Dynamic Coalitional TU Games: Distributed Bargaining among Players' Neighbors

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***Abstract*—We consider a sequence of transferable utility (TU) games where, at each time, the characteristic function is a random vector with realizations restricted to some set of values. The first part of the paper contributes to the definition of a robust (coalitional) TU game and the development of a distributed bargaining protocol. We prove the convergence with probability 1 of the bargaining process to a random allocation that lies in the core of the robust game under some mild conditions on the underlying communication graphs. The second part of the paper addresses the more general case where the robust game may have empty core. In this case, with the dynamic game we associate a dynamic average game by averaging over time the sequence of characteristic functions. Then, we consider an accordingly modified bargaining protocol. Assuming that the sequence of characteristic functions is ergodic and the core of the average game has a nonempty relative interior, we show that the modified bargaining protocol converges with probability 1 to a random allocation that lies in the core of the average game.**

I. Introduction

Coalitional games with transferable utilities (TU) have been introduced by von Neumann and Morgenstern [25]. A coalitional TU game constitutes of a set of players, who can form coalitions, and a characteristic function that provides a value for each coalition. Coalitional TU games have been used to model cooperation in supply chain applications [8], network flow applications [1] and communication networks [22].

In this paper, the game is played repeatedly over time thus generating a sequence of time varying characteristic functions. We refer to such a repeated game as *dynamic coalitional TU game*. Our main objective is to explore distributed agreement on solutions in the core of the game, where the players interact only with their neighbors. At every iteration, a player i observes the allocations of some of his neighbors. This is modeled using a directed graph with the set of players as the vertex set and a time-varying edge set composed of directed links (i, j) whenever player i observes the allocation vector proposed by player j at time t. We refer to this directed graph as players' *neighbor-graph*. Given a player's neighbor-graph, each player i negotiates allocations by adjusting the allocations he received from his neighbors through weight assignments. The player selects a new allocation by projecting the balanced allocation on his bounding set. We propose such bargaining protocols for solving both the robust and the average TU game. For each of these games, we use some mild assumptions on the connectivity of the players' neighbor-graph. Assuming that the core of the robust game is nonempty, we show that our bargaining protocol converges with probability 1 to a common (random) allocation in the core. In the case when the core of the robust game is empty, we consider an average game that can provide a meaningful solution under some conditions on the sequence of the characteristic functions. This is a new contribution with respect to the conference paper [12]. Specifically, in this case, we consider a dynamic average game by averaging over time the sequence of characteristic functions. We then modify accordingly the bounding set and the associated bargaining process by using the dynamic average game. This means that the value constraints are defined by using the time-averaged sequence rather than the sequence of random characteristic functions. Under the assumptions that the time-averaged sequence is ergodic and that the core of the average game has a nonempty relative interior, we show that the players' allocations generated by our bargaining protocol converge with probability 1 to a common (random) allocation in the core of the average game. Averaging coalitions' values is a way to capture players' *patience* and allocation *fairness*. Patience in bargaining games is a crucial aspect and has inspired a number of different models (see, for instance, Section 3.10.2 in [17] and Chapter 7, p.126 in [25]). Averaged (payoff) values are also among the foundations of the "approachability theory" [5].

The work in this paper is related to stochastic cooperative games [23]. However, the existence of a players' neighbor-graph, of multiple iterations in the bargaining process, and the consideration of robust game add new elements to the model. Bringing dy-

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namical aspects into the framework of coalitional TU games is an element in common with papers [7], [9]. However, unlike [7], [9], the values of the coalitions in this paper are realized exogenously and no relation is assumed between consecutive samples. Dynamic robust TU games have also been considered in [3], [2] and [4]. Convergence of allocation processes is a main topic in [11]. There, rewards are allocated by a game designer repeatedly in a centralized manner and based on the current excess rewards of the coalitions (accumulated reward up to current time minus the value of the coalition). Our approach, however, differs from that of [11] as we resort to a *decentralized scheme* where the allocation process is the result of a bargaining process among the players with local interactions. The work in this paper is also related to the literature on agreement among multiple agents, where an underlying communication graph for the agents and balancing weights have been used with some variations [24], [14] to reach an agreement on common decision variable, as well as in [15], [16], [20], [19] for distributed multi-agent optimization.

This paper is organized as follows. In Section II, we introduce the game and provide motivations and preliminary results. In Section III, we prove the convergence results for the robust game. In Section IV, we introduce the dynamic average game and the modified bargaining protocol. In Section V, we report some numerical simulations to illustrate our theoretical study, and we conclude in Section VI.

Notation. We view vectors as columns. For a vector x, we use x_i or $[x]_i$ to denote its ith coordinate component. For two vectors x and y, we use $x < y$ ($x \leq y$) to denote $x_i < y_i$ ($x_i \leq y_i$) for all coordinate indices i. We let x' denote the transpose of a vector x, and $\|x\|$ denote its Euclidean norm. An $n \times n$ matrix A is row-stochastic if the matrix has nonnegative entries a_{ij} and $\sum_{j=1}^{n} a_{ij} = 1$ for all $i = 1, \ldots, n$. For a matrix A, we use a_{ij} or $[A]_{ij}$ to denote its ijth entry. A matrix A is doubly stochastic if both A and its transpose A' are row-stochastic. Given two sets U and S, we write $U \subset S$ to denote that U is a proper subset of S. We use $|S|$ for the cardinality of a given finite set S. We write $P_X[x]$ to denote the projection of a vector x on a set X, and we write $\mathrm{dist}(x, X)$ for the distance from x to X, i.e., $P_X[x] = \arg\min_{y \in X} \|x - y\|$ and $\mathrm{dist}(x, X) = \|x - P_X[x]\|$, respectively. Given a set X and a scalar $\lambda \in \mathbb{R}$, the set λX is defined by $\lambda X \triangleq \{\lambda x \mid x \in X\}$. Given two sets $X, Y \subseteq \mathbb{R}^n$, the set sum $X + Y$ is defined by $X + Y \triangleq \{x + y \mid x \in X,\ y \in Y\}$.

II. Dynamic TU Game and Robust Game

In this section, we introduce the game and state some basic assumptions. We also provide an example motivating our development and establish some preliminary results.

A. Basic Concepts of TU Games

A coalitional TU game involves a set N of players and a value mapping $\eta : 2^N \to \mathbb{R}$ which is defined for each nonempty coalition $S \subseteq N$ (nonempty subset of N). The value mapping η is often referred to as *value* (or *characteristic*) *function*, as it fully characterizes the game. A TU game for a set N of players and a characteristic function η is denoted by $\langle N, \eta \rangle$.

We let η_S denote the scalar value associated with a nonempty coalition $S \subseteq N$ under the characteristic function η, i.e., η_S is the value of a coalition S in the game $\langle N, \eta \rangle$. The value η_S could be thought of as a monetary value that the players in coalition S need to somehow divide among themselves. One of the solution concepts in a TU game is the *core* of the game which works under the premise that the best option for players is to form a grand coalition N. The core of a TU game $\langle N, \eta \rangle$ is denoted by $C(\eta)$ and it is defined as a collection of possible allocation vectors among the players, as follows:

$$C(\eta) = \Big\{ x \in \mathbb{R}^{|N|} \,\Big|\, \sum_{i \in N} x_i = \eta_N,\ \sum_{i \in S} x_i \geq \eta_S \quad \text{for all nonempty } S \subset N \Big\},$$

where $x_i \in \mathbb{R}$ is an allocation value for player $i \in N$ and $x = (x_1, \ldots, x_{|N|})^T$ is an allocation vector. Thus, the core $C(\eta)$ of a game $\langle N, \eta \rangle$ consists of all possible allocations x_i to players $i \in N$ such that the total sum of allocation values x_i is equal to the value η_N of the grand coalition, which means that the grand coalition is efficient. Furthermore, for every other nonempty coalition S, the sum of allocations x_i for players $i \in S$ is at least as large as the coalition value η_S, which translates to interpretation of the grand coalition being stable (i.e., no player has incentive to leave the grand coalition). Given that the core $C(\eta)$ of a TU game $\langle N, \eta \rangle$ is nonempty, the question that one needs to address is to *determine a protocol (bargaining process) for the players that will render an allocation $\tilde{x}$ in the core on which all players agree*. We will investigate this problem for dynamic TU games played among players constrained to communicate locally over a communication graph. The precise setting is described in the next section.

B. Problem of our Interest and Bargaining Process

Consider a set of players $N = \{1, \ldots, n\}$ and the set of all (nonempty) *coalitions* $S \subseteq N$ arising among these players. Let $m = 2^n - 1$ be the *number of* *possible coalitions*. We assume that the time is discrete and use $t = 0, 1, 2, \ldots$ to index the time slots.

We consider a dynamic TU game, denoted $\langle N, \{v(t)\}\rangle$, where $\{v(t)\}$ is a sequence of characteristic functions. Thus, in the dynamic TU game $\langle N, \{v(t)\}\rangle$, the players are involved in a sequence of instantaneous TU games whereby, at each time t, the *instantaneous TU game* is $\langle N, v(t)\rangle$ with $v(t) \in \mathbb{R}^m$ for all $t \geq 0$. Further, we let $v_S(t)$ denote *the value assigned to a nonempty coalition* $S \subseteq N$ in the instantaneous game $\langle N, v(t)\rangle$. In what follows, we deal with dynamic TU games where each characteristic function $v(t)$ is a random vector with realizations restricted to some set of values.

In our development of robust TU game, we assume that the grand coalition value $v_N(t)$ is deterministic for every $t \geq 0$, while the values $v_S(t)$ of the other coalitions S have a common upper bound. These conditions are formally stated in the following assumption.

Assumption 1: There exists $v^{\max} \in \mathbb{R}^m$ such that for all $t \geq 0$, we have $v_N(t) = v_N^{\max}$ and $v_S(t) \leq v_S^{\max}$ for all nonempty coalitions $S \subset N$.

We refer to the game $\langle N, v^{\max}\rangle$ as *robust game*. In the first part of this paper, we rely on the assumption that the robust game has a nonempty core.

Assumption 2: We have $C(v^{\max}) \neq \emptyset$.

An immediate consequence of Assumptions 1 and 2 is that the core $C(v(t))$ of the instantaneous game is always nonempty. This follows from $C(v^{\max}) \subseteq C(\eta)$ for any η satisfying $\eta_N = v_N^{\max}$ and $\eta_S \leq v_S^{\max}$ for $S \subset N$, and the assumption that $C(v^{\max}) \neq \emptyset$.

Throughout the paper, we assume that each player i is *rational and efficient*. This translates to each player $i \in N$ choosing his allocation vector within the set of allocations satisfying value constraints of all the coalitions that include player i. This set is referred to as the *bounding set of player* i and, for a generic game $\langle N, \eta\rangle$, it is given by

$$X_i(\eta) = \Big\{x \in \mathbb{R}^n \mid \sum_{j \in N} x_j = \eta_N, \ \sum_{j \in S} x_j \geq \eta_S \quad \text{for all } S \subset N \text{ s.t. } i \in S\Big\}.$$

Note that each $X_i(\eta)$ is polyhedral.

We find it suitable to represent the bounding sets and the core in alternative equivalent forms. Let $e_S \in \mathbb{R}^n$ be the incidence vector for a nonempty coalition $S \subseteq N$, i.e., the vector with the coordinates given by

$$[e_S]_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{else.} \end{cases}$$

Then, the bounding sets and the core can be represented as follows:

$$\begin{aligned} X_i(\eta) \ &= \ \{x \in \mathbb{R}^n \mid e_N' x = \eta_N, \ e_S' x \geq \eta_S \\ &\qquad \text{for all } S \subset N \text{ with } i \in S\}, \end{aligned} \tag{1}$$

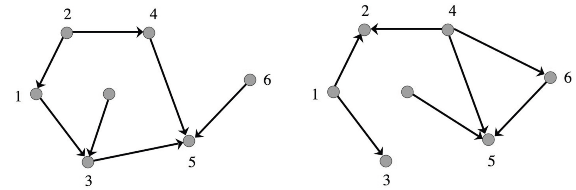


Fig. 1. Players' neighbor graphs for 6 players and two different time instances.

$$\begin{aligned} C(\eta) \ &= \ \{x \in \mathbb{R}^n \mid e_N' x = \eta_N, \ e_S' x \geq \eta_S \\ &\qquad \text{for all nonempty } S \subset N\}. \end{aligned} \tag{2}$$

Furthermore, observe that the core $C(\eta)$ of the game coincides with the intersection of the bounding sets $X_i(\eta)$ of all players $i \in N = \{1, \ldots, n\}$, i.e.,

$$C(\eta) = \cap_{i=1}^n X_i(\eta). \tag{3}$$

We now discuss the bargaining protocol where repeatedly over time each player $i \in N$ submits an allocation vector that the player would agree on. The allocation vector proposed by player i at time t is denoted by $x^i(t) \in \mathbb{R}^n$, where the jth component $x_j^i(t)$ represents the amount that player i would assign to player j. To simplify the notation in the dynamic game $\langle N, \{v(t)\}\rangle$, we let $X_i(t)$ denote the bounding set of player i for the instantaneous game $\langle N, v(t)\rangle$, i.e., for all $i \in N$ and $t \geq 0$,

$$\begin{aligned} &X_i(t) = \Big\{x \in \mathbb{R}^n \mid \textstyle\sum_{j \in N} x_j = v_N(t), \\ &\textstyle\sum_{j \in S} x_j \geq v_S(t) \text{ for all } S \subset N \text{ s.t. } i \in S\Big\}. \end{aligned} \tag{4}$$

We assume that each player may observe the allocations of a subset of the other players at any time, which are termed as the *neighbors of the player*. The players and their neighbors at time t can be represented by a directed graph $\mathcal{G}(t) = (N, \mathcal{E}(t))$, with the vertex set N and the set $\mathcal{E}(t)$ of directed links. A link $(i, j) \in \mathcal{E}(t)$ exists if player j is a neighbor of player i at time t. We always assume that $(i, i) \in \mathcal{E}(t)$ for all t, which is natural since every player i can always access its own allocation vector. We refer to graph $\mathcal{G}(t)$ as a *neighbor-graph* at time t. In the graph $\mathcal{G}(t)$, a player j is a neighbor of player i (i.e., $(i, j) \in \mathcal{E}(t)$) only if player i can observe the allocation vector of player j at time t. Figure 1 illustrates how the players' neighbor-graph may look at two time instances.

Given the players' neighbor-graph $\mathcal{G}(t)$, each player i negotiates allocations by averaging his allocation and the allocations he received from his neighbors. More precisely, at time t, the bargaining process for each player i involves the player's individual bounding set $X_i(t)$, its own allocation $x^i(t)$ and the observed allocations $x^j(t)$ of some of his neighbors j. Formally, we let $\mathcal{N}_i(t)$ be the set of neighbors of player i at time t (including himself), i.e., $\mathcal{N}_i(t) = \{j \in N \mid (i, j) \in$

$\mathcal{E}(t)\}$. With this notation, the bargaining process is given by, for all $i \in N$ and $t \geq 0$:

$$x^i(t+1) = P_{X_i(t)}\left[\sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)x^j(t)\right], \quad (5)$$

where $P_{X_i(t)}[\cdot]$ is the projection onto the player i bounding set $X_i(t)$ and $a_{ij}(t) \geq 0$ is a scalar weight that player i assigns to the proposed allocation $x^j(t)$ of a neighbor $j \in \mathcal{N}_i(t)$. The weights $a_{ij}(t), j \in \mathcal{N}_i(t)$ are assumed to be deterministic scalars chosen by player i (for example, see [15] for some specific possible choices of $a_{ij}(t), j \in \mathcal{N}_i(t)$). The initial allocations $x^i(0)$, $i = 1, \ldots, n$, are selected randomly and independently of $\{v(t)\}$.

The bargaining process in (5) can be written compactly by introducing zero weights for players j whose allocations are not available to player i at time t. Specifically by letting $a_{ij}(t) = 0$ for all $j \notin \mathcal{N}_i(t)$ and all t, we have the following equivalent representation of the bargaining process, for all $i \in N$ and $t \geq 0$:

$$x^i(t+1) = P_{X_i(t)}\left[\sum_{j=1}^{n} a_{ij}(t)x^j(t)\right]. \quad (6)$$

We now discuss the specific assumptions on the weights $a_{ij}(t)$ and the players' neighbor-graph that we use. We let $A(t)$ be the matrix with entries $a_{ij}(t)$.

Assumption 3: Each matrix $A(t)$ is doubly stochastic with positive diagonal, and there exists a scalar $\alpha > 0$ such that $a_{ij}(t) \geq \alpha$ whenever $a_{ij}(t) > 0$.

In view of the construction of matrices $A(t)$, we see that $a_{ij}(t) \geq \alpha$ for $j = i$ and perhaps for some players j that are neighbors of player i. The requirement that the positive weights are uniformly bounded away from zero is imposed to ensure that the information from each player diffuses in the network persistently in time. The requirement on the doubly stochasticity of the weights is used to ensure that in a long run each player has equal influence on the limiting allocation vector.

It is natural to expect that the connectivity of the players' neighbor-graphs $\mathcal{G}(t) = (N, \mathcal{E}(t))$ impacts the bargaining process. At any time, the instantaneous graph $\mathcal{G}(t)$ need not be connected. However, for the proper behavior of the bargaining process, the union of the graphs $\mathcal{G}(t)$ over a period of time is assumed to be connected.

Assumption 4: There is an integer $Q \geq 1$ such that the graph $\left(N, \bigcup_{\tau=tQ}^{(t+1)Q-1} \mathcal{E}(\tau)\right)$ is strongly connected for every $t \geq 0$.

Assumptions 3 and 4 together guarantee that the players communicate sufficiently often to ensure that the information of each player is persistently diffused over the network in time to reach every other player. Under these assumptions, we will study the dynamic bargaining process in (6). We want to provide conditions under which the process converges to an allocation in the core of the robust game. Before this, we give a motivating example in the following section.

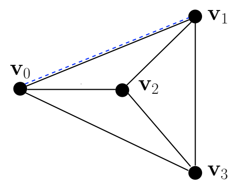
C. Motivations

Dynamic coalitional games capture coordination in a number of network flow applications. Network flows model flow of goods, materials, or other resources between different production/distribution sites [1]. In communication networks, or more generally in self-organizing, decentralized, and autonomic networks, *coalitional games prove to be a very powerful tool for designing fair, robust, practical, and efficient co-operation strategies* (see [22], p.93). A first example is the rate allocation in a multiple access channel. Here the bargaining process involves the users who aim at obtaining a fair allocation of the total transmission rate available. If a user, or coalition of users receives an unfair allocation, it will adopt a selfish behavior thus reducing the efficiency of the global network. Other applications arise in receivers and transmitters cooperation, or packet forwarding ad hoc networks. We refer the reader to the tutorial [22] for a more detailed discussion on the use of coalitional games in engineering applications.

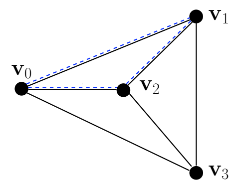
We next provide a supply chain application that justifies the model under study.

A single warehouse $\mathbf{v}_0$ serves a number of retailers $\mathbf{v}_i$, $i = 1, \ldots, n$, each one facing a demand $d_i(t)$ unknown but bounded by pre assigned values $d_i^{\min} \in \mathbb{R}$ and $d_i^{\max} \in \mathbb{R}$ at any time period $t \geq 0$. After demand $d_i(t)$ has been realized, retailer $\mathbf{v}_i$ must choose to either fulfill the demand or not. The retailers do not hold any private inventory and, therefore, if they wish to fulfill their demands, they must reorder goods from the central warehouse. Retailers benefit from joint re-orders as they may share the total transportation cost K (this cost could also be time and/or players dependent). In particular, if retailer $\mathbf{v}_i$ "plays" individually, the cost of reordering coincides with the full transportation cost K. Actually, when necessary a single truck will serve only him and get back to the warehouse. This is illustrated by the dashed path $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_0)$ in the network of Figure 2(a). The cost of not reordering is the cost of the unfulfilled demand $d_i(t)$.

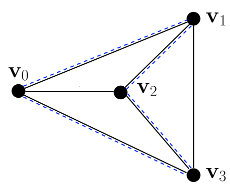
If two or more retailers "play" in a coalition, they agree on a joint decision ("everyone reorders" or "no one reorders"). The cost of reordering for the coalition also equals the total transportation cost that must be shared among the retailers. When necessary a single truck will serve all retailers in the coalition and get back to the warehouse. This is illustrated, with reference to coalition $\{\mathbf{v}_1, \mathbf{v}_2\}$ by the dashed path



(a) Truck leaving $\mathbf{v}_0$, serving $\mathbf{v}_1$ and returning to $\mathbf{v}_0$.



(b) Truck leaving $\mathbf{v}_0$, serving $\mathbf{v}_1$ and $\mathbf{v}_2$, and then returning to $\mathbf{v}_0$.



(c) Truck leaving $\mathbf{v}_0$, serving $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, and then returning to $\mathbf{v}_0$.

Fig. 2. Example of one warehouse $\mathbf{v}_0$ and three retailers $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

$(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_0)$ in Figure 2(b). A similar comment applies to the coalition $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and the path $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_0)$ in Figure 2(c). The cost of not reordering is the sum of the unfulfilled demands of all retailers. How the players will share the cost is a part of the solution generated by the bargaining process.

The cost scheme can be captured by a game with the set $N = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of players where the cost of a nonempty coalition $S \subseteq N$ is given by

$$c_S(t) = \min\left\{K, \sum_{i\in S} d_i(t)\right\}.$$

Note that the bounds on the demand $d_i(t)$ reflect into the bounds on the cost as follows: for all nonempty $S \subseteq N$ and $t \geq 0$,

$$\min\left\{K, \sum_{i\in S} d_i^{\min}\right\} \leq c_S(t) \leq \min\left\{K, \sum_{i\in S} d_i^{\max}\right\}. \tag{7}$$

To complete the derivation of the coalitions' values we need to compute the cost savings $v_S(t)$ of a coalition S as the difference between the sum of the costs of the coalitions of the individual players in S and the cost of the coalition itself, namely,

$$v_S(t) = \sum_{i\in S} c_{\{i\}}(t) - c_S(t).$$

Given the bound for $c_S(t)$ in (7), the value $v_S(t)$ is also bounded, as given: for any $S \subset N$ and $t \geq 0$,

$$v_S(t) \leq \sum_{i\in S} \min\{K, d_i^{\max}\} - \min\left\{K, \sum_{i\in S} d_i^{\min}\right\}.$$

Thus, the cost savings (value) of each coalition is bounded uniformly by a maximum value.

D. Preliminary Results

We derive some preliminary results pertinent to the core of the robust game and some error bounds for polyhedral sets applicable to the players' bounding sets $X_i(t)$. We later use these results to establish the convergence of the bargaining process in (6).

In our analysis we often use the following relation that is valid for the projection operation on a closed convex set $X \subseteq \mathbb{R}^n$: for any $w \in \mathbb{R}^n$ and any $x \in X$,

$$\|P_X[w] - x\|^2 \leq \|w - x\|^2 - \|P_X[w] - w\|^2. \tag{8}$$

This property of the projection operation is known as a strictly non-expansive projection property (see [6], volume II, 12.1.13 Lemma on page 1120).

We next prove a result that relates the distance $\mathrm{dist}(x, C(\eta))$ between a point x and the core $C(\eta)$ with the distances $\mathrm{dist}(x, X_i(\eta))$ between x and the bounding sets $X_i(\eta)$. This result will be crucial in our later development. The result relies on the polyhedrality of the bounding sets $X_i(\eta)$ and the core $C(\eta)$, as given in (1) and (2) respectively, and a special relation for polyhedral sets. This special relation states that for a nonempty polyhedral set $\mathcal{P} = \{x \in \mathbb{R}^n \mid a_\ell' x \leq b_\ell,\ \ell = 1, \ldots, r\}$, there exists a scalar $c > 0$ such that

$$\mathrm{dist}(x, \mathcal{P}) \leq c \sum_{\ell=1}^{r} \mathrm{dist}(x, H_\ell) \text{ for all } x \in \mathbb{R}^n, \tag{9}$$

where $H_\ell = \{x \in \mathbb{R}^n \mid a_\ell' x \leq b_\ell\}$ and the scalar c depends on the vectors $a_\ell, \ell = 1, \ldots, r$ only. Relation (9) has been established by Hoffman [10] and is known as *Hoffman bound*.

Aside from the Hoffman bound, in establishing the forthcoming Lemma 1, we also use the fact that the square distance from a point x to a closed convex set X contained in an affine set H is given by

$$\mathrm{dist}^2(x, X) = \|x - P_H[x]\|^2 + \mathrm{dist}^2(P_H[x], X), \tag{10}$$

which is illustrated in Figure 3.

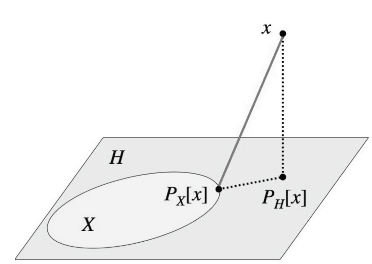


Fig. 3. Projection on a set X contained in an affine set H.

Now, we are ready to present the result relating the values $\mathrm{dist}^2(x, C(\eta))$ and $\mathrm{dist}^2(x, X_i(\eta))$.

Lemma 1: Let $\langle N, \eta\rangle$ be a TU game with a nonempty core $C(\eta)$. Then, there is a constant $\mu > 0$ such that, for all $x \in \mathbb{R}^n$,

$$\mathrm{dist}^2(x, C(\eta)) \le \mu \sum_{i=1}^{n} \mathrm{dist}^2(x, X_i(\eta)),$$

where μ depends on the collection of vectors $\{\tilde{e}_S \mid S \subset N,\ S \neq \emptyset\}$ with each $\tilde{e}_S$ being the projection of e_S on the hyperplane $H = \{x \in \mathbb{R}^n \mid e'_N x = \eta_N\}$.

Proof: Since the hyperplane H contains the core $C(\eta)$ (see (2)), by relation (10) we have for all $x \in \mathbb{R}^n$,

$$\begin{aligned}\mathrm{dist}^2(x, C(\eta)) =& \|x - P_H[x]\|^2 \\ &+ \mathrm{dist}^2(P_H[x], C(\eta)).\end{aligned} \tag{11}$$

The point $P_H[x]$ and the core $C(\eta)$ lie in the $(n-1)$-dimensional affine set H. By applying the Hoffman bound relative to the affine set H (cf. (9)), we obtain

$$\mathrm{dist}(P_H[x], C(\eta)) \le c \sum_{S \subset N} \mathrm{dist}(P_H[x], H \cap H_S),$$

where the summation is over nonempty subsets S and $H_S = \{x \in \mathbb{R}^n \mid e'_S x \ge \eta_S\}$. The constant c depends on the collection $\{\tilde{e}_S, S \subset N\}$ of projections of vectors e_S on the hyperplane H for $S \subset N$. Thus, it follows

$$\begin{aligned}&\mathrm{dist}^2(P_H[x], C(\eta)) \\ &\le c^2 \left(\textstyle\sum_{S \subset N} \mathrm{dist}(P_H[x], H \cap H_S)\right)^2 \\ &\le c^2(m-1) \textstyle\sum_{S \subset N} \mathrm{dist}^2(P_H[x], H \cap H_S),\end{aligned} \tag{12}$$

where m is the number of nonempty subsets of N. The last inequality follows by $(\sum_{j=1}^{\ell} a_j)^2 \le \ell \sum_{j=1}^{\ell} a_j^2$, which is valid for any collection of scalars $a_j, j = 1, \ldots, \ell$ with $\ell \ge 1$. Combining Eq. (12) with equality (11), we obtain for any $x \in \mathbb{R}^n$,

$$\begin{aligned}&\mathrm{dist}^2(x, C(\eta)) \le \|x - P_H[x]\|^2 \\ &+ c^2(m-1) \textstyle\sum_{S \subset N} \mathrm{dist}^2(P_H[x], H \cap H_S) \\ &\le c_1 \textstyle\sum_{S \subset N} \left(\|x - P_H[x]\|^2 + \mathrm{dist}^2(P_H[x], H \cap H_S)\right),\end{aligned}$$

where $c_1 = \max\{1,\ c^2(m-1)\}$. Since the set H is affine, in view of relation (10) we have $\|x - P_H[x]\|^2 + \mathrm{dist}^2(P_H[x], H \cap H_S) = \mathrm{dist}^2(x, H \cap H_S)$, implying that for any $x \in \mathbb{R}^n$,

$$\mathrm{dist}^2(x, C(\eta)) \le c_1 \sum_{S \subset N} \mathrm{dist}^2(x, H \cap H_S).$$

From the preceding relation it follows for all $x \in \mathbb{R}^n$,

$$\mathrm{dist}^2(x, C(\eta)) \le c_1 \sum_{S \subset N} |S| \, \mathrm{dist}^2(x, H \cap H_S), \tag{13}$$

where $|S|$ denotes the cardinality of the coalition S. Note that

$$\begin{aligned}&\textstyle\sum_{S \subset N} |S| \, \mathrm{dist}^2(x, H \cap H_S) \\ &= \textstyle\sum_{S \subset N} \sum_{i \in S} \mathrm{dist}^2(x, H \cap H_S) \\ &= \textstyle\sum_{i=1}^{n} \left(\sum_{\{S \subset N \mid i \in S\}} \mathrm{dist}^2(x, H \cap H_S)\right).\end{aligned} \tag{14}$$

We also note that $X_i(\eta) \subset H \cap H_S$ for each nonempty $S \subset N$ and $i \in S$, which follows by the definition of H_S and relation (1). Furthermore, since $\mathrm{dist}(x, Y) \le \mathrm{dist}(x, X)$ for any $x \in \mathbb{R}^n$ and for any two closed convex sets $X, Y \subseteq \mathbb{R}^n$ such that $X \subset Y$, it follows that, for all $x \in \mathbb{R}^n$

$$\mathrm{dist}(x, H \cap H_S) \le \mathrm{dist}(x, X_i(\eta)). \tag{15}$$

By combining relations (13)–(15), we obtain

$$\begin{aligned}&\mathrm{dist}^2(x, C(\eta)) \\ &\le c_1 \textstyle\sum_{i=1}^{n} \left(\sum_{\{S \subset N \mid i \in S\}} \mathrm{dist}^2(x, X_i(\eta))\right) \\ &= c_1 \kappa \textstyle\sum_{i=1}^{n} \mathrm{dist}^2(x, X_i(\eta)),\end{aligned} \tag{16}$$

where κ is the number of coalitions S that contain player i, which is the same number for every player (κ does not depend on i). The desired relation follows by letting $\mu = c_1 \kappa$, and by recalling that $c_1 = \max\{1, c^2(m-1)\}$, where c depends on the projections $\tilde{e}_S$ of vectors $e_S, S \subset N$, on the hyperplane H. ■

Note that the scalar μ in Lemma 1 does not depend on the coalitions' values η_S for $S \neq N$. It depends only on the vectors e_S, $S \subseteq N$, and the grand coalition value η_N.

As a direct consequence of Lemma 1, we have the following result for the instantaneous game $\langle N, v(t)\rangle$ under the assumptions of Section II-B.

Lemma 2: Let Assumptions 1 and 2 hold. We then have for all $t \ge 0$, $x \in \mathbb{R}^n$

$$\mathrm{dist}^2(x, C(v(t))) \le \mu \sum_{i=1}^{n} \mathrm{dist}^2(x, X_i(t))$$

where $C(v(t))$ is the core of the game $\langle N, v(t)\rangle$, $X_i(t)$ is the bounding set of player i, and μ is the constant from Lemma 1.

Proof: By Assumption 2, the core $C(v^{\max})$ is nonempty. Furthermore, under Assumption 1, we have $C(v^{\max}) \subseteq C(v(t))$ for all $t \ge 0$, implying that the core $C(v(t))$ is nonempty for all $t \ge 0$. Under Assumption 1, each core $C(v(t))$ is defined by the same affine equality corresponding to the grand coalition value, $e'_N x = v_N^{\max}$. Moreover, each core $C(v(t))$ is defined through the set of hyperplanes $H_S(t) = \{x \in \mathbb{R}^n \mid e'_S x \ge v_S(t)\}$, $S \subset N$, which have time invariant normal vectors e_S, $S \subseteq N$. Thus, the result follows from Lemma 1. ■

III. Convergence to Core of Robust Game

In this section, we prove convergence of the bargaining process in (6) to a random allocation that lies in the core of the robust game with probability 1. We find it convenient to re-write the bargaining protocol (6) by isolating a linear and a non-linear term. The linear term is the vector $w^i(t)$ defined as:

$$w^i(t) = \sum_{j=1}^{n} a_{ij}(t)x^j(t) \quad \text{for all } i \in N,\ t \geq 0. \quad (17)$$

Note that $w^i(t)$ is linear in players' allocations $x^j(t)$. The non-linear term is the error

$$e^i(t) = P_{X_i(t)}[w^i(t)] - w^i(t). \quad (18)$$

Now, using (17) and (18), we can rewrite protocol (6) as follows:

$$x^i(t+1) = w^i(t) + e^i(t) \quad \text{for all } i \in N,\ t \geq 0. \quad (19)$$

The main result of this section shows that, with probability 1, the bargaining protocol (17)–(19) converges to the core $C(v^{\max})$ of the robust game $\langle N, v^{\max}\rangle$, provided that $v(t) = v^{\max}$ happens infinitely often in time with probability 1. To establish this, we use some auxiliary results, as given in the following two lemmas.

The first lemma provides a property of the sequences $x^i(t)$ and shows that the errors $e^i(t)$ are diminishing.

Lemma 3: Let Assumptions 1 and 2 hold. Also, assume that each matrix $A(t)$ is doubly stochastic. Then, for the bargaining protocol (17)–(19), we have

(a) The sequence $\left\{\sum_{i=1}^n \|x^i(t+1) - x\|^2\right\}$ converges for every $x \in C(v^{\max})$.

(b) The errors $e^i(t)$ in (18) are such that $\sum_{t=0}^{\infty}\sum_{j=1}^{n} \|e^i(t)\|^2 < \infty$. In particular, $\lim_{t\to\infty} \|e^i(t)\| = 0$ for all $i \in N$.

Proof: By $x^i(t+1) = P_{X_i(t)}[w^i(t)]$ and by strictly non-expansive property of the Euclidean projection on a closed convex set $X_i(t)$ (see (8)), we have for any $i \in N$, $t \geq 0$ and $x \in X_i(t)$,

$$\|x^i(t+1) - x\|^2 \leq \|w^i(t) - x\|^2 - \|e^i(t)\|^2. \quad (20)$$

Under Assumptions 1 and 2, we have $C(v^{\max}) \subseteq C(v(t))$ for all $t \geq 0$. Furthermore, since $C(v(t)) = \cap_{i=1}^n X_i(t)$, it follows that $C(v^{\max}) \subseteq X_i(t)$ for all $i \in N$ and $t \geq 0$. Therefore, relation (20) holds for all $x \in C(v^{\max})$. Thus, by summing the relations in (20) over $i \in N$, we obtain for all $t \geq 0$ and $x \in C(v^{\max})$,

$$\sum_{i=1}^{n} \|x^i(t+1) - x\|^2 \leq \sum_{i=1}^{n} \|w^i(t) - x\|^2 - \sum_{i=1}^{n} \|e^i(t)\|^2). \quad (21)$$

By the definition of $w^i(t)$ in (17), using the stochasticity of $A(t)$ and the convexity of the squared norm, we obtain $\sum_{i=1}^n \|w^i(t) - x\|^2 \leq \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}(t)\right) \|x^j(t) - x\|^2$. Since $A(t)$ is doubly stochastic, we have $\sum_{i=1}^n a_{ij}(t) = 1$ for all j, implying $\sum_{i=1}^n \|w^i(t) - x\|^2 \leq \sum_{i=1}^n \|x^i(t) - x\|^2$. By substituting this relation in (21), we arrive at

$$\sum_{i=1}^{n} \|x^i(t+1) - x\|^2 \leq \sum_{i=1}^{n} \|x^i(t) - x\|^2 - \sum_{i=1}^{n} \|e^i(t)\|^2. \quad (22)$$

Relation (22) shows that the scalar sequence $\{\sum_{i=1}^n \|x^i(t+1) - x\|^2\}$ is non-increasing for any given $x \in C(v^{\max})$. Thus, the sequence must be convergent. Moreover, by summing the relations in (22) over $t = 0, \ldots, s$ and taking the limit as $s \to \infty$, we obtain $\sum_{t=0}^{\infty}\sum_{i=1}^n \|e^i(t)\|^2 \leq \sum_{i=1}^n \|x^i(0) - x\|^2$, implying that $\lim_{t\to\infty} e^i(t) = 0$ for all $i \in N$. ■

In our next result, we use the instantaneous average of players' allocations, defined as follows:

$$y(t) = \frac{1}{n}\sum_{j=1}^{n} x^j(t) \qquad \text{for all } t \geq 0.$$

The result shows that the difference between the bargaining payoff vector $x^i(t)$ for any player i and the average $y(t)$ of these payoffs converges to 0 as time goes to infinity. The proof essentially uses the line of analysis that has been employed in [16], where the sets $X_i(t)$ are static in time, i.e., $X_i(t) = X_i$ for all t. In addition, we also use the rate result for doubly stochastic matrices that has been established in [14].

Lemma 4: Let Assumptions 3 and 4 hold. Suppose that for the bargaining protocol (17)–(19) we have $\lim_{t\to\infty} \|e^i(t)\| = 0$ for all $i \in N$. Then, for every player $i \in N$ we have

$$\lim_{t\to\infty} \|x^i(t) - y(t)\| = 0, \qquad \lim_{t\to\infty} \|w^i(t) - y(t)\| = 0.$$

Proof: For any $t \geq s \geq 0$, define matrices

$$\Phi(t,s) = A(t)A(t-1)\cdots A(s+1)A(s),$$

with $\Phi(t,t) = A(t)$. Using the matrices $\Phi(t,s)$ and the expression for $x^i(t)$ in (19), we can relate the vectors $x^i(t)$ with the vectors $x^i(s)$ at a time s for $0 \leq s \leq t-1$, as follows:

$$\begin{aligned} x^i(t) &= \textstyle\sum_{j=1}^n [\Phi(t-1,s)]_{ij}\, x^j(s) \\ &+ \textstyle\sum_{r=s+1}^{t-1} \left(\sum_{j=1}^n [\Phi(t-1,r)]_{ij}\, e^j(r-1)\right) \\ &+ e^i(t-1). \end{aligned} \quad (23)$$

Under the doubly stochasticity of the matrices $A(t)$, using $y(t) = \frac{1}{n}\sum_{j=1}^n x^j(t)$ and relation (23), we obtain, for all $t \geq s \geq 0$

$$\begin{aligned} y(t) &= \tfrac{1}{n}\textstyle\sum_{j=1}^n x^j(s) \\ &+ \tfrac{1}{n}\textstyle\sum_{r=s+1}^{t} \left(\sum_{j=1}^n e^j(r-1)\right). \end{aligned} \quad (24)$$

By our assumption, we have $\lim_{t\to\infty} \|e^i(t)\| = 0$ for all i. Thus, for any $\epsilon > 0$, there is an integer $\hat{s} \geq 0$

such that $\|e^i(t)\| \le \epsilon$ for all $t \ge \hat{s}$ and all i. Using relations (23) and (24) with $s = \hat{s}$, we obtain for all i and $t \ge \hat{s}+1$,

$$\begin{aligned}\|x^i(t)-y(t)\| = \Bigg\| &\sum_{j=1}^n \Big([\Phi(t-1,\hat{s})]_{ij} - \frac{1}{n}\Big)x^j(\hat{s}) \\ &+ \sum_{r=\hat{s}+1}^{t-1}\sum_{j=1}^n \Big([\Phi(t-1,r)]_{ij} - \frac{1}{n}\Big)e^j(r-1) \\ &+ \Big(e^i(t-1) - \frac{1}{n}\sum_{j=1}^n e^j(t-1)\Big)\Bigg\| \\ \le &\sum_{j=1}^n \Big|[\Phi(t-1,\hat{s})]_{ij} - \frac{1}{n}\Big| \ \|x^j(\hat{s})\| \\ &+ \sum_{r=\hat{s}+1}^{t-1}\sum_{j=1}^n \Big|[\Phi(t-1,r)]_{ij} - \frac{1}{n}\Big| \|e^j(r-1)\| \\ &+ \|e^i(t-1)\| + \frac{1}{n}\sum_{j=1}^n \|e^j(t-1)\|.\end{aligned}$$

Since $\|e^i(t)\| \le \epsilon$ for all $t \ge \hat{s}$ and all i, it follows that

$$\begin{aligned}\|x^i(t)-y(t)\| \le &\sum_{j=1}^n \Big|[\Phi(t-1,\hat{s})]_{ij} - \frac{1}{n}\Big| \ \|x^j(\hat{s})\| \\ &+ \epsilon \sum_{r=\hat{s}+1}^{t-1}\sum_{j=1}^n \Big|[\Phi(t-1,r)]_{ij} - \frac{1}{n}\Big| \ + 2\epsilon.\end{aligned}$$

Under Assumptions 3 and 4, the following result holds for the matrices $\Phi(t,s)$, as shown in [13] (see there Corollary 1), for all $t \ge s \ge 0$:

$$\Big|[\Phi(t,s)]_{ij} - \frac{1}{n}\Big| \le \Big(1 - \frac{\alpha}{4n^2}\Big)^{\lceil \frac{t-s+1}{Q} \rceil - 2}.$$

Substituting the preceding estimate in the estimate for $\|x^i(t)-y(t)\|$, we obtain

$$\begin{aligned}\|x^i(t)-y(t)\| \le &\Big(1 - \frac{\alpha}{4n^2}\Big)^{\lceil \frac{t-\hat{s}}{Q} \rceil - 2} \sum_{j=1}^n \|x^j(\hat{s})\| \\ &+ n\epsilon \sum_{r=\hat{s}+1}^{t-1} \Big(1 - \frac{\alpha}{4n^2}\Big)^{\lceil \frac{t-r}{Q} \rceil - 2} + 2\epsilon.\end{aligned}$$

Letting $t \to \infty$, we see that

$$\begin{aligned}&\limsup_{t\to\infty} \|x^i(t)-y(t)\| \\ &\le n\epsilon \textstyle\sum_{r=\hat{s}+1}^{\infty} \left(1 - \frac{\alpha}{4n^2}\right)^{\lceil \frac{t-r}{Q} \rceil - 2} + 2\epsilon.\end{aligned}$$

Note that $\sum_{r=\hat{s}+1}^{\infty} \left(1 - \frac{\alpha}{4n^2}\right)^{\lceil \frac{t-r}{Q} \rceil - 2} < \infty$, which by the arbitrary choice of ϵ yields

$$\lim_{t\to\infty} \|x^i(t)-y(t)\| = 0 \qquad \text{for all } i \in N.$$

Now, we focus on $\sum_{i=1}^n \|w^i(t) - y(t)\|$. Since $w^i(t) = \sum_{j=1}^n a_{ij}(t)x^j(t)$ and since $A(t)$ is doubly stochastic, it can be seen that

$$\sum_{i=1}^n \|w^i(t)-y(t)\| \le \sum_{j=1}^n \|x^j(t)-y(t)\|.$$

Since $\lim_{t\to\infty} \|x^j(t)-y(t)\| = 0$ for all j, it follows that $\sum_{i=1}^n \|w^i(t)-y(t)\| \to 0$, thus implying $\|w^i(t)-y(t)\| \to 0$ for all $i \in N$. ∎

Lemma 4 captures the effects of the matrices $A(t)$ that represent players' neighbor-graphs. At the same time, Lemma 3 is basically a consequence of the projection property only. So far, the polyhedrality of the sets $X_i(t)$ has not been used at all. We now put all pieces together, namely Lemma 2 that exploits the polyhedrality of the bounding sets $X_i(t)$, Lemma 3 and Lemma 4. This brings us to the following result for the robust game.

Theorem 1: Consider a robust TU game $\langle N, v^{\max}\rangle$, and let Assumptions 1–4 hold. Also, assume that $\text{Prob}\,\{v(t) = v^{\max} \ \ i.o.\} = 1$, where $i.o.$ stands for infinitely often. Then, the players allocations $x^i(t)$ generated by bargaining protocol (17)–(19) converge with probability 1 to an allocation in the core $C(v^{\max})$, i.e., there is a random vector $\tilde{x} \in C(v^{\max})$ such that $\lim_{t\to\infty} \|x^i(t) - \tilde{x}\| = 0$ for all $i \in N$ with probability 1.

Proof: By Lemma 3, for each player $i \in N$, the sequence $\{\sum_{i=1}^n \|x^i(t)-x\|^2\}$ is convergent for every $x \in C(v^{\max})$ and the errors $e^i(t)$ are diminishing, i.e., $\|e^i(t)\| \to 0$. Then, by Lemma 4 we have $\|x^i(t) - y(t)\| \to 0$ for every i. Hence, for every $x \in C(v^{\max})$

$$\{\|y(t)-x\|\} \text{ is convergent.} \tag{25}$$

We want to show that $\{y(t)\}$ is convergent and that its limit is in the core $C(v^{\max})$ with probability 1. For this, we note that since $x^i(t+1) \in X_i(t)$, it holds, for all $t \ge 0$

$$\sum_{i=1}^n \text{dist}^2\left(y(t+1), X_i(t)\right) \le \sum_{i=1}^n \|y(t+1) - x^i(t+1)\|^2.$$

The preceding relation and $\|x^i(t)-y(t)\| \to 0$ for all $i \in N$ (cf. Lemma 4) imply that

$$\lim_{t\to\infty} \sum_{i=1}^n \text{dist}^2\left(y(t+1), X_i(t)\right) = 0.$$

Under Assumptions 1 and 2, by Lemma 2 we obtain, for all $t \ge 0$,

$$\text{dist}^2\left(y(t+1), C(v(t))\right) \le \mu \sum_{i=1}^n \text{dist}^2\left(y(t+1), X_i(t)\right).$$

By combining the preceding two relations we see that

$$\lim_{t\to\infty} \text{dist}^2\left(y(t+1), C(v(t))\right) = 0. \tag{26}$$

By our assumption, the event $\{v(t) = v^{\max} \text{ infinitely often}\}$ happens with probability 1. We now fix a realization $\{v_\omega(t)\}$ of the sequence $\{v(t)\}$ such that $v_\omega(t) = v^{\max}$ holds infinitely often (for infinitely many t's). Let $\{t_k\}$ be a sequence such that $v_\omega(t_k) = v^{\max}$ for all k. All the variables corresponding to the realization $\{v_\omega(t)\}$ are denoted by a subscript ω. By relation (25) the sequence $\{y_\omega(t)\}$ is bounded, therefore $\{y_\omega(t_k)\}$ is bounded. Without loss of generality (by passing to a subsequence of $\{t_k\}$ if necessary), we assume that $\{y_\omega(t_k)\}$ converges to some vector $\tilde{y}_\omega$, i.e., $\lim_{k\to\infty} y_\omega(t_k) = \tilde{y}_\omega$. This and Eq. (26) imply that $\tilde{y}_\omega \in C(v^{\max})$. Then, by relation (25), we have that $\{\|y_\omega(t) - \tilde{y}_\omega\|\}$ is convergent, from which we conclude that $\tilde{y}_\omega$ must be the unique accumulation point of the sequence $\{y_\omega(t)\}$, i.e.,

$$\lim_{t\to\infty} y_\omega(t) = \tilde{y}_\omega, \qquad \tilde{y}_\omega \in C(v^{\max}). \tag{27}$$

Since (27) is true for every realization ω such that $v_\omega(t) = v^{\max}$ holds infinitely often and since $\mathrm{Prob}\,\{v(t) = v^{\max} \;\; i.o.\} = 1$, it follows that the sequence $\{y(t)\}$ converges with probability 1 to a random point $\tilde{y} \in C(v^{\max})$. By Lemma 4 we have $\|x^i(t) - y(t)\| \to 0$ for every i. Thus, the sequences $\{x^i(t)\}, i = 1, \ldots, n$, converge with probability 1 to a common random point in the core $C(v^{\max})$. ∎

IV. Dynamic Average Game

When the core of the robust game is empty, the core of the instantaneous average game can provide a meaningful solution under some conditions on the distribution of the functions $v(t)$. Averaging coalitions' values is a way to capture players' *patience* and allocation *fairness*. Actually, if players have *memory*, they consider not only the coalitions' values at the present time, but also all past values. Past values are then used to filter out potential peaks of the values thus leading the players to accept less (more) than what they expect when peaks are positive (negative). Patience in bargaining games is a crucial aspect and has inspired a number of different models (see, for instance, Section 3.10.2 in [17] and Chapter 7, p.126 in [25]). The use of averaged (payoff) values intersects different areas of game theory. In two player repeated games with vector payoffs, for instance, one player looks for strategies that lead the long-run average payoff to converge to a so called approachable set, despite the influence of the opponent. This stream of literature, which builds upon the idea of "averaged payoff" goes under the name of "approachability theory" (see the seminal paper by D. Blackwell [5]) and the Blackwell's theorem is rewarded as the extension to an n-dimensional payoff space of the Von Neumann min-max theorem.

In what follows, we focus on the instantaneous average game associated with the dynamic TU game $\langle N, \{v(t)\}\rangle$. In the next sections, we define the instantaneous average game, we introduce a bargaining protocol for the game and investigate the convergence properties of the bargaining protocol.

A. Average Game and Bargaining Protocol

Consider a dynamic TU game $\langle N, \{v(t)\}\rangle$ with each $v(t)$ being a random characteristic function. With the dynamic game we associate a *dynamic average game* $\langle N, \{\bar{v}(t)\}\rangle$, where $\bar{v}(t)$ is the average of the characteristic functions $v(0), \ldots, v(t)$, i.e.,

$$\bar{v}(t) = \frac{1}{t+1}\sum_{k=0}^{t} v(k) \qquad \text{for all } t \geq 0.$$

An *instantaneous average game at time* t is the game $\langle N, \bar{v}(t)\rangle$. We let $C(\bar{v}(t))$ denote the core of the instantaneous average game at time t and let $\bar{X}_i(t)$ denote the bounding set of player i for the instantaneous game $\langle N, \bar{v}(t)\rangle$, i.e., for all $i \in N$ and $t \geq 0$,

$$\bar{X}_i(t) = \Big\{x \in \mathbb{R}^n \mid e_N' x = \bar{v}_N(t),\, e_S' x \geq \bar{v}_S(t) \\ \text{for all } S \subset N \text{ s. t. } i \in S\Big\}. \tag{28}$$

Note that $\bar{X}_i(0) = X_i(0)$ for all $i \in N$ since $\bar{v}(0) = v(0)$. In what follows, we *assume that* $\bar{X}_i(t)$ *are nonempty* for all $i \in N$ and all $t \geq 0$.

In this setting, the bargaining process for the players is given by, for all $i \in N$ and $t \geq 0$,

$$x^i(t+1) = P_{\bar{X}_i(t)}\left[\sum_{j=1}^{n} a_{ij}(t) x^j(t)\right], \tag{29}$$

where $a_{ij}(t) \geq 0$ is a scalar weight that player i assigns to the proposed allocation $x^j(t)$ received from player j at time t. The initial allocations $x^i(0)$, $i \in N$, are selected randomly and independently of $\{v(t)\}$. Regarding the weights $a_{ij}(t)$, recall that these weights are reflective of the players' neighbor-graph: $a_{ij}(t) = 0$ for all $j \notin \mathcal{N}_i(t)$, where $\mathcal{N}_i(t)$ is the set of neighbors of player i (including himself) at time t, while we may have $a_{ij}(t) \geq 0$ only for $j \in \mathcal{N}_i(t)$.

B. Assumptions and Preliminaries

In this section, we provide our assumptions for the average game, and discuss some auxiliary results that we need later on in the convergence analysis of the bargaining protocol. Regarding the random characteristic functions $v(t)$ we use the following assumption.

Assumption 5: We have $\lim_{t\to\infty} \bar{v}(t) = v^{\text{mean}}$ for some $v^{\text{mean}} \in \mathbb{R}^m$ and $v_N(t) = v_N^{\text{mean}}$ for all t, both with probability 1.

Assumption 5 basically says that $\{v(t)\}$ is ergodic, and the grand coalition value $v_N(t)$ is constant with probability 1. The assumption is satisfied, for example, when $\{v_S(t)\}$ is an independent identically distributed with a finite expectation $\mathsf{E}[v_S(t)]$ for all $S \subset N$.

We refer to the TU game $\langle N, v^{\text{mean}}\rangle$ as *average game*, which is well defined under Assumption 5. We let $C(v^{\text{mean}})$ be the core of the average game and $\bar{X}_i$ be the bounding set for player i in the game, i.e.,

$$C(v^{\text{mean}}) = \Big\{x \in \mathbb{R}^n \mid e_N' x = v_N^{\text{mean}},\ e_S' x \geq v_S^{\text{mean}} \\ \text{for all nonempty } S \subset N\Big\},$$

$$\bar{X}_i = \Big\{x \in \mathbb{R}^n \mid e_N' x = v_N^{\text{mean}}, e_S' x \geq v_S^{\text{mean}} \\ \text{for all } S \subset N \text{ with } i \in S\Big\}. \quad (30)$$

The average core $C(v^{\text{mean}})$ lies in the hyperplane $H = \{x \in \mathbb{R}^n \mid e_N' x = v_N^{\text{mean}}\}$. Hence, the dimension of the core is at most $n-1$. We will in fact assume that the dimension of the average core is $n-1$, by requiring the existence of a point that satisfies all other inequalities defining the core as strict inequalities. Specifically, we make use of the following assumption.

Assumption 6: There exists a vector $\hat{z} \in C(v^{\text{mean}})$ such that $e_S'\hat{z} > v_S^{\text{mean}}$ for all nonempty $S \subset N$.

Assumption 6 basically says that $\hat{z}$ is in the relative interior of the core $C(v^{\text{mean}})$ of the average game and the core $C(v^{\text{mean}})$ has dimension $n-1$. This assumption is used to ensure that the cores $C(\bar{v}(t))$ have a meaningful "limit set", as $t \to \infty$, and therefore, something can be said about the convergence of the bargaining protocol (29). In particular, the following result is an immediate consequence of Assumption 6 and the polyhedrality of the cores $C(\bar{v}(t))$.

Lemma 5: Let Assumptions 5 and 6 hold. Then, with probability 1, for every z in relative interior of $C(v^{\text{mean}})$ there exists t_z large enough such that z is in the relative interior of $C(\bar{v}(t))$ for all $t \geq t_z$ with probability 1.

Proof: Let z be in the relative interior of $C(v^{\text{mean}})$ which exists by Assumption 6. Thus, $e_N' z = v_N^{\text{mean}}$ and $e_S' z > v_S^{\text{mean}}$ for all $S \subset N$. By Assumption 5, with probability 1 we have $\bar{v}_N(t) = v_N^{\text{mean}}$ and $\bar{v}_S(t) \to v_S^{\text{mean}}$ for $S \subset N$. Hence, $\bar{v}_N(t)'z = (v_N^{\text{mean}})'z$ for all t with probability 1. Furthermore, there exists a random time t_z large enough so that with probability 1, $e_S' z > \bar{v}_S(t)$ for all $S \subset N$ and all $t \geq t_z$, implying that z is in the relative interior of $C(\bar{v}(t))$ for all $t \geq t_z$ with probability 1. ∎

Lemma 5 shows that the sets $C(\bar{v}(t)$ and $C(v^{\text{mean}})$ have the same dimension for large enough t with probability 1. In particular, this lemma implies that the cores $C(\bar{v}(t))$ are nonempty with probability 1 for all t sufficiently large.

Aside from Lemma 5, in our convergence analysis of the bargaining protocol, we use two additional results. One of them is the well-known super-martingale convergence theorem due to Robbins and Siegmund [21] (it can also be found in [18], Chapter 2.2, Lemma 11).

Theorem 2: Let $\{V_k\}, \{g_k\}$, and $\{h_k\}$ be non-negative random scalar sequences. Let F_k be the σ-algebra generated by $V_1, \ldots, V_k, g_1, \ldots g_k, h_1, \ldots h_k$. Suppose that almost surely, $\mathsf{E}[V_{k+1} \mid F_k] \leq V_k - g_k + h_k$ for all k, and $\sum_k h_k < \infty$ almost surely. Then, almost surely both the sequence $\{V_k\}$ converges to a non-negative random variable and $\sum_k g_k < \infty$.

The other result that we use is pertinent to two nonempty polyhedral sets whose description differs only in the right-hand side vector.

Lemma 6: Let X_b and $X_{\tilde{b}}$ be two polyhedral sets given by $X_b = \{x \in \mathbb{R}^n \mid Bx \leq b\}$ and $X_{\tilde{b}} = \{x \in \mathbb{R}^n \mid Bx \leq \tilde{b}\}$, where B is an $m \times n$ matrix and $b, \tilde{b} \in \mathbb{R}^m$. Then, there is a scalar $L > 0$ such that for every $b, \tilde{b} \in \mathbb{R}^m$ for which $X_b \neq \emptyset$ and $X_{\tilde{b}} \neq \emptyset$, we have, for any $x \in \mathbb{R}^n$

$$\text{dist}(x, X_b) \leq \text{dist}(x, X_{\tilde{b}}) + L\|b - \tilde{b}\|,$$

where the constant L depends on the matrix B.

Proof: We start from the triangle inequality $\|x - y\| \leq \|x - z\| + \|z - y\|$ for arbitrary vectors x, y, and z. By taking the infimum over $y \in X_b$, we have $\text{dist}(x, X_b) \leq \|x - z\| + \text{dist}(z, X_b)$ for all $x, z \in \mathbb{R}^n$. Then, by letting z be the projection of x on the set $X_{\tilde{b}}$, we obtain $\text{dist}(x, X_b) \leq \text{dist}(x, X_{\tilde{b}}) + \text{dist}(P_{X_{\tilde{b}}}[x], X_b)$. To estimate $\text{dist}(P_{X_{\tilde{b}}}[x], X_b)$, we use the following relation

$$\text{dist}(y, X_b) \leq L\|b - \tilde{b}\| \qquad \text{for any } y \in X_{\tilde{b}},$$

(see [6], 3.2.5 Corollary, pages 258–259). Since $P_{X_{\tilde{b}}}[x]$ is in the set $X_{\tilde{b}}$, we have $\text{dist}(P_{X_{\tilde{b}}}[x], X_b) \leq L\|b - \tilde{b}\|$, and the desired relation follows. ∎

C. Convergence to Core of Average Game

We show the convergence of the bargaining protocol to the core of the average game. In our analysis, we find it convenient to re-write the bargaining protocol (29) in an equivalent form by separating a linear and a non-linear term. The linear term is given by vector $\bar{w}^i(t)$:

$$\bar{w}^i(t) = \sum_{j=1}^n a_{ij}(t) x^j(t) \quad \text{for all } i \in N \text{ and } t. \quad (31)$$

The non-linear term is expressed by the error

$$\bar{e}^i(t) = P_{\bar{X}_i(t)}[\bar{w}^i(t)] - \bar{w}^i(t). \quad (32)$$

Now, using relations (31) and (32), we can rewrite (29) as follows, for all $i \in N$ and all $t \geq 0$:

$$x^i(t+1) = \bar{w}^i(t) + \bar{e}^i(t). \tag{33}$$

We first show some basic properties of the players' allocations by using the preceding equivalent description of the bargaining protocol (29). This properties hold under the doubly stochasticity of the weights $a_{ij}(t)$ that comprise the matrix $A(t)$.

Lemma 7: Let Assumptions 5 and 6 hold. Also, let the matrices $A(t)$ be doubly stochastic. Then, for bargaining protocol (31)–(33), we have with probability 1:

(a) The sequence $\{\sum_{i=1}^n \|x^i(t+1) - z\|^2\}$ converges for every z in the relative interior of $C(v^{\text{mean}})$.

(b) The errors $\bar{e}^i(t)$ in (18) are such that $\sum_{t=0}^{\infty} \sum_{j=1}^n \|\bar{e}^i(t)\|^2 < \infty$. In particular, $\lim_{t\to\infty} \|\bar{e}^i(t)\| = 0$ for all $i \in N$.

Proof: Let $\text{rint}Y$ denote the relative interior of a set Y. Let $z \in \text{rint}C(v^{\text{mean}})$ be arbitrary and fixed for the rest of the proof. By Lemma 5, there exists t_z large enough such that $z \in \text{rint}C(\bar{v}(t))$ for all $t \geq t_z$ with probability 1. Since $C(\bar{v}(t)) = \cap_{i=1}^n X_i(t)$, it follows that $z \in \bar{X}_i(t)$ for all $i \in N$ and $t \geq t_z$ with probability 1. From $x^i(t+1) = \bar{w}^i(t) + \bar{e}^i(t)$ (cf. (33)), the definition of $\bar{e}^i(t)$ in (32), and the projection property given in Eq. (8), we have with probability 1, for $i \in N$ and all $t \geq t_z$,

$$\|x^i(t+1) - z\|^2 \leq \|\bar{w}^i(t) - z\|^2 - \|\bar{e}^i(t)\|^2.$$

Summing these relations over $i \in N$, and using $\bar{w}^i(t) = \sum_{j=1}^n a_{ij} x^i(t)$, the convexity of the norm and the fact that $A(t)$ is doubly stochastic, we obtain for all $t \geq t_z$ with probability 1,

$$\sum_{i=1}^n \|x^i(t+1) - z\|^2 \leq \sum_{j=1}^n \|x^j(t) - z\|^2 - \sum_{j=1}^n \|\bar{e}^i(t)\|^2.$$

Applying the super-martingale convergence theorem (Theorem 2) (with an index shift), we see that the sequence $\{\sum_{i=1}^n \|x^i(t+1) - z\|^2\}$ is convergent and $\sum_{t=t_z}^{\infty} \sum_{j=1}^n \|\bar{e}^i(t)\|^2 < \infty$ with probability 1. Hence, the result in part (b) follows. ∎

We observe that Lemma 4 applies to protocol (31)–(33) in view of the analogy of the description of the protocol in (17)–(19) and the protocol in (31)–(33). We will re-state this lemma for an easier reference, but without the proof since it is almost the same as that of Lemma 4 (the proof in essence depends mainly on the matrices $A(k)$).

Lemma 8: Let Assumptions 3 and 4 hold. Suppose that for the bargaining protocol (31)–(33) we have $\lim_{t\to\infty} \|\bar{e}^i(t)\| = 0$ for all i with probability 1. Then, for every player $i \in N$ it holds with probability 1,

$$\lim_{t\to\infty} \|x^i(t) - y(t)\| = 0, \qquad \lim_{t\to\infty} \|\bar{w}^i(t) - y(t)\| = 0,$$

where $y(t) = \frac{1}{n} \sum_{j=1}^n x^j(t)$.

We are now ready to prove the convergence of the bargaining protocol. We show this by combining the protocol properties established in Lemmas 7 and 8.

Theorem 3: Let Assumptions 3–6 hold. Then, the bargaining protocol (31)–(33) converges to a random allocation in the core $C(v^{\text{mean}})$ of the average game with probability 1, i.e., $\lim_{t\to\infty} \|x^i(t) - \tilde{z}\| = 0$ for all i and some random vector $\tilde{z} \in C(v^{\text{mean}})$ with probability 1.

Proof: By Lemma 7, with probability 1, the sequence $\{\sum_{i=1}^n \|x^i(t) - z\|^2\}$ is convergent for every z in the relative interior of $C(v^{\text{mean}})$ and $\|\bar{e}^i(t)\| \to 0$ for each player $i \in N$. By Lemma 8, with probability 1 we have

$$\lim_{t\to\infty} \|x^i(t) - y(t)\| = 0 \qquad \text{for every } i \in N. \tag{34}$$

Hence, with probability 1, for every $z \in \text{rint}C(v^{\text{mean}})$

$$\{\|y(t) - z\|\} \text{ is convergent.} \tag{35}$$

We next show that $\{y(t)\}$ has accumulation points in the core $C(v^{\text{mean}})$ with probability 1. Since $x^i(t+1) \in \bar{X}_i(t)$, it follows

$$\begin{aligned} &\sum_{i=1}^n \text{dist}^2\left(y(t+1), \bar{X}_i(t)\right) \\ &\leq \sum_{i=1}^n \|y(t+1) - x^i(t+1)\|^2. \end{aligned} \tag{36}$$

Relations (36) and (34) imply, with probability 1

$$\lim_{t\to\infty} \sum_{i=1}^n \text{dist}^2\left(y(t+1), \bar{X}_i(t)\right) = 0. \tag{37}$$

The bounding sets $\bar{X}_i$ are nonempty by Assumption 6 and the fact $C(v^{\text{mean}}) \subset \bar{X}_i$ for all i, while $\bar{X}_i(t)$ are assumed nonempty (see the discussion after relation (28)). Furthermore, since $\bar{X}_i$ and $\bar{X}_i(t)$ are two polyhedral sets whose description differs only in the right-hand side vector, from Lemma 6 it follows that, for each $i \in N$, there is a scalar $L_i > 0$ such that

$$\begin{aligned} &\text{dist}(y(t+1), \bar{X}_i) \leq \text{dist}(y(t+1), \bar{X}_i(t)) \\ &+L_i\|\bar{v}(t) - v^{\text{mean}}\| \qquad \text{for all } t \geq 0. \end{aligned}$$

By letting $t \to \infty$ and using relations (37) and $\bar{v}(t) \to v^{\text{mean}}$ (Assumption 5), we see that for all $i \in N$ with probability 1,

$$\lim_{t\to\infty} \text{dist}(y(t+1), \bar{X}_i) = 0. \tag{38}$$

In view of relation (35), the sequence $\{y(t)\}$ is bounded with probability 1, so it has accumulation points with probability 1. By relation (38), all accumulation points of $\{y(t+1)\}$ lie in the set $\bar{X}_i$ for every $i \in N$ with probability 1. Therefore, the accumulation points of $\{y(t)\}$ must lie in the intersection $\cap_{i\in N} \bar{X}_i$ with probability 1. Since $\cap_{i\in N} \bar{X}_i = C(v^{\text{mean}})$, we conclude that all accumulation points of $\{y(t)\}$ lie in the core $C(v^{\text{mean}})$ with probability 1. Furthermore, according to relation (35) we have that, for any point $z \in \mathrm{rint}C(v^{\mathrm{mean}})$, the accumulation points of the sequences $\{y(t)\}$ are at the same (random) distance from z with probability 1. Since the accumulation points are in the set $C(v^{\mathrm{mean}})$, it follows that $\{y(k)\}$ is convergent with probability 1 and its limit point is in the core $C(v^{\mathrm{mean}})$ with probability 1. Now, since $\|x^i(t) - y(t)\| \to 0$ with probability 1 for all i (see (34)), the sequences $\{x^i(t)\}$, $i \in N$, have the same limit point as the sequence $\{y(t)\}$. Thus, the sequences $\{x^i(t)\}$, $i \in N$, converge to a common (random) point in $C(v^{\mathrm{mean}})$ with probability 1. ■

Theorem 3 shows the convergence of the allocations generated by the bargaining protocol in (31)–(33). The convergence relies on the properties of the matrices and the connectivity of the players' neighbor graphs, as reflected in Assumptions 3 and 4. It also critically depends on the fact that the core of the average game $C(v^{\mathrm{mean}})$ has dimension $n-1$ and that all bounding sets $\bar{X}_i(t)$ and, hence the cores $C(\bar{v}(t))$, lie in the same hyperplane with probability 1, the hyperplane defined through the constant value of the grand coalition.

V. Numerical Illustrations

In this section, we report some numerical simulations. We consider coalitional TU games with 3 players, so the number of possible nonempty coalitions is $m = 7$. We consider two different scenarios as shown respectively in rows I and II in Table I. The columns of Table I enumerate the coalitions. In each scenario, the characteristic functions $v_S(t)$ are generated independently with identical uniform distribution over an interval. Specifically, we suppose that the values of single player coalitions $\{1\}$ and $\{2\}$ are uncertain within the given interval. All the other coalitions' values are fixed and equal to zero except for the grand coalition, which has value 10 in both scenarios I and II. The two scenarios differ in that the core $C(v^{\max})$ of the robust game is nonempty in scenario I and empty in scenario II. For scenario I, we simulate the convergence behavior of the bargaining protocol (17)–(19) for the robust game, while for scenario II, we simulate bargaining protocol (31)–(33) for the average game.

For each scenario, we run 50 different Monte Carlo trajectories each one having 100 iterations (the figures below show only the first 60 iterations). The number of iterations is chosen long enough to show the convergence of the protocols. All plots include the sampled average and sampled variance for the 50 trajectories that were simulated. Each trajectory in each scenario is generated by starting with the same initial allocations, which are given by $x^1(0) = [10\ 0\ 0]'$, $x^2(0) = [0\ 10\ 0]'$, and $x^3(0) = [0\ 0\ 10]'$. The sampled average is computed for each time $t = 1, \ldots, 100$, by fixing the time t and computing the average value of the 50 trajectory sample values for that time. The sampled variance is computed as the variance of the samples with respect to their sampled average.

	$v_{\{1\}}$	$v_{\{2\}}$	$v_{\{3\}}$	$v_{\{i,j\}}$ for all i, j	$v_{\{1,2,3\}}$
I	$[4, 7]$	$[0, 3]$	0	0	10
II	$[4, 9]$	$[0, 5]$	0	0	10

TABLE I
Coalitions' values for the two simulations scenarios.

Regarding the players' neighbor-graphs, we assume that the graphs are deterministic but time-varying. The graphs for the times $t = 0, 1, 2$ are as follows: player 2 and 3 connected at time $t = 0$ (see Figure 4(a)), then player 3 and 1 connected at time $t = 1$ (Figure 4(b)), and finally player 1 and 2 connected at time $t = 3$ (Figure 4(c)). These graphs are then repeated consecutively in the same order. In this way, the players' neighbor-graph is connected every 3 time units (Assumption 4 is satisfied with $Q = 2$).

The matrices that we associate with these three graphs, are respectively given by:

$$A(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad A(1) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix},$$

$$A(2) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These matrices are also repeated in the same order for the rest of the time. Thus, at any time t, the matrix $A(t)$ is doubly stochastic, with positive diagonal, and every positive entry bounded below by $\frac{1}{2}$, so Assumption 3 is satisfied with $\alpha = \frac{1}{2}$. All simulations are carried out with MATLAB on an Intel(R) Core(TM)2 Duo, CPU P8400 at 2.27 GHz and a 3GB of RAM. The run time of each simulation is around 90 seconds.

A. Simulation Scenario I

In this scenario, the coalitions' values are generated as given in row I of Table I. In particular, at each

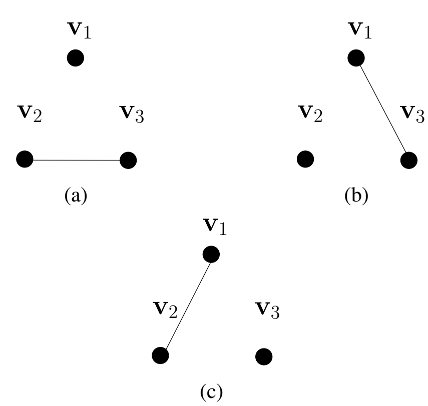


Fig. 4. Topology of players' neighbor-graph at three distinct times $t = 0$, 1 and 2.

time t, the value $v_{\{1\}}(t)$ is chosen randomly in the interval $[4, 7]$ with uniform probability independently of the other times. Similarly, the values $v_{\{2\}}(t)$ are generated in the interval $[0, 3]$. The grand coalition value is fixed to 10 at all times, and the other coalition values are 0. With this data, we consider the allocations as generated by players $i = 1, 2, 3$ according to the bargaining protocol in (17)–(19) for the robust game as reported in Subsection V-A1. We then consider the bargaining protocol in (31)–(33) for the average game in Subsection V-A2.

1) Robust game: For this specific example, the characteristic function $v^{\max}$ for the robust game is obtained by considering the highest possible coalition values (see row I of Table I), which results in $v^{\max} = [7\,3\,0\,0\,0\,0\,0\,10]'$. The resulting core of the robust game is given by

$$\begin{aligned} C(v^{\max}) = \{x \in \mathbb{R}^3 : \, & x_1 \geq 7, \, x_2 \geq 3, \, x_3 \geq 0, \\ & x_1 + x_2 \geq 0, \, x_1 + x_3 \geq 0, \\ & x_2 + x_3 \geq 0, \, x_1 + x_2 + x_3 = 10\}. \end{aligned}$$

This core contains a single point, namely $[7\,3\,0]'$. To ensure that $v(t) = v^{\max}$ infinitely often, as required by Theorem 1 for the convergence of the protocol, we adopt the following randomization mechanism. At each time $t = 1, \ldots, 100$, we flip a coin and if the outcome is "head" (probability 1/2), the coalitions' values $v_{\{1\}}(t)$ and $v_{\{2\}}(t)$ are extracted from the intervals $[4, 7]$ and $[0, 3]$, respectively, with uniform probability independently of the other times. If the outcome of the coin flip is "tail", then we assume that the robust game realizes and take $v(t) = v^{\max}$.

We next present the results obtained by the Monte Carlo runs for the bargaining protocol in (17)–(19). An illustration of a typical run with the allocations generated in periods $t = 0, 1, 2, 3$ is shown below:

$$\begin{array}{ll} v(0) = [6.8\ 2.7 \ldots 10]' & x^1(0) = [10\ 0\ 0]' \\ v(1) = [7\ 3 \ldots 10]' & x^1(1) = [10\ 0\ 0]' \\ v(2) = [4.4\ 1.1 \ldots 10]' & x^1(2) = [5\ 2.5\ 2.5]' \\ v(3) = [7\ 3 \ldots 10]' & x^1(3) = [7\ 1.5\ 1.5]' \end{array}$$

$$\begin{array}{ll} x^2(0) = [0\ 10\ 0]' & x^3(0) = [0\ 0\ 10]' \\ x^2(1) = [0\ 5\ 5]' & x^3(1) = [0\ 5\ 5]' \\ x^2(2) = [0\ 5\ 5]' & x^3(2) = [5\ 2.5\ 2.5]' \\ x^2(3) = [2.5\ 3.75\ 3.75]' & x^3(3) = [5\ 2.5\ 2.5]'. \end{array}$$

Recall that the initial allocations of the players are $x^1(0) = [10\ \ 0\ \ 0]'$, $x^2(0) = [0\ \ 10\ \ 0]'$, and $x^3(0) = [0\ \ 0\ \ 10]'$. At time $t = 1$, bargaining involves player 2 and 3 who update the allocations respectively as $x^2(1) = [0\ 5\ 5]'$ and $x^3(1) = [0\ 5\ 5]'$. These allocations are feasible for their bounding sets so the projections on these sets are not performed. At time $t = 2$, the bargaining involves player 1 and 3 who update their allocations, respectively, as $x^1(2) = [5\ \ 2.5\ \ 2.5]'$ and $x^3(2) = [5\ \ 2.5\ \ 2.5]'$. Again, these allocations are feasible for their bounding sets and the projections are not performed. Finally, at time $t = 3$, the bargaining involves player 1 and 2 who update their allocations resulting in $x^1(3) = [7\ 1.5\ 1.5]'$ and $x^2(3) = [2.5\ 3.75\ 3.75]'$. Notice that $x^1(3)$ is obtained after player 1 projects onto his bounding set.

In Figures 5 and 6, we report our simulation results for the average of the sample trajectories obtained by Monte Carlo runs. Figure 5 shows the sampled average and variance of the allocations $x^i(t)$, $i = 1, 2, 3$ per iteration t. In accordance with the convergence result of Theorem 1, the sampled averages of the players' allocations $x^i(t)$ converge to the same point, namely $x = [7\,3\,0]'$ which is in the core of the robust game

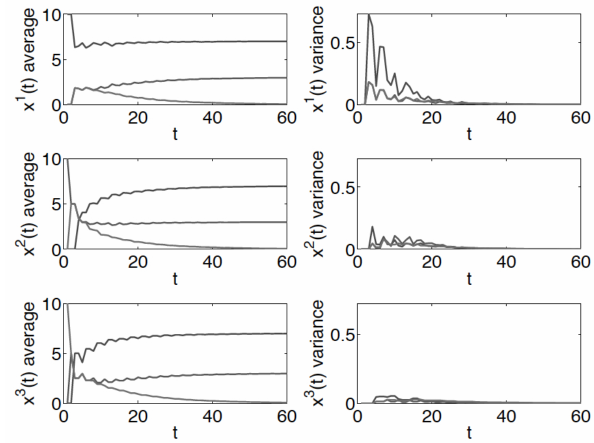


Fig. 5. Sampled average (left) and variance (right) of players' allocations $x^i(t)$, $i = 1, 2, 3$ for bargaining protocol (17)–(19) and the robust game associated with the data in row I of Table I. Sampled averages of the allocations $x^i(t)$ converge to the same point $\tilde{x} = [7\,3\,0]' \in C(v^{\max})$, while sampled variances go rapidly to zero.

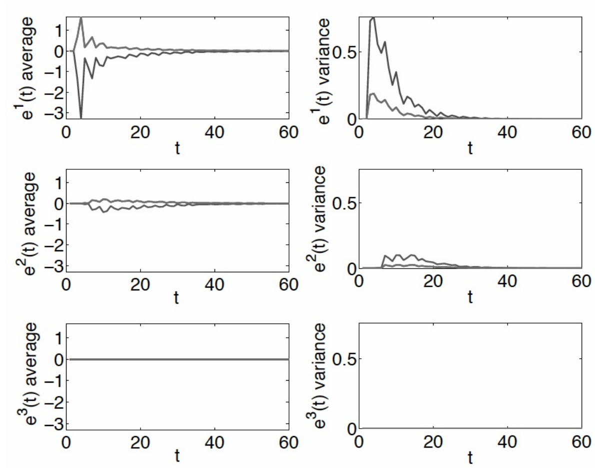


Fig. 6. Sampled average (left) and sampled variance (right) of the errors $e^i(t)$, $i = 1, 2, 3$, for the bargaining protocol (17)–(19) and the robust game associated with the data in row I of Table I. Sampled averages and the variances of the errors $e^i(t)$ converge to zero.

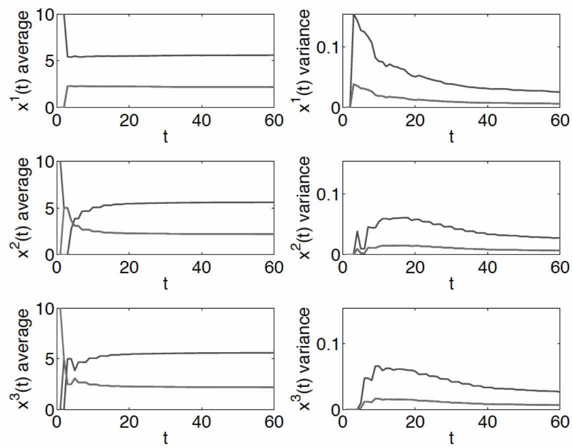


Fig. 7. Sampled averages (left) and variances (right) of players' allocations $x^i(t)$, $i = 1, 2, 3$, obtained by bargaining protocol (31)–(33) for the average game associated with the data in row I of Table I. Sampled averages converge to point $\tilde{x} = [5.6\ 2.2\ 2.2]' \in C(v^{\mathrm{mean}})$, which is the average of the limit points of the 50-sample trajectories.

$C(v^{\max})$. Figure 6 shows that the sample average and sampled variance of the errors $e^i(t)$ converge to 0, as expected in view of Lemma 3(b).

2) Average game: For scenario I data as given in row I of Table I, we consider the average TU game and its corresponding bargaining protocol (31)–(33). We have $v^{\mathrm{mean}} = [5.5\ 1.5\ 0\ 0\ 0\ 0\ 10]'$ and the core $C(v^{\mathrm{mean}})$ given by

$$C(v^{\mathrm{mean}}) = \{x \in \mathbb{R}^3 : x_1 \geq 5.5,\ x_2 \geq 1.5,\ x_3 \geq 0,\ x_1 + x_2 \geq 0,\ x_1 + x_3 \geq 0,\ x_2 + x_3 \geq 0,\ x_1 + x_2 + x_3 = 10\}.$$

In Figure 7 we depict our simulation results for bargaining protocol (31)–(33) for the average game. Figure 7 shows that the sampled average of the allocations $x^i(t)$, $i = 1, 2, 3$ converge to a common point $\tilde{x} = [5.6\ 2.2\ 2.2]'$ which belongs to the core $C(v^{\mathrm{mean}})$ of the average game, as guaranteed by Theorem 3. The sampled variance does not converge to zero as the common limit point of the allocations $x^i(t)$ can be different for different runs.

B. Simulation Scenario II

Here, we report the simulation results obtained by the bargaining protocol (31)–(33) for the average game corresponding to the data in row II of Table I. In this case the core of the robust game is empty, so we do not consider the robust game. The average game has characteristic function $v^{\mathrm{mean}} = [6.5\, 2.5\, 0\, 0\, 0\, 0\, 10]'$ and its core is

$$C(v^{\mathrm{mean}}) = \{x \in \mathbb{R}^3 : x_1 \geq 6.5,\ x_2 \geq 2.5,\ x_3 \geq 0,\ x_1 + x_2 \geq 0,\ x_1 + x_3 \geq 0,\ x_2 + x_3 \geq 0,\ x_1 + x_2 + x_3 = 10\}.$$

Figure 8 shows the results for the average game obtained in our simulations. In Figure 8, we report the sampled averages of the players' allocations $x^i(t)$, $i = 1, 2, 3$, obtained by bargaining protocol (31)–(33). In accordance with Theorem 3, the players' allocations converge to an allocation that lies in the core $C(v^{\mathrm{mean}})$ of the average game, precisely to the point $\tilde{x} = [6.6\ 2.6\ 0.8]' \in C(v^{\mathrm{mean}})$. Here, again, the sampled variance of the allocations does not diminish as the limit point is different for different runs.

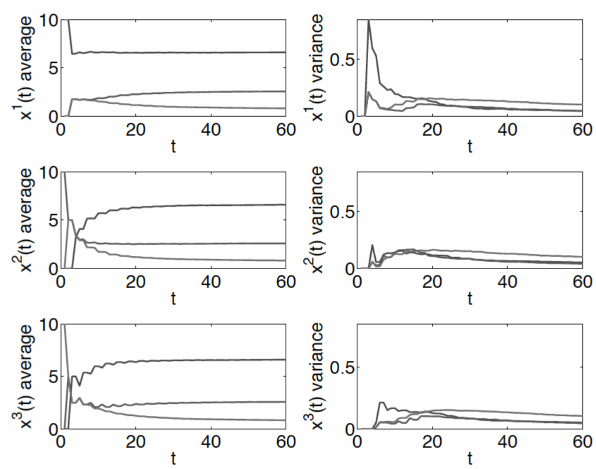


Fig. 8. Sampled averages (left) and variances (right) of the players' allocations $x^i(t)$, $i = 1, 2, 3$ obtained by the bargaining protocol (31)–(33) for the average game associated with the data in row II of Table I. Sampled averages converge to point $\tilde{x} = [6.6\ 2.6\ 0.8]' \in C(v^{\mathrm{mean}})$, which is the average of the limit points of the 50-sample trajectories.

VI. Conclusions

For a sequence of TU games, each with a random characteristic function, we design a decentralized allocation process defined over a communication graph of players. The proposed bargaining scheme is proven to converge, with probability 1, in either a robust game setting or an average game setting, under mild assumptions on the communication topology and the stochastic properties of the random characteristic function. The key properties that distinguish this work from the existing work on dynamic games are: (1) the introduction of a time-varying communication graph, termed players' neighbor-graph, over which the bargaining protocol takes place; and (2) the distributed bargaining protocol for players' allocation updates subject to local information exchange with neighboring players.

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