Regular star polyhedra in the nineteenth-century Italian treatise

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Abstract. During the Unification of Italy, great mathematicians were involved in the writing of works that were educationally effective and closer to the scientific results that were just being achieved at the time ([8]). In 1858 was published the *Trattato di Geometria Elementare*,([5]) Giovanni Novi's Italian translation of the French textbook by Antoine Amiot, *Leçons nouvelles de géométrie élémentaire* ([1]) Novi's substantial contribution was to integrate the French treatise with theories that had just been developed. A subject on which he dwells are regular star polihedra, starting from theories developed by L. Poinsot, J.L.F. Bertrand and A.L. Cauchy.

Keywords: history of education, regular star polyhedra.

1 Introduction

The climate of renewed Risorgimento fervour that pervaded pre-Unification Italy in the mid-nineteenth century made itself felt with particular vehemence in the universities, and mathematicians were no exception. This current involved leading figures who contributed to elevating the status of Italian mathematics to the level enjoyed beyond the Alps. Correlated to the relaunching of Italian mathematics was the problem of drafting textbooks and manuals for students aimed at raising the level of teaching in schools and universities to that of the other European countries. This was an objective that involved many mathematicians of the Risorgimento generation, who were occupied with both writing manuals, sometimes based on their own lessons, and with the translations of foreign textbooks.

In 1858 was published the *Trattato di Geometria Elementare*, Giovanni Novi's Italian translation of the French textbook of 1850 by Antoine Amiot, *Leçons nouvelles de géométrie élémentaire*. Novi's substantial contribution was to integrate the French treatise with the theories, some of which had just been developed, of Louis Poinsot, Girard Desargues, Lazare Carnot, Jean Victor Poncelet, Joseph Diez Gergonne, Jacob Steiner and Michel Chasles. Considered as a transitional undertaking, Novi's work of integrating new notions into Amiot's textbook turns out to be particularly interesting, a useful starting point for diffusing the new theories, and even though his book did not yet fully conform to the requirements that were being defined at the time, it was in any case an attempt that was almost unique at that time.

In Novi's Italian version, the text appears much more accurate and precise, rich in detailed historical references, which are also interesting from a didactic point of view, in that they provide on a perspective on the Italian authors intention of rendering geometry more enjoyable in the eyes of the students; some of them are starting points that lead to the idea of a 'creative mathematics'. In the introduction to the Italian version, Novi examines the didactic aims of the work and the reasons for the serious gap between the works of elementary geometry and the state of research in that field, tracing them back to the widely-held opinion that geometry is only of use if aimed at practical purposes. For some, he goes on to say, geometry is subordinate to analysis, and that in any case the only geometry that can be adopted is that of Euclid. Novi hopes of introducing the young students to the new and fruitful geometric theories, and wanting to improve studies of geometry in schools.

2 Polyhedra in Giovanni Novi's text

Novi added ten notes as a kind of 'Complement to Geometry'. The second note is aimed at introducing the Poinsot's polyhedra of a higher order. The first theorem presented in the work is the famous Euler formula among faces, vertices and edges of a polyhedron which Amiot has enunciated exclusively for convex polyhedra. Novi completes the notion, explaining that earlier Poinsot in the 1809 defined the most wide class of solid on which count the formulation, containing those polyhedra in which inside is possible to find a point that is the center of a sphere such that, projecting on its surface the faces of the solid (through segments which pass for the center) no one of this projection would end partly or entirely on the projection of another side. In the second part of the note, Novi decides on inserting "Some general considerations on the theory of polyhedra, which seem fairly simple and important as not having to leave out in a treatise of elementary geometry".[1]

These are the most recent theories of Poinsot contained in the *Note sur la théorie des polyedres* ([7]) based on considering a polyhedron as a network of triangles, each one of them is joined to the other by a common side and their set form a closed surface. From here, he shows the validity of Euler's theorem before enounced, which is also reviewed as a special case of the one demonstrated by Cauchy in ([3]): if a polyhedron is decomposed into others taking from the inside new vertices, indicating with P the number of new polyhedra, with C the total number of edges (ribs for Novi and Arêtes for Cauchy), with V the total number of vertices, with F the total number of sides, it follows that the V + F = C + P +1, in fact, Cauchy writes: "If all polyhedra are reduced to only one it follows that P = 1, and the equation will reduce to S + F = A +2 [where *S* stands for number of *sommets*, vertices]".[2]

Amiot continues by illustrating some theorems, aimed to demonstrate that the regular polyhedra are five.

[1] "Talune considerazioni generali sulla teorica dei poliedri, le quali ci sembrano abbastanza semplici ed importanti da non doversi tralasciare in un trattato elementare di Geometria".

[2] "Si l'on suppose tous les polyèdres réduits àun seul, on aura *P*=1, et l'équation se réduira à celle-ci *S*+*F*=*A*+2". ([3], p.16).

Novi demonstrates also the note “each angle of a regular polyhedron has only three or four or five plane angles”; Novi obtains the inequality $4+F \leq 2V \leq 4F-8$, which allows, with the first one, to answer the question about how many polyhedra could be built with a given number of faces. After having proved that exist just five regular polyhedra, in an annotation, he refers to the fact that “Apart from these five regular polyhedra, there are other four polyhedra of superior species due to Mr. Poinsot. Cauchy was the first to demonstrate that there are no other regular polyhedra out of these nine”.[3]

Afterwards he decides to insert, in the endnote, Bertrand’s demonstration ([2]) instead of Cauchy’s demonstration ([3]), about the uniqueness of the existence of just four new polyhedra. Novi refers to the notions treated by Poinsot, one more time, in *Mémoire sur le spolygones et les polyedres*, adding further regular polyhedra relative to the five “platonic solid” usually known: “The main difference of these solids from an ordinary polyhedron, is that in the latter, the faces are projected by the rays on the sphere inscribed or circumscribed polygons correspond in the cover once the sphere, whereas in the others, the corresponding polygons cover only one time the sphere unlike the others which recover it exactly twice, or three times, & c.”[4]

Poinsot defines “species” of a regular polyhedron the integer number which shows how many times the surface is covered by the circumscribed (or inscribed) sphere, projecting on it, from its center, the faces of the polyhedron.

As a consequence of the XI note, Novi concludes the demonstration, resumed from Poinsot, of the fact that there are only four polyhedron of the superior species. The demonstration is the result of the Novi’s theorem and of the corollary of the III theorem; to obtain regular polyhedron of superior species, it is necessary: to consider convex polyhedra , to chose a vertex above one of these polyhedral and to look for other vertices which joined to it, could create a regular polygon. This polygon is the only possible face of the regular polyhedron of superior species having the same vertices. The number of the faces which composes a solid angle of a new polyhedron, is the same as the number of the equal polygons that can belong to the same vertex.

Therefore he comply totally with Bertrand; Cauchy, explains Novi, has led his demonstration using only what he has entered as II theorem: “the regular polyhedra of superior species are the result of the extension of the ribs or faces of regular polyhedral pertaining to the same order and to the first species; showing also that the extension of the faces or edges of the five first species regular polyhedra cannot produce other regular polyhedra except for those owning to Poinsot”. The demonstration continues with: “Very ingenious, it appears unclear when one does not know the models in relief of the dodecahedron and of the regular icosahedrons”.[5]

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[3] “Oltre questi cinque poliedri regolari, ve ne sono altri quattro di specie superiore dovuti al sig. Poinsot. Cauchy ha dimostrato il primo che non vi ha altri poliedri regolari fuori di questi nove”.

[4] “La difference essentielle de ces solid es aux polyèdres ordinaries, est que, dans ceux-ci, les faces é tant projetées par des rayons sur la sphere inscrite ou circonscrite, les polygons corrispon dans recouvrent une seule fois la sphère; au lieu que dans les autres, ces polygones la recouvrent exactement ou deux fois, ou trois fois, &c”.

[5] “Assai ingegnosa, riesce poco chiara quando non si abbiano presenti i modelli in rilievo del dodecaedro e dell’icosaedro regolare”.

The regular polyhedral, which Poinsot has recognized , are described by Novi as in the note to the II theorem, and in the consecutive theorems, giving at the same time certain information about its fulfillment; prolonging the plane which contains each face of the regular dodecahedron of the first species up to the encounter with the five faces which surrounds the opposite face, it will be created regular pentagons of the first or second species.

The set of the first produces a regular dodecahedron of the third kind (nowadays also called the small stellated dodecahedron), and the set of second produces a stellated dodecahedron of the fourth species (great stellated dodecahedron). The first solid has twelve pentahedral angles of the second species and thirty edges; the second has twenty angles trihedral angles and thirty edges. It could be also obtained from the first solid described with the extension of the sides of its faces. By extending the sides of the twelve pentagons of the ordinary dodecahedron, it forms a stellated dodecahedron of the second species, (great dodecahedron) consisting of pentagons of the second species, gathered per five around each vertex.

Prolonging each face of the ordinary icosahedrons up to the encounter with the planes of the three triangles which surrounds the face opposite to that considered, it forms n icosahedron of the seventh species (great icosahedron) composed of twelve angles of the second species and thirty edges.

The book ends with five theorems added by the translator, to demonstrate (and to indicate the its construction) that with a given side it is always possible to create a regular tetrahedron, hexahedron, octahedron, dodecahedron and icosahedron.

Interesting from a didactic point of view is the corollary: to obtain regular polyhedra of superior species, it is necessary to consider the regular convex polyhedra and proceed as follows: to choose a vertex above one of these polyhedra and look for other vertices which reunited can create a regular polygon.

This polygon is the only possible face of the regular polyhedron of superior species having the same vertices of the proposed. The number of the equal polygons to which a vertex can belong, it will be the number of the faces that compose a solid angle of the new polyhedron.

3 The *Trattato* as a transitional work

The publication of the *Trattato di Geometria Elementare* was not an isolated incident, but was part of the larger historical context in which Italian mathematics was inserted. Those same years saw the publication of the *Trattato d'Aritmetica*, Novi's Italian translation of the *Traité d'arithmétique* by Joseph Bertrand; the *Trattato d'Algebra Elementare*, Enrico Betti's translation of Bertrand's *Traité élémentaire d'algèbre*; the *Trattato di Trigonometria*, Antonio Ferrucci's translation of Joseph Alfred Serret's *Traité de Trigonométrie*; the *Elementi d'Aritmetica*, written by Giovanni Novi, and others. One direct result of these early works, begun following the Unification of Italy was the involvement of the great mathematicians in the writing of works that were educationally effective and closer to the scientific results that were just being achieved at the time.

In the opinion of Luigi Cremona, expressed in a note published in 1860 in the journal *Il Politecnico* ([4]), the works just mentioned constituted "the best books, indeed, the only truly good ones that a conscientious teacher of elementary mathematics can adopt in his teaching."

A good part of the French work can be traced back to Legendre's *Eléments de géométrie*. Previous to the unification of Italy, and up to 1867 (the year in which Minister of Education Michele Coppino introduced the use of Euclid's *Elements* as a textbook in upper-level secondary schools), the majority of schools in the various Italian states had adopted translations of foreign volumes as textbooks, but only part of those books were genuinely appreciated, and with regard to projective geometry, Novi's translation was one of them. His book did not yet fully correspond to the requirements that were being outlined. His, however, was a first attempt, almost the only one at that time. In fact, the time seemed right for a renewal of those treatises that constituted the basis of school and university studies, bringing them into line with progress made in science in the previous thirty years; these textbooks constituted a response to questions that the illustrious authors asked themselves about the difficulty of teaching students about new scientific developments, using the new theories to go beyond the limits of elementary teaching ([9], [10]).

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