DOTTORATO DI RICERCA

in

Storia e Didattica delle Matematiche, Storia e Didattica della Fisica,

Storia e Didattica della Chimica

Ciclo XXIV, 2011/2013

SEDE AMMINISTRATIVA: UNIVERSITÀ DI PALERMO

A multimodal semiotic approach to investigate on synergies between Geometry and Chess

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TESI DI DOTTORATO DI RICERCA

SETTORE SCIENTIFICO DISCIPLINARE MAT/04

Palermo, 2014

Acknowledgments

This work is dedicated to my father, who has prematurely passed away. He taught me to be a man, he taught me to be a good man, and I hope that with this work and with my whole life I will make him proud of me.

By the way many persons helped me during these years, and I wish to thank them.

Firstly I have to thanks my family: my mother Emilia, my brothers Andrea and Alessio, my uncles Alberto and Augusto and my aunt Maria.

Secondly I wish to thank Emanuela that with her love saved me from the brink and helped me to produce this work.

Then I wish to thank my tutor Prof. Sona Ceretkova, my co-Tutor Prof. Luis Radford and Benedetto Di Paola for their help and availability.

Finally I wish to thank my colleagues and friends that helped me with some advices or with a simple smile: Alessio Cangemi, Luisa Raimondi, Cecilia Quattrocchi, Antonio Coppola, Giuliano D'Eredità, Pietro Milici and Munder Mohamed.

A particular thank goes to prof. Filippo Spagnolo who approached me to Mathematical Education and shared with me his challenging way to research.

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Chapter 1

Introduction

1.1 The research problem

The aim of this work is to study some processes of learning and teaching geometry and observe situations in which two or more individuals interact with each other in order to achieve a goal. In particular I want to investigate how and whether a non-disciplinary proposal, can help students in pose and solve geometric problems.

Over the past decades research in mathematics education has greatly studied aspects of geometrical thinking and mathematics in general, and given the enormous amount of literature and my purpose, I supported two lines of research based on two different lenses of observation:

1 . Semiotics - cognitive (Duval, 1998a; Fischbein, 1993; Van Hiele, 1986);

2 . Semiotics - cultural (Radford, 2006).

The delicate choice to combine these two streams of research has been done with the intention to investigate one of the "big research problems" proposed by Professor Presmeg which focuses attention on what can be the role of gestures in mathematics visualization (Presmeg, 2006).

In this direction, the first step (theoretical) of this work was to take note of such conflictual situations and obstacles present in the processes of visualization in geometry activities and to study some processes of mathematical thinking and learning in a semiotic point of view.

According to Duval (1998a), geometry involves three kind of cognitive processes that satisfied specific epistemological function: Visualization, Construction and Reasoning .

Duval considers a geometric figures as cognitive apprehension, and he made a distinction between four types of cognitive apprehensions: Perceptual Apprehension, Sequential Apprehension, Discursive Apprehension and Operational Apprehension.

For Duval, Visualization consists only of operative apprehension and measures are matter of discursive apprehension which put an obstacle for reasoning and for visualization.

Always Duval (Duval, 1993) raises a question regarding the representation of mathematical objects, calling cognitive paradox of mathematical thinking.

This paradox lies in the fact that unlike other fields of knowledge, there is no other way to have access to mathematical objects except through their semiotic representation. Therefore, students in the learning phase tend to confuse the subject and representation “there is noetic without semiotics” (Duval, 1998b), and also to be able to manipulate the mathematical treatment, necessarily linked to the representations, they need a conceptual learning of the objects represented “there is no semiotics without noetic” (Duval, 1998b).

With regard to the particular case of the visual representation of the geometric figures, a study of Fischbein (Fischbein , 1993) shows how these have both a conceptual nature and a figural intrinsic nature. In his theory of figural concepts, Fischbein states that the integration of conceptual and figural unitary mental structures, with the predominance of conceptual contents of the figural ones , is not a natural process, and therefore it should be a continuous, systematic and main activity of the teacher.

From the above discussion, Visualization processes appeared as crucial and delicate in geometrical thinking, and therefore in this work it was decided to focus attention on them.

Given the interest to analyze activities of the class, it was necessary to analyze these semiotic-cognitive aspects under a semiotic-cultural sphere, according to which the thinking is considered to be a mediated reflection on the world, in accordance with the shape or the way of activities of individuals (Radford, 2003), and then thought of as a social practice.

In this way, during the semiotic analysis of the activities of reasoning, thinking and construction, I will analyze the use that students and the teacher make of gestures, drawings, words, etc (Radford (2002) called them semiotic means of objectification).

The second point of this work (experimental) was seen to produce a didactical proposal, with the aim of improving certain visual spatial abilities of students, and help them to understand, ask and solve geometric problems.

This educational proposal consisted of a non-disciplinary workshop activity, which in this case was a laboratory of chess and mathematics.

The choice of non-disciplinary activity has been made in the interest of improving some soft skills and not only geometrical, and with the intent to avoid certain difficulties and epistemological and / or teaching obstacles found in a geometric context.

So the focus of this thesis is to understand how to use non-curricular activity to occur a transfer of skills in an environment desired by the researcher or teacher.

The research activity has seen strong synergy between the experimental stages, and therefore intrinsically linked to the activity of chess, and the theoretical phases in which an attempt was made to extend and generalize the results.
The research questions to which I have attempted to answer are the following:

1. What are the criteria for choosing a non-disciplinary activity in order to improve certain skill useful for the geometrical learning and thinking?
2. Is it possible make a theoretical proposal for the idea of transfer of skills according to the semiotic-cultural paradigm of the theory of objectification?
3. Chosen the chess laboratory as a non-disciplinary activity, which skills could be improved? What are the role of chess and math teacher in this multi-disciplinary didactical proposal?

In order to answer to the third question, the experimental activity wanted to investigate the possibility of transfer of skills between a chess course and a course in geometry.
The method of investigation has included the following stages:
A. Designing and control of some geometric items to assess specific visual spatial abilities;
B. Designing a chess-related work about 30 hours and the three phases of control: a quantitative analysis – before and after the chess course, a qualitative analysis – before and after the chess course, and a qualitative analysis during the chess course;
C. Control investigation on the validity of the survey method in which tasks are proposed in different environments and contexts.
The control phase related to point A provides a theoretical investigation about the items and a vertical investigation in which the items were offered to students from different grade school.
The quantitative survey phase relative to point B is useful to understand *what* kind of ability could be improved by a chess course, and test survey were both disciplinary and psychological.
The choice to make a control with a psychological lens has been taken to solve a problem of this type of investigation: the influence of geometry teacher. In fact, if the pre-test and post-test were only disciplinary, possible improvements of the students could be attributed to the action of the teacher of mathematics, not the chess course.
The decision to include disciplinary test was due to the need to have a feedback specification and then have the evidence on which math skills can be improved by this teaching practice.

Key step of this experimental work was, however, the choice of the survey sample and the actual structuring of the research and to do that you followed the paradigm of the theory of Van Hiele (Van Hiele, 1986) on geometry understanding.

The basic elements of this theory are three: the levels of thought, the phases of learning and the insight. Levels of thought, according to Van Hiele, are not a structure located within the study subjects, but they constitute a possible classification of moments belonging to the geometrical thinking.

When I talk about stages of learning, I mean those steps useful to the transition from a lower level to a higher one. With the term "insight" I mean the result of the perception of a structure, and according to Van Hiele the development of this should be understood as the real goal of the teaching.

In geometry van Hiele distinguishes five different levels of thought which are sequential and hierarchical. They are: 1. Visualization, 2. Analysis, 3. Abstraction, 4. Deduction, 5. Rigor.

Characteristics of these levels of thinking are: a) This classification is sequential and hierarchical; b) These levels have inductive nature; c) What is implied in one level to the next is explained; d) Each level has its own language, symbols and information network; e) Two people who think on different levels, they can not understand each other; f) The progresses from one level to another are more dependent on experience teaching that age or maturity.

Given that the aim of investigation are the visualization processes (according to Van Hiele Level 1), it was necessary to choose students whom school level is not too high, and so were chosen students of 6^{th} grade.

In relation to the first research question, it was necessary to make an epistemological analysis of the chess thought expanding the idea of figural concepts (Fischbein, 1993) and introducing a new theoretical idea for chess thinking: the configural concepts (Ferro, 2012).

I started defining chess objects and chess elements. I defined chess objects the elements of the artifact “chess” – the pieces, the squares and the chessboard – which therefore possess rules and scopes.

I defined chess elements the mental (i.e. personal – ideal) entities that a chess player uses and “builds” when he she thinks about a move, a position or a variant with a specific aim.

Then, I introduced the idea of configural concepts. A configural concept is made up of chess objects and their conceptual relationships. Its meaning comes from the hierarchical linkage of the conceptual relationships between the involved chess objects and from its position in the whole theoretical structure of the pieces in the chessboard.

The use of a configural concept depends on the goal that an individual is pursuing in a chess game. It involves the identification of general structure of the game at a certain moment and the role of the configural concept therein.
In terms of learning, configural concepts become noticed and valued through a process in which the student becomes conscious of the chess objects and their conceptual mutual relations. This process is what Radford calls objectification (Radford, 2010).
Then, I analyzed the processes of Objectification (Radford, 2009) of these configural concepts, studying multimodal semiotic activity during chess games, identifying some semiotic means – touch, pointing, linguistic devices – that shows different level of awareness of particular configural concepts.
Conclusive experimental step was to analyze some geometrical activities, with the aim to find out some evidence, using a semiotic approach, of what I have found out with a psychological one.
In parallel to this experimental work, it was made a theoretical one where I wanted to investigate and study the idea of skills and transfer of skills according to a semiotic-cultural approach.
It is defined (learning) skills, such as sensitivity to acknowledge or perceive the layer of generality of the object target of the reflection, and to establish a dialectical process mediated by culture between the subject and the object of which the subject himself becomes aware.
The sensitivity to which I refer is the Kantian sensitivity that is the faculty by which I perceive phenomena and is based on two a priori forms, space and time.
So I can think to the skill as the sensitivity to perceive a phenomenon, to become aware and to endow it with meaning in an active way and therefore not only meet but also modifying it in the thoughtful act. This reflective activity will be consistent with the goal of reflection and the context in which and with which it is acting subject.
What I mean is that if the context allows the expression of skills, it also allows the transfer of these, that happens to us when we are sensitive to deal with similar problems cognitively similar with a similar semiotic activity. The methodology for investigating possible transfer of skills, attempts to combine the theoretical constructs of the objectification of knowledge (diachronic analysis) and semiotic nodes (synchronic analysis), with the idea to investigate how they formed and developed certain cultural sensitivities of students.

1.2 Thesis structure

The thesis is composed of six chapters, and is organized as follows: from Chapter two to Chapter four is discussed the theoretical framework underlying the thesis and in Chapter five and six are presented experimental activities and discussed the experimental and theoretical results.

The second Chapter is an investigation with a semiotics-cognitive lens about which processes are involved in activities such as geometric and difficulties may have students in these activities.

The third Chapter discusses the relationship between semiotic analysis and mathematics education by introducing the theoretical framework of the theory of objectification of knowledge and the semiotic nodes. I also would to enter into the merits of *how* I intend to improve certain skills and my research is based on the idea of transfer of skills between different domains. Based on the theory of objectification of knowledge, the theoretical framework draws such a transfer as a domestication of the eye and then as a cultural transformation of the eye (and the rest of the body) of the students in an organ of perception more sophisticated.

In the fourth Chapter will be treated the theory of configural concepts and the analysis on the objectification of the configural concepts. Discussion a priori about the structuring of a chess course and about what was expected from the point of view of the transfer (gestures, deictic words, movement of the body).

In the fifth chapter I will discuss the research hypothesis, research question and research methodology. Also I will show in detail the design of chess course.

In the sixth chapter, starting from the research questions, were discussed and summarized the results, expressing the usefulness and potential of this work in teaching practices.

Chapter 2

Visualization and geometrical reasoning

In this chapter I will introduce some theoretical frameworks to study the cognitive processes in geometrical understanding and so investigate on the conflicts and obstacles that students have in geometrical activities.
My purpose is to find out some (visual-spatial) abilities useful for geometrical understanding, learning and reasoning.

2.1 Duval's Paradox of Mathematical thinking

Unlike other fields of knowledge, there is no other way to have access to mathematical objects except through their semiotic representation.
This means that students in the learning phase tend to confuse object and representation (there is no noetic without semiotic), and also to be able to manipulate the mathematical treatment, necessarily linked to the representations, they need a conceptual learning of the objects represented (there is no semiotic without noetic). This situation of "deadlock" has been called by Duval (Duval 1993) cognitive paradox of mathematical thinking.

> "(…) on the one hand, the learning of mathematical objects is a conceptual learning and, but on the other hand, it is only by semiotic representations that is possible an activity on mathematical objects.
> This paradox can be a vicious circle for learning. How the subjects in the learning phase could not confuse mathematical objects with their semiotic representations if they can have relationship only with the semiotic representations? The impossibility of direct access to mathematical objects outside of any semiotic representation, makes confusion almost inevitable.
> On the contrary, how can they acquire the mastery of mathematical treatments, necessarily linked to semiotic representations, unless they already have a conceptual learning of the objects represented? This paradox is even stronger if one identifies the conceptual activity and mathematical activity and if you consider the semiotic representations as secondary or extrinsic "(Duval, 1993).

By focusing on the geometrical thinking, the link between the object and its representation becomes thinner and thinner to perceive and discriminate when I talk about geometric figures.
In fact, both the name and its definition (of the geometrical figure) (Euclid, Book I, 13 final), according to which it is defined as *the one which is contained in one or more terms,* where with “terms” I mean *what is the end of it*, make direct reference to a figural representation or spatial properties.
In addition, it is quite clear, as in the learning phase is easier to draw and visualize geometric shapes rather than define them. Or more precisely, in the definition there are not appreciated certain terms that are summarized in universality, abstractness, idealism and absolute perfection of geometric figures.
Given the research sample (3rd - 6th grade), I have focused attention on the plane geometrical figures and to study the problem between conceptual and figural nature of geometric figures it was used the theory of figural concepts of Fischbein (Fischbein, 1993).

2.2 Theory of Figural Concepts

The theory of figural concepts wants to study the cognitive nature of geometric figures, arguing that in geometric activities we deal with a particular type of mental entities which are not reducible, neither to usual images nor to genuine concepts. We deal *with figures,* the properties of which are completely fixed -directly or indirectly - by definitions in the frame of a certain axiomatic System.

> “The figural concept is a mental reality, it is the construct handled by mathematical reasoning in the domain of geometry. It is devoid of any concrete-sensorial properties (like colour, weight, density, etc.) but displays figural properties. This figural construct is controlled and manipulated, in principle without residuals, by logical rules and procedures in the realm of a certain axiomatic System.”

Referred to didactical implication, Fischbein shown that Figural concepts constitute only the ideal limit of a process of fusion and integration between the logical and the figural facets and that processes of building figural concepts in the student's mind should not be considered a spontaneous effect of usual geometry courses.

> "All this leads to the conclusion that processes of building figural concepts in the student's mind should not be considered a spontaneous effect of usual geometry courses.
> The integration of conceptual and figural properties in unitary mental structures, with the predominance of the conceptual constraints over the figural ones, is not a natural process. It should constitute a continuous, systematic and main preoccupation of the teacher."

In this way Fischbein indicated conflictual situations to bring out this dual nature of geometrical figures, and he provided didactical indications consisting in training the students in mental activities in which they require cooperation between the conceptual and figural aspects.

> "What we claim is that this type of complex mental activities, which sometimes put a high strain on the intellectual process, represents an excellent opportunity for training the capacity of handling figural concepts in geometrical reasoning.
> Such a training is aimed to improve the following abilities: (a) the constructive cooperation of the figural and conceptual aspects in a geometrical problem solving activity ; (b) the ability to keep in mind and coordinate as many as possible figural conceptual items; (c) the ability to organize the mental process in meaningful subunits so as to reduce the memory load; and (d) the ability to predict and integrate the effect of each transformation on the road to the solution."

According to Fischbein (1993), the visual-spatial abilities on handling figural concepts take a crucial role on geometric didactic activities, and to better understand it I studied some theoretical framework about cognitive processes in mathematical visualization.

2.3 Cognitive model for geometrical reasoning

According to Duval (Duval 1998), visualizations processes are one of three kinds of cognitive processes involved in geometrical reasoning. These three cognitive processes are:

• visualization processes,

• construction processes,

• reasoning processes.

Duval points out that these different processes can be performed separately. Duval argues, however, that, "these three kinds of cognitive processes are closely connected and their synergy is cognitively necessary for proficiency in geometry".

Duval illustrates the connections between these three kinds of cognitive processes in the way represented in figure 1.

In Figure 1, each arrow represents the way one kind of cognitive process can support another kind in any geometrical activity. Duval makes arrow 2 dotted because, as argued above, visualization does not always help reasoning. Arrows 5A and 5B illustrate that reasoning can develop in a way independent of construction or visualisation processes.

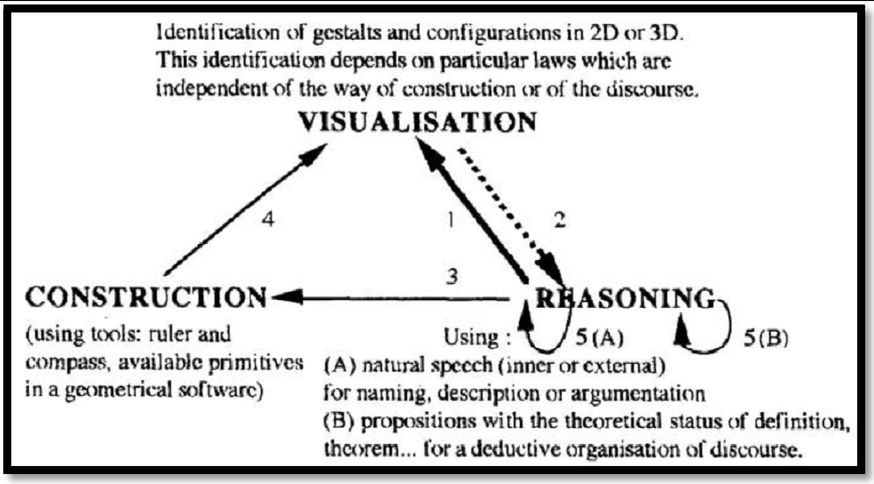

Figure 1 The underlying cognitive interactions involved in geometrical activity.

Now I can ask the following question: Which are the cognitive processes involved in visualization processes on handling of geometrical figures?

To answer this question I need clarify what is the meaning of “visualization” used in this work. According to Presmeg (Presmeg, 1997), visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics.[1]

[1] The term inscriptions is preferred by Presmeg – following the usage of Roth (Roth, 2004) – to that of representations, because the latter became imbued with various meanings and connotations in the

Duval (Duval, 1999) characterized visualization as a bi-dimensional organization of relations between some kinds of units. These units can be 1D or 2D shake (geometrical figures), coordinates (Cartesian graphs), propositions (propositional deductive graphs or "proof graphs") or words (semantic networks). And these units must be bi-dimensionally connected, because any organization requires at least two dimensions to become obvious.

About the cognitive processes in visualization of geometrical figures, Duval (Duval 1995) distinguishes four apprehensions for a geometrical figure: perceptual, sequential, discursive and operative.

To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three. Each has its specific laws of organization and processing of the visual stimulus array.

Perceptual apprehension refers to the recognition of a shape in a plane or in depth. In fact, one's perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures.

Sequential apprehension is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws

and cues, but on technical constraints and on mathematical properties.

Discursive apprehension is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu).

However, it is through operative apprehension that we can get an insight to a problem solution when looking at a figure.

Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way.

The mereologic way refer to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration).

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changing paradigms of the last two decades. However, it is not purpose of this work enter in this argument and we will use both terms on the same way.

> “You can divide the whole given figure into parts of various shapes (bands, rectangles...) and you can combine these parts in another whole figure or you can make
> appear new subfigures.
> In this way, you change the shapes that appeared at the first glance: a parallelogram is changed into a rectangle, or a parallelogram can appear by combining triangles... We call «reconfiguration» the most typical operation.” (Duval)

The optic way is when one made the figure larger or narrower, or slant,

> “You can make a shape larger or narrower, or slant, as if you would use lenses. In this way, without any change, the shapes can appear differently. Plane figures are seen as if they were located in a 3D space. The typical operation is to make two similar figures overlap in depth: the smaller one is seen as it was the bigger one at the distance.”

While the place way refer to its position or orientation variation.

> “You can change its orientation in the picture plane. It is the weakest change. It affects mainly the recognition of right angles, which visually are made up of vertical and
> horizontal lines.”

Always Duval affirms that visualization consists only of operative apprehension. Measures are a matter of discursive apprehension, and they put an obstacle in the way not only for reasoning but also for visualization.
Let’s return to the abilities that Fischbein individuate on handling of figural concepts and that could be trained through mental activities, in particular to the ability to organize the mental process in meaningful subunits so as to reduce the memory load and the ability to predict and integrate the effect of each transformation on the road to the solution.

According to Duval’s suggestions, I can affirm that a training to improve those abilities must involves mereologic and place way of modifying a given figure and so involves operative apprehension.

2.4 Van Hiele theory

Let’s expose briefly the theory of levels of geometric thought formulated by Van Hiele introduced in 1986. The basic elements of this theory are three:

- levels of thought
- the stages of learning
- the insight (intuition).

According to Van Hiele the levels of thought are not a structure located within the study subjects, but they constitute a possible classification of moments belonging to the geometrical thinking. For learning stages I mean those stages useful to the transition from a lower level to a higher one. The insight, instead is conceived as the result of the perception of a structure, and according to Van Hiele its development should be the true goal of education.

According to Van Hiele geometric thinking can be classified into 5 levels:

- Level 1 (Recognition / Visualization)
- Level 2 (Analyze / Descriptive)
- Level 3 (Sorting / Abstraction / Informal Deduction / Euclidean Geometry)
- Level 4 (Deduction / Formal Deduction / Formal Logic)
- Level 5 (Rigor)

Level 1: The student will work on geometric shapes, such as triangles and parallel lines through the identification, naming, and comparing them depending on their appearance. The perception is visual only. A student who reasons at level 1 recognizes certain figures without paying attention to its components.

Level 2: The student discovers empirically properties / rules of a class of shapes, analyzing the figures in terms of their components and relationships between components. At this level components and their characteristics are used to describe and characterize the figures. A figure at this level is presented as a set of its properties. A student may be able to indicate a definition, but will not have the comprehension.

Level 3: Following or giving informal reasoning the student logically connects property or rules saw previously. The student will work with these relationships both within a figure and between similar figures. There are two types of reasoning at this level. First, a student understands the relationships between abstract shapes and, secondly, he uses the deduction to justify the observations made at level 2. The role of the definition and the ability to construct formal proofs are not yet comprehended.

Level 4: The student proves theorems deductively and establishes interrelationships among networks of theorems. The student can manipulate the relations elaborated at level 3. It is included the need to justify the relationships and sufficient definitions can be developed. Reasoning at this level involves the study of geometry as a formal mathematical system rather than a set of forms.

Level 5: The study of the geometry at this level is very abstract and does not necessarily involve concrete models or figures. At this level the postulates or axioms themselves become the subject of intense rigorous examination. The abstraction is crucial.

Note that:

- This classification is sequential and hierarchical
- These levels have inductive nature.
- What is implied in one level is explained to the next.
- Each level has its own language, symbols and information network;
- Two people who think at different levels cannot understand each other.
- The progresses from one level to another are more dependent on teaching experience that age or maturity.

According to Van Hiele progress from one level to the next one involves 5 steps:

1. Information
2. Guided orientation
3. Explicitation
4. Free orientation
5. Integration

1) **Information:** The students become familiar with the work domain, using teaching materials. This material puts them in contact with a structure, and the discovery of the existence of this last gives them a common basis on which to establish a discussion, enabling them to have an exchange of views on the structure itself. The teacher enters the game of conversation with the students about the objects of study, he learns how the

students interpret the words and gives students some understanding of the objects of study; there will arise questions and comments using the vocabulary introduced on objects.

2) **Guided orientation:** The teacher carefully draws a succession of activities that the student must explore. Students explore the field of investigation through the material. They already understood in which direction the study is directed; the material will be chosen in such a way that the structures features will appear little by little. The activities in this phase should consist of easy tasks with a single direct question that requires a specific response.
3) **Explicitation:** The students, building on previous experience, with a minimum of teacher's help, refine their use of vocabulary and express their opinions about the objects of study. During this phase begin to form the system of relationships that need to be studied. It is essential at this stage that the student has to make explicit observations rather than receiving explanations by the teacher.
4) **Free orientation:** The student now meets tasks with more questions or that can be solved in multiple ways. Students gain experience in finding their way in solving tasks orienting themselves in research. Many of the relationships between the objects of study become explicit to the student.
5) **Integration:** the student now revising the methods at its disposal and it forms an overview of the study subjects. The objects and relationships are unified and internalized into a new domain of knowledge. The teacher helps this process by providing comprehensive guidelines on what students already know, taking care not to introduce new or discordant ideas.

At the end of the 5th stage a new level of thought is reached.

Chapter 3

Semiotics in mathematics and transfer of skills

In this chapter we will discuss the idea of transfer of learning skills according the theory of objectification. Initially we will introduce briefly the theory of objectification and then we will treat the concepts of learning as objectification process, semiotic nodes and domestication of the eye[2].

Secondly, we will propose a theoretical idea about a reworking of the idea of skills and transfer of skills and we will show the theoretical aspects of the methodology of experimentation.

3.1 Pierce, Saussure and Vygotskij

Mathematic activity is essentially a symbolic activity, but it is not just about symbols. It's also about objects. These objects are extern and precedent to the semiotic activity or are internal to mathematics? All these questions led to the reflection about the nature of mathematic objects and about their relationship with semiotic activity. When we talk about reflections we refer to three different semiotic traditions operated by Saussure (with his semiology), Peirce (with his semiotic) and Vygotskij.

Saussure focused on the distinction between "*langue*" and "*parole*" ("language" and "word"): "*parole*" is subjective, while langue is social. The "*langue"* is the most important sign system, and looking at this Saussure proposed a science (which included linguistic) that could study the signs in general. Signs have meaning only when they are elements of a system, i.e. when they are in relation with other signs. Saussure makes an analogy with chess: knight, as a material piece, it represent nothing, but this material object becomes real and concrete when he acquires the value that game rules give to him. With signs is the same, their meaning is based on their differential opposition.

Pierce, unlike Saussure, does not look at what the signs mean in social life, but at how people use signs in order to build new ideas and concepts to reach the truth. This pragmatic theory is

[2] The theory of objectification has a complete and wide literature, and in this work we will faithfully report Radford's statements and definitions with the aim to be complete as well.

the base of the semiotic. According to Pierce, we reach knowledge walking through three different experiences: Firstness, Secondness and Thirdness, that led to the production of brand new signs. Every thought is a signs, so the problem is about finding the right way to think, that is the right way to transform signs. This means that truth is just the last result at the end of this method of continuation of signs. Pierce believes that the only aim of thinking is letting things be intelligible.

Vygotskij pays his attention on the fact that the behavior is immersed in a series of artificial devices called artifacts. These devices are not simple helps, but change the natural development of psychic processes. According to Vygotskij the intellectual development is related to culture: during children development every function appears twice, at first on a social level and then on an individual one; at first between people (inter-psychologically) and then into child himself (intra-psychologically). Every superior function is originated as relation between human being. The sign has a mediating function between the individual and his context and allow the internalization on the action. Sign is converted into gesture only when another individual interprets an action (e.g. interprets the movement as an act to signal, because the gesture is a way to signal).

Unlike Saussure and Pierce, Vygotskij does not look at the sign as a component of a system of structure, not even a means of thought and formation of the ideas. He believes that the sign is above all a means of transformation of the psychic function of the individual.

3.2 Registers of Representation and Semiotic Systems

To study the contribution of gestures as material expression of semiotic means used by students during their activities we need to talk about semiotic systems (Duval). According to Duval, in order to produce a knowledge production (in mathematics) we need to coordinate different semiotic systems.

The first thing we have to talk about is the difference between semiotic and non-semiotic representations. A semiotic representation is intentionally produced by character with signs and rules of sign use (e.g. to speak, to write, to draw). A non-semiotic representation does not have an intentional character and is characterized by casual relation with an object that can be produced by physical devices (e.g. optical instruments) or by sensory and brain organization (e.g. visual memory). In the semiotic system of representation those providing specific possibilities of transformation of representations (e.g. computation) are called *register of representation* (e.g. algebraic formulas, graphs, geometrical figures). In mathematics the

register of representations has the function of allowing finding out results by manipulating the signs. As the same Duval stated “there is no knowledge that someone can mobilise without representation activity” (Duval, 1995), where with the term mobilise we mean essentially “to transform”. The connection between mathematics and representations is more understandable if we look at mathematical objects nature. These objects are abstract, unobservable, non-physical, so to be able to access to them representation is needed. Representation is the way through which we can learn about mathematical objects because we cannot directly perceive, manipulate or work with these objects.
When we talk about semiotic representation we are also referring to a cognitive operation that is called conversion. To convert a semiotic representation means to change the way of presentation of the knowledge, passing from on system to another.
A semiotic system of representation becomes a register when it satisfies the three cognitive activities that define a register:

- Formation of representations in a particular semiotic register, that is a set of perceptible marks that allow one to identify them as a representation of something on a giving system
- Treatment, that is a transformation within the register which constitutes a gain in knowledge
- Conversion, that is a transformation from on system to another which allows to make explicit other meanings related to the one represented.

By the way not all semiotic systems are registers of semiotic representations. Examples of semiotic system of representations (that satisfy the three cognitive activities) are graphs, geometrical figures, natural language and symbolic language. These are called register of representation.
The most important function in mathematics of signs and representations is the information’s treatment, the transformation of representations into other representations in order to produce new information or new knowledge.

3.3 The theory of objectification

The theory of objectification study the processes of the mathematics activity and learning of individuals from a cultural-semiotic perspective, arguing for a non-mentalist conception of thinking as well as an idea of learning, thematized as the communal acquisition of forms of reflecting upon the world, guided by historically formed epistemic-cultural modes of knowing.

The theory of objectification proposes a didactic anchored on principles according to which learning is viewed as a social activity deeply rooted in a cultural tradition that precedes it. Its fundamental principles are articulated according to five interrelated concepts.

The first of these is a concept of a psychological order: the concept of thinking. Thinking is intended as a form of active re-flection about the world, mediated by artifacts, the body (through perception, gestures, moveme0nts, etc.), language, signs, etc.

Where re-flection is a dialectical movement between a historically and culturally constituted reality and an individual who refracts it (as well as modifies it) according to his/her own subjective interpretations and feelings.

The second concept of the theory is of a socio-cultural order: the concept of learning. Learning is seen as an activity through which individuals enter into relationships not only with the world of cultural objects (subject-object plane) but also with other individuals (subject-subject plane or interaction plane) and acquire, in the joint pursuit of the objective and in the social use of signs and artifacts, human experience.

The third concept of the theory is of an epistemological nature: the semiotic systems of cultural signification, that are those symbolic superstructures that "naturalize" the ways that one questions and investigates the world.

The fourth concept of the theory is of an ontological nature: definition of mathematical object. In fact, according to Radford, mathematical objects are fixed patterns of reflexive activity incrusted in the ever-changing world of social practice mediated by artifacts.

The fifth concept of the theory is of a semiotic-cognitive nature: the idea of objectification, or a subjective awareness of the cultural object.

Now we try to treat three key concept of theory of objectification: Learning, objectification and semiotic nodes, and their relations.

3.4 Learning as objectification process, semiotic nodes and domestication of the eye

The theoretical construct of objectification of knowledge born from the conception of knowledge as movement, more precisely, as a culturally and historically codified sequence of actions that are continuously instantiated in social practice.

The term objectification derives from the Latin verb *obiectare*, meaning "to throw something in the way, to throw before". The suffix –tification comes from the verb *facere* meaning "to do" or "to make", so that in its etymology, objectification becomes related to those actions aimed at bringing or throwing something in front of somebody or at making something visible to the view.

Objectification is precisely the process of recognition of that which objects us – systems of ideas, cultural meanings, forms of thinking, etc, and the processes of objectification are those acts of meaningfully noticing something that unveils itself through our sensuous activity with material culture.

From a psychological viewpoint, the objectification of mathematical objects appears linked to the individuals' mediated and reflexive efforts aimed at the attainment of the goal of their activity. To arrive at it, usually the individuals have recourse to a broad set of means. They may manipulate objects, make drawings, employ gestures, write marks, use linguistic classificatory categories, and so on.

These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities, were called by Radford semiotic means of objectification.

Gestures, language and symbols thus become the very constituents of the cognitive act that positions the conceptual object, not inside the head, but on the social plane.

In the theory of objectification, learning is theorized as processes of objectification, that is to say, those social processes of progressively becoming critically aware of an encoded form of thinking and doing—something we gradually take note of and at the same time endow with meaning.

In other words, learning is the fusion between cultural modes of reflecting and doing and a consciousness which seeks to perceive them. In the course of this fusion, consciousness emerges and is continuously transformed. Thus, processes of objectification are entangled in processes of subjectification (processes of creation of a particular self).

Within the theory of objectification, consciousness is rather considered as a subjective reflection of the external world. Consciousness is the subjective process through which each one of us as individual subjects reflect on, and orient ourselves in, the world.

The individual consciousness is a specifically human form of subjective reflection of concrete reality in the course of which we come to form cultural sensibilities in order to ponder, reflect, understand, dissent, object and feel about others, ourselves and our world.

Consciousness can only be understood as the product of historical-cultural and emergent contingent relations and mediations that, rather than being given, “arise in the course of the establishment and development of society.” (Leont’ev, 1978, p. 79) Within this view, consciousness appears in concrete life, not as its origin, but as its result.
To sum up, we can individuate three key concept of theory of objectification:

- The dialectical relationship between human thinking and the material and cultural world,
- the central role of semiotic systems used within culturally and historically bound practices and social interaction in mathematical activity and learning,
- the twin dialectical processes of subjectification and objectification.

To investigate how gestures, words, and artifactual actions are mobilized by the students in order to objectify and endow with meaning the emerging mathematical content, Radford proposed the theoretical construct of semiotic nodes that are, pieces of the students’ semiotic activity where action, gesture, and word work together to achieve knowledge objectification.
Since knowledge objectification is a process of becoming aware of certain conceptual states of affairs, semiotic nodes are associated with the progressive course of becoming conscious of something. They are associated with layers of generality.
The investigation of semiotic nodes in classroom activity is a crucial point in understanding the students’ learning processes. The concept of semiotic node rests indeed on the idea that the understanding of multimodal action does not consist in making an inventory of signs and sensorial channels at work in a certain context.
From a methodological viewpoint, the problem is to understand how the diverse sensorial channels and semiotic signs (linguistic, written symbols, diagrams, etc.) are related, coordinated, and subsumed into a new thinking or psychic unity (Radford, 2012).
Now we want to introduce briefly an example of analysis with semiotic nodes: the idea of domestication of the eye.
According to Radford, the domestication of the eye is a lengthy process in the course of which individuals come to see and recognize things according to “efficient” cultural means. It is the process that converts the eye (and other human senses) into a sophisticated intellectual organ – a “theoretician” as Marx put it (Marx, 1998).

That capacity to perceive certain things in certain ways, the capacity to intuit and attend to them in certain manners rather than others, belongs to those sensibilities that students develop as they engage in processes of objectification.

The disclosing of the mathematical structure and the concomitant poetic moment of objectification resulted from the complex link of those semiotic means of objectification (gestures, words, rhythm) that accompanied and oriented the students' perceptual, aural, linguistic, and imaginative activity.

In fact, Radford's works about an interaction between students and teacher, show how the teacher coordinates eye, hand, and speech through a series of organized simultaneous actions that orient the students' perception and emergent understanding of the target mathematical ideas.

This is an example how the theoretical constructs of objectification and semiotic nodes, are useful to understand the phenomena accompanying the learning and teaching of mathematics. This awareness might lead in particular to revisiting the teacher-student relationship formed in the act of knowing.

3.5 Skills transfer and theory of objectification

On January 2007, was made an experiment by Washington Post with the aim to observe people reactions during the performances of one of the greatest violinist of the world, Joshua Bell.

The fact is that Bell played at the Washington Station dressed like a street musician for about 45 minutes.

The result was that most of the people ignored him and that he gained about 32 dollars (we have to notice that he played with a Stradivari of about 3.5 million dollars). Whether this is due to the usual chaotic rhythms that do not allow us to appreciate the beauty and its manifestations, the most interesting thing is that the same violinist stated that during the execution he thought "why they ignore me?"[3]

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[3] Quote verbatim:
With "Chaconne," the opening is filled with a building sense of awe. That kept him busy for a while. Eventually, though, he began to steal a sidelong glance. "It was a strange feeling, that people were actually, ah . . ."
The word doesn't come easily. ". . . *ignoring* me." Bell is laughing. It's at himself. "
At a music hall, I'll get upset if someone coughs or if someone's cell phone goes off.

That is the context in some way had questioned the counted and proved skill (absolute) of the musician, who appeared to be no appreciable in relation with the space and the time of the execution. In fact, the approval of the social context is what allows the individual to gain knowledge on the subject of reflection, and as well as *"the individual is an individual inasmuch as he/she is-with-others"* (Radford) he expresses certain skills and becomes aware of owning them when that individual is with others (and is accepted by the other - always quoting the words of Bell).

This example shows how a definition of skill must take into consideration the cultural and social situation in which the individual is acting and then define it in relation to the context and not in an absolute way.

To do this, we moved on the theoretical framework of the theory of knowledge objectification (Radford, 2006) according to which the learning process consists in finding out how to take note of or how to perceive these layers of generality[4].

Just as learning is a re-flection, to learn presupposes a dialectical process between subject and object mediated by culture; a process during which, through his/her actions (sensory or intellectual) the subject takes note of, or becomes aware of the object.

We then define (learning) skills, such as sensitivity to acknowledge or perceive the layer of generality of the target object of reflection, and to establish a dialectical process mediated by the culture between the subject and the object of which the subject himself becomes aware.

So we can think to the skill as to the sensitivity to perceive a phenomenon, to become aware and to endow it with meaning in an active way, and therefore not only meet but also modify it in the thoughtful act. This reflexive activity will be consistent with the goal of reflection and the context in which (and with whom) the person is acting.

But here, my expectations quickly diminished. I started to appreciate any acknowledgment, even a slight glance up. I was oddly grateful when someone threw in a dollar instead of change." This is from a man whose talents can command $1,000 a minute.
Before he began, Bell hadn't known what to expect. What he does know is that, for some reason, he was nervous.
"It wasn't exactly stage fright, but there were butterflies," he says. "I was stressing a little."
Bell has played, literally, before crowned heads of Europe. Why the anxiety at the Washington Metro?
"When you play for ticket-holders," Bell explains, "you are already validated. I have no sense that I need to be accepted. I'm already accepted. Here, there was this thought: *What if they don't like me? What if they resent my presence . . ."*

[4] This movement, which could be expressed as the movement from process to object (Sfard, 1991; Gray and Tall, 1994) has three essential characteristics. First, the object is not a monolithic or homogenous object. It is an object made up of layers of generality.

In the previous example Bell began playing in order to perform in the best way the pieces chosen by him using absolute skill and sensitivity. By the time he became sensitive to the reactions of the environment around him, his reflexive activity not lingered (only) to his ear or his hands, but he pushed them towards his ears and bodies of the people who watched him.
What we mean is that the environment has a direct impact on the implementation of a musician, on teacher's verbal expression, of the symbol produced by the student, thus allowing both the expression of the skills that the transfer of these.
Our research hypothesis is that the transfer of skills takes place when the student, sensitive in facing similar cognitive problems, reproduces a similar semiotic activity to the context. So, to be more precise, we don't talk about a transfer of skills, but about a maintenance of skills when the environment changes.
Our methodology for investigating possible transfer of skills, attempts to combine the theoretical constructs of the objectification of knowledge (diachronic analysis) and semiotic nodes (synchronic analysis), with the idea to investigate how they formed and developed certain cultural sensitivities of the students.
The scheme of investigation we adopted will therefore have the aim of identifying a common feature in the comparison of semiotic nodes belonging to a process of objectification.
In particular, we will analyze how the synchronization of semiotic means of objectifications into the semiotic nodes, change during the process of objectification.

Chapter 4

Chess thinking and configural concepts

The purpose of this chapter is to connect chess and mathematics education. First, we introduce the idea of configural concepts in chess thinking and then we outline a scheme to show the phases of chess reasoning and how to apply this idea to some conflictual situations. We conclude this work proposing two research problems in introducing chess in mathematical classroom activities.

4.1 Analyzing chess thinking: Configural Concepts

Inspired by Gestalt theory (Kohler, 1947) and the phenomenology of perception (M.M. Ponty, 1945), this article expands the idea of figural concepts (Fischbein, 1993) and introduces the concept of configural concepts to investigate chess thinking.

First we need to define chess objects and chess elements. We define *chess objects* the elements of the artifact "chess" – the pieces, the squares and the chessboard – which therefore possess rules and scopes.

Now, when a chess player moves a piece on the chessboard[5], he or she gives to the move a motivation linked to a judgment on the position. Thus, when a chess player moves, not only he or she "follows the rules", but he or she also deals with chess elements. We define *chess elements* the mental (i.e., personal-ideal) entities that a chess player uses and "builds" when he/she thinks about a move, a position or a variant with a specific aim.

A configural concept is made up of chess objects and their conceptual relationships. Its meaning comes from the hierarchical linkage of the conceptual relationships between the involved chess objects and from its position in the whole theoretical structure of the pieces in the chessboard.

The use of a configural concept depends on the goal that an individual is pursuing in a chess game. It involves the identification of general structure of the game at a certain moment and the role of the configural concept therein. In terms of learning, configural concepts become noticed and valued through a process in which the student becomes conscious of the chess objects and their conceptual mutual relations. This process is what Radford calls

[5] Or when he/she thinks to move it on the chessboard

objectification (Radford, 2010). The objectification of a configural concept requires the student to notice the chess objects' organization, often through a process mediated by artifacts, body, language, and signs (Radford, 2002). A configural concept embodies theoretical relationships that are objectified as the individual gain of a progressive awareness of their meanings, achievable by reasoned handling of elements during individual experience.

We have introduced the idea of configural concepts to analyze chess thinking for several reasons:

– Their meaning depends on their placement in the structure of the position;
– Their meaning depends on the configuration of its parts;
– They are "dynamic" because they depend on visual, spatial and temporal features that are always evolving in a chess game;
– They may have intersections between them;
– Chess elements are a particular type of mental entities that cannot be reduced, neither to usual images, or to traditional concepts.

For example, let us consider the following chess elements: the isolated pawn and the material concept. Now we want to show that only the isolated pawn is a configural concept.

A pawn is isolated if there isn't any pawn in the adjacent columns, and this configuration allows to assign it some features. For example, there is no pawn that could defend it or that could attack a "blocking piece".

The material concept consists of the static comparison of value of black and white pieces present on chessboard. We say static because it does not depend neither on the position of pieces on the chessboard, nor on possible future moves.

The difference between these chess elements consists of their nature. In fact the material comparison depends on all the pieces present on chessboard, but not on their positions and on the relationships between them.

The concept of "isolated pawn" is a configural concepts because its definition depends by the configuration of pawns in adjacent columns; also, its meaning (and judgment of it) depends on its position and on the complete structure of the pieces present on chessboard.

An isolated pawn could be an advantage or a disadvantage, it could be not recognized as relevant for chess player analysis or could be fundamental for an entire variation.

However, what we think about "isolated pawn" is a result of cultural process in which previous generations of chess players highlighted some important configurations of the chessboard.[6]

In this article we focus our attention on process of objectification of configural concepts and we will refer to those sensations and feelings that do not allow the chess player to recognize the isolated pawn in some positions, or that in other positions do not allow the chess player to "see" another element out of the isolated pawn.

It is clear how difficult is the task to correctly "find" and "evaluate" chess configural concepts, their relationships and their intersections. For these reasons, we have studied the phenomenology of perception and the phenomenology of judgment in recognizing chess configural concepts.

In this work, we divide the chess player thinking in two phases: the intuitive thinking and the reasoned thinking.

The intuitive thinking consists of the considerations and analysis that the chess player makes intuitively. In this phase the player recognizes some configural concepts and organizes them into a structure that is the mental image of the position (Figure 1).

This structure is focused on matching this mental image (and therefore, the position) with a class of mental images, allowing the player to express a judgment and one or more ways to play that position ("This position usually has to be played in this way").

The "reasoned" thinking consists of all activities that allow the player to gain awareness of the position and find *the* or *a* "correct" move. In fact, by knowing how to play in similar positions, the player begins to analyze accurately some variants and to verify their validity.

We call this phase the "analysis phase".

If the solutions are negative during the analysis phase, or during the matching with the generalized mental image, the chess player must find something else and re–pose the main problem. In this way, he or she becomes aware of other configural concepts and try to re–organize the structure of the position. In according to Gestalt theory, we call this phase the "insight phase" (Figure 2).

In this phase it is built the factual generalizations (Radford, 2003) in chess (that we will cover in the following paragraphs). In fact, in reasoned thinking the player tries to evaluate the

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[6] In line with Radford's conception of mathematical objects "More precisely, mathematical objects are fixed patterns of reflexive activity (in the explicit sense mentioned previously) incrusted in the ever-changing world of social practice mediated by artefacts ." Radford (2006, p.9)

considerations deducted from the generalized mental image and the intuitive or insight thinking.

Player's expertise, through the success or failing of this structure, gives to some mental images a "width", composing a hierarchical set of generalized mental images.

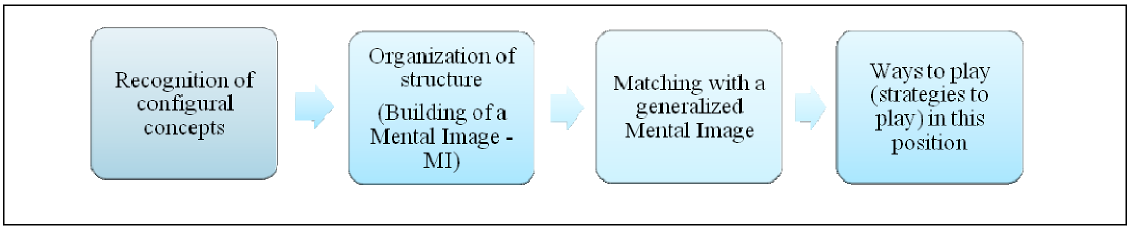

Figure 1. Specific mental activities in intuitive thinking

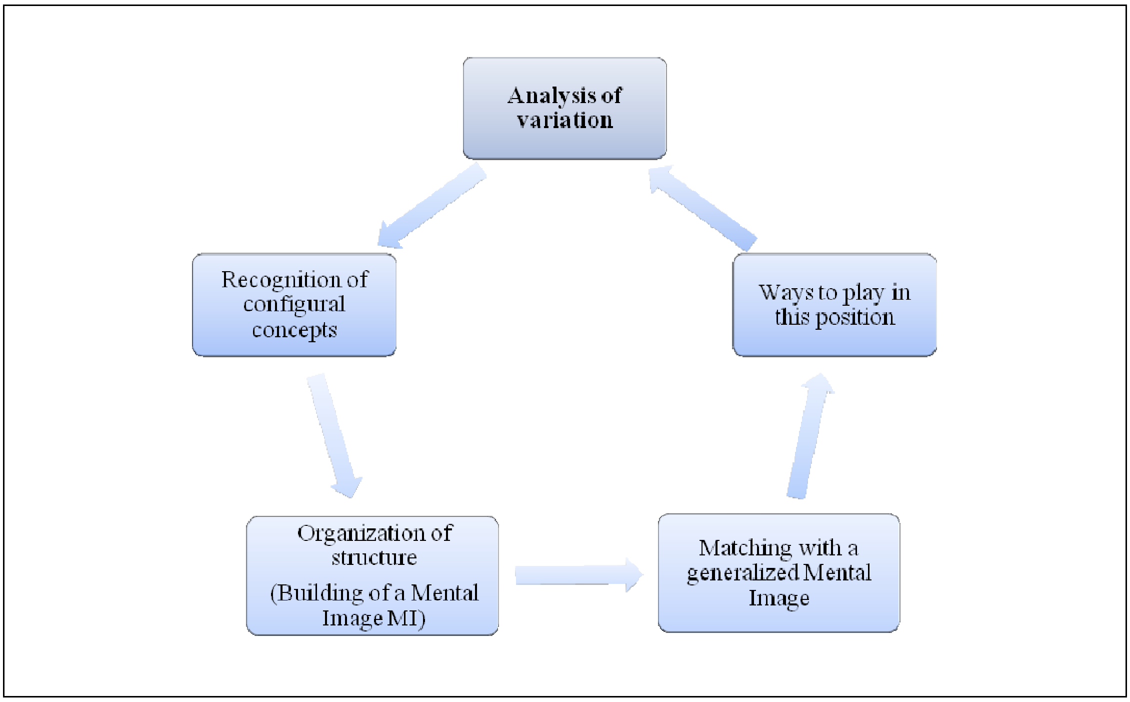

Figure 2. Cycle of mental activities in "Insight" thinking

4.2 Mental activities, obstacles and conflictual situations in chess

Previously we have showed the chess players' actions when facing a position: perceive, recognize, organize, match, remember, verify, re-recognize, re-organize, generalize, build a mental image, judge and choose. So we want to show that, by studying and playing chess, the player constantly solves and poses problems, and generalizes them.

Now we are going to deal with these chess activities, with emphasis on some conflictual situations.

4.2.1 Problem posing

A chess configural concept has conceptual and figural properties, and figural constraints could be seen as the first generators of conflictual situations.

For example, if we ask a chess player – that has just a little experience in the game – to observe the two positions in Figure 3, it is almost certain that he or she will intuitively prefer the position on the right. The term "intuitively" – as we suggested before – means that the chess player, looking at the position, can perceive some properties inherently contained in it.

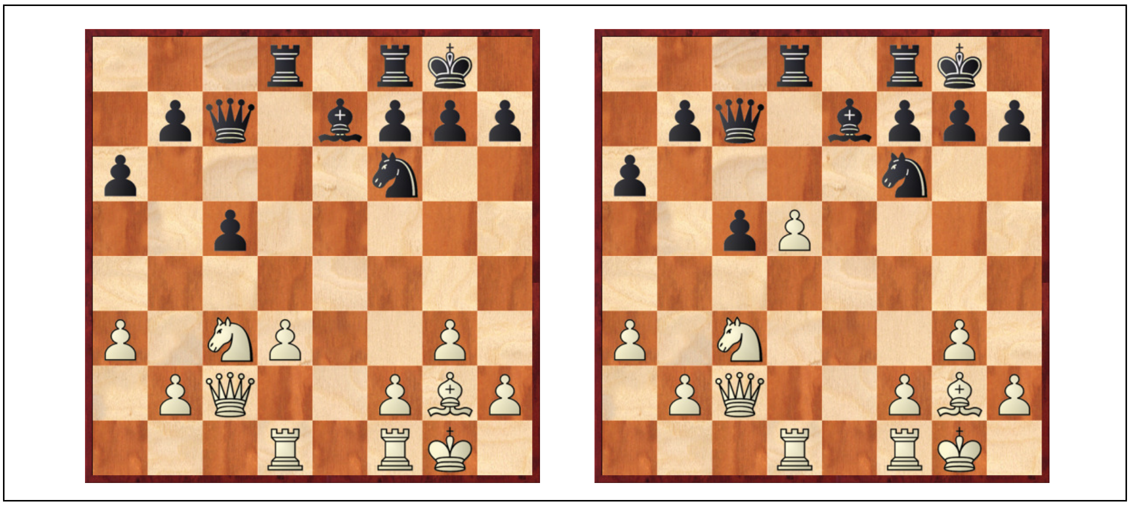

Figure 3. The instinctive property of space advantage could obscure the changed meaning of d pawn (and consequentially of all pieces) passing from d3 to d5

The problem that could occur is that some properties could overshadow the others, influencing the recognition of configural concepts. In fact, it is possible that in this way the player loses some important information, thus creating significant difficulties in problem posing.

In this case, the *instinctive property* is the space advantage[7] concealing the fact that, in the position on the right, the d pawn is a past pawn, it cannot be blocked in d4 and the outposts c6 and e6 can be more dangerous than e4 and c4.

The meaning of the configural concepts "isolated pawn" is more interesting, because its judgment depends on its position, by the pieces and by the game timing.

It is clear that the problem posing actions are influenced by the intuitive recognition, and it is

[7] The space advantage is a delicate element in chess, because while it is easily perceived and really gives a significant advantage for many reasons, sometimes it obscures other (more important) elements not allowing the correctly posing of the problem.

possible that the player is satisfied with a *figural sense* so that he/she does not need to analyze it more accurately and he/she does not enter on cycle of insight thinking.

Therefore, we can say that the intuitive recognition could be an obstacle to the posing of the problem, and by gaining experience, the chess player learns to never use just the intuitive recognition, but to repose the problem, to match it with another generalized mental image or to build a new generalized mental image altogether.

In chess thinking, the operation of re-posing the problem and generalizing new mental images is very important but, if these buildings are too rigid, it could hinder the process, and the chess player cannot return to the main problem.

This phenomenon could cause an "oversight", that is a situation in which the chess player doesn't start from the main position during the analysis, but he/she starts just from the re–posed problem. (Saariluoma, 1992). We are going to show some examples later, when we'll talk about problem solving.

4.2.2 Factual and contextual generalization

In this section, we are going to analyze the role of generalization in chess thinking and, according to Radford (2003), we are going to use the terms "factual" and "contextual"[8]. In this work these terms will refer to chess context.

In chess reasoning we can identify two kinds of generalizations; in the first one the player builds a mental image trough which he/she can affirm "in this position I can do these ideas/moves/plans", and it is associated to a figural sense and, therefore, to the intuitive thinking (visual memory and imagination).

When a chess player generalizes factually, sometimes he/she knows what he/she has to do in front of that specific position, but maybe he/she doesn't know why the sequence continues that way.

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[8] "A factual generalization is a generalization of actions in the form of an operational scheme (in a neo-Piagetian sense), that remains bound to the concrete level. In addition, this scheme enables the students to tackle virtually any particular case successfully.
In contrast to factual generalizations, contextual generalizations generalize not only the numerical actions but also the objects of the actions. They go beyond the realm of specific figures and deal with generic objects (like the figure) that cannot be perceived by our senses. They have to be objectified and produced within the realm of reasoned discourse, that which the Greeks called *logos*", Radford (2003, p.65).

What we want to emphasize is the fact that in this level of generalization he/she deals with *chess objects* and so with what he/she can see on the chessboard. In his/her argumentations about the reached pattern he/she uses mainly gestures (usually pointing) and his/her perception of the *chess elements* depends on his/her position in relation to the chessboard (point of view).

Also, their sentences in this level of generalization are never about the chess elements**,** much less the correct motives, hence generalizable to other similar positions.

The contextual generalization is not restricted to a particular instance in a game. A contextual generalization offers a kind of sensuous schema that can be repeated in, or applied to, other games. It usually occurs when the player studies chess, that he/she analyzes the game and try to explain the meaning of the chosen sequence (variation) to him/herself, to a friend or to a trainer.

In this level of generalization, a chess player uses less gestures and he/she refers to some chess elements and talks about "these positions". In fact, the position on the chessboard becomes a representation of those chess elements and, in order to improve his/her argumentation the chess player sometimes removes some "foggy pieces" from the chessboard or changes some pieces' placements.

In order to illustrate these concepts, we are going to focus our attention on chess endgames, trying to observe different ways to solve and generalize them and the relative obstacles that may be encountered.

The position in Figure 4 is called the *Réti endgame* and it shows the concept of diagonal in the endgames of King and pawns. In this endgame white seems to have no chance to draw the match because its King is too "far away" to help promoting its pawn and to stop the opponent pawn. On the contrary, by passing the squares g7, f6, e5, the King could draw the game promoting its pawn or taking the opponent pawn.

In this example, the figural constraint fails to accept what the "reason" is saying, and we can see how, in a factual generalization of the diagonal chess element, an existing generalized mental sensuous image, "the square rule", disregards this process.

In fact, the square rule is the configural concept that allows the player to understand how far away is the king compared to the opponent pawn. Its use, supported by a strongly figural sense (the square shape), gives to the square rule a hierarchical position on the mental image in this kind of endgames.

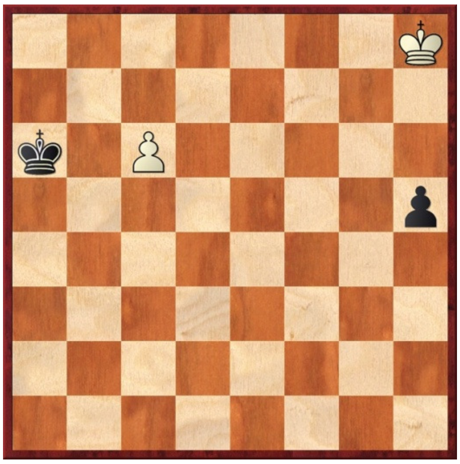


Figure 4. The Réti endgame show the diagonal idea in Endgame of King and pawns

In short, the player doesn't accept that he/she can reach the pawn despite staying out of the square, and so, it's possible that this endgame could be seen as an exception, special occurrence and often this is not easily accepted by the students.

So, although they have studied that this endgame is a draw, the students frequently try to find the win plan for the black.

This phenomenon could be found in every situations in which a new configural concept contrasts with the one previously accepted and this "figural reject" doesn't allow the player to factually generalize it.

In the endgame in Figure 5 we can observe an example of factually but not contextually generalization.

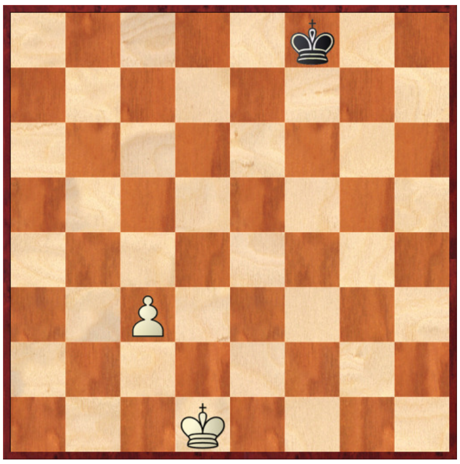


Figure 5. In this endgame white wins the game reaching with

the king the critical squares b5, d5 and c5 using the opposition idea.

In this task it's required to find a plan that allow white to win the game[9].

The objective of this endgame is to reach the critical squares with the king (for the pawn in c3 the critical squares are b5, c5 and d5) and to achieve this goal using a chess configural concept, the opposition.

The correct answer is: 1. Kc2 Ke7, 2. Kb3 Kd7, 3. Kb4 Kc6, 4. Kc4 Kd6, 5. Kb5 and white reaches a critical square and so he can win "easily" the game.

In our research, we have submitted this task to players with different expertise, and we have observed the fact that most of them know the way to win, but only few players (the better ones) know why it is this the only way to win (or which is the aim to achieve).

For example an expert chess player meeting this position has answered by moving immediately the king following the path c2-b3-b4-c4 and he has ended the task saying that the game is won because he can give opposition (correct answer) and gain space advantage (in this moment of the game and without pawn structures however, it is totally wrong to talk about space advantage).

What happened? Why doesn't he know the real objective and talks about "space advantage"?

According with what we have already shown, we can say that he/she knows how to continue this position and every position belonging to this type of positions. He/she has generalized factually the idea of opposition, but he/she doesn't have achieved a higher level of generalization (contextual).

We conclude this section saying that although the operation of generalization in chess is fundamental and it is present in every game, in most cases the chess player tries to contextualize it and achieve a higher level of generalization, but it is possible that he/she perceives only a sensuous meaning in the pieces and their relationships and not the proper theoretical chess meaning. The objectification of the relationships at the basis of the configural concepts may not have occurred.

[9] In endgame it is more easy to use the term "plan to win the game", because in many cases there is only one plan to achieve a victory. In fact, in other moments of the game it is difficult to talk about a plan to win.

4.2.3 Problem Solving

The chess elements are controlled by conceptual constraints, but the "conceptual way" to solve a problem could be deprived of creativity because in chess analysis the flow of productive ideas could be disturbed or even inhibited by looking constantly for analytical and formal justifications.

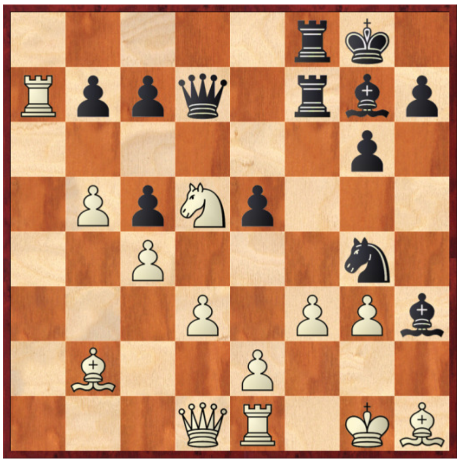


Figure 6. Choose a plan for black to attack the white king

In fact, in the position on the Figure 6 (move to black), the research of a conceptual way to find the correct move/plan could be very hard, and so the player must rely on creative thinking controlled prevalently by figural constraints.

For example, in my courses I have presented this position and asked the students to find a plan to attack the white king. Now we'll show the analysis of reflective activity of one student[10].

Initially, we noticed that the player starts by analyzing some elements and he has tried to find the plan with a logical structure. He saw the "aggressive" bishop in h3, the two rooks in the f file, the "weakness" of square e3 and the possibility to sacrifice the Knight in g4 to open the f file.

These elements do not give any information in order to find the attack plan[11], so he looked for other elements: the "strong" white rook in a7, the "weak" pawn b7 and the doubled pawns in c7 and c5, excluding the possibility to enter in an endgame[12].

[10] We have asked him to motivate his reasoning and to talk aloud.

[11] Probably because he can't relate them to a configural concept.

[12] It is interesting to observe the nature of the terms aggressive, strong and weak (for the b7 pawn) when related to the experience of the player. Only when he refers to the square e3 with the term weak this is accepted by chess literature (e.g. a square that can't be defended by a pawn).

During these few minutes, I stressed repeatedly the required task, and tried to put him on the right path.
It is interesting to observe that he used gestures only in some cases: pointing on square b8 to indicate the move Rb8, touching the pawn b7 to indicate its weakness and pointing on pawns b7, c7 and c5 when he considered to sacrifice them to attack the white king.
He introduced the term "candidate moves" that are the first moves of each variant.
After few minutes (5) he found (with our help) the move e4, but with the idea to attack the pawn f3 and thus freeing the f file or to move the pawn in e3. So he didn't realize to do Qd6 to attack the weak pawn in g3.
At this point, we asked him to analyze this particular plan:
1…e4, 2.BxBg7 Qd6, 3. f4 Rxf4, 4. gxRf4 RxRf4 (Figure 7).
The first thing that we noticed is that he used more gestures pointing on squares or simulating moves "in the air".
Finally, according to my predictions, the players built a mental image giving "weight" to some elements. This weight varies in function of the move that is controlled by conceptual constraints. In fact the player continued the variant with the following moves (now he always used the gestures to point the arrive-square of the moves) : 5.Ra8+ KxBg7, 6.Qa1+
…and Rd4, an illegal move! What's happened?
We think that in his mental image the pawn e4[13] disappeared (Saariluoma, 1991), maybe because its function is only to permit the black Queen to attack the white king.

[13] In our experience, this phenomenon has occurred in the following variant too: 5.Ra8+, KxAg7, 6.Qa1+ Kh6, 7.Qc1 without seeing the defence resource 7…e3!

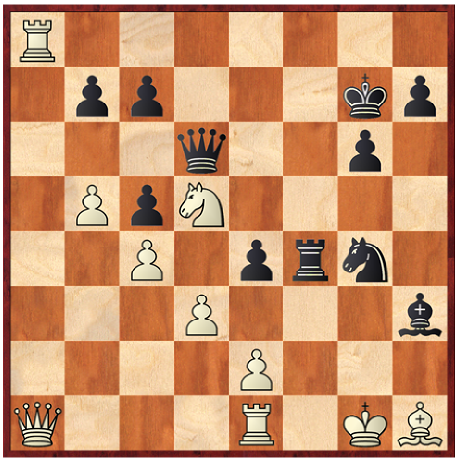


Figure 7 In this position the black pawn in e4 disappears from the chess player mind and he proposes Rd4

Therefore we gathered that conceptual constraints could be an obstacle for the chess player in creative thinking and long analysis in which the configural concepts could be disrupted by the function of its elements.

This phenomenon of disappearance could be explained by the fact that the pawn in e4 does not belong to any configural concepts. In our analysis in fact, the principal configural concepts are:

a) Weakness of square g3, Knight in g4 and Bishop in h3;
b) A pair of black rooks on f file;
c) Pawns structure;
d) Pieces activity.

And when the player moves the pawn in e4, it does not belong to any configural concepts.

4.3 Studying chess in classroom

In this section we consider the collaborative phases in chess studying in classroom or between two chess players who have finished a game.

During chess lessons, in the same way as mathematics ones, teaching attempts to provide knowledge and skills, computational techniques and criteria for evaluating positions. However there is a big difference between the two disciplines, because in chess, the concept of “best move” isn’t always unique.

In fact, during the game, it is common the effort of finding the best move or best variant, but it often happens that the players restrict their research to find a good move or, at least, not a bad one.

Since this assessment is often biased and not impartial, how do we define the best move during the time of study involving the interaction of many individuals?

We define best move (or best variant) as the one that is socially accepted by the collective, which is understood by all individuals and that could modify the zone of proximal development (Vygotsky, 1986) of each individual of the collective.

During our chess lessons there is no asymmetry between the students and the teacher, and everyone tries to find and share ideas to solve the problems. In fact, in situation of problem posing and problem solving, the teacher submits some tasks for certain reasons, but the students, with their answers and proposals, could lead the argumentation into other directions.

This comes from the fact that every students, in front of a position, can develop a different intuitively reasoning and recognize some configural concepts rather than others. They share ideas and conflictual situations, and thus generate opinions and questions that the teacher designing the tasks has not previously imagined.

However, for a chess player, an essential moment is the comparison of ideas and analysis not only with the rest of the class or with its game opponent, but also with him/herself.

In fact, there are games in which the chess player does not understand the reason of certain moves or why he/she did some mistakes, and other ones that he/she remember perfectly.

This phenomenon depends on the emotional sphere of the game, that does not affect the player during his/her analysis process.

For us, one of the main skills for a chess player is the ability to ask to himself/herself (and to other people) questions of a metacognitive type, to accept criticism and therefore, to extend the reflection activity not only to the game, but also to its analysis, and finally, to share them with the context.

The reason of this statement is based on the fact that in chess activities the chess player analyzes at different levels: recognition of configural concepts (with the theoretical levels with which they were generalized and so objectified), problem posing, problem solving and calculation of all variants.

It's for this reason that we propose chess like a classroom activity that could improve students' cognitive abilities (visual spatial abilities), students' metacognitive abilities and the capacity to interact between them with the purpose of reaching the goal of their activities.

Final Remarks

We can summarize that when a chess player thinks about a chess position, he/she deals with mental entities whose organization is mediated by the chess objects and the body (perception). When he/she thinks about a position, he/she first judges it through a logical interpretation of the signs presented by sensory perceptions (Merleau-Ponty, 1945)[14], so he/she uses those knowledge that are objectified in the patterns recognized by him/her to produce moves, ideas or variations and finally he/she tries to validate them.

When a chess player "moves" the pieces, he/she does not change the position of them, but he/she assigns to the square involved in the moves, the positions of the pieces.

Thus, in Figure 4, analyzing the variant 1. Kg7 h4, 2. Kf6 Kb6. 3. Ke5 h3, 4. Kd6 (Figure 8) he/she knows that in d6 there is the white king, in b6 there is the black king and in h3 there is the black pawn. The objects of reflection are the square and not the pieces.

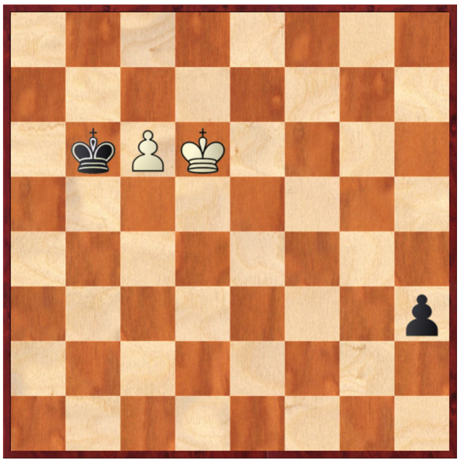


Figure 8. The position reached after 1. Kg7 h4,
2. Kf6 Kb6. 3. Ke5 h3, 4. Kd6

However, that is not all. The chess player in his/her sequence must ensures the moves alternation and who moves in Figure 8, white or black.

To cover the time features of the moves the chess players activate an intense bodily activity that includes: rhythmic breathing, rhythmic closing of the eyelids, rhythmic moving of the fingers and saying the moves.

[14] Merleau-Ponty shows that a judgment may be defined as a perception of a relationship between any objects of perception that it is neither a purely logical activity, nor a purely sensory activity.

These means of objectification become the real objective of reflection present in the multimodal semiotic analysis that allowing us to investigate on role of gesture in visualization in chess and geometry (See 6.3 and 6.4).

Chapter 5

The research process

In this chapter I will treat the Research Hypothesis, the Research Questions and the Methodology of the research. Then I will discuss on detail the chess course that I proposed in the experimental activity.

5.1 Research Hypothesis and questions

The research hypothesis and research questions that guided the whole experimentation phase were the following:

Research Hypothesis

RH_1: Cognitive paradox of mathematical thinking. The students in learning phase tend to confuse object and its representation, and also to able to manipulate the mathematical treatment, necessarily linked to the representations, they need a conceptual learning of the represented objects.

RH_2: Building Figural Concepts. The integration of conceptual and figural properties in unitary mental structures, with the predominance of the conceptual constraints over the figural ones, is not a natural process. It should constitute a continuous, systematic and main preoccupation of the teacher.

RH_3: Perceptual, Discursive and Operative Apprehension. Visualization consists only of operative apprehension. Measures are a matter of discursive apprehension, and in solving mathematical tasks students deal with inhibition of operative apprehension and a lack of interplay between perceptual and discursive apprehension.

RH_4: Perception of an object or figure can be radically affected by its orientation.

RH_5: Semiotic multimodality. Mathematics learning is based on activities involving body, artefact and sign, and they can be analyzed through a semiotic approach.

Research Questions

1. What are the criteria for choosing a non-disciplinary activity in order to improve certain skill[15] useful for the geometrical learning and thinking?
2. It's possible make a theoretical proposal for the idea of transfer of skills according to the semiotic-cultural paradigm of the theory of objectification?
3. Chosen the chess laboratory as a non-disciplinary activity, which skills could be improved? What are the role of chess and math teacher in this multi-disciplinary didactical proposal?

5.2 Methodology

The research investigation was divided in the following phases: a) A theoretical investigation on the historical evolution of the concepts of height, base and area of a triangle (see 6.1); b) An experimental investigation on synergies between chess and geometrical thinking based on protocols inscription analysis (see 6.2); c) Two experimental investigation on synergies between chess and geometrical thinking based on a multimodal semiotic analysis (see 6.3 and 6.4).

The first investigation was necessary to analyze the geometrical tasks in point b that involve recognition of geometrical plane figures, and the segments that need to calculate their area.

This historical investigation start from Euclidian Elements and ends into Hilbert's Axiomatization, then we did an investigation on the evolution of this concepts till the end of first half of the twenty century.

In the second, third and fourth investigation we analyzed the results of a chess course in school context using two different approach: analysis of inscriptions and analysis of multimodal activity.

The two classrooms of 6th grade that we analyzed belonged to two school: "Silvio Boccone" and "Michelangelo Buonarroti" of Palermo, and the duration of chess laboratory was of about 24 hours[16].

[15] In this work we focused the visual-spatial abilities.
[16] The experimental classroom had 16 students, the control classroom had 19 students.

To one classroom (experimental classroom) was introduced both the independent variable (the classical geometry course) and the dependent variable (a chess course), while in the other classroom (control classroom) was introduced only the independent variable.

Both groups were subjected to a geometry pre-test that was compared with a geometrical post-test.

Both the geometrical tasks and the chess activities were recorded with audio-video devices.

These recordings allowed us to study with a semiotic approach the phase of learning and reasoning of a case study involving a student of the experimental classroom.

In the paragraph below we will describe in detail the chess course proposed.

5.2.1 Chess course

The aim of the course was to observe how a structured chess course can help the students to improve the perception and handling of geometrical elements.

The realization of didactical activities relates, under an epistemological point of view, to the analysis of the semiotic systems of cultural signification as theorized by theory of objectification, and under an ontological point of view, to the theory of configural concepts (see Chapter 4).

According to the research discussed in this context (Giaquinto 2007, Mach 1897, Kohler 1947, etc.), often students of primary and lower secondary school prefer a thought in which they focus on the shape of geometric figures instead of the elements of the figure and the relationships between them.

According to these problems, in the designing of the course, we have chosen activities that include the study and play of chess to encourage in students:

- The research and production of “self-made” patterns occurrences encouraging the attention and the memory;
- The need to challenge their own judgment, the opponent’s one and the classmates’ ones;
- The ability to make local and global evaluations;
- The ability to perceive the fundamental elements of a spatial configuration depending by the context and request.

The activities of problem posing and problem solving proposed during the eight lab meetings, gradually integrating the geometrical and chess context , were then designed with the aim of encouraging development of students visual spatial abilities.

In the “free game” moments the students played between them and solved particular tasks. This moments were aimed specifically to favor the development of meta-cognitive and emotional abilities, related to the learning, favoring an active and positive attitude.
The workshop was divided into eight three-hours lesson[17]. In the first three meetings we introduced the formal rules of chess, with strong attention to some basic geometrical and visual concepts (Horizontal, Vertical, Diagonal).
In the next five meeting we introduced more advanced contents (Elementary chess mate, tactical and strategically ideas) in parallel with the introduction and practice of geometrical concepts.
The lessons were designed as follow:
Lesson 1:

- Chess and geometry pre test;
- Fundaments of chess rules (Movement of pieces);
- Chess exercise “Movement of pieces”;

Lesson 2:

- Fundaments of chess rules (Purpose of the game, special moves and other rules)
- Free play;

Lesson 3:

- Relationships between pieces in the “space chessboard” (Center concept, opened column, “good and bad” pieces;
- Recognition of the properties and characteristics of chess configurations on vary of the “point of view”;
- Free play;

Lesson 4:

- Strategy activities and identification of distinguishing and recurrent characteristics;
- Strategy Elements: Plans and goals in chess;

[17] The design of the course was did using the personal experience of the researcher as chess player and chess instructor (Mario Ferro is a Federal Chess Instructor and a Chess Master) and as researcher in Chess and Math relationships (Mario Ferro collaborated for the ministerial Project SAM (D’Eredità, 2012))

- Free play;

Lesson 5:

- Relationships between pieces (Double attack, Discovered check, Pin and Skewer);
- Dynamic geometry (Construction and deconstruction of geometric figures);
- Free play;

Lesson 6:

- Chess Task (Perception and recognition of parts of a chess position);
- Free Play;

Lesson 7:

- Geometrical Task (Perception and recognition of parts of a geometrical problem);
- Free Play;

Lesson 8:

- Geometrical Post-test;
- Chess tournament.

The free play time varied from 30 minutes in 2nd lesson to 1 hour and half in 7th lesson with the aim to improve the moments of collaboration and challenge. All the activities were audio-video recorded and the analysis that we will show in 6.3 and 6.4 will be extracted from “Free play” moments.

Chapter 6

Experimental and theoretical research project

In this chapter are shown the experimental activities of this work involving an historical investigation on the use of base, height and area of triangle, an analysis of inscription before and after a chess course, and two multimodal semiotic analysis of students interactions during chess course.

6.1 An Historical investigation on use of base, height and area of triangle.

In this paragraph I will study the evolution of the meaning and terminology of the concepts of base, height and area of triangle with the purpose to study the 1st item of geometrical test that we will propose in next experimentation (see 6.2).

In this way we introduced the Euclid's Elements and we discussed some remarks. Then we did a brief discussion on the post-Euclidian geometry and an investigation on evolution of the concepts of base, height and area of triangle in Italian education between nineteenth and twentieth centuries.

Euclid's Elements.

The Elements of Euclid, in thirteen books, covers various topics of pure mathematics. It also contains the first axiomatic accommodation that has ever been given to a branch of the mathematics, the geometry.

The work was composed in Alexandria around 300 BC. The extraordinary merit of its author is highlighted by two facts of primary importance.

Firstly, it must be remembered that the first attempts to give a similar arrangement to the arithmetic have been made only at the end of the nineteenth century.

Secondly, it should be noted that The Elements of Euclid were the main work of reference for the geometry until the nineteenth century. In it we find a treatment, covering simultaneously what is today known as affine geometry, metric geometry and conformal geometry.

Euclid turns his attention to geometric entities such as objects that can be constructed with ruler and compass. The question of design seems to be central in Greek geometry: the famous three problems of duplication of the cube, the squaring of the circle and the trisection of the angle concern, of course, the ability to draw shapes that possess exactly certain measures assigned. Euclid's work combines his interest in the practical application to the rigor of abstract mathematical reasoning: it in fact includes both propositions (statements provable) and resolutions of problems. The doctrinal part of the first book contains all the theorems and problems which are necessary and sufficient to reach the Pythagorean theorem. In fact, for Euclid, the construction of the figures has the same role of the demonstrations of the existence of modern mathematics.

In Book II of the elements are laid the foundations of the process that made the ancients able to resolve, through images and geometric arguments all the problems of grade I and II.

The book III teaches the main theorems concerning the circumference which are not involved in the ratios and proportions. Of the results obtained, Euclid operates an important application in the sixth book of his Elements by establishing the theory of similarity of plane figures and with it is closed the planimetric part of the work in question.

The last three books of the Elements are devoted for the most part to the geometry of the space. It is precisely the book XI that teaches the fundamental relationships between straight lines and planes, parallelism and perpendicularity etc.. The next book instead is about the size of the area and volume of solids considered in elementary geometry.

To the convex regular polyhedral is dedicated to the last book of the Elements. By instituting a comparison between as they contain Euclid's Elements and the totality of geometrical knowledge possessed by previous geometers, we see easily that he did not belong Books I, II and IV, which would be known to the Pythagoreans; not the third which in all probability, is the main work of contemporary geometers of Hippocrates from Chios (fifth century BC), or the V, which we know to be the result of the meritorious labors of Eudoxus.

In the sixth he probably helped to establish with full rigor truth revealed by the testimony of the senses, and in the following three he merely coordinated and completed truth already known in the school of Pythagoras. In composing the Book X exploited work of Theaetetus enriching them of important developments, while the materials of the last three books he was given by Archytas, Aristaeus and Theaetetus, which progressed in the way opened by Pythagoras and Plato.

Following what was said by Frajese we can say that The Elements of Euclid are the culmination of a period of processing along three centuries of the mathematics (which is usually called the period of pre-Euclidean geometry). But The Elements of Euclid are also a point of departure; immediate successors of Euclid are the chief mathematicians Archimedes and Apollonius.

This work summarizes, uses, coordinates and arranges the work of predecessor mathematicians, offering one very valuable synthesis, which is at the same time analytical in the vastness of its framework.

Or as Enriques summarizes: “The work can not be considered original construction of Euclid, but appears reduction in an organic treaty of what the greek genius has built in the past three centuries.”

Remarking some aspects of Euclide’s Elements

In this section, I shall not dwell on the “classic” ambiguity extracted for centuries in the analysis of the elements, such as the V postulate or in ‘”darkness” of certain terms in the treatment of straight lines from which it seems clear to Euclid as the straight line is not given in the its actual infinity, but only in the potential of the segment can be extended as you want.

My goal will be to observe some definitions and propositions concerning the base and heights given by Euclid and notice if certain ambiguities are perpetuated in the use of students, including:

- Book I Proposition 4 (def base)
- Book I Proposition 5 (Theorem isosceles triangles)
- Book VI Definition 4 (polygon high)

Following a statement of Proclus on the proposition 4 it can be said that this is the first time it is mentioned, the term “base (of a triangle)” in the Elements, and he (Proclus) states that, according to Euclid, the base is the third side of a triangle after they have been defined before two, or “ the side on the point of view.” Also refers to the base and not to a base, which “clashes” with the modern theory.

A similar ambiguity is noted in the 5th proposition, in fact, this is said in an isosceles triangle, the base angles are equal. Even here refers to (and therefore unique) basis, and also if the isosceles triangle was “resting” on one of the sides equal, the proposition should not apply?

This question could respond by interpreting the comment of Proclus, and therefore, when considering the isosceles triangles are probably introduced the two equal sides, and then the base would be the remaining side. Despite this, it remains the “problem” of the alleged uniqueness of the base.

Finally, in the definition of height of any figure is the perpendicular drawn from the vertex to the base, then lacks in this definition that must be perpendicular to the base or *to the extension of it*.

In fact, if you do not consider the extension of the base, how can you determine the height from C in the triangle in Figure 1? It should be noted that the two segments are defined perpendicular if they form a right angle, so if you do not use the term “extension of the base” you should then talk of the orthogonality between the directions of the base and the height.

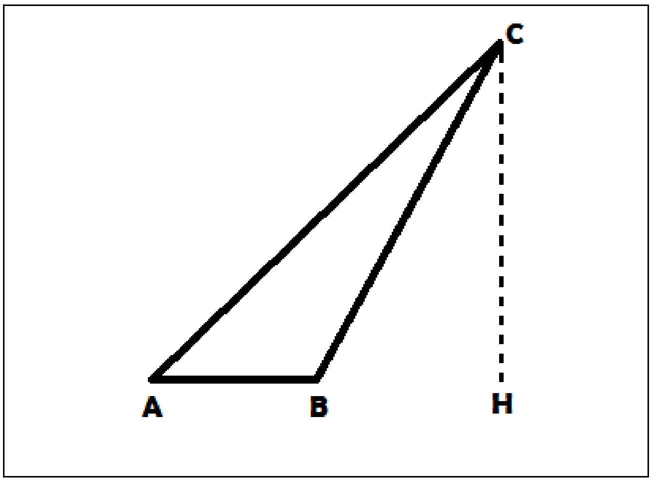


Figure 1 Triangle ABC, and CH is the height from the vertex C

Is interesting to note that these ambiguities recur, and they are now obstacles in learning situations.

The post-Euclidean geometry

At this point, we tried to see if there had been developments in terminology by the mathematicians that followed Euclid, but we have to take a big step forward.

In fact, after it, took root in Greece the "commentators" rather than "thinkers" giving rise to the silver period to which they belong Eutocio, Proclus, Sereno of Antissa and Pappus of Alexandria which caused extensive contributions to the works that commented on the famous *Mathematical Collection*.

Among them stood out the figure of Heron of Alexandria that inspired by the works of Archimedes and Euclid, wrote several treatises on mathematics and engineering (mechanical).

In his *Metric* (lost for a long time and rediscovered in Constantinople in 1896 in a manuscript dating back to 1100) and *Geometric* (known only indirectly) gives theorems and rules for flat areas, on the surface areas and volumes of many figures , and thanks to these can be understood as the greatest exponent of the "practical geometry".

Among these, stands out the so-called Heron's formula for calculating the area of a triangle as a function of the perimeter, and then the sides.

An indisputable mathematical regression on the scene was attributable to the appearance of the Roman people. Since the Romans, conquerors and legislators of the world, seem to have been free from any tendency to research in pure science. If the geometry did not fall into complete oblivion, it is only thanks to the people involved in the measurement of fields, who, however, had the only purpose to achieve sufficient accuracy in their operations for the daily needs of civil life. This sad period for mathematics, it can be said has to end 1200 with the appearance of Leonardo Fibonacci da Pisa (about 1180-1250) with the writing of Liber Abbaci and Practice Geometriae. This work was inspired by and modeled on the style of The Elements of Euclid (and likewise the Size of Erone), and has the intention to teach the procedures to be followed whenever you want to evaluate an area, a volume, or to divide a figure under certain conditions. In the writing of this book, Fibonacci begins by defining the technical terms used by him and sets out the principles to be applied in later. Then he teaches you how to calculate the areas of the squares and rectangles, and the inverse problem according to which will find the side of a square whose area is known, of course, devoting a section to the extraction of the square root. After this he comes to the calculation of the areas of triangles defining it as semi-product of a side for the corresponding height, to be applied to any triangle. The work follows the discussion of rhomb, rhomboids (parallelograms oblique-angles), trapezoid (isosceles, scalene or rectangles), not excluding the non-convex quadrangles.

What you immediately notice in this work is that although it is "vulgar" (the numbers are widely used in the explanations), this has undoubtedly less ambiguity of the work of Euclid.

In fact there is not here any reference to the base of a triangle (and thus the point of view) denoting perhaps by the Pisan mathematician the inadequacy of the term "base" simultaneously with the calculation of the area.

In fact, in agreement with Euclid, Fibonacci probably meant the meaning of the base as dependent on the relationship with the observer and then variant for rigid transformations, which however does not apply to the surface of a figure. At this point the previous doubts are "revealed", according to Euclid the term basis is related to the observer's viewpoint, but this term can not be used, as done by Fibonacci, together with the calculation of the areas. In this way seems “rehabilitated” (in part) also the definition of height from a vertex (whose opposite side is placed on the base).

Now, however, other questions arise: Today, why we missed the Euclidean meaning of the basis? Why the formula for calculating the area of a triangle is *base* times height divided by two, and not *side* times height divided by two? Is it possible that this mix between the expressions of Euclid and the Fibonacci is now a generator of the difficulties faced by students? Is it possible that the position of Euclid on triangles can generate the phenomenon in which all the triangles are designed as isosceles? Is it possible that the Euclidean concept of "base" can generate difficulties in the recognition of the figures? (see figure 2)

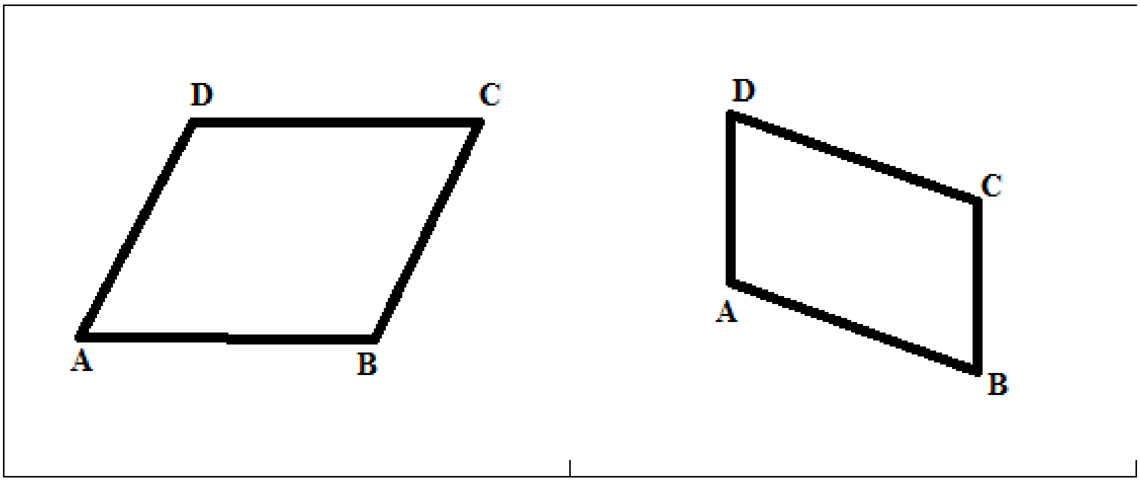


Figure 2 Two parallelograms, the second of which was recognized as a rectangle because the sides AD and BC were “not twisted side”

Indeed I have observed in a trial that the second parallelogram in Figure 2 is recognized as a rectangle and this was motivated by saying that the sides AD and BC are not "twisted" as those of the first figure. In this case then what you notice is how in these students is nestled the Euclidean conception of the base as a side point of view. It is therefore possible that these students have "objectified" the rectangle as the quadrilateral that has vertical sides perpendicular to the base (understood, however, in a Euclidean).

Returning to our historical analysis, we move in the fifteenth and sixteenth centuries AD in which there was a shift of interest towards the development of analytical research that he saw in the Italian Tartaglia, Cardano, Scipione Ferro and Ferrari the developers of the theories of the equations. After these, the primacy of mathematics is reached from France, especially

thanks to Viète. At the same time Mydorge, Pascal and Desargues consolidate the hegemony of France with the increase of the geometry of the original views, new methods and new propositions.
Even in Germany there was a revival, as evidenced by the research of geometric Johannes Kepler which can be traced back to the concept of the infinite point of the straight line.
With the passage of time it was strong the emergence of analytical researches, and so it was that in the seventeenth century gave birth to a new topic of research, analytical geometry.
Their founders and followers were Descartes and Fermat, who were the first to have seen the opportunity to represent with the signs of algebraic calculation the shapes of the space constructed in accordance with any law and have received all the skill that the analysis and geometry could have by their unexpected union.
Descartes in 1637 published the Gèomètrie, work in which is made explicit reference to the equation of the curve, it suggests the distinction between algebraic and transcendental curves, it is taught a method to draw the tangents to plane curves.
Fermat came perhaps before Descartes to this new branch of mathematics, and in his memory "Ad locos plano set solidos isagoge" has made known to the concept of equation of a curve, also has used the equation of the straight line, is discussed the general equation of second degree with two indeterminate, and applied the method of the coordinates to the solution of the equations. Also in the seventeenth century Italians Cavalieri (1598-1647) and Torricelli (1608-1647) invented the so-called "method of indivisible" who worked on the figures as solid as composed of an infinite number of infinitely thin planes and plans to endless straight lines. Although this type of geometry was founded on bases not very rigorous and therefore subject to much criticism, using it came to important results such as the theorem of Pappus Guldino and the principle of Cavalieri. The method was actually a first formulation of integral geometry but still concepts that were the basis of the analysis were not very clear.
The eighteenth century was the century of Euler who focused on the analysis but that gave birth to the topology and graph theory.
In the nineteenth century geometry, after a century in which he had made virtually no progress, it returned to be an important subject of study.
Gauss (1777-1855) found the conditions under which a regular polygon could be built using only ruler and compass, solving a problem that was open for millennia. Always Gauss started a new branch of geometry, differential geometry, introducing the notion of curvature of a surface.

Jakob Steiner (1796-1863) showed that all the figures constructible using ruler and compass can be constructed using only the straight line and an initial circle. Then Moebius introduced the Moebius function and studied the topology (Moebius strip).
But the most important innovation of the century in geometry were the non-Euclidean geometry. Gauss, trying to prove Euclid's fifth postulate, he came to the revolutionary conclusion that could exist geometry independent of the postulate and began studying hyperbolic geometry.
Janos Bolyai came to the same conclusion. However, the true developer of hyperbolic geometry was the Russian Lobachevsky (1792-1856).
Riemann made a fundamental contribution to the study of non-Euclidean geometry by defining the concept of a straight line as geodesic of a space.
Then studied the geometry built on the surface of a sphere; elliptical or Riemannian geometry. Riemann also dealt with topology introducing the Riemann surfaces. He anticipated the concept of the metric and of the tensor.
The revolutionary concept that was the basis of these geometries struggled a lot to be accepted. Henri Poincaré (1854-1912), Felix Klein (1849-1925) and Eugenio Beltrami showed the consistency and independence from the fifth postulate of these geometries, sanctioning their acceptance.
The text "Grundlagen der Geometrie" (translation: Foundations of Geometry) published by Hilbert in 1899, replaces the axioms of Euclid with a formal set, consisting of 21 axioms, which avoids the contradictions arising from that of Euclid. Independently and simultaneously, an American student of 19 years, Robert Lee Moore published an equivalent set of axioms. It is interesting that, although some axioms are the same, some axiom proposed by Moore is a theorem in Hilbert system, and vice versa. Hilbert uses concepts undefined and specifies their property exclusively through the axioms. These elements point, straight line, plane and others, could be replaced, as Hilbert says, by tables, chairs, beer mugs and other items. Of course, if the geometry deals with "things", the axioms are certainly not self-evident truths in themselves, but must be regarded as arbitrary. Hilbert first enumerates the undefined concepts; they are: point, straight line, plane, lying on (a relation between point and plane), stand between, congruence of pairs of points, and congruence of angles. The system of axioms (connection, order, congruence, of parallels and continuity) brings together into one whole the Euclidean plane and solid geometry. What we see is that in Theorem 1 (of the

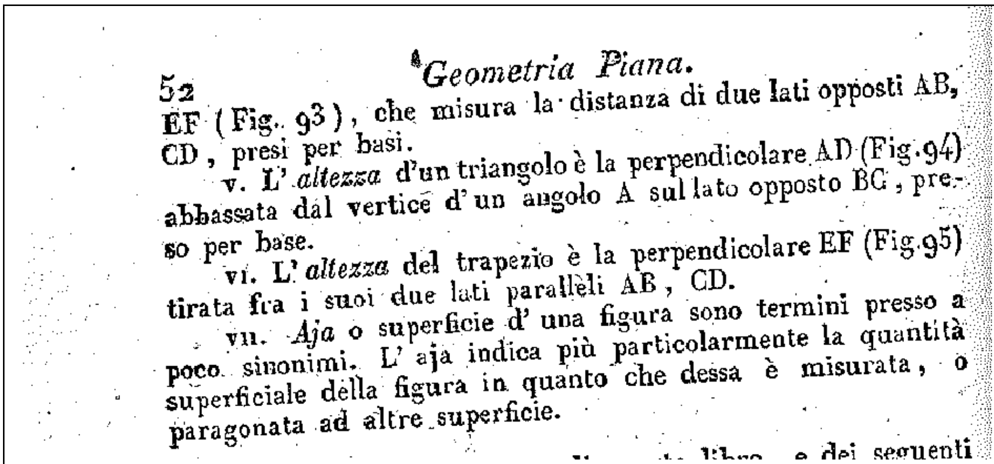
isosceles triangle) of the 3rd group of axioms (congruence), it is present the Euclidean concept of basis (unique and the side on the point of view).

The evolution of the concept of base and triangle in Italy between the nineteenth and twentieth centuries[18]

In this section, we made a survey on the historical evolution of mathematical terminology in reference to calculate the area of a triangle and the base, watching some Italian texts of geometry from 1842 to 1949.

Let's start with the book "Elementi di Geometria Piana" written by Legendre in 1842 (Fig. 3), and we show the definition of the height of a triangle (Fig. 3a) and the theorem to calculate the area of a triangle (fig. 3c).

> *The height of a triangle is the perpendicular AD (fig.3c) lowered from the vertex of an angle A on the opposite side BC, taken as a basis.*
> *The aja (area) of a triangle is equal to the product of its base times the half of its height.*



52 *Geometria Piana.*

EF (Fig. 93), che misura la distanza di due lati opposti AB, CD, presi per basi.

v. L'*altezza* d'un triangolo è la perpendicolare AD (Fig. 94) abbassata dal vertice d'un angolo A sul lato opposto BC, preso per base.

vi. L'*altezza* del trapezio è la perpendicolare EF (Fig. 95) tirata fra i suoi due lati paralleli AB, CD.

vii. *Aja* o superficie d'una figura sono termini presso a poco sinonimi. L'aja indica più particolarmente la quantità superficiale della figura in quanto che dessa è misurata, o paragonata ad altre superficie.

Figure 3a

[18] This is intended as an indicative survey and obviously not exhaustive seen that the research was carried out from the materials available in the library of the Department of Mathematics, University of Palermo.

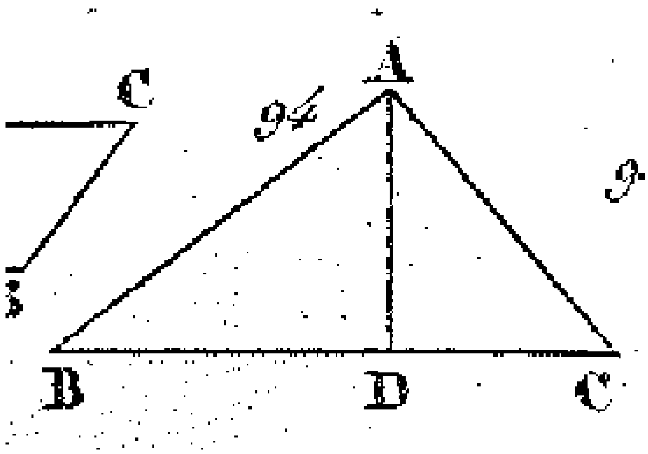


Figure 3b

PROPOSIZIONE VI.

TEOREMA.

L' aja d' un triangolo è eguale al prodotto della sua base per la metà della sua altezza.

Poichè il triangolo ABC (Fig. 104) è la metà del parallelogrammo ABCE, che ha la medesima base BC, e la medesima altezza AD (***); ora la superficie del parallelogrammo = BC × AD (****), dunque quella del triangolo = $\frac{1}{2}$ BC × AD, o BC × $\frac{1}{2}$ AD.

Corollario. Due triangoli della medesima altezza stanno fra loro come le loro basi, e due triangoli della medesima base stanno fra loro come le altezze.

Figure 3c

Figure 3 Scanned parts of Book "Elementi di Geometria Piana":
Definition of Height and base of triangle in fig.3a, the drawn triangle in fig.3b and the theorem to calculate the area of a triangle in fig. 3c.

The second book is " Elementi di geometria" written by Vittone in 1869 (fig. 4) and we show the definition of the height of a triangle (fig.4a) and a corollary of a theorem to calculate the area of a triangle (Fig. 4b).

> *The height of a triangle is the length of the perpendicular dropped from the vertex of an angle on the opposite side considered as the base, or on its prolongation.*
> *The area of a triangle is equal to the product of its base times half of its height.*

139. *L'altezza* di un triangolo è la lunghezza della perpendicolare abbassata dal vertice di un angolo sul lato opposto preso per base, o sul suo prolungamento.

Il vertice dell'angolo opposto alla base si chiama il vertice del triangolo.

Figure 4a

157. *Corollario* 3°. L'area di un triangolo è eguale al prodotto della sua base per la metà della sua altezza; perchè ogni triangolo è la metà di un parallelogrammo di egual base e di eguale altezza (147).

158. *Scolio*. L'area di un triangolo rettangolo è eguale al semiprodotto dei cateti.

Figura 4b

Figure 4 Scanned parts of Book “Elementi di Geometria”: Definition of Height and Base of triangle definition in fig.4a, and the corollary to calculate the area of a triangle in fig.4b.

The third text is " Fondamenti di Geometria" by Veronese, 1891, and we show the basic definition of an isosceles triangle (fig. 5).

If two sides eg. (AB) and (BC) of the isosceles triangle are equal it is said isoscleles, the third side (AC) is called the base of the triangle.

Def. III. Se due lati ad es. (*AB*) e (*BC*) del triangolo sono uguali esso dicesi *isoscele*, il terzo lato (*AC*) si chiama *base* del triangolo.
Def. IV. Se tutti i tre lati sono uguali il triangolo dicesi *equilatero.*

Figure 5 Scanned parts of Book "Fondammenti di Geometria":
Definition of Base of an isosceles triangle

The fourth text " Elementi di Geometria Intuitiva" by Lo Monaco - April 1926 (fig. 6) and we show the basic definition of the triangle (fig.6a) and a corollary to calculate the area of a triangle (Fig.6b).

Height of a triangle is the perpendicular segment led by a vertex to the straight line belongs to the opposite side (to which we give the name of base relative to that height). In a triangle, there are therefore three bases and three heights: they contribute to the same point.
Since a triangle is half of a parallelogram of equal base and equal height, as follows: The area of a triangle is given by half the product of the values of one side and the relative height.

The fifth text is " Elementi di geometria" by Amato - Usalini (1931) and we show the basic definition of a triangle (fig. 7).

Given a triangle, we say height relative to a straight line the segment perpendicular dropped from the opposite corner, on the straight line of the side and included between the vertex and the straight line of the side. The side, to whose straight line is perpendicular the height, is called the base of the triangle relative to that height. Every triangle has three heights.

Altezza di un triangolo è il segmento di perpendicolare condotta da un vertice alla retta cui appartiene il lato opposto (al quale si dà il nome di *base relativa* a quell'altezza).

In un triangolo vi sono dunque tre basi e tre altezze: queste concorrono in un medesimo punto.

L'altezza relativa ad un lato può incontrare questo lato o uno dei suoi prolungamenti; può darsi anche che l'altezza relativa ad un lato coincida con un altro lato.

Le unite figure danno un esempio di ciascuno dei tre casi.

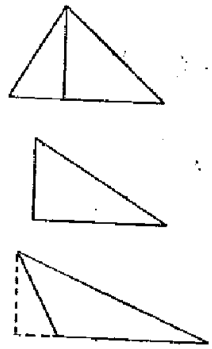


Figure 6a

198. Siccome un parallelogramma è equivalente al rettangolo che ha la stessa base e la stessa altezza, così:

L'area di un parallelogramma è eguale al prodotto della lunghezza della base e dell'altezza.

199. Poichè un triangolo è metà di un parallelogramma di eguale base ed eguale altezza, così:

L'area di un triangolo è data dal semiprodotto dei valori di un lato e dell'altezza relativa.

Figure 6b

Figure 6 Scanned parts of Book "Elementi di Geometria Intuitiva": Definition of Height and Base of triangle definition in fig.6a, and the corollary to calculate the area of a triangle in fig.6b.

147 — Dato un triangolo, diremo *altezza* relativa ad un lato il segmento perpendicolare abbassato dal vertice opposto, sulla retta del lato e compreso fra il vertice e la retta del lato.

Il lato, alla cui retta è perpendicolare l'altezza, dicesi *base* del triangolo relativa a quell'altezza.

Ogni triangolo ha tre altezze.

Figure 7 Scanned parts of Book "Elementi di Geometria":

Definition of Height and Base of triangle definition

The sixth and final text "Elementi di Geometria" by Tortorici in 1941 and we show the definition of the height of the triangle (Fig.8A) and a corollary to calculate the area of a triangle (Fig.8b).

> *Is called height of a triangle the distance of a vertex from the opposite side. This is often called the base (compared to the height). In any triangle will therefore have three heights.*
> *The area of a triangle is obtained by multiplying the (length) of the base times the (length) of the height and dividing the product by two.*

Let's summarize and discuss the definitions and theorems that we shown before, starting by the height definition.
The height was defined as: a perpendicular, a length of the perpendicular, a perpendicular segment and a distance.
The base was not defined by Euclide, and in the previous books was defined as follow: The opposite side of the height and the side to whose straight line is perpendicular the height.
Although there are several ways to name the height, one thing is more clear: that the base is defined and constructed by the height.
There is no reference to the position of the side in relation to the observer (e.g. the side in the point of view). So, in finding the segments that need to calculate the area of a triangle, the students have to individuate one of the heights and so the relative base.
The question now is, why in all the books[19] the investigation of base and height is inverse? So, first it is searched the base (as the side in the point of view) and then the height as the perpendicular segments to the base.
In fact the problem stands on the fact that with the term "base of a polygon" we have to refer to the base of one height of the polygon. And then, the criteria for the calculation of the area of triangle should be Area=(length of) height (one of the heights) times the (length of) relative base divided by two.
So, an obstacle that born as an epistemological obstacle now becomes a didactical obstacle, the language does not help the students investigation and their perceptive activities.

[19] All the scholastic books seen by the author

Se un triangolo è rettangolo i lati dell'angolo retto si dicono *cateti* e il lato opposto all'angolo retto dicesi *ipotenusa*.

Si chiama *altezza* di un triangolo la distanza di un vertice dal lato opposto. Questo dicesi spesso la *base* (rispetto a quella altezza). In ogni triangolo si hanno perciò tre altezze.

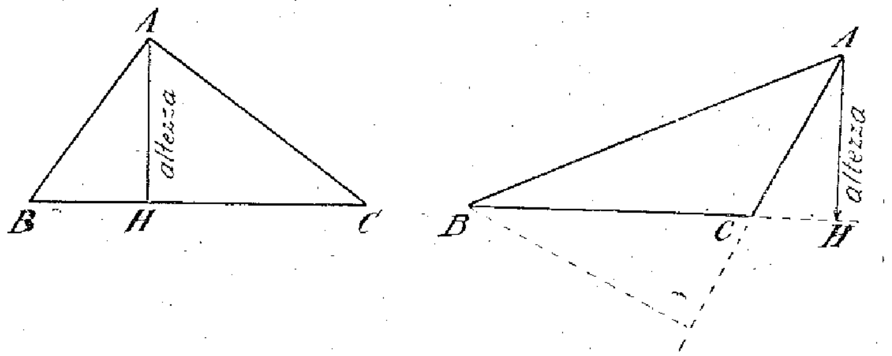


Figure 8a

121. — *L'area di un triangolo si ottiene moltiplicando la* (lunghezza della) *base per la* (lunghezza della) *altezza e dividendo il prodotto per due.*

Infatti (cfr. n. 111) noi abbiamo imparato che un triangolo è la metà di un parallelogrammo che ha la stessa base e la stessa altezza e perciò la sua area è la metà di quella del parallelogrammo.

Dunque, si hanno le formule:

$$S = \frac{1}{2} b \cdot h, \qquad b = 2S : h, \qquad h = 2S : b,$$

denotando sempre con S l'area del triangolo, con b ed h le lunghezze della sua base e della sua altezza.

Figure 8a

Figure 8 Scanned parts of Book "Elementi di Geometria": Definition of Height and Base of triangle definition in fig.8a, and the criteria to calculate the area of a triangle in fig.8b.

6.2 Study of a case study: inscriptions analysis of pre-test and post-test.

In this paragraph we will show a semiotic analysis that involves two 6th grade classrooms, one of which will participate to a twenty-four hours chess course. We call it experimental classroom, and the other one control classroom.

We made the classrooms selection proposing some geometrical tasks to five different classrooms of the same school, and choosing the ones with similar results.

After chess course we proposed geometrical task to both classroom and we analyzed the difference between pre-tests and post-tests.

In following sections we will analyze the pre-tests protocols evaluating the mistakes and obstacles that students dealt with and then we will propose the semiotic analysis of the post test protocols.

Semiotic analysis of pre-test protocols

In this chapter we will study the significant answers to the geometrical task (see appendix A) of the students of experimental and control classroom (that are similar for choose).

1st Item

With this item, we want to observe the constructive cooperation of the figural and conceptual aspects in geometrical problem solving activities. In particular we want to understand how this ability works unless the position and the orientation of geometrical figures on the students.

In fact, according to the H_4 hypothesis we expect that students can perceive the proposed figures depending on their orientation on paper (the fifth figure is a well known problem in Math Education), but how they perceive basis, eights, sides and so on?

The first three polygons are placed in the usual position and we guess that students do not have great difficulty on perceive them and mark the segments that need to calculate the area.

The other polygons are not placed in the usual way and we expect that they use term as "rotate", "long" or "twisted"[20] to characterize them, but in the case of fifth polygon (the square) with a rotation of a 45°, we expect that it can be perceived as a rhombus and do not as a rotated square. In this analysis we will focus on fifth, sixth and eighth polygon.

[20] In Italian rotate=ruotato, long=lungo, twisted=storto

Let's start the analysis studying the answers to fifth geometrical figure.

We remember that we asked to recognize the figures and to mark the segments that need to calculate the area.

Forty-eight students (twenty-six of experimental classroom and twenty-two of control classroom) recognized the figure as a rhombus and only five students (two of experimental classroom and three of control classroom) recognized the figure as a square.

This phenomena was expected on designing of item, but we can't expect that all of the students that recognized the figure as rhombus, in marking the segment for calculation of the area, do not search, mark or mentionate the diagonals as shown in figure 1 where we put the most significant answers.

In fig.1a students marked the angles but they wrote "No base sotto" – literally No base below – arguing that they can't mark a base below the figure. In this case the students were searching for base[21] when it wasn't asked by task. They do not need of measurement of base to calculate the area of rhombus or square.

In fig.1b students signed two sides of polygon with letter "l" – that in Italian stand for *lato* (side) – and in this example it is clear how the students recognize the figure as a rhombus but marked two sides for the calculation of the square area A=s*s[22].

In fig.1c students marked a dashed segment and a letter h that stand for height, but not only it was not needed in the task request, but they marked it on the diagonal position.

In fig.1d students marked two segments, both without letters and extern of the figure, and obviously these segments are not the diagonals of rhombus.

In this case we can suppose that the students marked a base and an height of an hypothetic polygon, because they are usually used to calculate the area of common polygons and they have usually that positions.

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[21] Now we do not care if they were searching in the right way the base, but we care on the reason of this research.

[22] In Italian A=l*l

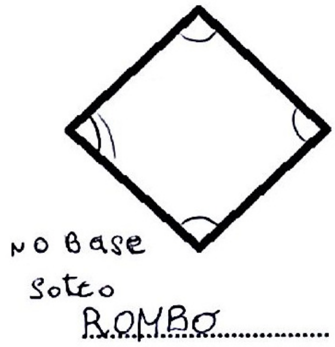


Fig.1a

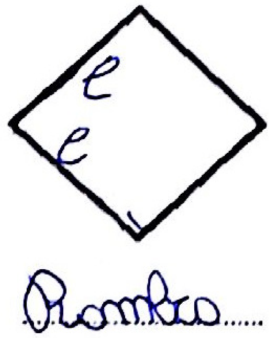


Fig.1b

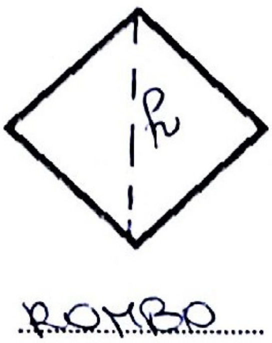


Fig.1c

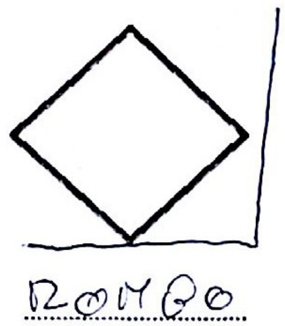


Fig.1d

Figure 1 The significant answers of students of experimental classroom to the 5th polygon of 1st item of pre-test

The other polygons were solved in the right way (more or less), and the students had problems only with sixth and eighth polygon, and we will shown the most significant answers in figure 2.

The half of students of both experimental classroom and control classroom answered in this way, and it is clear how there is no cooperation between conceptual and figural aspect of height in all of that cases.

The figure 2d, similar to figure 1d show a need to mark segments parallel and perpendicular to student view.

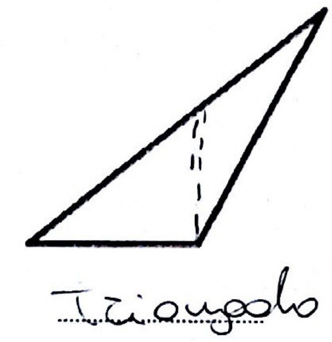


Fig.2a

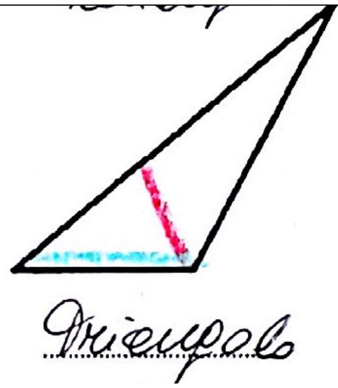


Fig.2b

Fig.2c

Fig.2d

Figure 2 The significant answers of students of experimental classroom to the 6th polygon of 1st item of pre-test (fig.2a,2b) and to 8th polygon of 1st item of pre-test(fig.2c,fig.2d)

2nd item

In the 2nd item was required to construct some geometrical figures starting by some sides. In this item was required a cooperation of figural and conceptual aspects too with the prevalence of discursive apprehension.

This item was designed to evaluate the answers on fifth polygon of first item. In fact it was placed on purpose on same place of first item and although the side was “twisted” or “rotate” they had no difficulty on construct a square. Only one student affirm that he can’t construct a square for the position of side (fig.3a).

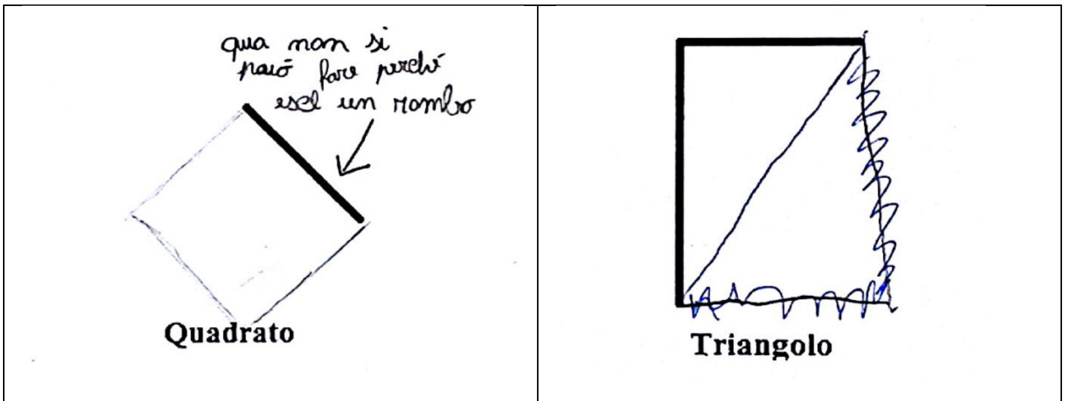


Fig. 3a **Fig. 3b**

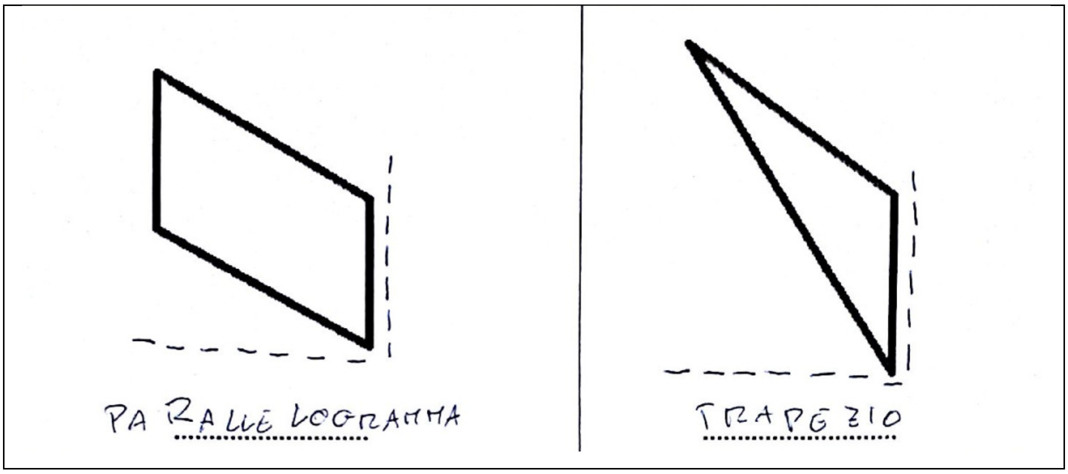
Figure 3 The significant answers of students of experimental classroom to the 2^{nd} item of pre-test

What we don’t expect was that many students made the mistake in fig.3b, in fact they started constructing a rectangle, and then they changed it in triangle, that we can attribute to distraction.

3^{rd} item

In the 3^{rd} item we put the requests of previous items and in this item too the students shown a non cooperation between figural and conceptual aspects.

In the following figure (fig.4) we will show the most significant answers.



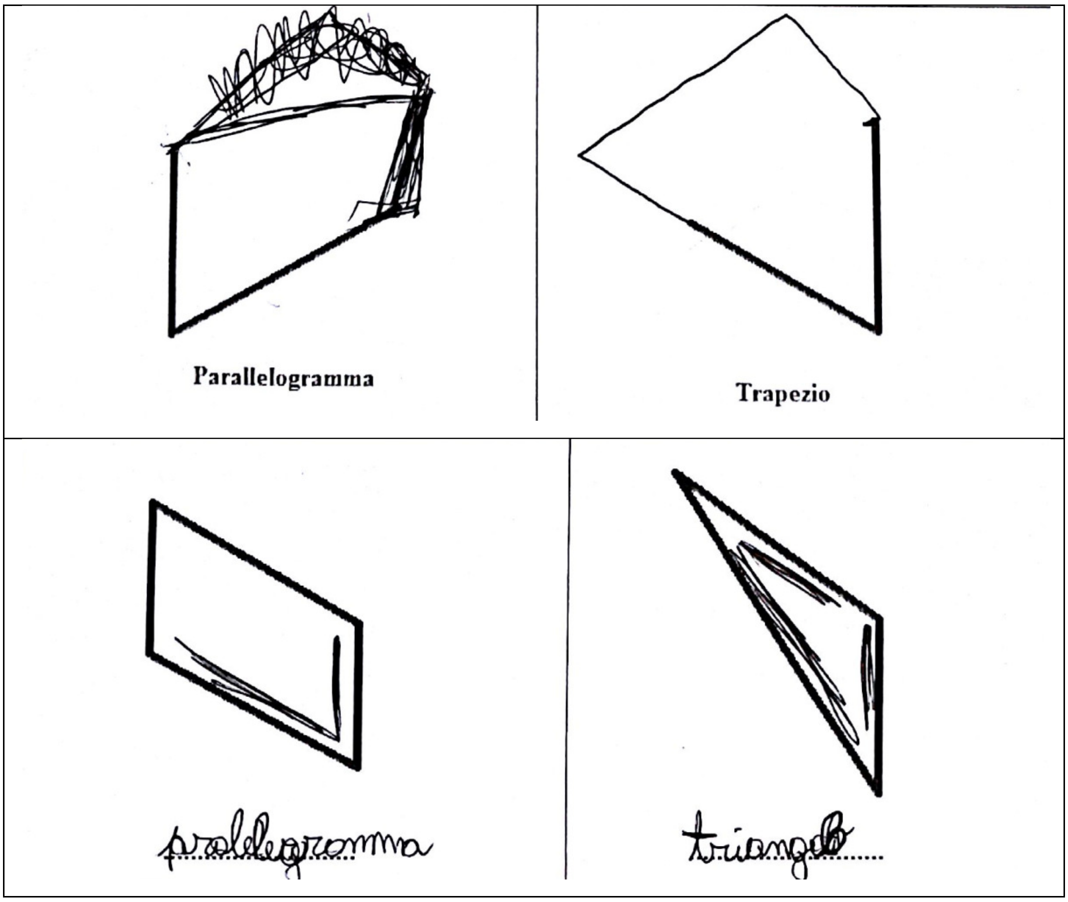


Figure 4 The significant answers of students of experimental classroom to the 3rd item of pre-test

4th item

With this item we want to observe the ability to keep in mind and coordinate as many as possible figural conceptual items and the ability to predict and integrate the effect of each transformation on the road to the solution.

In fact the core of this item was the request to draw an equilateral right – angle triangle and an equilateral – scalene triangle. At 6th grade students can't prove that it is not possible draw a triangle that satisfy both conditions, with chains of conceptual deductions. So the strategy that the students used to solve this task was to draw a right – angle triangle and then manipulate the magnitude of the sides to obtain (in vane) an equilateral triangle.

In figure 5 we will show the most significant answers.

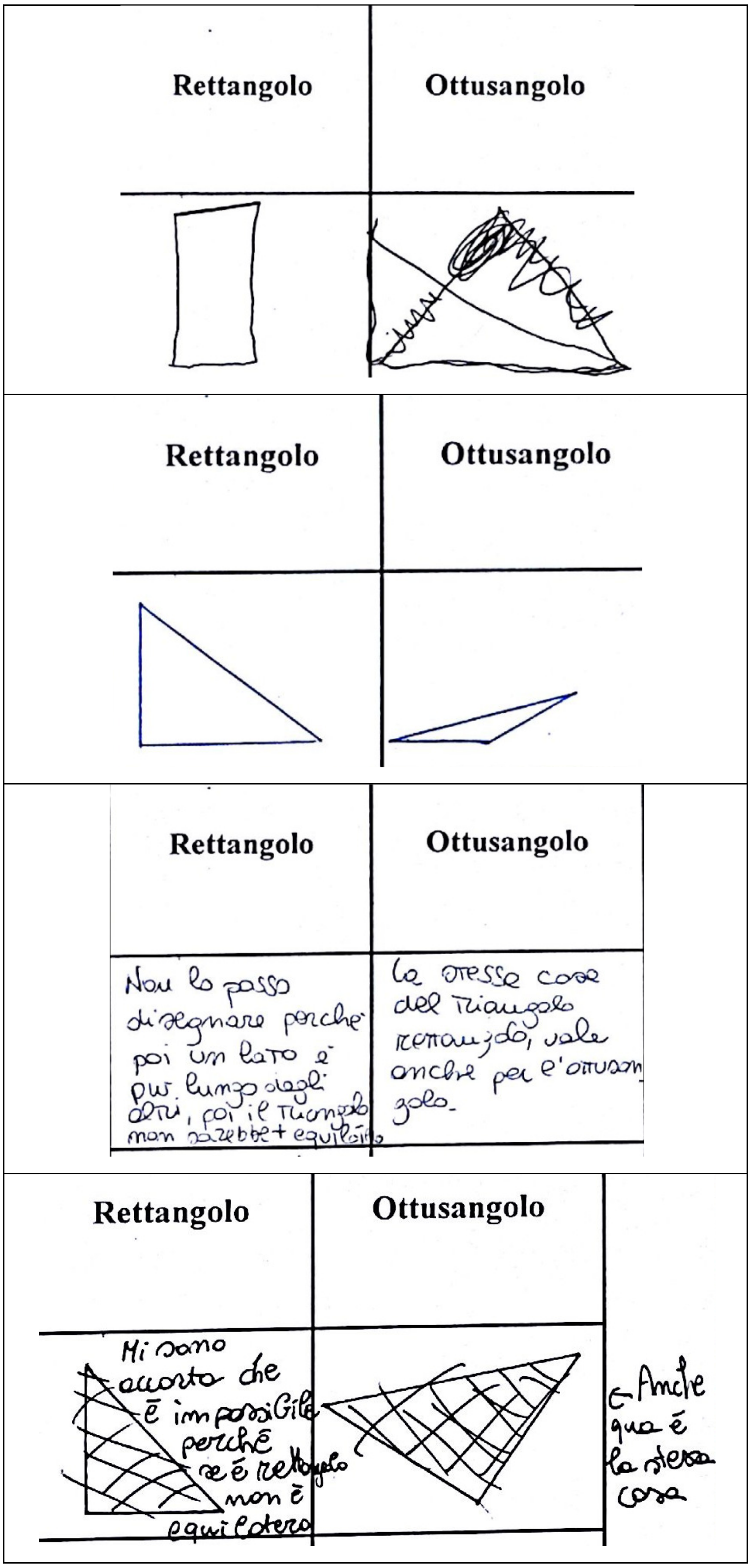


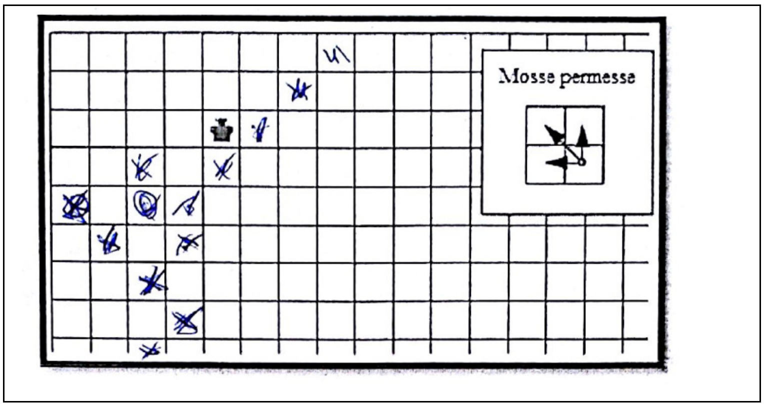
Figure 5 The significant answers of students of experimental classroom to the 4th item of pre-test

In fig.5a there is a result of a linguistic ambiguity. In fact right-angled triangle in Italian is "triangolo rettangolo" – literally "triangle rectangle", so the students drawn a rectangle. Always in fig.5a there is a confused draw where the students designed a right-angled triangle and an equilateral triangle.

In fig.5b there is the most frequent wrong answer, in which the students drawn a right-angled triangle and an ottusangle triangle. In fig 5c and fig 5d the students affirm that they can't draw the triangle because one side become longest of other sides.

5th item

This item[23] require the ability to keep in mind and coordinate as many as possible figural conceptual items and the ability to predict and integrate the effect of each transformation on the road to the solution and the ability to organize the mental process in meaningful subunits so as to reduce the memory load. This item was totally failed by all the students and in figure 6 we will show the best answers.



[23] This item was designed using a similar task that I seen in a Math B-Day in Nitra (Slo)

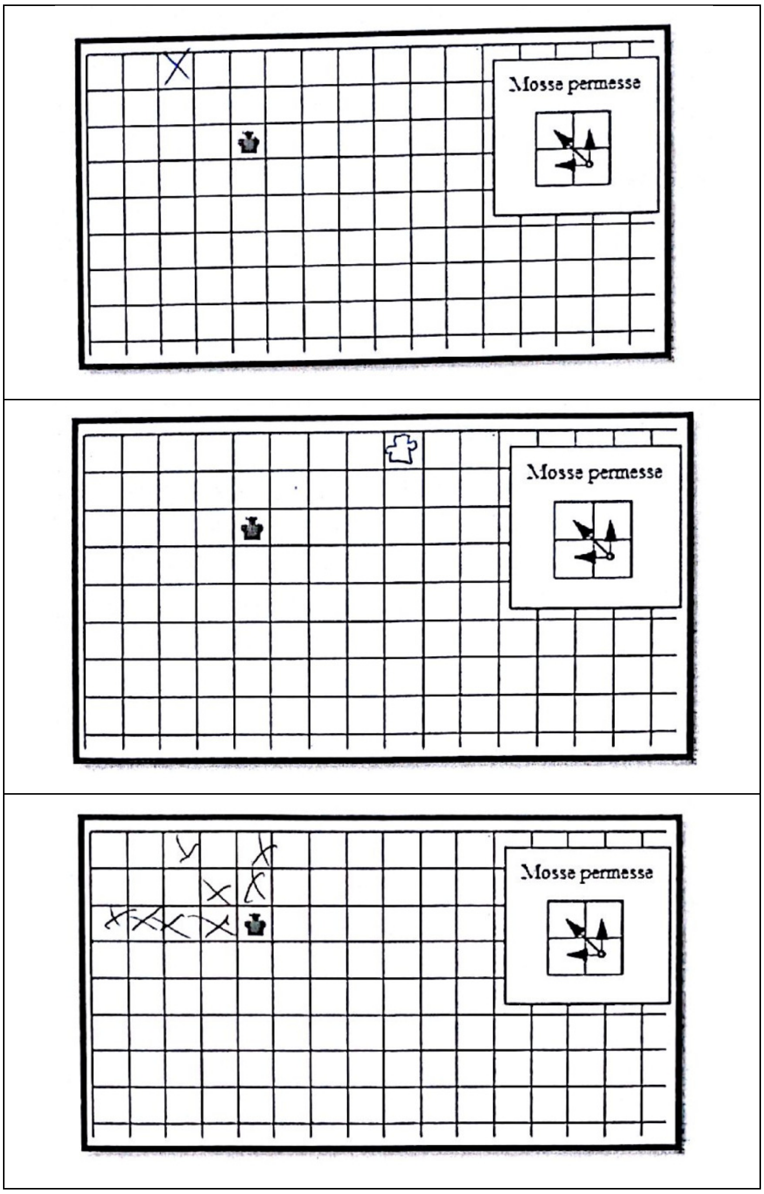


Figure 6 The significant answers of students of experimental classroom to the 5th item of pre-test

Semiotic analysis of post-test protocols

The post-test was proposed about two months later of pre-test, and only students of experimental classroom improved significantly their answers about the 1st and 5th item. (Only the 2nd item was solved correctly by both experimental and control classroom).

In this section we will show the most significant answers of experimental classroom in the 1st item and 5th of geometrical task.

1st item

We will focus our attention on fifth polygon, because it was the one with more interesting answers and the students of experimental classroom solved correctly the sixth and eighth polygons.

The answer to the fifth polygon were of four kinds (fig.7). The most "popular" was the one in fig.7a where the students recognized that polygon as a rhombus , but differently by pre-test, they looked for the diagonals and don't for the sides.

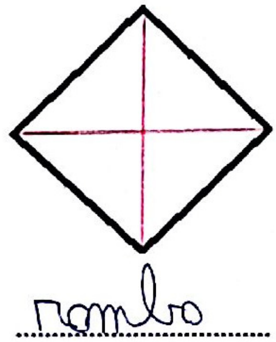


Fig7a

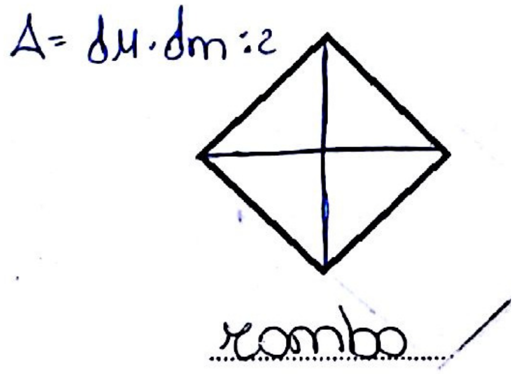


Fig7b

Fig7c

Fig7d

Figure 7 The significant answers of students of experimental classroom to the 5th polygon of 1st item of post-test

In the answer in figure 7b the students marked the diagonals and wrote the formula for the calculation of the area of a Rhombus $A=D_M*D_m/2$[24].

[24] "Area=Major Diagonal times minor Diagonal divided by two"

In the answer in figure 7c the students recognized the polygon as a rhombus, but they wrote square (*quadrato* in Italian language) on brackets, they marked the diagonals and marked ":2" out the figure in completing the formula for the calculation of the area.
In the answer in figure 7d the students recognized the figure as a rotated square and they marked the side (*lato* in Italian language).
These answers show how the way to look a geometrical figure of the students is significantly changed. After perceiving the figure as a rhombus their objective of reflection becomes the diagonals. The transcriptions in fig. 7b, 7c and 7d show a certain synergy between figural and conceptual aspects. So we can claim that students improved the constructive cooperation of the figural and conceptual aspects in a geometrical problem solving activity (Fischbein).

5th item

The student of control classroom, similarly to the post test, did not produce any significant answer and only few of them tried to solve the problem.
Instead, all the students of experimental classroom tried to solve the problem and the majority of the answers were of two kinds (fig.8).
In the answer in fig. 8a students founded correctly the "game ending squares" and marked with the number "1" and "2" the positions of player 1 and player 2 respectively that they reached in some variations.
In the answer in fig. 8b students founded correctly the "game ending squares" and marked the path moves of players that they reached in some variations.
These answers show that the students improved the ability to predict and integrate the effect of each transformation on the road to the solution (Fischbein).

Final Remarks

Whit the analysis of inscriptions we observed that after the chess course of twenty-four hours (its duration was of about two months), the students of experimental classroom improved the constructive cooperation of the figural and conceptual aspects in a geometrical problem solving activity, the ability to keep in mind and coordinate as many as possible figural conceptual items and the ability to predict and integrate the effect of each transformation on the road to the solution (Fischbein, 1993).
Now we can state that the students of experimental classroom improved their ways to answer to the proposed geometrical post-test. Can we say that this "transformation" depend of an

improving of visual spatial abilities? In order to answer to this question we were helped from some psychologist experts[25] that proposed psychological test to the students of experimental and control classrooms. This tests submission was with the geometrical pre-test and post-test submission.

The analysis of these psychological tests (entirely made by Department of Psychology of University of Palermo) show that the students of Experimental classroom improved the following abilities (D'Amico, Ferro, et. ali, 2012):

- Serial recognition of visual pattern;
- Recall of Visual-Spatial pattern;
- Recall of Visual-Spatial sequences;
- Visual Spatial Working Memory;
- Listening Span;
- Counting Span;

Anyway we would to understand if the analyzed case was only an isolated case and we would to proof the influence of the chess teacher that was the researcher too.

So we did an investigation in other School of another Italian city (E. Fieramosca in Barletta) with another chess teacher, and we proposed the post test to three 6th grade classroom that does not did chess and two 6th grade classroom that did chess.

The results were that the students that played and studied chess did similar answer to which ones of the students of our experimental classroom.

To improve this investigation we would observe how a chess course could improve the abilities mentioned above. In this way we did a multimodal semiotic analysis of specific chess activities: Reconstruction of chess position (see 6.3) and Objectification of Movement of pieces (see 6.4).

[25] Prof. D'Amico Antonella, Department of Psychology, Palermo.

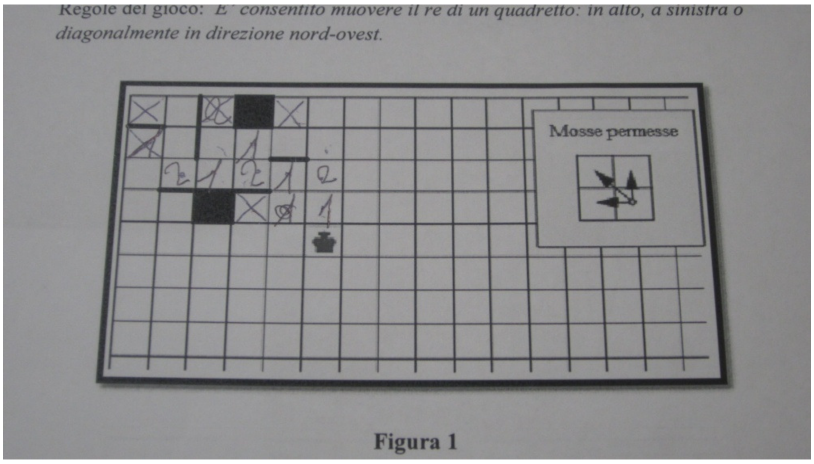


Fig.8a

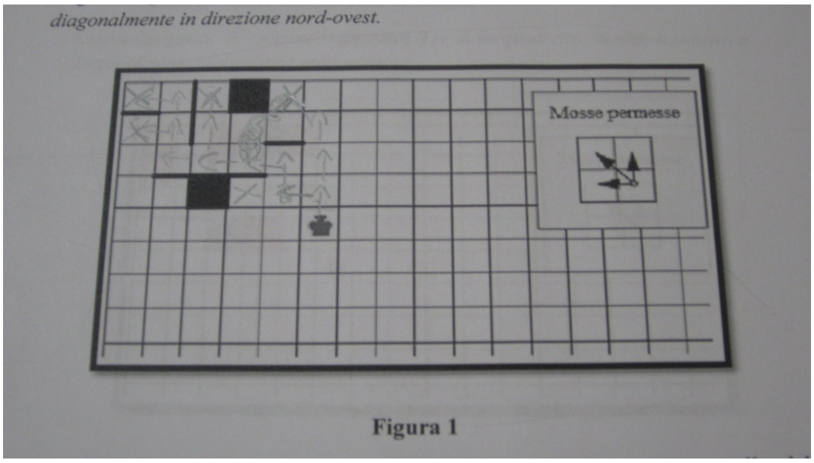


Fig.8b

Figure 8 The significant answers of students of experimental classroom to the 5st item of post-test

6.3 Construction of chess position and geometrical figure: Multimodal semiotic analysis of a case study

Usually, in chess course at school or at chess clubs we use the so-called wall chessboard that consists of a board large enough to serve as a blackboard for chess instructor. In this chapter we want to study an exercise that can be trivial to chess players at a certain level but that turns out to be an obstacle for many students who are approaching the game of chess: the reconstruction of a chess position.

The exercise of reconstruction of a chess position consists in placing the pieces on the chessboard in front of him, according to those prepared by the teacher on the wall board. This type of activity has interesting connections with the reconstruction of the figure of Rey in fig.1 (a psychological test used for the detection of LD – Learning Disability) and can be highlighted the following common characteristics: 1. The position of chess is equipped with a complex structure; 2. Meaningless in its entirety for non-expert players and students.

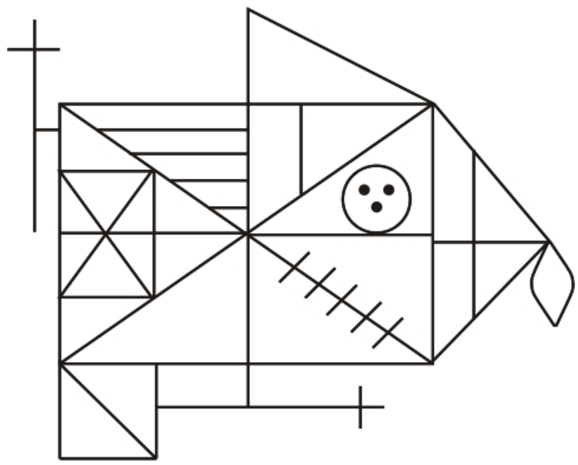


Figure 1 Figure of Rey

These aspects make the student, looking at the position on the mural board, did not manage to build a mental image and then he tries to reproduce it on the board in front of him. In fact, the only way to perform this work is to reproduce only a few pieces at a time and iteratively reconstruct the entire position.

These are the processes that we identified in this task of reconstruction:

- Recognizing the pieces and the board (eg: Knowing the name of each piece, recognize the different pieces and identify other characteristics of the board);
- Recognizing the artifact "wall chessboard" and "table chess board" (eg: in the mural one is possible that the pieces and boxes have different color from white and black, or knowing that the box at the bottom right should be white);

In the case of integral construction to the wall board (in the arrangement on the wall chessboard in the bottom there is white, and the student has white in the table chessboard):

1. Visual analysis of the pieces and boxes of the mural chessboard (elements) and evaluation of the relationships and the spatial constraints between elements (eg, "there is a pawn in the lower left corner on a black box in front of his pawn");
2. Reproduction of the previous evaluation on the board and control the execution (with one hand holding the piece and the eyes move from the wall to the board to check the correctness of the placement - the persistence of the hand is used to draw your attention to the point of mural chessboard, hand and eye are coordinated in this "de facto" control);
3. Eventual self-correction;
4. Repeat steps 1-2-3 using as a reference the pieces already placed until the completion of the board (Figure)
5. Eventual self-correction of 4.

N.B. It 'possible that the student is not able to perform this procedure because it does not know how and where to start (in one case the teacher had to give an idea providing a technique or just a starting point), everything, always in case the student does not have LD and therefore he does not have specific difficulties in copying from the blackboard.
In the case of non-interal construction with the wall board (in the arrangement on the wall chessboard in the bottom there is white, and the student has black in the table chessboard), the situation is different. In fact, the mental image constructed by the position of the wall chessboard differs from the position on the table board of a rotation of 180 °.
For example, suppose that the position placed on the wall chessboard is the one in Fig. 2a and the student must arrange the pieces blacks and then reach the position shown in Fig. 2b.
What the student should do is:

1. Visual analysis of the pieces and boxes of the mural chessboard (elements) and the evaluation of the relationships and the spatial constraints between elements (eg, "there is a Rook at the top right of the black box in the middle of the pawn");
2. Change these relationships and spatial constraints in order to be consistent with the 180 ° rotation of the board (eg, "the rook goes to the bottom left, on the black box, in the middle of the pawn");
3. Reproduction of the previous evaluation on the board and control the execution;

This is the moment of rupture, in fact, the control of reproduction is performed on a visual level, and this also implies the construction of a mental image of the position arranged on the table board, the rotation of this and finally the comparison with the mural one.
In terms of cognitive processes according to Duval, the arrangement of the pieces on the board requires discursive apprehension, however the control and possible corrections require an operative one.

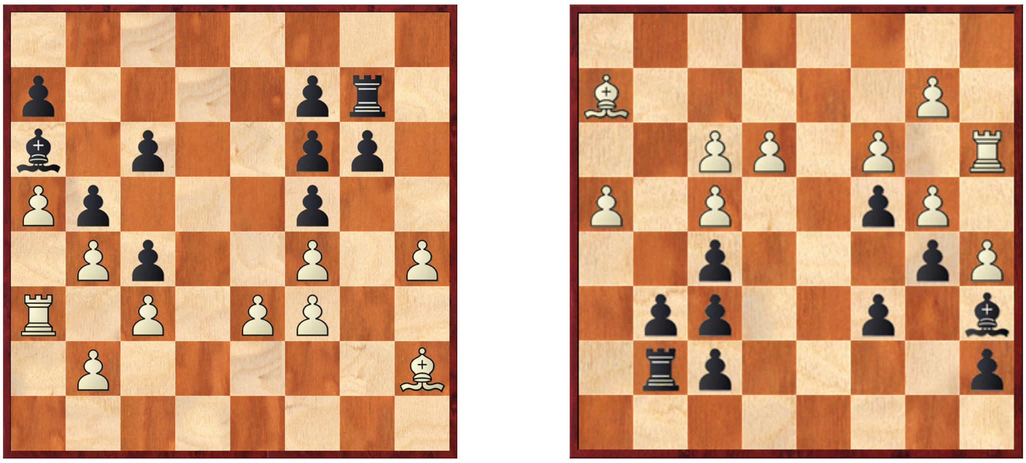


Figure 2a **Figure 2b**

Figure 2 The position placed on wall chessboard (fig.2a) and the position that the student with black pieces will construct in the table chessboard (fig.2b).

Seen in these terms, the reconstruction of a chess position could be very complex given the unreliability of the spatial relations used in the examples above.
In these examples, it was missing an element of the artifact chessboard that is initially ignored by the students, the coordinates. The "true" position offered to students is in fact the one in fig.3, in which the coordinates define unequivocally the sqaures on the board.

Figure 3 The position seen by white player

Under these conditions, the construction becomes much easier, because now I can identify the black rook on the mural chessboard, and thanks to the coordinates I can pinpoint its exact position on the board (g7), then I take a black tower in hand, I identify the box g7on the board, and there I will place the rook.

Nevertheless, the phase of control of the execution will not be only discursive, but still requires a significant manipulation of visual-spatial.

To want to be more accurate, and in accordance with Fischbein, at this stage are required the following skills:

- The ability to organize the mental process into meaningful subunits so as to reduce the memory load;
- The ability to keep in mind and coordinate as many as possible figural conceptual items;
- the constructive cooperation of the figural and conceptual aspects in a (geometrical) problem solving activity;

One of the crucial points in the experimental activity, which will show in later chapters, was to understand if the task of reconstruction of chess positions could improve those skills deemed essential by Fischbein for solving geometric problems.

To do this, the authors have carried out a semiotic multimodal analysis, assuming that certain cognitive processes in viewing activities, are characterized by specific semiotic developments.

6.3.1 Reconstruction of the position and configural concepts

Although we have previously made references to the theory of figural concepts of Fischbein, surely we cannot say that a chess position is a figural concept. In fact, according to Fischbein, Figural concepts are abstract, general, ideal, pure, logically determinable entities, though they still reflect and manipulate mentally representations of spatial properties.

A chess position must ensure respect an axiomatic system of rules, has conceptual and figural, but definitely it is not an ideal and abstract entity. To overcome this problem and taking inspiration from the theory of figural concepts we have introduced a theoretical idea that could be applied in the context of chess: the configural concepts.

We have defined a configural concept as a made up of chess objects and their conceptual relationships. Its meaning comes from the hierarchical linkage of the conceptual relationships between the involved chess objects and from its position in the whole theoretical structure of the pieces in the chessboard.

The chess objects were defined as the elements of the artifact "chess" – the pieces, the squares and the chessboard – which therefore possess rules and scopes.

At this point it seems clear how the comparison with the figure of Rey made at the beginning of the chapter may be inappropriate, since the arrangement of the pieces on the board cannot be completely random. For example, in the above examples both Kings were missing, or pawns cannot be arranged in 1° crossbar and so.

A (regular) chess position possesses a meaning that depends on the arrangement of the pieces on the board, whose revelation is not linear and complete, and depends on the purpose of reflection. It is for these reasons that a chess position is a prime example of configural concept.

These conceptual connections between chess objects (pieces and boxes) objectified in chess experience, develop more sophisticated ways for the reconstruction of chess positions, such as the arrangement of the pieces by location or configuration (see fig.4) or arrangement of parts by type (see fig.4).

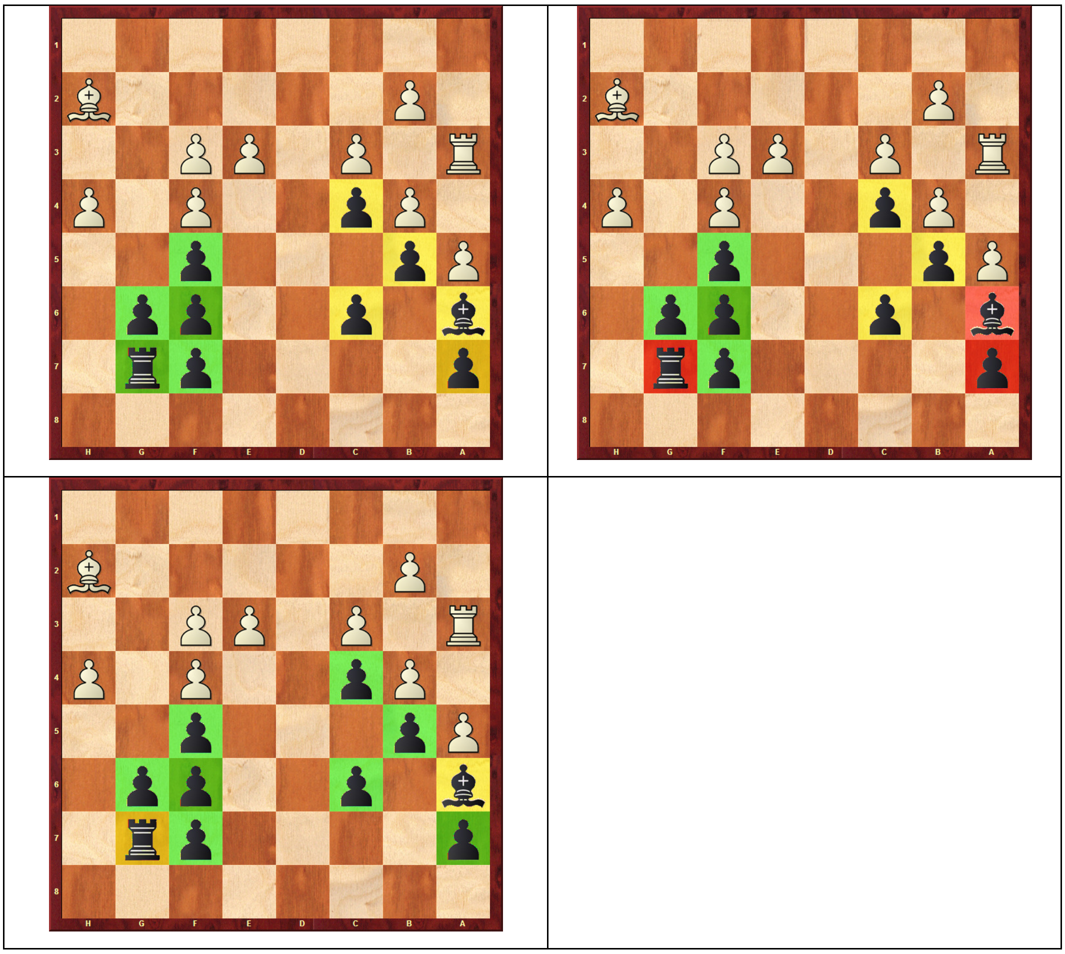


Figure 4. In the first position the pieces on the side of King (green) are arranged before and then from the side of Queen (yellow). In the second position are arranged configurations (chunk) of pieces (green and yellow) and then the "seams" (red). In the third position are arranged before the pawns (green) and after the pieces (yellow).

These methods of reconstruction (the ones above are just examples and we could also enumerate others) are an example of one of the distinguishing characteristics of the chess player: the ability to control. In fact, the chess player develops methods of control over their thoughts that require intense cooperation between the discursive apprehension and operational one, between conceptual aspect and figural aspect of the position.
Our interest is now to investigate on the semiotic means of objectification involved in this process of become aware of the chessboard, the pieces and their relationships with the aim to reconstruct a chess position.

Then, we want investigate on the semiotic means of objectification involved in problem of reconstruction of a geometrical figure with the aim to find some relationships between these two activities. To do that we will study a case study that involves a 6^{th} grade student.

6.3.2 Case Study Analysis

In this section we will analyze the multimodal semiotic activity of one student in situations in which he constructs a chess position and solve geometrical tasks.
The scheme of investigation we adopted will therefore have the aim of identifying a common feature in the comparison of semiotic nodes belonging to a process of objectification.
To discuss it we will show some frames of video recording and stylized drawings in different moments of a chess/geometry course.
Aim of our analysis is to investigate on how the semiotic activities of the student change in gaining awareness on construction chess position and so acquiring and improving those visual spatial abilities that we discussed above.
The observed skills had visual-spatial nature, and in particular were observed activity based on the construction and deconstruction of geometric and chess figures.
The geometric activities included Tangram exercises of varying complexity in which the complete figure is projected in a wall and the students try to manipulate the subfigures using drawings, linguistic devices, gestures and so on. The chess course had as main objectives that of teaching the axiomatic system on the rules of the game of chess, moving correctly a piece and do not making illegal moves.
In this analysis we will treat five episodes: four episodes in reconstruction of chess position and two episodes in reconstruction of geometrical figures (Tangram).
In every episode we will investigate on how the students use the following semiotic means of objectification:

- Artefacts (e.g. The student move a piece on the chessboard or make a draw in the paper);
- Kinesthetic actions (e.g. The student turn his head to look the blackboard or go stand up);
- Gestures (e.g. The student taps the piece on chessboard or simulate the rotation of a geometrical figure);
- Pointing Gestures (e.g. The student point at a square of the chessboard or at a subfigure of a configuration);

- Linguistic Devices (e.g. The student say the coordination of a square of the chessboard or the combination of some subfigures).

and how they are distributed in each episode.

To calculate the distribution of semiotic means of objectification, we used a file excel (fig.5) in which we indicated the timeline with a span of five seconds in the first column and in the first row we indicated the semiotic means observed.

	SEMIOTIC MEANS OF OBJECTIFICATION				
TIME	Gestures	Pointing Gest	Ling. Dev	Kinaest. Act.	Artefacts
00:33:30					
00:33:35					
00:33:40					
00:33:45					
00:33:50					
00:33:55					
00:34:00					
00:34:05					
00:34:10					
00:34:15					
00:34:20					
00:34:25					
00:34:30					
00:34:35					
00:34:40					
00:34:45					
00:34:50					
00:34:55					
00:35:00					
00:35:05					
00:35:10					
00:35:15					
00:35:20					
00:35:25					
00:35:30					

Figure 5 An example of file excel that we used to calculate the distribution of semiotic means of objectification

The analysis consisted in checking in videotape recordings the semiotic means of objectification every five seconds, and then mark it on the Table. In situation in which there were different individuals we used different colors for marks.

So, in this way we can express how many time (in an approximate way) a semiotic mean occurred in whole episode, obtaining an absolute frequency.

In the graph that we will show below we used a relative frequency calculated dividing the absolute frequency by the number of time spans.

Episode 1

In this episode of 2 minutes and 30 seconds, we will observe the student while he is trying to reconstruct a chess position (fig.2a) in a chessboard in front of him, copying by a wall chessboard (that he had in his back).

Having the black pieces, in his chessboard he will reconstruct the position in fig.2b. This episode belonged to the 1st chess lesson[26] and we observed great difficult in this position reconstruction (that he cannot completed).

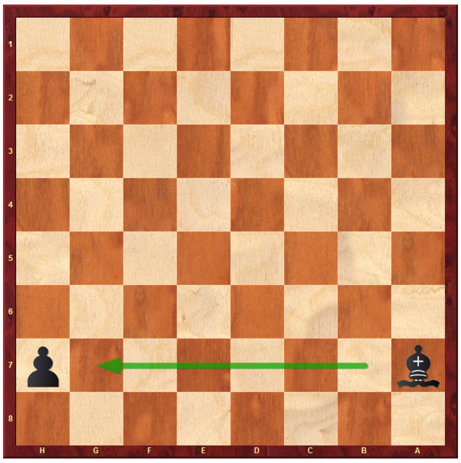


Fig. 6a

Fig. 6b

Figure 6 The student gesture to connect the pawn in h7 and the bishop in a7. In fig.6a the finger path on the chessboard and in fig.6b a photo of this gesture.

[26] The duration of very chess lesson was of three hours. The whole chess course consisted of eight lessons.

Initially he putted the pawn in h7 and the bishop in a7 (fig.6a), then he did the gesture in fig.6b linking the putted pieces.

Successively, he moves the head continuously from table chessboard to wall chessboard (he is back to it), putting pieces on chessboard and changing them placement obtaining the positions in figure 7 (in time order).

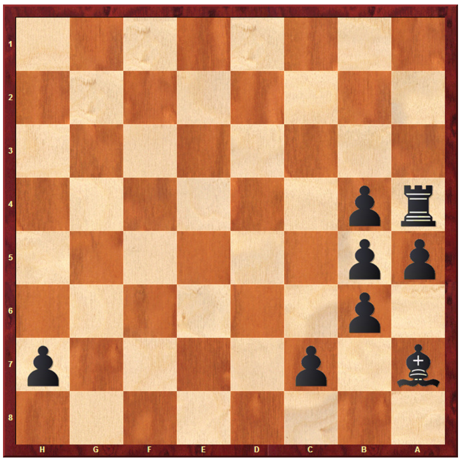


Fig.7a

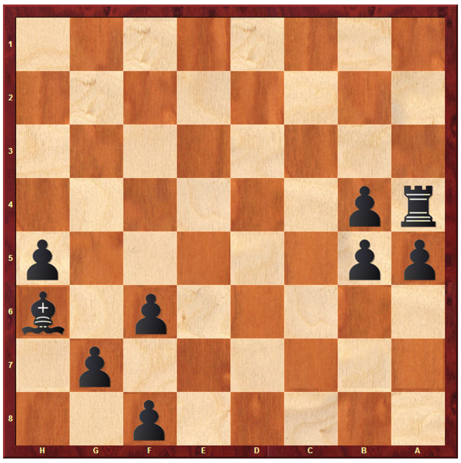


Fig.7b

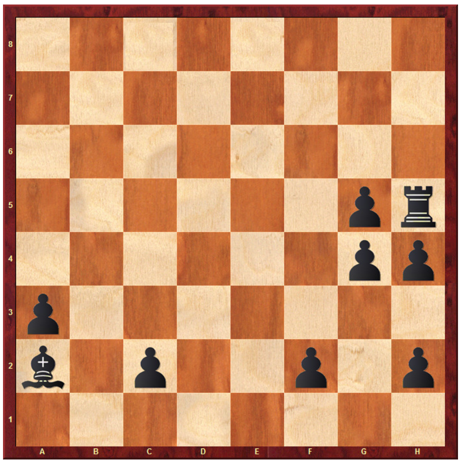


Fig.7c

Fig.7d

Figure 7 In fig 7a,7b,7c the positions that student built with the attempt to obtain the position in fig.3. In fig.7d he is saying *uno* “one”.

When he reached the position in fig.7c he stopped some seconds and realized that the position wasn’t good because missed one piece on the chessboard (in fig.7d he is saying “uno” – one in Italian, with a “raised finger” that means *one*).

So it feels like he doesn’t care that all the positions were totally wrong, but the question is “I can’t continue because I don’t know where I put the missing piece”.

This episode shows that the student has great difficulty on copying and construct the position in the table chessboard. He can’t keep in mind any figural and conceptual items, and he doesn’t care about the coordinate on the chessboard sides.

In figure 8 we show the distribution of semiotic means of objectification of this episode.

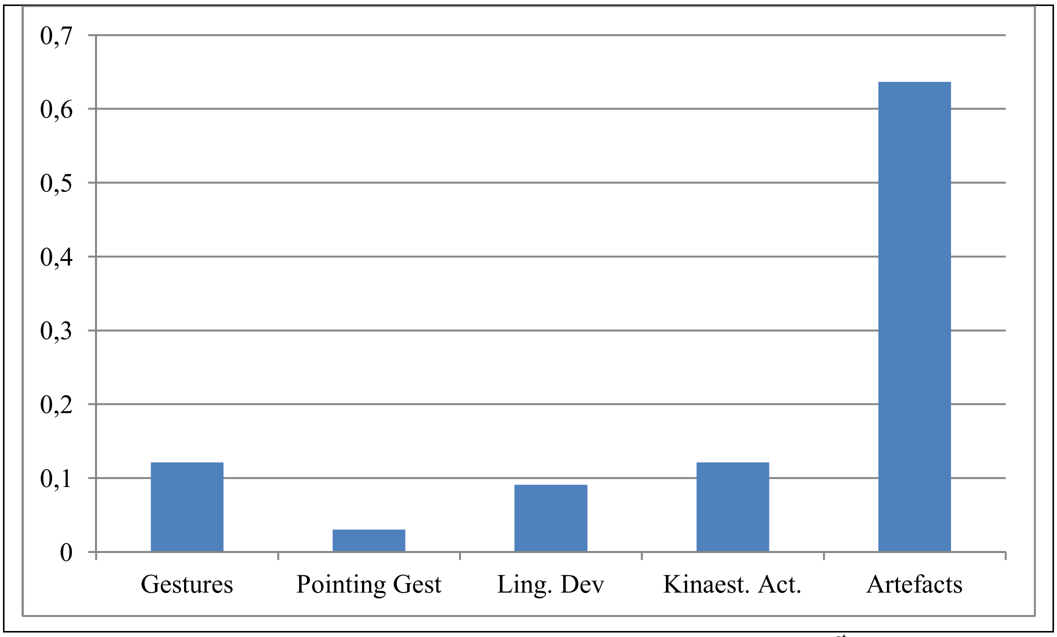


Figure 8 The distribution of semiotic means of objectification of 1st episode

Episode 2

In this episode of 5 minutes and 20 seconds, like the episode 1, we will observe the student while he is trying to reconstruct a chess position (fig.2a) in a chessboard in front of him, copying by a wall chessboard (that he had in his back).

Having the black pieces, in his chessboard he will reconstruct the position in fig.2b. This episode belonged to the 2nd lesson and we observed some little improvements by 1st episode.

Also in this episode he moved his head continuously toward the wall chessboard (fig.9a), after he chosen a piece, he hold it with the hands and look alternatively the wall chessboard (fig.9b) and table chessboard (fig.9c).

After some attempts he obtained the positions in figure 10 (fig.10a and fig.10b in time order).

Fig.9a

Fig.9b

Fig.9c

Figure 9 The student moves his head toward the wall chessboard (fig.9a), after he chosen a piece, he hold it with the hands and look alternatively the wall chessboard (fig.9b) and table chessboard (fig.9c).

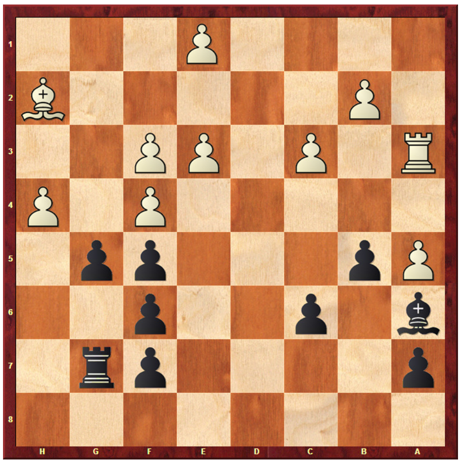


Fig. 10a

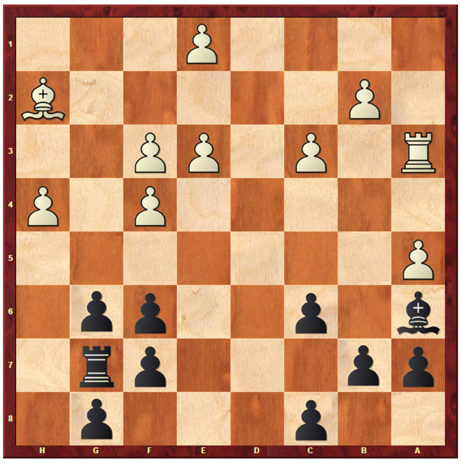


Fig. 10b

Figure 10 The sequence of positions that students built in attempts to obtained the required position.

So the teacher let students notice that both black and white pieces are placed in wrong way. After few seconds the white player took the pawn in e1 and placed it correctly in b4.

Now started an interesting dialogue to adjust the placement of black pieces:

1. T: (3.18) “Also the black did some mistakes”;
2. B: (3.19) “What?”;
3. T: (3.20) “Let’s try to check it”;
4. B: (3.20) He pointed the b8 square;
5. B: (3.27) He takes the pawn placed in a7 and he tapped it;
6. B: (3.36) “This!”, He took the pawn placed in g8 and he moved it in g5;

7. T: (3.38) “It is there?”

8. B: (3.40) He moved the pawn from g5 to f5;

9. B: (3.43) “f5”;

10. T: (3.45) “And the others? There are also other mistakes”;

11. B: (3.50) He pointed the finger in b8 and he moved it below the square c8;

12. B: (3.53) “This is well placed, it is right” Referring on pawn placed in c8;

13. T: (3.56) “Yes…why?”;

14. B: (4.00) He look the wall chessboard;

15. B: (4.05) “5b and there is 5b” pointing the square b8; (fig.11a)

16. T: (4.08) “What is in 5b? …In b5...”

17. T: (4.11) “In b5 here, there is nothing”

18. B: (4.14) He pointed with finger the letter b (fig.11b), he moves it in b7 (fig.11c) and he moved the head to look toward the square b5 (fig.11d);

19. B: (4.18) He took the pawn placed in c6 and moved it in b5.

So, he adjusted the pieces and obtained the right position.

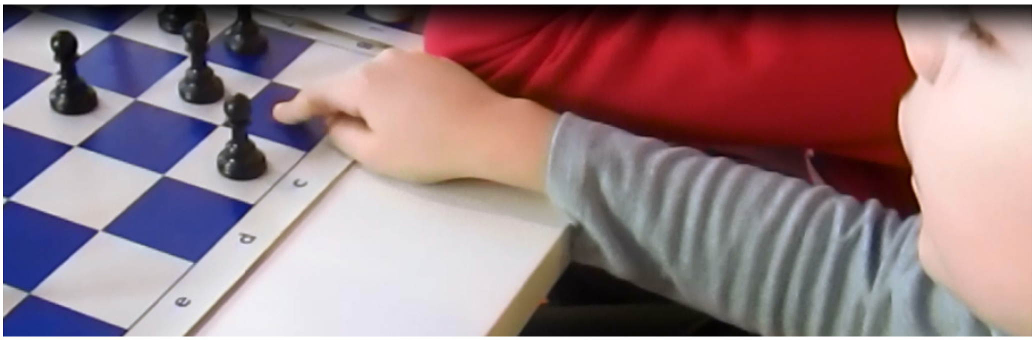


Fig. 11a

Fig.11b

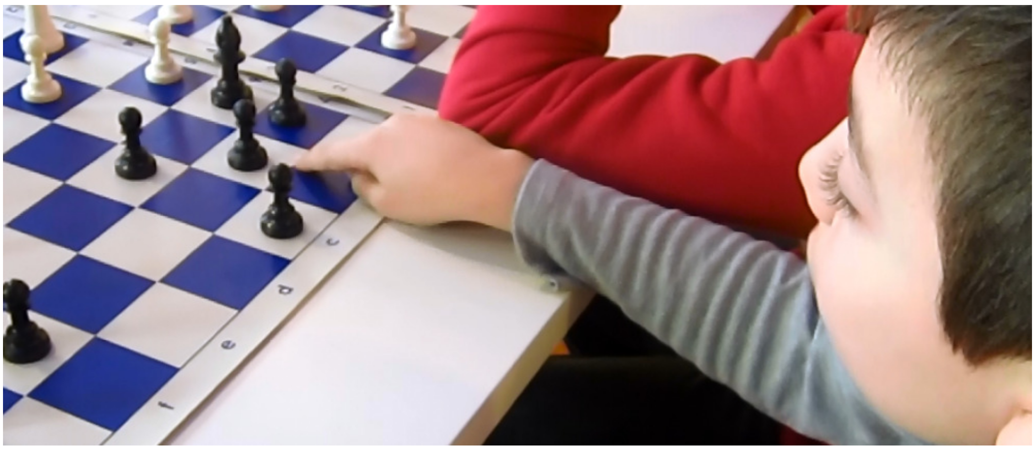


Fig.11c

Fig.11d

Figure11 The student is pointing the square b8 (fig.11a), The student is pointing the letter b (fig.11b), The student is pointing the pawn b7 (fig.11c), The student is moving his head toward the square b5

In figure 12 we show the distribution of semiotic means of objectification of this episode.

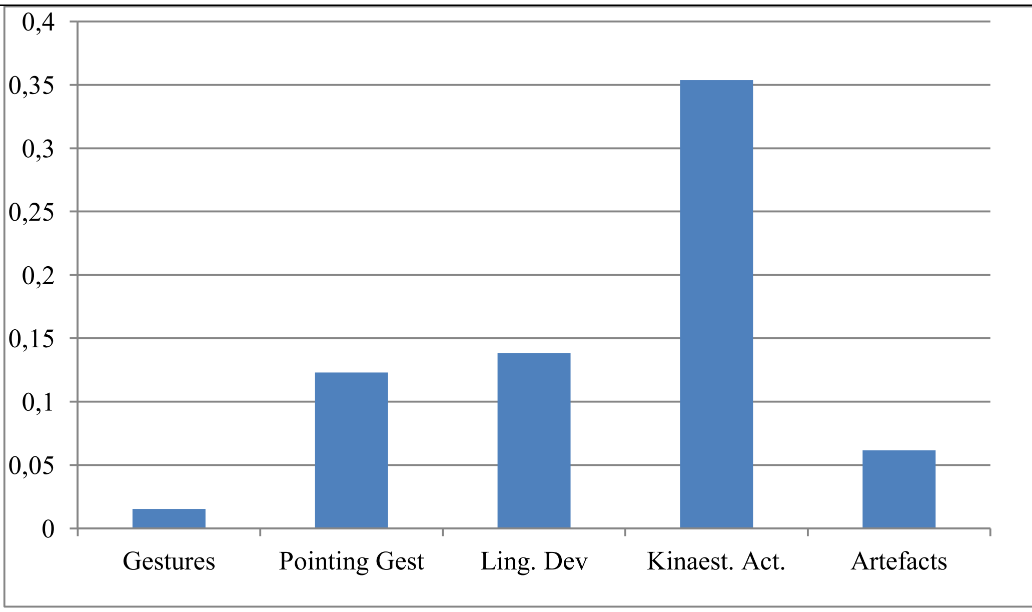


Figure 12 Distribution of semiotic means of objectification of 2nd episode

Episode 3

In this episode of 2 minutes and 10 seconds, like the episodes 1 and 2, we will observe the student while he is trying to reconstruct a chess position (fig.2a) in a chessboard in front of him, copying by a wall chessboard (that he had in his back).

Having the black pieces, in his chessboard he will reconstruct the position in fig.2b. This episode belonged to the 3rd lesson and we observed a significant improvement in reconstructing the position.

During the whole episode he stand up and he move his head toward the wall chessboard to the table chessboard. After few seconds he placed the pawn c6, b5 and c7 and the bishop in a6 and started an intense dialogue:

1. B: (0.49) He pointed the squares b7 and b8 (fig.13a);
2. B: (0.54) He took the pawns c6 and b5 holding them in that positions for some seconds (fig.13b);
3. B: (1.00) “Sorry teacher, in the chessboard there is (looking to wall chessboard)…a6 and there is the bishop (pointing on bishop in table chessboard) (fig.14a) …then there is 5b (looking to the wall chessboard) and..”;
4. I: (1.10) “and there is the pawn”;
5. B: (1.10) “here” (pointing the square b5) (fig.14b);
6. B: (1.14) “then there is 4c” (looking to the wall chessboard);
7. I: (1.14) “c4!”;
8. B: (1.16) “…c4…” He pointed the square c8 (fig.14 c), he moved the finger to the square c6 and moved the pawn from c6 to c4 “here” (fig.14d);
9. B: (1.20) He took a pawn “c…c…c…” and placed it on c6;
10. B: (1.26) He counted the placed pieces;
11. B: (1.38) “Rook in 7g…g7!” (Looking the wall chessboard with a rook in one hand);
12. B: (1.40) “g7…g7…g7” With the right hand he hold the rook near the square a8 and with the left hand at first (fig.15a), with eyes then (fig.15b), and finally with the right hand holding the rook he founded the square g7, where placed the rook after a fast check on wall chessboard (fig.15c)”
13. B: (1.52) “f7” He placed into square h7 and then he moved it into square f7;
14. B: (1.58) “f6” He placed a pawn into square f8 (fig. 15d) and moved it into square f6;

15. B: (2.06) He took a pawn, he pointed the square g8, looked the number 6 in his right (fig.15e), moved the head toward square g6 and placed the pawn in g6;
16. B: (2.18) “f5” He looked the square f8, he looked the number 5 in his right (fig.15f), moved the head toward the square f5 and placed the pawn in f5;

Fig.13a

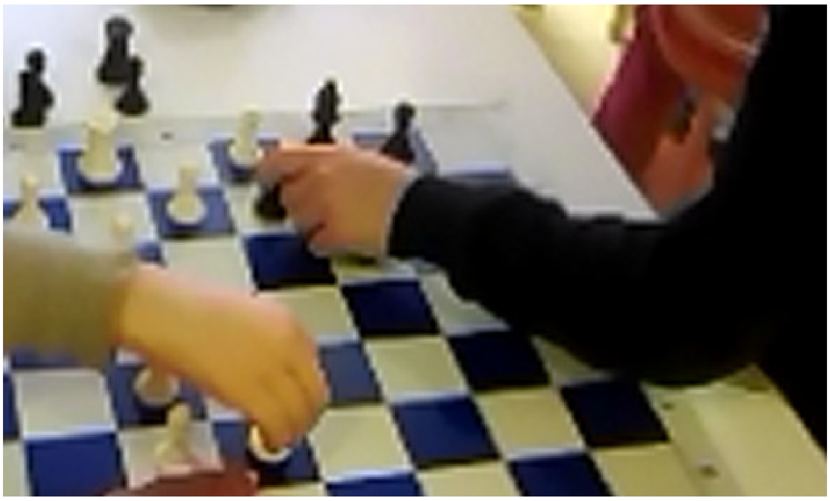


Fig.13b

Figure 13 The student is pointing the square b8 (fig.13a),
The student is keeping in his hand the pawns placed in c6 and b5.

Fig.14a

Fig.14b

Fig.14c

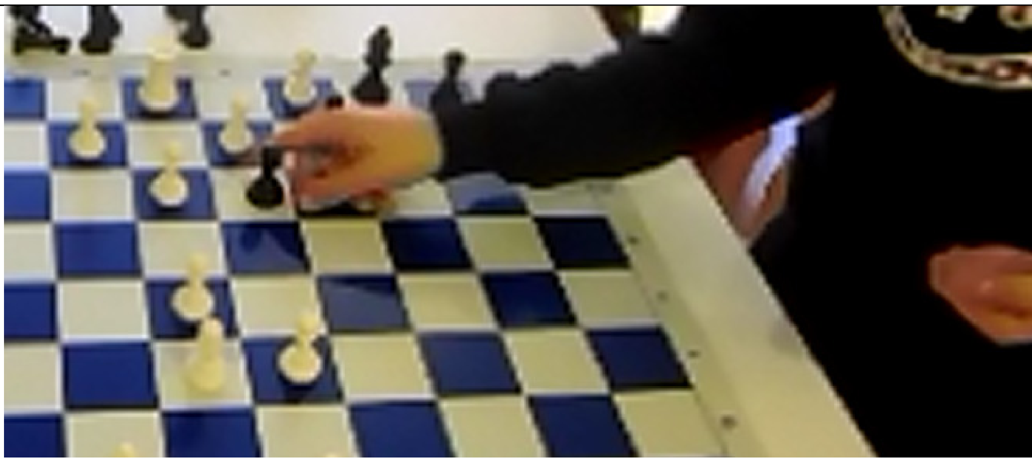


Fig.14d

Figure 14 The student is pointing the Bishop placed in a6 (fig.14a), The student is pointing the square b5 (fig.14d), The student is pointing the square c8 (fig.14c), The student moved the finger to the square c6 and he is moving the pawn from c6 to c4 (Fig.14d).

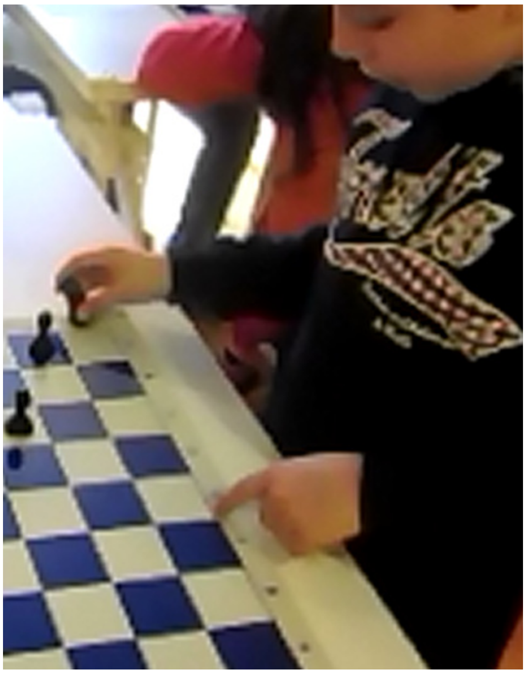


Fig.15a

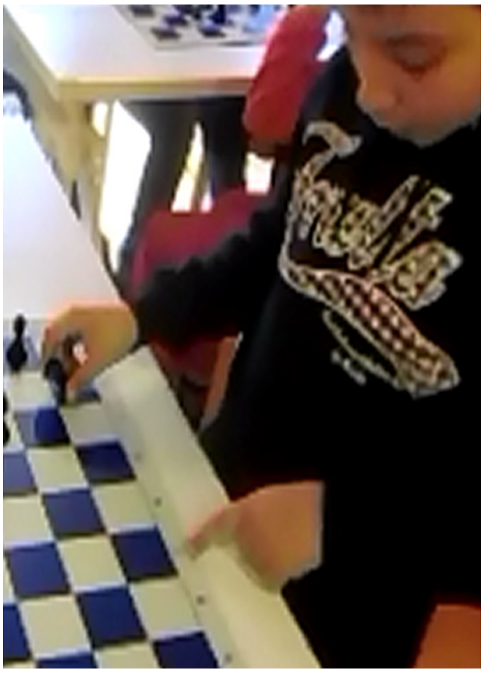


Fig.15b

Fig.15c

Fig.15d

Fig.15e

Fig.15f

Figure 15 The student coordinate the hands to find the square g7 in Fig.15a and Fig.15b, Then he checked it looking the wall chessboard in fig.15c, The student putted a pawn in square f8 in fig.15d, The student pointed the square g8 in Fig.15e, The student is looking for the number 5 in his right

Figure 16 show the distribution of semiotic means of objectification of this episode

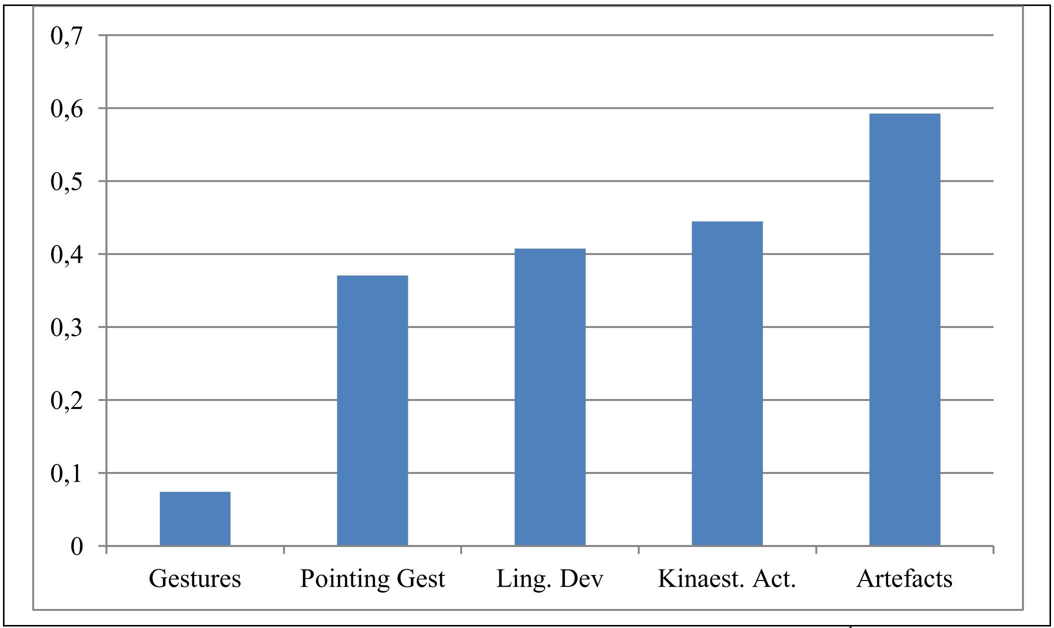


Figure 16 The distribution of semiotic means of objectification of 3rd episode

Episode 4

In this episode of 2 minutes, we will observe the student while he is trying to reconstruct a chess position (fig. 17a) in a wall chessboard, copying by a table chessboard. This episode is the inverse of previous and it belonged to 5th lesson.

This task is a little bit difficult than the previous ones because his position of the body change continuously in positioning of the pieces, and because it is not easy recognize the whole configuration of the pieces near the wall chessboard.

By the way he was sufficiently fast in constructing the position showing few uncertainties.

It is interesting to notice that he spoke only to answer to the following questions that the teacher did during the student construction: “Speak, what are you doing?”, “What are you looking for?” and he said “I am doing that one!” referring to the position on table chessboard.

What we can observe is that after few lessons he can keep in mind figural and conceptual item. Now he is more aware of the artifact chess. He cares about the coordinates and use them to place the pieces.

The figure 17b shows the distribution of semiotic means of objectification in this episode.

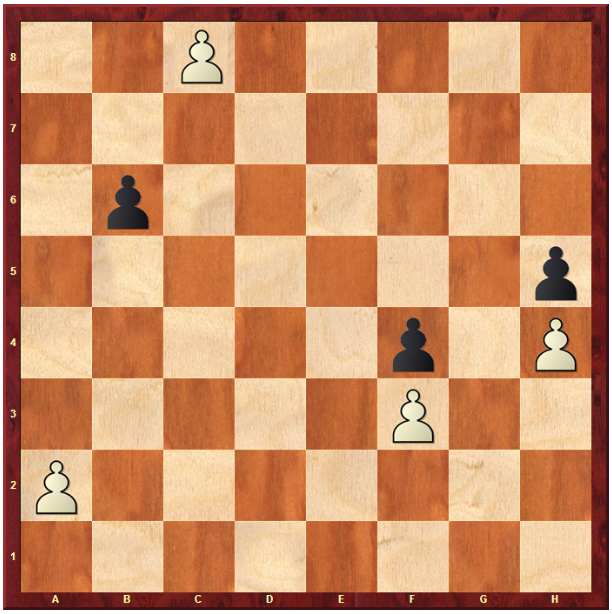


Fig.17a

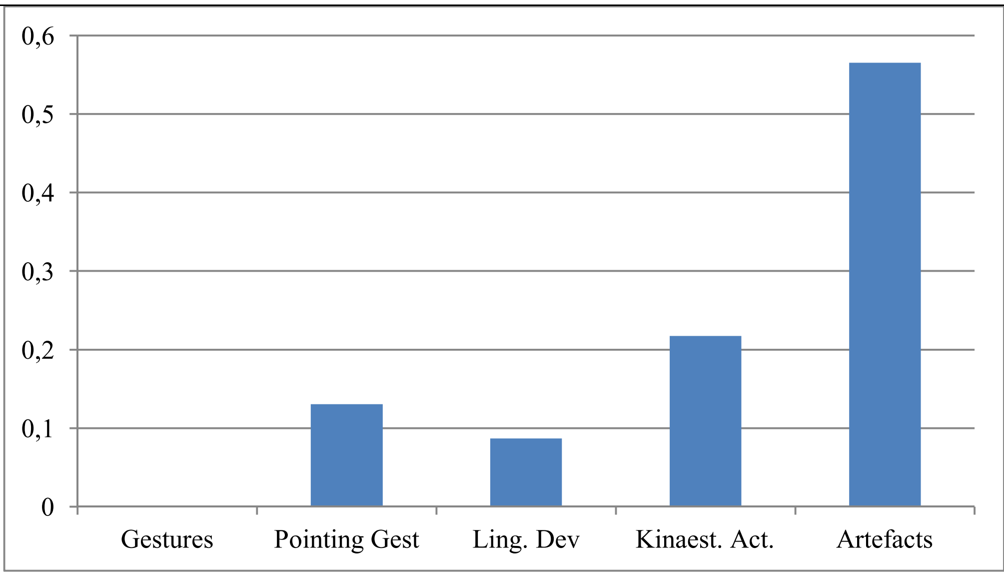


Fig.17b

Figure 17 The position that student have to construct in 4th episode in Fig.17a, The distribution of semiotic means of objectification of 4th episode.

Episode 5

In this episode of 2 minutes and 20 seconds we proposed a geometrical task involving constructing of geometrical figures. The students were disposed in groups and the teacher projected in a wall a geometrical figure involving seven sub-figures: a standard Tangram (fig.18a).

To the students were required to combine these sub-figures to obtain the biggest number of squares. In order to help the classroom communication the sub-figures were numbered.

The students can talk between them and they can use papers and pencils (fig.18b).

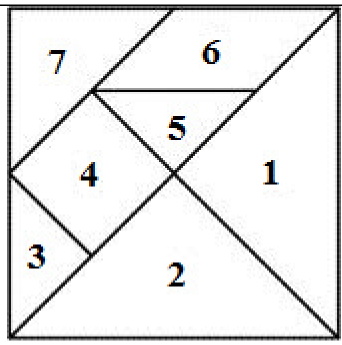


Fig.18a

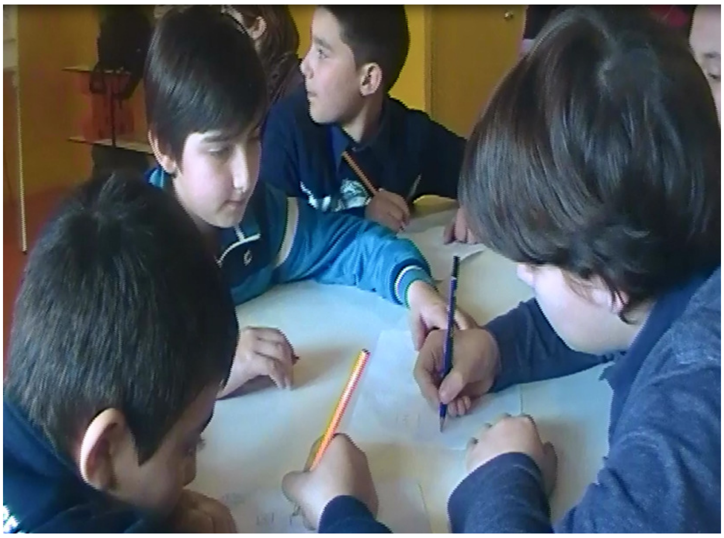


Fig.18b

Figure 18 The Tangram proposed to the students in Fig.18a, Interaction between students in attempting to solve the Tangram.

In this episode we want to observe how the student (Giuseppe) of previous episodes, try to solve this task.

In the same way of 4th episode, he do not spoke so much and he preferred to look the wall where was projected the Tangram and to draw squares combining the subfigures. Inside the squares he wrote the number of the sub-figures (fig.19a).

His classmate use gestures in the air to show the combination of sub-figures and use the numbers to name them.

After few seconds of elaboration started an interesting dialogue between Giuseppe (G), Costantino (C) (in his left) and Alessandro (A) (in his right):

1. C: (0.00) “Can we use three of them?” (fig.19b);
2. A: (0.05) “Can we combine three of them together?” (Asking to teacher);
3. T: (0.10) “You can do what do you want”;
4. C: (0.12) “So, I got it!...7…plus 4…plus 3, plus 5” (He refer to polygons 7, 4, 5 and 3)
5. A: (0.15) He did a grimace to express his disapproval;
6. C: (0.18) “Wait..” He took the sheet and pencil and start drawn the subfigures;
7. G: (0.19) “But…what are you saying?” He stopped drawing and looked Costantino reconfigurations”
8. C: (0.22) “Seven here…four here…five here and three here” Meanwhile he is drawing something in the paper;
9. A: (0.32) “And this is a square?”;

They laughing about it.

10. G: (0.45) “I wrote 7 plus six plus five plus four plus three plus two plus one”;
11. C: (0.50) “Yes, true…all of them form a square”.

After few minutes the teacher stopped all the students and asked to compare their answers, asking to Giuseppe to explain his answers on the wall.

12. G: (0.01) “The four is already a square” pointing on polygon 4 (fig.19c)
13. G: (0.05) “This” pointing on polygon 1 (fig.19d), “if you turn it” simulating its turning (fig.19e), “you can add to it” pointing on polygon 2 (fig.19f), and become a square.
14. G: (0.10) “This” pointing on polygon 3 “If you add it to this” pointing on polygon 5, “became a square”;
15 G: (0.15) “All of these” He putted the opened hand in the wall to “reach” all of them (fig.19g) “make a square”.

Fig.19a

Fig.19b

Fig.19c

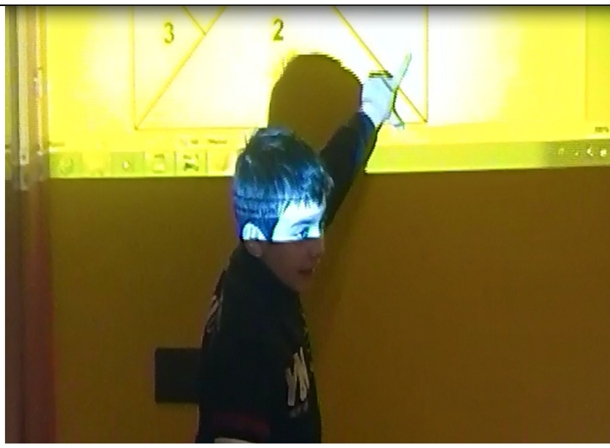


Fig.19d

Fig.19e

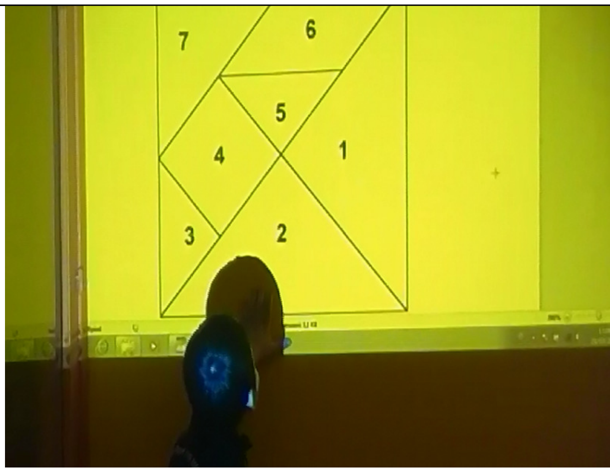


Fig.19f

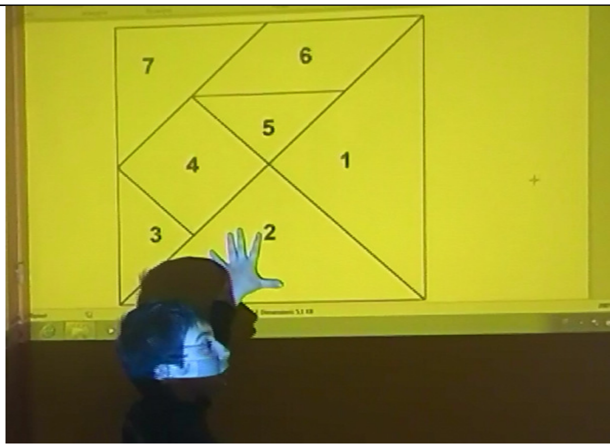


Fig.19g

Figure 19 Giuseppte is writing the number of the sub-figures inside the square in fig.19a, Costantino is asking if they can use three subfigures in fig.19b, Giuseppe is pointing the subfigure 4 in Fig.19.c, Giuseppe is pointing the subfigure 1 in Fig.19d, Giuseppe is simulating the turning of subfigure 1 in Fig.19e, Giuseppe is pointing the subfigure 2 in Fig.19f, Giuseppe is putting an opened hand to reach all of the subfigure in Fig.19 g

The figure 20 show the distribution of semiotic means of objectification in this episode.

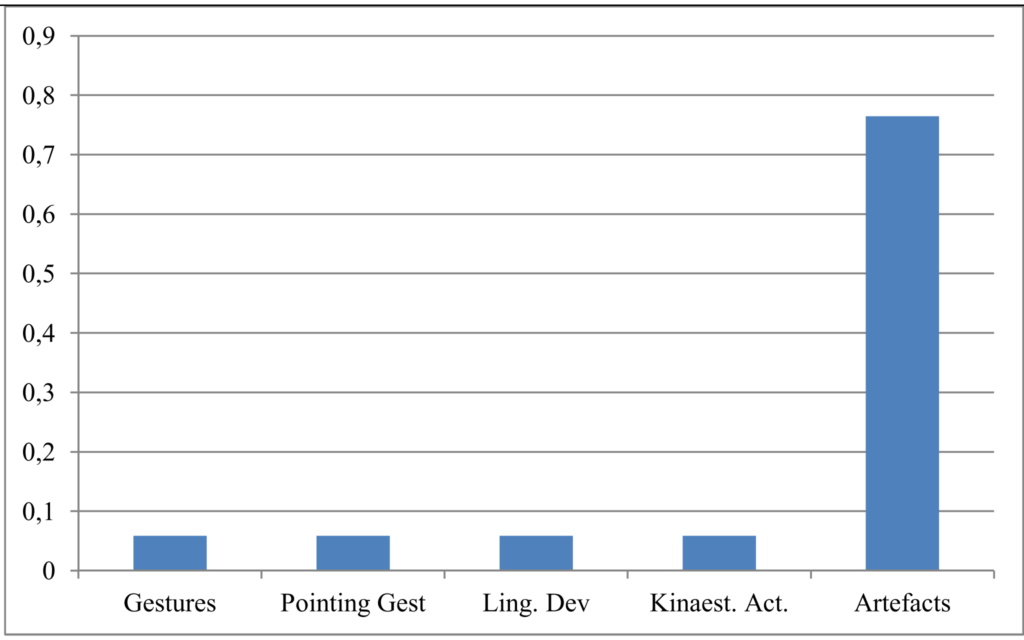


Figure 20 The distribution of semiotic means of objectification in 5th episode

Final Remarks

In section 6.2 we observed that students of experimental classroom improved some visual spatial abilities, so our purpose was to observe this transformation with a more precise lens. To do it we chosen one of the students while he is dealing with reconstruction of chess positions and observe how changed the semiotic means of objectification during the process of objectification.

In this way we choose 4 chess episodes in which the student reconstruct a chess position and a geometrical one in which then students try to solve a Tangram and we analyzed the distribution of semiotic means of objectification in each episode. In figure 21 we will show how they changed in the episodes.

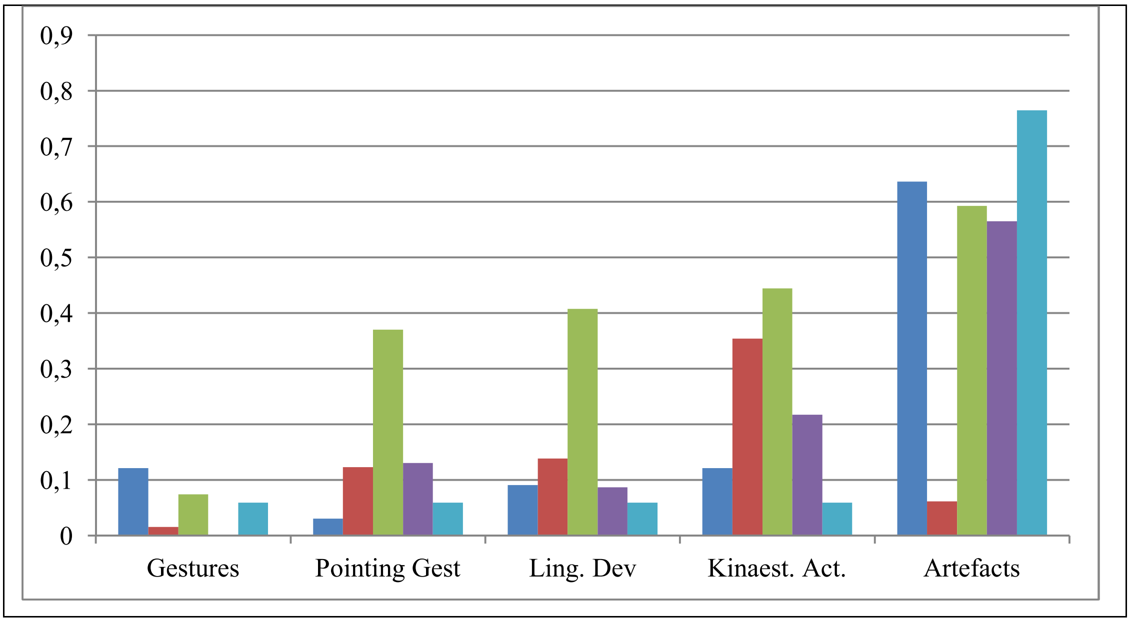


Figure 21 The distribution of the semiotic means of objectification in the five episodes. For every SMO the first column refer to the first episode, the second column to the second episode and so on.

We observed that in lower level of awareness the student use the gesture to point squares, to keep in hand pieces or to tap them over the chessboard.

These gestures (in particular the pointing gestures) were modified (or simply contracted) into his eyes action. (fig.21) When he achieved highest level of awareness he moved his eyes and his head to individuate the squares on the chessboard without using gestures.

By the way this eyes motions were not “alone” it was coordinated to the language that the student improved in “calling” the columns or the squares. As shown in figure 21 this SMO had the same behavior of pointing gestures.
What we want emphasize is that in fifth episode in which the student is solving a Tangram the distribution of Pointing gestures and Linguistic devices continue in the same way of the chess episode.
So, in this work we can affirm that the following phenomena:

- The contraction and synchronization of pointing gestures and linguistic devices;
- The acquisition and or an improvement of the constructive cooperation of the figural and conceptual aspects in a geometrical problem solving activity, the ability to keep in mind and coordinate as many as possible figural conceptual items and the ability to predict and integrate the effect of each transformation on the road to the solution

occurred at the same time, but we want to argue that these phenomena are strictly linked. With the aim to establish some evident relationships between chess activities and geometrical tasks in next section we will study the objectification of configural concepts and we will show the similarities between semiotic activities in solving Tangram and solving problems in which the students have to move after a check.

6.4 Going deep on objectification of configural concepts: Multimodal semiotic analysis of a case study

This thesis work was carried out from the idea that a transfer of skills between two disciplines, or more precisely between some activities of these disciplines, could occurs if these ones share similar cognitive processes (Singley, 1989). For us this is like a "necessary condition" for transfer of skills, but this is not enough. In fact, to obtain a transfer of skills, the teachers need to know the semiotic dimension[27] of both disciplines to understand which level of awareness about a knowledge object is reached by their students. For the particular case of chess we introduced a new theoretical proposal to study chess thinking: the configural concepts (see Chapter 4).

The educational needs of this research, addressed our study to understand what are first configural concepts that students encounter in a typical chess course and how they are objectified (Radford, 2006).

So, In this section we did a theoretical investigation on the learning phases of pieces movement and we studied the relationships between figural concepts and configural concepts.

Then, we analyzed the processes of Objectification (Radford, 2009) of these configural concepts, studying multimodal semiotic activity that occurred in a particular case study in which two students of experimental classroom of the experimentation in 6.2 and 6.3 played and interacted each other during a chess game.

Objectification of Configural Concepts: pieces movement.

Each chess courses start presenting his axiomatic system: chessboard elements, pieces movement and game scope. In this work we will focus our attention on learning processes of pieces movement.

The movement of each piece has its concept definition (Tall and Vinner, 1981), that it is rigorous and culturally accepted, and in phase of learning, are built many cognitive structures about that definition.

[27] With semiotic dimension we mean the epistemological statute of semiotic interactions and mediations.

For example of rook movement: “When the rook is not castling, it can moves any number of vacant squares horizontally and vertically”. Students link this definition to some concrete images like the sign “+” or teacher gestures.

At epistemological level, a piece becomes an artifact when we give to it a movement rule. The chessboard, although its physical structure is always been the same, its meaning was culturally modified by individuals that in the past played chess. In this way we suggest that pieces, gestures, sings and linguistic devices are semiotic means of signification of some chessboard areas (from square to the whole chessboard).

The purpose to move a piece placed over the chessboard require a visualization and dynamic manipulation of the squares including its potential movement, guided by the definition movement and its feelings.

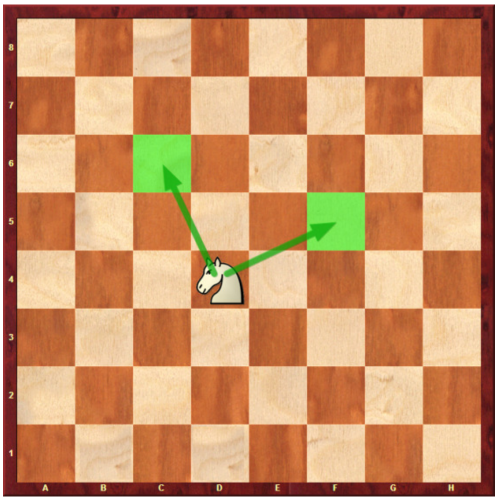
Since that chess objects aren’t ideal like geometrical ones, it is not possible to talk about figural concept (Fischbein, 1993), so we had introduce a new theoretical idea that involves the empirical nature of chess thinking: the configural concepts.

Objectification processes of a chess configural concept are that processes (physical and mental) that allow students to acquire an higher level of awareness that shift the objective of reflection from pieces and their movements to the chessboard and complex ensemble of rules. For example, in low level of awareness of pieces movement, the chess player starts by the piece and try to find where it can be placed in one or more moves (fig.21a). Instead, in high level of awareness, the chess player start by the square in which he/she wants to place the chosen piece, and then find the sequence of moves to reach the square (fig.21b).

So, through the movement of a piece, the chess player modifies the meaning of chessboard. So he/she objectifies some relationships between pieces and chessboard. For example he/she learn which squares are reachable with fewer moves (fig.21c).

Here we want to say that in chess activities, students gain the sensibility to disclosure the structures inside a position (configural concepts), handle them to reach a purpose and, consequently, to shift the objective of reflection from pieces to chessboard.

To study objectification processes of pieces movement, we have analyzed a case study in which two students (6^{th} grade) play chess after few chess lessons, with the aim to show that at lower level of awareness, pieces are the objective of reflection. Usually it is not easy to understand what chess players are looking for, but in this game, black player generates an intense semiotic activity that left *traces* of his analysis.



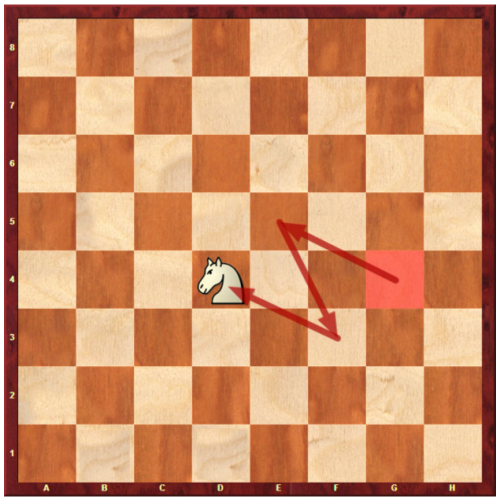


Fig.21a **Fig.21b**

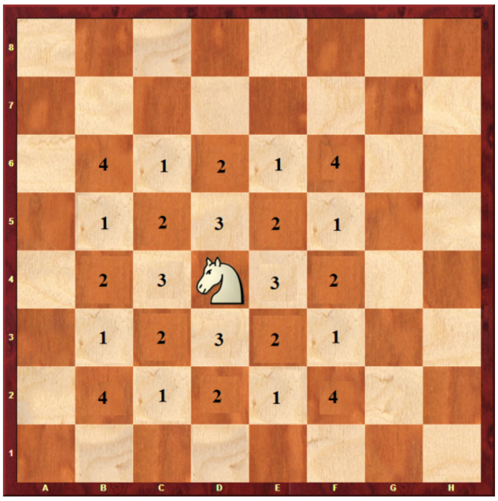


Fig.21c

Figure 21 The squares f5 and c6 are an example of reachable squares by a knight placed in d4 (fig.21a), Starting by square g4 this is an example of variation to reach it by a knight placed in d4 (fig.21b), The numbers show how many moves need a knight placed in d4 to reach that squares (fig.21c)

We presented to students the position showed in Fig.3 and the purpose of the game was to capture all of the opponent's pieces (in the transcript, the numbers in brackets refer to the elapsed seconds).

1. White: (0.00) He moves the tower from a3 to a1;

2. Black: (0.02) He moves his hand to move a piece on the side of the King, stops; (0.04) He touches pc6, holds her hand on pawn two seconds and moves on d5 (incorrectly);
3. White: (0.09) He is going to move the rook on a1 but the instructor points out that there has been an irregularity in the move of the black (without saying what it was)]; (0.12) He says*," not diagonally"*;
4. Black: (0.14) He withdraws the move saying *"Ah ... true!"*, and moves immediately it on c5;
5. White: (0.16) He moves pawn b4 x c5;
6. Black: (0.20) He touch tower on g7; (0.21) He approaches his hand to the pf6; (0.24) He moves his hand on the board and touch again pf6; (0.25) He has the hand closest to the pf6, and then touches it and leaves it intermittently; (0.27) He touch pf5; (0.29) He raise his hand and mumbles something; (0.31) He touch and move pa5 (an opponent's pawn);
7. White: (0.32) He reacts immediately saying *"it's my white, you're black!"*;
8. Black: (0.33) He responds laughing and saying *"Oh, right!"*; (0.37) He touches the pb5 and holds very strong (enough to move the chessboard with a sharp movement); (0.40) He touches the pc4 and then does it "jump" on the square a couple of times; (0.44) He touches the pa7;
9. White: (0.44) He asks to black deciding, says *"c'mon, play something"*;
10. Black: (0.47) He touches and moves the tower from g7 to h7.

Firstly, we note that when in line 2 black player moves the pawn from c6 to d5, he doesn't want to move a pawn into the square d5, but he only try to move correctly (failing it) the chosen pawn. Secondly, in the 27 seconds that black player uses to make a move (from line 6 to line 10), it seems that his gestures show us what he is looking at. He uses intensely gestures because is required to him to be able to handle the potential movement of all pieces on the chessboard, then to build in his mind something different that it is really shown on the chessboard.

This case study show that the student has many difficulties to handle these visual spatial elements touching all pieces that he is looking for, and even he moves an opponent piece (line 6).

The movement of piece is a configural concept because its meaning depends by the relationships (figural and conceptual) between chess objects involved and from its position in the whole theoretical structure of pieces in the chessboard.

Students, when move a piece on chessboard, use all cognitive structures that involve the piece itself, portion of chessboard, linguistic and informal rules etc.

In objectification of pieces movement chess player acquire different levels of awareness on what he/she is reflecting for:

- Pieces (choose a piece and try to move it correctly);
- From pieces to chessboard (reflection on pieces shift the attention on chessboard – For example try best way to react to an opponent check);
- From chessboard to pieces (reflection on pieces guide how to move pieces – For example "square rule" show if a pawn can be promoted without that it will be reached by opponent king).

All in all, when a chess player is in front of a position, he has to configure some chess objects (pieces or square) with the aim to imaging new possibilities similar to a well known configural concept.

In this imagination process he/she can "invoke" different configural concepts and so different evaluation of a position. So chess players improve the ability to disclosure the right (and useful) configural concepts and handle them with the aim to reach a purpose.

In amateur chess courses, the "fundamental purposes" are to learn the axiomatic system, to move correctly a piece and don't do illegal moves. A clear argument that need all of these fundamental purposes is to move a piece when the opponent player attack the our king.

To show the relationships between chess and geometry we observed in the following two episodes the students while they tried to solve the previous chess and task and we compared these actions with which they produce during Tangram activity.

In this group there are three students of experimental classroom, Marco (M) in the middle, Valerio (V) in Marco's right and Pietro (P) in Marco's left (fig. 22).

Episode 1

1. P (0.00): "Mmm…there is check by the…" (pointing on the Queen in h4);
2. P (0.03): He moves the white king in f1 "ok…done";
3. M (0.08): "No, we have to defend the king";
4. M (0.11): He took the rook in b3 and putted it in e3;
5. V (0.15): "Eh?"…(0.18) "You have to defend the king by the queen!"…(0.22) "We have to put a piece here!" pointing on square f2 (fig.22a);

6. M (0.24): “Ah…ok…” he pointed the square f2 with left hand and moving the right hand over the pieces… (0.31) “This!” referring to bishop placed in a7 and holding with right hand; He stayed in that position (fig.22a) for about 4 seconds and then he moved the bishop from a7 to f2.

Episode 2[28]

1. P (0.00): “Here we can put this” He took the piece of sheet with a square shape, and he is putting it in a area of the figure;
2. M (0.01): “Wait...don’t move it!” He removed the piece of sheet by the figure;

Some seconds of confusion because Pietro didn’t accepted the Marco’s attitude.

3. V (0.13): “Ok, there we can put the square” pointing in the position proposed by Pietro…(0.18): “and here?” pointing in a close zone (fig.22c);
4. M (0.22): He pointed with finger of right hand the same zone pointed by Valerio, then he moved his left hand to search the subfigure to put there, (0.33) “This!” He pointed a subfigure and held it for some seconds (fig.22c), then he moved the piece of sheet in the pointed place.

What we can observe is that similar to Giuseppe gestures (fig.15a, fig.15b and fig.15 b), they coordinate the hands to find the piece (chess piece or paper piece) that can be putted in a place. In this case we are observing that students, in achieving an higher level of awareness are using the pointing gestures and synchronization of their hands.

Although we have not a recorded data, we can say that an higher level of awareness is achieved when the student contract his semiotic activities, and in this case, transform the pointing gestures in eyes movement.

However what we want emphasize is that this transformation seems doesn’t care of the context and the student (Marco in this case) use the same semiotic activity with the purpose to solve two tasks that are cognitive similar but in different context.

[28] In this episode Pietro is in Marco’s right and Valerio in Marco’s left. This episode was in same lesson of episode 1, the seventh lesson of the course.

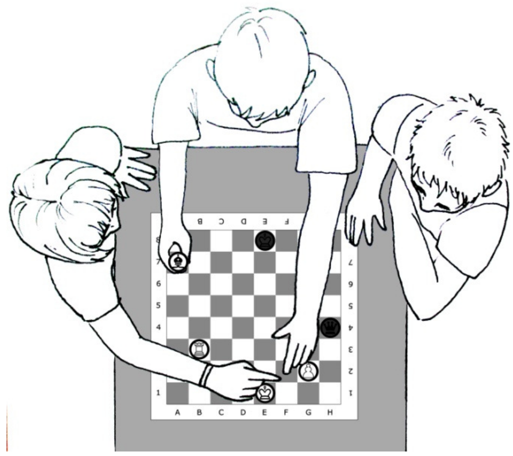


Fig.22a

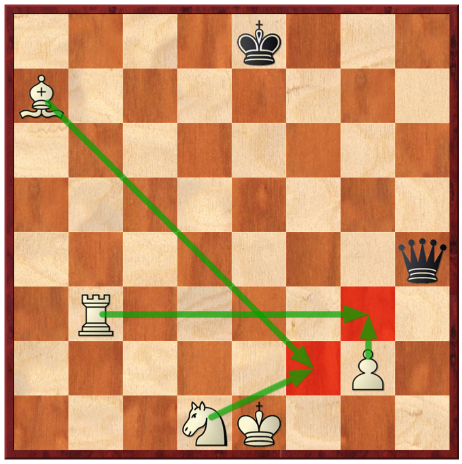


Fig.22b

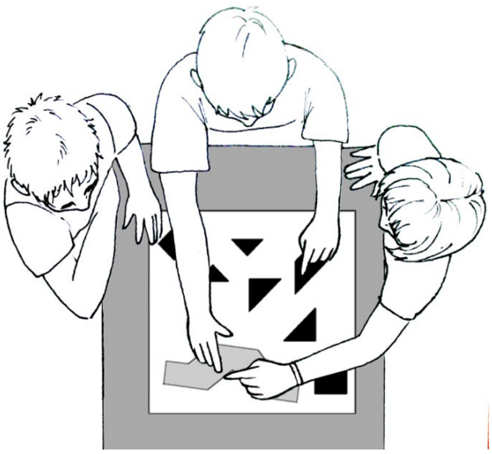


Fig.22c

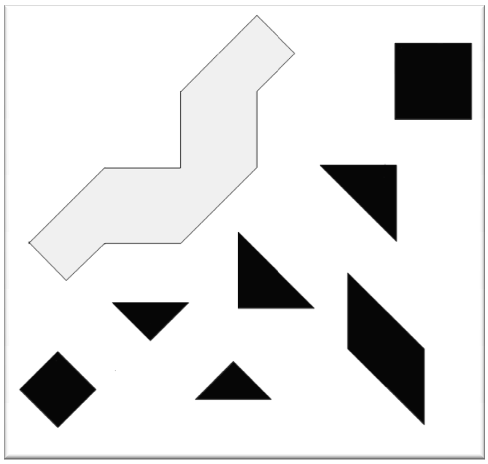


Fig.22d

Figure 22 The students try to find which piece they have to put in the pointed square (fig.22a), The position that they are dealing with (fig.22b), The students try to find the sub-figure to put in the pointed empty space in order to complete a Tangram (fig.22c), The Tangram that they are dealing with (fig.22d)

Conclusions

In this chapter I will summarize the principal outputs of my work. The aim of this work was to investigate how and whether a non-disciplinary proposal, can help students in pose and solve geometric problems.

In particular I want investigate on the educational potentiality of play chess and analyze the synergies between chess and geometry.

The research questions to which I have attempted to answer are the following:

4. What are the criteria for choosing a non-disciplinary activity in order to improve certain skill useful for the geometrical learning and thinking?
5. It's possible make a theoretical proposal for the idea of transfer of skills according to the semiotic-cultural paradigm of the theory of objectification?
6. Chosen the chess laboratory as a non-disciplinary activity, which skills could be improved? What are the role of chess and math teacher in this multi-disciplinary didactical proposal?

First step was to investigate which abilities could be useful for the students in posing and solving geometrical problems, to take note of such conflictual situations and obstacles are present in the processes of visualization in geometry activities and to study some processes of mathematical thinking and learning in a semiotic point of view.

Focusing our attention on visualization in geometry I started this treatment clarifying what is the meaning of "visualization" used in this work. According to Presmeg (Presmeg, 1997), visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics.

Duval (Duval, 1999) characterized visualization as a bi-dimensional organization of relations between some kinds of units. These units can be 1D or 2D shake (geometrical figures), coordinates (Cartesian graphs), propositions (propositional deductive graphs or "proof graphs") or words (semantic networks). And these units must be bi-dimensionally connected, because any organization requires at least two dimensions to become obvious.

About the cognitive processes in visualization of geometrical figures, according to Duval, I assume that visualization consists only of operative apprehension and measures are matter of discursive apprehension which put an obstacle for reasoning and for visualization.

With regard to the particular case of the visual representation of the geometric figures, a study of Fischbein (Fischbein , 1993) shows how these have both a conceptual nature that a figural intrinsic nature. In his theory of figural concepts, Fischbein states that the integration of conceptual and figural unitary mental structures, with the predominance of conceptual contents of the figural ones , is not a natural process, and therefore it should be a continuous, systematic and main activity of the teacher.

In particular he pointed out that to help the students in this operation, a well designed training is aimed to improve the following abilities: (a) the constructive cooperation of the figural and conceptual aspects in a geometrical problem solving activity ; (b) the ability to keep in mind and coordinate as many as possible figural conceptual items; (c) the ability to organize the mental process in meaningful subunits so as to reduce the memory load; and (d) the ability to predict and integrate the effect of each transformation on the road to the solution.

So, to answer to RQ1, I argue that a non-disciplinary activity in order to improve geometrical visualization, should improve at least one of the abilities mentioned above (a-d) and involves operative apprehension as a cognitive process.

The choice of non-disciplinary activity has been made in the interest of improving some transversal skills and not only geometrical, and with the intent to avoid certain difficulties and epistemological and / or didactic obstacles found in a geometric context.

An activity that could satisfies the criteria shown above is the chess. To understand the relations between chess and geometry I did an epistemological analysis of the chess thinking expanding the idea of figural concepts (Fischbein, 1993) and introducing a new theoretical idea for chess thinking: the configural concepts (Ferro, 2012).

I started defining chess objects and chess elements. I defined chess objects the elements of the artifact "chess" – the pieces, the squares and the chessboard – which therefore possess rules and scopes.

I defined chess elements the mental (i.e. personal – ideal) entities that a chess player uses and "builds" when he she thinks about a move, a position or a variant with a specific aim.

Then, I introduced the idea of configural concepts. A configural concept is made up of chess objects and their conceptual relationships. Its meaning comes from the hierarchical linkage of the conceptual relationships between the involved chess objects and from its position in the whole theoretical structure of the pieces in the chessboard.

The use of a configural concept depends on the goal that an individual is pursuing in a chess game. It involves the identification of general structure of the game at a certain moment and the role of the configural concept therein.

In terms of learning, configural concepts become noticed and valued through a process in which the student becomes conscious of the chess objects and their conceptual mutual relations. This process is what Radford calls objectification (Radford, 2010).

Then, I analyzed the processes of Objectification (Radford, 2009) of these configural concepts, studying multimodal semiotic activity during chess games, identifying some semiotic means – touch, pointing, linguistic devices – that shows different level of awareness of particular configural concepts.

The delicate step of this work was to establish how the chess activity could effectively improve certain visual spatial abilities and how I can check this phenomena.

So I had answer to RQ2 trying to elaborate an idea of transfer of skill according to the theory of objectification.

Then I defined (learning) skills, such as sensibility to acknowledge or perceive the layer of generality of the target object of reflection, and to establish a dialectical process mediated by the culture between the subject and the object of which the subject himself becomes aware.

So I can think to the skill as to the sensibility to perceive a phenomenon, to become aware of it and to endow it with meaning in an active way, and therefore not only meet but also modify it in the thoughtful act. This reflexive activity will be consistent with the goal of reflection and the context in which (and with whom) the person is acting.

The research hypothesis is that the transfer of skills takes place when the student, sensitive in facing similar cognitive problems, reproduces a similar semiotic activity to the context. So, to be more precise, I don't talk about a transfer of skills, but about a maintenance of skills when the environment changes.

The methodology for investigating possible transfer of skills, attempts to combine the theoretical constructs of the objectification of knowledge (diachronic analysis) and semiotic nodes (synchronic analysis), with the idea to investigate how they formed and developed certain cultural sensitivities of the students.

The scheme of investigation adopted will therefore have the aim of identifying a common feature in the comparison of semiotic nodes belonging to a process of objectification.

In particular, I analyzed how the synchronization of semiotic means of objectifications into the semiotic nodes, change during the process of objectification.

To answer to RQ3 I wanted to investigate the possibility of transfer of skills between a chess course and a course in geometry.

The method of investigation has included the following stages:

A. Designing a chess-related work about 24 hours and the three phases of control: a quantitative analysis – before and after the chess course, a qualitative analysis – before and after the chess course, and a qualitative analysis during the chess course;

B. Control investigation on the validity of the survey method in which tasks are proposed in different environments and contexts.

The quantitative survey phase relative to point B was useful to understand *what* kind of ability could be improved by a chess course, and test survey were both disciplinary and psychological.

The choice to make a control with a psychological lens has been taken to solve a problem of this type of investigation: the influence of geometry teacher. In fact, if the pre-test and post-test were only disciplinary, possible improvements of the students could be attributed to the action of the teacher of mathematics, not to the chess course.

The decision to include disciplinary test was due to the need to have a feedback specification and then have the evidence on which math skills can be improved by this teaching practice.

Key step of this experimental work was, however, the choice of the survey sample and the actual structuring of the research and to do that you followed the paradigm of the theory of Van Hiele (Van Hiele, 1986) on geometry understanding.

Given that the aim of investigation are the visualization processes (according to Van Hiele Level 1), it was necessary to choose students to a not high school level, more precisely between 3rd and 8th grade. Then I have chosen students of 6th grade.

Conclusive experimental step was to analyze some geometrical activities, with the aim to find out some evidences, using a semiotic approach.

The geometrical tasks that I proposed to the students require the handling of the terms base, height and area and to analyze the answers of the students I studied the evolution of the meaning and terminology of the concepts of base, height and area of triangle.

In this way I introduced the Euclid's Elements and I discussed some remarks. Then I did a brief discussion on the post-Euclidian geometry and an investigation on evolution of the concepts of base, height and area of triangle in Italian education between nineteenth and twentieth century's.

I observed that their definition was not clear and unique, but for example the height was defined as: a perpendicular, a length of the perpendicular, a perpendicular segment and a distance. The base was not defined by Euclide, and in Italian studied books was defined as follow: The opposite side of the height and the side to whose straight line is perpendicular the height.

In fact the problem stands on the fact that with the term "base of a polygon" I have to refer to the base of one height of the polygon. And then, the criteria for the calculation of the area of triangle should be Area=(length of) height (one of the heights) times the (length of) relative base divided by two.

In this way, an obstacle that born as an epistemological obstacle now becomes a didactical obstacle, and the language does not help the students investigation and their perceptive activities.

Successively I did a semiotic analysis that involves two 6th grade classrooms, one of which will participate to a twenty-four hours chess course (see 5.2.1). I call it experimental classroom, and the other one control classroom.

I choose these two classes proposing some geometrical tasks to five different classrooms of the same school, and choosing the ones with similar results.

After chess course I proposed geometrical task to both classroom and I analyzed the difference between pre-tests and post-tests.

Whit the analysis of inscriptions I observed that after the chess course of twenty-four hours (its duration was of about two months), seems[29] that the students of experimental classroom improved:

- the constructive cooperation of the figural and conceptual aspects in a geometrical problem solving activity;
- the ability to keep in mind and coordinate as many as possible figural conceptual items;
- the ability to predict and integrate the effect of each transformation on the road to the solution.

[29] The term "seems" it is used because we only observed that they improved their answers in tasks that involve those abilities, and we can only argue that they improved that abilities.

The term “seems” it is used because I only observed that they improved their answers in tasks that involve those abilities, and I can only argue that the students improved them. Can I say that this “transformation” depend of an improving of the mentioned geometrical abilities?
In order to answer to this question I were helped from some psychologist experts that proposed psychological test to the students of experimental and control classrooms in the same period of geometrical pre-test and post-test submission.
The analysis of these psychological tests (entirely made by Department of Psychology of University of Palermo) show that the students of Experimental classroom improved the following abilities (D’Amico, Ferro, et. ali, 2012):

- Serial recognition of visual pattern;
- Recall of Visual-Spatial pattern;
- Recall of Visual-Spatial sequences;
- Visual Spatial Working Memory;
- Listening Span;
- Counting Span;

These results (especially the improvement of VSWM) could allow us to affirm that the students improved their answer in the proposed tasks because they improved the geometrical abilities mentioned above.
Anyway I would to understand if the analyzed case was only an isolated case and I would to check the influence of the chess teacher that was the researcher too.
So I did an investigation in other School of another Italian city (E. Fieramosca in Barletta) with another chess teacher, and I proposed the post test to three 6^{th} grade classroom that does not did chess and two 6^{th} grade classroom that did chess.
The results were that the students that played and studied chess did similar answer to which ones of the students of our experimental classroom.
That chess teacher followed two guidelines: 1. The course must includes the content shown in 5.2.1, 2. The chess course must be designed and proposed in synergy with the mathematical teacher.
In fact I do not think that the chess course could improve “alone” some abilities as a “mental gym”, but I think that the chess course must be designed following disciplinary timing and requests and controlled by disciplinary tasks. Then, the role of chess teacher and math teacher is to find a meeting point between what a discipline (geometry in this case) needs (in a

specific place, time and context) and what the non-disciplinary activity (chess in this case) can offer.

However I want to justify these improvement of skills and so I did a multimodal semiotic analysis of specific chess activities: Reconstruction of chess position (see 6.3) and Objectification of Movement of pieces (see 6.4).

The exercise of reconstruction of a chess position consists in placing the pieces on the chessboard in front of him, according to those prepared by the teacher on the wall board.

Our analysis consisted to choose one of the students of experimental classroom while he is dealing with reconstruction of chess positions and observe how changed the semiotic means of objectification during the process of objectification.

In this way I choose 4 chess episodes in which the student reconstruct a chess position and a geometrical one in which then students try to solve a Tangram and I analyzed the distribution of semiotic means of objectification in each episode. In figure 21 I will show how they changed in the episodes.

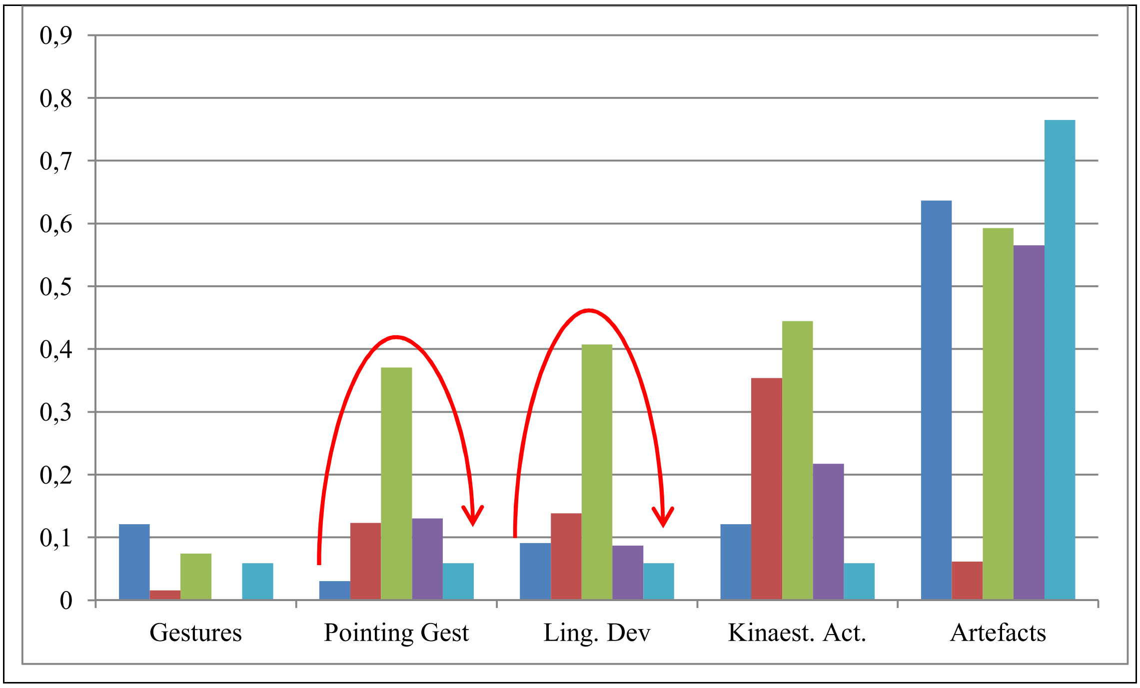


Figure 21 The distribution of the semiotic means of objectification on the five episodes. For every SMO the first column refer to the first episode, the second column to the second episode and so on. The red arrows show how the use of Pointing Gestures and Linguistic Devices changed in the five episodes.

I observed that in lower level of awareness the student use the gesture to point squares, to keep in hand pieces or to tap them over the chessboard.

These gestures (in particular the pointing gestures) were modified (or simply contracted) into his eyes action. (fig.21) When he achieved highest level of awareness he moved his eyes and his head to individuate the squares on the chessboard without using gestures.

By the way this eyes motions were not "alone" it was coordinated to the language that the student improved in "calling" the columns or the squares. As shown in figure 21 this SMO had the same behavior of pointing gestures.

What I want emphasize is that in fifth episode in which the student is solving a Tangram the distribution of Pointing gestures and Linguistic devices continue in the same way of the chess episode.

So, in this work I can affirm that the following phenomena:

- The contraction and synchronization of pointing gestures and linguistic devices;
- The acquisition and/or an improvement of the constructive cooperation of the figural and conceptual aspects in a geometrical problem solving activity, the ability to keep in mind and coordinate as many as possible figural conceptual items and the ability to predict and integrate the effect of each transformation on the road to the solution

occurred at the same time, and I want to argue that these phenomena are strictly linked.

The last experimentation step was to study the objectification of the configural concepts "movement of pieces", analyzing a case study, with the aim to show a strong similarity between a chess activity and a geometrical activity.

The movement of piece is a configural concept because its meaning depends by the relationships (figural and conceptual) between chess objects involved and from its position in the whole theoretical structure of pieces in the chessboard.

Students, when move a piece on chessboard, use all cognitive structures that involve the piece itself, portion of chessboard, linguistic and informal rules etc.

In objectification of pieces movement chess player acquire different levels of awareness on what he/she is reflecting for:

- Pieces (choose a piece and try to move it correctly);
- From pieces to chessboard (reflection on pieces shift the attention on chessboard – For example try best way to react to an opponent check);

- From chessboard to pieces (reflection on pieces guide how to move pieces – For example "square rule" show if a pawn can be promoted without that it will be reached by opponent king).

All in all, when a chess player is in front of a position, he has to configure some chess objects (pieces or square) with the aim to imaging new possibilities similar to a well known configural concept.
In this imagination process he/she can "invoke" different configural concepts and so different evaluation of a position. So chess players improve the ability to disclosure the right (and useful) configural concepts and handle them with the aim to reach a purpose.
In amateur chess courses, the "fundamental purposes" are to learn the axiomatic system, to move correctly a piece and don't do illegal moves. A clear argument that need all of these fundamental purposes is to move a piece when the opponent player attack the our king.
To show the relationships between chess and geometry I observed the students while they tried to solve the previous chess and task and I compared these actions with which they produce during Tangram activity.
What I can observe is that they coordinate the hands to find the piece (chess piece or paper piece) that can be putted in a place. In this case I observed that students, in achieving an higher level of awareness are using the pointing gestures and synchronization of their hands.
Although I have not a recorded data, I can say that an higher level of awareness is achieved when the student contract his semiotic activities, and in this case, transform the pointing gestures in eyes movement.
However what I want emphasize is that this transformation seems doesn't care of the context and the students use the same semiotic activity with the purpose to solve two tasks that are cognitive similar but in different context. This is what I have defined transfer of skills, or more precisely, a maintenance of skills on changing of the context.
I can affirm that a chess course designed following our guidelines could be used as an intense training for the constructive cooperation of the figural and conceptual aspects in a geometrical problem solving activity, the ability to keep in mind and coordinate as many as possible figural conceptual items and the ability to predict and integrate the effect of each transformation on the road to the solution.
This training must be designed by the collaboration of chess teacher and math teacher, the last of which must control the students transformation by a disciplinary lens. An example of the need of this control is that some students of experimental classroom had a wrong idea of

the concept of diagonal. In fact they thought that the diagonal has only an “x” form (like for bishop movement) refusing the correct definition.

In fact, if the math teacher did not controlled this phenomena, maybe in the 1^{st} item of geometrical post-test these students would marked in wrong way the diagonals of the 5^{th} polygon.

I conclude this work synthesizing and clarifying some aspects:

1. In this work I affirmed that discipline “Chess” inserted in the school curricula could be an intense training for specific geometrical abilities;
2. In this work the discipline Chess is thought as a supporting discipline, so the design of an activity including chess must follow the requests of the ordinary disciplines;
3. I reject the proposition “Chess is good for learning mathematics”. In fact the results that I obtained were possible because of an intense work of collaboration between the researcher/instructor and school;
4. With this work I observed the improvement of certain skills through a synergic action between chess and mathematics, but I can suppose that there are other skills that could be improved by a similar action. For example the History teacher of experimental classroom confirmed that after this research action the students can handle the idea that events in different chapters of history book could be contemporary.
5. With this work I gave an example on how design a non-disciplinary activity to improve certain disciplinary skills. In this case I chosen the discipline Chess, but I believe that other disciplines like “go” or the role games can have similar effect of which I obtained with chess.

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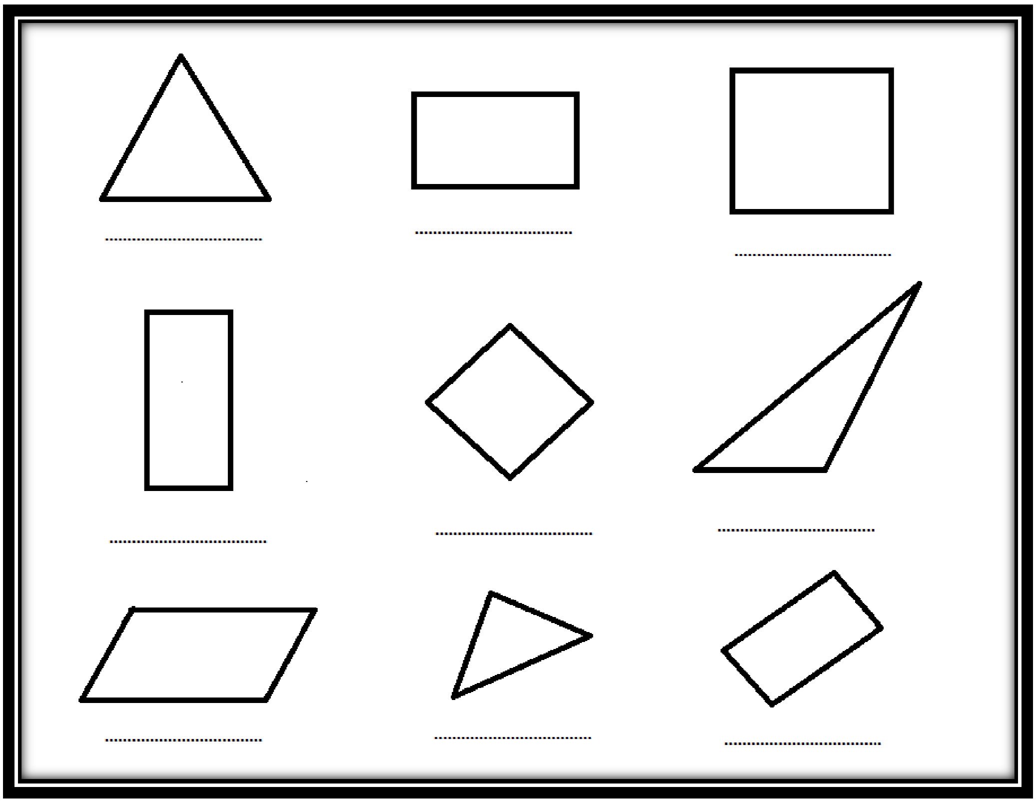
Appendix A

GEOMETRICAL TASK – PRE TEST

Item1

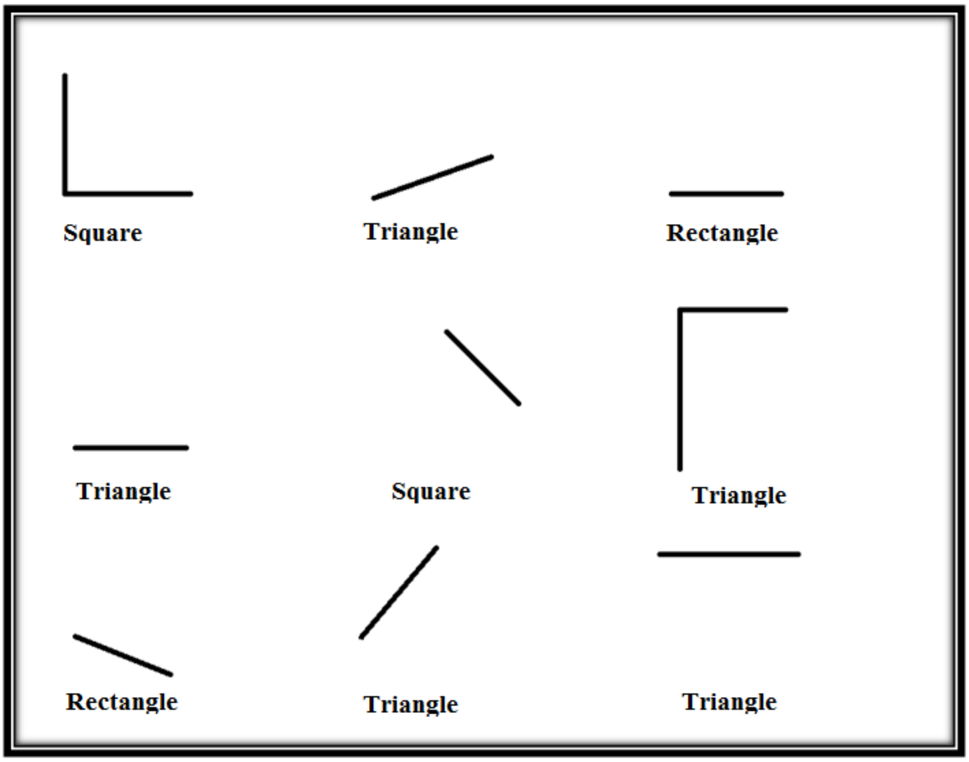
Recognize the following geometrical figures and write their name below each of them:



Also, mark the segments that you need to measure in order to calculate their area

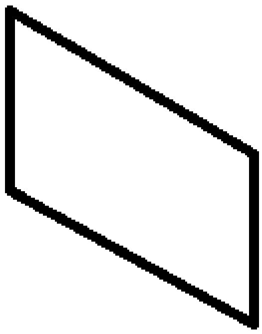
Item 2

Complete the following geometrical figures



Item 3

Recognize the following geometrical figures and write their name below each of them:



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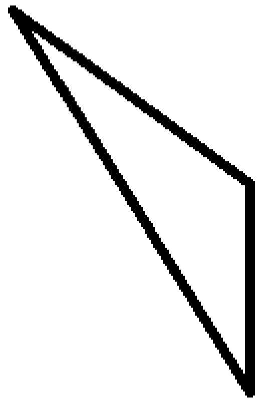
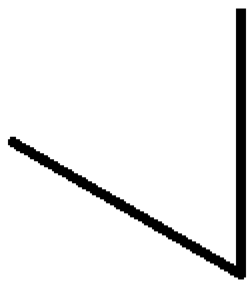


....

Also, mark the segments that you need to measure in order to calculate their area

Complete the following geometrical figures

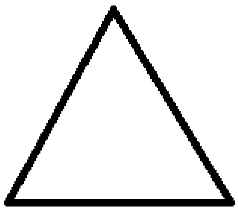
Parallelogram



Trapezoid

<u>Item 4</u>

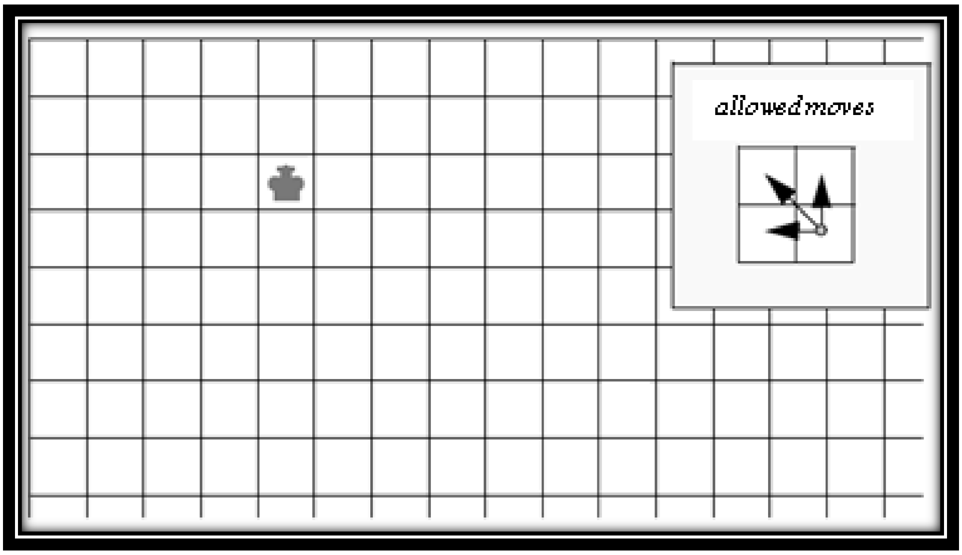
Draw in the following table a triangle that satisfies both the conditions

	Acutangle	**Right-angled**	**Obtusangle**
Equilateral			
Isosceles			
Scalene			

In the example, the triangle drawn is both equilateral and acutangle.

Item 5

Rules of the game: An allowed move consists of moving the king by one square: up, left or diagonally to the upper left. The players take turns to move. Starting player: alternate.
Win/lose rule: The player who cannot make a move, loses.



The game is played as follows: The king is placed on a square of the board and two players move alternately. *The first player that moves will be the player 1, and the other one will be the player 2.*

- In which square will it be at the end of the game? Marks it on figure 1
- Placing the king like in figure 1, who will win the game: the player 1 or the player 2?

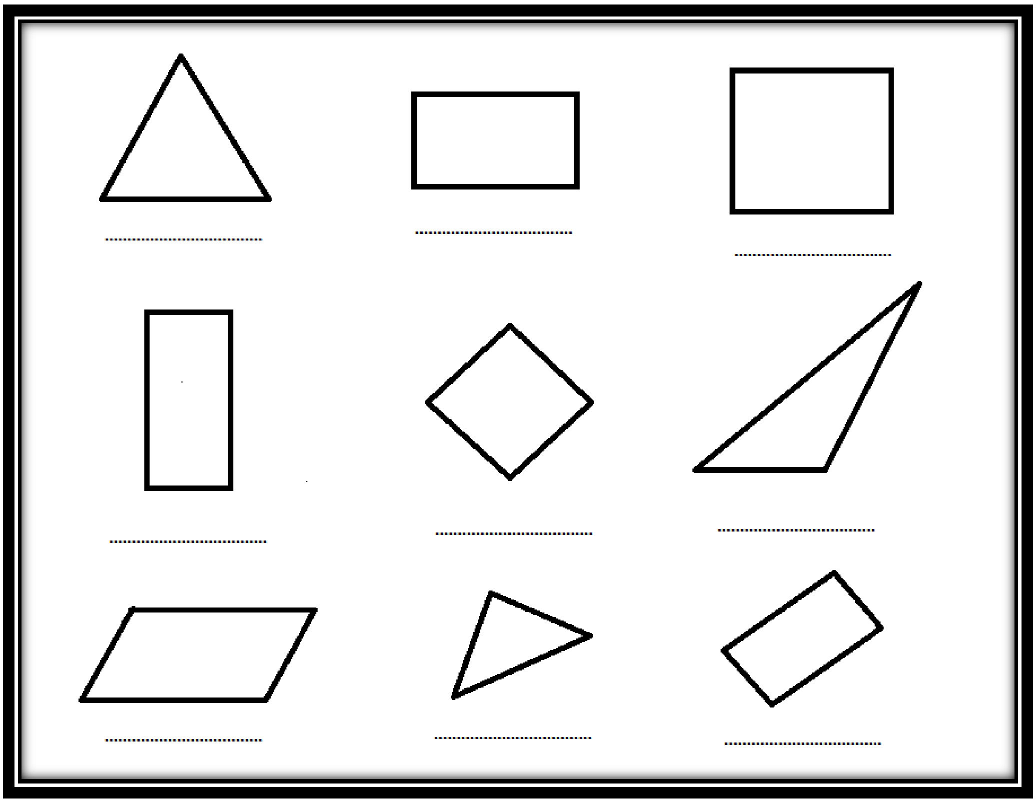
Remember that you have to think the best moves for both players, and so it does not matter how well the opponent plays.

Appendix B

GEOMETRICAL TASK – POST TEST

<u>Item1</u>

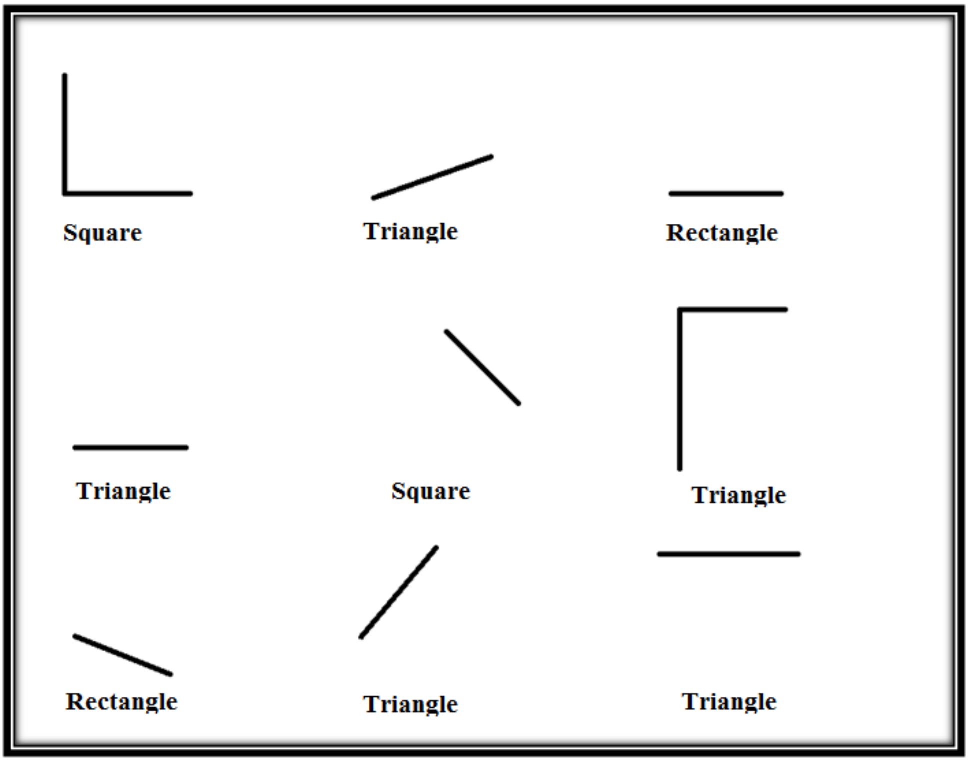
Recognize the following geometrical figures and write their name below each of them:



Also, mark the segments that you need to measure in order to calculate their area

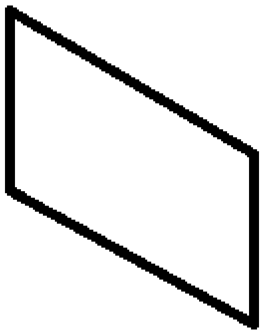
Item 2

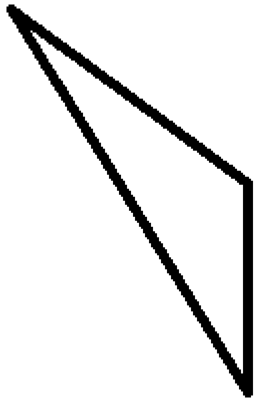
Complete the following geometrical figures



Item 3

Recognize the following geometrical figures and write their name below each of them:



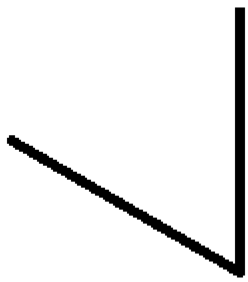


....

Also, mark the segments that you need to measure in order to calculate their area

Complete the following geometrical figures

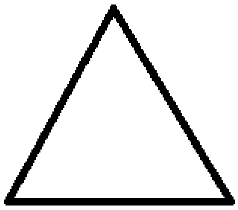
Parallelogram



Trapezoid

Item 4

Draw in the following table a triangle that satisfies both the conditions

	Acutangle	**Right-angled**	**Obtusangle**
Equilateral			
Isosceles			
Scalene			

In the example, the triangle drawn is both equilateral and acutangle.

Item 5

Rules of the game: An allowed move consists of moving the king by one square: up, left or diagonally to the upper left. The king cannot "jump" the obstacles on the map. The players take turns to move. Starting player: alternate.

Win/lose rule: The player who cannot make a move, loses.

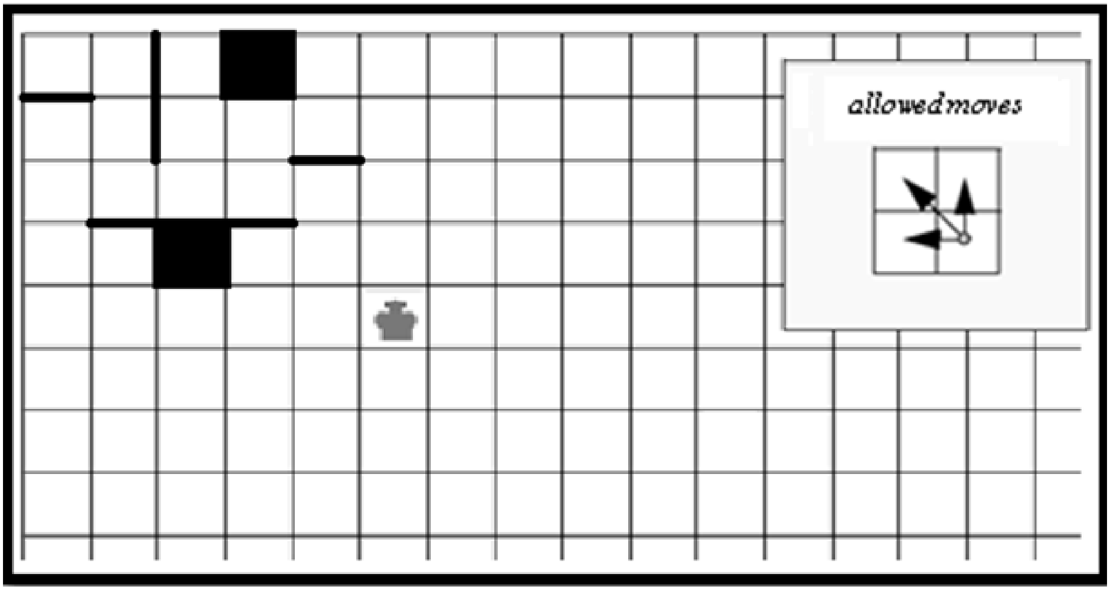


Figure 7

The game is played as follows: The king is placed on a square of the board and two players move alternately. *The first player that moves will be the player 1, and the other one will be the player 2.*

- In which square will it be at the end of the game? Marks it on figure 1
- Placing the king like in figure 1, who will win the game: the player 1 or the player 2?

Remember that you have to think the best moves for both players, and so it does not matter how well the opponent plays.