Università degli Studi di Palermo

Dipartimento di Matematica e Informatica

Dottorato di Ricerca in Matematica e Informatica

- XXIII ciclo -

Polynomial Identities of the Jordan algebra $UJ_2(F)$

Author
Fabrizio Martino

Coordinator
Prof.ssa Luisa Di Piazza

Supervisor
Prof. Antonio Giambruno

Settore Scientifico Disciplinare MAT/02

Contents

Introduction

Let A be an associative algebra over a field F of characteristic 0, and denote by $Id(A)$ its T-ideal. Since $\operatorname{char} F = 0$ it suffices to study only the multilinear polynomial identities of A. Let P_n be the vector space of the multilinear polynomials in $x_1, \ldots, x_n$ in the free associative algebra $F\langle X\rangle$. We assume that the set X of free generators is countable and infinite. Thus in order to study the identities of A one studies the intersections $P_n \cap Id(A)$, $n \geq 1$. But for practical purposes these intersections are not suitable since they tend to become very large as $n \to \infty$. Therefore one is led to study the quotients $P_n(A) = P_n/(P_n \cap Id(A))$. The dimension $c_n = c_n(A) = \dim P_n(A)$ is called the n-th *codimension* of A; the sequence of codimensions of a given algebra is one of the most important characteristics of the identities of A. In [8, 10] Giambruno and Zaicev proved that the sequence $(c_n(A))^{1/n}$ converges, and its limit is always an integer, called the PI-*exponent* of A. Since then an extensive research on the exponent of PI algebras has been conducted. It is of interest to study the minimal algebras with respect to their PI-exponent. Recall that A is minimal of exponent ≥ 2 when for every algebra B such that $Id(A) \subset Id(B)$ (a proper inclusion), the PI exponent of B is less than that of A. The interested reader may wish to consult Chapters 7 and 8 of the monograph [9] for further reading about minimal algebras and varieties.

One may define and study analogous concepts for large classes of nonassociative algebras as well. Here we mention only that the PI exponent of a nonassociative algebra need not be an integer.

The algebras $UT_n(F)$ of upper triangular matrices are one of the first classes of algebras to have their identities completely described. They are crucial in classifying the subvarieties of the variety of algebras generated by the matrix algebra of order 2. Here we mention that concrete bases of identities for an algebra are known in few cases. The identities of the matrix algebra $M_2(F)$ are known over any field as long as $\operatorname{char} F \neq 2$, see [28, 26, 17]; bases for the Grassmann algebra E are also known over any field, see [24, 22, 32, 35]. In [29] the identities of $E \otimes E$ when $\operatorname{char} F = 0$ were described. The identities of $UT_n(F)$ are also known. Concerning Lie algebras, a basis of the identities of $sl_2(F)$ is known, see [28, 36]. The identities of the Lie algebra $UT_n(F)$ are easy to describe. In the case of

Jordan algebras, the only nontrivial cases where the identities are known are those of the algebras B_n and B of a nondegenerate symmetric bilinear form, to be defined later. These results are due to Vasilovsky [37]. Recall that earlier Iltyakov [14] had developed methods to study the identities in these algebras and had proved that the variety generated by B_n is Spechtian. Apart from the results mentioned above one does not know the concrete form of the identities satisfied by a given algebra.

Gradings on algebras and the corresponding graded identities have become an area of extensive study. We refer the interested reader to the survey [2] for further reading and reference (see also [18]) concerning gradings and graded identities. Let us mention that graded identities are "easier" to study than the ordinary ones, and quite a lot is known about bases of graded identities in large classes of associative algebras. These include all T-prime algebras with their natural gradings, upper triangular matrices, etc.

In contrast with the associative case, graded identities for Lie and Jordan algebras have seldom been studied. Among the few known results we mention [18] and [20]. In the first of these papers, the graded identities for the Lie algebra $sl_2(F)$ were described, under every possible grading. The second paper dealt with the graded identities of the Jordan algebra of the symmetric matrices of order two.

The first chapter of the thesis is introductory. We introduce the algebras with polynomial identity by giving their basic definitions and properties. We only deal with associative algebras over a field F of characteristic zero. Recall that a polynomial $f(x_1, \ldots, x_n)$ in non-commuting variables is a polynomial identity for the algebra A if $f(a_1, \ldots, a_n) = 0$, for all $a_1, \ldots, a_n \in A$. The set of all polynomial identities of A, $Id(A)$, is a T-ideal of the free associative algebra $F\langle X\rangle$, where $X = \{x_1, x_2, \ldots\}$ is a countable set. Then we introduce the sequences of codimensions and colengths. Finally, we give the definition of exponent of a PI-algebra and we introduce a special kind of polynomials, namely proper polynomials.

The second chapter gives a complete view of Jordan algebras, with their definitions and properties. We also talk about the corresponding free Jordan algebra and we relate it with the associative free algebra. For instance, we shall give a criterion that, under some conditions, enable us to identify a Jordan polynomial, i.e. a polynomial written using the non-associative multiplication, as an associative polynomial of the free associative algebra.

At the end of the chapter, we introduce a particular Jordan algebra, namely the Jordan algebra of a symmetric bilinear form, which is in strict relation with the main object of our study.

Finally, in the fourth chapter we study the polynomial identities for the Jordan algebra $UJ_2(F)$ of the upper triangular matrices of order 2 over an infinite field of characteristic different from 2. We describe all gradings on $UJ_2(F)$ by the group $\mathbb{Z}_2$. Moreover we obtain bases of the corresponding graded identities in each one of the three cases. For these results we need an infinite field of characteristic different from 2. Finally we describe a basis of the ordinary identities satisfied by $UJ_2(F)$, under the restriction char $F \neq 2$, 3. Recall that one may consider the ordinary identities as the graded ones for $UJ_2(F)$ equipped with the trivial grading. Our methods yield somewhat more general result in this last situation. In fact we find a basis of the polynomial identities of the Jordan algebra of a symmetric bilinear form on a vector space in the case when the form is of rank one.

Chapter 1

Preliminaries

In this first chapter we introduce the main object of our study, i.e. PI-algebras and their basic properties. We give some typical examples of PI-algebras, involving the Grassman algebra and $UT_n(F)$, the upper triangular matrices of order n over a field F. We also introduce the related notion of variety of algebras.

Finally, we introduce the definition of G-grading of an algebra, talking about the G-graded polynomial identities, and the definition of proper polynomials, a very useful tool that we employ in order to compute the basis of the T-ideal of any unitary algebra.

Remark that in this first part of the thesis, where not specified, all the objects involved are associative.

1.1 Some basic definitions

We start with the basic definition of free algebra. Let F be a field and X a countable set. The free associative algebra on X over F is the algebra $F\langle X\rangle$ of polynomials in the non-commuting indeterminates $x \in X$. A basis of $F\langle X\rangle$ is given by all words in the alphabet X, adding the empty word 1. Such words are called monomials and the product of two monomials is defined by juxtaposition. The elements of $F\langle X\rangle$ are called polynomials and if $f \in F\langle X\rangle$, then we write $f = f(x_1, \ldots, x_n)$ to indicate that $x_1, \ldots, x_n \in X$ are the only variables occurring in f.

We define $\deg u$, the degree of a monomial u, as the length of the word u. Also $\deg_{x_i} u$, the degree of u in the indeterminate x_i, is the number of the occurrences of x_i in u. Similarly, the degree $\deg f$ of a polynomial $f = f(x_1, \dots, x_n)$ is the maximum degree of a monomial in f and $\deg_{x_i} f$, the degree of f in x_i, is the maximum degree of $\deg_{x_i} u$, for u monomial in f.

The algebra $F\langle X\rangle$ is defined, up to isomorphism, by the following universal property: given an associative F-algebra A, any map $X \to A$ can be uniquely extended to a homomorphism of algebras $F\langle X\rangle \to A$. The cardinality of X is called the rank of $F\langle X\rangle$. Finally, we give the definition of the main object of the thesis.

Definition 1.1.1 Let A be an associative F-algebra and $f = f(x_1, \dots, x_n) \in F\langle X\rangle$. We say that $f \equiv 0$ is a *polynomial identity* for A if

$$f(a_1, \dots, a_n) = 0, \qquad \text{for all} \quad a_1, \dots, a_n \in A.$$

We shall usually say also that A satisfies $f \equiv 0$ or, sometimes, that f itself is an identity of A.

Remark that if Φ denotes the set of all homomorphism $\varphi : F\langle X\rangle \to A$, then it is clear that $f \equiv 0$ is true if and only if $f \in \bigcap_{\varphi\in\Phi} \ker\varphi$. Since the trivial polynomial $f = 0$ is an identity for any algebra A, we make the following:

Definition 1.1.2 If the associative algebra A satisfies a non-trivial polynomial identity $f \equiv 0$, we call A a *PI-algebra*.

For $a, b \in A$, let $[a, b] = ab - ba$ denote the Lie commutator of a and b. Now we are able to give some examples of PI-algebras.

Example 1.1.3 If A is a commutative algebra, then A is a PI-algebra. In fact, it satisfies the polynomial identity $[x_1, x_2] \equiv 0$.

Example 1.1.4 If A is a nilpotent algebra of index $n \geq 1$, then A satisfies the polynomial identity $x_1 \cdots x_n \equiv 0$ since $A^n = 0$.

Example 1.1.5 If A is an n-dimensional associative algebra, then A satisfies the standard identity of degree $n + 1$:

$$St_{n+1}(x_1, \dots, x_{n+1}) = \sum_{\sigma\in S_{n+1}} (\operatorname{sgn}\sigma) x_{\sigma(1)} \cdots x_{\sigma(n+1)} \equiv 0.$$

Example 1.1.6 Let $UT_n(F)$ be the algebra of $n \times n$ upper triangular matrices over F. Then $UT_n(F)$ satisfies the identity:

$$[x_1, x_2] \cdots [x_{2n-1}, x_{2n}] \equiv 0$$

This is easy to prove, because the commutator of any two upper triangular matrices is a strictly upper triangular matrix. Moreover, the set of strictly upper triangular matrices forms a nilpotent two-sided ideal of $UT_n(F)$ of index equal to n.

Example 1.1.7 Let G be the Grassmann algebra on a countable dimension vector space over a field F of characteristic different from 2. Then G satisfies the identity $[[x, y], z] \equiv 0$.

1.2 T-ideals and varieties of algebras

If A is an algebra, we define

$$Id(A) = \{f \in F\langle X \rangle \mid f \equiv 0 \text{ on } A\},$$

the set of polynomial identities of A. It is easy to check that $Id(A)$ is a two-sided ideal of $F\langle X \rangle$. Moreover, if $f = f(x_1, \ldots, x_n)$ is any polynomial in $Id(A)$, and $g_1, \ldots, g_n$ are arbitrary polynomials in $F\langle X \rangle$, then $f(g_1, \ldots, g_n)$ is still in $Id(A)$. Since any endomorphism of $F\langle X \rangle$ is determined by mapping $x \mapsto g$, $x \in X$, $g \in F\langle X \rangle$, it follows that $Id(A)$ is an ideal invariant under all endomorphism of $F\langle X \rangle$. This is the property that characterize the so-called T-ideals:

Definition 1.2.1 An ideal I of $F\langle X \rangle$ is called a *T-ideal* if $\varphi(I) \subseteq I$ for all endomorphisms $\varphi \in F\langle X \rangle$.

It follows that $Id(A)$ is a T-ideal of $F\langle X \rangle$. Moreover, since given a T-ideal I, it is easily proved that $Id(F\langle X \rangle / I) = I$, all T-ideals of $F\langle X \rangle$ are actually ideals of polynomial identities for a suitable algebra A.
Since many algebras may correspond to the same set of polynomial identities (or T-ideal) we need to introduce the notion of variety of algebras.

Definition 1.2.2 Let $S \subseteq F\langle X\rangle$ be a non-empty set of polynomials of $F\langle X\rangle$. The class of all algebras A for which S is a set of polynomial identities, i.e. $f \equiv 0$ on A for all $f \in S$, is called the *variety* $\mathcal{V} = \mathcal{V}(S)$ determined (or generated) by S.

A variety $\mathcal{V}$ is called non-trivial if $S \neq 0$ and $\mathcal{V}$ if it is non-trivial and contains a non-zero algebra.

Example 1.2.3 The class of all commutative algebras forms a proper variety defined by the polynomial $[x, y]$.

Example 1.2.4 The class of all associative algebras is a variety defined by the polynomial $(x, y, z) := (xy)z - x(yz)$.

Example 1.2.5 The class of all nil algebras of exponent bounded by n forms a variety defined by the polynomial x^n.

Remark that if $\mathcal{V}$ is the variety determined by the set S and $\langle S\rangle_T$ is the T-ideal of $F\langle X\rangle$ generated by S, then $\mathcal{V}(S) = \mathcal{V}(\langle S\rangle_T)$ and $\langle S\rangle_T = \bigcap_{A\in\mathcal{V}} Id(A)$. We can write $\langle S\rangle_T = Id(\mathcal{V})$. Thus to each variety corresponds a T-ideal of $F\langle X\rangle$. The converse is also true, as we can see by the following theorem:

Theorem 1.2.6 *There is a one-to-one correspondence between T-ideals of $F\langle X\rangle$ and varieties of algebras. In this correspondence a variety $\mathcal{V}$ corresponds to the T-ideal of identities $Id(\mathcal{V})$ and a T-ideal I corresponds to the variety of algebras satisfying all the identities of I.*

One can find a proof of this theorem in [9], Theorem 1.2.5.

Notice that if $\mathcal{V}$ is a variety determined by the set S and A is a PI-algebra whose T-ideal coincides with S, then we say that the variety $\mathcal{V}$ is generated by A and we write $\mathcal{V} = var(A)$.

1.3 Multihomogeneous and multilinear polynomials

If the ground field F is infinite, then the study of polynomial identities of an algebra over F can be reduced to the study of the so-called homogeneous or multilinear polynomials. As we shall see in this section, this reduction is very advantageous because these kinds of polynomials are easier to deal with.

So, let $F_n = F\langle x_1, \ldots, x_n \rangle$ be the free algebra of rank $n \geq 1$ over F. This algebra can be naturally decomposed as

$$F_n = F_n^{(1)} \oplus F_n^{(2)} \oplus \cdots$$

where, for every $k \geq 1$, $F_n^{(k)}$ is the subspace spanned by all monomials of total degree k. The $F_n^{(i)}$'s are called the homogeneous components of F_n. The previous decomposition can be rewritten as follows: for every $k \geq 1$ write

$$F_n^{(k)} = \bigoplus_{i_1+\ldots+i_n=k} F_n^{(i_1,\ldots,i_n)}$$

where $F_n^{(i_1,\ldots,i_n)}$ is the subspace spanned by all monomials of degree i_1 in x_1,..., i_n in x_n. Finally, we can give the definition of homogeneous polynomial as follow:

Definition 1.3.1 A polynomial f belonging to $F_n^{(k)}$ for some $k \geq 1$, is called *homogeneous* of degree k. Moreover, if f belongs to some $F_n^{(i_1,\ldots,i_n)}$, then it is called *multihomogeneous* of multidegree $(i_1, \ldots, i_n)$. We also say that a polynomial f is homogeneous in the variable x_i, if x_i appears with the same degree in every monomial of f.

If $f(x_1, \ldots, x_n) \in F\langle X \rangle$ is a polynomial, then f can be always decomposed into a sum of multihomogeneous polynomials. In fact, it can be written as:

$$f = \sum_{k_1 \geq 0, \ldots, k_n \geq 0} f^{(k_1,\ldots,k_n)}$$

where $f^{(k_1,\ldots,k_n)}$ is the sum of all monomials in f where $x_1, \ldots, x_n$ appear at degree $k_1, \ldots, k_n$, respectively. The polynomials $f^{(k_1,\ldots,k_n)}$ which are non-zero are called the multihomogeneous components of f.

The important role that multihomogeneous polynomials play, is underlined by the following theorem:

Theorem 1.3.2 *Let F be an infinite field. If $f \equiv 0$ is a polynomial identity* *for the algebra A, then every multihomogeneous component of f is still a polynomial identity for A.*

Proof. For every variable x_t, $1 \leq t \leq n$, we can decompose $f = \sum_{i=0}^{m} f_i$, where f_i is the sum of all monomials of f in which x_t appears at degree i and $m = \deg_{x_i} f$ is the degree of f in x_i. By an obvious induction argument, in order to prove the theorem, it is enough to prove that, for every variable x_t, $f_i \equiv 0$ for all $i \geq 0$.

Let $\alpha_0, \ldots, \alpha_m$ be distinct elements of F. Clearly, for every $j = 0, \ldots, m$,

$$f(x_1, \ldots, \alpha_j x_t, \ldots, x_n) \equiv 0$$

is still an identity for A. Since each f_i is homogeneous in x_t of degree i,

$$f_i(x_1, \ldots, \alpha_j x_t, \ldots, x_n) = \alpha_j^i f_i(x_1, \ldots, x_t, \ldots, x_n).$$

Hence

$$f(x_1, \ldots, \alpha_j x_t, \ldots, x_n) = \sum_{i=0}^{m} \alpha_j^i f_i(x_1, \ldots, x_n) \equiv 0 \tag{1.1}$$

on A, for all $j = 0, \ldots, m$.

Write Vandermonde matrix

$$\Delta = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \alpha_0 & \alpha_1 & \ldots & \alpha_m \\ \vdots & \vdots & & \vdots \\ \alpha_0^m & \alpha_1^m & \ldots & \alpha_m^m \end{pmatrix}.$$

Then (1.1) says that for every $a_1, \ldots, a_n \in A$, if we write $f_i(a_1, \ldots, a_n) = \overline{f}_i$, then

$$(\overline{f}_0 \cdots \overline{f}_m)\Delta = 0.$$

Since Vandermonde determinant $\det(\Delta) = \prod_{0 \leq i < j \leq m}(\alpha_j - \alpha_i)$ is non-zero, it follows that $f_0 \equiv 0, \ldots, f_m \equiv 0$ are identities of A. The proof is therefore complete. □

The previous theorem has an important consequence, given by the following

Corollary 1.3.3 *Let A be an algebra over an infinite field F. Then $Id(A)$*

is generated by its multihomogeneous polynomials.

The proof is trivial.

Among multihomogeneous polynomials a special role is played by the multilinear ones.

Definition 1.3.4 A polynomial f is called *linear* in the variable x_i if x_i occurs with degree 1 in every monomial of f (equivalently, if f is homogeneous in the variable x_i of degree $k_i = 1$).
A polynomial f is called *multilinear* if f is linear in each of its variables (equivalently, if f is multihomogeneous of multidegree $(1, \ldots, 1)$).

Since in a multilinear polynomial $f(x_1, \ldots, x_n)$ each variable appears in each monomial at degree 1, it is obvious that

$$f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)},$$

where $\alpha_\sigma \in F$ and S_n is the symmetric group on $\{1, \ldots, n\}$.
Observe that if $f(x_1, \ldots, x_n)$ is a polynomial linear in one variable, say x_i, then:

$$f(x_1, \ldots, x_{i-1}, \sum_{j=1}^{m} \alpha_j y_j, \ldots, x_n) = \sum_{j=1}^{m} \alpha_j f(x_1, \ldots, x_{i-1}, y_j, \ldots, x_n)$$

for every $\alpha_j \in F$ and $y_j \in F\langle X \rangle$.

Remark 1.3.5 Let A be an F-algebra. If a multilinear polynomial f vanishes on a basis of A, then f is a polynomial identity of A.

Proof. This is a direct consequence of the argumentation above. $\square$

It is always possible to reduce an arbitrary polynomial to a multilinear one. Such a process can be found, for instance, in [9].

Definition 1.3.6 Let S be a set of polynomials in $F\langle X \rangle$ and $f \in F\langle X \rangle$. We say that f is a consequence of the polynomials in S (or follows from the polynomials in S) if $f \in \langle S \rangle_T$, the T-ideal generated by the set S.

Definition 1.3.7 Two sets of polynomials are equivalent if they generate the same T-ideal.

Theorem 1.3.8 *If the algebra A satisfies an identity of degree k, then it satisfies a multilinear identity of degree less then k.*

Proof. Let $f(x_1, \ldots, x_n) \in F\langle X\rangle$ be a polynomial identity for the algebra A. If each variable x_i appears at degree ≤ 1, in each monomial of f, then by eventually specializing some of variables to zero, we obtain a multilinear polynomial. Hence we may assume that there exists a variable, say x_1, such that $\deg_{x_1} f = d > 1$.
Compute the polynomial

$$\begin{aligned} h(y_1, y_2, x_2, \ldots, x_n) = f(y_1 + y_2, x_2, \ldots, x_n) - f(y_1, x_2, \ldots, x_n) \\ - f(y_2, x_2, \ldots, x_n). \end{aligned}$$

Notice that h is still a polynomial identity for A. Moreover h is a non-zero polynomial. In fact, suppose $h = 0$. Since any map $X \to X$ can be extended to an endomorphism of $F\langle X\rangle$, replacing y_1 and y_2 with x_1 in h we also get a zero polynomial, i.e.

$$h(x_1, x_1, x_2, \ldots, x_n) = f(2x_1, x_2, \ldots, x_n) - 2f(x_1, x_2, \ldots, x_n) = 0.$$

If we decompose f into the sum $f = f_1 + \ldots + f_d$ where f_k is the sum of monomials of degree k in x_1, then the previous relations imply

$$-f_0 + (2^2 - 2)f_2 + \ldots + (2^d - 2)f_d = 0$$

which contradicts the inequality $d > 1$.
Since $\deg_{y_1} h = d - 1 < \deg_{x_1} f$, by an induction argument we obtain a multilinear polynomial which is an identity on A. □

The previous theorem has a very important consequence.

Theorem 1.3.9 *If* $\operatorname{char} F = 0$, *every non-zero polynomial $f \in F\langle X\rangle$ is equivalent to a finite set of multilinear polynomials.*

Proof. By Theorem 1.3.2, f is equivalent to the set of its multihomogeneous components. Hence we may assume that $f = f(x_1, \ldots, x_n)$ is multihomogeneous. Now we apply the process of multilinearization to f: if

$\deg_{x_1} f = d > 1$, then we write

$$f(y_1 + y_2, x_2, \ldots, x_n) = \sum_{i=0}^{d} g_i(y_1, y_2, x_2, \ldots, x_n)$$

where $\deg_{y_1} g_i = i$, $\deg_{y_2} g_i = d - i$ and $\deg_{x_t} g_i = \deg_{x_t} f$, for all $t = 2, \ldots, n$. Then all polynomials $g_i = g_i(y_1, y_2, x_2, \ldots, x_n)$, $i = 1, \ldots, d-1$, are consequences of f.
Notice that for every i,

$$g_i(y_1, y_1, x_2, \ldots, x_n) = \binom{d}{i} f(y_1, x_2, \ldots, x_n).$$

Since $\operatorname{char} F = 0$, $\binom{d}{i} \neq 0$, hence f is a consequence of any g_i, $i = 1, \ldots, d-1$. We now apply induction in order to complete the proof. □

The above result still holds if $\operatorname{char} F > \deg f$. So, if $\operatorname{char} F = 0$ we get a stronger result then in Corollary 1.3.3:

Corollary 1.3.10 *If* $\operatorname{char} F = 0$, *every T-ideal is generated, as a T-ideal, by the multilinear polynomials it contains.*

1.4 S_n-action on multilinear polynomials, codimensions, colengths

Let A be a PI-algebra and $Id(A)$ its T-ideal of identities.
We introduce

$$P_n = \operatorname{span}\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$$

the *vector space of multilinear polynomials* in the variables $x_1, \ldots, x_n$ in the free algebra $F\langle X \rangle$.

According to Corollary 1.3.10, if $\operatorname{char} F = 0$ then $Id(A)$ is determined by its multilinear polynomials, so it suffices to study the multilinear identities of A, that is $\{P_n \cap Id(A)\}_{n \geq 1}$.

Moreover, it is possible to define an action of the symmetric group S_n

on P_n in the following way: if $\sigma \in S_n$ and $f(x_1, \ldots, x_n) \in P_n$, then

$$\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

that is σ acts by permuting the variables. Since T-ideals are invariant under permutations of the variables, we obtain that the subspace $P_n \cap Id(A)$ is invariant under this action, that is $P_n \cap Id(A)$ is a left S_n-submodule of P_n. Hence

$$P_n(A) = \frac{P_n}{P_n \cap Id(A)}$$

has an induced structure of left S_n-module.

Definition 1.4.1 The non-negative integer

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap Id(A)}$$

is called the *nth codimension* of the algebra A. The sequence $\{c_n(A)\}_{n \geq 1}$ is the *codimension sequence* of A.

Clearly, $\dim(P_n \cap Id(A)) = n! - c_n(A)$. It is also obvious that A is a PI-algebra if and only if $c_n(A) < n!$ for some $n \geq 1$.

Example 1.4.2 Let A be a nilpotent algebra and let $A^m = 0$. Then $c_n(A) = 0$, for all $n \geq m$.

Example 1.4.3 Let A be a commutative algebra, then $c_n(A) \leq 1$ for all $n \geq 1$.

Example 1.4.4 Let $A = UT_2(F)$ be the associative algebra of 2×2 upper triangular matrices. Then $c_n(A) = 2^{n-1}(n-2)+2$. The proof of this results is in the third chapter of the thesis.

Definition 1.4.5 For $n \geq 1$, the S_n-character of $P_n(A) = P_n/(P_n \cap Id(A))$, denoted by $\chi_n(A)$, is called the *nth cocharacter* of A.

If we decompose the nth cocharacter into irreducibles, then we obtain:

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{1.2}$$

where χ_λ is the irreducible S_n-character associated to the partition $\lambda \vdash n$ and $m_\lambda \geq 0$ is the corresponding multiplicity.

We can define another numerical sequence, the sequence of colengths, that counts the number of irreducible S_n-modules appearing in the decomposition of $P_n(A)$.

Definition 1.4.6 The non-negative integer

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda$$

is called the *nth colength* of A. The sequence $\{l_n(A)\}_{n\geq 1}$ is the *colength sequence* of A.

1.5 Group-graded algebras

In this section we introduce the notation of an algebra graded by a group and we give some examples. Remember that all the stuffs about upper triangular matrices will be given in the third chapter.

Let A be an algebra over a field F and let G be any group.

Definition 1.5.1 The algebra A is *G-graded* if A can be uniquely written as the direct sum of subspaces $A = \bigoplus_{g\in G} A_g$ such that for all $g, h \in G$, $A_g A_h \subseteq A_{gh}$.
The subspaces A_g are called *homogeneous components* of A.

It is clear that any $a \in A$ can be written uniquely as a sum $a = \sum_{g\in G} a_g$, where $a_g \in A_g$. The a_g's are called *homogeneous of degree g*. According to this, one says that a is homogeneous if and only if $a \in A_g$ for some $g \in G$ and write $\deg a = g$.

Definition 1.5.2 Let A be a G-graded algebra. A subspace $B \subseteq A$ is a *graded or homogeneous* if $B = \bigoplus_{g\in G}(B \cap A_g)$.

In other words, B is G-graded if for any $b \in B$, $b = \sum_{g\in G} b_g$ implies that $b_g \in B$ for all $g \in G$. Similarly one can define the structure of subalgebras, ideals and so on. Remark that if H is a subgroup of G, then $B = \bigoplus_{h\in H} A_h$

is a graded subalgebra of A. In particular, A_e, where e it the unit element of G, is a subalgebra of A.

Definition 1.5.3 The *support* of a graded algebra is defined as

$$\mathrm{Supp}\, A = \{g \in G \mid A^{(g)} \neq 0\}.$$

The G-grading is called *trivial* if $\mathrm{Supp}_G = \{e\}$.

Definition 1.5.4 A map $\varphi : A = \bigoplus_{g\in G} A_g \to B = \bigoplus_{g\in G} B_g$ is called a *homomorphism* (*isomorphism*) *of graded algebras* if φ is an ordinary homomorphism (isomorphism) and $\varphi(A_g) \subseteq B_g$, for all $g \in G$.

It is easy to observe that $\ker\varphi$ is a graded ideal of A.
Now, we give some examples of G-graded algebras.

Example 1.5.5 The free associative algebra of countable rank $A = F\langle X\rangle$ has a natural $\mathbb{Z}$-grading by setting $A = \bigoplus_{n\in\mathbb{Z}} A_n$ where $A_n = 0$ if $n \leq 0$ and A_n is the linear span of all monomials of total degree n, if $n > 0$.
Moreover, if $X' = \{x_1, \ldots, x_k\}$ is a finite set, then $A = F\langle X'\rangle$, the free associative algebra of rank k, can be graded by the group $\mathbb{Z}^k = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ by setting

$$A_{(n_1,\ldots,n_k)} = \{f \in F\langle X'\rangle \mid \text{f is multihomogeneous, } \deg_{x_i} f = n_i,\ i = 1, \ldots, k\}.$$

Example 1.5.6 The group algebra $A = FG$ of a group G is naturally graded by G by setting: $A_g = \mathrm{span}_F\{g\}$.

Example 1.5.7 Let $A = M_k(F)$ be the algebra of $k \times k$ matrices over the field F and let G be an arbitrary group. Given any k-tuple $(g_1, \ldots, g_k)$ of elements of G, one can define a G-grading of A by setting

$$A_g = \mathrm{span}_F\{e_{ij} \mid g_i^{-1}g_j = g\},$$

where the e_{ij}'s are the usual matrix units. This kind of G-grading of matrices is called *elementary* grading.

Example 1.5.8 Let $A = M_2(F)$ and let $G = \langle a, b \mid a^2 = b^2 = e\rangle$ the Klein group. Then define

$$A_e = \mathrm{span}_F\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\},\ A_a = \mathrm{span}_F\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\}$$
$$A_b = \mathrm{span}_F\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\},\ A_{ab} = \mathrm{span}_F\left\{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\}.$$

An easy computation shows that $A = \bigoplus_{g\in G} A_g$ and $A_g A_h = A_{gh}$, for any $g, h \in G$.

1.6 Identities of G-graded algebras

We are now able to discuss about graded polynomial identities of a G-graded algebra A and their relationship with the ordinary identities. We shall introduce also, the analogous of $P_n(A)$ and codimensions in the case of a graded algebra.

Let $F\langle X\rangle$ be the free algebra over F on a countable set X and let G be a finite group. One can write X as $X = \bigcup_{g\in G} X_g$, where $X_g = \{x_1^{(g)}, x_2^{(g)}, \ldots\}$ are disjoint sets. The indeterminates of X_g are said to be of homogeneous degree g. The homogeneous degree of a monomial $x_{i_1}^{(g_1)} \cdots x_{i_t}^{(g_t)} \in F\langle X\rangle$ is defined to be $g_1 g_2 \cdots g_t$, as opposed to its total degree, which is defined to be t. Denote by $F\langle X\rangle_g$ the subspace of the algebra $F\langle X\rangle$ spanned by all the monomials having homogeneous degree g. Remark that $F\langle X\rangle_g F\langle X\rangle_h \subseteq F\langle X\rangle_{gh}$, for every $g, h \in G$. It follows that

$$F\langle X\rangle = \bigoplus_{g\in G} F\langle X\rangle_g$$

is a G-grandig. We denote with $F\langle X\rangle^{gr}$ the algebra $F\langle X\rangle$ with this grading.

Definition 1.6.1 $F\langle X\rangle^{gr}$ is called *the free G-graded algebra of countable rank over F.*

It is easy to prove that the following universal property holds: for any G-graded algebra $A = \bigoplus_{g\in G} A_g$ and for any set-theoretical map $\psi : X \to A$ such that $\psi(X_g) \subseteq A_g$, there exists a homomorphism of G-graded algebras $\bar{\psi} : F\langle X\rangle \to A$ such that $\bar{\psi}_{|X} = \psi$. Let $\bar{\Psi}$ be the set of all such homomorphisms, then $Id^{gr}(A) = \bigcap_{\bar{\psi}\in\bar{\Psi}} \ker\bar{\psi}$ is called the ideal of G-graded polynomial identities of A. This means that a graded polynomial $f(x_1^{(g_1)},\ldots,x_n^{(g_n)}) \in F\langle X\rangle^{gr}$ is a *graded identity* for the algebra A, and we write $f \equiv 0$ in A, if $f(a_1^{(g_1)},\ldots,a_n^{(g_n)}) = 0$ for all $a_1^{(g_1)} \in A^{(g_1)},\ldots,a_n^{(g_n)} \in A^{(g_n)}$.

Definition 1.6.2 $Id^{gr}(A) = \{f \in F\langle X\rangle^{(gr)} \mid f \equiv 0 \quad \text{on} \quad A\}$ is the *ideal of graded identities of* A.

Clearly $Id^{gr}(A)$ is stable under all graded endomorphisms of $F\langle X\rangle$, i.e. $Id^{gr}(A)$ is a T^{gr}-ideal. Given a finitely generated T^{gr}-ideal, one interesting problem of PI-theory is to find a finite generating set.

It can be easily proved that any non-trivial graded identity has non-trivial graded multilinear consequence and, in particular, if $\operatorname{char} F = 0$ then $Id^{gr}(A)$ is uniquely determined by all the multilinear polynomials it contains.

There is an obvious way of relating ordinary identities and graded identities of the algebra A. Recalling that the indeterminates from X_g are denoted $x_i^{(g)}$, then any multilinear graded polynomial can be written as:

$$f = f(x_1^{(g_1)},\ldots,x_n^{(g_n)}) = \sum_{\sigma\in S_n} \alpha_\sigma x_{\sigma(1)}^{(g_{\sigma(1)})}\cdots x_{\sigma(n)}^{(g_{\sigma(n)})}.$$

For a fixed n-tuple $(g_1,\ldots,g_n) \in G^n$, all multilinear polynomials in the variables $x_1^{(g_1)},\ldots,x_n^{(g_n)}$ form the $n!$-dimensional subspace:

$$P_n^{(g_1,\ldots,g_n)} = \operatorname{span}_F\{x_{\sigma(1)}^{(g_{\sigma(1)})}\cdots x_{\sigma(n)}^{(g_{\sigma(n)})} | \sigma \in S_n\}.$$

The intersection

$$P_n^{(g_1,\ldots,g_n)} \cap Id^{gr}(A)$$

consists of all multilinear graded identities of A in the variables $x_1^{(g_1)},\ldots,$ $x_n^{(g_n)}$. Define

$$\bar{x}_i = \sum_{g\in G} x_i^{(g)}$$

for every $i = 1,2,\ldots$. Then it is clear that the set $\{\bar{x}_1,\bar{x}_2,\ldots\}$ generates the free associative algebra of countable rank. Moreover, given a polynomial $f(x_1,\ldots,x_n) \in F\langle X\rangle$, f is an ordinary polynomial identity of A if and only if

$$f(\bar{x}_1, \dots, \bar{x}_n) \in Id^{gr}(A).$$

This basic observation implies the following technical result. For $(g_1, \dots, g_n) \in G^n$,

$$c_n^{(g_1,\dots,g_n)}(A) = \dim \frac{P_n^{(g_1,\dots,g_n)}}{P_n^{(g_1,\dots,g_n)} \cap Id^{gr}(A)}$$

is called the *homogeneous nth codimension* associated to $(g_1, \dots, g_n)$.

Obviously, existence of a graded identity on a graded algebra is a much weaker condition than the existence of an ordinary polynomial identity. For example, if B is an arbitrary algebra with trivial G-grading, that is $B_g = 0$ for all non-unitary $g \in G$, then B satisfies any graded identity of the type $x \equiv 0$ with $x \in X_g$, $g \neq e$.

1.7 The PI-Exponent of an algebra

One of the main line of research about PI-algebras is to understand how "quickly" the codimension sequence of a PI-algebra grows. This is not a trivial problem, since in most cases we can only have information about asymptotic behavior of $c_n(A)$. We now introduce an useful tool that concerns the codimension growth of an algebra.

Let A be a PI-algebra over a field F of characteristic zero and let $\{c_n(A)\}_{n \geq 1}$ be its sequence of codimensions. The following result holds.

Theorem 1.7.1 (Regev) *If the algebra A satisfies an identity of degree $d \geq 1$, then $c_n(A) \leq (d-1)^{2n}$.*

If A is a nilpotent algebra i.e., $x_1 \cdots x_N \equiv 0$ is a polynomial identity of A for some $N \geq 1$, then $c_n(A) = 0$ for any $n \geq N$. But if A is a non-nilpotent algebra, then $c_n(A) \neq 0$ for $n \geq 1$ and by Theorem 1.7.1,

$$1 \leq c_n(A) \leq a^n$$

for some costant a. Hence the sequence of nth roots $\sqrt[n]{c_n(A)}$, $n = 1, 2, \dots$ is bounded and we can consider its upper and lower limit.

Definition 1.7.2 Let A be any PI-algebra. Then

$$\underline{exp}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n(A)}$$

is called *the lower exponent of* A and

$$\overline{exp}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n(A)}$$

is called *the upper exponent of* A.

For any bounded sequence the lower limit and the upper limit may not coincide. Nevertheless, in case they do, we can define the exponent of A.

Definition 1.7.3 Let A be a PI-algebra. Then the *exponent* (or *PI-exponent*) of A is

$$exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)},$$

provided $\underline{exp}(A) = \overline{exp}(A)$. In case $\mathcal{V}$ is a variety of algebras, we write $exp(\mathcal{V}) = exp(A)$ and we call $exp(A)$ the exponent of the variety $\mathcal{V}$.

An important result about this topic, is the proof of the existence of the PI-exponent and its integrality.

Theorem 1.7.4 *Let A be a PI-algebra over a field F of characteristic zero. Then there exists an integer $q \geq 0$ and constants C_1, C_2, r_1, r_2 such that $C_1 \neq 0$ and*

$$C_1 n^{r_1} q^n \leq c_n(A) \leq C_2 n^{r_2} q^n.$$

As a consequence $exp(A)$ exists and is a non-negative integer.

For a proof of this Theorem we remand to [9]. However, it is clear that this definition of PI-exponent is not so useful in order to compute it. In fact in this moment we are able to compute the exponent of those algebras for which we already know the codimension sequence. Luckily there exists a method that helps us to compute the exponent for any finite dimensional algebra.

Let A be a finite dimensional PI-algebra over a algebraically closed field F. From the structure theorems follow that one can write $A = A_{ss}+J$, where A_{ss} is a maximal semisimple subalgebra and J is the Jacobson radical of A.

Let $A_{ss} = A_1 \oplus \ldots \oplus A_k$ where $A_1, \ldots, A_k$ are simple subalgebras. Then

$$exp(A) = \max \dim_F \{A_{i_1} \oplus \ldots \oplus A_{i_r} \mid A_{i_1} J \cdots J A_{i_r} \neq 0\},$$

where $A_{i_1}, \ldots, A_{i_r}$ are distinct among $A_1, \ldots, A_k$. We make use of this reduction in the proof of the next theorem. For an algebra A let us denote by $Z = Z(A)$ the center of A.

Theorem 1.7.5 *Let A be a finite dimensional PI-algebra over a field F of characteristic zero. Then*

1. $exp(A) \leq \dim_F A$.
2. *If A is semisimple, $exp(A) = \dim_{Z(B)} B$ where B is a simple subalgebra of A of the greatest dimension over its center $Z(B)$. In particular, if A is simple $exp(A) = \dim_{Z(A)} A$.*
3. *A is central simple over F if and only if $exp(A) = \dim_F A$.*

The previous theorem is a strong result in PI-theory and one can find the proof, for instance, in [9].

Example 1.7.6 An important class of finite dimensional algebras is given by the upper block triangular matrix algebra $UT(d_1, \ldots, d_m)$, that is

$$UT(d_1, \ldots, d_m) = \begin{pmatrix} M_{d_1}(F) & B_{12} & \ldots & B_{1m} \\ 0 & M_{d_2}(F) & \ldots & B_{2m} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & M_{d_m}(F) \end{pmatrix}$$

where all the B_{ij}'s are rectangular matrices over F of corresponding size. Then $UT(d_1, \ldots, d_m) \cong M_{d_1}(F) \oplus \ldots \oplus M_{d_m}(F) + J$ where $\bigoplus_{i,j} B_{ij} \cong J$ is the Jacobson radical. In order to compute the exponent we may assume that F is algebraically closed. Then, since in this case,

$$M_{d_1}(F) B_{12} M_{d_2}(F) B_{23} \cdots B_{m-1,m} M_{d_m}(F) \neq 0,$$

we obtain that

$$exp(UT(d_1, \ldots, d_m)) = \dim M_{d_1}(F) + \ldots + \dim M_{d_m}(F) = d_1^2 + \ldots + d_m^2.$$

Example 1.7.7 By Theorem 1.7.5 is clear that $exp(M_k(F)) = k^2$. Moreover, $exp(M_k(G)) = 2k^2$, where G is the Grassmann algebra.

It is clear that the definition and the properties of the PI-exponent can be translated in the case of a G-graded algebra, in a naturally way.

1.8 The Specht Problem

One of the most important argument about polynomial identities, concerns the investigation on the basis of varieties of algebras.

Definition 1.8.1 A variety of algebras $\mathcal{V}$ is called *finitely based* (or has a *finite basis of its polynomial identities*) if $\mathcal{V}$ can be defined by a finite system of polynomial identities. If $\mathcal{V}$ cannot be defined by a finite system of identities, it is called *infinite based.* If all subvarieties of $\mathcal{V}$, including $\mathcal{V}$ itself, are finite based, then $\mathcal{V}$ satisfies the *Specht property.*

This is known as the *Specht Problem*: is every variety of associative, Lie or Jordan algebras finitely based?
In 1987 Kemer gave a solution of the Specht Problem for associative algebras over a field of characteristic 0:

Theorem 1.8.2 (Kemer) *Every variety of associative algebras over a field of characteristic 0 has a finite basis for its polynomial identities.*

We have two main directions in order to analyze the Specht Problem:

(i) To show that some varieties satisfy the Specht Problem;

(ii) To construct counterexamples to the Specht problem, i.e. examples of varieties which have finite basis for their identities.

1.9 Proper Polynomial Identities

Now we introduce a special kind of polynomial identities that will help us in order to solve many problems concerning the computation of the basis of some T-ideals.

Definition 1.9.1 A polynomial $f \in F\langle X\rangle$ is called *proper polynomial* (or *commutator polynomial*), if it is a linear combination of products of commutators

$$f(x_1, \ldots, x_n) = \sum \alpha_{i,\ldots,j}[x_{i_1}, \ldots, x_{i_p}] \cdots [x_{j_1}, \ldots, x_{j_q}]$$

where $\alpha_{i,\ldots,j} \in F$.

Here, with $[x_1, \ldots, x_n]$ we mean the Lie commutator with left normalized brackets, so:

$$[x_1, \ldots, x_n] = [\ldots[[[x_1, x_2], x_3], \ldots,], x_n].$$

We denote by B the set of all proper polynomials in $F\langle X\rangle$ and similarly

$$B_m = B \cap F\langle x_1, \ldots, x_m\rangle,\ m = 1, 2, \ldots \text{ and } \Gamma_n = B \cap P_n,\ n = 0, 1, 2, \ldots$$

respectively the set of the proper polynomials in m variables and the set of all proper multilinear polynomials of degree n.

Similarly to the case of ordinary and graded identities, we can define the proper codimension sequence as follows.

Definition 1.9.2 Let A be a unitary PI-algebra over a field of characteristic 0. We introduce the *vector subspace of proper polynomial identities* for A

$$\Gamma_n(A) = \frac{\Gamma_n}{\Gamma_n \cap Id(A)},$$

and the *proper codimension sequence*

$$\gamma_n(A) = \dim \Gamma_n(A),\ n = 0, 1, 2, \ldots$$

The special role that proper polynomials play in PI-theory, is underlined by the theorems that follow.

Proposition 1.9.3 *If A is a unitary PI-algebra over an infinite field F, then all polynomial identities of A follow from the proper ones. Moreover, if* $\operatorname{char} F = 0$ *then the polynomial identities follow from the proper multilinear identities.*

Proof. Let $f(x_1, \ldots, x_m) \equiv 0$ be a polynomial identity for A. We may assume that f is homogeneous in each of its variable. We write f in the form

$$f = \sum \alpha_a x_1^{a_1} \dots x_m^{a_m} w_a(x_1, \dots, x_m),\ \alpha_a \in F,$$

where $w_a(x_1, \dots, x_m)$ is a linear combination of

$$[x_{i_1}, x_{i_2}]^b \cdots [x_{l_1}, \dots, x_{l_p}]^c.$$

Clearly, if we replace by 1 a variable in a commutator $[x_{i_1}, \dots, x_{i_p}]$, the commutator vanishes. Since $f(1 + x_1, x_2, \dots, x_m) \equiv 0$ is also a polynomial identity for A, we get that

$$f(1{+}x_1, x_2, \dots, x_m) = \sum \alpha_a \sum_{i=0}^{a_1} \binom{a_1}{i} x_1^i x_2^{a_2} \cdots x_m^{a_m} w_a(x_1, \dots, x_m) \in Id(A).$$

The homogeneous component of minimal degree with respect to x_1 is obtained from the summands with a_1 maximal among those with $\alpha_a \neq 0$. Since the T-ideal $Id(A)$ is homogeneous, we obtain that

$$\sum_{a_1 max} \alpha_a x_2^{a_2} \cdots x_m^{a_m} w_a(x_1, \dots, x_m) \in Id(A),$$

that is $w_a(x_1, \dots, x_m) \in Id(A)$ and this completes the proof. The multilinear part of the statement is also trivial. Starting with any multilinear polynomial identity for A, and doing exactly the same procedure as above, we obtain that the identity follows from some proper identities which are also multilinear. $\square$

Lemma 1.9.4 *Let $f(x_1, \dots, x_n) \in F\langle X\rangle$ be a multihomogeneous polynomial. Then $f \in B$ if and only if $\partial f/\partial x_i = 0$ for all $i = 1, \dots, n$.*

Proof. Observe that $\partial f/\partial x_1$ can be viewed as the multihomogeneous component of $f(x_1 + y, x_2, \dots, x_n)$ of degree 1 in y where we substitute $y = 1$. Denote by $L(X)$ the free Lie algebra freely generated by the set X. Then $L(X)$ is isomorphic to the Lie subalgebra (under the usual commutator operation) of $F\langle X\rangle$ generated by X. Take an ordered basis of $L(X)$ consisting of multihomogeneous elements and such that the elements of X precede all of the remaining basic elements, thus the basis will look like $x_1 < x_2 < \cdots < u_1 < u_2 < \cdots$ where the u_i are multihomogeneous and of degree ≥ 2. But $F\langle X\rangle$ is the universal enveloping algebra of $L(X)$. According to the Poincaré–Birkhoff–Witt theorem (see [6], theorem 1.3.2) one forms a basis of $F\langle X\rangle$ consisting of 1 and all elements $x_{i_1}\cdots x_{i_k}u_{j_1}\cdots u_{j_m}$ where $i_1\leq\cdots\leq i_k$ and $j_1\leq\cdots\leq j_m$. Writing f as a linear combination of these basis elements it is clear that $f\in B$ if and only if f is a linear combination of the $u_{j_1}\cdots u_{j_m}$. But this happens if and only if $\partial f/\partial x_i=0$ since substituting 1 in a commutator vanishes the commutator. $\square$

In order to prove the next theorem, that is the main result of this section, we need to specify the notation as follows. If A is an algebra, for any S subset of $F\langle X\rangle$, we denote with $S(A)$ the image of S under the canonical homomorphisms

$$F\langle X\rangle\to F(A)=var(A)=\frac{F\langle X\rangle}{Id(A)}$$

Theorem 1.9.5 *Let A be a unitary PI-algebra over an infinite field F.*

(i) Let

$$\{w_j(x_1,\ldots,x_m)\mid j=1,2,\ldots\}$$

be a basis of the vector space $B_m(A)$ of the proper polynomials in the relatively free algebra $F_m(A)$ of rank m, i.e.

$$B_m(A)=\frac{F\langle x_1,\ldots,x_m\rangle\cap B}{Id(A)\cap F\langle x_1,\ldots,x_m\rangle\cap B}.$$

Then $F_m(A)$ has a basis

$$\{x_1^{a_1}\cdots x_m^{a_m}w_j(x_1,\ldots,x_m)\mid a_i\geq 0,\ j=0,1,\ldots\}.$$

(ii) If

$$\{u_{jk}(x_1,\ldots,x_k)\mid j=1,2,\ldots,\gamma_k(A)\}$$

is a basis of the vector space $\Gamma_k(A)$ of the proper multilinear polynomials of degree k in $F(A)$, then $P_n(A)$ has a basis consisting of all multilinear polynomials of the form

$$x_{p_1}\cdots x_{p_{n-k}}u_{jk}(x_{q_1},\ldots,x_{q_k}),\ j=1,\ldots,\gamma_k(A),\ k=0,1,2,\ldots,n,$$

such that $p_1<\ldots<p_{n-k}$ and $q_1<\ldots<q_k$.

Proof. (i) Let $w_j'(x_1,\ldots,x_m)\in B_m$ be homogeneous preimages of the element $w_j(x_1,\ldots,x_m)\in B_m(A)$, $j=1,2,\ldots$ We choose an arbitrary homogeneous basis $\{v_k \mid k = 1, 2, \ldots\}$ of $B_m \cap Id(A)$. Then

$$\{w'_j(x_1, \ldots, x_m), v_k \mid j = 1, 2, \ldots, \ k = 1, 2, \ldots\}$$

is a homogeneous basis of B_m. Applying Proposition 1.9.3, we see that $F_m(A)$ is spanned by

$$x_1^{a_1} \cdots x_m^{a_m} w_j(x_1, \ldots, x_m), \ a_i \geq 0, \ j = 1, 2, \ldots$$

and these elements are linearly independent. The proof of (ii) is similar. $\square$

The previous theorem tells that in order to compute a basis for the T-ideal of the polynomial identities for a unitary algebra, we need to study only the proper identities that are, of course, easier to deal with. Moreover, we get the following relationship among proper codimension and ordinary codimension sequence:

Corollary 1.9.6 *The codimension sequence $c_n(A)$ of an algebra A and the corresponding proper codimension $\gamma_k(A)$ are related by the condition*

$$c_n(A) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k(A).$$

Proof. It is sufficient to count the basis elements of $P_n(A)$ given in the first part of Theorem 1.9.5. $\square$

Chapter 2

Jordan Algebras

Jordan algebras arose from the search for an "exceptional" setting for quantum mechanics. In the usual interpretation of quantum mechanics (the so-called "Copenhagen model"), the physical observables are represented by self-adjoint or hermitian matrices (or operators on Hilbert space). The basic operations on matrices or operators are multiplication by a complex scalar, addition, multiplication of matrices (composition of operators), and forming the complex conjugate transpose matrix (adjoint operator). But these underlying matrix operations are not "observable": the scalar multiple of hermitian matrix is not again hermitian unless the scalar is real, the product is not hermitian unless the factor happen to commute, and the adjoint is just the identity map on hermitian matrices.

In 1932 the physicist Pascual Jordan (Hannover, 18 October 1902 – Hamburg, 31 July 1980) proposed a program to *discover a new algebraic setting for quantum mechanics*. He wished to study the intrinsic algebraic properties of hermitian matrices, to capture these properties in formal algebraic properties, and then to see what other possible non-matrix system satisfied these axioms.

So, Jordan algebras were created to illuminate a particular aspect of physics, quantum-mechanical obsevables, but turned out to have illuminating connections with many areas of mathematics. Jordan system arises naturally as "coordinates" for Lie algebras having a grading into 3 parts. The physical investigation turned up one unexpected system, an "exceptional" 27-dimensional simple Jordan algebra, and it was soon recognized that this exceptional Jordan algebra could help us understand the five exceptional

Lie algebras.
Later came surprising applications to differential geometry, first to certain symmetric spaces, the self-dual homogeneous cones in n-real space, and then a deep connection with bounded symmetric domains in complex n-spaces. In these cases the algebraic structure of Jordan algebra system encode the basic geometric information for the associated space or domain. Once more the exceptional geometric spaces turned out to be connected with exceptional Jordan algebras.
Another surprising application of exceptional Jordan algebra was to octonion planes in projective geometry; once these planes were realized in terms of exceptional Jordan algebra, it became possible to describe their automorphisms. For more information about the story and the several applications of Jordan algebras in sciences, see for instance [27].
In this chapter, F will always denote a field of characteristic different from 2.

2.1 Some definitions and examples

We start with the definition of an operator that we will use very frequently.

Definition 2.1.1 Let U be a non-associative algebra over a field F. For all $a, b, c \in U$, we define the *associator* among a, b and c as follows:

$$(a, b, c) = (ab)c - a(bc).$$

The associator measures how far three elements are from associating: a, b, c associate if their associator is zero. In these terms an algebra is associative if all its associators vanish (see Example 1.2.4).

Definition 2.1.2 An algebra J is called *Jordan algebra* if it satisfies the identities

1. Commutative law: $xy = yx$;
2. Jordan identity: $(x^2, y, x) = 0$.

As we saw, Jordan algebras are not associative, but they recover the commutative property. It is also clear that the role that the associators play in Jordan algebras' theory is similar to the one of commutators in Lie algebras' theory.

One can always obtain a Jordan algebra, starting with an associative algebra. In fact, if A is any associative algebra over a field F, $\operatorname{char} F \neq 2$, then let define a new multiplication

$$a \circ b = \frac{1}{2}(ab + ba).$$

After replacing the old multiplication with the new one, a new algebra is obtained which is denoted by $A^{(+)}$. It is easy to verify that $A^{(+)}$ is Jordan. If a Jordan algebra J is a submodule of the algebra A which is closed with respect to the operation $\circ$, the J together with this operation is a subalgebra of $A^{(+)}$, and consequently a Jordan algebra. Such Jordan algebra is called *special*. The subalgebra A_0 of the algebra A generated by the set J is called the *associative enveloping algebra* for J. There's exist also some Jordan algebras that are nonspecial: we call them *exceptional*. We note that the algebra $A^{(+)}$ can also be defined for a non-associative algebra A, but in this case it is not a Jordan algebra.

There exists a result due to Shirshov that gives a criterion on special Jordan algebras.

Theorem 2.1.3 (Shirshov) *Every Jordan algebra from two generators is special.*

We remand to [31] in order to find the proof of Shirshov's theorem.

Here, there are two main examples of special Jordan algebras. Let V a vector space over a field F with a symmetric bilinear form $f = f(x, y)$ defined on V. We consider the direct sum $B = F \cdot 1 \oplus V$ of the vector space V and the one-dimensional vector space $F \cdot 1$ with basis 1, and we define a multiplication on B by the rule:

$$(\alpha \cdot 1 + x) \cdot (\beta \cdot 1 + y) = (\alpha\beta + f(x, y)) \cdot 1 + (\beta x + \alpha y),$$

where $\alpha, \beta \in F$ and $x, y \in V$. We note at once that the element 1 is an identity element for the algebra B, and that in view of symmetry of the form f the multiplication on the algebra B is commutative. An easy computation shows that actually B satisfies the Jordan identity, i.e. is a Jordan algebra. It is called the *Jordan algebra of the symmetric bilinear form f*. Remember that if $\dim_F V = n < \infty$, then we denote this algebra with B_n. We'll give more details about B in the next sections.

Now let U be some algebra with an involution $*$. The set $H(U, *) = \{a \in U \mid u = u^*\}$ of symmetric elements, with respect to $*$, is closed under the operation $\circ$, and is a subalgebra of $U^{(+)}$. If U is associative, then $U^{(+)}$ is a Jordan algebra and $H(U, *)$ is a special Jordan algebra. We call such an algebra *Hermitian Jordan algebra*. An important special case of Hermitian Jordan algebra is when U is the algebra $M_n(F)$ of $n \times n$ matrices over the field F. In fact, we can define two different involutions:

- the *transpose* involution t: if $A \in M_n(F)$ then A^t is the usual transpose matrix of A.
- the *symplectic* involution s: if $A \in M_{2n}(F)$ then A^s is induced by the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

In this way, we get two new special Jordan algebras, i.e. $H(M_n(F), t)$ and $H(M_n(F), s)$, respectively the symmetric matrices under transpose involution and the symmetric matrices under symplectic involution. It is clear that

$$H(M_n(F), t) = \{A = (a_{ij})_{i,j} \in M_n(F) \mid a_{ij} = a_{ji} \text{ for all } i, j = 1, \ldots, n\}.$$

Moreover,remark that a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n}(F)$, where $A, B, C, D \in M_n(F)$, is symplectic if it satisfies

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \cdot \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

By an easy computation, it turns out that

$$H(M_{2n}(F), s) = \left\{ \begin{pmatrix} A & B \\ C & A^t \end{pmatrix} \mid A, B, C \in M_n(F),\ B = -B^t,\ C = -C^t \right\}.$$

The following theorem classifies the class of simple Jordan algebras:

Theorem 2.1.4 *Let F be a field algebraically closed and let J be finite dimensional simple Jordan algebra. Then J is isomorphic to one of the* *followings:*

1. $M_n(F)^{(+)}$;
2. $(M_n(F), t)$;
3. $(M_{2n}(F), s)$;
4. B_n, *with* $n \geq 2$ *and* f *non-degenerate.*

We refer the reader to [15] for more details about simple Jordan algebras and their classification. Furthermore, in [11] and in [13] some properties about codimension growth of a special Jordan algebra, are given:

Theorem 2.1.5 *Let* J *be a finite dimensional simple Jordan algebra over a field of characteristic zero. Then* $exp(J)$ *exists and equals the dimension of* J *over its center.*

The following result generalizes the previous theorem:

Theorem 2.1.6 *Let* J *be a finite dimensional Jordan algebra over a field of characteristic zero. Then* $exp(J)$ *exists and is a non-negative integer. Moreover if* J *is nilpotent, then* $exp(J) = 0$*. If* J *is not nilpotent, then either* $exp(J) \geq 2$ *or* $exp(J) = 1$*, and* $c_n(J)$ *is polynomially bounded.*

Moreover, in [12] Giambruno and Zaicev compared the exponent of a special Jordan algebra with one of its associative enveloping.

Theorem 2.1.7 *If* A *is a finitely generated PI-algebra, then* $exp(A^{(+)}) = exp(A)$*.*

2.2 Free Special Jordan Algebra

In this section we introduce the analogue of $F\langle X\rangle$ for non associative variables and we give a relationship among polynomials, written using the ordinary associative multiplication, and the so-called Jordan polynomials, i.e. polynomials in which the multiplication is the Jordan multiplication.

Consider the free associative algebra $F\langle X\rangle$ from the set of free generators $\{x_1, x_2, \ldots\}$. We shall call the subalgebra generated by the set X in the algebra $F\langle X\rangle^{(+)}$ the *free special Jordan algebra from the set of free generators*

X, and we shall denote it by $SJ(X)$. As we shall see at the end of the section, special Jordan algebras do not form a variety since for $|X| \geq 3$ there exist homomorphic images of the algebra $SJ(X)$ which are exceptional algebras. However, all special Jordan algebras from a set of generators with cardinality n are homomorphic images of the algebra $SJ(X)$ where $|X| = n$. It is also in this sense that we speak about the freedom of the algebras $SJ(X)$.

Proposition 2.2.1 *Let J be a special Jordan algebra. Then every mapping of X into J can be uniquely extended to a homomorphism of $SJ(X)$ into J.*

Proof. Let $\sigma : X \to J$ be some mapping and A be an associative enveloping algebra for J. Then the mapping σ can be extended to an homomorphism $\overline{\sigma} : F\langle X\rangle \to A$. It is now clear that the restriction of $\overline{\sigma}$ to $SJ(X)$ is a homomorphism of $SJ(X)$ into J which extends σ. This proves the proposition. $\square$

Definition 2.2.2 An element of the free associative algebra $F\langle X\rangle$ is called *Jordan polynomial* (or *j-polynomial*) if it belongs to $SJ(X)$, that is, can be expressed from the elements of the set X by means of the operations $+$ and $\circ$.

There is still not a single appropriate criterion known that enables one to recognize from the expression of a polynomial in $F\langle X\rangle$ whether or not it is a j-polynomial. Cohn's theorem, which will be formulated later, gives such a criterion for $|X| = 3$.

There is yet one more Jordan algebra connected with the free associative algebra $F\langle X\rangle$. For this we define an involution $*$ on $F\langle X\rangle$, which is defined on monomials by the rule:

$$(x_{i_1}x_{i_2}\cdots x_{i_k})^* = x_{i_k}\cdots x_{i_2}x_{i_1},$$

and which is extended to polynomials by linearity: if $f = \sum \alpha_s u_s$ where $\alpha_s \in F$ and u_s are monomials, then $f^* = \sum \alpha_s u_s^*$. We denote by $H(X)$ the Jordan algebra $H(F\langle X\rangle, *)$ of symmetric elements with respect to $*$ of the algebra $F\langle X\rangle$.

First we give this technical result:

Lemma 2.2.3 *Let A an associative algebra, then the following equation holds:*

$$\frac{1}{2}(abc + cba) = (a \circ b) \circ c + (b \circ c) \circ a - (c \circ a) \circ b.$$

Proof. The proof consists of a straightforward and easy computation and we omit it. $\square$

Theorem 2.2.4 (Cohn) *The containment $H(X) \supseteq SJ(X)$ is valid for any set X. Moreover, the equality holds for $|X| = 3$, but this is a strict containment for $|X| > 3$.*

Proof. The generators $x_\alpha \in X$ are symmetric with respect to $*$ and are in $H(X)$. In addiction, if $a, b \in H(X)$, then also $a \circ b \in H(X)$ since

$$(a \circ b)^* = \frac{1}{2}[(ab)^* + (ba)^*] = \frac{1}{2}(b^*a^* + a^*b^*) = \frac{1}{2}(ab + ba) = a \circ b.$$

Therefore $H(X) \supseteq SJ(X)$.
Let $|X| = 3$, that is $X = \{x_1, x_2, x_3\}$. For the proof of the fact that $H(X) = SJ(X)$, it suffices to show that elements from $H(X)$ of the form $u+u^*$, where u is a monomial, are j-polynomials. In fact, if $f = \sum u_i$ is symmetric with respect to $*$, then $f = \frac{1}{2}(f + f^*) = \frac{1}{2}\sum(u_i + u_i^*)$.
Let $u = x_{i_1}^{l_1} x_{i_2}^{l_2} \cdots x_{i_h}^{l_h}$, where $i_r \neq i_{r+1}$ for $r = 1, \dots, h-1$. We call the number h the *height* of the monomial u.
To prove that $u + u^* \in SJ(X)$, we carry out a double induction. The first induction is on the length of the monomial u, with the initial case being obvious. Assume the insertion has been proved for monomials of length $< k$. We consider the set of monomials of height 1. These are x_1^k, x_2^k and x_3^k. For them our assertion is obvious, and so we have a basis for a second induction. Assume that the assertion is valid for all monomials of length k and height $< h$. We first consider the case when a monomial u begins and ends with the same generating element, namely $u = a^p v a^q$ where $a \in \{x_1, x_2, x_3\}$ and v is a monomial. In this case

$$\begin{aligned} u + u^* &= a^p v a^q + a^q v a^p \\ &= a^p(va^q + a^q v^*) + (a^q v^* + va^q)a^p - (a^{p+q}v^* - va^{p+q}). \end{aligned}$$

We observe that the sum of the first two summands is a j-polynomial by our first induction assumption, and since the monomial $a^{p+q}v^*$ has height $h-1$,

the last summand is a j-polynomial by our second assumption.
A second case which it is necessary to consider is the following: $u = a^p b^r a^q v$ where $a, b \in \{x_1, x_2, x_3\}$, $a \neq b$ and v is a monomial. We have

$$\begin{aligned} u + u^* &= a^p b^r a^q v + v^* a^q b^r a^p \\ &= a^p b^r (a^q v + v^* a^q) + (v^* a^q + a^q v) b^r a^p - (a^p b^r v^* a^q + a^q v b^r a^p), \end{aligned}$$

where in the view of Lemma 2.2.3 the sum of the first two summands is a Jordan polynomial, and in the last parentheses we have our first case.
A third case is $u = a^p v a^q b^r$, where $a, b \in \{x_1, x_2, x_3\}$, $a \neq b$, and v is a monomial. In this case

$$\begin{aligned} u + u^* &= a^p v a^q b^r + b^r a^q v^* a^p \\ &= a^p (v a^q + a^q v^*) b^r + b^r (a^q v^* + v a^q) a^p - (a^{p+q} v^* b^r + b^r v a^{p+q}). \end{aligned}$$

In view of Lemma 2.2.3 and our first induction assumption, the sum of the first two terms is a j-polynomial, while the subtracted term satisfies the condition of our second induction assumption.
It only remains to consider as a fourth case two possibilities for the monomial u: $u = a^p b^r c^s$ and $u = a^p c^q b^t v a^l c^r b^s$, where v is a monomial which is possibly missing, and $a, b, c \in \{x_1, x_2, x_3\}$ are pairwise distinct. If $u = a^p b^r c^s$, then by Lemma 2.2.3 $u + u^* = a^p b^r c^s + c^s b^r a^p \in SJ(X)$. Now let $u = a^p c^q b^t v a^l c^r b^s$. Then

$$\begin{aligned} u + u^* &= a^p c^q b^t v a^l c^r b^s + b^s c^r a^l v^* b^t c^q a^p \\ &= a^p (c^q b^t v a^l c^r + c^r a^l v^* b^t c^q) b^s + b^s (c^r a^l v^* b^t c^q + c^q b^t v a^l c^r) a^p \\ &- (a^p c^r a^l v^* b^t c^q b^s + b^s c^q b^t v a^l c^r a^p). \end{aligned}$$

Because of Lemma 2.2.3 and the first induction assumption, the sum of the first two terms is a j-polynomial, and by already considered second case, the third is likewise a j-polynomial.
Thus the equality $H(X) = SJ(X)$ is proved in the case when $|X| = 3$. Of course from this follows the validity of the equality $H(X) = SJ(X)$ for $|X| = 2$.
We shall now prove that $H(X) \neq SJ(X)$ for $|X| > 3$. For this it suffices to show that the element $f(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4 + x_4 x_3 x_2 x_1$ is not a Jordan polynomial.

We consider the free F-module E from generators e_1, e_2, e_3, e_4 and the exterior algebra $\bigwedge(E)$ of this module (for its definition and properties see [23]). The F-module of the algebra $\bigwedge(E)$ is also free with basis $\{1, e_{i_1} \wedge \ldots \wedge e_{i_s} \mid i_1 < \ldots < i_s;\ s = 1, 2, 3, 4\}$ and in addition $e_i \wedge e_j = 0$ in $\bigwedge(E)$. Let $\sigma : F\langle X\rangle \to \bigwedge(E)$ be the homomorphism such that $\sigma(x_i) = e_i$ for $i = 1, 2, 3, 4$, and $\sigma(x_i) = 0$ for all $i > 4$. All Jordan polynomials are mapped to zero under this homomorphism. However, $\sigma[f(x_1, x_2, x_3, x_4)] = 2e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0$. This means that $f(x_1, x_2, x_3, x_4)$ is not a Jordan polynomial, and therefore the theorem is completely proved. $\square$

Theorem 2.2.5 *If $|X| > 1$, then the free Jordan algebra $SJ(X)$ is not isomorphic to an algebra $A^{(+)}$ for any associative algebra A.*

Proof. Let us assume that $SJ(X)$ is isomorphic to an algebra $A^{(+)}$, where A is an associative algebra. We can then assume that an associative operation (denoted by a dot) which distributes with addition is defined on the set $SJ(X)$, such that for any $a, b \in SJ(X)$

$$a \circ b = \frac{1}{2}(a \cdot b + b \cdot a), \tag{2.1}$$

and that $A = \langle SJ(X), +, \cdot\rangle$. By (2.1) the set X is also a set of generators for A. Let $\sigma : F\langle X\rangle \to A$ be the homomorphism such that $\sigma(x_\alpha) = x_\alpha$ for all $x_\alpha \in X$. We consider the kernel I of this homomorphism. By (2.1) the restriction of σ to $SJ(X)$ is the identity mapping. Therefore

$$I \cap SJ(X) = (0). \tag{2.2}$$

In addition, for any $f(x) \in F\langle X\rangle$ there can be found a $j(x) \in SJ(X)$ such that $f(x) - j(x) \in I$. In particular, this is true for the polynomial $f(x) = x_1x_2$. But then by Lemma 2.2.3

$$[x_1x_2 - j(x)][x_2x_1 - j(x)] = x_1x_2^2x_3 - x_1x_2j(x) - j(x)x_2x_1 + j(x)^2 \in I \cap SJ(X).$$

By (2.2) this means $[x_1x_2j(x)][x_2x_1 - j(x)] = 0$, and so we obtain that either $j(x) = x_1x_2$ or $j(x) = x_2x_1$. But neither one is possible, which proves the theorem. $\square$

We turn now to the study of homomorphic images of free special Jordan algebras.

Lemma 2.2.6 *Let I be an ideal of a special Jordan algebra J with associative enveloping algebra A, and let $\widehat{I}$ be the ideal of the algebra A generated by the set I where*
$$\widehat{I} \cap J = I.$$
Then the quotient algebra J/I is special.

Proof. We first note an obvious isomorphism: for any ideal B of the algebra A
$$\left(\frac{A}{B}\right)^{+} \cong \frac{A^{(+)}}{B^{(+)}}.$$
Then by the second homomorphism theorem
$$\frac{J}{I} = \frac{J}{J \cap \widehat{I}^{(+)}} \cong \frac{J + \widehat{I}^{(+)}}{\widehat{I}^{(+)}} \subseteq \frac{A^{(+)}}{\widehat{I}^{(+)}},$$
which by our previous remark gives us that the algebra $\frac{J}{I}$ is special. □

Lemma 2.2.7 (Cohn) *Let I be an ideal of the free special Jordan algebra $SJ(X)$ and $\widehat{I}$ be the ideal generated by the set I in $F\langle X\rangle$. The quotient algebra $SJ(X)/I$ is special if and only if $\widehat{I} \cap SJ(X) = I$.*

Proof. If the condition $\widehat{I} \cap SJ(X) = I$ is satisfied, then the quotient algebra $SJ(X)/I$ is special by Lemma 2.2.6. Now let assume that the quotient algebra $SJ(X)/I$ is special and that A is an associative enveloping algebra for it. We denote by τ the canonical homomorphism of the algebra $SJ(X)$ onto $SJ(X)/I$. The algebra A is generated by the elements $a_\alpha = \tau(x_\alpha)$, where $x_\alpha \in X$. Let σ the homomorphism of the algebra $F\langle X\rangle$ onto A such that $\sigma(x_\alpha) = a_\alpha$. Then the restriction of σ to $SJ(X)$ is a homomorphism of $SJ(X)$ to $SJ(X)/I$ which coincides with τ on the generators, and consequently equals τ. But then $\ker\sigma \cap SJ(X) = \ker\tau$. We now note that $\widehat{I} \subseteq \ker\sigma$ and $\ker\tau = I$. Consequently $\widehat{I} \cap SJ(X) \subseteq I$, and since the reverse containment is obvious, we have that $\widehat{I} \cap SJ(X) = I$. This proves the lemma. □

The following theorem gives an example of exceptional Jordan algebra.

Theorem 2.2.8 (Cohn) *Let $SJ(x, y, z)$ be the free special Jordan algebra from generators x, y and z, and let I be the ideal in $SJ(x, y, z)$ generated* *by the element $k = x^2 - y^2$. Then the quotient algebra $SJ(x, y, z)/I$ is exceptional.*

Proof. The element $v = kxyz + zyxk$ is in $\widehat{I} \cap SJ(x, y, z)$, where $\widehat{I}$ is the ideal of the algebra $F\langle X\rangle$ generated by the set I. We shall prove that $v \notin I$, and then by Lemma 2.2.7 everything will be proved. Assume that $v \in I$. Then there exists a Jordan polynomial $j(x, y, z, t)$, each monomial of which contains t, such that $v = (x, y, z, k)$. It is clear we can consider that all of the monomials in $j(x, y, z, t)$ have degree 4 and are linear in z. Comparing the degrees of the element v and $j(x, y, z, x^2-y^2)$ in the separate variables, we conclude that $j(x, y, z, t)$ is linear in t, and consequently in the other variables as well. The polynomial $j(x, y, z, t)$ is symmetric and is in $H(x, y, z, t)$. Therefore it is a linear combination of the 24 quadruples of the form $\{xyzt\} := xyzt + tzyx$ with all permutation of the elements x, y, z, t. However, the form of the element v indicates that in this linear combination there are non-zero coefficients only for those quadruples which end (or begin) with a z:

$$\begin{aligned} j(x, y, z, t) = {} & \alpha_1\{txyz\} + \alpha_2\{xtyz\} + \alpha_3\{tyxz\} \\ & + \alpha_4\{ytxz\} + \alpha_5\{xytz\} + \alpha_6\{yxtz\}. \end{aligned}$$

Substituting $x^2 - y^2$ for t in this equality, we obtain the relation

$$\begin{aligned} \{x^3yz\} - \{y^2xyz\} = {} & \alpha_1\{x^3yz\} - \alpha_1\{y^2xyz\} + \alpha_2\{x^2yz\} \\ & - \alpha_2\{xy^3z\} + \alpha_3\{x^2yxz\} - \alpha_3\{y^3xz\} \\ & + \alpha_4\{yx^3z\} - \alpha_4\{y^3xz\} + \alpha_5\{xy^3z\} \\ & - \alpha_5\{yx^3z\} + \alpha_6\{yx^3z\} - \alpha_6\{yxy^2z\}. \end{aligned}$$

Comparing the coefficients for $\{y^2xyz\}$, we conclude that $\alpha_1 = 1$. We next compare the coefficients for $\{x^3yz\}$ and obtain $\alpha_2 = 0$. Comparison of the coefficients for $\{x^2yxz\}$ gives us $\alpha_3 = 0$. Hence, it follows that as the coefficient for $\{y^3xz\}$, $\alpha_4 = 0$. Furthermore, comparing the coefficients for $\{xyx^2z\}$ and for $\{yxy^2z\}$, we obtain that $\alpha_5 = \alpha_6 = 0$. But this means that

$$j(x, y, z, t) = \{xyzt\}.$$

This is a contradiction because, as we saw in the proof of Theorem 2.2.4,

$\{xyzt\}$ is not a Jordan polynomial. This proves the theorem. □

Theorem 2.2.9 *All homomorphic images of the free special Jordan algebra $SJ(x,y)$ from two generators are special.*

Proof. Let I be an arbitrary ideal of the algebra $SJ(x,y)$ and $\widehat{I}$ be the ideal generated by the set I in $F\langle X\rangle$. By Lemma 2.2.7 it suffices to show that $\widehat{I} \cap SJ(x,y) = I$. Let $u \in \widehat{I}$. Then $u = \sum v_i k_i w_i$, where $k_i \in I$ and v_i, w_i are monomials. Let us assume that $u \in SJ(x,y)$. For the proof that $u \in I$, it suffices to show the inclusion $vkw + w^*kv^* \in I$ is valid for all $k \in I$ and any monomials v and w. But this inclusion, in fact, holds, since by Theorem 2.2.4 $vzw + w^*zv^*$ is a Jordan polynomial from x, y, z. This proves the theorem. □

In order to prove the last property of free special Jordan algebras, we need first to prove two technical results.

Lemma 2.2.10 *In the free associative algebra $F\langle x,y\rangle$, an elements belongs to the subalgebra $\langle Z\rangle$ generated by the set $Z = \{z_i = xy^ix \mid i \geq 1\}$ if and only if it is a linear combination of monomials of the form*

$$xy^{j_1}x^2y^{j_2}x^2 \cdots x^2y^{j_n}x,\ j_k \geq 1. \tag{2.3}$$

This subalgebra is a free associative algebra with set of free generators Z.

Proof. The proof of the first assertion of the lemma is obvious. For the second assertion we consider the homomorphism

$$\varphi : F\langle x_1, \ldots, x_n, \ldots\rangle \to \langle Z\rangle$$

sending x_i to z_i. It is easy to see that φ is an isomorphism, since

$$f(z_1, \ldots, z_n) = 0$$

implies $f = 0$. We shall now denote the algebra $\langle Z\rangle$ by $F(Z)$. □

Lemma 2.2.11 *Let I be an ideal of the algebra $F(Z)$ and $\widehat{I}$ be the ideal generated by the set I in $F\langle x,y\rangle$. Then*

$$\widehat{I} \cap F(Z) = I.$$

Proof. Every element u from $\widehat{I}$ is representable in the form

$$u = \sum_i v_i k_i w_i, \tag{2.4}$$

where $k_i \in I$ and v_i, w_i are monomials. If $u \in F(Z)$, then u is linear combination of monomials of the form (2.3). The elements k_i can also be represented analogously. We note that if k and vkw are monomials of the indicated form, then the monomials v, w likewise have this same form. Therefore, if $u \in F(Z)$, then all the terms $v_i k_i w_i$ on the right side of equality (2.4) for which either v_i or w_i does not belong to $F(Z)$ must cancel. But in this case $u \in I$. This proves the lemma. $\square$

Finally we give the following

Theorem 2.2.12 (Shirshov) *Every special Jordan algebra with not more than a countable number of generators is imbeddable in a special Jordan algebra with two generators.*

Proof. Let J be a special Jordan algebra as in the hypothesis of the theorem. Then J is isomorphic to some quotient algebra $SJ(Z)/I$ of the algebra $SJ(Z)$. We note that by Lemma 2.2.7 the relation

$$\widehat{I} \cap SJ(Z) = I$$

holds for the ideal $\widehat{I}$ of the algebra $F(Z)$ generated by the set I. We now consider the ideal $\widetilde{\widehat{I}}$ of the algebra $F\langle x, y\rangle$ generated by the set $\widehat{I}$. In view of the Lemma 2.2.11

$$\widetilde{\widehat{I}} \cap F(Z) = \widehat{I},$$

whence $SJ(Z) \cap \widetilde{\widehat{I}} = SJ(Z) \cap (F(Z) \cap \widetilde{\widehat{I}}) = SJ(Z) \cap \widetilde{\widehat{I}} = I$. The image of the algebra $SJ(Z)$ in the quotient algebra $\overline{A} = F\langle x, y\rangle/\widetilde{\widehat{I}}$ is the subalgebra $SJ(Z)/(SJ(Z) \cap \widetilde{\widehat{I}}) = SJ(Z)/I$ isomorphic to J. Since $xy^i x = 2(y^i \circ x) \circ x - y^i \circ x^2$, this subalgebra is in the subalgebra of the algebra $\overline{A}^{(+)}$ generated by the elements $\overline{x} = x + \widetilde{\widehat{I}}$ and $\overline{y} = y + \widetilde{\widehat{I}}$.
This proves the theorem. $\square$

2.3 The Jordan algebra of a symmetric bilinear form

In this section we make a quick description of the Jordan algebra B that we defined in Section 2.1. Recall the definition: let V be a vector space, over a field F, equipped with a symmetric bilinear form $f(x, y)$, and let $B = F \oplus V$. Define a multiplication on B by

$$(\alpha \cdot 1 + x) \cdot (\beta \cdot 1 + y) = (\alpha\beta + f(x, y)) \cdot 1 + (\beta x + \alpha y),$$

where $\alpha, \beta \in F$ and $x, y \in V$. Then B is a Jordan algebra. Moreover, if $\dim_F V = n < \infty$, we shall denote it by B_n.

Definition 2.3.1 Let V be a vector space over a field F. For any non-negative integer k, we define the k^{th} *tensor power of* V to be the tensor product of V with itself k times:

$$T^k V = V^{\otimes k} = V \otimes \ldots \otimes V.$$

Moreover, the *tensor algebra* $T(V)$ is defined as the direct sum of $T^k V$ for $k = 0, 1, 2, \ldots$

$$T(V) = \bigoplus_{k=0}^{\infty} T^k V.$$

Definition 2.3.2 Let V a vector space over a field F and let q a quadratic form defined over V. The *Clifford algebra* $C(V, q)$ is defined as the quotient of the tensor algebra $T(V)$ modulo the two-sided ideal I generated by the elements of the form

$$v \otimes v - q(v) \cdot 1, \text{ for all } v \in V.$$

If the dimension of V is n and $\{e_1, \ldots, e_n\}$ is a basis of V over F, then the set

$$\{e_{i_1} e_{i_2} \cdots e_{i_k} \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq n \text{ and } k = 0, \ldots, n\}$$

is a basis for the Clifford algebra $C(V,q)$. As a consequence, we get that

$$\dim_F C(V,q) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

The relevance of Clifford algebras is justified by the following theorem, a proof of which is given for instance in [23].

Theorem 2.3.3 *The Jordan algebra $B = F \oplus V$ of symmetric bilinear form f is special. In particular its associative enveloping is the Clifford algebra $C(V,q)$, where q is the quadratic form induced by f.*

What about polynomial identities? In a paper due to Vasilovsky (see [37]), a basis for the T-ideal of the polynomial identities for $var(B)$ is computed. The following theorems hold:

Theorem 2.3.4 *Let $B = F \oplus V$, with $dim_F V = \infty$, be the Jordan algebra of the symmetric bilinear form f. The identities*

$$([x,y]^2, z, t) \equiv 0, \tag{2.5}$$

$$\sum_{\sigma \in S_3} \operatorname{sgn} \sigma (x_{\sigma(1)}, (x_{\sigma(2)}, x, x_{\sigma(3)}), x) \equiv 0 \tag{2.6}$$

form a basis for polynomial identities of the variety $var(B)$ of Jordan algebras over an infinite field of characteristic different from 2,3,5 and 7.

Theorem 2.3.5 *Let $B_n = F \oplus V$ with $dim_F V = n < \infty$ be the Jordan algebra of the symmetric bilinear form f. The identities (2.5), (2.6),*

$$\sum_{\sigma \in S_{n+1}} \operatorname{sgn} \sigma (x_{\sigma(1)}, y_1, x_{\sigma(2)}, \ldots, y_n, x_{\sigma(n+1)}) \equiv 0, \tag{2.7}$$

$$\sum_{\sigma \in S_{n+1}} \operatorname{sgn} \sigma (x_{\sigma(1)}, y_1, x_{\sigma(2)}, \ldots, y_{n-1}, x_{\sigma(n)})(y_n, x_{\sigma(n+1)}, y_{n+1}) \equiv 0 \tag{2.8}$$

form a basis for polynomial identities of the variety $var(B_n)$ of Jordan algebras over an infinite field of characteristic different from 2,3,5 and 7.

Remark that in the previous theorem, we denote with $(x_1, x_2, \ldots, x_n)$ the so-called *long associators*, where the brackets are left-normalized. Moreover, if the ground field has characteristic zero, then (2.8) follows from (2.5)–(2.7); that is, (2.5)–(2.7) is a minimal basis for identities of Jordan algebra B_n where $3 \leq n < \infty$.

By Theorem 2.3.4 follow two important corollaries.

Corollary 2.3.6 *If the characteristic of the ground field is zero, then the variety $var(B)$ of unitary algebras possesses the Specht property.*

Corollary 2.3.7 *Every Jordan algebra over a field of characteristic zero which satisfies the identities (2.5) and (2.7) is a special one.*

In [5] Drensky gave a complete description of the codimension growth of the varieties generated by B and B_n when the bilinear form is non-degenerate. In particular, he proved a formula for an asymptotical estimation of the codimensions.

Theorem 2.3.8 $c_n(var(B)) = \mathcal{O}((cn)^{\frac{n}{2}})$, *where* $c = \exp(\pi - 1)$.

He also proved that if $n > 1$, i.e B_n is simple, then $exp(var B_n) = n + 1$. As last result of this section, we classify all possible G-grading on B_n.

Theorem 2.3.9 (Bahturin, Shestakov) *Any grading $J = \bigoplus_{g\in G} J_g$ of $J = B_n = F \oplus V$ by a group G over a field F of characteristic different from 2 can be described as follows. There exists a graded basis $\mathcal{B}$ of V, which is disjoint union $\mathcal{B} = \mathcal{E} \cup \mathcal{E}' \cup \mathcal{F}$, and a bijection $\mathcal{E} \ni e \leftrightarrow e' \in \mathcal{E}'$ such that $\deg e = (\deg e')^{-1} \neq 1_G$ for any $e \in \mathcal{E}$ and $(\deg f)^2 = 1_G$ for any $f \in \mathcal{F}$. The sets $\mathcal{E}$ and $\mathcal{E}'$ are dual each other, the duality established by the same mapping $e \leftrightarrow e'$, $\mathcal{F}$ is orthonormal and orthogonal to both $\mathcal{E}$ and $\mathcal{E}'$. Conversely, any choice of a basis as just described and any collection of elements*

$$\{g_e, h_f \mid e \in \mathcal{E},\ f \in \mathcal{F}\} \subset G$$

such that $(g_e)^2 \neq 1_G$ and $(h_f)^2 = 1_G$ defines a grading on J if one sets $\deg e = g_e$, $\deg e' = (g_e)^{-1}$ and $\deg f = h_f$, for all respective $e \in \mathcal{E}$ and $f \in \mathcal{F}$.

Proof. If we start with the choice of the canonical basis and the group elements as above then the verification of all conditions to be satisfied is easy because we have

$$e_1e_2 = (e_1)'(e_2)' = ef = e'f = 0,$$
$$e_1(e_2)' = \begin{cases} 0 & \text{if } e_1 \neq e_2 \\ 1 & \text{otherwise} \end{cases}, \quad f_1f_2 = \begin{cases} 0 & \text{if } f_1 \neq f_2 \\ 1 & \text{otherwise} \end{cases},$$

for $e, e_1, e_2 \in \mathcal{E}$ and $f, f_1, f_2 \in \mathcal{F}$.

Conversely, let us assume that we are given a G-grading on J. Obviously, $1 \in J_{1_G}$. We first have to show that for any $g \neq 1_G$ we have $J_g \subset V$. Let us assume the contrary, that is $x = \lambda + u \in J_g$, $u \in V$ and $g \neq 1_G$. Then $u \neq 0$ and

$$(\lambda + u)^2 \in J_{g^2}, \text{ or } \lambda^2 \cdot 1 + f(u,u) \cdot 1 + 2\lambda u \in J_{g^2}. \tag{2.9}$$

Here $(\lambda^2 + f(u,u)) \cdot 1 \in J_{1_G}$ and $2\lambda u \in J_g$. Then it follows from (2.9) that $2\lambda u \in J_g \cap J_{g^2} = \{0\}$. Hence $\lambda = 0$, a contradiction.

Now let us look at J_{1_G}. Suppose $\lambda + u \in J_{1_G}$. Then also $u \in J_{1_G}$. Pick any $g \neq 1_G$ and suppose $v \in J_g$, $v \neq 0$. Then $uv \in J_g$, but also $uv = f(u,v) \cdot 1 \in J_{1_G}$. It follows that $f(u,v) = 0$ for all $v \in J_g$ with $g \neq 1_G$. Hence we have the following: $J_{1_G} = F \oplus \widetilde{J_{1_G}}$ where $\widetilde{J_{1_G}} \subset V$, and also $\widetilde{J_{1_G}}$ is orthogonal to every J_g, with $g \neq 1_G$. Thus we have an orthogonal decomposition

$$V = \widetilde{J_{1_G}} \oplus \sum_{g \neq 1_G} J_g.$$

One more observation is as follows. If $g, h \in G - \{1_G\}$, $gh \neq 1_G$ then $f(J_g, J_h) = 0$. Since, if $u \in J_g$ and $v \in J_h$ then $uv \in J_{gh}$ and also $uv = f(u,v) \cdot 1 \in J_{1_G}$. Hence $f(u,v) = 0$. Now let us look at the support Σ of J. We have that $\Sigma = \Xi \cup \{1_G\} \cup \Psi$ where the elements of Ξ are not elements of order 2, while the elements of Ψ are all of order 2. We split arbitrarily Ξ into disjoint union $\Xi = \Pi \cup \Pi'$ where the elements of Π' are the inverses of those in Π. Then if we pick, say, $\pi \in \Pi$ we have that J_π is orthogonal to $\widetilde{J_{1_G}}$ and J_g where g is different from π^{-1}. Because f is non-degenerate we must have $J_{\pi^{-1}} \neq \{0\}$ and, moreover, one can choose a basis $\mathcal{E}_\pi$ in J_π and ${\mathcal{E}'}_\pi$ of $J_{\pi^{-1}}$ in bijective correspondence given by $\mathcal{E} \ni e \leftrightarrow e' \in \mathcal{E}'$ such that $f(e_1, (e_2)') = 0$ unless $e_1 = e_2$, in which case the value of the form is 1. Now let us set $\mathcal{E} = \bigcup_{\pi \in \Pi} \mathcal{E}_\pi$ and $\mathcal{E}' = \bigcup_{\pi \in \Pi} {\mathcal{E}'}_\pi$. Thus we have constructed the first two portions of the required basis $\mathcal{B} = \mathcal{E} \cup \mathcal{E}' \cup \mathcal{F}$.

To construct $\mathcal{F}$ let us pick $\psi = 1_G$ or $\psi \in \Psi$. That is, with $\psi^2 \neq 1_G$. Then the restriction of f to $\widetilde{J_{1_G}}$ or to J_ψ must be non-degenerate. Choose an orthonormal basis inside each J_ψ or $\widetilde{J_{1_G}}$, and their union $\mathcal{F}$ will be the remaining portion of the desired basis for J. The proof is now complete. $\square$

Chapter 3

The Associative Upper Triangular Matrices $UT_2(F)$

The role of $UT_2(F)$ in PI-theory is highlighted in the study of the codimension growth of a PI-algebra. In fact, given an algebra A, by Theorem 1.7.4, it follows that $c_n(A)$ is polynomially bounded if and only if $exp(A) \leq 1$: in this case, we say that A has *polynomial growth*, otherwise A has *exponential growth*. Moreover, a variety $\mathcal{V}$ has *almost polynomial growth* if it has exponential growth but any proper subvariety of $\mathcal{V}$ has polynomial growth. A study made in [16], states that in the associative case, a variety $\mathcal{V}$ has polynomial growth if and only if the Grassmann algebra and $UT_2(F)$ do not lie in $\mathcal{V}$. Furthermore in [9] a classification of varieties of almost polynomial growth is given, in fact it holds the theorem that claims that $var(G)$ and $var(UT_2(F))$ are the only varieties of almost polynomial growth.

In order to give a complete description of $UT_2(F)$, we shall talk about the properties and the polynomial identities in the associative case, while in the next chapter we will talk about the analogous stuffs in the Jordan case. Moreover, to simplify the notation, throughout this chapter we assume that $UT_2(F)$ is the associative algebra of 2×2 upper triangular matrices over F, while $UJ_2(F)$ is the corresponding special Jordan algebra.

3.1 The ordinary identities of $UT_2(F)$

First of all, let compute a basis for the T-ideal of ordinary polynomial identities of $UT_2(F)$. The following theorem gives a basis for the general case of $UT_n(F)$.

Theorem 3.1.1 *Let F be any infinite field and $UT_n(F)$ the associative algebra of $n \times n$ upper triangular matrices.*

(i) The polynomial identity

$$[x_1, x_2] \cdots [x_{2n-1}, x_{2n}] \equiv 0$$

forms a basis of the polynomial identities of $UT_n(F)$.

(ii) The relatively free algebra $F\langle X\rangle/Id(UT_n(F))$ has a basis consisting of all products

$$x_1^{a_1} \cdots x_m^{a_m} [x_{i_{11}}, x_{i_{22}}, \ldots, x_{i_{p_1 1}}] \cdots [x_{i_{1r}}, x_{i_{2r}}, \ldots, x_{i_{p_r r}}],$$

where the number r of participating commutators is $\leq n-1$ and the indices in each commutator $[x_{i_{1s}}, x_{i_{2s}}, \ldots, x_{i_{p_s s}}]$ satisfy the relation $i_{1s} > i_{2s} \leq \ldots \leq i_{p_s s}$.

Proof. We have seen in Example 1.1.6 that $UT_n(F)$ satisfies the polynomial identity

$$[x_1, x_2] \cdots [x_{2n-1}, x_{2n}] \equiv 0 \tag{3.1}$$

and we shall show that all other polynomial identities for $UT_n(F)$ follow from this identity. We shall se that modulo this identity every element of $F\langle X\rangle$ is a linear combination of the elements from part (ii) of the theorem and that any nontrivial linear combination of these elements does not vanish on the algebra $UT_n(F)$. According to Theorem 1.9.5, we shall consider proper polynomial identities only. It is also convenient to work in the relatively free algebra

$$F(\mathcal{T}_n) = F\langle X\rangle/I$$

where $I = \langle [x_1, x_2] \cdots [x_{2n-1}, x_{2n}]\rangle_T$ is the T-ideal generated by the identity (3.1). For simplicity of the exposition we give the proof for the cases

$n=2$ and $n=3$ only. The general case is similar.
Let $n=2$. Then in $F(\mathcal{T}_2)$

$$[x_1, x_2][x_3, x_4] \equiv 0$$

and $B(\mathcal{T}_2)$ is spanned by 1 and by all commutators

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_k}],\ k \geq 2.$$

Using the identity

$$\begin{aligned} 0 \equiv [x_1, x_2][x_3, x_4] - [x_3, x_4][x_1, x_2] &= [[x_1, x_2], [x_3, x_3]] \\ &= [x_1, x_2, x_3, x_4] - [x_1, x_2, x_4, x_3], \end{aligned}$$

we see that in $F(\mathcal{T}_n)$

$$[y_1, y_2, x_{\sigma(1)}, \ldots, x_{\sigma(p)}] = [y_1, y_2, x_1, \ldots, x_p],\ \sigma \in S_p.$$

Additionally, the Jacobi identity and the anticommutativity regarding Lie's commutators, allow to change the places of the variables in the first three positions:

$$\begin{aligned} &[x_1, x_2] = -[x_1, x_2], \\ &[x_3, x_2, x_1] = [x_3, x_1, x_2] - [x_2, x_1, x_3]. \end{aligned}$$

In this way, we can assume that $B(\mathcal{T}_2)$ is spanned by 1 and

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_k}],\ i_1 > i_2 \leq \ldots \leq i_k.$$

We shall show that these elements are linearly independent modulo $Id(UT_2)$.
Let

$$f(x_1, \ldots, x_m) = \sum_i \alpha_i [x_{i_1}, x_{i_2}, \ldots, x_{i_k}] \equiv 0,\ i_1 > i_2 \leq \ldots \leq i_k,\ \alpha_i \in F,$$

be a nontrivial polynomial identity for $UT_2(F)$. We fix i_1 maximal with the property $\alpha_i \neq 0$ and consider the elements

$$\overline{x}_{i_1} = e_{12} + \xi_{i_1} e_{22},$$
$$\overline{x}_j = \xi_j e_{22}$$

where $j \neq i_1$ and $\xi_{i_1}, \xi_j \in F$. Concrete calculations show that

$$f(\overline{x}_1, \ldots, \overline{x}_m) = \left(\sum_{i_1 \text{ fixed}} \alpha_i \xi_{i_2} \ldots \xi_{i_k} \right) \cdot e_{12}$$

which can be chosen different from 0 because the ground field F is infinite. Hence all coefficients α_i are equal to 0 and this complete the proof for $n = 2$. Now let $n = 3$. Then in $F(\mathcal{T}_3)$

$$[x_1, x_2][x_3, x_4][x_5, x_6] \equiv 0$$

and $B(\mathcal{T}_3)$ is spanned by 1 and by all commutators

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_k}], \ k \geq 2$$

and all products of two commutators

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_p}][x_{j_1}, x_{j_2}, \ldots, x_{i_q}].$$

Applying the identity

$$[[y_1, y_2], [y_3, y_4], y_5] = [[y_1, y_2, y_5], [y_3, y_4]] + [[y_1, y_2], [y_3, y_4, y_5]],$$

we see that $[[y_1, \ldots, y_a], [z_1, \ldots, z_b], t_1, \ldots, t_c]$ is a linear combination of products of two commutators. As in the case $n = 2$ we see that $B(\mathcal{T}_3)$ is spanned by 1,

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_k}], \ i_1 > i_2 \leq \ldots \leq i_k,$$

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_p}][x_{j_1}, x_{j_2}, \ldots, x_{i_q}], \ i_1 > i_2 \leq \ldots \leq i_p, \ j_1 > j_2 \leq \ldots \leq j_q,$$

and it is sufficient to show that these elements are linearly independent. Considering a linear combination and replacing $x_1, x_2, \ldots$ by 2×2 upper triangular matrices (regarded as 3×3 upper triangular matrices with zero entries in the third row and in the third column) we may assume that

$$f(x_1,\dots,x_m)=\sum\alpha_{ij}[x_{i_1},x_{i_2},\dots,x_{i_p}][x_{j_1},x_{j_2},\dots,x_{i_q}]=0,\ \alpha_{ij}\in F.$$

Now we consider a maximal pair (i_1, j_1) with α_{ij} (first maximal in i_1 and between all such pairs, maximal in j_1). Let

$$\begin{aligned}\overline{x}_{i_1}&=e_{12}+\xi_{i_1}e_{22}+\eta_{i_1}e_{33},\\ \overline{x}_{j_1}&=e_{23}+\xi_{j_1}e_{22}+\eta_{j_1}e_{33},\\ \overline{x}_l&=\xi_l e_{22}+\eta_l e_{33},\end{aligned}$$

where $l\neq i_1, j_1$ and $\xi_{i_1},\xi_{j_1},\xi_l,\eta_{i_1},\eta_{j_1},\eta_l\in F$. If $i_1=j=1$, then we assume that

$$\overline{x}_{i_1}=e_{12}+e_{23}+\xi_{i_1}e_{22}+\eta_{i_1}e_{33}.$$

Again,

$$f(\overline{x}_1,\dots,\overline{x}_m)=\left(\sum\alpha_{ij}\xi_{i_2}\dots\xi_{i_p}(\eta_{j_2}-\xi_{j_2})\dots(\eta_{j_q}-\xi_{j_q})\right)\cdot e_{13}$$

and this can be made different from 0. Therefore the above products of commutators are linearly independent modulo the polynomial identities of $UT_3(F)$ and this complete the proof of the case $n=3$. □

Remark 3.1.2 Let A be a finite generated PI-algebra over an infinite field F, satisfying a *nonmatrix polynomial identity*, i.e. an identity which does not hold for 2×2 matrix algebra $M_2(F)$. Then Latyshev in [24] proved that A satisfies some polynomial identity of the form

$$[x_1,x_2]\cdots[x_{2n-1},x_{2n}]\equiv 0.$$

From this point of view the polynomial identities of the upper triangular matrices serve as a measure how complicated are the polynomial identities of A.

3.2 Graded identities of $UT_2(F)$

First, we shall next give a complete description of all gradings on $UT_2(F)$.

Definition 3.2.1 Let G be an arbitrary group, we say that $A = UT_2(F)$ has the *classical G-grading* if there exists $g \in G$, $g \neq 1$ such that $A = A_1 \oplus A_g$, where $A_1 = Fe_{11} \oplus Fe_{22}$ and $A_g = Fe_{12}$.

We have the following:

Theorem 3.2.2 *Any G-grading on $UT_2(F)$ is, up to isomorphism, either trivial or classical.*

Proof. Write again $A = UT_2(F)$ and let $e \in G$ the unit element of G. If $\dim_F A_e = 3$ then A has the trivial grading and we are done. Hence we may assume that $\dim_F A_e \leq 2$.
Suppose first that $\dim_F A_e = 2$. We may clearly assume that $e_{11} + e_{22}$ and $ae_{11}+be_{12}$ form a basis of A_e over F, for suitable $a, b \in F$. Since $\dim_F A = 3$, there exists $g \in G$ such that $\dim_F A_g = 1$ and let $A_g = F(a'e_{11}+b'e_{12}+c'e_{22})$. In case $a = 0$, then the inclusions $A_gA_e \subseteq A_g$ and $A_eA_g \subseteq A_g$ lead to $a' = c' = 0$. Hence $A_g = Fe_{12} \subseteq A_e$, a contradiction. Thus $a \neq 0$. It follows that the element $e_{11} + be_{12}$ and $e_{22} - be_{12}$ span A_e over F.
Suppose first that $b \neq 0$. Since

$$(a'e_{11} + b'e_{12} + c'e_{22})(e_{11} + be_{12}) = a'(e_{11} + be_{12}) \in A_g \cap A_e = 0$$

we obtain that $a' = 0$. Similarly, by multiplying $b'e_{12} + c'e_{22}$ on the left by $e_{22} - be_{12}$, we obtain $c' = 0$. Hence $A_g = Fe_{12}$, $A_e = F(e_{11} + e_{22}) \oplus F(e_{11} + be_{12})$ and $A_e \oplus A_g$ is isomorphic to $UT_2(F)$ with the classical G-grading. In case $b = 0$ we get $A_e = Fe_{11} + Fe_{22}$ and it easily follows that $A_g = Fe_{12}$. Thus we are done in this case too.
Suppose now that $\dim A_e = 1$ that is $A_e = F(e_{11}+be_{12})$. So either $A = A_e \oplus A_g \oplus A_h$ where $\dim_F A_g = \dim_F A_h = 1$ or $A = A_e \oplus A_g$, with $\dim_F A_g = 2$. Let $A = A_e \oplus A_g \oplus A_h$ and suppose first that $gh \neq e$. Then $A_gA_h = 0$ and, in case $g^2 \neq e$ and $h^2 \neq e$, we get that $A_g \oplus A_h$ is a two-dimensional nilpotent ideal of A, contradicting the fact that $\dim_F J(A) = 1$, where $J(A)$ is the Jacobson radical of A. Hence either $g^2 \neq e$ and $h^2 = e$ or $g^2 = h^2 = e$. In the first case one easily gets that $A_g = J(A)$ and let $A_h = F(ae_{11}+be_{12}+$

ce_{22}). From $A_gA_h = A_hA_g = 0$ we easily show obtain $a = c = 0$. Hence $A_h = A_g$, a contradiction.
In case $g^2 = h^2 = e$, since $A_g = Fu$ and $A_h = Fv$, where $u^2 = v^2 = 1$, we get that $0 \neq uv \in A_gA_h$, a contradiction.
Suppose now that $gh = e$. If $g^3 \neq e$ then $g^2 \neq g^{-1}$ and $g^{-1} \neq g$, hence $A_g^2 = A_h^2 = 0$, a contradiction.
Moreover, in case $g^3 = e$ we obtain that $A_g = Fa$ where $a = \alpha e_{11} + e_{22}$ with α a third root of the unity. Then we would get that $\dim_F(Fa + Fa^2 + Fe) = 2$, a contradiction.
We are left with $A = A_e \oplus A_g$ and $\dim_F A_g = 2$. If $g^2 \neq e$ it follows that A_g is a nilpotent ideal, hence $A_g \subseteq J(A)$, and this is a contradiction. Thus $g^2 = e$. Since $A_gA_g \subseteq A_e$ we easily obtain a contradiction also in this case.
$\square$

In case of a finite abelian group G all possible G-grading of $UT_n(F)$, the algebra of $n \times n$ upper triangular matrices are described in [34] provided that F is an algebraically closed field of characteristic zero. In fact, the following theorem holds:

Theorem 3.2.3 *Let F an algebraically closed field of characteristic zero and let $A = UT_n(F)$ be graded by a finite group G. Then A as G-graded algebra is isomorphic to $UT_n(F)$ with some elementary G-grading.*

Recall Example 1.5.7 for the definition of elementary grading on matrix algebras.

Now let's compute a basis for the T_2-ideal of the $\mathbb{Z}_2$-graded polynomial identities of $A = UT_2(F)$ when A has the classical grading. Denote with y's the variables corresponding to A_0, i.e. the even part of the grading, and with z's the variables corresponding to A_1, i.e. the odd part of the grading.

Lemma 3.2.4 *The polynomials*

$$z_1 z_2 \tag{3.2}$$

and

$$[y_1, y_2] \tag{3.3}$$

are graded identities identities for the classical grading.

Proof. A direct and easy verification. □

Now, let I the T_2-ideal of $F\langle X\rangle$ generated by the identities (3.2) and (3.3).

Lemma 3.2.5 *For any variable $x = y + z$, we have that $z_1xz_2 \in I$.*

Proof. Write $F\langle X\rangle = \mathcal{F}_0 \oplus \mathcal{F}_1$, where $\mathcal{F}_0$ is the subspace of $F\langle X\rangle$ generated by all monomials in the variables of X having even degree in the variables Z and $\mathcal{F}_1$ the subspace of $F\langle X\rangle$ generated by all monomials of odd degree in Z. Since $z_1y \in \mathcal{F}_1$, it follows that $z_1yz_2 \in \langle z_1z_2\rangle_{T_2}$. Hence $z_1(y+z) \in \langle z_1z_2\rangle_{T_2}$ and so $z_1xz_2 \in \langle z_1z_2\rangle_{T_2} \subset I$. □

Theorem 3.2.6 *The identities (3.2) and (3.3) generate $Id^{gr}(UT_2(F))$ as T_2-ideal.*

Proof. Let $f(y_1,\dots,y_t,z_1,\dots,z_t)$ be a multilinear polynomial in $Id^{gr}(UT_2)$. We wish to show that modulo I, f is the zero polynomial. From the previous Lemma it is clear that we can write

$$f(y_1,\dots,y_t,z_1,\dots,z_t) = f_1(y_1,\dots,y_s) + f_2(z,y_1,\dots,y_s) \pmod I$$

and, by multihomogeneity of T_2-ideals, it follows that f_1 and f_2 are both identities of A. Since $[y_1,y_2] \in I$, we obtain that $f_1 = \alpha y_1\cdots y_s$. But then, by substituting $y_1 = \ldots = y_s = e_{11}$ we obtain $\alpha = 0$ and, so, $f_1 = 0 \pmod I$. Write

$$f_2 = \sum \alpha y_{i_1}\cdots y_{i_t} z y_{j_1}\cdots y_{j_{n-t}}$$

where $i_1 < \ldots < i_t$ and $j_1 < \ldots < j_{n-t}$. Fix one nonzero monomial of f_2, let it be $\alpha y_1\cdots y_s z y_{s+1}\cdots y_n$. By substituting $y_1 = \ldots = y_s = e_{22}$ and $z = e_{12}$, we get $f = \alpha e_{12}$. Hence $\alpha = 0$, a contradiction. It follows that $f_2 = 0 \pmod I$ and we are done. □

The last result, together with the details about exponent and codimension growth that we described in the introduction of this chapter, give us a complete view of $UT_2(F)$ as an associative algebra.

Chapter 4

The Jordan Upper Triangular Matrices $UJ_2(F)$

This fourth chapter is the core of the thesis. In fact, here we give all the results obtained concerning the polynomial identities of the Jordan algebra of upper triangular matrices of order two over an infinite field. One can find these results in [19].

The first step of our study is to choose a basis of $UJ_2(F)$ in order to get a "nice" table of multiplication, i.e. a table where the Jordan product of two elements is not a linear combination of the elements of the basis, but exactly an element of the same basis. This can be obtained fixing $\mathcal{B} = \{1, a, b\}$, where

$$\begin{aligned} 1 &= e_{11} + e_{22}, \\ a &= e_{11} - e_{22}, \\ b &= e_{12}. \end{aligned}$$

In fact, by an easy computation, we get that $a^2 = 1$ and $ab = b^2 = 0$. Sometimes, we denote the element 1 also with I. According to Lemma 1.9.4, we give the following definition:

Definition 4.0.7 The polynomial $f(x_1, \ldots, x_n) \in J(X)$ is a *proper polynomial* if $\partial f / \partial x_i = 0$ for every i.

Here, recall that $J(X)$ is the free Jordan algebra generated over F by the set X and $B(X)$ is the set of all proper polynomials in $F\langle X\rangle$. Moreover, we shall denote with $L(X)$ the free Lie algebra generated over F by the set X.

It is well-known that the Jordan algebras $UJ_n(F)$ are special. Moreover, according to a theorem due to Slinko, see [33], the variety of Jordan algebras generated by $UJ_2(F)$ is special. In other words if a Jordan algebra satisfies the identities of $UJ_2(F)$ then it is special.

4.1 Graded identities for $UJ_2(F)$

Throughout this section we assume F an infinite field and $\operatorname{char} F \neq 2$. First we describe all possible $\mathbb{Z}_2$-gradings on the Jordan algebra $J = UJ_2(F)$ of the upper triangular 2×2 matrices.

Lemma 4.1.1 *The following decompositions $J = J_0 \oplus J_1$ are $\mathbb{Z}_2$-gradings on $J = UJ_2(F)$.*

1. *(The associative grading) $J_0 = F \oplus Fb$, $J_1 = Fa$;*
2. *(The scalar grading) $J_0 = F$, $J_1 = Fa \oplus Fb$;*
3. *(The classical grading) $J_0 = F \oplus Fa$, $J_1 = Fb$.*

Here we identify F with the scalar matrices in J.

Proof. The proof consists of a straightforward and easy computation with 2×2 matrices. □

Lemma 4.1.2 *The three gradings from Lemma 4.1.1 are pairwise nonisomorphic.*

Proof. In the scalar grading one has $\dim J_0 = 1$ while $\dim J_0 = 2$ in the remaining two gradings. Hence the scalar grading cannot be isomorphic to any of the other two. On the other hand in the classical grading one has $J_1^2 = 0$ while in the associative grading $J_1^2 = F$ hence they cannot be isomorphic either. □

Proposition 4.1.3 *The three gradings from Lemma 4.1.1 are, up to a graded isomorphism, the only nontrivial $\mathbb{Z}_2$-gradings on $J = UJ_2(F)$.*

Proof. Let $J = J_0 \oplus J_1$ be a nontrivial grading on J, then either $\dim J_0 = 1$ or $\dim J_0 = 2$. Observe that $1 \in J_0$. Assume first $\dim J_0 = 1$, then $J_0 = F$. Let $a = \alpha 1 + u$, $b = \beta 1 + v$, α, $\beta \in F$, u, $v \in J_1$. Then $1 = a^2 = \alpha^2 + 2\alpha u + u^2$, and since $u^2 \in J_0$ and $u \notin J_0$ one gets $\alpha = 0$, thus $a \in J_1$. Similarly $0 = b^2 = \beta^2 + 2\beta v + v^2$, $v^2 \in J_0$ and hence $\beta = 0$ and $b \in J_1$. Therefore if $\dim J_0 = 1$ the grading is exactly the scalar one.

Now assume $\dim J_0 = 2$ and therefore $\dim J_1 = 1$. Let 1 and u form a basis of J_0 and let v be a basis of J_1. We can write $u = \alpha a + \beta b$, $v = \lambda 1 + \mu a + \nu b$. By the multiplication table among 1, a, b one gets $u \circ v = \lambda u + \alpha\mu$. As $u \circ v \in J_1$ this implies $\lambda = 0$ and $\alpha\mu = 0$.

Consider first the case $\alpha = 0$, then without loss of generality $u = b$. Also we must have $\mu \neq 0$, and we can choose $\mu = 1$. Hence $v = \begin{pmatrix} 1 & \nu \\ 0 & -1 \end{pmatrix}$. If $\nu = 0$ we have exactly the associative grading. If, otherwise, $\nu \neq 0$ then the matrix v can be diagonalized by means of a conjugation by an upper triangular matrix. Such conjugation preserves the identity matrix, and sends b to a scalar multiple of b. Therefore this conjugation gives the graded isomorphism between our grading and the associative one.

Now let $\mu = 0$, then we can take $v = b$. But $u = \alpha a + \beta b$ and necessarily $\alpha \neq 0$. Dividing by α we consider $\alpha = 1$, and then we repeat the above diagonalization procedure exchanging v and u. In this case we have a grading isomorphic to the classical one. $\square$

In the next three subsections we describe the graded identities for each one of the three gradings on $UJ_2(F)$. In fact one may see that the graded identities for these gradings are pairwise different. This may give another (indirect and much longer, though) proof of the fact that they are nonisomorphic.

4.1.1 The associative grading

Here we describe generators of the ideal of graded identities for the associative grading. Recall that in it $J_0 = F \oplus Fb$, $J_1 = Fa$. Recall also that the letters y, with or without lower indices, stand for even variables; z are odd variables in the free $\mathbb{Z}_2$-graded Jordan algebra.

Lemma 4.1.4 *The following polynomials are graded identities for the as-*

sociative grading on $UJ_2(F)$.

$$(y_1, y_2, y_3), \quad (z_1, y, z_2), \quad (z_1, z_2, z_3), \quad (y_1, z, y_2),$$
$$(z, y_1, y_2), \quad (z_1 z_2, x_1, x_2), \quad (x_1, z_1 z_2, x_2).$$

Here x_1 and x_2 are any variables; that is they may be even or odd.

Proof. The proof consists of a direct verification and we omit it. □

Proposition 4.1.5 *The seven identities of the preceding lemma generate the ideal of graded identities for the associative grading.*

Proof. Denote by I the ideal of graded identities generated by the identities of Lemma 4.1.4. We shall work in the free graded Jordan algebra modulo the ideal I. In order to simplify the notation we use the same letters y for $y+I$, and analogously for z. The algebra generated by the variables y_i in $J(X)/I$ is associative and commutative. If $f(y_1, \ldots, y_k, z)$ is a multihomogeneous polynomial that is linear in z then the first, fourth and fifth identities from Lemma 4.1.4 yield that $f \equiv \alpha y_1^{i_1} \cdots y_k^{i_k} z$, $\alpha \in F$ (the congruence is modulo I). Analogously, the second, third and last two identities yield that every multihomogeneous polynomial $f(y, z_1, \ldots, z_k)$ that is linear in y can be written modulo I as $\beta y(z_1^{j_1} \ldots z_k^{j_k}) + \gamma (y z_1) z_1^{j_1 - 1} \ldots z_k^{j_k}$ for some β, $\gamma \in K$. Also $z_1 z_2$ associates and commutes with every element. Therefore if $j_1 + \cdots + j_k$ is odd then $f \equiv \beta y(z_1^{j_1} \ldots z_k^{j_k})$ only.

Let $f(y_1, \ldots, y_r, z_1, \ldots, z_s)$ be multihomogeneous. We shall prove that modulo I one can write f as a linear combination of elements of the types: YZ, $(Yz_1)Z$, $((Yz_1)z_2)Z$, $(((Y_1 z_1)z_2)Y_2)Z$.
Here the uppercase Y stands for a product $y_1^{i_1} \ldots y_r^{i_r}$, Y_1 and Y_2 are again products of the variables Y (we assume these products ordered as the algebra generated by the y_i is associative and commutative), and Z is an ordered product of the variables z_j, $\deg Z$ is even. We assume f is a monomial and we induct on $\deg f = n$. If $n = 2$, 3, 4, the statement obviously holds.
If m is a monomial of some of the given types, $\deg m < n$, we must show that my and mz are again of the given type. We shall discard the Z as it lies in the associative center.

This is clearly the case when $m = Y$. If $m = Yz_1$ then $my = (Yz_1)y = (Yy)z_1$, and $mz = (Yz_1)z$. Let $m = (Yz_1)z_2$, then my is of the fourth type; $mz = ((Yz_1)z_2)z = (Yz_1)(z_2 z)$, and we get an element of the second type (plus the product $z_2 z$ to the Z part). So suppose $m = ((Y_1 z_1)z_2)Y_2$. Then $my = ((Y_1 z_1)z_2)(Y_2 y)$ which is again of the fourth type. Finally

$$mz = (((Y_1 z_1)z_2)Y_2)z = ((Y_1 z_1)z_2)(Y_2 z) = ((Y_1 z_1)(Y_2 z))z_2$$
$$= (((Y_1 Y_2)z_1)z)z_2 = ((Y_1 Y_2)z_1)(zz_2)$$

which is of the second type.

We shall show that the four types above are linearly independent modulo the graded identities of $UJ_2(F)$. Substitute y_i for the "generic" matrix $\alpha_i I + \beta_i b$ and z_i for $\gamma_i a$. Here the α_i, β_i, γ_i are commuting variables. The elements $(Yz_1)Z$ are the only odd elements so we consider the remaining three types. The elements $((Yz_1)z_2)Z$ only evaluate to scalar matrices thus we can discard them as well. Now let $Y = Y_1 Y_2$ and let $Z = Z' z_1 z_2$. Then we discard Z' and consider $Y(z_1 z_2)$ and $((Y_1 z_1)z_2)Y_2$. The coefficients of the identity matrix are equal but the ones of e_{12} are independent (the latter includes the $(1,1)$-entries of the part Y_2 only). Hence our types are independent and we are done. □

4.1.2 The scalar grading

Here we fix the grading $J_0 = F$, $J_1 = Fa \oplus Fb$ which we called the scalar one.

Lemma 4.1.6 *The polynomials*

$$(y_1, y_2, y_3), \quad (y, z_1, z_2), \quad (z_1, y, z_2), \quad (z_1, z_2, z_3)z_4$$

are graded identities for the scalar grading.

Proof. A direct and easy verification. □

Proposition 4.1.7 *The graded identities from Lemma 4.1.6 generate the ideal of graded identities for $UJ_2(F)$ with respect to the scalar grading.*

Proof. The proof is similar to that of Proposition 4.1.5. Denote by I the ideal of graded identities generated by the ones from Lemma 4.1.6. The even variables y lie in the associative and commutative centre of the relatively free graded algebra $J(X)/I$. Therefore every multihomogeneous polynomial

$f(y_1, \ldots, y_r, z_1, \ldots, z_s)$ can be written as

$$f(y_1, \ldots, y_r, z_1, \ldots, z_s) \equiv y_1^{i_1} \cdots y_r^{i_r} g(z_1, \ldots, z_s) \pmod{I}$$

for some multihomogeneous polynomial $g(z_1, \ldots, z_s)$ depending only on the odd variables. Substituting all y_k by the identity matrix we get that f is a graded identity if and only if g is.

Denote by M the subalgebra of $J(X)/I$ generated by all odd variables z_i, then M is $\mathbb{Z}_2$-graded by putting $M = M_0 \oplus M_1$. Here M_0 is spanned by all monomials of the type $(z_{i_1} z_{j_1}) \cdots (z_{i_t} z_{j_t})$ and M_1 is spanned by $z_{i_0}(z_{i_1} z_{j_1}) \cdots (z_{i_t} z_{j_t})$.

Using the last identity from Lemma 4.1.6 we obtain $(z_1 z_2)(z_3 z_4) = (z_1 z_3)(z_2 z_4)$. Thus if $\deg g$ is even we can write

$$g(z_1, \ldots, z_s) \equiv \alpha(z_{i_1} z_{j_1}) \cdots (z_{i_t} z_{j_t}), \alpha \in F$$

where $i_1 \le j_1 \le \cdots \le i_t \le j_t$. Therefore g is a graded identity if and only if $\alpha = 0$, that is if and only if $g \in I$.

If on the other hand $\deg g$ is odd we write it as

$$g(z_1, \ldots, z_s) \equiv \sum \gamma_{i_0} z_{i_0}(z_{i_1} z_{j_1}) \cdots (z_{i_t} z_{j_t}), \gamma_{i_0} \in F$$

where as above $i_1 \le j_1 \le \cdots \le i_t \le j_t$. Now we substitute $z_{i_k} = \alpha_{i_k} a + \beta_{i_k} b$ and $z_{j_k} = \alpha_{j_k} a + \beta_{j_k} b$ where the α_{i_k}, β_{i_k} and α_{j_k}, β_{j_k} are (associative) commutative independent variables. Then g evaluates at

$$\begin{aligned} &\sum_{i_0} \gamma_{i_0}(\alpha_{i_0} a + \beta_{i_0} b)(\alpha_{i_1} \alpha_{j_1}) \cdots (\alpha_{i_t} \alpha_{j_t}) \\ =&\sum_{i_0} \gamma_{i_0}(\alpha_{i_1} \alpha_{j_1}) \cdots (\alpha_{i_t} \alpha_{j_t})(\alpha_{i_0} a + \beta_{i_0} b). \end{aligned}$$

The coefficient β_{i_0} comes only from the summand starting with z_{i_0}. Therefore g is a graded identity for J if and only if all $\gamma_{i_0} = 0$ which clearly means $g \in I$. $\square$

4.1.3 The classical grading

Here we fix the classical grading on $J = UJ_2(F)$: $J_0 = F \oplus Fa$, $J_1 = Fb$.

Lemma 4.1.8 *The polynomial*

$$(x_1x_2, x_3, x_4) - x_1(x_2, x_3, x_4) - x_2(x_1, x_3, x_4) \tag{4.1}$$

is an ordinary polynomial identity for J.

Proof. The proof is a straightforward computation and therefore will be omitted. We shall return to the above identity in the next section. $\square$

Corollary 4.1.9 *Every monomial in the free Jordan algebra $J(X)$ can be written, modulo the identity (4.1) as a linear combination of monomials where the brackets are right-normed (or, respectively, left-normed).*

Proof. By expanding (4.1) one gets

$$\begin{aligned}((x_1x_2)x_3x_4) - (x_1x_2)(x_3x_4) \equiv x_1((x_2x_3)x_4) - x_1(x_2(x_3x_4))\\ + x_2((x_1x_3)x_4) - x_2(x_1(x_3x_4))\end{aligned}$$

and by the commutativity we rewrite the above as

$$\begin{aligned}(x_1x_2)(x_3x_4) \equiv x_1(x_2(x_3x_4)) + x_2(x_1(x_3x_4)) - x_1(x_4(x_2x_3))\\ - x_2(x_4(x_1x_3)) + x_4(x_3(x_1x_2)).\end{aligned}$$

This shows that the product $(x_1x_2)(x_3x_4)$ can be written as a linear combination of right-normed products.

Observe that using once again the commutativity for the right-hand side we can write $x_1(x_2(x_3x_4)) = ((x_3x_4)x_2)x_1$. $\square$

Now we continue as in the previous two gradings. The next lemma is immediate.

Lemma 4.1.10 *The following polynomials are graded identities for the classical grading on J.*

$$(y_1, y_2, y_3), \quad z_1z_2, \quad (y_1, z, y_2).$$

Proposition 4.1.11 *The graded identities from Lemma 4.1.10 together with the identity (4.1) generate the ideal of graded identities for the classical grading on $J = UJ_2(F)$.*

Proof. Denote by I the ideal of graded identities generated by the identities from the statement and (4.1). Assume f is multihomogeneous polynomial. Then the graded identity $z_1z_2 = 0$ implies that f contains only one odd variable z and f is linear in z (or otherwise f depends only on even variables). Moreover according to Corollary 4.1.9 we can express f as a linear combination of right-normed monomials. Furthermore by applying several times the graded identities (y_1, y_2, y_3) and (y_1, z, y_2) to a right-normed monomial with only one z we can put that z at the rightmost position and keeping the right-normed brackets. In addition we can reorder the variables y at will.

In this way we obtain $f(y_1, \ldots, y_s, z) \equiv \alpha y_{i_1}(y_{i_2}(\ldots(y_{i_{r-1}}(y_{i_r}z))\ldots))$ (mod I) where $i_1 \leq \cdots \leq i_r$, and f is a graded identity if and only if $\alpha = 0$ that is $f \in I$. □

4.2 The ordinary identities of $UJ_2(F)$

In this section we complete the study of the graded identities of $UJ_2(F)$ and deal with the trivial grading on this algebra. Notwithstanding this turns out to be the most difficult case. In what follows we assume K is an infinite field, char $F \neq 2$, 3. We shall work in the free Jordan algebra $J(X)$ and sometimes we shall denote the free generators by the letters x, y, z, t, u, v with or without indices.

Definition 4.2.1 Let Ω be the least subset in $J(X)$ such that if f, g, $h \in \Omega \cup X$ then $(f, g, h) \in \Omega$. The elements of Ω are called *long associators.* If the parentheses in such a long associator are left normed we shall call it a *regular* associator.

In the case of regular associators we shall omit the inner parentheses: (x, y, z, t, u) stands for $((x, y, z), t, u)$, and so on.

Lemma 4.2.2 *The polynomials*

$$[x, y]^2, \qquad (u, (x, y, z), v) \tag{4.2}$$

are identities for the Jordan algebra $J = UJ_2(F)$.

Proof. The first polynomial is obviously an identity for J. As for the second, observe that (x, y, z) when evaluated on J, yields a multiple of e_{12}, and then for any g, $h \in J$ one gets $(g, e_{12}, h) = 0$. $\square$

Remark 4.2.3 The polynomial $[x, y]^2$ lies in the free special Jordan algebra in two generators $SJ(x, y)$; it is well known that $SJ(x, y) = J(x, y)$. Therefore we view $[x, y]^2$ as the corresponding Jordan element.

We shall prove later on that the two identities from (4.2) generate the ideal of all identities of J. First we collect a list of identities for the algebra J and follow from those of (4.2). We begin with some facts about proper polynomials in Jordan algebras. We shall use results and ideas from [14] and from [37].

Let $T(x, y, z, t) = (xy, z, t) - x(y, z, t) - y(x, z, t)$ for all x, y, z, $t \in J(X)$. We already established that $T(x, y, z, t)$ is an identity for J, see Lemma 4.1.8. Moreover in the free special Jordan algebra $SJ(X)$ one has the equality

$$T(x, y, z, t) = \frac{1}{4}([z, x] \circ [y, t] - [y, z] \circ [x, t]).$$

But the right-hand side is readily seen to be, up to a scalar, the linearization of the polynomial $[x, y]^2$. Therefore we can substitute the identity $[x, y]^2$ for $T(x, y, z, t)$, the latter being multilinear. We shall denote by I the ideal of identities in the free Jordan algebra generated by the polynomials $T(x, y, z, t)$ and $(u, (x, y, z), v)$. We set $R(X) = J(X)/I$ the corresponding relatively free algebra; we shall work in it.

Lemma 4.2.4 *The equality* $(x_1, x_2, x_3)(y_1, y_2, y_3) = 0$ *holds in* $R(X)$.

Proof. It suffices to note that

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = \tfrac{1}{3}\big(T((x_1, x_2, x_3), y_3, y_2, y_1) - T((x_1, x_2, x_3), y_1, y_2, y_3)\big)$$

modulo I, see [37, Eq. 3.37]. $\square$

The ideal of identities of J is unitarily closed hence the identities of J are generated by some set of proper polynomials. Recall that a multihomogeneous polynomial is proper if all its partial derivatives vanish. The next lemma describes the proper polynomials in $R(X)$. It is a particular case of a more general result established in [14, Lemma 2].

Lemma 4.2.5 *The subalgebra $P(X)$ of the proper polynomials in $R(X)$ is spanned by all regular associators.*

Proof. It follows from [14, Lemma 2] that $P(X)$ is spanned by all $T(x, y, z, t)$, by all long associators and by all products of two long associators. (In such products one of the associators may be taken of length 3.) Since T vanishes on $R(X)$ and the same does the product of two associators we are left with long associators only. Now according to [15, pp. 343, 344] every associator can be represented as a linear combination of regular (that is left-normed) ones, and the proof is complete. □

Remark 4.2.6 The proof of the fact that every associator is a linear combination of regular ones given in [15] requires passing to *Lie triple systems*.

Every Jordan algebra satisfies the identities

$$(x, y, z) = -(z, y, x), \quad (x, y, z) + (y, z, x) + (z, x, y) = 0 \qquad (4.3)$$

Lemma 4.2.7 *The identity $(x, y, z, t, u) = (x, y, z, u, t)$ holds in $R(X)$.*

Proof. Applying the second identity of (4.3) to $(u, (x, y, z), v)$ we get

$$\begin{aligned} (u, (x, y, z), t) &= -((x, y, z), t, u) - (t, u, (x, y, z)) \\ &= -(x, y, z, t, u) + (x, y, z, u, t) \end{aligned}$$

(we used also the first identity of (4.3) in the second line). Thus we have the identity of the lemma. □

Lemma 4.2.8 *The identity $(x, y, z, t, u) = (x, t, z, u, y)$ holds in $R(X)$.*

Proof. Write $f = (x, y, z, t, u) - (x, t, z, u, y)$ as

$$f = ((x, y, z)t)u - (x, y, z)(tu) - ((x, t, z)u)y + (x, t, z)(yu).$$

The identity $(x, yz, t) = (x, y, t)z + (x, z, t)y$ holds in every Jordan algebra. Applying this identity we have

$$(x, t, z)(yu) = (x, t(yu), z) - (x, yu, z)t,$$
$$-(x, y, z)(tu) = -(x, y(tu), z) + (x, tu, z)y.$$

But $(x,t(yu),z)-(x,y(tu),z)=(x,(y,u,t),z)=0$ in $R(X)$ and we get

$$\begin{aligned}
f &= ((x,y,z)t)u-(x,y,z)(tu)-((x,t,z)u)y+(x,t,z)(yu)\\
&= (x,tu,z)y-(x,yu,z)t+((x,y,z)t)u-((x,t,z)u)y\\
&= ((x,t,z)u)y+((x,u,z)t)y-((x,y,z)u)t-((x,u,z)y)t\\
&+ ((x,y,z)t)u-((x,t,z)u)y\\
&= ((x,u,z)t)y-((x,u,z)y)t+((x,y,z)t)u-((x,y,z)u)t\\
&= (t,(x,u,z),y)+(t,(x,y,z),u).
\end{aligned}$$

The last expression vanishes in $R(X)$, and the lemma is proved. □

Corollary 4.2.9 *The identity* $(x,y,z,t,u)=(x,t,z,y,u)$ *holds in* $R(X)$.

Proof. Apply first Lemma 4.2.8 and then Lemma 4.2.7. □

So, now let f be a regular associator in $R(X)$ depending on the variables $x_1, \ldots, x_n$. Then by applying several times the identities from (4.3) we can write f in $R(X)$ as a linear combination of associators starting with x_1. Then applying if necessary Lemmas 4.2.7 and 4.2.8 we can write every such associator (starting with x_1) in the form

$$h=(x_{i_1},x_{i_2},x_{i_3},\ldots,x_{i_m}),\quad i_1=1, i_2\leq i_4\leq\cdots\leq i_m.$$

We cannot say much about i_3 (if $i_3=i_1$ then clearly $h=0$).

On the other hand let us substitute, in the above associator, the generic matrices $g_j=\begin{pmatrix} a_j & b_j\\ 0 & c_j\end{pmatrix}$ where the a_j, b_j, c_j are independent (associative and commutative) variables. A simple computation shows that

$$h(g_1,\ldots,g_n)=\pm((a_{i_1}-c_{i_1})b_{i_3}-(a_{i_3}-c_{i_3})b_{i_1})(a_{i_2}-c_{i_2})\prod_{j=4}^{m}(a_{i_j}-c_{i_j}).$$

Therefore different nonzero associators of the type of h are linearly independent modulo the identities $Id(J)$ of J. Thus they must be independent in $R(X)$ as well. Since $I\subset J$ and these associators span the proper elements of $R(X)$ they must span the proper elements of $J(X)/Id(J)$, too. All this gives us proof of the following theorem.

Theorem 4.2.10 *The identities from (4.2) form a basis of the ordinary polynomial identities for the Jordan algebra of the upper triangular matrices of order two, over any infinite field of characteristic different from 2 and from 3.*

Corollary 4.2.11 *Let V be a vector space over an infinite field F,* $\operatorname{char} F \neq$ *2, 3, and suppose V is equipped with a symmetric bilinear form $\langle u, v\rangle$ of rank one. Then the polynomials from (4.2) form a basis of the identities of the Jordan algebra $J = F \oplus V$ of this form.*

Proof. Let v_1, v_2, ..., be a basis of V such that $\langle v_1, v_1\rangle = 1$ and the span W of v_2, v_3, ..., is an isotropic subspace of V. Then clearly W is an ideal of J and $W^2 = 0$. It is immediate that the identities from (4.2) hold for the Jordan algebra J. Thus $Id(UJ_2(F)) \subseteq Id(J)$. The opposite inclusion is obvious since one can find a copy of $UJ_2(F)$ inside J. □

Corollary 4.2.12 *Let $h_m = (x_1, x_2, \ldots, x_m)$, $m = 2k + 1$, and denote by W_m the variety of unitary Jordan algebras determined by the identities (4.2) and h_m. Then W_m, $m \geq 3$, are the only subvarieties of $var(UJ_2(F))$.*

Proof. The proof follows from the fact that we consider unitary algebras. □

Let $\operatorname{char} F = 0$ and consider the variety of associative F-algebras with 1 defined by the identity $[x, y]^2 = 0$. Drensky in [4] gave a complete description of its subvarieties. That description is somewhat more complicated than in our case; here we have no proper elements of even degree.

It is clear that h_{m+2} is a consequence of the identity h_m, and that in characteristic 0 the polynomial h_m generates an irreducible S_m-module corresponding to the partition $(2k, 1)$. Therefore its dimension equals $m - 1$. Now we use the relationship between the proper and the ordinary codimensions, see for example [6, p. 47] for the associative case, or [5] for the case of Jordan algebras. Let γ_m be the m-th proper codimensions of the variety $var(UJ_2(F))$, then $\gamma_m = m - 1$ when m is odd, and $\gamma_m = 0$ otherwise. A straightforward computation then shows that for the ordinary codimensions c_n we have $c_n = \sum_{m=0}^{n} \binom{n}{m} \gamma_m$, see [5, Corollary 4.2]. Therefore the codimensions of every proper unitary subvariety of $var(UJ_2(F))$ have polynomial growth. On the other hand the codimensions of $UJ_2(F)$ behave like

2^n. More precisely, $c_n = \sum \binom{n}{2k+1} 2k$ where the sum runs over all k. Writing $(1+x)^n = \sum_{t=0}^n \binom{n}{t} x^t$ and $(1-x)^n = \sum_{t=0}^n \binom{n}{t} (-1)^t x^t$, differentiating and putting $x = 1$, we obtain $c_n = (n-2)2^{n-2}$.

Acknowledgements

This thesis would have never been written without the help and the suggestions of my supervisor prof. Antonio Giambruno and prof. Plamen Koshloukov, who gave me a very big help in the period of three months that I spent in Brazil, at University of Campinas. Then I must thank my family and my friends, especially my teammates of Sharks Palermo American Football Team: they always remember me that clear eyes and full hearts can't lose. I can't imagine what would have been my life without American Football.

Bibliography

[1] Yu. Bahturin, I. Shestakov, *Gradings of simple Jordan algebras and their relation to the gradings of simple associative algebras*, Commun. Algebra **29** (2001), no. 9, 4095–4102.

[2] Yu. A. Bahturin, M. Zaicev, *Graded algebras and graded identities*, in *Polynomial Identities and Combinatorial Methods*, Lect. Notes Pure Appl. Math., **235**, Marcel Dekker, New York, 2003, 101–139.

[3] O.M. Di Vincenzo, V. Drensky, *The basis of the graded polynomial identities for superalgebras of triangular matrices*, Comm. in Algebra, **24 (2)**, (1996), 727–736.

[4] V. Drensky, *Lattices of varieties of associative algebras*, Serdica **8**, No. 1 (1982), 20–31 (in Russian).

[5] V. Drensky, *Polynomial identities for the Jordan algebra of a symmetric bilinear form*, J. Algebra **108**, no. 1 (1987), 66–87.

[6] V. Drensky, *Free algebras and PI-algebras. Graduate course in algebra*, Springer, Singapore, 1999.

[7] V. Drensky, *Combinatorial Approach to Algebras with Polynomial Identities*, Doctor of Sciences Thesis, Sofia, 1999.

[8] A. Giambruno, M. Zaicev, *On codimension growth of finitely generated associative algebras*, Adv. Math. **140** (1998), 145–155.

[9] A. Giambruno, M. Zaicev, *Polynomial identities and asymptotic methods*, Math. Surveys Monographs **122**, Amer. Math. Soc., Providence, RI, 2005.

[10] A. Giambruno, M. Zaicev, *Exponential codimension growth of P.I. algebras: an exact estimate*, Adv. Math. **142** (1999), 221–243.

[11] A. Giambruno, M. Zaicev, *Codimesion growth of special simple Jordan algebras*, Transactions of the Amer. Math. Soc. **362** (2009), no. 6, 3107–3123.

[12] A. Giambruno, M. Zaicev, *Lie, Jordan and proper codimensions of associative algebras*, Rendiconti del Circ. Mat. di Palermo **57** (2008), 161–171.

[13] A. Giambruno, I. Shestakov, M. Zaicev, *Finite-dimensional non-associative algebras and codimension growth*, Adv. in Appl. Math. (2010).

[14] A. Iltyakov, *The Specht property of ideals of identities of certain simple nonassociative algebras*, (Russian) Algebra i Logika **24** (1985), no. 3, 327–351, 374–375; English transl. Algebra and Logic **24** (1985), no. 3, 210–228.

[15] N. Jacobson, *Structure and representations of Jordan algebras*, Amer. Math. Soc. Colloq. Publ. **39**, Amer. Math. Soc., Providence, RI, 1968.

[16] A. R. Kemer, *Varieties of finite rank*, Proc. 15-th All the Union Algebraic Conf., Krasnoyarsk, Vol. **2**, p. 73, 1979, (Russian).

[17] P. Koshlukov, *Basis of the identities of the matrix algebra of order two over a field of characteristic* $p \neq 2$, J. Algebra **241** (2001), 410–434.

[18] P. Koshlukov, *Graded polynomial identities for the Lie algebra* $sl_2(K)$, Internat. J. Algebra Comput. **18 (5)** (2008), 825–836.

[19] P. Koshlukov, F. Martino, *Polynomial identities for the Jordan algebra of upper triagular matrices of order two*, 2010 (to appear).

[20] P. Koshlukov, D. Silva, *2-Graded polynomial identities for the Jordan algebra of the symmetric matrices of order two*, J. Algebra **327 (1)** (2011), 236–250.

[21] P. Koshlukov, *Polynomial identities for a family of simple Jordan algebras*, Com. in Algebra, **16 (7)** (1988), 1325–1371.

[22] D. Krakowski, A. Regev, *The polynomial identities of the Grassmann algebra*, Trans. AMS **181** (1973), 429–438.

[23] S. Lang, *Algebra*, Grad. Text in Math. **211**, Springer-Verlag, New York, 2002.

[24] V. N. Latyshev, *On the choice of basis in a T-ideal*, Sibirsk. Matem. Zh. **4**, no. 5 (1963), 1122–1126 (Russian).

[25] V. N. Latyshev, *Generalization of Hilbert's theorem of the finiteness of bases* (Russian), Sirbisk. Math. Zh. **7** (1966), 1422–1424. Translation: Sib. Math. J. **7** (1966), 1112–1113.

[26] Yu. N. Maltsev, E. N. Kuzmin, *A basis for identities of the algebra of second order matrices over a finite field*, Algebra Logic **17** (1978), 17–21.

[27] K. McCrimmon, *A Taste of Jordan Algebras*, Springer-Verlag, New York, 2004.

[28] Yu. Razmyslov, *Identities of algebras and their representations*, Translations Math. Monographs, **138**, AMS, Providence, RI, 1994.

[29] A. Popov, *Identities of the tensor square of a Grassmann algebra*, Algebra Logic **21** (1982), 296–316.

[30] R. D. Schafer, *An Introduction to Nonassociative Algebras*, Pure and Applied Mathematics **22**, Academic Press, New York, 1966.

[31] A. I. Shirshov, I. P. Shestakov, A. M. Slin'ko, K. A. Zhevlakov, *Rings that are nearly associative*, Academic Press, 1982.

[32] P. Zh. Chiripov, P. N. Siderov, *On bases for identities of some varieties of associative algebras*, PLISKA Stud. Math. Bulgar. **2** (1981), 103–115 (Russian).

[33] A. Slinko, *Specal varieties of Jordan algebras*, Matem. Zametki **26 (3)** (1979),337–344 (Russian); English translation: Math. Notes **26 (3-4)** (1979), 661–665 (1980).

[34] A. Valenti, M. Zaicev, *Abelian gradings on upper triangular matrices*, Arch. Math. **80**, No. 1 (2003), 12–17.

[35] A. H. Stojanova–Venkova, *Bases of identities of Grassmann algebras*, Serdica **6**, No. 1 (1980), 63–72 (Russian).

[36] S. Yu. Vasilovskii, *The basis of identities of a three-dimensional simple Lie algebra over an infinite field*, Algebra i Logika **28**, No. 5 (1989), 534–554 (Russian); English translation, Algebra and Logic **28**, No. 5 (1989), 355–368.

[37] S. Vasilovsky, *A finite basis for polynomial identities of the Jordan algebra of a bilinear form*, Trudy Inst. Mat. (Novosibirsk) **16** (1989), Issled. po Teor. Kolets i Algebr, 5–37; English transl. Siberian Adv. Math. **1** (1991), no. 4, 142–185.