

DIFFERENTIATION OF AN ADDITIVE INTERVAL MEASURE WITH VALUES IN A CONJUGATE BANACH SPACE

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: We present a complete characterization of finitely additive interval measures with values in conjugate Banach spaces which can be represented as Henstock-Kurzweil-Gelfand integrals. If the range space has the weak Radon-Nikodým property (WRNP), then we precisely describe when these integrals are in fact Henstock-Kurzweil-Pettis integrals.

Keywords: Kurzweil–Henstock integral, Pettis integral, variational measure.

1. Notations and preliminaries.

Let $[0, 1]$ be the unit interval of the real line equipped with the usual topology and the Lebesgue measure λ . We denote by \mathcal{I} the family of all nontrivial closed subintervals of $[0, 1]$, by \mathcal{L} the family of all Lebesgue measurable subsets of $[0, 1]$ and by \mathcal{L}^+ the family of all Lebesgue measurable subsets of $[0, 1]$ of positive measure.

If $E \subset [0, 1]$, then its Lebesgue measure is denoted by $|E|$ or $\lambda(E)$. Throughout X is a Banach space with its dual X^* . The closed unit ball of X is denoted by $B(X)$. A mapping $\nu: \mathcal{L} \rightarrow X$ is said to be an X -valued measure if ν is countably additive in the norm topology of X . If μ is a positive measure on \mathcal{L} or an X -valued measure, then by $\mu \ll \lambda$ we mean that $|E| = 0$ implies $\mu(E) = 0$. We say then that μ is λ -continuous. The variation of an X -valued measure ν is denoted by $|\nu|$.

$\tau(X^*, X)$ is the Mackey topology on X^* and $\tau_c(X^*, X)$ is the topology of uniform convergence on compact subsets of X . It is known (cf. [12]) that $\tau_c(X^*, X)$ coincides on $B(X^*)$ with the weak*-topology $\sigma(X^*, X)$.

A partition in $[0, 1]$ is a finite collection of pairs $\mathcal{P} = \{(I_1, t_1), \dots, (I_p, t_p)\}$, where I_1, \dots, I_p are non-overlapping subintervals of $[0, 1]$ and $t_i \in I_i$, for all $i \leq p$. Given a subset E of $[0, 1]$, we say that the partition \mathcal{P} is anchored on E if $t_i \in E$

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for each $i = 1, \dots, p$. If $\cup_{i=1}^p I_i = [0, 1]$ we say that \mathcal{P} is a *partition of* $[0, 1]$. A *gauge* on $E \subset [0, 1]$ is a positive function on E . For a given gauge δ , we say that a partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ is δ -*fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1, \dots, p$.

Given two real numbers a, b , we denote by the symbol $\langle a, b \rangle$ the interval $[\min\{a, b\}, \max\{a, b\}]$.

Definition 1.1. A function $f: [0, 1] \rightarrow \mathbb{R}$ is said to be *Henstock-Kurzweil integrable*, or simply *HK-integrable*, on $[0, 1]$ if there exists $w \in \mathbb{R}$ with the following property: for every $\epsilon > 0$ there exists a gauge δ on $[0, 1]$ such that

$$\left| \sum_{i=1}^p f(t_i) |I_i| - w \right| < \epsilon,$$

for each δ -fine partition $\mathcal{P} = \{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0, 1]$.

We set $(HK) \int_0^1 f d\lambda := w$. By $HK[0, 1]$ is denoted the set of all *HK-integrable* functions $f: [0, 1] \rightarrow \mathbb{R}$.

It is well known that if $f \in HK[0, 1]$ then f is *HK-integrable* on each $I \in \mathcal{I}$. We call the additive interval function $F(I) := (HK) \int_I f d\lambda$ the *HK-primitive* of f .

Definition 1.2. A function $f: [0, 1] \rightarrow X$ is said to be *scalarly Henstock-Kurzweil integrable* if, for each $x^* \in X^*$, the function $x^* f$ is Henstock-Kurzweil integrable. A scalarly Henstock-Kurzweil integrable function f is said to be *Henstock-Kurzweil-Pettis integrable* (or simply *HKP-integrable*) if for each $I \in \mathcal{I}$ there exists $w_I \in X$ such that

$$\langle x^*, w_I \rangle = \int_I \langle x^*, f(t) \rangle dt, \quad \text{for every } x^* \in X^*.$$

We call w_I the *Henstock-Kurzweil-Pettis integral* of f over I and we write $w_I := (HKP) \int_a^b f(t) dt$.

We denote by $HKP([0, 1], X)$ the set of all X -valued Henstock-Kurzweil-Pettis integrable functions on $[0, 1]$ (functions that are scalarly equivalent are identified).

Definition 1.3. A function $f: [0, 1] \rightarrow X^*$ is said to be *w^* -scalarly Henstock-Kurzweil integrable* if, for each $x \in X$, the function $x f$ is Henstock-Kurzweil integrable. A w^* -scalarly Henstock-Kurzweil integrable function $f: [0, 1] \rightarrow X^*$ is said to be *Henstock-Kurzweil-Gelfand integrable* (or simply *HKG-integrable*) if, for each interval $I \in \mathcal{I}$, there exists a vector $\Psi(I) \in X^*$ such that for every $x \in X$

$$\langle x, \Psi(I) \rangle = (HK) \int_I \langle x, f(t) \rangle dt.$$

We call $\Psi(I)$ the *Henstock-Kurzweil-Gelfand integral* of f over I and we write $\Psi(I) := (HKG) \int_I f(t) dt$. Ψ is called the *HKG-primitive* of f .

Following the proof of [9, Theorem 3] (with suitable changes), it is easy to see that a function $f: [0, 1] \rightarrow X^*$ is *HKG*-integrable if and only if f is w^* -scalarly Henstock-Kurzweil integrable.

Throughout, we identify a function $\Psi: [0, 1] \rightarrow X$ (resp. $\Psi: [0, 1] \rightarrow X^*$) with the additive interval function $\Psi: \mathcal{I} \rightarrow X$ (resp. $\Psi: \mathcal{I} \rightarrow X^*$) defined by $\Psi(I) = \Psi(b) - \Psi(a)$, if $I = [a, b]$. And conversely, with each $\Psi: \mathcal{I} \rightarrow X$, (resp. $\Psi: \mathcal{I} \rightarrow X^*$) we associate $\Psi: [0, 1] \rightarrow X$ (resp. $\Psi: [0, 1] \rightarrow X^*$) by setting $\Psi(t) = \Psi([0, t])$.

Definition 1.4. A function $f: [0, 1] \rightarrow X$ is said to be *scalarly* measurable (*scalarly* integrable) if, for each $x^* \in X^*$, the function x^*f is Lebesgue measurable (integrable). A scalarly integrable function $f: [0, 1] \rightarrow X$ is said to be *Pettis integrable* if, for each set $A \in \mathcal{L}$ there exists a vector $\nu_f(A) \in X$ such that for every $x^* \in X^*$

$$\langle x^*, \nu_f(A) \rangle = \int_A \langle x^*, f(t) \rangle dt.$$

We call $\nu_f(A)$ the *Pettis integral* of f over A and we write $\nu_f(A) := (P)\int_A f(t) dt$. It is known (see [15]) that $\nu_f: \mathcal{L} \rightarrow X$ is a measure of σ -finite variation.

Definition 1.5. A function $f: [0, 1] \rightarrow X^*$ is said to be w^* -scalarly measurable (resp. w^* -scalarly integrable) if, for each $x \in X$, the function xf is Lebesgue measurable (resp. integrable). It is well known that each w^* -scalarly integrable function $f: [0, 1] \rightarrow X^*$ is *Gelfand integrable*, that is, for each set $A \in \mathcal{L}$, there exists a vector $\nu(A) \in X^*$ such that

$$\langle x, \nu(A) \rangle = \int_A \langle x, f(t) \rangle dt,$$

for every $x \in X$.

We call the set function $\nu: \mathcal{L} \rightarrow X^*$ the *Gelfand integral* of f on $[0, 1]$ and we write $\nu(A) := (G)\int_A f(t) dt$.

Definition 1.6. A function $f: [0, 1] \rightarrow X^*$ is said to be *weak*-scalarly bounded* on E if

$$\exists M > 0 \forall x \in B(X) |\langle x, f \rangle| \leq M \quad \text{a.e. on } E.$$

A function $f: [0, 1] \rightarrow X$ is said to be *scalarly bounded on E* , if it is weak*-scalarly bounded, when considered as an X^{**} -valued function.

Definition 1.7. Let $\Phi: [0, 1] \rightarrow X$ be a function. If there is a function $\Phi'_p: [0, 1] \rightarrow X$ such that for each $x^* \in X^*$

$$\lim_{h \rightarrow 0} \frac{x^*(\Phi < t, t + h >)}{|h|} = x^*(\Phi'_p(t)),$$

for almost all $t \in [0, 1]$ (the exceptional sets depend on x^*), then Φ is said to be *pseudo-differentiable* on $[0, 1]$, with *pseudo-derivative* Φ'_p (see [16], p. 300).

Let $\Phi : [0, 1] \rightarrow X^*$ be a function. If there is a function $\Phi'_p : [0, 1] \rightarrow X^*$ such that for each $x \in X$

$$\lim_{h \rightarrow 0} \frac{x(\Phi < t, t + h >)}{|h|} = x(\Phi'_p(t)) ,$$

for almost all $t \in [0, 1]$ (the exceptional sets depend on x^*), then Φ is said to be *w*-pseudo-differentiable on $[0, 1]$* , with *w*-pseudo-derivative Φ'_p* .

2. Variational measures.

Definition 2.1. Given an additive interval function $\Phi : \mathcal{I} \rightarrow X$, a gauge δ and a set $E \subset [0, 1]$ we define

$$\text{Var}(\Phi, \delta, E) = \sup \left\{ \sum_{i=1}^p \|\Phi(I_i)\| : \begin{array}{l} \{(I_i, t_i) : i = 1, \dots, p\} \text{ } \delta\text{-fine} \\ \text{partition anchored on } E \end{array} \right\}$$

if $E \neq \emptyset$ and $\text{Var}(\Phi, \delta, \emptyset) = 0$. Then we set

$$V_\Phi(E) = \inf \{ \text{Var}(\Phi, \delta, E) : \delta \text{ is a gauge on } E \}$$

if $E \neq \emptyset$ and $V_\Phi(\emptyset) = 0$.

We call V_Φ the *variational measure generated by Φ* . V_Φ is known to be a metric outer measure in $[0, 1]$ (see [17]). In particular, V_Φ restricted to Borel subsets of $[0, 1]$ is a measure. We say that V_Φ is absolutely continuous with respect to λ (we write then $V_\Phi \ll \lambda$), if $\lambda(E) = 0$ yields $V_\Phi(E) = 0$, for all $E \in \mathcal{L}$. Notice that if $V_\Phi \ll \lambda$, then given $\varepsilon > 0$ and $\emptyset \neq E \in \mathcal{L}$ with $|E| = 0$, there exists a gauge δ such that $\text{Var}(\Phi, \delta', E) < \varepsilon$, for every $\delta' \leq \delta$.

If Φ is continuous, then $V_\Phi(I) \leq |\Phi|(I)$ for every $I \in \mathcal{I}$, where

$$|\Phi|(I) = \sup \left\{ \sum_{i=1}^p \|\Phi(I_i)\| : I_i \text{ are non-overlapping subintervals of } I \right\} .$$

We would like to remark that if Φ is discontinuous the inequality $V_\Phi(I) \leq |\Phi|(I)$ may fail. As an example consider Φ on $[0, 1]$ defined in the following way: $\Phi(t) = 1$ for $t \in [0, 1/2)$, $\Phi(t) = 0$ for $t \in [1/2, 1]$. Φ is not continuous, and $V_\Phi([1/2, 1]) = 1 > |\Phi|([1/2, 1]) = 0$.

Moreover we say that a variational measure V_Φ is *σ -finite* if there is a sequence of (pairwise disjoint) sets F_n covering $[0, 1]$ and such that $V_\Phi(F_n) < \infty$, for every $n \in \mathbb{N}$.

By a result of Thomson (see [17, Theorem 3.15]) it follows that the sets F_n in the previous definition can be taken from \mathcal{L} .

We recall that a function $\Phi : [0, 1] \rightarrow X$ is said to be *BV** on a set $E \subseteq [0, 1]$ if $\sup \sum_{i=1}^n \omega(\Phi(J_i)) < +\infty$, where the supremum is taken over all finite collections $\{J_1, \dots, J_n\}$ of non overlapping intervals in \mathcal{I} with end-points in E , and the symbol $\omega(\Phi(J))$ stands for $\sup\{\|\Phi(u) - \Phi(z)\| : u, z \in J\}$. The function Φ is said to be *BVG** on $[0, 1]$ if $[0, 1] = \bigcup_n E_n$ and Φ is *BV** on each E_n .

In the following we will use the following results proved in [2].

Proposition 2.2. *Let $\Phi: \mathcal{I} \rightarrow X$ be an additive interval function.*

1. *If $V_\Phi \ll \lambda$, then Φ is continuous on $[0, 1]$ and V_Φ is σ -finite.*
2. *V_Φ is σ -finite if and only if Φ is BVG_* on $[0, 1]$.*

In case of a separable Banach space X and Φ being an HKP-integral we are able to describe the variational measure V_Φ more precisely. Our result generalizes a well known fact for real valued functions.

Proposition 2.3. *Assume that X is a separable Banach space, $\Phi: \mathcal{I} \rightarrow X$ is additive and*

$$\Phi(I) = (HKP) \int_I f(t) dt.$$

If $V_\Phi \ll \lambda$, then

$$V_\Phi(E) = \int_E \|f\| dt, \quad \text{for every } E \in \mathcal{L}.$$

Proof. By Proposition 2.2, V_Φ is σ -finite and so Φ is a BVG^* function. Moreover, by [13, Theorem 9], for each measurable set E , we have

$$V_\Phi(E) = \int_E |\overline{D}|\Phi(t) dt$$

where the symbol $|\overline{D}|\Phi(t)$ denotes the upper absolute derivative of Φ in t , that is

$$|\overline{D}|\Phi(t) = \limsup_{h \rightarrow 0} \frac{\|\Phi \langle t, t+h \rangle\|}{|h|}.$$

Let us observe that since Φ is the HKP-primitive of f , then f is a pseudo-derivative of Φ . Now, since X is separable, then by a result in an unpublished paper of Gordon [11] (see also [13]), Φ is differentiable a. e. on $[0, 1]$ with derivative f . So $|\overline{D}|\Phi(t) = \|f\|$ a.e. on $[0, 1]$ and this completes the proof. \blacksquare

Question 2.4. Do we have always $V_\Phi(E) = \int_E \|f(t)\| dt$ or $V_\Phi(E) \leq \int_E \|f\| d\lambda$, for every $E \in \mathcal{L}$, if the function $\|f\|$ is measurable?

Besides the above variational measure we define the following two outer measures, introduced for technical reasons only:

$$W_\Phi^w(E) = \sup_{x^* \in B(X^*)} V_{x^*\Phi}(E), \quad \text{if } \Phi: \mathcal{I} \rightarrow X$$

and

$$W_\Phi^*(E) = \sup_{x \in B(X)} V_{x\Phi}(E), \quad \text{if } \Phi: \mathcal{I} \rightarrow X^*.$$

In general, the two outer measures are not metric and not all Borel subsets of $[0, 1]$ are measurable with respect to them.

Let us observe that if $\Phi : \mathcal{I} \rightarrow X^*$ is an additive interval function, then by the definitions of variational measures we have:

$$W_\Phi^*(E) \leq W_\Phi^w(E) \leq V_\Phi(E) \quad (1)$$

for every $E \subset [0, 1]$. In fact, for every $I \in \mathcal{I}$, $x^{**} \in B(X^{**})$ and $x \in B(X)$, we have: $|x^{**}\Phi(I)| \leq \|\Phi(I)\|$ and $|x\Phi(I)| \leq \|\Phi(I)\|$. So $V_{x^{**}\Phi}(E) \leq V_\Phi(E)$, $V_{x\Phi}(E) \leq V_\Phi(E)$ and inequalities (1) follow.

Definition 2.5. Let V be one of the above introduced outer measures and let $AV := \{\frac{V(E)}{|E|} : |E| > 0\}$ be the *average range* of V . We say that AV is *locally bounded* if there are sets $E_n \in \mathcal{L}$ such that $|\bigcup_n E_n| = 1$ and $V(E_n \cap E) \leq n|E_n \cap E|$, for every $n \in \mathbb{N}$ and $E \in \mathcal{L}$.

Proposition 2.6. *Let $\Phi : \mathcal{I} \rightarrow X$. If $V_\Phi \ll \lambda$, then AV_Φ is locally bounded.*

Proof. By Proposition 2.2 we have that V_Φ is σ -finite. Since $V_\Phi|_{\mathcal{L}}$ is a measure, applying the Radon-Nikodým Theorem, we conclude that AV_Φ is locally bounded. \blacksquare

Remark 2.7. Assume that

$$\Phi(I) = (HKP) \int_I f(t) dt.$$

In general V_Φ is neither σ -finite nor absolutely continuous. In fact, if V_Φ is σ -finite, then by Proposition 2.2, Φ is a BVG_* function. So, if X has the RNP, then Φ is a.e. differentiable (see [1, Theorem 3.6]). But by a result in [5] we know that in each infinite dimensional Banach space (in particular in a conjugate space with the RNP) there exist strongly measurable Pettis (and then Henstock-Kurzweil-Pettis) integrable functions whose Pettis integrals are nowhere differentiable. Each such a function is HKP-integrable and induces a non- σ -finite variational measure V_Φ .

In the general case the following characterization holds.

Proposition 2.8. *A function $\Phi : [0, 1] \rightarrow X$ is an HKP-primitive (of a function f) if and only if $W_\Phi^w \ll \lambda$ and Φ is pseudo-differentiable (with pseudo-derivative f).*

Proof. The proof follows at once from the characterization of the primitives of real valued HK-integrable functions (see [3]). \blacksquare

3. Henstock-Kurzweil-Gelfand integral.

The following result gives a full description of X^* -valued additive interval measures that can be represented as an HKG-integral.

Theorem 3.1. *An additive function $\Phi : \mathcal{I} \rightarrow X^*$ is an HKG-primitive if and only if $W_\Phi^* \ll \lambda$ and AW_Φ^* is locally bounded.*

Proof. Assume first that $f: [0, 1] \rightarrow X^*$ is *HKG*-integrable and let $\Phi(I) = (HKG) \int_I f(t) dt$, for every $I \in \mathcal{I}$. Since $xf \in HK[0, 1]$ for every $x \in X$, we have $V_{x\Phi} \ll \lambda$, and so also $W_{\Phi}^* \ll \lambda$. Moreover, according [14, Corollary 3.1] there are pairwise disjoint sets $E_n \in \mathcal{L}$ such that $\bigcup_n E_n = [0, 1]$ and $|xf\chi_{E_n}| \leq n$ a.e., for each $x \in B(X)$ (the exceptional sets depend on x). It follows that every $f\chi_{E_n}$ is Gelfand integrable.

According to [4] and [6] we have also

$$V_{x\Phi}(E \cap E_n) = \int_{E \cap E_n} |xf(t)| dt \leq n|E \cap E_n| \|x\|$$

for every $E \in \mathcal{L}$ and $n \in \mathbb{N}$. Hence $W_{\Phi}^*(E \cap E_n) \leq n|E \cap E_n|$ and consequently AW_{Φ}^* is locally bounded.

Assume now that $W_{\Phi}^* \ll \lambda$ and AW_{Φ}^* is locally bounded. Then $V_{x\Phi} \ll \lambda$ for every $x \in X$. According to [3], for every $x \in B(X)$, let $f_x \in HK[0, 1]$ be such that

$$\langle x, \Phi(I) \rangle = (HK) \int_I f_x(t) dt \quad \text{for every } I \in \mathcal{I}.$$

Let ρ be a lifting on $L_{\infty}[0, 1]$. Since AW_{Φ}^* is locally bounded, there are pairwise disjoint sets $E_n = \rho(E_n) \in \mathcal{L}$ such that $|\bigcup_n E_n| = 1$ and

$$W_{\Phi}^*(E_n \cap E) \leq n|E_n \cap E|, \quad \text{for every } n \in \mathbb{N} \text{ and } E \in \mathcal{L}. \quad (2)$$

According to [4] and [6], then

$$V_{x\Phi}(E) = \int_E |f_x(t)| dt \quad \text{for every } E \in \mathcal{L} \text{ and } x \in X. \quad (3)$$

In particular (3) holds true for measurable $E \subseteq E_n$. It follows from (2) and (3) that for every $n \in \mathbb{N}$ and $x \in B(X)$ we have $|f_x|\chi_{E_n} \leq n\chi_{E_n}$, a.e. In particular

$$|\rho(f_x)|(t)\chi_{E_n}(t) = \rho(|f_x|)(t)\chi_{E_n}(t) \leq n \quad \text{for every } t \in [0, 1], x \in B(X) \text{ and } n \in \mathbb{N}.$$

Define now a function $f: [0, 1] \rightarrow X^*$ by setting for each $x \in X$

$$\langle x, f(t) \rangle = \begin{cases} \rho(f_x)(t)\chi_{E_n}(t) & \text{if } t \in E_n \\ 0 & \text{if } t \notin \bigcup_n E_n \end{cases}$$

For each $t \in E_n$ the function $x \rightarrow \langle x, f(t) \rangle$ is linear and $|\langle x, f(t) \rangle| \leq n\|x\|$. If $t \notin \bigcup_n E_n$, then $f(t) = 0$. It follows that $f(t) \in X^*$, for every t .

Since $\langle x, f \rangle \stackrel{\text{a.e.}}{=} f_x \in HK[0, 1]$, we get the representation

$$\langle x, \Phi(I) \rangle = (HK) \int_I \langle x, f(t) \rangle dt \quad \text{for every } I \in \mathcal{I}. \quad (4)$$

of Φ as an *HKG*-integral of f . ■

It follows from the construction of f that it is w^* -scalarly bounded, hence Gelfand integrable on every E_n . It is a consequence of lifting measurability properties that $\|f\|$ is measurable on every E_n , and so on $[0, 1]$.

If X^* has the WRNP, then according to [14, Proposition 12.3] and [14, Corollary 3.1.], f is Pettis integrable and scalarly bounded on each E_n . Thus, we can formulate the following consequence of the proof of Theorem 3.1:

Corollary 3.2. *Assume that $\Phi : \mathcal{I} \rightarrow X^*$ is an HKG-primitive. Then there exists a function $f : [0, 1] \rightarrow X^*$ such that f is a weak*-pseudo-derivative of Φ and there exists a sequence of pairwise disjoint sets $E_n \in \mathcal{L}$ such that $\bigcup_n E_n = [0, 1]$, f is weak*-scalarly bounded and Gelfand integrable on every E_n , $n \in \mathbb{N}$, $AW_{\Phi}^*(E_n) < \infty$ and $\|f\|$ is measurable.*

If X^ has the WRNP, then f and the sets E_n $n \in \mathbb{N}$ can be taken in such a way that f is Pettis integrable and scalarly bounded on each E_n .*

If $V_{\Phi} \ll \lambda$, then by Proposition 2.6, AV_{Φ} is locally bounded. Consequently, in view of (1), AW_{Φ}^* is locally bounded. Thus, the following result is a direct consequence of Theorem 3.1.

Proposition 3.3. *Let $\Phi : \mathcal{I} \rightarrow X^*$ be additive and such that $V_{\Phi} \ll \lambda$. Then Φ is an HKG-primitive.*

4. Henstock-Kurzweil-Pettis integral.

We begin with the following characterization of Pettis integrability that holds true in case of an arbitrary perfect measure in place of the Lebesgue one.

Proposition 4.1. *For a scalarly integrable function $f : [0, 1] \rightarrow X$ the following conditions are equivalent:*

1. f is Pettis integrable;
2. the mapping $X^* \ni x^* \rightarrow x^* f \in L_1[0, 1]$ is $\tau_c(X^*, X)$ -norm continuous;
3. the mapping $X^* \ni x^* \rightarrow x^* f \in L_1[0, 1]$ is $\tau(X^*, X)$ -norm continuous.

Proof. (i) \Rightarrow (ii) Since f is Pettis integrable, the functional $x^* \rightarrow \int_E \langle x^*, f(t) \rangle dt$ is, for each $E \in \mathcal{L}$, weak*-continuous (cf. [14]). Due to Stegall's result [8], the set $\nu_f(\mathcal{L})$ is norm relatively compact. Hence, if $x_{\alpha}^* \xrightarrow{\tau_c(X^*, X)} x_0^*$, then $x_{\alpha}^* \rightarrow x_0^*$ uniformly on $\nu_f(\mathcal{L})$. It follows that $\lim_{\alpha} \int_0^1 |x_{\alpha}^* f(t) - x_0^* f(t)| dt = 0$.

(i) \Rightarrow (iii) The proof is almost the same.

(iii) \Rightarrow (i) If $x_{\alpha}^* \xrightarrow{\tau(X^*, X)} x_0^*$, then $\int_E \langle x_{\alpha}^*, f(t) \rangle dt \rightarrow \int_E \langle x_0^*, f(t) \rangle dt$ for each $E \in \mathcal{L}$. Thus, the functional $x^* \rightarrow \int_E \langle x^*, f(t) \rangle dt$ is, for each $E \in \mathcal{L}$, weak*-continuous. Consequently, f is Pettis integrable (see [14]).

(ii) \Rightarrow (i) The proof is the same, but now we assume that $B(X^*) \ni x_{\alpha}^* \xrightarrow{\sigma(X^*, X)} x_0^*$. We obtain now the weak* continuity of the functionals $x^* \rightarrow \int_E \langle x^*, f(t) \rangle dt$ on $B(X^*)$, but due to the Banach-Dieudonné Theorem (see [12, p. 154]) this yields its weak* continuity. Consequently, f is Pettis integrable (see [14]). \blacksquare

In order to obtain a complete characterization of the HKP-primitive of functions taking values in a dual space with the WRNP, we need some preliminary results.

Proposition 4.2. *Assume that $\Phi : \mathcal{I} \rightarrow X$ is of the form*

$$\Phi(I) = (\text{HKP}) \int_I f(t) dt, \quad \text{for each } I \in \mathcal{I}.$$

Then, for each $I \in \mathcal{I}$, the mapping $x^ \rightarrow \int_I \langle x^*, f(t) \rangle dt$ is weak*-continuous. Moreover, there exists a partition $[0, 1] = \bigcup_k H_k$ such that, for every $k \in \mathbb{N}$, f is Pettis integrable and scalarly bounded on H_k , $AW_\Phi^w(H_k) < \infty$ and the functional $x^* \rightarrow V_{x^*\Phi}(H_k)$ is $\tau_c(X^*, X)$ -continuous.*

Proof. The first continuity fact has been proven in [7]. Exactly as in the proof of Theorem 3.1 one can obtain a sequence of pairwise disjoint sets $E_n \in \mathcal{L}$ such that $AW_\Phi^w(E_n) < \infty$, for each $n \in \mathbb{N}$. It follows also from [7, Corollary 1] that there exists a decomposition $[0, 1] = \bigcup_k F_k$ into sets of positive measure such that f is Pettis integrable and scalarly bounded on each F_k . Denote by $\{H_k : k \in \mathbb{N}\}$ the collection of all intersections $E_n \cap F_m$ of positive measure. Then, by Proposition 4.1, for each k , the function $x^* \rightarrow x^* f|_{H_k}$ is $\tau_c(X^*, X)$ -norm continuous as a map from X^* to $L_1(\lambda|_{H_k})$, because f is Pettis integrable on H_k . Consequently, if $x_\alpha^* \xrightarrow{\tau_c(X^*, X)} x_0^*$, then according to [4] and [6] we have

$$\lim_\alpha V_{(x_\alpha^* - x_0^*)\Phi}(H_k) = \lim_\alpha \int_{H_k} |x_\alpha^* f(t) - x_0^* f(t)| dt = 0. \quad \blacksquare$$

Lemma 4.3. *(see [1, Lemma 3.3]) Let X be a Banach space and let $\nu : \mathcal{L} \rightarrow Y$ be a λ -continuous measure of finite variation. If $\Phi : \mathcal{I} \rightarrow X$ is defined by $\Phi(I) := \nu(I)$, for all $I \in \mathcal{I}$, then V_Φ is finite, $V_\Phi \ll \lambda$ and $V_\Phi(E) \leq |\nu|(E)$, whenever $E \in \mathcal{L}$.*

Theorem 4.4. *Let X be a Banach space. Consider the following two properties of an additive interval function $\Phi : \mathcal{I} \rightarrow X$:*

- (k) $W_\Phi^w \ll \lambda$ and there exists a decomposition $[0, 1] = \bigcup_k H_k$ of $[0, 1]$ into sets of positive measure such that for every $k \in \mathbb{N}$ the function $x^* \rightarrow V_{x^*\Phi}(H_k)$ is $\tau(X^*, X)$ -continuous and $AW_\Phi^w(H_k) < \infty$.
- (kk) There is an HKP-integrable function $f : [0, 1] \rightarrow X$ such that

$$\langle x^*, \Phi(I) \rangle = (\text{HK}) \int_I \langle x^*, f(t) \rangle dt \quad \text{for every } I \in \mathcal{I}.$$

If (k) \Rightarrow (kk) for every additive $\Phi : \mathcal{I} \rightarrow X$, then X has the WRNP.

Proof. Let $\nu : \mathcal{L} \rightarrow X$ be a λ -continuous measure of finite variation. Define $\Phi : \mathcal{I} \rightarrow X$ by $\Phi(I) := \nu(I)$. It follows from Lemma 4.3 that $V_\Phi \ll \lambda$ and V_Φ is finite. So $\Phi : \mathcal{I} \rightarrow X$ is an additive interval measure such that $V_{x^*\Phi} \ll \lambda$ for every $x^* \in X^*$. Moreover, $V_{x^*\Phi}(E) \leq |x^*\nu|(E)$, for every $E \in \mathcal{L}$. Let $\langle x_\alpha^* \rangle \subset B(X^*)$ be a net of functionals that is $\tau(X^*, X)$ -convergent to 0. Since $\nu(\mathcal{L})$ is a weakly

relatively compact subset of X , the net $\langle x_\alpha^* \nu \rangle$ is uniformly convergent to zero on \mathcal{L} . Hence, $\lim_\alpha |x_\alpha^* \nu|[0, 1] = 0$. By the inequality $V_{x_\alpha^* \Phi}(E) \leq |x_\alpha^* \nu|(E)$, for every $E \in \mathcal{L}$, we have also $\lim_\alpha V_{x_\alpha^* \Phi}[0, 1] = 0$, what proves the weak*-continuity of the map $x^* \rightarrow V_{x^* \Phi}[0, 1]$.

We are going to prove yet the local boundedness of W_Φ^w . To do it notice that the classical Radon-Nikodým Theorem yields the existence of a decomposition $[0, 1] = \bigcup_k H_k$ such that $|\nu|(E) \leq k|E|$, for every measurable $E \subset H_k$. It follows that

$$\frac{V_{x^* \Phi}(E)}{|E|} \leq \frac{|x^* \nu|(E)}{|E|} \leq k$$

and hence $AW_\Phi^w(H_k) < \infty$.

Thus, condition (k) is satisfied. Hence, there is a Henstock-Kurzweil-Pettis integrable function $f : [0, 1] \rightarrow X$ such that

$$\Phi(I) = (HKP) \int_I f(t) dt, \quad \text{for every } I \in \mathcal{I}.$$

Proceeding as in the proof of [2, Theorem 4.5] we see that f is also Pettis integrable and ν is its indefinite Pettis integral. \blacksquare

Proposition 4.5. *Let X be an arbitrary Banach space and $\Phi : \mathcal{I} \rightarrow X$ be an additive interval function such that $W_\Phi^w \ll \lambda$. Assume that there is a decomposition $[0, 1] = \bigcup_k H_k$ into measurable sets of positive measure such that $V_{x^* \Phi}(H_k) < \infty$ for every $k \in \mathbb{N}$ and every $x^* \in X^*$ and, for every $k \in \mathbb{N}$, the function $x^* \rightarrow V_{x^* \Phi}(H_k)$ is sequentially weak*-continuous.*

If $f : [0, 1] \rightarrow X$ is a scalarly measurable function, then the set

$$K = \left\{ x^* \in X^* : x^* f \in HK[0, 1] \text{ and } x^* \Phi(I) = (HK) \int_I \langle x^*, f(t) \rangle dt, \forall I \in \mathcal{I} \right\}$$

is sequentially weak-closed.*

If for every $k \in \mathbb{N}$, the function $x^ \rightarrow V_{x^* \Phi}(H_k)$ is $\tau(X^*, X)$ -continuous and f is Pettis integrable on H_k , then K is weak*-closed.*

Proof. It is obvious that $K \neq \emptyset$ and K is convex. Notice first that if $x^* \in K$, then $(x^* \Phi)' = x^* f$ a.e. (see [10]). Let $\{x_n^*\} \subset K$ be such that $x_n^* \rightarrow x_0^*$ in the w^* -topology. We may assume, without loss of generality, that all x_n^* , $n = 0, 1, 2, \dots$ belong to $B(X^*)$. By hypothesis $V_{x_n^* \Phi} \ll \lambda$, and so there exists $g \in HK[0, 1]$ such that $x_0^* \Phi(I) = (HK) \int_I g(t) dt$, for all $I \in \mathcal{I}$ (cf. [3]).

By the assumption and by [6, Corollary 3] we have, for each $k \in \mathbb{N}$,

$$\lim_n \int_{H_k} |x_n^* f(t) - g(t)| dt = \lim_n V_{(x_n^* - x_0^*) \Phi}(H_k) = 0.$$

Hence, there is a subsequence $\{x_{k, n_m}^*\}_m$ of $\{x_n^*\}$ with $\lim_m x_{k, n_m}^* f = g$, a.e. on H_k . It follows that $g = x_0^* f$ a.e. and so $x_0^* f \in HK[0, 1]$. Moreover

$$\lim_m \int_I \langle x_{k, n_m}^*, f(t) \rangle dt = \lim_m \langle x_{k, n_m}^*, \Phi(I) \rangle = \langle x_0^*, \Phi(I) \rangle = \int_I \langle x_0^*, f(t) \rangle dt.$$

This yields $x_0^* \in K$ and so K is weak* sequentially closed.

Assume now that f is Pettis integrable on every H_k . We are going to prove that K is weak*-closed. We know that for each $k \in \mathbb{N}$ the function $x^* \rightarrow x^*g|_{H_k}$ is $\tau(X^*, X)$ -norm continuous as a map from X^* to $L_1(\lambda|_{H_k})$. Consequently, if $x_\alpha^* \xrightarrow{\tau(X^*, X)} x_0^*$, then

$$\lim_{\alpha} \int_{H_k} |x_\alpha^* f(t) - x_0^* f(t)| dt = 0.$$

By hypothesis $V_{x_0^* \Phi} \ll \lambda$, and so there exists $g \in HK[0, 1]$ such that $x_0^* \Phi(I) = (HK) \int_I g(t) dt$, for all $I \in \mathcal{I}$ and so [6, Corollary 3] we have

$$\lim_{\alpha} \int_{H_k} |x_\alpha^* f(t) - g(t)| dt = \lim_{\alpha} V_{(x_\alpha^* - x_0^*) \Phi}(H_k) = 0.$$

It follows that $x_0^* f = g \in HK[0, 1]$. Moreover

$$\lim_{\alpha} \int_I \langle x_\alpha^*, f(t) \rangle dt = \lim_{\alpha} \langle x_\alpha^*, \Phi(I) \rangle = \langle x_0^*, \Phi(I) \rangle = \int_I \langle x_0^*, f(t) \rangle dt$$

and so $x_0^* \in K$. Thus, K is $\tau(X^*, X)$ -closed, and as it is convex, it is also weak*-closed. \blacksquare

Now we are ready to prove the main result of this section.

Theorem 4.6. *Let X be a Banach space such that X^* has the WRNP and let $\Phi : \mathcal{I} \rightarrow X^*$ be an additive interval measure. Then the following two conditions are equivalent:*

- (j) $W_{\Phi}^w \ll \lambda$ and there exists a decomposition $[0, 1] = \bigcup_k H_k$ of $[0, 1]$ into sets of positive measure such that for every $k \in \mathbb{N}$ the function $x^{**} \rightarrow V_{x^{**} \Phi}(H_k)$ is weak*-continuous and $AW_{\Phi}^*(H_k) < \infty$.
- (jj) There is an HKP-integrable function $f : [0, 1] \rightarrow X^*$ such that

$$\langle x^{**}, \Phi(I) \rangle = (HK) \int_I \langle x^{**}, f(t) \rangle dt \quad \text{for every } I \in \mathcal{I}.$$

Moreover, f can be chosen in such a way that $\|f\|$ is a measurable function.

Proof. The implication (jj) \Rightarrow (j) is a particular case of Proposition 4.2. In order to prove the implication (j) \Rightarrow (jj), we may apply Theorem 3.1 to conclude that there exists a function $f : [0, 1] \rightarrow X^*$ that is HKG-integrable on $[0, 1]$ and Pettis integrable on each H_k , $k \in \mathbb{N}$. Proposition 4.5 yields the HKP-integrability of f on $[0, 1]$. \blacksquare

Remark 4.7. According to Remark 2.7 each strongly measurable Pettis integrable (and hence also Henstok-Kurzweil-Pettis integrable) function with nowhere differentiable Pettis integral satisfies the conditions (j) and (jj) of Theorem 4.6 and has non- σ -finite variational measure V_{Φ} .

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