomial of degree r. Note that by definition s_λ is multi-homogeneous of multidegree $(\lambda_1, \ldots, \lambda_k)$.

Theorem 2.5.4 *1. The element $s_\lambda(x_1, \ldots, x_k)$, defined above, generates an irreducible GL_m-submodule W of F_m^n isomorphic to W^λ.*

2. Every submodule $W^\lambda \subseteq F_m^n$ is generated by a non-zero polynomial called highest weight vector of W^λ, of the type:

$$f_\lambda = s_\lambda \sum_{\sigma \in S_n} \alpha_\sigma \sigma, \quad \alpha_\sigma \in F.$$

The highest weight vector f_λ is unique up to a multiplicative constant and it is contained in the one-dimensional vector space $W^{\lambda,(\lambda_1,\ldots,\lambda_k)} = W^\lambda \cap F_m^{\lambda,(\lambda_1,\ldots,\lambda_k)}$.

3. Let $\sum_{\sigma \in S_n} \alpha_\sigma \sigma \in FS_n$. if $s_\lambda \sum_{\sigma \in S_n} \alpha_\sigma \sigma \neq 0$, then it generates an irreducible submodule $W \cong W^\lambda$, $W \subseteq F_m^n$.

Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in P_m(n)$ and let T_λ be a Young tableau. We denote by f_{T_λ} the highest weight vector obtained from (1) by considering the only permutation $\sigma \in S_n$ such that the first column of T_λ is filled in from top to bottom with the integers $\sigma(1), \ldots, \sigma(h_1(\lambda))$, in this order, the second column is filled in with $\sigma(h_1(\lambda)+1), \ldots, \sigma(h_1(\lambda)+h_1(\lambda))$, etc.

Proposition 2.5.5 *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in P_m(n)$ and let $W^\lambda \subseteq F_m^n$. The highest weight vector f_λ of W^λ can be expressed uniquely as a linear combination of the polynomials f_{T_λ} with T_λ standard tableau.*

2.6 Group gradings on algebras

In this section we introduce the notion of an algebra graded by a group and we give some examples; moreover we extend the asymptotic methods developed in the previous sections.

Definition 2.6.1 *Let G be a group and A be an associative algebra over an infinite field F. A G-grading on A is a decomposition of A as direct sum of F-vector subspaces $A = \bigoplus_{g \in G} A_g$ such that $A_g A_h \subseteq A_{gh} \quad \forall g, h \in G$.*

From the definition it is clear that any $a \in A$ can be uniquely written as a finite sum $a = \sum_{g \in G} a_g$ with $a_g \in A_g$. The subspaces A_g are called the homogeneous components of A. Accordingly, an element $a \in A$ is homogeneous (or homogeneous of degree g) if $a \in A_g$. A subspace $B \subseteq A$ is graded or homogeneous if $B = \bigoplus_{g \in G}(B \cap A_g)$. In other words, B is graded if, for any $b \in B$, $b = \sum_{g \in G} b_g$ implies that $b_g \in B$ for all $g \in G$. Similarly, one can define graded subalgebras, graded ideals, etc. Notice that if H is a subgroup of G, then clearly $B = \bigoplus_{h \in H} A_h$ is a graded subalgebra of A. In particular, if e is the unit of G, A_e is a subalgebra of A. Next we give some examples of graded algebras.

Example 2.6.2 *Any algebra A can be graded by any group G by setting $A = A_e$ and $A_g = 0$ for any $g \neq e$. This grading is called trivial.*

Example 2.6.3 *The group algebra $A = FG$ of a group G is naturally graded by G by setting $A_g = span_F g$.*

Example 2.6.4 *Let $A = M_k(F)$ be the algebra of $k \times k$ matrices over F and let G be an arbitrary group. Give any k-tuple $(g_1, \ldots, g_k) \in G^k$, one can define a G-grading of A by setting*

$$A_g = span_F\{e_{ij} \mid g_i^{-1} g_j = g\},$$

where e_{ij} are the usual matrix units.

Example 2.6.5 *Let UT_n be the algebra of $n \times n$ upper-triangular matrices. A G-grading on $A = UT_n(F)$ is called elementary if there exists $\overline{g} = (g_1, \ldots, g_n) \in G^n$ an n-tuple of elements of G such that $A_g = span\{e_{ij} \mid g_i^{-1}g_j = g\} \quad \forall g \in G$, i.e.: the homogeneous degree of e_{ij} is equal to $g_i^{-1}g_j$ for every i and j such that $1 \leq i \leq j \leq n$.*

Suppose that the algebra A is graded by a finite group G. Let $G = \{g_1 = e, g_2, \ldots, g_s\}$ and let $A = \bigoplus_{i=1}^{s} A_{g_i}$ be the decomposition of A into its homogeneous components. Hence $A_{g_i}A_{g_j} \subseteq A_{g_ig_j}$, for all $i, j = 1, \ldots, s$.

We denote by $F\langle X, G\rangle$ the free associative G-graded algebra of countable rank over F. Here the set X decomposes as $X = \bigcup_{i=1}^{s} X_{g_i}$, where the sets $X_{g_i} = \{x_{1,g_i}, x_{2,g_i}, \ldots\}$ are disjoint, and the elements of X_{g_i} have homogeneous degree g_i. The algebra $F\langle X, G\rangle$ has a natural G-grading $F\langle X, G\rangle = \bigoplus_{g\in G} \mathcal{F}_g$, where $\mathcal{F}_g$ is the subspace of $F\langle X, G\rangle$ spanned by the monomials $x_{i_1,g_{j_1}} \cdots x_{i_t,g_{j_t}}$ of homogeneous degree $g = g_{j_1} \cdots g_{j_t}$.

Recall that an element of $F\langle X, G\rangle$ is called a graded polynomial. Also, f is a graded (polynomial) identity of the algebra A, and we write $f \equiv 0$, in case f vanishes under all graded substitutions $x_{i,g} \to a_g \in A_g$. Let $Id^G(A) = \{f \in F\langle X, G\rangle \mid f \equiv 0 \text{ on } A\}$ be the ideal of graded identities of A. Clearly $Id^G(A)$ is invariant under all graded endomorphism of $F\langle X, G\rangle$.

Notice that if for $i \geq 1$ we set $x_i = x_{i,g_i} + \cdots + x_{i,g_s}$, then the free algebra $F\langle X\rangle$ is naturally embedded in $F\langle X, G\rangle$ and we can regard the ordinary identities of A as a special kind of graded identities.

Since $\mathrm{char}F = 0$, the multilinear polynomials of $Id^G(A)$ determine all of $Id^G(A)$. Hence for $n \geq 1$ we define

$$P_n^G = \mathrm{span}_F\{x_{\sigma(1),g_{i_{\sigma(1)}}} \cdots x_{\sigma(n),g_{i_{\sigma(n)}}} \mid \sigma \in S_n, g_{i_1}, \ldots, g_{i_n} \in G\}$$

to be the space of multilinear G-graded polynomials in the variables

$$x_{1,g_{i_1}}, \ldots, x_{n,g_{i_n}}, g_{i_j} \in G.$$

The ideal $Id^G(A)$ is determined by the sequence of subspaces $P_n^G \cap Id^G(A), n = 1, 2, \ldots$, but we can consider even smaller spaces.

Let $n \geq 1$ and write $n = n_1 + \cdots + n_s$ as a sum of non-negative integers. Define $P_{n_1,\ldots,n_s} \subseteq P_n^G$ to be the space of multilinear graded polynomials in which the first n_1 variables have homogeneous degree g_1, the next n_2 variables have homogeneous degree g_2 and so on. Notice that given such $n_1, \ldots, n_s$, there are $\binom{n}{n_1,\ldots,n_s}$ subspaces isomorphic to $P_{n_1,\ldots,n_s}$ where $\binom{n}{n_1,\ldots,n_s}$ denotes the multinomial coefficient. It is clear that P_n^G is the direct sum of such subspaces with $n_1 + \cdots + n_s = n$. Moreover such decomposition is inherited by $P_n^G \cap Id^G(A)$ and we consider the spaces $P_{n_1,\ldots,n_s} \cap Id^G(A)$. At the light of these observations, one defines

$$P_{n_1,\ldots,n_s}(A) = \frac{P_{n_1,\ldots,n_s}}{P_{n_1,\ldots,n_s} \cap Id^G(A)}.$$

The space $P_{n_1,\ldots,n_s}(A)$ is naturally endowed with a structure of $S_{n_1} \times \cdots \times S_{n_s}$- module in the following way: the group $S_{n_1} \times \cdots \times S_{n_s}$ acts on the left on $P_{n_1,\ldots,n_s}$ by permuting the variables of the same homogeneous degree; hence S_{n_1} permutes the variables of homogeneous degree g_1, S_{n_2} those of homogeneous degree g_2, etc. Since $Id^G(A)$ is invariant under this action, $P_{n_1,\ldots,n_s}(A)$ has a structure of $S_{n_1} \times \cdots \times S_{n_s}$- module and we denote by $\chi^G_{n_1,\ldots,n_s}(A)$ its character.

If $\lambda(1) \vdash n_1, \ldots, \lambda(s) \vdash n_s$, are partitions, then we write $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s)) \vdash (n_1, \ldots, n_s)$ and we say that $\langle\lambda\rangle$ is a multipartition of $n = n_1 + \cdots + n_s$.

Since $\mathrm{char}F = 0$, by complete reducibility $\chi^G_{n_1,\ldots,n_s}(A)$ can be written as a sum of irreducibles characters in the following way:

$$\chi^G_{n_1,\ldots,n_s}(A) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}, \qquad (1)$$

where $m_{\langle\lambda\rangle}$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$ in $\chi^G_{n_1,\ldots,n_s}(A)$. We call $\chi^G_{n_1,\ldots,n_s}(A)$ the $(n_1, \ldots, n_s)$th cocharacter of A.

Recall (see for instance [30]) that the wreath product of G and S_n is the group defined by

$$G \wr S_n = \{(g_1, \ldots, g_n; \sigma) \mid g_1, \ldots, g_n \in G, \sigma \in S_n\}$$

with multiplication given by

$$(g_1, \ldots, g_n; \sigma)(h_1, \ldots, h_n; \tau) = (g_1 h_{\sigma^{-1}(1)}, \ldots, g_n h_{\sigma^{-1}(n)}; \sigma\tau).$$

We remark that if G is an abelian group, there is a well-known duality between G-gradings and G-actions on the algebra A (one needs to assume that the base field has enough roots of 1). Through this duality one can define an action of the wreath product $G \wr S_n$ on P_n^G (see [29]). Since this action preserves $Id^G(A)$, the space $P_n^G(A) = P_n^G/(P_n^G \cap Id^G(A))$ becomes a $G \wr S_n$- module and let $\chi_n^G(A)$ be its character. The irreducible characters of $G \wr S_n$ are indexed by multipartition of n. Hence one writes

$$\chi_n^G(A) = \sum_{\langle\lambda\rangle \vdash n} m'_{\langle\lambda\rangle} \chi_{\langle\lambda\rangle}, \qquad (2)$$

and by an obvious generalization of [25], we have that if $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s))$ with $\lambda(1) \vdash n_1, \ldots, \lambda(s) \vdash n_s$, then in (1) and (2), $m_{\langle\lambda\rangle} = m'_{\langle\lambda\rangle}$.

Chapter 3

Group Graded Algebras and Multiplicities bounded by a constant

3.1 Preliminaries

Throughout this chapter F will denote a field of characteristic zero and A an associative F-algebra satisfying a non-trivial polynomial identity (PI-algebra).

Let E be the infinite dimensional Grassmann algebra generated by a countable set $\{e_1, e_2, \ldots\}$ subject to the condition $e_ie_j = -e_je_i$, for all i, j. Then E has a natural $\mathbb{Z}_2$-grading, $E = E_0 \bigoplus E_1$ where

$$E_0 = \operatorname{span}\{e_{i_1} \cdots e_{i_{2k}} \mid 1 \leq i_1 < \ldots < i_{2k}, k \geq 0\}$$

and

$$E_1 = \operatorname{span}\{e_{i_1} \cdots e_{i_{2k+1}} \mid 1 \leq i_1 < \ldots < i_{2k+1}, k \geq 0\}.$$

If $A = A_0 \oplus A_1$, is a $\mathbb{Z}_2$-graded algebra, then the Grassmann envelope of A is defined as $E(A) = (E_0 \otimes A_0) \oplus (E_1 \otimes A_1)$. Notice that if A is a $G \times \mathbb{Z}_2$-graded algebra, $A = \bigoplus_{(g,i) \in G \times \mathbb{Z}_2} A_{(g,i)}$ is a $G \times \mathbb{Z}_2$-graded algebra, we can consider the induced $\mathbb{Z}_2$-grading on A, $A = A_0 \oplus A_1$ where $A_0 = \bigoplus_{g\in G} A_{(g,0)}$ and $A_1 = \bigoplus_{g\in G} A_{(g,1)}$. Hence in this case the Grassmann envelope of A can be regarded as a G-graded algebra via $E(A) = \bigoplus_{g\in G} E(A)_g$ where $E(A)_g = (E_0 \otimes A_{(g,0)}) \oplus (E_1 \otimes A_{(g,1)})$.

Next we recall an important theorem of Aljadeff and Belov [3], proved independently by Sviridova in [48] for abelian groups.

Theorem 3.1.1 *Let G be a finite group and A a G-graded PI-algebra over a field of characteristic zero. Then there exists a finite dimensional $G \times \mathbb{Z}_2$-graded algebra B such that $Id^G(A) = Id^G(E(B))$ where $E(B)$ is the Grassmann envelope of B.*

Now let A be a finite dimensional $G \times \mathbb{Z}_2$-graded algebra. By the Wedderburn-Malcev theorem ([19]), we can write $A = B + J$ where B is a maximal semisimple subalgebra of A and J is its Jacobson radical. It is well known that J is a graded ideal, moreover by [49] we assume, as we may, that B is a $G \times \mathbb{Z}_2$-graded subalgebra of A. Hence we can write $B = B^{(1)} \oplus \cdots \oplus B^{(m)}$ where every $B^{(i)}$ is a $G \times \mathbb{Z}_2$-graded simple algebra. There is an important theorem of Bahturin, Sehgal and Zaicev in [16] that gives a characterization of all $G \times \mathbb{Z}_2$-simple algebras.

Theorem 3.1.2 *Let B be a finite dimensional $G \times \mathbb{Z}_2$-graded simple algebra over an algebraically closed field F. Then B has the following structure: there exist a subgroup H of $G \times \mathbb{Z}_2$, a 2-cocycle $\alpha : H \times H \to F^*$ where the action of H on F is trivial, an integer k and a k-tuple $(a_1 = e, a_2, \ldots, a_k) \in (G \times \mathbb{Z}_2)^k$ such that B is $G \times \mathbb{Z}_2$- isomorphic to $C = F^\alpha H \otimes M_k(F)$ where for $a \in G \times \mathbb{Z}_2$, $C_a = span_F\{u_h \otimes e_{ij} : a = a_i^{-1} h a_j\}$. Here $u_h \in F^\alpha H$ is a representative of $h \in H$ and the e_{ij}'s are the matrix units of $M_k(F)$.*

We recall that if $\mathcal{V} = var^G(A)$ is a variety of G-graded algebras generated by A, we write $\chi_{n_1,\ldots,n_s}(\mathcal{V})$ for $\chi_{n_1,\ldots,n_s}(A)$.

Recall also that UT_2 is the algebra of 2×2 upper triangular matrices. By [52, Theorem 1], any G-grading on UT_2 is up to isomorphism, the elementary grading determined by (e, g), for some $g \in G$. When $g = e$, then we get the trivial grading. So by $var^G(UT_2^G)$ we denote the variety of G-graded algebras generated by UT_2 with an elementary G-grading. Recall that $G = \{g_1 = e, g_2, \dots, g_s\}$. Then we can restate the following.

Theorem 3.1.3 *([52, Theorem 3]) Let UT_2^G be endowed with a non-trivial G-grading determined by (e, g_i), for some $i \neq 1$. Then the T_G-ideal of graded identities of UT_2^G is generated by the polynomials $[x_{1,e}, x_{2,e}]$, $x_{1,g_i}x_{2,g_i}$ and x_{1,g_j} for all $g_j \in G$, $g_j \neq e, g_i$. Moreover if*

$$\chi^G_{n_1,\dots,n_s}(UT_2^G) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \dots \otimes \chi_{\lambda(s)}$$

is the $(n_1, \dots, n_s)$th cocharacter of UT_2^G, we have:

1. *$m_{\langle\lambda\rangle} = q + 1$, if $\lambda(1) = (p + q, p), \lambda(i) = (1)$, and $\lambda(j) = \emptyset$, $j \neq i, 1$.*
2. *$m_{\langle\lambda\rangle} = 1$, if $\langle\lambda\rangle = ((n), \emptyset, \dots, \emptyset)$.*
3. *$m_{\langle\lambda\rangle} = 0$ in all other cases.*

Proof. Let $A = UT_2^G$ be graded by the pair (e, g_i). Then $A = A_e \oplus A_{g_i}$. If we consider the canonical $\mathbb{Z}_2$- grading on UT_2, we get $A = A_{\bar{0}} \oplus A_{\bar{1}}$ and $A_{\bar{0}} = A_e$, $A_{\bar{1}} = A_{g_i}$. It follows that $f(x_{1,e}, \dots, x_{k,e}, x_{1,g_i}, \dots, x_{l,g_i}) \in Id^G(UT_2^G)$ if and only if $f(x_{1,\bar{0}}, \dots, x_{k,\bar{0}}, x_{1,\bar{1}}, \dots, x_{l,\bar{1}}) \in Id^{\mathbb{Z}_2}(UT_2^{\mathbb{Z}_2})$. We then apply [52, Theorem 3] and we obtain conditions 1), 2) and 3) also for the $(n_1, \dots, n_s)$th cocharacter of UT_2^G. $\square$

We now give some preliminary results needed in what follows. We start with the following remarks.

Remark 3.1.4 *Let A, B be G-graded algebras such that $Id^G(A) \subseteq Id^G(B)$. If*

$$\chi^G_{n_1,\dots,n_s}(A) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \dots \otimes \chi_{\lambda(s)}$$

and

$$\chi^G_{n_1,\dots,n_s}(B) = \sum_{\langle\lambda\rangle\vdash n} m'_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(s)}$$

are the $(n_1,\dots,n_s)$*th cocharacter of* A *and* B *respectively, then*

$$m_{\langle\lambda\rangle} \geq m'_{\langle\lambda\rangle},$$

for all $\langle\lambda\rangle \vdash n$*.*

Proof. Let $I_1 = Id^G(B)$, $I_2 = Id^G(A)$; since $I_1 \supseteq I_2$ then

$$\frac{P_n^G}{P_n^G\cap I_1} \cong \frac{P_n^G}{P_n^G\cap I_2}\Big/\frac{P_n^G\cap I_1}{P_n^G\cap I_2}.$$

Thus we have an embedding (of FH_n-modules)

$$\frac{P_n^G}{P_n^G\cap I_1} \hookrightarrow \frac{P_n^G}{P_n^G\cap I_2}$$

and this implies that $m_{\langle\lambda\rangle} \leq m'_{\langle\lambda\rangle}$ for every $\langle\lambda\rangle \vdash n$. □

Remark 3.1.5 *Let* A*,* B *be two* G*-graded algebras and let* $A\oplus B$ *be their direct sum. If* $\chi^G_{n_1,\dots,n_s}(A) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(s)}$ *,* $\chi^G_{n_1,\dots,n_s}(B) = \sum_{\langle\lambda\rangle\vdash n} m'_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(s)}$ *and* $\chi^G_{n_1,\dots,n_s}(A\oplus B) = \sum_{\langle\lambda\rangle\vdash n} \overline{m}_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(s)}$ *are the corresponding* $(n_1,\dots,n_s)$*-th cocharacters, then*

$$\overline{m}_{\langle\lambda\rangle} \leq m_{\langle\lambda\rangle} + m'_{\langle\lambda\rangle},$$

for all $\langle\lambda\rangle \vdash n$*.*

Proof. Let $I_1 = Id^G(A)$, $I_2 = Id^G(B)$ and $I = Id^G(A\oplus B)$; clearly $I = I_1\cap I_2$. Consider the following linear map:

$$f : P_n^G \to \frac{P_n^G}{P_n^G\cap I_1} + \frac{P_n^G}{P_n^G\cap I_2}$$

such that $a \mapsto (a + (P_n^G\cap I_1), a + (P_n^G\cap I_2))$. It is easy to see that its kernel is $Ker(f) = P_n^G\cap I_1\cap I_2 = P_n^G\cap I$ thus we have an embedding (of FH_n-modules)

$$\frac{P_n^G}{P_n^G\cap I} \hookrightarrow \frac{P_n^G}{P_n^G\cap I_1} + \frac{P_n^G}{P_n^G\cap I_2}.$$

It follows that $\overline{m}_{\langle\lambda\rangle} \leq m_{\langle\lambda\rangle} + m'_{\langle\lambda\rangle}$, for all $\langle\lambda\rangle \vdash n$. □

3.2 Some lemmas

In this section we prove some important lemmas.

Lemma 3.2.1 *Let $\mathcal{V}$ be a variety of G-graded PI-algebras and suppose that $UT_2^G \notin \mathcal{V}$, for any G-grading on UT_2. Then $\mathcal{V} = var^G(E(A))$ for some finite dimensional $G \times \mathbb{Z}_2$-graded algebra A such that $A = B + J$ where $B \cong B^{(1)} \oplus \cdots \oplus B^{(m)}$ with $B^{(i)} \cong F^{\alpha_i}H_i$, $1 \leq i \leq m$, and $J = J(A)$. Here H_i is a subgroup of $G \times \mathbb{Z}_2$ and $\alpha_i : H_i \times H_i \to F^*$ is a 2-cocycle.*

Proof. By Theorem 3.1.1, we can write $\mathcal{V} = var^G(E(A))$ where $E(A)$ is the Grassmann envelope of a finite dimensional $G \times \mathbb{Z}_2$-graded algebra A. As we have previously seen, $A = B^{(1)} \oplus \cdots \oplus B^{(m)} + J$, with the $B^{(i)}$'s G-simple algebras for every $i = 1, \ldots, m$. Now, by Theorem 3.1.2, for every i, $B^{(i)} \cong M_{k_i}(F) \otimes F^{\alpha_i}H_i$ for some subgroup H_i of $G \times \mathbb{Z}_2$ and 2-cocycle $\alpha_i : H_i \times H_i \to F^*$. We need to prove that for every i, $1 \leq i \leq m$, $k_i = 1$, i.e., $B^{(i)} \cong F^{\alpha_i}H_i$.

Suppose by contradiction that $B^{(i)} \cong D = M_k(F) \otimes F^{\alpha}H$ with $k > 1$. Then $D = \bigoplus_{(g,a)\in G\times\mathbb{Z}_2} D_{(g,a)}$. Let $((h_1, a_1), \ldots, (h_k, a_k)) \in (G\times\mathbb{Z}_2)^k$ be the k-tuple inducing the elementary grading on $M_k(F)$. Then for any $(g, a) \in G \times \mathbb{Z}_2$ we have

$$D_{(g,a)} = \operatorname{span}\{u_{(h,b)} \otimes e_{ij} \mid (h_i, a_i)(h, b)(h_j, a_j)^{-1} = (g, a)\},$$

where $\{u_{(h,b)} \mid (h, b) \in H\}$ is the canonical basis of the twisted group algebra $F^{\alpha}H$. Note that $u_{(e,0)} \otimes e_{ii} \in D_{(e,0)}$ since $(h_i, a_i)(e, 0)(h_i, a_i)^{-1} = (e, 0)$ and $u_{(e,0)} \otimes e_{12} \in D_{(h_1h_2^{-1},a_1-a_2)}$. Hence $L \cong Fe_{11} \oplus Fe_{22} \oplus Fe_{12}$ is a subalgebra of D with induced $G \times Z_2$ grading

$$\begin{pmatrix} (e,0) & (h_1h_2^{-1}, a_1 - a_2) \\ 0 & (e,0) \end{pmatrix}.$$

We write $L = L_{(e,0)} \oplus L_{(h_1h_2^{-1},a_1-a_2)}$ where $L_{(e,0)} \cong Fe_{11} + Fe_{22}$, $L_{(h_1h_2^{-1},a_1-a_2)} \cong Fe_{12}$ with induced $G\times\mathbb{Z}_2$-grading. Consider now the Grassmann envelope

$E(L)$ of L. If $a_1 - a_2 = 0$ we have

$$E(L) = E_0 \otimes (L_{(e,0)} \oplus L_{(h_1h_2^{-1},0)}),$$

and if $a_1 - a_2 = 1$ we have

$$E(L) = (E_0 \otimes L_{(e,0)}) \oplus (E_1 \otimes L_{(h_1h_2^{-1},1)}).$$

Therefore if $a_1 - a_2 = 0$, $E(L) \cong \begin{pmatrix} E_0 & E_0 \\ 0 & E_0 \end{pmatrix}$ with trivial grading. It follows that if UT_2 denotes UT_2^G with trivial grading, then $UT_2 \in var^G(E(L)) \subseteq var(E(A)) = \mathcal{V}$, a contradiction.

Suppose now that $a_1 - a_2 = 1$. Then

$$E(L)_e = (E_0 \otimes L_{(e,0)}) \oplus (E_1 \otimes L_{(e,1)}) = E_0 \otimes L_{(e,0)},$$

$$E(L)_{h_1h_2^{-1}} = (E_0 \otimes L_{(h_1h_2^{-1},0)}) \oplus (E_1 \otimes L_{(h_1h_2^{-1},1)}) = E_1 \otimes L_{(h_1h_2^{-1},1)}$$

and $E(L)_g = 0$, for all $g \neq e$, $g \neq h_1h_2^{-1}$. Thus $E(L) \cong \begin{pmatrix} E_0 & E_1 \\ 0 & E_0 \end{pmatrix}$ with grading $\begin{pmatrix} e & h_1h_2^{-1} \\ 0 & e \end{pmatrix}$. We will show that in this case $Id^G(E(L)) = Id^G(UT_2^G)$ where UT_2^G has grading $\begin{pmatrix} e & h_1h_2^{-1} \\ 0 & e \end{pmatrix}$.

In fact it is easy to verify that $E(L)$ satisfies the identities, $[x_{1,e}, x_{2,e}] \equiv 0$, $x_{1,g}x_{2,g} \equiv 0$, for $g = h_1h_2^{-1}$ and $x_{1,h} \equiv 0$, for all $h \neq e, g$. Thus $Id^G(UT_2^G) \subseteq Id^G(E(L))$. On the other hand, let $f \in Id^G(E(L))$ be a multilinear polynomial. Let $\langle [x_{1,e}, x_{2,e}], x_{1,g}x_{2,g}, x_{1,h} \mid h \neq e, g\rangle_T$ be the T-ideal generated by the polynomials $[x_{1,e}, x_{2,e}]$, $x_{1,g}x_{2,g}$ and $x_{1,h}$. If we reduce f mod$\langle [x_{1,e}, x_{2,e}], x_{1,g}x_{2,g}, x_{1,h} \mid h \neq e, g\rangle_T$, we may clearly assume that only one variable of homogeneous degree g appears in f. Hence we may assume that the polynomial f can be written in the form:

$$f = \sum_{i_1 < \cdots < i_h, j_1 < \cdots j_{n-h-1}} \alpha_{i_1 \ldots i_h} x_{i_1,e} \cdots x_{i_h,e} x_{1,g} x_{j_1,e} \cdots x_{j_{n-h-1},e}.$$

We shall prove that for any $\{i_1, \ldots, i_h\} \subseteq \{1, \ldots, n\}$, $\alpha_{i_1 \ldots i_h} = 0$. In fact if we specialize $x_{i_1,e} = \cdots = x_{i_h,e} = e_{11}$, $x_{1,g} = e_{12}$, $x_{j_1,e} = \cdots = x_{j_{n-h-1},e} = e_{22}$, f takes value $\alpha_{i_1 \ldots i_h} e_{12} = 0$. This proves that $\alpha_{i_1 \ldots i_h} = 0$ for all $i_1, \ldots, i_h$ and, so, $f = 0$. Hence $E(L) \cong \begin{pmatrix} E_0 & E_1 \\ 0 & E_0 \end{pmatrix}$ has the same G-graded identities as UT_2^G with grading $\begin{pmatrix} e & h_1 h_2^{-1} \\ 0 & e \end{pmatrix}$. It follows that $var(UT_2^G) \subseteq var(E(L)) \subseteq var(E(A))$, a contradiction. So, for all $i \geq 1$, $B^{(i)} \cong F^\alpha H$ and the proof is complete. $\square$

Lemma 3.2.2 *Under the hypotheses of the previous lemma, $\mathcal{V} = var^G(E(A_1) \oplus \cdots \oplus E(A_n))$ where for every $i \in \{1, \ldots, n\}$, A_i is a finite dimensional $G \times \mathbb{Z}_2$-graded algebra with Jacobson radical J_i. Moreover $A_i = B_i + J_i$, where B_i is a $G \times \mathbb{Z}_2$-graded simple algebra isomorphic to $F^{\alpha_i} H_i$ for some $H_i \leq G \times \mathbb{Z}_2$ and 2-cocycle $\alpha_i : H_i \times H_i \to F^*$.*

Proof. By the previous lemma $\mathcal{V} = var^G(E(A))$ where $A = B^{(1)} \oplus \cdots \oplus B^{(m)} + J$ and, for every $i \in \{1, \ldots, m\}$, $B^{(i)} \cong F^{\alpha_i} H_i$ for some $H_i \leq G \times \mathbb{Z}_2$ and $\alpha_i : H_i \times H_i \to F^*$ a 2-cocycle.

Suppose that $B^{(i)} J B^{(k)} \neq 0$, for some $i \neq k$. Then there exist homogeneous elements $b_i \in B^{(i)}$, $b_k \in B^{(k)}$, $c \in J$ such that $b_i c b_k \neq 0$. But $b_i = b_i 1_{B^{(i)}}, b_j = b_j 1_{B^{(j)}}$ implies $b_i 1_{B^{(i)}} c b_j 1_{B^{(j)}} \neq 0$. Set $f = 1_{B^{(i)}}$, $g = 1_{B^{(j)}}$, $h = 1_{B^{(i)}} c 1_{B^{(j)}}$ and note that h is homogeneous and $f^2 = f$,$g^2 = g$, $fh = hg = h$, $hf = fg = gf = gh = 0$. Also f and g have homogeneous degree $(e, 0)$ and h has homogeneous degree (g, a) ($a = 0$ or 1). Thus if N is the algebra generated by f, g and h we have that $N \cong UT_2$ with $G \times \mathbb{Z}_2$ grading $\begin{pmatrix} (e,0) & (g,a) \\ 0 & (e,0) \end{pmatrix}$.

As we have seen in the proof of Lemma 3.2.1,

$$E(N) \cong \begin{pmatrix} E_0 & E_0 \\ 0 & E_0 \end{pmatrix} \text{ or } \begin{pmatrix} E_0 & E_1 \\ 0 & E_0 \end{pmatrix}$$

with induced G-grading and it follows that $var(UT_2^G) = var(E(N)) \subseteq \mathcal{V}$, a contradiction.

Thus $B^{(i)}JB^{(k)} = 0$ for all $i \neq k$. Recall also that $B^{(i)}B^{(k)} = 0$ for all $i \neq k$. Clearly these relations imply that the same relations hold for the Grassmann envelope: $E(B^{(i)})E(J)E(B^{(j)}) = 0$, $E(B^{(i)})E(B^{(j)}) = 0$, for all $i \neq j$. Set $A_i = B^{(i)} + J$, for $1 \leq i \leq m$. Then $A = B^{(1)} \oplus \cdots \oplus B^{(m)} + J = (B^{(1)}+J)+\cdots+(B^{(m)}+J) = A_1+\cdots+A_m$ where $B^{(i)}+J = A_i$, $1 \leq i \leq m$. We claim that $Id^G(E(A_1)\oplus\cdots\oplus E(A_m)) = Id^G(E(A_1))\cap\cdots\cap Id^G(E(A_m))$.

In fact, if $f = f(x_1, \ldots, x_n) \in Id^G(E(A_1))\cap\cdots\cap Id^G(E(A_m))$ is multilinear, we shall prove that $f \equiv 0$ on $E(A_1)+\cdots+E(A_m)$. To this end it suffices to check evaluations such that $\varphi(x_{i,g}) = \bar{x}_{i,g} \in E(A_1) \cup \cdots \cup E(A_m)$. Now if $\bar{x}_{1,g_1}, \ldots, \bar{x}_{n,g_n} \in E(A_j)$ for some j, then $f(\bar{x}_{1,g_1}, \ldots, \bar{x}_{n,g_n}) = 0$. If, say, $\bar{x}_{1,g_1} \in E(A_i)$ and $\bar{x}_{2,g_2} \in E(A_j)$ with $i \neq j$ then $\bar{x}_{\sigma(1),g_{\sigma(1)}} \cdots \bar{x}_{\sigma(n),g_{\sigma(n)}} = 0$, for all $\sigma \in S_n$, by the previous relations. Thus $f \in Id^G(E(A_1) \oplus \cdots \oplus E(A_m))$. Since the other inclusion is obvious we get equality. It follows that $var^G(E(B)) = var^G(E(A_1) \oplus \cdots \oplus E(A_m))$. □

3.3 The main result

In this section we prove our main theorem. First we need to recall and prove some more results.

First we need the following result which was proved in [28].

If $d \geq 1, l, t$ are integers as in [28] we define a hook shaped diagram of arm d and leg l as

$$h(d,l,t) = (\underbrace{l+t, \ldots, l+t}_{d}, \underbrace{l, \ldots, l}_{t}).$$

Also we define

$$H(d,l) = \bigcup_{n\geq 1} \{\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n \quad | \lambda_{d+1} \leq l\}.$$

Finally for any integer $a \geq 1$ we define:

$$H(d,l) \cup (a^a) = \bigcup_{n\geq 1} \{\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n \quad | \lambda_{d+1} \leq l + a, \lambda_{d+a+1} \leq l\}.$$

We recall that if A is a $G \times \mathbb{Z}_2$-graded algebra and we consider the G-graded structure on A, then we write $A = \bigoplus_{j=1}^{s} A_{g_j}$ where $A_{g_j} = A_{(g_j,0)} \oplus A_{(g_j,1)}$.

Lemma 3.3.1 *Let $A = B + J$ be a finite dimensional $G \times \mathbb{Z}_2$-graded algebra, $\dim A = m$, with B a maximal $G \times \mathbb{Z}_2$-graded semisimple subalgebra. Let $\langle \lambda \rangle = (\lambda(1), \ldots \lambda(s)) \vdash n$ be a multipartition of n such that for some j, $1 \leq j \leq s$,*

$$\lambda(j) \geq h(d, p_j - d, (m+1)^2),$$

where d is an integer and $p_j = dimB_{g_j}$. Then $m_{\langle\lambda\rangle} = 0$ in the $(n_1, \ldots, n_s)$th cocharacter $\chi^G_{n_1,\ldots,n_s}(E(A))$ of $E(A)$.

The following result can be essentially found in [28]. Here we give the proof for completeness.

Lemma 3.3.2 *Let $A = B + J$ be a finite dimensional $G \times \mathbb{Z}_2$-graded algebra with B a maximal semisimple graded subalgebra. Let $p_j = \dim B_{g_j}$, $1 \leq j \leq s$, and $m = \dim A$. If $\chi^G_{n_1,\cdots,n_s}(E(A)) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$, then $m_{\langle\lambda\rangle} \neq 0$ implies that $\langle \lambda \rangle = (\lambda(1), \ldots, \lambda(s))$ where for $1 \leq j \leq s$, $\lambda(j) \subseteq H(d_j, p_j - d_j) \cup (u^u)$, for suitable integers $0 < d_j \leq p_j$, with $u = (m+1)^2 + m$.*

Proof. Let $\langle \lambda \rangle \vdash n$ and suppose that $m_{\langle\lambda\rangle} \neq 0$. Write $\lambda(j) = (\lambda(j)_1, \lambda(j)_2, \ldots)$, $1 \leq j \leq s$ and suppose that for some j there exists i such that $\lambda(j)_i > (m+1)^2 + m$. Let k be the integer such that $\lambda(j)_k > (m+1)^2 + m$ and $\lambda(j)_{k+1} \leq (m+1)^2 + m$. If $k > p_j$ then $\lambda(j) \geq h(p_j + 1, 0, (m+1)^2)$ and we reach a contradiction by the previous lemma. Thus $k \leq p_j$.

Set $u = (m+1)^2 + m$. If $\lambda(j)_{u+1} \geq p_j - k + 1$, then $\lambda(j) \geq \mu$ where $\mu = (\mu_1, \ldots \mu_{u+1}) = ((u+1)^k, (p_j - k + 1)^{u+1-k})$. Since $(m+1)^2 + m + 1 - (p_j - k + 1) \geq (m+1)^2$ and $(m+1)^2 + m + 1 - k \geq (m+1)^2$ we see that

$\mu \geq h(k, p_j - k + 1, (m+1)^2)$, hence $\lambda(j) \geq h(k, p_j - k + 1, (m+1)^2)$, again a contradiction by the previous lemma.

Thus $\lambda(j)_{u+1} \leq p_j - k$ and $\lambda(j) \subseteq H(k, p_j - k) \cup (u^u)$. Therefore we may assume that $\lambda(j)_1 \leq (m+1)^2 + m$. Clearly $\lambda(j)_{u+1} \leq p_j$ since otherwise $\lambda(j) \geq h(0, p_j + 1, (m+1)^2)$ contrary to the previous lemma. This says that $\lambda(j) \subseteq H(0, p_j) \cup (u^u)$ and we are done. $\square$

As an immediate consequence of the previous lemma we get.

Corollary 3.3.3 *Let $A = B + J$ be defined as in the previous lemma, $m = \dim A$, $p_j = \dim B_{g_j}$. If in $\chi_{n_1,\cdots,n_s}(E(A)) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$, $m_{\langle\lambda\rangle} \neq 0$, then for every $j \in \{1, \ldots, s\}$, $\lambda(j) \in H(r_j, r_j)$, where $r_j = (m+1)^2 + m + p_j$.*

Proof. We can remark that, for every $j = 1, \ldots, s$, $H(d, p_j - d) \cup (r^u) \subseteq H(d + u, l + r)$; therefore, since $d > p_j \quad \forall j = 1, \ldots, s$, $r > u$ and then $H(d + u, l + r) \subseteq H(d + r, l + r)$. Recall that d, l are integers such that $d + l > \dim B_{g_j} \leq 1$. So we can consider $d = l = 1$. Hence for every $j = 1, \ldots, s$, $\lambda(j) \subseteq H(r + 1, r + 1)$ with r depending on j. If we set $r_1 = (m+1)^2 + m + d$, then $r_1 > r$ for every $j = 1, \ldots, s$ and so if we consider $\overline{r} > r + 1$ then $\lambda(j) \subseteq H(\overline{r}, \overline{r})$, $\forall j \in \{1, \ldots, s\}$. $\square$

Lemma 3.3.4 *Let $A = B + J$ be a $G \times \mathbb{Z}_2$-graded algebra with J the Jacobson radical of A and $B \cong F^\alpha H$ for some $H \leq G \times \mathbb{Z}_2$ and $\alpha : H \times H \to F^*$ a 2-cocycle. Then there exists a constant M such that $\chi^G_{n_1,\ldots,n_s}(E(A)) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$ and $m_{\langle\lambda\rangle} \leq M$, for all $\langle\lambda\rangle \vdash n$ and for all $n \geq 1$.*

Proof. As we have seen in Corollary 3.3.3, the G-graded cocharacter of $E(A)$ lies in the union of s hooks $H(r_j, r_j)$, $1 \leq j \leq s$. Choose a basis of A of $G \times \mathbb{Z}_2$- homogeneous elements. Let $m_j = \dim A_{(g_j,0)}$, $\overline{m}_j = \dim A_{(g_j,1)}$, $1 \leq j \leq s$. We have $A_{(g_j,k)} = B_{(g_j,k)} + J_{(g_j,k)}$ where $k = 0$ or $k = 1$, and since $B \cong F^\alpha H$, $\dim B_{(g_j,k)} \leq 1$. So let

$$A_{(g_1,0)} = \mathrm{span}\{a_0^1, \ldots, a_{m_1-1}^1\} \quad ; \quad A_{(g_1,1)} = \mathrm{span}\{b_0^1, \ldots, b_{\overline{m}_1-1}^1\}$$

$$\vdots \qquad \vdots$$

$$A_{(g_s,0)} = \mathrm{span}\{a_0^s, \ldots, a_{m_s-1}^s\} \quad ; \quad A_{(g_s,1)} = \mathrm{span}\{b_0^s, \ldots, b_{\overline{m}_s-1}^s\},$$

where $a_0^j \in B_{(g_j,0)}$ if $B_{(g_j,0)} \neq 0$ and similarly $b_0^j \in B_{(g_j,1)}$ if $B_{(g_j,1)} \neq 0$, $1 \leq j \leq s$. All other elements a_i^j, b_l^j lie in J, $1 \leq i \leq m_j - 1$, $1 \leq l \leq \overline{m}_j - 1$.

Let q be the least positive integer such that $J^q = 0$ and set

$$N_0 = ((2q)^{\sum_{j=1}^s m_j + \overline{m}_j})^{2sr_j}.$$

We shall prove that every $m_{\langle\lambda\rangle}$ in $\chi^G_{n_1,\ldots,n_s}(E(A))$ is bounded by $M = (\sum_{j=1}^s m_j + \overline{m}_j)N_0$.

To this end let $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s))$ be such that $\lambda(j) \in H(r_j, r_j)$, $1 \leq j \leq s$, and consider $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, \ldots, T_{\lambda(s)})$, a corresponding multitableau. For every tableau $T_{\lambda(j)}$ let $R_{T_{\lambda(j)}}$ and $C_{T_{\lambda(j)}}$ be the row stabilizer and the column stabilizer of $T_{\lambda(j)}$, respectively. Let $R^+_{T_{\lambda(j)}} = \sum_{\sigma \in R_{T_{\lambda(j)}}} \sigma$, $C^-_{T_{\lambda(j)}} = \sum_{\tau \in C_{T_{\lambda(j)}}} (\mathrm{sgn}\tau)\tau$ and let $e_{T_{\lambda(j)}} = R^+_{T_{\lambda(j)}} C^-_{T_{\lambda(j)}}$ denote the corresponding essential idempotent of the group algebra FS_{n_j}. Then $e_{T_{\langle\lambda\rangle}} = e_{T_{\lambda(1)}} \cdots e_{T_{\lambda(s)}}$ is an essential idempotent of $F(S_{n_1} \times \ldots \times S_{n_s})$. For every $j = 1, \ldots, s$, consider the group $K_j = \{\sigma \in C_{T_{\lambda(j)}} \mid \sigma(i) = i, \quad \text{for all } i \text{ out of the first } r_j \text{ columns}\}$, and let $K_j^- = \sum_{\sigma \in K_j} (-1)^\sigma \sigma$. Then define $K^- = K_1^- \cdots K_s^-$ and notice that, since each $e_{T_{\lambda(j)}}$ is an essential idempotent, then $K^- e_{T_{\langle\lambda\rangle}} \neq 0$ and $e_{T_{\langle\lambda\rangle}}$ generate the same minimal left ideal of $F(S_{n_1} \times \ldots \times S_{n_s})$.

For every j, let Y_i^j be the set of variables of homogeneous degree g_j, whose indeces lie in the i-th column of $\lambda(j)$, let also X_i^j be the set of variables of homogeneous degree g_j whose indeces lie in the i-th row of $\lambda(j)$ but do not belong to the first r_j columns. Then, for every polynomial $f \in P_{n_1,\ldots,n_s}$, $K_j^- e_{T_{\lambda(j)}} f$ is alternating on each of the sets $Y_1^j, \ldots, Y_{r_j}^j$ and is symmetric on each of the sets $X_1^j, \ldots, X_{r_j}^j$. Thus, if we now consider the polynomial

$g = K^- e_{T_{\langle\lambda\rangle}} f$, the variables of g are partitioned into $2r_1 + \cdots + 2r_s$ disjoint subsets:

$$X_1^1, \ldots, X_{r_1}^1, Y_1^1, \ldots, Y_{r_1}^1, \ldots, X_1^s, \ldots, X_{r_s}^s, Y_1^s, \ldots, Y_{r_s}^s$$

and g is symmetric or alternating on each set as described above. Note that for every $j = 1, \ldots, s$ and $i = 1, \ldots, r_j$ if $\lambda(j) = (\lambda(j)_1, \lambda(j)_2, \ldots)$ then X_i^j is empty if $\lambda(j)_i \leq r_j$, i.e, if the length of the i-th row of $T_{\lambda(j)}$ is less than or equal to r_j. On the other hand if $\lambda(j)_i > r_j$ then $|X_i^j| = \lambda(j)_i - r_j$. Moreover $|Y_i^j| = \lambda(j)'_i$ where $\lambda(j)' = (\lambda(j)'_1, \lambda(j)'_2, \ldots)$ is the conjugate partition of $\lambda(j)$.

Notice that for any $\rho_j \in S_{n_j}$ we also have $\rho_j K_j^- e_{T_{\lambda(j)}} \neq 0$, and so if $\rho = \rho_1 \cdots \rho_s \in S_{n_1} \times \cdots \times S_{n_s}$, $\rho K^- e_{T_{\langle\lambda\rangle}} = \rho_1 K_1^- e_{T_{\lambda(1)}} \cdots \rho_s K_s^- e_{T_{\lambda(s)}} \neq 0$. It follows that if $f \in P_{n_1,\ldots,n_s}$ is such that $e_{T_{\langle\lambda\rangle}} f \neq 0$, then the polynomials $e_{T_{\langle\lambda\rangle}} f$ and $g' = \rho K^- e_{T_{\langle\lambda\rangle}} f$ generate the same irreducible $S_{n_1} \times \ldots \times S_{n_s}$-module. Now we choose ρ_j, $1 \leq j \leq s$, in such a way that $\rho_j K_j^- e_{T_{\lambda(j)}} f$ is symmetric separately on the first $\lambda(j)_1 - r_j$ variables, on the next $\lambda(j)_2 - r_j$ variables and so on. A similar condition holds for the alternating sets of variables Y_i^j, $1 \leq i \leq r_j$. The corresponding property of the polynomial g' is clear.

Let now $f_1, \ldots, f_M \in P_{n_1,\ldots,n_s}$ be multilinear polynomials such that $F(S_{n_1} \times \cdots \times S_{n_s}) f_i \cong F(S_{n_1} \times \cdots \times S_{n_s}) f_j$, for all $i, j = 1, \ldots, M$, i.e., $f_1, \ldots, f_M$ generate irreducible $S_{n_1} \times \ldots \times S_{n_s}$-modules corresponding to the same multipartition $\langle\lambda\rangle$. By what we remarked above, we can choose permutations $\rho_1, \ldots, \rho_M \in S_{n_1} \times \cdots \times S_{n_s}$ and a decomposition $X^1 \cup \ldots \cup X^s \cup Y^1 \cup \ldots \cup Y^s$, where for every $j = 1, \ldots, s$, $X^j = X_1^j \cup \ldots \cup X_{r_j}^j$, $Y^j = Y_1^j \cup \ldots \cup Y_{r_j}^j$ are sets of variables of homogeneous degree g_j and $\rho_1 f_1, \ldots, \rho_M f_M$ are simultaneously symmetric on X_i^j and alternating on Y_i^j, for all $j = 1, \ldots, s$, $i = 1, \ldots, r_j$.

Assume by contradiction that $m_{\langle\lambda\rangle} = M \geq \sum_{j=1}^{s} (m_j + \overline{m}_j) N_0$. We shall prove that $E(A)$ satisfies an identity of the type

$$f = \gamma_1 f_1 + \cdots + \gamma_M f_M \equiv 0, \qquad (1)$$

where $\gamma_1, \ldots, \gamma_M \in F$ are not all zero. Clearly it is sufficient to verify that f has only zero values on elements of the form $a_k^j \otimes e$ and $b_l^j \otimes e'$, where $a_k^j \in A_{(g_j,0)}, e \in E_0, b_l^j \in A_{(g_j,1)}, e' \in E_1$ and $k \in \{0, \ldots, m_j - 1\}$, $l \in \{0, \ldots \overline{m}_j - 1\}$.

First we define special substitutions as follows. Let

$$0 \leq \alpha_0^{ji}, \alpha_1^{ji}, \ldots, \alpha_{m_j-1}^{ji}, \beta_0^{ji}, \beta_1^{ji}, \ldots, \beta_{\overline{m}_j-1}^{ji}$$

be integers satisfying the following equalities:

$$\sum_{k=0}^{m_j-1} \alpha_k^{ji} + \sum_{k=0}^{\overline{m}_j-1} \beta_k^{ji} = |X_i^j|$$

$$\sum_{k=0}^{m_j-1} \alpha_k^{j(r_j+i)} + \sum_{k=0}^{\overline{m}_j-1} \beta_k^{j(r_j+i)} = |Y_i^j|$$

$$1 \leq j \leq s, 1 \leq i \leq r_j.$$

We say that a substitution φ has type $0 \leq \alpha_0^{ji}, \alpha_1^{ji}, \ldots, \alpha_{m_j-1}^{ji}, \beta_0^{ji}, \beta_1^{ji}, \ldots,$ $\beta_{\overline{m}_j-1}^{ji}$, $1 \leq j \leq s, 1 \leq i \leq r_j$, if we replace the variables in the following way: for fixed i and j, we replace the first α_0^{ji} variables from X_i^j by elements $a_0^j \otimes e$ (with distinct elements e for distinct $x \in X_i^j$), the next α_1^{ji} variables by elements $a_1^j \otimes e$ and so on, where all elements e lie in E_0. Now substitute the following β_0^{ji} variables from X_i^j by elements $b_0^j \otimes e'$, the next by $b_1^j \otimes e'$, and so on where all elements e' lie in E_1. We apply the same procedure in order to replace the variables in Y_i^j by elements of the type $a_k^j \otimes e$ and $b_k^j \otimes e'$.

In order to obtain a non zero value of the polynomials in (1), any substitution above should satisfy the following restrictions:

1. $\beta_k^{ji} \leq 1, \quad 1 \leq j \leq s,\ 1 \leq i \leq r_j$, where $0 \leq k \leq \overline{m}_j - 1$,

2. $\alpha_1^{ji} + \cdots + \alpha_{m_j-1}^{ji} \leq q-1, \quad 1 \leq j \leq s,\ 1 \leq i \leq r_j,$

3. $\alpha_0^{ji} = |X_i^j| - (\alpha_1^{ji} + \cdots + \alpha_{m_j-1}^{ji} + \beta_0^{ji} + \cdots + \beta_{\overline{m}_j-1}^{ji}).$

The first property follows since f_j is symmetric on X_i^j and, so, it becomes zero when we evaluate two variables of X_i^j in $b_u^j \otimes e'$, $b_u^j \otimes e''$, for some $e', e'' \in E_1$. The second property follows since $J^q = 0$.

Similarly we replace the variables from Y_i^j by elements of the form $a_u^j \otimes e$, $b_u^j \otimes e'$ as above and we obtain the following restrictions of the integers $\alpha_k^{j(r_j+i)}$, $\beta_k^{j(r_j+i)}$, for $1 \leq j \leq s$, $1 \leq i \leq r_j$ and $0 \leq k \leq m_j - 1$.

1. $\alpha_k^{ji} \leq 1, \quad 1 \leq j \leq s,\ r_j + 1 \leq i \leq 2r_j$ where $0 \leq k \leq m_j - 1,$

2. $\beta_1^{ji} + \cdots + \beta_{\overline{m}_j-1}^{ji} \leq q-1, \quad 1 \leq j \leq s,\ r_j + 1 \leq i \leq 2r_j$

3. $\beta_0^{ji} = |Y_i^j| - (\beta_1^{ji} + \cdots + \beta_{\overline{m}_j-1}^{ji} + \alpha_0^{ji} + \cdots + \alpha_{m_j-1}^{ji}).$

Now, from the restrictions $1, 2, 3$ above we get that for each $j = 1, \ldots, s$, $i = 1, \ldots, r_j$, the number of distinct $\overline{m}_j$ - tuples $(\beta_0^{ji}, \ldots, \beta_{\overline{m}_j-1}^{ji})$ is at most $2^{\overline{m}_j}$ and the number of distinct m_j-tuples $(\alpha_0^{ji}, \ldots, \alpha_{m_j-1}^{ji})$ is at most q^{m_j}. Thus the number of distinct $m_j + \overline{m}_j$-tuples $(\alpha_0^{ji}, \ldots, \beta_{\overline{m}_j-1}^{ji})$ is at most $2^{\overline{m}_j} q^{m_j} < (2q)^{m_j + \overline{m}_j}$. Similarly, from the other three conditions, we get that the number of distinct $m_j + \overline{m}_j$-tuples $(\alpha_0^{j(r_j+i)}, \ldots, \beta_{\overline{m}_j-1}^{j(r_j+i)})$ is bounded by $(2q)^{m_j+\overline{m}_j}$. It follows that the total number N of distinct types of substitutions is less than $((2q)^{\sum_{j=1}^{s} m_j + \overline{m}_j})^{\sum_{j=1}^{s} 2r_j} = N_0$.

Note that if φ, φ' are two substitutions of the same type and $\varphi(z) = u \otimes p$ for some $z \in X$, $u \in A$, $p \in E$, then $\varphi'(z) = u \otimes p'$ with the same grading of the elements p, p'. Hence if $X = \{z_1, \ldots, z_n\}$, $\varphi(z_i) = u_i \otimes p_i$ and $\varphi'(z_i) = u_i \otimes p_i'$, then

$$\varphi(f) = f(u_1 \otimes p_1, \ldots, u_n \otimes p_n) = w \otimes p_1 \cdots p_n$$

$$\varphi'(f) = f(u_1 \otimes p_1', \ldots, u_n \otimes p_n') = w \otimes p_1' \cdots p_n'.$$

In this case we say that φ and φ' are similar. Let N be the number of similarity classes. Now let $\varphi_1, \dots, \varphi_N$ be substitutions, chosen one from each similarity class of distinct types. If φ is one of these substitutions, and h_1, h_2 are two multilinear polynomials of degree n, then by multilinearity and supercommutativity $\varphi(h_1) = r_1 \otimes p_1 \cdots p_n$ and $\varphi(h_2) = r_2 \otimes p_1 \cdots p_n$, where $p_1, \dots, p_n \in E$ and $r_1,\, r_2 \in A$. Therefore for each $j = 1, \dots, N$ and $i = 1, \dots, M$ we get

$$\varphi_j(f_i) = a_{ij} \otimes p_{j1} \cdots p_{jn},$$

where $a_{ij} \in A$ and $p_{j1}, \dots, p_{jn}$ depend on φ_j only.

We consider the matrix (a_{ij}), $1 \le i \le M$, $1 \le j \le N$, whose elements a_{ij} lie in A. Since $M = (\sum_{k=1}^{s}(m_k + \overline{m}_k))N_0$, where $\dim A = \sum_{k=1}^{s}(m_k + \overline{m}_k)$, the rows of (a_{ij}) are linearly dependent. Hence there exist $\gamma_1, \dots, \gamma_M \in F$ not all zero, such that:

$$\sum_{i=1}^{M} \gamma_i a_{ij} = 0 \quad 1 \le j \le N.$$

From the above we get $\varphi_j(\sum_{i=1}^{M} \gamma_i f_i) = \sum_{i=1}^{M} \gamma_i \varphi_j(f_i) = (\sum_{i=1}^{M} \gamma_i a_{ij}) \otimes p_{j1} \cdots p_{jn} = 0$, for all $1 \le j \le N$.

We claim that this implies that $f = \sum_{i=1}^{M} \gamma_i f_i$ is an identity of $E(A)$. In fact by multilinearity it is enough to check only substitutions φ^* where the variables are evaluated into elements of the type $u \otimes p$, where $u = a_i^j$ or b_i^j, for some i, j and $p \in E_0 \cup E_1$. Now, there exists a permutation σ of the variables (preserving the homogeneous degree) such that $\varphi^\star \sigma = \varphi'$ is similar to some $\varphi_j, 1 \le j \le N$. Thus $\varphi'(f_i) = a_{ij} \otimes p'_{j1} \cdots p'_{jn}$ and, so, $\varphi'(f) = 0$. We remark that the above σ satisfies $\sigma(X_i^j) = X_i^j$ and $\sigma(Y_i^j) = Y_i^j$ $1 \le j \le s$, $1 \le i \le r_j$. Since f is symmetric on X_i^j and alternating on Y_i^j, it follows that $\varphi'(f) = \varphi^* \sigma(f) = \varphi(\pm f) = \pm \varphi^*(f) = 0$. Thus $\varphi^*(f) = 0$. We have shown that modulo the identities of $E(A)$, any M polynomials corresponding to the same multitableau are linearly dependent and this is equivalent to say that $m_{\langle \lambda \rangle} \le M$ for all $\langle \lambda \rangle \vdash n$.

$\square$

Lemma 3.3.5 *Let $A = B + J$, $B \cong F^\alpha H$ for some H subgroup of $G \times \mathbb{Z}_2$ and $\alpha : H \times H \to F^*$ a 2-cocycle. Then there exists a constant M such that $\chi^G_{n_1,\dots,n_s}(E(A)) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$ and*

$$n_j - \lambda(j)_1 - \lambda(j)'_1 \leq M \quad 1 \leq i \leq s.$$

Proof. Let $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s)) \vdash n$ be a multipartition of n and let q be such that $J^q = 0$. We claim that if $m_{\langle\lambda\rangle} \neq 0$, then $\lambda(j)_2 \leq q+1$, for all j, $1 \leq j \leq s$, that is the diagram of each $\lambda(j)$ contains at most $q+1$ boxes in the second row.

In fact, suppose by contradiction that there exists j, $1 \leq j \leq s$, such that $\lambda(j)_2 \geq q+2$ and $m_{\langle\lambda\rangle} \neq 0$. Then there exists a multitableau $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, \dots, T_{\lambda(s)})$, a corresponding essential idempotent $e_{T_{\langle\lambda\rangle}} = e_{T_{\lambda(1)}} \cdots e_{T_{\lambda(s)}}$ and a polynomial $f \in P_{n_1,\dots,n_s}$ such that $e_{T_{\langle\lambda\rangle}} f \notin Id^G(E(A))$. Recall that $e_{T_{\lambda(j)}}$ acts on n_j variables of homogeneous degree $g_j \in G$. Since $e_{T_{\lambda(j)}}$ is an essential idempotent, there exists $\tau \in R_{T_{\lambda(j)}}$ such that $\tau C^-_{T_{\lambda(j)}} e_{T_{\langle\lambda\rangle}} f \notin Id^G(E(A))$. Let $i_1, \dots, i_{q+2}$ denote the integers in the first $q+2$ boxes of the first row of the diagram of $\lambda(j)$ written from left to right. Similarly, let $k_1, \dots, k_{q+2}$ be the integers in the first $q+2$ boxes of the second row of $\lambda(j)$. Then the polynomial $g = \tau C^-_{T_{\lambda(j)}} e_{T_{\langle\lambda\rangle}} f$ is alternating on each of the following sets: $\{x_{\tau(i_1),g_j}, x_{\tau(k_1),g_j}\}, \dots, \{x_{\tau(i_{q+2}),g_j}, x_{\tau(k_{q+2}),g_j}\}$.

Notice that these variables are evaluated in

$$E(A)_{g_j} = ((E_0 \otimes B_{(g_j,0)}) \oplus (E_0 \otimes J_{(g_j,0)})) \oplus ((E_1 \otimes B_{(g_j,1)}) \oplus (E_1 \otimes J_{(g_j,1)}))$$

and, since $B \cong F^\alpha H$, the spaces $B_{(g_j,0)}$ and $B_{(g_j,1)}$ are at most 1-dimensional. Now, if at least q of the above variables are evaluated in $E_0 \otimes J_{(g_j,0)} \cup E_1 \otimes J_{(g_j,1)}$, then we get that g vanishes in $E(A)$ since $J^q = 0$. Therefore there exist three sets among $\{x_{\tau(i_1),g_j}, x_{\tau(k_1),g_j}\}, \dots, \{x_{\tau(i_{q+2}),g_j}, x_{\tau(k_{q+2}),g_j}\}$ that are evaluated in $(E_0 \otimes B_{(g_j,0)}) \cup (E_1 \otimes B_{(g_j,1)})$. If one of these sets, say

$\{x_{\tau(i_1),g_j}, x_{\tau(k_1),g_j}\}$, is evaluated in the commutative algebra $(E_0 \otimes B_{(g_j,0)})$, then we will get $g \equiv 0$ on $E(A)$, since g is alternating in $x_{\tau(i_1),g_j}$ and $x_{\tau(k_1),g_j}$. Then we deduce that there are at least two variables corresponding to indeces in the same first row or second row of $T_{\lambda(j)}$, say $x_{\tau(i_1),g_j}$ and $x_{\tau(i_2),g_j}$ that are evaluated in $E_1 \otimes B_{(g_j,1)}$.
Now the polynomial $e_{T_{\langle\lambda\rangle}} f$ is symmetric on the set $\{x_{i_1,g_j}, \dots, x_{i_{q+2},g_j}\}$; hence, since $\tau \in R_{T_{\lambda(j)}}$, it is also symmetric on $\{x_{\tau(i_1),g_j}, \dots, x_{\tau(i_{q+2}),g_j}\}$. Since the variables $x_{\tau(i_1),g_j}$ and $x_{\tau(i_2),g_j}$ are evaluated in $E_1 \otimes B_{(g_j,1)}$, which is anticommutative, we get that $e_{T_{\langle\lambda\rangle}} f \equiv 0$ on $E(A)$ and the claim is proved.

Next we claim that if $m_{\langle\lambda\rangle} \neq 0$ then $\lambda(j)'_2 \leq 2q$, for all j, $1 \leq j \leq s$. This is the same as to say that the diagram of each $\lambda(j)$ contains at most $2q$ boxes in the second column.

In fact, suppose to the contrary that there exists j, $1 \leq j \leq s$, such that $\lambda(j)'_2 \geq 2q+1$ and $m_{\langle\lambda\rangle} \neq 0$. As above, this says that there exists a multitableau $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, \dots, T_{\lambda(s)})$, an essential idempotent $e_{T_{\langle\lambda\rangle}} = e_{T_{\lambda(1)}} \cdots e_{T_{\lambda(s)}}$ and a polynomial $f \in P_{n_1,\dots,n_s}$ such that $e_{T_{\langle\lambda\rangle}} f \notin Id^G(E(A))$.

Let $\tau \in R_{T_{\lambda(j)}}$ be such that $g = \tau C_{T_{\lambda(j)}} e_{T_{\langle\lambda\rangle}} f \notin Id^G(E(A))$. Let $i_1, \dots, i_{2q+1}$ be the first integers in the first column of $T_{\lambda(j)}$ written from top to bottom and $k_1, \dots, k_{2q+1}$ the corresponding integers of the second column. Then g is alternating on $\{x_{\tau(i_1),g_j}, \dots, x_{\tau(i_{2q+1}),g_j}\}$ and on $\{x_{\tau(k_1),g_j}, \dots, x_{\tau(k_{2q+1}),g_j}\}$. In order to get a non zero value of g, since $E_0 \otimes B_{(g_j,0)}$ is commutative, we can evaluate at most one variable of each set in $E_0 \otimes B_{(g_j,0)}$. Moreover since $J^q = 0$, we have to evaluate at most $q-1$ variables of each set into $E_1 \otimes B_{(g_j,1)}$. It follows that two variables corresponding to indeces in the same row, say $x_{\tau(i_1),g_j}$ and $x_{\tau(k_1),g_j}$, are evaluated into $E_1 \otimes B_{(g_j,1)}$. Since g is symmetric on these two variables and $E_1 \otimes B_{(g_j,1)}$ is anticommutative, we get $g \equiv 0$, a contradiction. This proves the second claim.

As a result of the above two claims we get that if $m_{\langle\lambda\rangle} \neq 0$, then $\lambda(j)_2 \leq q+1$ and $\lambda(j)'_2 \leq 2q$. This implies that $n_j - \lambda(j)_1 - \lambda(j)'_1 \leq q(2q-1) = M$.

□

Lemma 3.3.6 *Let A, B be two G-graded algebras and let $A \oplus B$ be their direct sum. Consider $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s)) \vdash n$ multipartition of n, such that $\lambda(1) \vdash n_1, \ldots, \lambda(s) \vdash n_s$ and $n_1 + \cdots + n_s = n$. Let $\chi^G_{n_1,\ldots,n_s}(A) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$, $\chi^G_{n_1,\ldots,n_s}(B) = \sum_{\langle\lambda\rangle\vdash n} m'_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$ and $\chi^G_{n_1,\ldots,n_s}(A \oplus B) = \sum_{\langle\lambda\rangle\vdash n} \overline{m}_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$ be the $(n_1, \ldots, n_s)$th cocharacters of A, B, $A \oplus B$, respectively. Suppose that for every $\langle\lambda\rangle$ multipartition of n, we have that $m_{\langle\lambda\rangle} \leq C$ and $m'_{\langle\lambda\rangle} \leq C'$, for some constants C, C'; Then, for all $\langle\lambda\rangle \vdash n$,*

$$\overline{m}_{\langle\lambda\rangle} \leq \overline{C} = C + C'.$$

Proof. By lemma 3.1.5, for all $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s)) \vdash n$ multipartition of n we have that $\overline{m}_{\langle\lambda\rangle} \leq m_{\langle\lambda\rangle} + m'_{\langle\lambda\rangle}$. Therefore $\overline{m}_{\langle\lambda\rangle} \leq \overline{C} = C + C'$ for all $\langle\lambda\rangle \vdash n$. □

Lemma 3.3.7 *Let $\mathcal{V}$ be a G-graded variety of algebras such that $UT_2^G \notin \mathcal{V}$ for any canonical G-grading on UT_2. Then there exists a constant M such that $\chi^G_{n_1,\ldots,n_s}(\mathcal{V}) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \chi_{\lambda(s)}$ and $m_{\langle\lambda\rangle} \leq M$ for every $n \geq 1$.*

Proof. In Lemma 3.2.2 we proved that $\mathcal{V} = var(E(A_1) \oplus \cdots \oplus E(A_n))$ where for every $i \in \{1, \ldots, n\}$, $\dim A_i < \infty$ and A_i is a $G \times \mathbb{Z}_2$- graded algebra with Jacobson radical J_i. Therefore for every $i \in \{1, \ldots, n\}$, $A_i = B_i + J_i$, where B_i is a G-graded simple algebra isomorphic to $F^{\alpha_i}H_i$ for some $H_i \leq G \times \mathbb{Z}_2$ and $\alpha_i : H_i \times H_i \to F^*$ 2-cocycle. Then using lemma 3.3.4, this lemma is proved by induction on n.

□

We are now ready to prove our main theorem:

Theorem 3.3.8 *Let A be a G-graded PI-algebra, and*

$$\chi^G_{n_1,\ldots,n_s}(A) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$$

its $(n_1, \dots, n_s)$th cocharacter. Then the following conditions are equivalent.

1. *There exists a constant M such that for all n and $\langle\lambda\rangle \vdash n$, the inequality*
$$m_{\langle\lambda\rangle} \leq M$$
holds.

2. *$Id^G(A) \nsubseteq Id^G(UT_2^G)$ for any G-grading on UT_2.*

3. *There exists a constant N such that for all n and $\langle\lambda\rangle \vdash n$, the inequalities*
$$n_i - \lambda(i)_1 - \lambda(i)'_1 \leq N$$
hold, for all $1 \leq i \leq s$.

Proof. 1) $\Rightarrow$ 2) Let $\mathcal{V} = \mathrm{var}^G(A)$. Now if $UT_2^G \in \mathcal{V}$ for some G-grading on UT_2, then by Theorem 3.1.3 the multiplicities in $\chi^G_{n_1,\dots,n_s}(UT_2^G)$, and so, in $\chi^G_{n_1,\dots,n_s}(\mathcal{V})$ are not bounded by a constant. This proves 2).

2) $\Rightarrow$ 1) Suppose that $UT_2^G \notin \mathcal{V}$, for any G-grading on UT_2. Now by Lemma 3.2.2 we can write $\mathcal{V} = \mathrm{var}^G(E(A_1) \oplus \cdots \oplus E(A_n))$ where for every $i \in \{1, \dots n\}$, $A_i = B_i + J_i$, with B_i a $G \times \mathbb{Z}_2$-graded simple algebra isomorphic to $F^{\alpha_i} H_i$ for some $H_i \leq G \times \mathbb{Z}_2$ and $\alpha_i : H_i \times H_i \to F^*$ a 2-cocycle.

Now let $\chi^G_{n_1,\dots,n_s}(E(A_i)) = \sum_{\langle\lambda\rangle\vdash n} m^{(i)}_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$, $1 \leq i \leq s$. Then

$$\sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)} = \chi^G_{n_1,\dots,n_s}(\mathcal{V}) = \chi^G_{n_1,\dots,n_s}(E(A_1) \oplus \cdots \oplus E(A_s))$$

$$\leq \sum_{\langle\lambda\rangle\vdash n} (\sum_{i=1}^{s} m^{(i)}_{\langle\lambda\rangle}) \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}.$$

Since by Lemma 3.3.4, $m^{(i)}_{\langle\lambda\rangle} \leq M_i$, for some constant M_i, we get that $m_{\langle\lambda\rangle} \leq \sum_{i=1}^{s} M_i$ is bounded by a constant, for all $\langle\lambda\rangle \vdash n$. This proves 1).

2) $\Rightarrow$ 3) This implication was proved in Lemma 3.3.5.

3) $\Rightarrow$ 2) Suppose by contradiction that $UT_2^G \in \mathcal{V}$ for some G-grading on UT_2. If

$$\chi^G_{n_1,\ldots,n_s}(UT_2^G) = \sum_{\langle\lambda\rangle\vdash n} m'_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(s)}$$

and

$$\chi^G_{n_1,\ldots,n_s}(\mathcal{V}) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(s)},$$

then since $UT_2^G \in \mathcal{V}$, we get that $m'_{\langle\lambda\rangle} \leq m_{\langle\lambda\rangle}$, for all $\langle\lambda\rangle \vdash n$. So for every $\langle\lambda\rangle \vdash n$ such that $m'_{\langle\lambda\rangle} \neq 0$, we have that $n_i - \lambda(i)_1 - \lambda(i)'_1 \leq N$ for some constant N and for all i, $1 \leq i \leq s$.

Now take $n = 2N+5$ and $\langle\lambda\rangle = ((N+2,N+2),(1),\emptyset,\ldots,\emptyset) \vdash n$; hence $\lambda(1) = (N+2,N+2) \vdash 2N+4$, $\lambda(2) = (1)$ and $\lambda(i) = \emptyset$ for all $i \geq 3$. Then, according to Theorem 3.1.3, $m'_{\langle\lambda\rangle} = 1 \neq 0$, but $2N+4-(N+1)-2 = N+1 > N$. Thus $m_{\langle\lambda\rangle} \geq m'_{\langle\lambda\rangle} > N$, a contradiction. $\square$

Chapter 4

Gradings on $UT_3(F)$

4.1 Introduction

Let F be a field of characteristic zero. The algebra

$$UT_3(F) = \{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{ij} \in F\}$$

is the algebra of upper triangular matrices of order 3.

We recall first this definition:

Definition 4.1.1 *A G-grading on $UT_n(F)$ is called elementary if there exists $\overline{g} = (g_1, \ldots, g_n) \in G^n$ an n-tuple of elements of G such that $A_g = span\{e_{ij} \mid g_i^{-1}g_j = g\} \quad \forall g \in G$, i.e., the homogeneous degree of e_{ij} is equal to $g_i^{-1}g_j$ for every i and j such that $1 \leq i \leq j \leq n$.*

Di Vincenzo, Koshlukov and Valenti in [20] proved that if G is a finite group, then there are $|G|^{n-1}$ nonisomorphic elementary G-gradings on $UT_n(F)$, so in case $n = 3$ there are $|G|^2$ nonisomorphic elementary G-gradings on $UT_3(F)$.

Valenti and Zaicev in [53] proved that if G is a finite abelian group and F is algebraically closed of characteristic zero, then all gradings on $UT_n(F)$

are, up to isomorphism, elementary.

4.2 Gradings on $UT_3(F)$

In this section we want to classify all elementary, non isomorphic, G-gradings on $UT_3(F)$ in terms of triples that induce G-grading.

Theorem 4.2.1 *Let G be a finite abelian group. The different elementary G-gradings on $UT_3(F)$, non isomorphic and non trivial, are induced by the following triples of elements of G:*

1. $(e, g, h) \qquad g \neq h,\ h \neq g^2,\ g, h \neq e$
2. $(e, g, g^2) \qquad g \neq e$
3. $(e, g, g) \qquad g \neq e$
4. $(e, e, g) \qquad g \neq e$
5. $(e, g, e) \qquad g \neq e.$

Proof. Let $A = UT_3^G(F)$, then the homogeneous components of A are $A_g = \operatorname{span}\{e_{ij} \mid g_i^{-1}g_j = g\}$ and $A = \bigoplus_{g\in G} A_g$. If $\dim A_e = 6$ then the corresponding G-grading is trivial. Hence $\dim A_e < 6$. We observe that $e_{ii} \in A_e \quad \forall i = 1, 2, 3$, then $\dim A_e \geq 3$. Now we can consider all different triples of elements of G with at least one element equal to e. We can suppose that the first element of the triple is always e, in fact if it is equal to some $g \neq e$, then we can multiply all elements of that triple by g^{-1}, obtaining the same elementary G-grading.

Hence we can suppose that the triples that induce elementary G-grading are (e, g, h) with $g, h \in G$ not both equal to e. If in this triple $g \neq h$ and $g, h \neq e$ then $\dim A_e = 3$, otherwise, if $g = h \neq e$ or $g = e$, $h \neq e$ or $h = e$, $g \neq e$, $\dim A_e = 4$. We are now ready to study the corresponding cases:

1. $\dim A_e = 3$, triple (e, g, h) $\quad g \neq h \quad g, h \neq e$.

   If $h \neq g^2$ then $A_e = \text{span}\{e_{11}, e_{22}, e_{33}\}$, $A_g = \text{span}\{e_{ij} \mid g_i^{-1}g_j = g\} = \text{span}\{e_{12}\}$, $A_h = \text{span}\{e_{ij} \mid g_i^{-1}g_j = h\} = \text{span}\{e_{13}\}$, $A_{g^{-1}h} = \text{span}\{e_{ij} \mid g_i^{-1}g_j = g^{-1}h\} = \text{span}\{e_{23}\}$.

   If $h = g^2$ then $A_e = \text{span}\{e_{11}, e_{22}, e_{33}\}$, $A_g = \text{span}\{e_{ij} \mid g_i^{-1}g_j = g\} = \text{span}\{e_{12}, e_{23}\}$, $A_{g^2} = \text{span}\{e_{ij} \mid g_i^{-1}g_j = h\} = \text{span}\{e_{13}\}$.

2. $\dim A_e = 4$. If the triple is (e, g, g) $\quad g \neq e$, then $A_e = \text{span}\{e_{11}, e_{22}, e_{33}, e_{23}\}$, $A_g = \text{span}\{e_{ij} \mid g_i^{-1}g_j = g\} = \text{span}\{e_{12}, e_{13}\}$.

3. $\dim A_e = 4$. If the triple is (e, e, g) $\quad g \neq e$, then $A_e = \text{span}\{e_{11}, e_{22}, e_{33}, e_{12}\}$, $A_g = \text{span}\{e_{ij} \mid g_i^{-1}g_j = g\} = \text{span}\{e_{23}, e_{13}\}$.

4. $\dim A_e = 4$. If the triple is (e, g, e) $\quad g \neq e$, then $A_e = \text{span}\{e_{11}, e_{22}, e_{33}, e_{13}\}$, $A_g = \text{span}\{e_{ij} \mid g_i^{-1}g_j = g\} = \text{span}\{e_{12}\}$, $A_{g^{-1}} = \text{span}\{e_{23}\}$.

□

4.3 Graded identities of $UT_3(F)$

Now we want to find the T-ideal of G-graded polynomial identities of $UT_3(F)$ for each elementary G-grading described in the previous section.

We will use the techniques described in [20]. We recall these techniques in the following results.

Definition 4.3.1 *Let $\overline{\eta} = (\eta_1, \ldots, \eta_m)$ be an element of G^m. We say that $\overline{\eta}$ is a good sequence with respect to the elementary G-grading ϵ if there exists a sequence of m matrix units $(r_1, \ldots, r_m)$ in the Jacobson radical of $UT_n(F)$ such that the product $r_1 \cdots r_m$ is not zero and also the homogeneous degree of r_i is η_i for all $i = 1, \ldots, m$. In this case we say that $\overline{\eta}$ is ϵ-good, otherwise $\overline{\eta}$ is called ϵ-bad sequence.*

For any sequence $\overline{\eta} \in G^m$ we consider the polynomial $f_{\overline{\eta}} = f_{\overline{\eta},1} f_{\overline{\eta},2} \cdots f_{\overline{\eta},m}$ where $f_{\overline{\eta},i} = [x_{e,2i-1}, x_{e,2i}]$ if $\eta_i = e$ while $f_{\overline{\eta},i} = x_{\eta_i}$ if $\eta_i \neq e$.

Theorem 4.3.2 *Let G be a group and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ be an elementary G-grading on $UT_n(F)$ with F infinite field. Then the T-ideal $Id^G(UT_n(F), \epsilon)$ of G-graded polynomial identities of $UT_n(F)$ is generated by all multilinear polynomials $f_{\overline{\eta}}$ where $\overline{\eta} = (\eta_1, \ldots, \eta_m)$ lies on the set of all ϵ-bad sequences and $m \leq n$.*

Now we are going to apply this theorem to $UT_3(F)$.

Theorem 4.3.3 *Let G be a group and $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ be an elementary G-grading on $UT_3(F)$ with F infinite field. If we denote by x_i the variables of homogeneous degree e, and by y_j the variables of homogeneous degree g, then we have*

1. *if $\epsilon = (e, g, h)$, $g \neq h$, $h \neq g^2$,$g, h \neq e$ then*

$$Id^G(UT_3(F)) = \langle [x_1, x_2], y_1 y_2, z_1 z_2, t_1 t_2, yt, ty, zy, zt, tz \rangle$$

where z_i are variables of homogeneous degree $g^{-1}h$ and t_j are variables of homogeneous degree h.

2. *if $\epsilon = (e, g, g^2)$, $g^2 \neq e$ then*

$$Id^G(UT_3(F)) = \langle [x_1, x_2], z_1 z_2, yz, zy, y_1 y_2 y_3 \rangle$$

where z_i are variables of homogeneous degree g^2.

3. *if $\epsilon = (e, e, g)$, $g \neq e$ then*

$$Id^G(UT_3(F)) = \langle [x_1, x_2][x_3, x_4], y_1 y_2, y[x_1, x_2] \rangle.$$

4. *if $\epsilon = (e, g, g)$, $g \neq e$ then*

$$Id^G(UT_3(F)) = \langle [x_1, x_2][x_3, x_4], y_1 y_2, [x_1, x_2] y \rangle.$$

5. *if* $\epsilon = (e, g, e)$, $g \neq e$ *then* $Id^G(UT_3(F)) =$

$$\langle [x_1, x_2][x_3, x_4], y_1y_2, z_1z_2, zy, [x_1, x_2]y, y[x_1, x_2], [x_1, x_2]z, z[x_1, x_2] \rangle$$

where z_i *are variables of homogeneous degree* g^{-1}.

Proof. The Jacobson radical of $UT_3(F)$ is $J = span\{e_{12}, e_{13}, e_{23}\}$, and any product of three elements of J is equal to zero; then all ϵ-good sequences have 2 elements at most. Let us consider all distinct elementary G-gradings:

1. $\epsilon = (e, g, h)$, $g \neq h$, $g, h \neq e$, $h \neq g^2$. $\overline{\eta} = (\eta_1, \eta_2)$ is ϵ-good if there exist $r_1, r_2 \in J$ such that $r_1r_2 \neq 0$ and deg$r_1 = \eta_1$, deg$r_2 = \eta_2$. So $r_1 = e_{12}$ and $r_2 = e_{23}$.

   Since deg$e_{12} = g$ and deg$e_{23} = g^{-1}h$, $\overline{\eta} = (\eta_1, \eta_2)$ is ϵ-good if and only if $\overline{\eta} = (g, g^{-1}h)$.

   $\overline{\eta} = (\eta_1)$ is ϵ-good if there exists $r_1 \in J$ such that deg$r_1 = \eta_1$. So $\overline{\eta} = (\eta_1)$ is ϵ-good if and only if $\overline{\eta} = (g)$, $\overline{\eta} = (h)$, $\overline{\eta} = (g^{-1}h)$.

   All other sequences are ϵ-bad. If we set $x_e = x$, $x_g = y$, $x_h = t$, $x_{g^{-1}h} = z$, then

   $$Id^G(UT_3(F)) = \langle [x_1, x_2], y_1y_2, z_1z_2, t_1t_2, yt, zy, ty, tz, zt \rangle.$$

   We remark that all other identities obtained from ϵ-bad sequences are consequences of these ones.

2. $\epsilon = (e, g, g^2)$, $g^2 \neq e$. $\overline{\eta} = (\eta_1, \eta_2)$ is ϵ-good if there exist $r_1, r_2 \in J$ such that $r_1r_2 \neq 0$ and deg$r_1 = \eta_1$, deg$r_2 = \eta_2$. So $r_1 = e_{12}$ and $r_2 = e_{23}$.

   Since deg$e_{12} =$ deg$e_{23} = g$, $\overline{\eta} = (\eta_1, \eta_2)$ is ϵ-good if and only if $\overline{\eta} = (g, g)$.

   $\overline{\eta} = (\eta_1)$ is ϵ-good if there exists $r_1 \in J$ such that deg$r_1 = \eta_1$. So $\overline{\eta} = (\eta_1)$ is ϵ-good if and only if $\overline{\eta} = (g)$, $\overline{\eta} = (g^2)$.

All other sequences are ϵ-bad. If we set $x_e = x$, $x_g = y$, $x_{g^2} = z$, then

$$Id^G(UT_3(F)) = \langle [x_1, x_2], z_1 z_2, yz, zy, y_1 y_2 y_3 \rangle.$$

We remark that all other identities obtained from ϵ-bad sequences are consequences of these ones.

3. $\epsilon = (e, e, g)$, $g \neq e$. By repeating the same arguments we obtain that

$$Id^G(UT_3(F)) = \langle [x_1, x_2][x_3, x_4], y_1 y_2, y[x_1, x_2] \rangle.$$

4. $\epsilon = (e, g, g)$, $g \neq e$. By repeating the same arguments we obtain that

$$Id^G(UT_3(F)) = \langle [x_1, x_2][x_3, x_4], y_1 y_2, [x_1, x_2]y \rangle.$$

5. $\epsilon = (e, g, e)$, $g \neq e$. By repeating the same arguments we obtain that

$$Id^G(UT_3(F)) =$$
$$\langle [x_1, x_2][x_3, x_4], y_1 y_2, z_1 z_2, zy, [x_1, x_2]y, y[x_1, x_2], [x_1, x_2]z, z[x_1, x_2] \rangle.$$

□

Now we are ready to calculate the multiplicities in $\chi_n^G(UT_3(F))$, for some elementary G-grading on $UT_3(F)$.

4.4 Cocharacter sequence of $UT_3(F)$ with elementary G-grading induced by (e, g, h), $g \neq h$, $h \neq g^2$, $g, h \neq e$.

We recall that for every $\lambda \vdash n$, T_λ is a Young tableau of shape λ and e_{T_λ} is the corresponding essential idempotent of the group algebra FS_n. We recall also that $e_{T_\lambda} = \sum_{\substack{\sigma \in R_{T_\lambda} \\ \tau \in C_{T_\lambda}}} \mathrm{sgn}\tau \sigma \tau$, where R_{T_λ} and C_{T_λ} are the subgroups of row and column permutations of T_λ, respectively.

Because we want to study $UT_3(F)$ with elementary G-grading induced by the triple (e, g, h), $g \neq h \neq g^2$, let $\lambda(1) \vdash n_1, \ldots, \lambda(4) \vdash n_4$, $n_1 + \cdots + n_4 = n$ and let $W_{\lambda(1),\ldots,\lambda(4)}$ be an $S_{n_1} \times \cdots \times S_{n_4}$-irreducible left module.

It is well known that if $T_{\lambda(1)}$ is a Young tableau of shape $\lambda(1)$, ..., $T_{\lambda(4)}$ a Young tableau of shape $\lambda(4)$, then $W_{\lambda(1),\ldots,\lambda(4)} \cong F(S_{n_1} \times \cdots \times S_{n_4})e_{T_{\lambda(1)}} \cdots e_{T_{\lambda(4)}}$, where $S_{n_1}, \ldots, S_{n_4}$ act on distinct sets of integers.

Hence we can write the explicit decomposition of the nth G-graded cocharacter of $UT_3(F)$ into irreducibles. Recall that if $\lambda \vdash n$, $h(\lambda)$ denote the height of the Young diagram associated to λ.

Theorem 4.4.1 *Let $\chi_n^G(UT_3(F)) = \sum_{n_1+\ldots+n_4=n} \sum_{\lambda(i)\vdash n_i} m_{\lambda(1),\ldots,\lambda(4)}\chi_{\lambda(1)} \otimes \ldots \otimes \chi_{\lambda(4)}$ be the nth graded cocharacter of $UT_3(F)$ with elementary G-grading induced by the triple (e, g, h), $g \neq h$, $h \neq g^2$. If we set for simplicity $m_{\lambda(1),\lambda(2),\lambda(3),\lambda(4)} = m$, then:*

1. *$m = \frac{(q+1)(r+1)(q+r+2)}{2}$ if $\lambda(1) = (p+q+r, p+q, p)$ $p, q, r \geq 0$, $\lambda(2) = \lambda(3) = (1)$, $\lambda(4) = \emptyset$.*

2. *$m = (r+1)$ if $\lambda(1) = (q+r, q)$ $q, r \geq 0$ and $\lambda(2) = (1)$, $\lambda(3) = \lambda(4) = \emptyset$, or $\lambda(3) = (1)$, $\lambda(2) = \lambda(4) = \emptyset$, or $\lambda(4) = (1)$, $\lambda(2) = \lambda(3) = \emptyset$.*

3. *$m = 1$ if $\lambda(1) = (n)$ and $\lambda(2) = \lambda(3) = \lambda(4) = \emptyset$.*

4. *$m = 0$ in all other cases.*

Proof. We recall that for our elementary G-grading induced by the triple $\epsilon = (e, g, h)$, $g \neq h \neq g^2$, $\dim A_e = 3$, $\dim A_g = \dim A_h = \dim A_{g^{-1}h} = 1$. If we set $x_e = x$, $x_g = y$, $x_h = t$, $x_{g^{-1}h} = z$, then any polynomial alternating on four variables x or in two variables y, z, or t vanishes in $A = UT_3^G(F)$.

From the general form of the element $e_{T_{\lambda(1)}}e_{T_{\lambda(2)}}e_{T_{\lambda(3)}}e_{T_{\lambda(4)}}$, it follows that $m = 0$ if either $h(\lambda(1)) > 3$ or $h(\lambda(i)) > 1$ $\forall i = 2, 3, 4$. Moreover, by the previous construction, for every variable x_1, x_2,

$$y_1x_1y_2, z_1x_1z_2, t_1x_1t_2, yx_1zx_2t, yx_1t, zx_1t, zx_1y \in Id^G(A)$$

and this implies that $m = 0$ whenever $|\lambda(i)| \geq 2$, $i = 2, 3, 4$ and also whenever $\lambda(2) = \emptyset$, $\lambda(3), \lambda(4) \neq \emptyset$; $\lambda(3) = \emptyset$, $\lambda(2), \lambda(4) \neq \emptyset$ and $\lambda(i) \neq \emptyset$ $i = 2, 3, 4$.

So let us assume that $|\lambda(i)| \leq 1 \quad i = 2, 3, 4$. Suppose first that $\lambda(i) = \emptyset$ $i = 2, 3, 4$, then $[x_1, x_2] \equiv 0$ on A implies that $x_1 \cdots x_n$ is a basis of $P_{n,0,0,0}$ $(\mathrm{mod} Id^G(A))$. Hence $m = 1$ if $\lambda(1) = (n)$ and $m = 0$ if $\lambda(1) \neq (n)$.

Suppose now that $h(\lambda(1)) \leq 3$ and $\lambda(2) = \lambda(3) = (1)$, $\lambda(4) = \emptyset$. Let $\lambda(1) = (p + q + r, p + q, p)$, $p, q, r \geq 0$, we want to prove that $m = \frac{(q+1)(r+1)(r+2)+q(q+1)(q+2)}{2} = \frac{(q+1)(r+1)(q+r+2)}{2}$. For simplicity we will study first the case $p = 0$.

Let $r_1, r_2, r_3 \geq 0$ such that $r = r_1 + r_2 + r_3$ and $q_1, q_2, q_3 \geq 0$ such that $q = q_1 + q_2 + q_3$. It is obvious that for every choice of r_1, r_2, q_1, q_2 we will obtain a different tableau and hence a different polynomial associated to the tableau.

We remark that $r_1 = 0, \ldots, r$ and so if $r_1 = 0$ then $r_2 = 0, \ldots, r$; if $r_1 = 1$ then $r_2 = 0, \ldots r - 1$; $\ldots$; if $r_1 = r$ then $r_2 = 0$. So the total number of possible choices for r_1 and r_2 is $\sum_{i=1}^{r+1} i = \frac{(r+1)(r+2)}{2}$. By repeating the same argument for q_1 and q_2, we obtain that the total number of possible choices for q_1 and q_2 is $\sum_{j=1}^{q+1} j = \frac{(q+1)(q+2)}{2}$. So we will obtain $\frac{(q+1)(q+2)(r+1)(r+2)}{4}$ different tableaux.

Now we shall consider a few of these tableaux, in particular we shall consider all tableaux in which $q_3 = 0$ and all tableaux in which $r_1 = 0$ and $q_3 \neq 0$. For every r, if $q_3 = 0$, then the total number of such tableaux is $\frac{(q+1)(r+1)(r+2)}{2}$; if $r_1 = 0$ and $q_3 \neq 0$, similar arguments show that the number of tableaux is $\frac{(r+1)(q)(q+1)}{2}$. So we will consider $\frac{(q+1)(r+1)(r+2)}{2} + \frac{(r+1)(q)(q+1)}{2} \leq \frac{(q+1)(q+2(r+1)(r+2)}{4}$ different tableaux.

For every $i = 0, \ldots, r_1$, $j = 0, \ldots, r_2$, $\bar{i} = 0, \ldots, q_1$ define the following tableaux:

$$T^{i,j,\bar{i}}_{\lambda(1)} =$$

$i+1$	...	$i+\bar{i}$	$i+\bar{i}$ $+1$	...	$i+$ q	1	...	i	$i+q$ $+2$	...	$i+q$ $+j+1$	$i+q+$ $\bar{i}+j+3$	...	$q+\bar{i}$ $+r+2$
$i+q$ $+j+2$	...	$i+q+$ $\bar{i}+j+1$	$q+r$ $+\bar{i}+3$	...	n									

$$T^{i,j,\bar{i}}_{\lambda(2)} = \boxed{i+q+1}$$

$$T^{i,j,\bar{i}}_{\lambda(3)} = \boxed{i+q+\bar{i}+j+2}$$

Now for every $j = 0, \ldots, r_2$, $\bar{i} = 0, \ldots, q_1$, $\bar{j} = 0, \ldots, q_2$, $q_1 + q_2 < q$, define the following tableaux

$$T^{j,\bar{i},\bar{j}}_{\lambda(1)} =$$

1	...	$\bar{i}$	$\bar{i}+1$	...	$\bar{i}+\bar{j}$	$\bar{i}+\bar{j}$ $+j+2$	...	$q+$ $j+1$	$\bar{i}+$ $\bar{j}+2$	...	$\bar{i}+\bar{j}$ $+j+1$	$q+\bar{i}$ $+j+3$	...	$q+\bar{i}$ $+r+1$
$q+$ $j+2$	...	$q+\bar{i}$ $+j+1$	$q+r$ $+\bar{i}+2$	...	$q+\bar{i}+$ $\bar{j}+r+1$	$q+\bar{i}+\bar{j}$ $+r+2$	...	n						

$$T^{j,\bar{i},\bar{j}}_{\lambda(2)} = \boxed{\bar{i}+\bar{j}+1}$$

$$T^{j,\bar{i},\bar{j}}_{\lambda(3)} = \boxed{q+\bar{i}+j+2}$$

If $q_3 = 0$ we associate to $T^{i,j,\bar{i}}_{\lambda(1)}$, $T^{i,j,\bar{i}}_{\lambda(2)}$, $T^{i,j,\bar{i}}_{\lambda(3)}$ the polynomials: $a^{i,j,\bar{i},q-\bar{i}}$

$$(x_1, x_2, y, z) = x_1^i \underbrace{\acute{x}_1 \cdots \acute{x}_1}_{\bar{i}} \underbrace{\dot{x}_1 \cdots \dot{x}_1}_{q-\bar{i}} y x_1^j \underbrace{\acute{x}_2 \cdots \acute{x}_2}_{\bar{i}} z x_1^{r-i-j} \underbrace{\dot{x}_2 \cdots \dot{x}_2}_{q-\bar{i}}$$

and if $r_1 = 0$ and $q_3 \neq 0$ we associate to $T^{j,\bar{i},\bar{y}}_{\lambda(1)}$, $T^{j,\bar{i},\bar{j}}_{\lambda(2)}$, $T^{j,\bar{i},\bar{j}}_{\lambda(3)}$ the polynomials:

$$a^{0,j,\bar{i},\bar{j}}(x_1, x_2, y, z) = \underbrace{\acute{x}_1 \cdots \acute{x}_1}_{\bar{i}} \underbrace{\dot{x}_1 \cdots \dot{x}_1}_{\bar{j}} y x_1^j \underbrace{\tilde{x}_1 \cdots \tilde{x}_1}_{q-\bar{i}-\bar{j}} \underbrace{\acute{x}_2 \cdots \acute{x}_2}_{\bar{i}} z x_1^{r-j} \underbrace{\dot{x}_2 \cdots \dot{x}_2}_{\bar{j}} \underbrace{\tilde{x}_2 \cdots \tilde{x}_2}_{q-\bar{i}-\bar{j}}$$

where $'$, $\cdot$, $\sim$ mean alternation on the corresponding elements. Notice that the polynomials $a^{i,j,\bar{i},q-\bar{i}}$, $a^{0,j,\bar{i},\bar{j}}$ are obtained from the essential idempotents corresponding to the triple of tableaux $(T^{i,j,\bar{i}}_{\lambda(1)}, T^{i,j,\bar{i}}_{\lambda(2)}, T^{i,j,\bar{i}}_{\lambda(3)})$ and to the triple of tableaux $(T^{j,\bar{i},\bar{j}}_{\lambda(1)}, T^{j,\bar{i},\bar{j}}_{\lambda(2)}, T^{j,\bar{i},\bar{j}}_{\lambda(3)})$ respectively, by identifying all the elements in each row of $\lambda(1)$. We shall prove that $(\mathrm{mod} Id^G(A))$ these $\frac{(q+i)(r+1)(q+r+2)}{2}$ are linearly independents over F. Suppose not.

Let

$$\sum_{i,j,\bar{i}} \alpha_{i,j,\bar{i},q-\bar{i}} a^{i,j,\bar{i},q-\bar{i}}(x_1, x_2, y, z) + \sum_{j,\bar{i},\bar{j}} \alpha_{0,j,\bar{i},\bar{j}} a^{0,j,\bar{i},\bar{j}}(x_1, x_2, y, z) = 0$$

$(\mathrm{mod} Id^G(A))$ with scalars $\alpha_{i,j,\bar{i},q-\bar{i}}$ and $\alpha_{0,j,\bar{i},\bar{j}}$ not all zero. So it is a polynomial identity for A and then it should be zero for every substitution of x_1, x_2, y, z.

Consider the substitution $x_1 = \alpha e_{22} + \beta e_{33}$, $x_2 = \gamma e_{11} + \delta e_{33}$, $y = e_{12}$ and $z = e_{23}$, $\alpha, \beta, \gamma, \delta \in F$. With this substitution all the polynomials with $r_1 \neq 0$ take zero value, so the polynomials with non-zero evaluations are the following:

$$\underbrace{\acute{x}_1 \cdots \acute{x}_1}_{\bar{i}} \underbrace{\dot{x}_1 \cdots \dot{x}_1}_{\bar{j}} y x_1^j \underbrace{\tilde{x}_1 \cdots \tilde{x}_1}_{q-\bar{i}-\bar{j}} \underbrace{\acute{x}_2 \cdots \acute{x}_2}_{\bar{i}} z x_1^{r-j} \underbrace{\dot{x}_2 \cdots \dot{x}_2}_{\bar{j}} \underbrace{\tilde{x}_2 \cdots \tilde{x}_2}_{q-\bar{i}-\bar{j}}.$$

With this substitution every monomial of each polynomial takes zero value except one that takes value

$$\alpha^{q+j-\bar{j}} \beta^{r-j+\bar{j}} \gamma^{\bar{i}+\bar{j}} \delta^{q-\bar{i}-\bar{j}}.$$

We can observe that

$$\alpha^{q+j-\bar{j}} \beta^{r-j+\bar{j}} \gamma^{\bar{i}+\bar{j}} \delta^{q-\bar{i}-\bar{j}} = \alpha^{q+j'-\bar{j}'} \beta^{r-j'+\bar{j}'} \gamma^{\bar{i}'+\bar{j}'} \delta^{q-\bar{i}'-\bar{j}'}$$

if and only if

$$\bar{i} + \bar{j} = \bar{i}' + \bar{j}' \qquad j - \bar{j} = j' - \bar{j}'.$$

So there are four different possibilities:

1. $\bar{i} = \bar{i}' - k \quad \bar{j} = \bar{j}' + k \quad j = j' + h \quad \bar{j} = \bar{j}' + h$

2. $\bar{i} = \bar{i}' + k \quad \bar{j} = \bar{j}' - k \quad j = j' + h \quad \bar{j} = \bar{j}' + h$

3. $\bar{i} = \bar{i}' - k \quad \bar{j} = \bar{j}' + k \quad j = j' - h \quad \bar{j} = \bar{j}' - h$

4. $\bar{i} = \bar{i}' + k \quad \bar{j} = \bar{j}' - k \quad j = j' - h \quad \bar{j} = \bar{j}' - h.$

From cases 1) or 4) we obtain that $k = h$. From cases 2) or 3) we obtain that all valuations obtained from the considered substitution are distinct. So we have to consider case 1) and case 4) (we consider only case 1), because case 4) is the same). In case 1) the polynomial with the same valuations are:

$$\underbrace{\acute{x}_1 \cdots \acute{x}_1}_{\bar{i}} \underbrace{\dot{x}_1 \cdots \dot{x}_1}_{\bar{j}} y x_1^j \underbrace{\tilde{x}_1 \cdots \tilde{x}_1}_{q-\bar{i}-\bar{j}} \underbrace{\acute{x}_2 \cdots \acute{x}_2}_{\bar{i}} z x_1^{r-j} \underbrace{\dot{x}_2 \cdots \dot{x}_2}_{\bar{j}} \underbrace{\tilde{x}_2 \cdots \tilde{x}_2}_{q-\bar{i}-\bar{j}}$$

and

$$\underbrace{\acute{x}_1 \cdots \acute{x}_1}_{\bar{i}+k} \underbrace{\dot{x}_1 \cdots \dot{x}_1}_{\bar{j}-k} y x_1^{j-k} \underbrace{\tilde{x}_1 \cdots \tilde{x}_1}_{q-\bar{i}-\bar{j}} \underbrace{\acute{x}_2 \cdots \acute{x}_2}_{\bar{i}+k} z x_1^{r-j+k} \underbrace{\dot{x}_2 \cdots \dot{x}_2}_{\bar{j}-k} \underbrace{\tilde{x}_2 \cdots \tilde{x}_2}_{q-\bar{i}-\bar{j}}$$

$k = 1, 2, \ldots$.

For every choice of $j, \bar{i}, \bar{j}$ the corresponding polynomial $a^{0,j,\bar{i},\bar{j}}$ takes the same value of other k (eventually zero) polynomials $a^{0,j-k,\bar{i}+k,\bar{j}-k}$.

Now consider the substitution $x_1 = \alpha e_{11} + \beta e_{22} + \gamma e_{33}$, $x_2 = \varepsilon e_{11} + \delta e_{22}$, $y = e_{12}$ and $z = e_{23}$ to evaluate the polynomials $a^{0,j-k,\bar{i}+k,\bar{j}-k}(x_1, x_2, x_3, y, z)$ for every $k \geq 0$ and $a^{i_1 j_1,\bar{i}_1,q-\bar{i}}(x_1, x_2, x_3, y, z)$. With this substitution these polynomials takes value respectively

1. $\varepsilon^{\bar{j}-k}\beta^{j-k}\delta^{q-\bar{i}-\bar{j}}\gamma^{q+r-j-\bar{i}}(\sum_{t=0}^{\bar{i}+k}(-1)^{\bar{i}+k-t}\alpha^t\delta^t\varepsilon^{\bar{i}+k-t}\beta^{\bar{i}+k-t})$.

2. $\alpha^{i_1}\varepsilon^{q-\bar{i}_1}\beta^{j_1}\gamma^{q+r-i_1-j_1-\bar{i}_1}(\sum_{t=0}^{\bar{i}_1}(-1)^{\bar{i}_1-t_1}\alpha^{t_1}\delta^{t_1}\varepsilon^{\bar{i}_1-t_1}\beta^{\bar{i}_1-t_1})$.

Now we want to see if substitution 1) is equal to substitution 2)for any choice of $\bar{i}, \bar{j}, j, t$ and $\bar{i}_1, \bar{j}_1, i_1, j_1, t_1$ and for any $k \geq 0$. But if 1) is equal to 2) then, for example, $\alpha^t = \alpha^{i_1+t_1}$ and $\varepsilon^{\bar{i}+\bar{j}-t} = \varepsilon^{q-t_1}$. Hence $t = i_1 + t_1$, so $\bar{j} + \bar{i} - i_1 - t_1 = q - t_1 \Rightarrow \bar{i} + \bar{j} = i_1 + q \geq q$. But if $i_1 + q \geq q$, then $\bar{i} + \bar{j} \geq q$, a contradiction because $\bar{i} + \bar{j} \leq q$ (recall that in the first case $q_3 \neq 0$). So every substitution 1) is different to any substitution 2).

Then we want to prove that for any choice of $k \geq 0$ all evaluation 1) are different and so all the polynomials $a^{0,j-k,\bar{i}+k,\bar{j}-k}(x_1, x_2, x_3, y, z)$ are linearly independent. Set $\overline{k} = \max\{k \mid k \geq 0\}$. For any $k \in \{0, \ldots, \overline{k}\}$ any evaluation of 1) has $\delta^{q-\bar{i}-\bar{j}}\gamma^{q+r-j-\bar{i}}$ in common; so, in what follows, we can exclude it. So we obtain:

$$\varepsilon^{\bar{j}-\overline{k}}\beta^{j-\overline{k}}((-1)^{\bar{i}+\overline{k}}\varepsilon^{\bar{i}+\overline{k}}\beta^{\bar{i}+\overline{k}} + (-1)^{\bar{i}+\overline{k}-1}\varepsilon^{\bar{i}+\overline{k}-1}\beta^{\bar{i}+\overline{k}-1}\alpha\delta + \cdots + \alpha^{\bar{i}+\overline{k}}\delta^{\bar{i}+\overline{k}})$$

$$\varepsilon^{\bar{j}-\overline{k}+1}\beta^{j-\overline{k}+1}((-1)^{\bar{i}+\overline{k}-1}\varepsilon^{\bar{i}+\overline{k}-1}\beta^{\bar{i}+\overline{k}-1}+(-1)^{\bar{i}+\overline{k}-2}\varepsilon^{\bar{i}+\overline{k}-2}\beta^{\bar{i}+\overline{k}-2}\alpha\delta+\cdots+\alpha^{\bar{i}+\overline{k}-1}\delta^{\bar{i}+\overline{k}-1})$$

$$\vdots \qquad \vdots$$

$$\varepsilon^{\bar{j}}\beta^{j}((-1)^{\bar{i}}\varepsilon^{\bar{i}}\beta^{\bar{i}} + \cdots + \alpha^{\bar{i}}\delta^{\bar{i}}).$$

Now let make a linear combination of these evaluations with scalars $\alpha_{0,j-k,\bar{i}+k,\bar{j}-k} \in F$ for every $k \in \{0, \ldots, \overline{k}\}$. We obtain the following relations

$$\begin{cases} \alpha_{0,j-\overline{k},\bar{i}+\overline{k},\bar{j}-\overline{k}} = 0 \\ \alpha_{0,j-\overline{k}+1,\bar{i}+\overline{k}-1,\bar{j}-\overline{k}+1} - \alpha_{0,j-\overline{k},\bar{i}+\overline{k},\bar{j}-\overline{k}} = 0 \\ \vdots \\ (-1)^{\bar{i}+\overline{k}}\alpha_{0,j-\overline{k},\bar{i}+\overline{k},\bar{j}-\overline{k}} + \cdots + (-1)^{\bar{i}}\alpha_{0,j,\bar{i},\bar{j}} = 0 \end{cases}$$

So they are all equal to zero and then the considered polynomials are linearly independent.

Now consider evaluations 2)

$$\alpha^{i}\varepsilon^{q-\bar{i}}\beta^{j}\gamma^{q+r-i-j-\bar{i}}((-1)^{\bar{i}}\varepsilon^{\bar{i}}\beta^{\bar{i}} + (-1)^{\bar{i}-1}\varepsilon^{\bar{i}-1}\beta^{\bar{i}-1}\alpha\delta + \cdots + \alpha^{\bar{i}}\delta^{\bar{i}})$$

for every $\bar{i} = 0, \ldots, q$, $i = 0, \ldots, r$ and $j = 0, \ldots, r-i$, and we want to prove that all the polynomials $a^{i,j,\bar{i},q-\bar{i}}(x_1, x_2, x_3, y, z)$ are linearly independent.

We will write now the explicit value for every $\bar{i} = 0, \ldots, q$.

$$\alpha^i \varepsilon^q \beta^j \gamma^{q+r-i-j}$$
$$\alpha^i \varepsilon^{q-1} \beta^j \gamma^{q+r-i-j-1}(-\varepsilon\beta + \alpha\delta)$$
$$\alpha^i \varepsilon^{q-2} \beta^j \gamma^{q+r-i-j-2}(\varepsilon^2\beta^2 - \alpha\delta\varepsilon\beta + \alpha^2\delta^2)$$
$$\vdots \qquad \vdots$$
$$\alpha^i \beta^j \gamma^{r-i-j}((-1)^q \varepsilon^q \beta^q + (-1)^{q-1}\varepsilon^{q-1}\beta^{q-1}\alpha\delta + \cdots + \alpha^q\delta^q).$$

Let consider the evaluation obtained for $\bar{i} = q$; then for every $i = 0, \ldots, r$ and for every $j = 0, \ldots, r-i$ all the monomials of degree q in δ are distinct. So $\alpha_{i,j,q,0} = 0$ for every i, j. Then consider the evaluation obtained for $\bar{i} = q-1$. With same arguments $\alpha_{i,j,q-1,1} = 0$ for every i, j and so on. So all coefficients $\alpha_{i,j,\bar{i},q-\bar{i}}$ are equal to zero for every $\bar{i} = 0, \ldots, q$, $i = 0, \ldots, r$ and $j = 0, \ldots, r-i$.

Notice that for every $i, j, \bar{i}, \bar{j}$,

$$e^{i,j,\bar{i}}_{T_{\lambda(1)}} e^{i,j,\bar{i}}_{T_{\lambda(2)}} e^{i,j,\bar{i}}_{T_{\lambda(3)}} e^{i,j,\bar{i}}_{T_{\lambda(4)}}(x_1, \ldots, x_{n-2}, y, z)$$

is the complete linearization of $a^{i,j,\bar{i}}_{q,r}(x_1, x_2, y, z)$, and

$$e^{j,\bar{i},\bar{j}}_{T_{\lambda(1)}} e^{j,\bar{i},\bar{j}}_{T_{\lambda(2)}} e^{j,\bar{i},\bar{j}}_{T_{\lambda(3)}} e^{j,\bar{i},\bar{j}}_{T_{\lambda(4)}}(x_1, \ldots, x_{n-2}, y, z)$$

is the complete linearization of $a^{j,\bar{i},\bar{j}}_{q,r}(x_1, x_2, y, z)$. It follows that the polynomials

$$e^{i,j,\bar{i}}_{T_{\lambda(1)}} e^{i,j,\bar{i}}_{T_{\lambda(2)}} e^{i,j,\bar{i}}_{T_{\lambda(3)}} e^{i,j,\bar{i}}_{T_{\lambda(4)}},$$

$i + j = 0, \ldots, r$, $\bar{i} = 0, \ldots, q$; and the polynomials

$$e^{j,\bar{i},\bar{j}}_{T_{\lambda(1)}} e^{j,\bar{i},\bar{j}}_{T_{\lambda(2)}} e^{j,\bar{i},\bar{j}}_{T_{\lambda(3)}} e^{j,\bar{i},\bar{j}}_{T_{\lambda(4)}}$$

$j = 0, \ldots, r$, $\bar{i} + \bar{j} = 0, \ldots, q$ are linearly independent $(\mathrm{mod} Id^G(A))$, and this implies that $m \geq \frac{(q+1)(r+1)(q+r+2)}{2}$.

Then we want to prove that $m \leq \frac{(q+1)(r+1)(q+r+2)}{2}$. Let $T_{\lambda(1)}, T_{\lambda(2)}, T_{\lambda(3)}$, $T_{\lambda(4)}$ be any four tableaux and $f = e_{T_{\lambda(1)}} e_{T_{\lambda(2)}} e_{T_{\lambda(3)}} e_{T_{\lambda(4)}}(x_1, \ldots, x_{n-2}, y, z)$

the corresponding polynomial. If $f \notin \langle [x_1, x_2], y_1y_2, z_1z_2, t_1t_2, yt, zt\rangle$, then any two alternating variables have to be separated by y or z (i.e $\tilde{x}_1 \cdots y\dot{x}_1\tilde{x}_2 \cdots z\dot{x}_2 \cdots$). Since f is a linear combination (mod $\quad Id^G(A)$) of polynomials each alternating on q pairs of x_j's, we obtain that f is a polynomial of this type:

$$x_1^{r_1}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2}yx_1^{r_2}\underbrace{\tilde{x}_1\cdots\tilde{x}_1}_{q_3}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1}zx_1^{r_3}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2}\underbrace{\tilde{x}_2\cdots\tilde{x}_2}_{q_3} \quad (1).$$

With $r_1 + r_2 + r_3 = r$ and $q_1 + q_2 + q_3 = q$. Now we can prove that any polynomial of the type (1) can be written in the following way:

$$x_1^{r_1-q_3}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2+q_3}yx_1^{r_2+q_3}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1}zx_1^{r_3}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2+q_3}+$$

$$-q_3x_1^{r_1-q_3}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1+1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2+q_3-1}yx_1^{r_2+q_3-1}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1+1}zx_1^{r_3+1}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2+q_3-1}+$$

$$\binom{q_3}{2}x_1^{r_1-q_3}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1+2}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2+q_3-2}yx_1^{r_2+q_3-2}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1+2}zx_1^{r_3+2}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2+q_3-2}+$$

$$\vdots \qquad \vdots$$

$$(-1)^{q_3}x_1^{r_1-q_3}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1+q_3}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2}yx_1^{r_2}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1+q_3}zx_1^{r_3+q_3}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2}.$$

We can prove this formula using an induction on q_3.

If $q_3 = 1$ then:

$$x_1^{r_1}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2}yx_1^{r_2}\tilde{x}_1\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1}zx_1^{r_3}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2}\tilde{x}_2 =$$

$$x_1^{r_1-1}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2+1}yx_1^{r_2+1}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1}zx_1^{r_3}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2+1}+$$

$$-x_1^{r_1-1}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1+1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2}yx_1^{r_2}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1+1}zx_1^{r_3+1}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2}.$$

Now suppose it true for q_3 and we prove it for q_3+1.

$$x_1^{r_1}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2}yx_1^{r_2}\underbrace{\tilde{x}_1\cdots\tilde{x}_1}_{q_3+1}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1}zx_1^{r_3}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2}\underbrace{\tilde{x}_2\cdots\tilde{x}_2}_{q_3+1}=$$

$$=x_1^{r_1}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2}yx_1^{r_2}\underbrace{\tilde{x}_1\cdots\tilde{x}_1}_{q_3}x_1\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1}zx_1^{r_3}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2}\underbrace{\tilde{x}_2\cdots\tilde{x}_2}_{q_3}x_2+$$

$$-x_1^{r_1}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2}yx_1^{r_2}\underbrace{\tilde{x}_1\cdots\tilde{x}_1}_{q_3}x_2\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1}zx_1^{r_3}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2}\underbrace{\tilde{x}_2\cdots\tilde{x}_2}_{q_3}x_1.$$

We obtain the desired formula by applying induction to these two polynomials.

Notice that if $r_1-q_3=-c<0$, then we can apply this formula to the first q_3-c variables. In this case we'll obtain a linear combination of polynomials with $q_3=0$ and $r_1\neq 0$ and polynomials with $q_3\neq 0$ and $r_1=0$.

The following is an example of the previous formula when $q_3=2$:

$$x_1^{r_1}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2}yx_1^{r_2}\tilde{x}_1\tilde{x}_1\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1}zx_1^{r_3}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2}\tilde{x}_2\tilde{x}_2=$$

$$x_1^{r_1-1}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2+1}yx_1^{r_2+1}\tilde{x}_1\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1}zx_1^{r_3}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2+1}\tilde{x}_2$$

$$-x_1^{r_1-1}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1+1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2}yx_1^{r_2}\tilde{x}_1\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1+1}zx_1^{r_3+1}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2}\tilde{x}_2=$$

$$x_1^{r_1-2}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2+2}yx_1^{r_2+2}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1}zx_1^{r_3}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2+2}$$

$$-x_1^{r_1-2}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1+1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2+1}yx_1^{r_2+1}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1+1}zx_1^{r_3+1}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2+1}$$

$$-x_1^{r_1-2}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1+1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2+1}yx_1^{r_2+1}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1+1}zx_1^{r_3+1}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2+1}$$

$$+x_1^{r_1-2}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1+2}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2}yx_1^{r_2}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1+2}zx_1^{r_3+2}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2}=$$

$$x_1^{r_1-2}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2+2}yx_1^{r_2+2}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1}zx_1^{r_3}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2+2}$$

$$-2x_1^{r_1-2}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1+1}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2+1}yx_1^{r_2+1}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1+1}zx_1^{r_3+1}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2+1}$$

$$+x_1^{r_1-2}\underbrace{\acute{x}_1\cdots\acute{x}_1}_{q_1+2}\underbrace{\dot{x}_1\cdots\dot{x}_1}_{q_2}yx_1^{r_2}\underbrace{\acute{x}_2\cdots\acute{x}_2}_{q_1+2}zx_1^{r_3+2}\underbrace{\dot{x}_2\cdots\dot{x}_2}_{q_2}.$$

So we have written f as a linear combination of the polynomials

$$e^{i,j,\bar{i}}_{T_{\lambda(1)}}e^{i,j,\bar{i}}_{T_{\lambda(2)}}e^{i,j,\bar{i}}_{T_{\lambda(3)}}e^{i,j,\bar{i}}_{T_{\lambda(4)}}\text{and}e^{j,\bar{i},\bar{j}}_{T_{\lambda(1)}}e^{j,\bar{i},\bar{j}}_{T_{\lambda(2)}}e^{j,\bar{i},\bar{j}}_{T_{\lambda(3)}}e^{j,\bar{i},\bar{j}}_{T_{\lambda(4)}}.$$

Hence $m \leq \frac{(q+1)(r+1)(q+r+2)}{2}$. If $p \neq 0$ we can repeat the same arguments adding to each polynomial p triples of alternating variables x's (i.e: $\underbrace{\tilde{x}_1\cdots\tilde{x}_1}_{p}\cdots y\underbrace{\tilde{x}_2\cdots\tilde{x}_2}_{p}\cdots z\underbrace{\tilde{x}_3\cdots\tilde{x}_3}_{p}\cdots$), and all results still hold.

Then we can consider the remaining cases:

1. $\lambda(1) \neq 0, \quad \lambda(2) = (1), \quad \lambda(3) = \lambda(4) = \emptyset$
2. $\lambda(1) \neq 0, \quad \lambda(3) = (1), \quad \lambda(2) = \lambda(4) = \emptyset$
3. $\lambda(1) \neq 0, \quad \lambda(4) = (1), \quad \lambda(2) = \lambda(3) = \emptyset.$

These three cases are similar, so we can study only the first. If $\lambda(1) = (q+r, q)$ then $m = r+1$ (see [52, Theorem 3]).

If $\lambda(1) = (p+q+r, p+q, p)$ then the p triples of x_i's are not separated by y and z and so the polynomial obtained is a polynomial identity for A. Then $m = 0$ in this case.

$\square$

4.5 Cocharacter sequence of $UT_3(F)$ with elementary G-grading induced by (e, g, g^2), $g^2 \neq e$.

We consider a cyclic group of order three, $G = \{e, g, g^2\}$ and the elementary G-grading induced by (e, g, g^2), $g^2 \neq e$, is the same of Vasilovsky grading. Hence if $A' = UT_3(F)$, $A'_e = span\{e_{11}, e_{22}, e_{33}\}$, $A'_g = span\{e_{12}, e_{23}\}$ and $A'_{g^2} = span\{e_{13}\}$. Set $\lambda(1) \vdash n_1, \lambda(2) \vdash n_2, \lambda(3) \vdash n_3$, $n_1 + n_2 + n_3 = n$ and let $W_{\lambda(1),\lambda(2),\lambda(3)}$ be an $S_{n_1} \times S_{n_2} \times S_{n_3}$-irreducible left module. It is well known that if $T_{\lambda(1)}$ is a Young tableau of shape $\lambda(1)$, ..., $T_{\lambda(3)}$ a young tableau of shape $\lambda(3)$, then $W_{\lambda(1),\lambda(2),\lambda(3)} \cong F(S_{n_1} \times S_{n_2} \times S_{n_3})e_{T_{\lambda(1)}}e_{T_{\lambda(2)}}e_{T_{\lambda(3)}}$, where $S_{n_1}, S_{n_2}, S_{n_3}$ act on distinct sets of integers.

Let $h, k \in G$, $h, k \neq e$, $h \neq k$, $k \neq h^2$, and consider the elementary G-grading induced by (e, h, k) Note that $A'_g = A_h \oplus A_{h^{-1}k} = \text{span}\{e_{12}, e_{23}\}$, where A_h and $A_{h^{-1}k}$ are the homogeneous components of homogeneous degree h and $h^{-1}k$ in the elementary grading induced by (e, h, k), $g \neq h$, $g, h \neq e$. Moreover $A'_g A'_{g^2} = A'_{g^2} A'_g = 0$ and $(A_h \oplus A_{h^{-1}k})A_h = A_h(A_h \oplus A_{h^{-1}k}) = 0$.

Hence we can regard the space of multilinear G-graded polynomials P_{n_1,n_2,n_3,n_4} $(\text{mod}Id^G(A'))$ in the first grading as the space P_{n_1,n_2,n_3} $(\text{mod}Id^G(A'))$ in the second grading.

Since $\dim A'_e = 3$, $\dim A'_g = 2$ and $\dim A'_{g^2} = 1$, if $\chi_n^G(UT_3(F)) = \sum_{n_1+n_2+n_3=n} \sum_{\lambda(i) \vdash n_i} m_{\lambda(1),\lambda(2),\lambda(3)} \chi_{\lambda(1)} \otimes \chi_{\lambda(2)} \otimes \chi_{\lambda(3)}$ is the nth graded cocharacter of $UT_3(F)$ with this grading, it follows that $m_{\lambda(1),\lambda(2),\lambda(3)} = 0$ if either $h(\lambda(1)) > 3$ or $h(\lambda(2)) > 2$ or $h(\lambda(3)) > 1$. It is easy to prove that in order to have $m_{\lambda(1),\lambda(2),\lambda(3)} \neq 0$, $\lambda(2) = (1, 1)$.

With the same techniques of the previous grading we obtain the following

Theorem 4.5.1 *Let* $\chi_n^G(UT_3(F)) = \sum_{n_1+n_2+n_3=n} \sum_{\lambda(i)\vdash n_i} m_{\lambda(1),\lambda(2),\lambda(3)}$ $\chi_{\lambda(1)} \otimes \chi_{\lambda(2)} \otimes \chi_{\lambda(3)}$ *be the* n*th graded cocharacter of* $UT_3(F)$ *with elementary* G*-grading induced by* (e, g, g^2)*,* $g^2 \neq e$*. If we set for simplicity* $m_{\lambda(1),\lambda(2),\lambda(3)} = m$*, then:*

1. $m = \frac{(q+1)(r+1)(q+r+2)}{2}$ *if* $\lambda(1) = (p+q+r, p+q, p)$ $p, q, r \geq 0$*,* $\lambda(2) = (1, 1)$*,* $\lambda(3) = \emptyset$*.*

2. $m = (r+1)$ *if* $\lambda(1) = (q+r, q)$ $q, r \geq 0$ *and* $\lambda(2) = (1)$*,* $\lambda(3) = \emptyset$*, or* $\lambda(3) = (1)$*,* $\lambda(2) = \emptyset$*.*

3. $m = 1$ *if* $\lambda(1) = (n)$ *and* $\lambda(2) = \lambda(3) = \emptyset$*.*

4. $m = 0$ *in all other cases.*

4.6 Cocharacter sequence of $UT_3(F)$ with elementary G-grading induced by (e, e, g), $g \neq e$.

In this case if $A = UT_3(F)$ then $A_e = \text{span}\{e_{11}, e_{22}, e_{33}, e_{12}\}$, $A_g = \text{span}\{e_{13}, e_{23}\}$. Consider $\lambda(1) \vdash n_1, \lambda(2) \vdash n_2$, $n_1 + n_2 = n$ and let $W_{\lambda(1),\lambda(2)}$ be an $S_{n_1} \times S_{n_2}$-irreducible left module. As in the previous cases if $T_{\lambda(1)}$ is a Young tableau of shape $\lambda(1)$ and $T_{\lambda(2)}$ a Young tableau of shape $\lambda(2)$, then $W_{\lambda(1),\lambda(2)} \cong F(S_{n_1} \times S_{n_2})e_{T_{\lambda(1)}}e_{T_{\lambda(2)}}$, where S_{n_1}, S_{n_2} act on distinct sets of integers.

We can write the explicit decomposition of the nth graded cocharacter of A into irreducibles and calculate the corresponding multiplicities.

First recall (see [24] section 4.3) this definition

Definition 4.6.1 *A polynomial* $f \in F\langle X\rangle$ *is called proper, if it is a linear combination of products of long commutators*

$$f(x_1, \ldots, x_m) = \sum \alpha_{i,\ldots,j}[x_{i_1}, \ldots, x_{i_p}] \cdots [x_{j_1}, \ldots, x_{j_q}]$$

where $\alpha_{i,\dots,j} \in F$.

Recall that the multiplicities in the cocharacter sequence are equal to the maximal number of highest weight vectors linearly independent according to the representation theory of GL_m. Since in our case $\dim A_e = 4$ and $\dim A_g = 2$ we can work with the group $GL_4 \times GL_2$ and write all possible highest weight vectors.

We remark that the general method of constructing an highest weight vector is that of writing a multihomogeneous polynomial obtained by putting a variable for each box of the Young tableaux $T_{\lambda(1)}$ and $T_{\lambda(2)}$, then by identifying all variables in the same rows and alternating all variables in the same columns. Then reducing every polynomial modulo $Id(A)$.

In our case we work with proper polynomials, compute proper multiplicities and then we obtain the ordinary multiplicities using the Littlewood Richardson rule.

We begin with a technical lemma.

Lemma 4.6.2 *Let* $UT_3(F)^G$ *be equipped with the elementary* G*-grading induced by the triple* (e, e, g), $g \neq e$. *If we denote by* x_i *the variables of homogeneous degree* e *and by* y_j *the variables of homogeneous degree* g*; then*

1. $[x_{i_1}, \dots, x_{i_h}, y, x_{j_1}, \dots, x_{j_k}] \equiv \sum_{l \in I} \alpha_l g_l \quad mod \quad Id(UT_3(F)^G)$ *where* I *is a finite set of indeces,* $h \geq 0$, $k \geq 2$, $\alpha_l \in F$ *and* g_l *is a product of two long commutators for all* $l \in I$. *The first is in the* x_i*'s only, and the second is a commutator in the* x_i*'s and* y *shifted in the last position (i.e.:* $g_l = [x_{k_1}, \dots, x_{k_t}][x_{r_1}, \dots, x_{r_m}, y]$.

2. $[x_{i_1}, \dots, x_{i_k}, x_2, x_1, x_{j_1}, \dots, x_{j_h}, y] \equiv$
$$[x_{i_1}, \dots, x_{i_k}, x_1, x_2, x_{j_1}, \dots, x_{j_h}, y] + \sum_{l \in I} \alpha_l g_l \quad mod \quad Id(UT_3(F)^G)$$
where $i_1, \dots, i_h$ *are not necessarily ordered indeces,* $h, k \geq 0$, I *is a finite set of indeces,* $\alpha_l \in F$ *and* g_l *is a product of two long commu-*

tators. The first commutator is in the x_i's only, and the second is a commutator in the x_i's and y, for all $l \in I$.

3. $[x_{i_1}, \dots, x_{i_h}, \overline{x}_1, x_{j_1}, \dots, x_{j_l}, \overline{x}_2, x_{t_1}, \dots, x_{t_m}, y] \equiv \sum_{l \in I} \alpha_l g_l$ *mod* $Id(UT_3(F)^G)$ *where* $h, l, m \geq 0$, *I is a finite set of indeces,* $\alpha_l \in F$ *for all* $l \in I$ *and* g_l *is a product of two long commutators with no alternating variables.*

Proof. 1) if $k = 0, 1$, for every $h \geq 0$ we have nothing to do. Then we can suppose that $k \geq 2$.

If $h = 0$, for every $k \geq 2$, then

$$[y, x_{j_1}, \dots, x_{j_k}] \equiv -[x_{j_1}, \dots, x_{j_k}]y \mod Id(UT_3(F)^G).$$

Suppose first $h = 1$. For every $k \geq 2$, $[x_{i_1}, y, x_{j_1}, \dots, x_{j_k}]$. So we can apply the Jacoby identity ($[a, [b, c]] = [b, [a, c]] - [c, [a, b]]$) with $a = x_{i_1}$, $b = y$, $c = [x_{j_1}, \dots, x_{j_k}]$. Then we obtain

$$[x_{i_1}, y, x_{j_1}, \dots, x_{j_k}] = [y, x_{i_1}, x_{j_1}, \dots, x_{j_k}] - [[x_{j_1}, \dots, x_{j_k}], [x_{i_1}, y]] \quad (1)$$

and reducing mod $Id(UT_3(F)^G)$ we obtain

$$[x_{i_1}, y, x_{j_1}, \dots, x_{j_k}] \equiv [x_{i_1}, x_{j_1}, \dots, x_{j_k}]y - [x_{j_1}, \dots, x_{j_k}][x_{i_1}, y]$$

and this is a linear combination of the required form.

Suppose now $h = 2$. For every $k \geq 2$, $[x_{i_1}, x_{i_2}, y, x_{j_1}, \dots, x_{j_k}] =$ from (1)

$$[x_{i_1}, y, x_{i_2}, x_{j_1}, \dots, x_{j_k}] - [x_{i_1}, [x_{j_1}, \dots, x_{j_k}], [x_{i_2}, y]].$$

We apply again the Jacoby identity to the first and to the second commutator with $a = x_{i_1}$, $b = y$, $c = [x_{i_2}, x_{j_1}, \dots, x_{j_k}]$ in the first case and $a = x_{i_1}$, $b = [x_{j_1}, \dots, x_{j_k}]$, $c = [x_{i_2}, y]$ in the second case. Hence we obtain

$$[y, x_{i_1}, x_{i_2}, x_{j_1}, \dots, x_{j_k}] - [[x_{i_2}, x_{j_1}, \dots, x_{j_k}], [x_{i_1}, y]] -$$

$$[[x_{j_1}, \dots, x_{j_k}], [x_{i_1}, x_{i_2}, y]] + [[x_{i_2}, y], [x_{i_1}, x_{j_1}, \dots, x_{j_k}]] \equiv$$

mod $\quad Id(UT_3(F)^G) \quad - [x_{i_1}, x_{i_2}, x_{j_1}, \ldots, x_{j_k}]y - [x_{i_2}, x_{j_1}, \ldots, x_{j_k}][x_{i_1}, y] -$

$$[x_{j_1}, \ldots, x_{j_k}][x_{i_1}, x_{i_2}, y] - [x_{i_1}, x_{j_1}, \ldots, x_{j_k}][x_{i_2}, y]$$

and it is a linear combination of the required form. Obviously we can apply the same procedure for every $h, k \geq 2$, always obtaining a linear combination in the form (1).

2) Suppose first that $k = 0$. For every $h \geq 0$

$$[x_2, x_1, x_{j_1}, \ldots, x_{j_h}, y] =$$

$$[x_1, x_2, x_{j_1}, \ldots, x_{j_h}, y] - [[x_{j_1}, \ldots, x_{j_h}, y], [x_2, x_1]] \equiv$$

$$[x_1, x_2, x_{j_1}, \ldots, x_{j_h}, y] + [x_1, x_2][x_{j_1}, \ldots, x_{j_h}, y] \text{mod} \quad Id(UT_3(F)^G$$

Now suppose $k = 1$. For every $h \geq 0$

$$[x_{i_1}, x_2, x_1, x_{j_1}, \ldots, x_{j_h}, y] =$$

$$[x_{i_1}, x_1, x_2, x_{j_1}, \ldots, x_{j_h}, y] - [x_{i_1}, [[x_{j_1}, \ldots, x_{j_h}, y], [x_2, x_1]]]$$

Now we can apply the Jacoby identity to the second commutator.

$$[x_{i_1}, [[x_{j_1}, \ldots, x_{j_h}, y], [x_2, x_1]]] =$$

$$[[x_{j_1}, \ldots, x_{j_h}, y], [x_{i_1}, [x_2, x_1]]] - [[x_2, x_1], [x_{i_1}, x_{j_1}, \ldots, x_{j_h}, y]]$$

and reducing mod $\quad Id(UT_3(F)^G)$ we obtain

$$[x_{i_1}, [x_2, x_1]][x_{j_1}, \ldots, x_{j_h}, y] + [x_1, x_2][x_{i_1}, x_{j_1}, \ldots, x_{j_h}, y].$$

It is easy to see that is a linear combination of the required form. Now we can apply the same procedure for every $k > 1$ and we obtain the required result.

3)

$$[x_{i_1}, \ldots, x_{i_h}, \overline{x}_1, x_{j_1}, \ldots, x_{j_l}, \overline{x}_2, x_{t_1}, \ldots, x_{t_m}, y] =$$

$$[x_{i_1}, \ldots, x_{i_h}, x_1, x_{j_1}, \ldots, x_{j_l}, x_2, x_{t_1}, \ldots, x_{t_m}, y] -$$

$$[x_{i_1}, \dots, x_{i_h}, x_2, x_{j_1}, \dots, x_{j_l}, x_1, x_{t_1}, \dots, x_{t_m}, y].$$

Suppose first $l, h = 0$. For every $m \geq 0$

$$[\overline{x}_1, \overline{x}_2, x_{t_1}, \dots, x_{t_m}, y] =$$

$$[x_1, x_2, x_{t_1}, \dots, x_{t_m}, y] - [x_2, x_1, x_{t_1}, \dots, x_{t_m}, y] \quad (1).$$

We can apply the Jacoby identity to the second commutator

$$[x_1, x_2, x_{t_1}, \dots, x_{t_m}, y] - [x_1, x_2, x_{t_1}, \dots, x_{t_m}, y] + [[x_{t_1}, \dots, x_{t_m}, y], [x_2, x_1]] \quad (2).$$

Then reducing mod $Id(UT_3(F)^G)$ we get $[x_1, x_2][x_{t_1}, \dots, x_{t_m}, y]$ and this is a product of two commutators of the required form.

If $h > 0$ we have to apply h times the Jacoby identity to the third commutator of (2) and then reduce mod $Id(UT_3(F)^G)$.

If $l > 0$ we have to apply $l + 1$ times at most the Jacoby identity to the second commutator of (1). In all cases we get a linear combination of products of commutators of the required form.

$\square$

Remark 4.6.3 *Let G be a finite group, $G = \{g_1 = e, g_2, \dots, g_s\}$ and A be a finitely generated PI-algebra, graded by G. Let $\dim A_e = p_1, \dots, \dim A_{g_s} = p_s$, ($p_1 + \dots + p_s = \dim A$). If $\chi^G_{n_1,\dots,n_s}(A) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \dots \otimes \chi_{\lambda(s)}$ is its $(n_1, \dots, n_s)$th cocharacter, with $h(\lambda(1)) \leq p_1, \dots, h(\lambda(s)) \leq p_s$, then, if $1 \in A$ the proper G-graded cocharacter sequence of A is $\overline{\chi}^G_{n_1,\dots,n_s}(A) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\overline{\lambda(1)}} \otimes \dots \otimes \chi_{\lambda(s)}$ with $h(\overline{\lambda(1)}) \leq p_1 - 1$.*

Proof. Let us consider proper polynomials of A.

Recall that A_e is a subalgebra of A so we can suppose that $A_e = \mathrm{span}\{a_1 = 1_A, a_2, \dots, a_{p_1}\}$. If $h(\overline{\lambda(1)}) \leq p_1$, then every proper polynomial is alternating in p_1 variables of homogeneous degree e at most. If we consider multilinear polynomials, we can evaluate this polynomial with basis elements of A_e and, since $1_A \in A_e$, it takes zero value; so it is a polynomial identity, a contradiction. Then $h(\overline{\lambda(1)}) \leq p_1 - 1$.

$\square$

Theorem 4.6.4 *Set $A = UT_3(F)$ and let*

$$\overline{\chi}_n^G(A) = \sum_{n_1+n_2=n} \sum_{\lambda(i)\vdash n_i} m_{\lambda(1),\lambda(2)} \chi_{\lambda(1)} \otimes \chi_{\lambda(2)}$$

be the nth graded proper cocharacter of A with elementary G-grading induced by (e, e, g), $g \neq e$. If we set for simplicity $m_{\lambda(1),\lambda(2)} = m$, then:

1. *$m = 1$ if $\lambda(1) = (q)$, $q \geq 0$ and $\lambda(2) = (1)$.*

2. *$m = q + 1$ if $\lambda(1) = (p + q, p)$, $p > 0$, $q \geq 0$ and $\lambda(2) = (1)$.*

3. *$m = 2(q + 1)$ if $\lambda(1) = (p + q, p, 1)$, $p > 0$, $q \geq 0$ and $\lambda(2) = (1)$.*

4. *$m = q(p + 1) - 1$ if $\lambda(1) = (p + q, p, r)$ $p, q \geq 0$, $r = 0, 1$, $\lambda(2) = \emptyset$.*

5. *$m = 0$ in all other cases.*

Proof.

By Remark 4.6.3 and by obvious arguments, we can deduce that any polynomial alternating on four variables of homogeneous degree e or on three variables of homogeneous degree g vanishes in A. By the identity $y_1y_2 \equiv 0$ we can deduce that $\lambda(2) = \emptyset, (1)$. It is also easy to prove that $m = 0$ if $\lambda(1)_3 > 1$.

By the Poincarè-Birkhoff-Witt theorem (see [24] section 4.3) and using the same arguments of [25] , we can deduce that the vector space $F\langle X \mid G\rangle$ is generated by elements of the type

$$y_1 \cdots y_k[z_{i_1}, \ldots, z_{i_l}] \cdots [z_{j_1}, \ldots, z_{j_m}] \quad (1)$$

where y_i, $i = 1, \ldots, k$, are variables of homogeneous degree g and all z's are variables of homogeneous degree e or g.

Recall that, with our elementary G-grading,

$$Id^G(UT_3(F)) = \langle [x_1, x_2][x_3, x_4], y_1y_2, y[x_1, x_2]\rangle,$$

so in (1) there are no y's before commutators, moreover any commutator takes value αe_{12} or $\beta e_{23} + \gamma e_{13}$ for some $\alpha, \beta, \gamma \in F$, so in (1) there are at most two long commutators.

Suppose first that $\lambda(2) = \emptyset$, then in this case $UT_3(F)^G \cong UT_2(F) \oplus F$ with trivial grading, then the proper multiplicities of $UT_3(F)^G$ are the same of the ordinary multiplicities of $UT_2(F)$.

Suppose now $\lambda(2) = (1)$.

Let now consider multilinear polynomials associated to Young tableaux $T_{\lambda(1)}, T_{\lambda(2)}$ where $\lambda(1) = (p + q, p, r) \quad p, q, r \geq 0$.

When we evaluate these polynomials on the elements e_{11}, e_{22}, e_{33} of the basis of A_e, they always take zero value, so we can suppose that the elements of the third row of $T_{\lambda(1)}$ are evaluated in αe_{12}, $\alpha \in F$. This implies that $r \leq 1$, so the third row of the partition λ has length at most one.

Now we want to prove that all variables in commutators can be written in the following way:

1. $[x_1, \dots, x_1, x_2, \dots, x_2, x_3, y]$
2. $[x_2, \dots, x_2, x_1, \dots, x_1, x_3] \cdot [x_1, \dots, x_1, x_2, \dots, x_2, y]$
3. $[x_3, x_2, \dots, x_2, x_1, \dots, x_1, x_2] \cdot [x_1, \dots, x_1, x_2, \dots, x_2, y]$

Consider first a product of two commutators.

The commutator with only x's can be written $[x_1, \dots, x_1, x_2, \dots, x_2, x_3]$ or $[x_3, x_2, \dots, x_2, x_1, \dots, x_1, x_2]$, because $(UT_3(F)^G)_e \cong UT_2(F) \oplus F$, and it is well known that the variables of $UT_2(F)$ are in commutators of these types:

$$[x_{j_1}, \dots, x_{j_m}, x_k] \quad k > j_m \leq j_{m-1} \leq \dots \leq j_1.$$

By applying condition 1) of Lemma 4.6.2 to the commutator where y appears, and reducing modulo $Id(UT_3(F)^G)$, we can deduce that the only y is in the last position of the second commutator. By condition 2) of the same lemma we can order all the other variables.

Now consider only one commutator with all x's and one y. Using condition 1) and 2) of Lemma 4.6.2 we can suppose that all variables are in ordered commutator.

By condition 3) of Lemma 4.6.2 we deduce that all the alternating variables are into different long commutators.

Suppose first that $\lambda(1)_3 = 0$. Then $\lambda(1) = (p+q, p)$, $p, q \geq 0$.

If $p = 0$, for every $q \geq 0$, the only proper polynomial is $[\underbrace{x_1, \dots, x_1}_{q}, y]$ and so $m = 1$.

If $p > 0$, for every $q \geq 0$, the proper polynomials are the following:

$$a_{q_1}(x_1, x_2, y) = [\underbrace{\overline{x}_1, \dots, \overline{x}_1}_{p-1}, \underbrace{x_1, \dots, x_1}_{q_1}, x_2] \cdot [\underbrace{x_1, \dots, x_1}_{q-q_1+1}, \underbrace{\overline{x}_2, \dots, \overline{x}_2}_{p-1}, y],$$

where $0 \le q_1 \le q$. For every choice of $q_1 = 0, \ldots, q$ we obtain $q+1$ different polynomials. So $m \le q+1$.

Now we want to prove that these $q+1$ polynomials are linearly independent. To do this, let us consider a linear combination of $a_{q_1}(x_1, x_2, y)$ for every $q_1 = 0, \ldots, q$. Let

$$\sum_{q_1=0}^{q} \alpha_{q_1} a_{q_1}(x_1, x_2, y)$$

be this linear combination. Then we have to prove that it is a polynomial identity; so we have to prove that it is equal to zero for every G-graded substitution with linear combinations of basis elements of $UT_3(F)^G$.

Let us consider this substitution: $x_1 = \alpha e_{11} + \beta e_{22} + \gamma e_{12}$, $\alpha, \beta, \gamma \in F$, $\alpha, \beta, \gamma \neq 0$, $x_2 = e_{11}$, and $y = e_{23}$.

With this substitution every monomial is equal to zero except the monomial with all alternating x_2's in the first commutator. In this case we have $[x_1, x_2] = -\gamma e_{12} = -\gamma(\alpha - \beta)^0 e_{12}$, $[x_1, [x_1, x_2]] = -\gamma(\alpha - \beta)e_{12}$. In general $[\underbrace{x_1, \ldots, x_1}_{t}, x_2] = -\gamma(\alpha - \beta)^{t-1} e_{12}$.

$[x_1, y] = \beta e_{23} + \gamma e_{13}$, $[x_1, [x_1, y]] = \gamma(\alpha + \beta)e_{13} + \beta^2 e_{23}$, in general $[\underbrace{x_1, \ldots, x_1}_{t}, y] = \gamma \sum_{i=0}^{t-1} \alpha^i \beta^{t-i-1} e_{13} + \beta^t e_{23}$.

So the complete evaluations of these polynomials are:

$$\gamma(\alpha - \beta)^{q_1 - 1} \beta^{p+q-q_1} e_{13}.$$

We obtain that

$$\sum_{q_1=0}^{q} \alpha_{q_1} (\alpha - \beta)^{q_1 - 1} \beta^{p+q-q_1} e_{13} = 0.$$

We prove that $\alpha_{q_i} = 0$ for every $q_i = 0, \ldots, q$.

We use induction on q, for every $p > 0$. If $q = 0$, $\alpha_0 \gamma(\alpha - \beta)^{-1} \beta^p e_{13} = 0$ implies that $\alpha_0 = 0$ since $\alpha, \beta \neq 0$.

Now suppose that $\alpha_{q_1} = 0$ for every $q_1 = 0, \ldots, q$ and prove that $\alpha_{q+1} = 0$.

When we consider $q+1$ we obtain all the relations of case q with a coefficient more (α_{q+1}) and an other relation, $\alpha_{q+1}=0$, corresponding to the monomial in commutative variables $\gamma\alpha^q\beta^{p-1}$ which is the only monomial of degree q in α. So $\alpha_{q_1}=0$ for every $q_1=0,\dots,q$, then $m\geq q+1$. Thus $m=q+1$ and we are done.

Suppose now that $\lambda(1)_3=1$. Then $\lambda(1)=(p+q,p,1)$, $p\geq 1$, $q\geq 0$ and proper polynomials are only the following:

$$a_{q_1}(x_1,x_2,x_3,y)=[\underbrace{\overline{x}_1,\dots,\overline{x}_1}_{p},\underbrace{x_1,\dots,x_1}_{q_1},x_3]\cdot[\underbrace{x_1,\dots,x_1}_{q-q_1},\underbrace{\overline{x}_2,\dots,\overline{x}_2}_{p},y]$$

and

$$a'_{q_1}(x_1,x_2,x_3,y)=[x_3,\underbrace{\overline{x}_1,\dots,\overline{x}_1}_{p-1},\underbrace{x_1,\dots,x_1}_{q_1},x_2]\cdot[\underbrace{x_1,\dots,x_1}_{q-q_1+1},\underbrace{\overline{x}_2,\dots,\overline{x}_2}_{p-1},y],$$

where $0\leq q_1\leq q$, and $-$ means alternation on corresponding elements.

We remark that for every choice of $q_1=0,\dots,q$ we obtain $2(q+1)$ different polynomials.

So we have proved that $m\leq 2(q+1)$.

Now we want to prove that these $2(q+1)$ polynomials are linearly independent.

To do this let us consider a linear combination of $a_{q_1}(x_1,x_2,x_3,y)$ and $a'_{q_1}(x_1,x_2,x_3,y)$ for every $q_1=0,\dots,q$.

Let
$$\sum_{q_1=0}^{q}\alpha_{q_1}a_{q_1}(x_1,x_2,x_3,y)+\alpha'_{q_1}a'_{q_1}(x_1,x_2,x_3,y)$$
be this linear combination.

Then we have to prove that it is a polynomial identity; so we have to prove that it is equal to zero for every G-graded substitution with linear combinations of basis elements of $UT_3(F)^G$.

Let us consider this substitution: $x_1 = \alpha e_{11} + \beta e_{22}$, $\alpha, \beta \in F$, $\alpha, \beta \neq 0$, $x_2 = e_{11}$, $x_3 = e_{12}$ and $y = e_{23}$. With this substitution all the polynomials $a'_{q_1}(x_1, x_2, x_3, y)$ take zero value. So the polynomials with non-zero evaluations are $a_{q_1}(x_1, x_2, x_3, y)$, for every $q_1 = 0, \dots, q$. Moreover every monomial of $a_{q_1}(x_1, x_2, x_3, y)$ is equal to zero, except the monomial with all alternating x_2's in the first long commutators and all x_1's in the second long commutators.

We obtain $[x_1, y] = \beta e_{23}$, $[x_2, x_3] = e_{12}$ and $[x_1, x_3] = (\alpha - \beta)e_{12}$.

So the complete evaluations of these polynomials are:

$$(\alpha - \beta)^{q_1} \beta^{p+q-q_1} e_{13}.$$

We obtain that

$$\sum_{q_1=0}^{q} \alpha_{q_1} (\alpha - \beta)^{q_1} \beta^{p+q-q_1} e_{13} = 0.$$

Now we want to prove that $\alpha_{q_i} = 0$ for every $q_i = 0, \dots, q$. We use again induction on q.

If $q = 0$, then $\alpha_0 \beta^p = 0$ implies that $\alpha_0 = 0$ since $\beta \neq 0$.

Now suppose that $\alpha_{q_1} = 0$ for every $q_1 = 0, \dots, q$. We want to prove that $\alpha_{q+1} = 0$.

When we consider $q + 1$ we obtain all the relations of case q with a coefficient more (α_{q+1}) and an other relation, $\alpha_{q+1} = 0$, corresponding to the monomial in commutative variables $\alpha^{q+1}\beta^p$ which is the only monomial of degree $q + 1$ in α. So $\alpha_{q_1} = 0$ for every $q_1 = 0, \dots, q$.

Now consider the substitution

$x_1 = \alpha e_{11} + \beta e_{22} + \gamma e_{12}$, $\alpha, \beta, \gamma \in F$, $\alpha, \beta, \gamma \neq 0$, $x_2 = e_{11}$, $x_3 = e_{11}$ and $y = e_{23}$. With this substitution every monomial of $a'_{q_1}(x_1, x_2, x_3, y)$ is equal to zero, except the monomial with all alternating x_2's in the first long commutators and all x_1's in the second long commutators.

In this case we have $[x_1, x_2] = -\gamma e_{12} = -\gamma(\alpha-\beta)^0 e_{12}$, $[x_1,[x_1,x_2]] = -\gamma(\alpha-\beta)e_{12}$. In general $[\underbrace{x_1,\ldots,x_1}_{t}, x_2] = -\gamma(\alpha-\beta)^{t-1}$.

$[x_1, y] = \beta e_{23} + \gamma e_{13}$, $[x_1,[x_1,y]] = \gamma(\alpha+\beta)e_{13} + \beta^2 e_{23}$, in general $[\underbrace{x_1,\ldots,x_1}_{t}, y] = \gamma\sum_{i=0}^{t-1}\alpha^i\beta^{t-i-1}e_{13} + \beta^t e_{23}$.

So the complete evaluations of these polynomials are:

$$\gamma(\alpha-\beta)^{q_1-1}\beta^{p+q-q_1}e_{13}.$$

With same arguments of the previous evaluation we can prove that $\alpha'_{q_1} = 0$ for every $q_1 = 0,\ldots,q$.

Then we have proved that $m \geq 2(q+1)$ and so $m = 2(q+1)$. □

4.7 Cocharacter sequence of $UT_3(F)$ with elementary G-grading induced by (e, g, g), $g \neq e$.

If $A = UT_3(F)$ then in this case $A_e = \text{span}\{e_{11}, e_{22}, e_{33}, e_{23}\}$, $A_g = \text{span}\{e_{12}, e_{13}\}$.

We can apply the same techniques of section 4.6; so we are able to calculate proper multiplicities in the sequence of G-graded cocharacters.

First we note that we can restate Lemma 4.6.2 in this way.

Lemma 4.7.1 *Let A be equipped with the elementary G-grading induced by the triple (e, e, g), $g \neq e$. If we denote by x_i the variables of homogeneous degree e and by y_j the variables of homogeneous degree g; then*

1. $[x_{i_1},\ldots,x_{i_h}, y, x_{j_1},\ldots,x_{j_k}] \equiv \sum_{l\in I}\alpha_l g_l \mod Id(UT_3(F)^G)$ *where I is a finite set of indeces, $h \geq 0$, $k \geq 2$, $\alpha_l \in F$ and g_l is a product of two long commutators for all $l \in I$. The first is in the variables x_i's and y shifted in the last position (i.e.: $g_l = [x_{k_1},\ldots,x_{k_t}][x_{r_1},\ldots,x_{r_m}, y]$, and the second is a commutator in the x_i's only.*

2.
$$[x_{i_1},\dots,x_{i_k},x_2,x_1,x_{j_1},\dots,x_{j_h},y]\equiv$$
$$[x_{i_1},\dots,x_{i_k},x_1,x_2,x_{j_1},\dots,x_{j_h},y]+\sum_{l\in I}\alpha_l g_l \quad mod \quad Id(UT_3(F)^G)$$
where $i_1,\dots,i_h$ are not necessarily ordered indeces, $h,k\geq 0$, I is a finite set of indeces, $\alpha_l\in F$ and g_l is a product of two long commutators. The first is in the variables x_i's and y shifted in the last position, and the second is a commutator in the x_i's only.

3. $[x_{i_1},\dots,x_{i_h},\overline{x}_1,x_{j_1},\dots,x_{j_l},\overline{x}_2,x_{t_1},\dots,x_{t_m},y] \equiv \sum_{l\in I}\alpha_l g_l$ *mod* $Id(UT_3(F)^G)$ *where $h,l,m\geq 0$, I is a finite set of indeces, $\alpha_l\in F$ for all $l\in I$ and g_l is a product of two long commutators with no alternating variables.*

Proof. The proof is the same of that of Lemma 4.6.2, but when we reduce mod $Id(UT_3(F)^G)$ we use the identity $[x_1,x_2]y$ instead of $y[x_1,x_2]$. □

In the same way we can follow the proof of Theorem 4.6.4 obtaining the same result; in this case the only differences are that all variables in commutators can be written in the following way:

1. $[x_1,\dots,x_1,x_2,\dots,x_2,x_3,y]$
2. $[x_1,\dots,x_1,x_2,\dots,x_2,y]\cdot[x_2,\dots,x_2,x_1,\dots,x_1,x_3]$
3. $[x_1,\dots,x_1,x_2,\dots,x_2,y]\cdot[x_3,x_2,\dots,x_2,x_1,\dots,x_1,x_2]$

and that the considered substitution are $x_1=\alpha e_{11}+\beta e_{22}$, $x_2=e_{11}$, $x_3=e_{23}$, $y=e_{12}$ and $x_1=\alpha e_{11}+\beta e_{22}+\gamma e_{23}$, $x_2=e_{11}$, $x_3=e_{11}$,$y=e_{12}$, $\alpha,\beta\in F$, $\alpha,\beta\neq 0$.

So if
$$\overline{\chi}_n^G(A)=\sum_{n_1+n_2=n}\sum_{\lambda(i)\vdash n_i} m'_{\lambda(1),\lambda(2)}\chi_{\lambda(1)}\otimes\chi_{\lambda(2)}$$
is the nth graded proper cocharacter of A with the considered G-grading, then $m'_{\lambda(1),\lambda(2)}=m_{\lambda(1),\lambda(2)}$ for every $\lambda(1)\vdash n_1$ and $\lambda(2)\vdash n_2$.

4.8 Cocharacter sequence of $UT_3(F)$ with elementary G-grading induced by (e, g, e), $g \neq e$.

If $A = UT_3(F)$ then $A_e = \text{span}\{e_{11}, e_{22}, e_{33}, e_{13}\}$, $A_g = \text{span}\{e_{12}\}$ and $A_{g^{-1}} = \text{span}\{e_{23}\}$. We can construct the space $W_{\lambda(1),\lambda(2),\lambda(3)} \cong F(S_{n_1} \times S_{n_2} \times S_{n_3})e_{T_{\lambda(1)}}e_{T_{\lambda(2)}}e_{T_{\lambda(3)}}$ as in the previous cases.

Since $\dim A_e = 4$, $\dim A_g = \dim A_{g^{-1}} = 1$, if

$$\chi_n^G(UT_3(F)) = \sum_{n_1+n_2+n_3=n} \sum_{\lambda(i)\vdash n_i} m_{\lambda(1),\lambda(2),\lambda(3)}\chi_{\lambda(1)} \otimes \chi_{\lambda(2)} \otimes \chi_{\lambda(3)} \quad (1)$$

is the nth graded cocharacter of $UT_3(F)$ with this grading, it follows that $m_{\lambda(1),\lambda(2),\lambda(3)} = 0$ if either $h(\lambda(1)) > 4$ or $h(\lambda(2)) \geq 2$ or $h(\lambda(3)) \geq 2$; also it is easy to prove that $m_{\lambda(1),\lambda(2),\lambda(3)} = 0$ if $\lambda(1)_4 \geq 1$; so the fourth row of $T_{\lambda(1)}$ has one box at most.

If we denote with y_i any variable of homogeneous degree g, with z_j any variable of homogeneous degree g^{-1}, the identities $y_1y_2 \equiv 0$ and $z_1z_2 \equiv 0$ show that $\lambda(2), \lambda(3) = \emptyset$ or (1).

It is easy to prove that $m_{\lambda(1),\lambda(2),\lambda(3)} = 0$ if $\lambda(1) = (l_1, l_2, l_3, 1)$, $l_1 \geq l_2 \geq l_3 \geq 1$ and $\lambda(2) = \lambda(3) = (1)$ because any evaluation with basis elements of any multilinear polynomial in variables x, y, z takes zero value.

Suppose now that $\lambda(2) = \lambda(3) = \emptyset$. Since $A_e \cong UT_2(F) \oplus F$ with trivial grading, then the proper multiplicities of A are the same of the ordinary multiplicities of $UT_2(F)$. More precisely if $\chi_n(UT_2(F)) = \sum_{\lambda\vdash n} m_\lambda\chi_\lambda \quad (2)$ is the nth cocharacter of $UT_2(F)$, then we get that $m_{\lambda(1),\emptyset,\emptyset} = m_{\lambda(1)}$ i.e.: the multiplicities of $\chi_{\lambda(1)} \otimes \chi_\emptyset \otimes \chi_\emptyset$ in (1) is the same as the multiplicity of $\chi_{\lambda(1)}$ in (2).

Now suppose that $\lambda(2) = \lambda(3) = (1)$ and $h(\lambda(1)) \leq 3$. In these cases we can apply the same techniques of Theorem 4.4.1 and obtain the same multiplicities, since with the identities $[x_1, x_2]y, y[x_1, x_2], [x_1, x_2]z, z[x_1, x_2]$

we can order all x's before and after any y or z. More precisely if

$$\chi_n^G(UT_3(F)) = \sum_{n_1+\ldots+n_4=n} \sum_{\lambda(i)\vdash n_i} m_{\lambda(1),\ldots,\lambda(4)} \chi_{\lambda(1)} \otimes \ldots \otimes \chi_{\lambda(4)} \quad (3)$$

is the nth graded cocharacter of $UT_3(F)$ with elementary G-grading induced by the triple (e, g, h), $g \neq h$, $h \neq g^2$, then $m'_{\lambda(1),(1),(1),\emptyset} = m_{\lambda(1),(1),(1)}$, i.e. the multiplicity of $\chi_{\lambda(1)} \otimes \chi_{(1)} \otimes \chi_{(1)} \otimes \chi_{\emptyset}$ in (3) is the same as the multiplicity of $\chi_{\lambda(1)} \otimes \chi_{(1)} \otimes \chi_{(1)}$ in (1).

In all other cases $m_{\lambda(1),\lambda(2),\lambda(3)} = 0$.

Bibliography

[1] E. Aljadeff, A. Giambruno, D. La Mattina, *Graded polynomial identities and exponential growth*, J.Reine Angew.Math **650**, (2011), 83-100.

[2] E. Aljadeff, A. Giambruno, *Multialternating graded polynomials and growth of polynomial identities*, Proc. Amer. Math. Soc. (to appear).

[3] E. Aljadeff, A. Kanel-Belov *Representability and Specht problem for G-graded algebras*, Adv. Math. **225**,(2010), 2391-2428.

[4] S.S. Azevedo, *Graded identities for the matrix algebra of order n over an infinitive field*, Comm. Algebra **30**, (2002), 5849-5860.

[5] S.S. Azevedo, *A basis for $\mathbb{Z}$-graded identities of matrices over infinitive fields*, Serdica **29**, (2003), 149-158.

[6] Yu. Bahturin, V. Drensky, *Identities of bilinear mappings and graded polynomial identities of matrices*, Linear Algebra and its Applications **369**, (2003), 95-112.

[7] Yu. Bahturin, A. Giambruno, D. Riley, *Group graded algebras satisfying a polynomial identity*, Israel J. Math. **104**, (1998), 145-155.

[8] Yu. Bahturin, A. Giambruno, M. Zaicev, *Codimension growth and graded identities*, in: Algebra (Moscow, 1998), de Gruyter, Berlin, (2000), 57-76.

[9] Yu. Bahturin, M. Kotchetov, *Classification of group gradings on simple Lie algebras of types A, B, C and D*, arXiv:0908.0906v1.

[10] Yu. A. Bahturin, S. Montgomery, M.V. Zaicev, *Generalized Lie solvability of associative algebras*, Groups, rings, Lie and Hopf algebras (St. John's, NF, 2001) Math. Appl. **555**, Kluwer Acad. Publ., Dordrecht, (2003).

[11] Yu. A. Bahturin, M. V. Zaicev, *Graded algebras and graded identities.* Polynomial identities and combinatorial methods (Pantelleria, 2001), 101-139, Lecture Notes in Pure and Appl. Math., **235**, Dekker, New York, (2003).

[12] Yu. A. Bahturin, M. V. Zaicev, *Group Gradings on Matrix Algebras*, Canadian Math. Bull. Vol. **45**, (2002), 499-508.

[13] Yu. A. Bahturin, M. V. Zaicev, *Identities of graded algebras and codimension growth*, Trans. of Amer. Math. Soc. **356**, (2004), 3939-3950.

[14] Yu. A. Bahturin, M. V. Zaicev and S. K. Sehgal, *Group Gradings on Associative Algebras*, J. Algebra **241** (2001), 677-698.

[15] Y. A. Bahturin, V. Drensky, *Graded polynomial identities of matrices*, Linear Algebra Appl. **357**, (2002), 15-34.

[16] Y. A. Bahturin, M. V. Zaicev, S. K. Sehgal *Finite-dimensional simple graded algebras*, Sb. Math. **199** (7), (2008), 965-983.

[17] J. Bergen, M. Cohen, *Actions of commutative Hopf algebras*, Bull. London Math. Soc. **18** (2), (1986), 159-164.

[18] A. Berele, *Magnum PI*, Israel J. Math. **51** (1-2), (1985), 13-19.

[19] C. W. Curtis, I. Reiner, "Representation Theory of Finite Groups and Associative Algebras", Wiley Classics Lib.,John Wiley & Sons, Inc.,New York, 1988.

[20] O. M. Di Vincenzo, P. Koshlukov, A. Valenti *Gradings on the algebra of upper triangular matrices and their graded identities*, J. Algebra. **275**, (2004), 550-566.

[21] O. M. Di Vincenzo, *On the graded identities of $M_{1,1}(E)$*, Isr. J. Math. **80**, (1992), 323-335.

[22] V. S. Drensky, *A minimal basis for the identities of a second order matrix algebra over a field of chararcteristic 0*, Algebra i Logika **20**, (1981), 282-290.

[23] V. Drensky, *Codimensions of T-ideals and Hilbert series of relatively free algebras*, J. Algebra **91**, (1984), 1-17.

[24] V. Drensky *Free Algebras and PI-algebras, Graduate course in Algebra*, Springer (Singapore), (1999).

[25] V. Drensky, A. Giambruno, *Cocharacters, codimensions and Hilbert series of the polynomial identities for 2×2 matrices with involution*, Canadian J. Math. **46**, (1994), 718-733.

[26] V. Drensky, T. G. Rashkova, *Weak polynomial identities for the matrix algebras*, Commun. Algebra **21**, (1993), 3779-3795.

[27] E. Formanek, *Invariants and the ring of generic matrices*, J. Algebra **89**, (1984), 178-223.

[28] A. Giambruno, D. La Mattina, *Graded polynomial identities and codimensions: Computing the exponential growth*, Adv. Math. **225**, (2010), 859-881.

[29] A. Giambruno, A. Regev, *Wreath products and PI-algebras*, J.Pure Appl. Algebra. **35**, (1985), 133-149.

[30] A. Giambruno, M. Zaicev, *Polynomial Identities and asymptotic methods*, Mathematical Surveys and Monographs, vol. **122**, AMS, Providence, RI, (2005).

[31] A. Giambruno, M. Zaicev, *A characterisation of algebras with polynomial growth of the codimension*, Proc.Amer.Math.Soc 129 , (2001), 59-67.

[32] T. W. Hungerford, *Algebra*, Springer.

[33] G. James, A. Kerber, *The Representation Theory of the symmetric Group, Encyclopedia of Mathematics and its applications*, Addison-Wesley, London, Vol 16, (1981).

[34] A. R. Kemer, *Varieties of* $\mathbb{Z}_2$*-graded algebras*, Izv. Akad. Nauk SSSR Ser. Mat **48**, (1984), 1042-1049.

[35] A. R. Kemer, *Varieties of finite rank*, Proc. 15-th All the Union Algebraic Conf., Krasnoyarsk, Vol 2, **73**, (1979).

[36] A. Kemer, *Ideals of Identities of Associative Algebras*, Transl. Math. Monogr., vol. **87**, Amer. Math. Soc., Providence, RI, (1991).

[37] P. Koshlukov, S. Azevedo, *Graded identities for T-prime algebras over fields of positive characteristic*, Israel J. Math. **128**, (2002), 157-176.

[38] D.Krakowski, A.Regev, *The polynomial identities of the Grassmann Algebra*, Trans. Amer. Math. Soc. **181**, (1973), 429-438.

[39] D. La Mattina, *On the graded identities and cocharacters of the algebra of* 3×3 *matrices*, Linear Algebra and its Applications **384**, (2004), 55-75.

[40] S. P. Mishchenko, A. Regev, M. Zaicev, *A characterization of P.I. algebras with bounded multiplicities of the cocharacters*, J. Algebra **219**, (1999), 356-368.

[41] J. B. Olson, A. Regev, *An application of representation theory to PI-algebras*, Proc. Amer. Math. Soc. **55**, (1976), 253-257.

[42] C. Procesi, *Computing with* 2×2 *matrices*, J. Algebra **87**, (1984), 342-359.

[43] Yu. P. Razmyslov, *Finite basing of the identities of a matrix algebra of second order over a field of characteristic zero* (Russian), Algebra i Logika **12**, (1973), 83-113 [Translation: Algebra and Logic 12 (1973) 47-63].

[44] A. Regev, *On the codimensions of matrix algebras*, in: Algebra - Some Current Trends (Varna 1986), Lecture Notes in Mathematics, vol. **1352**, (1988), 162-172.

[45] A. Regev, *Existence of identities in* $A \otimes B$, Israel J. Math. **11**, (1972), 131-152.

[46] A. Regev, *Algebras satisfying a Capelli identity*, Israel J. Math. **33**, (1979), 149-154.

[47] S. H. Sehgal, M. V. Zaicev, *Graded identities and induced gradings on group algebras, In Groups, rings, Lie and Hopf algebras*, Mathematics and its application Vol. **555**, Kluwer Acad. Press, (2003), 211-219.

[48] I. Sviridova, *Identities of PI-algebras graded by a finite abelian group*, Comm.Algebra, in press.

[49] E. Taft, *Orthogonal conjugacies in associative and Lie algebras*, Trans.Amer.Math.Soc **113**, (1964), 18-29.

[50] S. Yu. Vasilovsky, $\mathbb{Z}$*-graded polynomial identities of the full matrix algebra*, Comm. Algebra **26** (2), (1998), 601-612.

[51] S. Yu. Vasilovsky, $\mathbb{Z}_n$*-graded polynomial identities on the full matrix algebra of order* n, Proc. AMS 1 **27**, (1999), 3517-3524.

[52] A. Valenti, *The graded identities of upper triangular matrices of size two*, J. Pure Appl. Algebra **172**, (2002), 325-335.

[53] A. Valenti, M. Zaicev, *Abelian Gradings on Upper Triangular Matrices*, Arch.Math.**80**, (2003), 12-17.