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MULTICRITERIA OPTIMAL DESIGN OF CONTINUOUS CIRCULAR PLATES

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Abstract. *The paper is devoted to a quite general version of the multicriteria optimal (minimum volume) design of axisymmetric circular plates. The constitutive material is considered as elastic perfectly plastic without any ductility limit and the actions are assumed as quasi-statically variable within a given load domain. In the design problem formulation different resistance criteria are considered, in order to investigate all the possible structural limit responses, and for each one a suitably chosen safety factor is chosen. The optimal design problem is formulated as the search for the minimum structure volume according with a statical approach. The features of the optimal structures will be studied through the relevant Euler-Lagrange equations. A numerical application is presented utilizing an appropriate discretization of the minimum volume problem.*

Sommario. *Si presenta una formulazione generale del progetto ottimale multicriterio di minimo volume di piastre circolari assialsimmetriche. Al materiale si assegna un comportamento costitutivo elastico perfettamente plastico e non si impone alcun limite sulla sua duttilità. Le azioni si assumono come variabili in modo quasi-statico e ci si riferisce ad un opportuno dominio dei carichi all'interno del quale trovano definizione le loro varie possibili combinazioni. Nella formulazione del progetto compaiono vincoli relativi a diversi criteri di resistenza in modo tale che la struttura ottimale verifichi la sicurezza per diverse opportunamente scelte condizioni limite; per ciascuna di esse si sceglie un adeguato coefficiente di sicurezza. La formulazione del progetto ottimale è rivolta alla ricerca del minimo volume della struttura sulla base di un approccio statico. Le proprietà della struttura ottimale sono determinate attraverso la deduzione e l'interpretazione delle equazioni di Eulero-Lagrange. In ambito computazionale il problema di minimo volume viene discretizzato ed applicato ad una semplice struttura.*

1 INTRODUCTION

As it is well known, more and more the interest of the designers is devoted to the use of structure constituted by materials exhibiting good and comparable elastic properties in traction and compression, exhibiting adequate resistance reserve capacity after reaching the yield stress and possessing suitably wide ductility properties. Actually, in such a case, even exhibiting different behaviours, the structure can be optimally designed for the different load conditions which can occur during its lifetime and, furthermore, the mechanical problem can be easily formulated.

The definition of appropriate models able to describe the real load conditions for the structure is a fundamental topic in order to obtain a good design. In general, it is possible to assume that the load can be represented as a combination of fixed and variable load, and such position is certainly adequate to the structure considered in the present paper. In particular, it can be assumed that the variable actions doesn't involve the dynamic behaviour and, as a consequence, they reduce to be quasi statically variable; so, even if the load history is usually unknown, it is possible to define a suitable admissible load domain. Furthermore, in the great part of engineering applications, the variable loads can be modelled as cyclic loads. As a consequence, in the following we will refer to combinations of fixed and cyclic loads.

Under such conditions, if the elastic limit is overpassed but the load intensities doesn't exceed suitable limits, the elastic shakedown theory provides useful tools in studying the behaviour of the relevant structure [1,7]. On the contrary, if the load multiplier exceeds the elastic shakedown limit, then the structure is addressed towards a collapse condition, either due to a plastic shakedown behaviour (alternating plasticity) or to a ratchetting behaviour (incremental collapse). Finally, for further increasing values of the loads, the structure is eventually addressed towards an instantaneous collapse condition.

The above defined structure behaviours can be represented in the so called Bree-like diagram, whose knowledge is of crucial importance in order to establish if the assigned structure/load system safely operates with potentially different load conditions. On denoting with ξ_0 and ξ_c two appropriate multipliers of the fixed and the cyclic load respectively, in the plane (ξ_0, ξ_c) the relevant diagram has the form plotted in Figure 1.

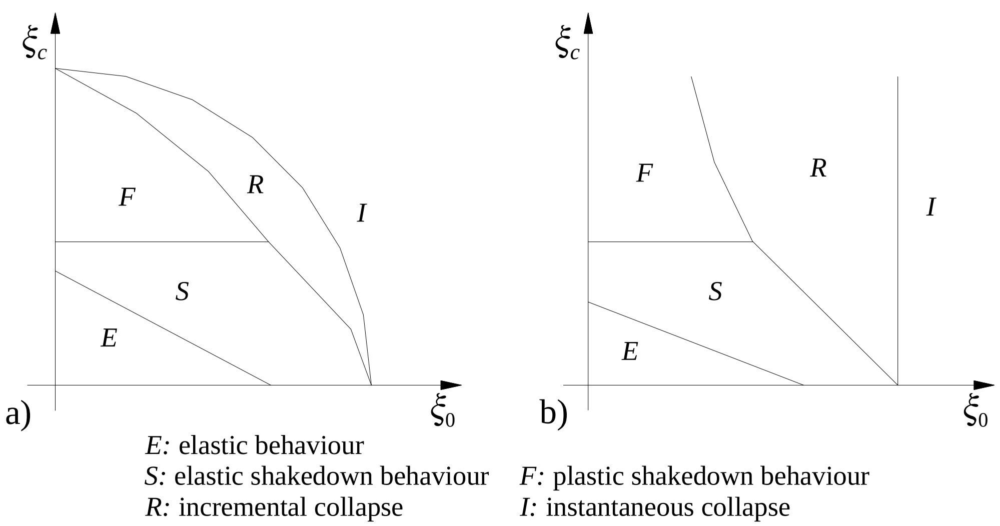

Figure 1: Typical Bree-like diagrams: a) mechanical cyclic load; b) thermal cyclic load.

Imposing suitable limit values for the loads multipliers, depending on the special chosen resistance criterion, several authors have formulated the optimal design problem for structures constituted by elastic plastic material and subjected to loads variable inside a given domain and they have investigated both theoretical and computational aspects [8,27].

In all the cases each optimal structure shows a safe behaviour just with regard to the special limit state accounted by the chosen resistance criterion, but no information can be deduced in order to ascertain the safety requisites with respect to the other possible limit states of the designed structure.

Such an occurrence can be avoided by making use of suitable multicriteria optimal design formulations, in which the optimal structure is constrained to simultaneously satisfy different criteria with appropriate safety factors [28,32].

The present paper is mainly devoted to the formulation of a complete multicriteria optimal design problem imposing simultaneously constraints on the purely elastic limit, on the elastic shakedown limit, on the plastic shakedown limit and on the instantaneous collapse limit, obviously accounting for different suitably selected safety factor values. In particular, the optimal design of axisymmetric circular plates in bending will be formulated according with the described criteria.

In this way it is possible to take into account constraint on the purely elastic behaviour and on the elastic shakedown of the structure subjected just to the fixed load and limited variable actions (with appropriate different amplifiers) in order to ensure a good performance in serviceability conditions, to investigate the structural behaviour above the elastic shakedown region considering simultaneously plastic shakedown limit conditions and the instantaneous collapse limit conditions for exceptional intensity variable loads.

A continuous elastic perfectly plastic model will be adopted for the plate; the loads will be treated as arbitrarily and quasi-statically variable inside a given load domain and in particular, the acting loads will be described as the combination of fixed and cyclic loads; furthermore, the restrictive hypothesis that the cyclic load is a perfect one, namely for each basic load condition an opposite one exists in the load space, will be accepted. In author's opinion, the relevant formulation devoted to the case of circular plates possesses a practical interest cause the frequent use of such a structure in industrial and civil engineering.

The optimal design problem is formulated as the search for the minimum volume design whose elastic limit load multiplier, elastic shakedown limit load multiplier, plastic shakedown limit load multiplier and ultimate limit load multiplier are not smaller than suitably assigned values. A statical approach is utilized. The Euler-Lagrange equations related to the optimization problem are deduced and interpreted in order to point out the special features of the optimal design.

The minimum volume design can also be formulated on the grounds of a kinematical approach and it is possible to prove that it provides the same solution [16,17]. Such formulation is here skipped for the sake of brevity.

At the same way the optimal design, under the described behavioural constraints can be formulated as the search for the maximum load multiplier for the structure of assigned volume and performed following a statical or a kinematical approach. Even in this case it can be proved that under adequate conditions the two resulting problem provide the same solution as in the case of the minimum volume design [16,17].

In order to utilize the optimal design formulation for computational aims it is necessary to introduce a suitable discretization. In particular, in the present paper a discretization procedure, applied to the minimum volume design problem formulated utilizing the statical approach, is proposed. Even this last formulation is treated as problem in the calculus of variations and the related Kuhn-Tucker equations are discussed.

A simple numerical application concludes the paper; in particular, a steel plate is designed taking into account an elastic shakedown behaviour in serviceability conditions and a limit state of impending collapse in the cases of exceptional very high loads. The obtained results allow us to confirm the theoretical expectations in terms of behavioural features of the obtained optimal design.

2 THE ELASTIC PLASTIC MODEL

Let us consider an axisymmetric solid circular plate of radius R, referred to a cylindrical co-ordinate system r, ϑ, z with the origin in the centre of the plate middle plane and with the $z-$axis normal to this plane and oriented downward (Figure 2). The plate has variable thickness $H(r)$ and let us suppose that it is subjected to a radially symmetric, transverse, mechanical, variable load $P(r,t)$, being $t \geq 0$ the time variable.

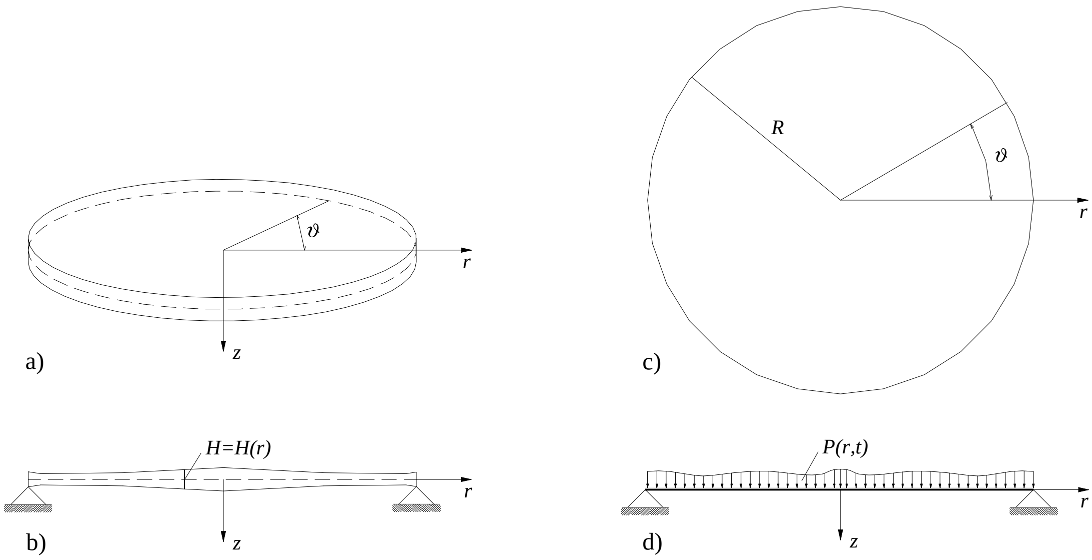

Figure 2: Circular plate a) reference system; b), c), d) geometry and acting load.

The classical Love-Kirchhoff plate kinematical model is adopted, together with the assumption of small displacements and negligible inertia and viscous forces; as a consequence, t is not a physical time but just a pseudo-time and $P(r,t)$ is a quasi-statical load. Furthermore, the material exhibits an elastic perfectly plastic behaviour and no ductility limit are considered.

Let us assume now the simplifying hypothesis that both elastic and plastic strains are linearly distributed through the plate thickness, such that the total (bending) curvatures can be decomposed into an elastic and a plastic part, namely:

$$\boldsymbol{K} = \begin{vmatrix} K_r \\ K_\vartheta \end{vmatrix} = \begin{vmatrix} K_r^e \\ K_\vartheta^e \end{vmatrix} + \begin{vmatrix} K_r^p \\ K_\vartheta^p \end{vmatrix} = \boldsymbol{K}^e + \boldsymbol{K}^p \quad \text{in } [0,R], \quad t \geq 0 \tag{1}$$

The equations governing the plate response are as follow:

- *compatibility*

$$K_r = -w'' \quad \text{in } [0,R], \quad t \geq 0 \tag{2a}$$

$$K_\vartheta = -\frac{1}{r} w' \quad \text{in } [0,R], \quad t \geq 0 \tag{2b}$$

- *equilibrium*

$$(rM_r)'' - (M_\vartheta)' + rP = 0 \quad \text{in } [0,R], \quad t \geq 0 \tag{2c}$$

- *constitutive law*

$$\boldsymbol{M} = \mathbf{l}\,\boldsymbol{B}\boldsymbol{K}^e \quad \text{in } [0,R], \quad t \geq 0 \tag{2d}$$

$$\boldsymbol{\Phi}(\boldsymbol{M}) = \tilde{\boldsymbol{N}}\boldsymbol{M} - M_y\,\boldsymbol{\eta} \leq \mathbf{0} \quad \text{in } [0,R], \quad t \geq 0 \tag{2e}$$

$$\dot{\boldsymbol{\lambda}} \geq \mathbf{0}, \quad \tilde{\boldsymbol{\Phi}}\dot{\boldsymbol{\lambda}} = 0 \quad \text{in } [0,R], \quad t \geq 0 \tag{2f}$$

$$\dot{\boldsymbol{K}}^p = \boldsymbol{N}\dot{\boldsymbol{\lambda}} \quad \text{in } [0,R], \quad t \geq 0 \tag{2g}$$

together with the following mechanical and kinematical boundary conditions (BCs):

$$\text{at } r = 0: \; w'(0) = 0 \text{ and } Q(0) = \left[(rM_r)' - M_\vartheta\right]'_{r=0} = 0 \tag{3a}$$

$$\text{at } r = R: \; w(R) = 0 \text{ and either} \tag{3b}$$

$$M_r(R) = 0 \text{ and } w'(R) \text{ is free (simply supported plate) or} \tag{3c}$$

$$w'(R) = 0 \text{ and } M_r(R) \text{ is free (clamped plate).} \tag{3d}$$

In Eqns. (2-3) w is the deflection, $\tilde{\boldsymbol{M}} = |M_r \;\; M_\vartheta|$ the bending moment vector and Q the shear force, $\mathbf{l} = EH^3/12(1-\nu^2)$ the plate-bending stiffness, with E Young's modulus and ν Poisson's ratio, while matrix

$$\boldsymbol{B} = \begin{vmatrix} 1 & \nu \\ \nu & 1 \end{vmatrix} \tag{4}$$

represents the specific stiffness. Furthermore, $M_y = \sigma_y H^2/4$ is the plate yield bending moment, and σ_y the material yield stress. Assuming, for the sake of simplicity, the Tresca yield criterion, the matrix $\boldsymbol{N}$ of the unit external normals to the yield surface assumes the form:

$$\boldsymbol{N} = \begin{vmatrix} \tilde{\boldsymbol{N}}_r \\ \tilde{\boldsymbol{N}}_\vartheta \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 1 & -1 & -1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} \tag{5a}$$

while

$$\tilde{\boldsymbol{\eta}} = \left| 1 \quad 1 \quad 1 \quad 1 \quad 1/\sqrt{2} \quad 1/\sqrt{2} \right| \tag{5b}$$

is the vector of the specific plastic resistances. Finally, $\boldsymbol{\Phi}$ is the plastic potential vector and $\dot{\boldsymbol{\lambda}}$ the plastic activation vector.

As already stated, the load $P(r,t)$ is a function of the parameter t and usually it is an unknown variable action, namely the real load history can't be expected. On the other side it is possible to establish reasonable limits for the load intensity during the lifetime of the structure (such a criterion is largely used by most of the structural international codes). Furthermore, we are interested to evaluate the structural behaviour for increasing load values. As a consequence, let us denote by $\xi > 0$ a suitable load multiplier and let us indicate by $\xi P(r,t)$ the relevant amplified load as a function of time t. Let us assume that $\xi P(r,t)$ is represented as a path arbitrarily shaped within a given domain Ω of the load space, namely, any chosen path within Ω is an admissible load history. Taking into account that the load function is a suitable combination of different single load conditions potentially simultaneously acting (basic load conditions), the domain Ω can be shaped as a convex hyperpolyhedron, whose vertices $\xi P_i(r)$, $i \in I(b) \equiv \{1,2,\ldots b\}$, are the basic loads [33]. As a consequence, any load inside Ω can be modelled as a linear combination of the basic loads, namely:

$$\xi P(r,t) = \xi \sum_{i=1}^{b} \beta_i(t) P_i(r) \quad \text{in } [0,R], \quad t \geq 0 \tag{6a}$$

where the coefficients β_i must satisfy the following conditions:

$$\beta_i(t) \geq 0, \quad \sum_{i=1}^{b} \beta_i(t) = 1 \quad \text{in } [0,R], \quad t \geq 0 \tag{6b,c}$$

Due to the presence of the load multiplier ξ, the load domain Ω just possess a constant shape, while it can expand or shrink homothetically on increasing or decreasing ξ, respectively.

The above position extremely simplify the problem solution; actually, by virtue of Eqns. (6), it is sufficient to satisfy Eqns. (2) in the discrete space $I(b)$ of the basic loads, instead of in the continuous space t.

3 CYCLIC LOADS AND STEADY-STATE ELASTIC PLASTIC STRUCTURAL RESPONSE

Let us introduce now a further simplifying hypothesis, i.e. that the acting load is defined as a combination of fixed mechanical load and cyclic mechanical load. According to the previously introduced loading scheme, let us denote with $P_0(r)$ and $P_{ci}(r)$, $\forall i \in I(b)$, the reference fixed mechanical load and the reference cyclic mechanical load, respectively. Furthermore, let us assume the hypothesis that the cyclic load is a perfect one, namely for each basic load condition an opposite one exists in the load space (actually, a generic cyclic load can be always decomposed in the sum of a fixed load and a perfect cyclic load).

Finally, according to the previously defined symbols, $\xi_0 \, P_0(r)$ and $\xi_c \, P_{ci}(r)$, $\forall i \in I(b)$, represent the amplified fixed and perfect cyclic loads, being $\xi_0 \geq 0$ and $\xi_c \geq 0$ the fixed and the cyclic load multipliers, respectively.

Due to such a load combination, after a transient phase which also depends on the initial conditions and on the special real loading path, the structure eventually exhibits a steady-state response which is characterized by the same periodicity features as the loads and it is independent of the above referred initial conditions [34,38].

Actually, for the described load conditions the steady-state response of the plate, in terms of generalized stresses and strains, just depends on the sequence of the b amplified basic load conditions $P_i(r)=\xi_0\, P_0(r)+\xi_c\, P_{ci}(r)$, $\forall\, i\in I(b)$, obtained as combination of the amplified reference fixed and cyclic loads. As a consequence, the elastic plastic steady-state response of the plate in the cycle can be obtained by an analysis effected just for the b basic load conditions.

For the purposes of the present paper it can be very useful to consider the steady-state elastic plastic response of the plate subjected just to the amplified perfect cyclic loads $\bar{\xi}_c\, P_{ci}$, $\forall\, i\in I(b)$, where $\bar{\xi}_c$ is a selected cyclic load multiplier such that $0\le\bar{\xi}_c<\xi_c^I$ results, being ξ_c^I the ultimate purely cyclic load multiplier (Figure 3).

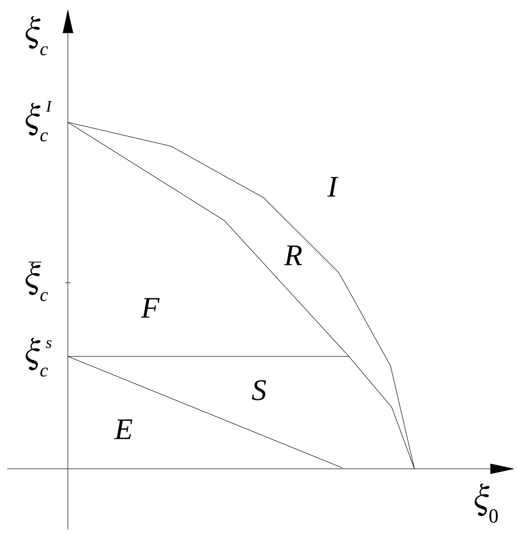

Figure 3: Selected cyclic load multiplier on the Bree diagram.

According to the previously described hypotheses, the elastic plastic steady-state behaviour of the plate to the cyclic loading is described by the following equations:

$$K^E_{rci}+\left(w^E_{ci}\right)''=0 \quad \text{in } (0,R) + \text{BCs on } w^E_{ci}, \quad \forall\, i\in I(b) \tag{7a}$$

$$K^E_{\vartheta ci}+\frac{1}{r}\left(w^E_{ci}\right)'=0 \quad \text{in } (0,R) + \text{BCs on } w^E_{ci}, \quad \forall\, i\in I(b) \tag{7b}$$

$$\left(rM^E_{rci}\right)''-\left(M^E_{\vartheta ci}\right)'+rP_{ci}=0 \quad \text{in } (0,R) + \text{BCs on } \boldsymbol{M}^E_{ci}, \quad \forall\, i\in I(b) \tag{7c}$$

$$\boldsymbol{M}^E_{ci}=\mathbf{I}\,\boldsymbol{B}\boldsymbol{K}^E_{ci} \quad \text{in } [0,R] \tag{7d}$$

$$Z_{ci} \equiv -\boldsymbol{\Phi}_{ci} = M_y \boldsymbol{\eta} - \bar{\xi}_c \boldsymbol{N} \boldsymbol{M}^E_{ci} + \boldsymbol{S} \boldsymbol{\lambda}_{ci} \quad \text{in } [0,R], \quad \forall i \in I(b) \tag{8a}$$

$$\boldsymbol{Z}_{ci} \geq \boldsymbol{0}, \quad \boldsymbol{\lambda}_{ci} \geq \boldsymbol{0}, \quad \boldsymbol{Z}^T_{ci} \boldsymbol{\lambda}_{ci} = 0 \quad \text{in } [0,R], \quad \forall i \in I(b) \tag{8b}$$

$$\boldsymbol{K}^p_{ci} = \boldsymbol{N} \boldsymbol{\lambda}_{ci} \quad \text{in } [0,R] \tag{8c}$$

In Eqns. (7) $\boldsymbol{K}^E_{ci}$, w^E_{ci} and $\boldsymbol{M}^E_{ci}$ are the purely elastic response to the reference cyclic loads in terms of (bending) curvatures, deflection and bending moment field vector, respectively. In Eqns. (8) $\boldsymbol{Z}_{ci}$ represents the opposite of the plastic potential $\boldsymbol{\Phi}_{ci}$ and $\boldsymbol{S}$ is a time independent symmetric structural matrix which transforms the plastic activation intensities $\boldsymbol{\lambda}_{ci}$ into plastic potentials.

If $0 \leq \bar{\xi}_c \leq \xi^s_c$ is assumed, being ξ^s_c (Figure 4a) the elastic shakedown limit load multiplier, Eqns. (7)-(8) admit the vanishing solution $\boldsymbol{\lambda}_{ci} = \boldsymbol{0}$, $\forall i \in I(b)$, and in the steady-state phase the whole structure behaves elastically. If $\xi^s_c < \bar{\xi}_c < \xi^I_c$ is assumed (Figure 4b), Eqns. (7)-(8) admit a non-vanishing solution $\boldsymbol{\lambda}_{ci}$ and the plate exhibits an elastic plastic behaviour.

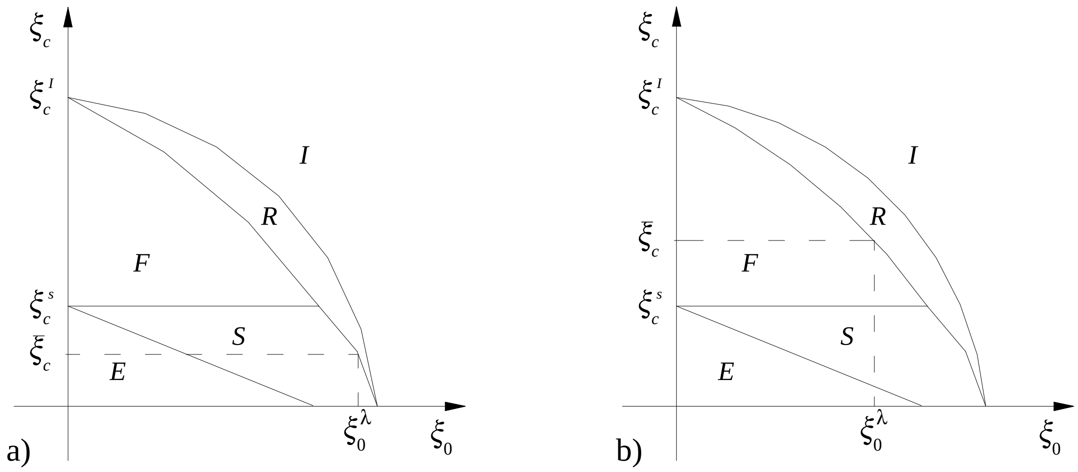

Figure 4: Load multipliers at the limit state of: a) elastic shakedown; b) plastic shakedown.

Taking into account Eqns. (7)-(8), for a selected value of the cyclic load multiplier $\bar{\xi}_c$, the fixed load multiplier related to the elastic/plastic shakedown limit, ξ^{l}_0, can be determined solving the following problem [39]:

$$K^E_{r0} + \left(w^E_0\right)'' = 0 \quad \text{in } (0,R) \text{ + BCs on } w^E_0 \tag{9a}$$

$$K^E_{\vartheta 0} + \frac{1}{r}\left(w^E_0\right)' = 0 \quad \text{in } (0,R) \text{ + BCs on } w^E_0 \tag{9b}$$

$$\left(rM_{r0}^{E}\right)'' - \left(M_{\vartheta 0}^{E}\right)' + rP_0 = 0 \quad \text{in } \left(0,R\right) \text{ + BCs on } \boldsymbol{M}_0^E \tag{9c}$$

$$\boldsymbol{M}_0^E = \mathbf{I}\,\boldsymbol{B}\boldsymbol{K}_0^E \quad \text{in } \left[0,R\right] \tag{9d}$$

$$\xi_0^{l}\left(\overline{\xi}_c\right) = \max_{(\xi_0,\boldsymbol{\lambda}_0)} \xi_0 \qquad \text{subject to:} \tag{10a}$$

$$\boldsymbol{Z}_{ci}\left(\overline{\xi}_c\right) - \xi_0 \tilde{\boldsymbol{N}}\boldsymbol{M}_0^E + \boldsymbol{S}\boldsymbol{\lambda}_0 \geq \mathbf{0} \quad \text{in } \left[0,R\right], \quad \forall\, i \in I\left(b\right) \tag{10b}$$

$$\boldsymbol{\lambda}_0 \geq \mathbf{0} \quad \text{in } \left[0,R\right] \tag{10c}$$

In Eqns. (9)-(10) $\boldsymbol{K}_0^E$, w_0^E and $\boldsymbol{M}_0^E$ are the purely elastic response to the reference fixed loads in terms of (bending) curvatures, deflection and bending moment field vector, respectively; $\boldsymbol{\lambda}_0$ represents a fictitious plastic activation intensity vector related to the elastic/plastic shakedown limit. If Eqns. (10) provide the vanishing solution $\boldsymbol{\lambda}_{ci} = \mathbf{0}$, $\forall\, i \in I\left(b\right)$, they become a classic elastic shakedown limit load multiplier problem, otherwise the elastic plastic response to the purely cyclic load is involved and the problem becomes a plastic shakedown limit load multiplier one.

4 MULTICRITERIA OPTIMAL DESIGN OF MINIMUM VOLUME

Let us consider the plate described at the previous section and let us choose the plate thickness $H\left(r\right)$ as design variable. A typical design of the relevant plate can be performed for any choice of the function $H\left(r\right)$ defined in $\left[0,R\right]$. With the plate subjected to a loading scheme as previously described, we are interested to determine the/a special design $H\left(r\right)$ of minimum volume and whose elastic limit load, elastic shakedown limit load, plastic shakedown limit load and ultimate limit load are not smaller than appropriate assigned values, i.e. the structure basic loads (combinations of fixed and perfect cyclic loads) alternatively amplified by $\overline{\xi}\overline{\xi}_0^E$ and $\overline{\xi}\overline{\xi}_c^E$, $\overline{\xi}\overline{\xi}_0^S$ and $\overline{\xi}\overline{\xi}_c^S$, $\overline{\xi}\overline{\xi}_0^F$ and $\overline{\xi}\overline{\xi}_c^F$, $\overline{\xi}\overline{\xi}_0^I$ and $\overline{\xi}\overline{\xi}_c^I$, where $\overline{\xi}_0^E$, $\overline{\xi}_c^E$, $\overline{\xi}_0^S$, $\overline{\xi}_c^S$, $\overline{\xi}_0^F$, $\overline{\xi}_c^F$, $\overline{\xi}_0^I$, $\overline{\xi}_c^I$ and $\overline{\xi}$ are positive assigned parameters, behaves elastically, eventually shakes down, exhibits an alternating plasticity behaviour or prevents the instantaneous collapse, respectively.

On the ground of the statical approach the minimum volume multicriteria optimal design can be formulated as follows:

$$\min_{\left(H,\xi,w_0^E,w_{ci}^E,\boldsymbol{K}_0^E,\boldsymbol{K}_{ci}^E,\boldsymbol{M}_0^E,\boldsymbol{M}_{ci}^E,\boldsymbol{\lambda}_0^S,\boldsymbol{\lambda}_{ci}^F,\boldsymbol{\lambda}_0^F,\boldsymbol{\lambda}_{0i}^I\right)} \frac{1}{2\pi} V = \tag{11a}$$

$$\min_{\left(H,\xi,w_0^E,w_{ci}^E,\boldsymbol{K}_0^E,\boldsymbol{K}_{ci}^E,\boldsymbol{M}_0^E,\boldsymbol{M}_{ci}^E,\boldsymbol{\lambda}_0^S,\boldsymbol{\lambda}_{ci}^F,\boldsymbol{\lambda}_0^F,\boldsymbol{\lambda}_{0i}^I\right)} \int_0^R rH\left(r\right)\mathrm{d}\,r$$

subject to:

$$H\left(r\right) \geq 0 \quad \text{in } \left[0,R\right] \tag{11b}$$

$$\bar{\xi} - \xi \le 0 \tag{11c}$$

$$K_{r0}^{E} + \left(w_0^E\right)'' = 0 \quad \text{in } (0,R) \text{ + BCs on } w_0^E \tag{11d}$$

$$K_{\vartheta 0}^{E} + \frac{1}{r}\left(w_0^E\right)' = 0 \quad \text{in } (0,R) \text{ + BCs on } w_0^E \tag{11e}$$

$$\left(rM_{r0}^{E}\right)'' - \left(M_{\vartheta 0}^{E}\right)' + rP_0 = 0 \quad \text{in } (0,R) \text{ + BCs on } \boldsymbol{M}_0^E \tag{11f}$$

$$\boldsymbol{M}_0^E = \mathbf{l}\, \boldsymbol{BK}_0^E \quad \text{in } [0,R] \tag{11g}$$

$$K_{rci}^{E} + \left(w_{ci}^E\right)'' = 0 \quad \text{in } (0,R) \text{ + BCs on } w_{ci}^E, \quad \forall i \in I(b) \tag{11h}$$

$$K_{\vartheta ci}^{E} + \frac{1}{r}\left(w_{ci}^E\right)' = 0 \quad \text{in } (0,R) \text{ + BCs on } w_{ci}^E, \quad \forall i \in I(b) \tag{11i}$$

$$\left(rM_{rci}^{E}\right)'' - \left(M_{\vartheta ci}^{E}\right)' + rP_{ci} = 0 \quad \text{in } (0,R) \text{ + BCs on } \boldsymbol{M}_{ci}^E, \quad \forall i \in I(b) \tag{11j}$$

$$\boldsymbol{M}_{ci}^E = \mathbf{l}\, \boldsymbol{BK}_{ci}^E \quad \text{in } [0,R], \quad \forall i \in I(b) \tag{11k}$$

$$-\boldsymbol{\Phi}_i^E = M_y \boldsymbol{\eta} - \tilde{\boldsymbol{N}}\left(\xi\bar{\xi}_0^E \boldsymbol{M}_0^E + \xi\bar{\xi}_c^E \boldsymbol{M}_{ci}^E\right) \ge \mathbf{0} \quad \text{in } [0,R], \quad \forall i \in I(b) \tag{11ℓ}$$

$$-\boldsymbol{\Phi}_i^S = M_y \boldsymbol{\eta} - \tilde{\boldsymbol{N}}\left(\xi\bar{\xi}_0^S \boldsymbol{M}_0^E + \xi\bar{\xi}_c^S \boldsymbol{M}_{ci}^E\right) + \boldsymbol{S\lambda}_0^S \ge \mathbf{0} \quad \text{in } [0,R], \quad \forall i \in I(b) \tag{11m}$$

$$\boldsymbol{\lambda}_0^S \ge \mathbf{0} \quad \text{in } [0,R] \tag{11n}$$

$$\boldsymbol{Z}_{ci} \equiv -\boldsymbol{\Phi}_{ci}^F = M_y \boldsymbol{\eta} - \tilde{\boldsymbol{N}}\xi\bar{\xi}_c^F \boldsymbol{M}_{ci}^E + \boldsymbol{S\lambda}_{ci}^F \quad \text{in } [0,R], \quad \forall i \in I(b) \tag{11o}$$

$$\boldsymbol{Z}_{ci} \ge \mathbf{0}, \quad \boldsymbol{\lambda}_{ci}^F \ge \mathbf{0}, \quad \tilde{\boldsymbol{Z}}_{ci}\boldsymbol{\lambda}_{ci}^F = 0 \quad \text{in } [0,R], \quad \forall i \in I(b) \tag{11p}$$

$$-\boldsymbol{\Phi}_i^F = \boldsymbol{Z}_{ci} - \xi\bar{\xi}_0^F \tilde{\boldsymbol{N}}\boldsymbol{M}_0^E + \boldsymbol{S\lambda}_0^F \ge \mathbf{0} \quad \text{in } [0,R], \quad \forall i \in I(b) \tag{11q}$$

$$\boldsymbol{\lambda}_0^F \ge \mathbf{0} \quad \text{in } [0,R] \tag{11r}$$

$$-\boldsymbol{\Phi}_i^I = M_y \boldsymbol{\eta} - \tilde{\boldsymbol{N}}\left(\xi\bar{\xi}_0^I \boldsymbol{M}_0^E + \xi\bar{\xi}_c^I \boldsymbol{M}_{ci}^E\right) + \boldsymbol{S\lambda}_{0i}^I \ge \mathbf{0} \quad \text{in } [0,R], \quad \forall i \in I(b) \tag{11s}$$

$$\boldsymbol{\lambda}_{0i}^I \ge \mathbf{0} \quad \text{in } [0,R], \quad \forall i \in I(b) \tag{11t}$$

where, besides the already defined symbols, V is the plate volume, $\boldsymbol{\Phi}_i^E$, $\boldsymbol{\Phi}_i^S$, $\boldsymbol{\Phi}_{ci}^F$, $\boldsymbol{\Phi}_i^F$ and $\boldsymbol{\Phi}_i^I$, $\forall i \in I(b)$, are the plastic potentials related to the elastic behaviour, to the elastic shakedown behaviour, to the plastic shakedown for purely cyclic load, to the plastic shakedown for a combination of fixed and cyclic loads, and to the instantaneous collapse, respectively. Finally, $\boldsymbol{\lambda}_{ci}^{F}$, $\forall\, i \in I(b)$, are plastic activation intensities related to the cyclic loads in the region of the plastic shakedown and $\boldsymbol{\lambda}_{0}^{S}$, $\boldsymbol{\lambda}_{0}^{F}$, $\boldsymbol{\lambda}_{0i}^{I}$, $\forall\, i \in I(b)$, are fictitious plastic activation intensities related to the elastic shakedown limit, to the plastic shakedown limit and to the instantaneous collapse, respectively.

Any solution to the constraint Eqns. (11b-t) specifies a feasible design $H(r)$ admitting elastic, shakedown and ultimate statical load multipliers not smaller than the above mentioned values. The equation $\xi = \overline{\xi}$ holds for the optimal design, as it will be shown later on.

The Euler-Lagrange equations related to problem (11) provide necessary conditions for the optimal design and useful information about the obtained design features. Applying the Lagrange multiplier method, denoting by $c \geq 0$, $\xi\tilde{\boldsymbol{\mu}}_0(r) = \xi|\mu_{r0} \quad \mu_{\vartheta 0}|$, $\xi\, u_0(r)/r$, $\xi\tilde{\boldsymbol{\chi}}_0(r) = \xi|\chi_{r0} \quad \chi_{\vartheta 0}|$, $\xi\tilde{\boldsymbol{\mu}}_{ci}(r) = \xi|\mu_{rci} \quad \mu_{\vartheta ci}|$, $\xi\, u_{ci}(r)/r$, $\xi\tilde{\boldsymbol{\chi}}_{ci}(r) = \xi|\chi_{rci} \quad \chi_{\vartheta ci}|$, $\boldsymbol{\lambda}_i^E(r) \geq \mathbf{0}$, $\boldsymbol{\lambda}_i^S(r) \geq \mathbf{0}$, $\boldsymbol{f}^S \leq \mathbf{0}$, $\boldsymbol{y}_i \geq \mathbf{0}$, a, $\boldsymbol{\lambda}_i^F(r) \geq \mathbf{0}$, $\boldsymbol{f}^F \leq \mathbf{0}$, $\boldsymbol{\lambda}_i^I(r) \geq \mathbf{0}$ and $\boldsymbol{f}_i^I \leq \mathbf{0}$ the Lagrange multipliers (with ξ a scaling factor not subjected to variations), the Lagrangian reads:

$$
\begin{aligned}
\Psi = {} & \gamma \int_0^R r H(r)\, dr + c\left(\overline{\xi} - \xi\right) \qquad (12) \\
& + \int_0^R \left\{ (\xi\mu_{r0}) \left[K_{r0}^E + \left(w_0^E\right)'' \right] + (\xi\mu_{\vartheta 0}) \left[K_{\vartheta 0}^E + \frac{1}{r}\left(w_0^E\right)' \right] \right\} r\, \mathrm{d}r \\
& + \int_0^R (\xi u_0) \left[\left(r M_{r0}^E\right)'' - \left(M_{\vartheta 0}^E\right)' + r P_0 \right] \mathrm{d}r + \int_0^R (\xi\tilde{\boldsymbol{\chi}}_0) \left[\boldsymbol{M}_0^E - \mathbf{I}\, \boldsymbol{B} \boldsymbol{K}_0^E \right] r\, \mathrm{d}r \\
& + \sum_{i=1}^{b} \int_0^R \left\{ (\xi\mu_{rci}) \left[K_{rci}^E + \left(w_{ci}^E\right)'' \right] + (\xi\mu_{\vartheta ci}) \left[K_{\vartheta ci}^E + \frac{1}{r}\left(w_{ci}^E\right)' \right] \right\} r\, \mathrm{d}r \\
& + \sum_{i=1}^{b} \int_0^R (\xi u_{ci}) \left[\left(r M_{rci}^E\right)'' - \left(M_{\vartheta ci}^E\right)' + r P_{ci} \right] \mathrm{d}r \\
& + \sum_{i=1}^{b} \int_0^R (\xi\tilde{\boldsymbol{\chi}}_{ci}) \left[\boldsymbol{M}_{ci}^E - \mathbf{I}\, \boldsymbol{B} \boldsymbol{K}_{ci}^E \right] r\, \mathrm{d}r \\
& - \sum_{i=1}^{b} \int_0^R \tilde{\boldsymbol{\lambda}}_i^E \left[M_y \boldsymbol{\eta} - \tilde{\boldsymbol{N}} \left(\xi\overline{\xi}_0^E \boldsymbol{M}_0^E + \xi\overline{\xi}_c^E \boldsymbol{M}_{ci}^E \right) \right] r\, \mathrm{d}r \\
& - \sum_{i=1}^{b} \int_0^R \tilde{\boldsymbol{\lambda}}_i^S \left[M_y \boldsymbol{\eta} - \tilde{\boldsymbol{N}} \left(\xi\overline{\xi}_0^S \boldsymbol{M}_0^E + \xi\overline{\xi}_c^S \boldsymbol{M}_{ci}^E \right) + \boldsymbol{S} \boldsymbol{\lambda}_0^S \right] r\, \mathrm{d}r \\
& + \int_0^R \boldsymbol{f}^S \boldsymbol{\lambda}_0^S\, r\mathrm{d}r \\
& - \sum_{i=1}^{b} \int_0^R \tilde{\boldsymbol{y}}_i \left(M_y \boldsymbol{\eta} - \tilde{\boldsymbol{N}} \xi\overline{\xi}_c^F \boldsymbol{M}_{ci}^E + \boldsymbol{S} \boldsymbol{\lambda}_{ci}^F \right) r\, \mathrm{d}r \\
& + a \sum_{i=1}^{b} \int_0^R \tilde{\boldsymbol{\lambda}}_{ci}^F \left(M_y \boldsymbol{\eta} - \tilde{\boldsymbol{N}} \xi\overline{\xi}_c^F \boldsymbol{M}_{ci}^E + \boldsymbol{S} \boldsymbol{\lambda}_{ci}^F \right) r\, \mathrm{d}r
\end{aligned}
$$

$$-\sum_{i=1}^{b}\int_0^R \tilde{\boldsymbol{\lambda}}_i^F \left[\boldsymbol{Z}_{ci} - \tilde{\boldsymbol{N}}\xi\bar{\xi}_0^F \boldsymbol{M}_0^E + \boldsymbol{S}\boldsymbol{\lambda}_0^F\right] r\,\mathrm{d}r$$

$$+\int_0^R \tilde{\boldsymbol{f}}^F \boldsymbol{\lambda}_0^F \, r\mathrm{d}r$$

$$-\sum_{i=1}^{b}\int_0^R \tilde{\boldsymbol{\lambda}}_i^I \left[M_y\boldsymbol{\eta} - \tilde{\boldsymbol{N}}\left(\xi\bar{\xi}_0^I \boldsymbol{M}_0^E + \xi\bar{\xi}_c^I \boldsymbol{M}_{ci}^E\right) + \boldsymbol{S}\boldsymbol{\lambda}_{0i}^I\right] r\,\mathrm{d}r$$

$$+\sum_{i=1}^{b}\int_0^R \tilde{\boldsymbol{f}}_i^I \boldsymbol{\lambda}_{0i}^I \, r\mathrm{d}r$$

where $\gamma = 1$ is a dimensional constant.

With the assumption that the unknown functions are as smooth as necessary, with integration by parts where appropriate and remembering that $\boldsymbol{\Psi}$ is required to take a minimum with respect to the variables of problem (11) and a maximum with respect to the Lagrange multipliers, the Euler-Lagrange equations are as follows:

$$\bar{\xi} - \xi \le 0\,, \quad c \ge 0\,, \quad c\left(\bar{\xi} - \xi\right) = 0 \tag{13a}$$

$$K_{r0}^E + \left(w_0^E\right)'' = 0 \quad \text{in } (0,R) \text{ + BCs on } w_0^E \tag{13b}$$

$$K_{\vartheta 0}^E + \frac{1}{r}\left(w_0^E\right)' = 0 \quad \text{in } (0,R) \text{ + BCs on } w_0^E \tag{13c}$$

$$\left(rM_{r0}^E\right)'' - \left(M_{\vartheta 0}^E\right)' + rP_0 = 0 \quad \text{in } (0,R) \text{ + BCs on } \boldsymbol{M}_0^E \tag{13d}$$

$$\boldsymbol{M}_0^E = \mathbf{l}\,\boldsymbol{B}\boldsymbol{K}_0^E \quad \text{in } [0,R] \tag{13e}$$

$$K_{rci}^E + \left(w_{ci}^E\right)'' = 0 \quad \text{in } (0,R) \text{ + BCs on } w_{ci}^E\,, \quad \forall\, i \in I(b) \tag{13f}$$

$$K_{\vartheta ci}^E + \frac{1}{r}\left(w_{ci}^E\right)' = 0 \quad \text{in } (0,R) \text{ + BCs on } w_{ci}^E\,, \quad \forall\, i \in I(b) \tag{13g}$$

$$\left(rM_{rci}^E\right)'' - \left(M_{\vartheta ci}^E\right)' + rP_{ci} = 0 \quad \text{in } (0,R) \text{ + BCs on } \boldsymbol{M}_{ci}^E\,, \quad \forall\, i \in I(b) \tag{13h}$$

$$\boldsymbol{M}_{ci}^E = \mathbf{l}\,\boldsymbol{B}\boldsymbol{K}_{ci}^E \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13i}$$

$$-\boldsymbol{\Phi}_i^E = M_y\boldsymbol{\eta} - \tilde{\boldsymbol{N}}\left(\xi\bar{\xi}_0^E \boldsymbol{M}_0^E + \xi\bar{\xi}_c^E \boldsymbol{M}_{ci}^E\right) \ge \boldsymbol{0} \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13j}$$

$$-\boldsymbol{\Phi}_i^E \ge \boldsymbol{0}\,, \;\; \boldsymbol{\lambda}_i^E \ge \boldsymbol{0}\,, \;\; \tilde{\boldsymbol{\Phi}}_i^E \boldsymbol{\lambda}_i^E = 0 \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13h}$$

$$-\boldsymbol{\Phi}_i^S = M_y\boldsymbol{\eta} - \tilde{\boldsymbol{N}}\left(\xi\bar{\xi}_0^S \boldsymbol{M}_0^E + \xi\bar{\xi}_c^S \boldsymbol{M}_{ci}^E\right) + \boldsymbol{S}\boldsymbol{\lambda}_0^S \ge \boldsymbol{0} \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13ℓ}$$

$$-\boldsymbol{\Phi}_i^S \geq \mathbf{0}\,,\;\; \boldsymbol{\lambda}_i^S \geq \mathbf{0}\,,\;\; \tilde{\boldsymbol{\Phi}}_i^S \boldsymbol{\lambda}_i^S = 0 \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13m}$$

$$-\boldsymbol{f}^S \geq \mathbf{0}\,,\;\; \boldsymbol{\lambda}_0^S \geq \mathbf{0}\,,\;\; \tilde{\boldsymbol{f}}^S \boldsymbol{\lambda}_0^S = 0 \quad \text{in } [0,R] \tag{13n}$$

$$-\boldsymbol{\Phi}_{ci}^F = M_y\,\boldsymbol{\eta} - \tilde{\boldsymbol{N}}\xi\overline{\xi}_c^F \boldsymbol{M}_{ci}^E + \boldsymbol{S}\boldsymbol{\lambda}_{ci}^F \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13o}$$

$$-\boldsymbol{\Phi}_{ci}^F \geq \mathbf{0}\,,\;\; \boldsymbol{y}_i \geq \mathbf{0}\,,\;\; \tilde{\boldsymbol{\Phi}}_{ci}^F \boldsymbol{y}_i = 0 \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13p}$$

$$-\boldsymbol{\Phi}_i^F = \boldsymbol{Z}_{ci} - \tilde{\boldsymbol{N}}\xi\overline{\xi}_0^F \boldsymbol{M}_0^E + \boldsymbol{S}\boldsymbol{\lambda}_0^F \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13q}$$

$$-\boldsymbol{\Phi}_i^F \geq \mathbf{0}\,,\;\; \boldsymbol{\lambda}_i^F \geq \mathbf{0}\,,\;\; \tilde{\boldsymbol{\Phi}}_i^F \boldsymbol{\lambda}_i^F = 0 \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13r}$$

$$-\boldsymbol{f}^F \geq \mathbf{0}\,,\;\; \boldsymbol{\lambda}_0^F \geq \mathbf{0}\,,\;\; \tilde{\boldsymbol{f}}^F \boldsymbol{\lambda}_0^F = 0 \quad \text{in } [0,R] \tag{13s}$$

$$-\boldsymbol{\Phi}_i^I = M_y\,\boldsymbol{\eta} - \tilde{\boldsymbol{N}}\left(\xi\overline{\xi}_0^I \boldsymbol{M}_0^E + \xi\overline{\xi}_c^I \boldsymbol{M}_{ci}^E\right) + \boldsymbol{S}\boldsymbol{\lambda}_{0i}^I \geq \mathbf{0} \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13t}$$

$$-\boldsymbol{\Phi}_i^I \geq \mathbf{0}\,,\;\; \boldsymbol{\lambda}_i^I \geq \mathbf{0}\,,\;\; \tilde{\boldsymbol{\Phi}}_i^I \boldsymbol{\lambda}_i^I = 0 \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13u}$$

$$-\boldsymbol{f}_i^I \geq \mathbf{0}\,,\;\; \boldsymbol{\lambda}_{0i}^I \geq \mathbf{0}\,,\;\; \tilde{\boldsymbol{f}}_i^I \boldsymbol{\lambda}_{0i}^I = 0 \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13v}$$

$$\begin{aligned}\sum_{i=1}^{b}\int_0^R \Big[& \overline{\xi}_0^E \tilde{\boldsymbol{M}}_0^E \boldsymbol{N}\boldsymbol{\lambda}_i^E + \overline{\xi}_c^E \tilde{\boldsymbol{M}}_{ci}^E \boldsymbol{N}\boldsymbol{\lambda}_i^E + \overline{\xi}_0^S \tilde{\boldsymbol{M}}_0^E \boldsymbol{N}\boldsymbol{\lambda}_i^S + \overline{\xi}_c^S \tilde{\boldsymbol{M}}_{ci}^E \boldsymbol{N}\boldsymbol{\lambda}_i^S + \overline{\xi}_0^F \tilde{\boldsymbol{M}}_0^E \boldsymbol{N}\boldsymbol{\lambda}_i^F \\ & + \overline{\xi}_c^F \left(\tilde{\boldsymbol{M}}_{ci}^E \boldsymbol{N}\boldsymbol{y}_i - a\,\tilde{\boldsymbol{M}}_{ci}^E \boldsymbol{N}\boldsymbol{\lambda}_{ci}^F + \tilde{\boldsymbol{M}}_{ci}^E \boldsymbol{N}\boldsymbol{\lambda}_i^F \right) + \overline{\xi}_0^I \tilde{\boldsymbol{M}}_0^E \boldsymbol{N}\boldsymbol{\lambda}_i^I + \overline{\xi}_c^I \tilde{\boldsymbol{M}}_{ci}^E \boldsymbol{N}\boldsymbol{\lambda}_i^I \Big] r\,dr = c\end{aligned} \tag{13w}$$

$$\left(r\,\mu_{r0}\right)'' - \left(\mu_{\vartheta 0}\right)' = 0 \quad \text{in } (0,R) \tag{13x}$$

$$\left(r\,\mu_{rci}\right)'' - \left(\mu_{\vartheta ci}\right)' = 0 \quad \text{in } (0,R), \quad \forall\, i \in I(b) \tag{13y}$$

$$\boldsymbol{\mu}_0 - \mathbf{l}\,\boldsymbol{B}\boldsymbol{\chi}_0 = \mathbf{0} \quad \text{in } [0,R] \tag{13z}$$

$$\boldsymbol{\mu}_{ci} - \mathbf{l}\,\boldsymbol{B}\boldsymbol{\chi}_{ci} = \mathbf{0} \quad \text{in } [0,R], \quad \forall\, i \in I(b) \tag{13a'}$$

$$\chi_{r0} + \boldsymbol{N}_r \sum_{i=1}^{b}\left(\overline{\xi}_0^E \boldsymbol{\lambda}_i^E + \overline{\xi}_0^S \boldsymbol{\lambda}_i^S + \overline{\xi}_0^F \boldsymbol{\lambda}_i^F + \overline{\xi}_0^I \boldsymbol{\lambda}_i^I \right) + \left(u_0\right)'' = 0 \quad \text{in } (0,R) \tag{13b'}$$

$$\chi_{\vartheta 0} + \boldsymbol{N}_\vartheta \sum_{i=1}^{b}\left(\overline{\xi}_0^E \boldsymbol{\lambda}_i^E + \overline{\xi}_0^S \boldsymbol{\lambda}_i^S + \overline{\xi}_0^F \boldsymbol{\lambda}_i^F + \overline{\xi}_0^I \boldsymbol{\lambda}_i^I \right) + \frac{1}{r}\left(u_0\right)' = 0 \quad \text{in } (0,R) \tag{13c'}$$

$$\chi_{cri} + \boldsymbol{N}_r \left[\overline{\xi}_c^{\,E} \boldsymbol{\lambda}_i^E + \overline{\xi}_c^{\,S} \boldsymbol{\lambda}_i^S + \overline{\xi}_c^{\,F} \left(\boldsymbol{y}_i - a\, \boldsymbol{\lambda}_{ci}^F + \boldsymbol{\lambda}_i^F \right) + \overline{\xi}_c^{\,I} \boldsymbol{\lambda}_i^I \right] + \left(u_i \right)'' = 0 \quad \text{in } (0,R), \quad \forall\, i \in I(b) \tag{13d'}$$

$$\chi_{c\vartheta i} + \boldsymbol{N}_\vartheta \left[\overline{\xi}_c^{\,E} \boldsymbol{\lambda}_i^E + \overline{\xi}_c^{\,S} \boldsymbol{\lambda}_i^S + \overline{\xi}_c^{\,F} \left(\boldsymbol{y}_i - a\, \boldsymbol{\lambda}_{ci}^F + \boldsymbol{\lambda}_i^F \right) + \overline{\xi}_c^{\,I} \boldsymbol{\lambda}_i^I \right] + \frac{1}{r} \left(u_i \right)' = 0 \quad \text{in } (0,R), \quad \forall\, i \in I(b) \tag{13e'}$$

$$-\boldsymbol{f}^{\,S} + \boldsymbol{S\lambda}_i^S \geq \boldsymbol{0} \quad \text{in } (0,R) \tag{13f'}$$

$$-\boldsymbol{f}^{\,F} + \boldsymbol{S\lambda}_i^F \geq \boldsymbol{0} \quad \text{in } (0,R) \tag{13g'}$$

$$a\, \boldsymbol{M}_y \eta - a\, \tilde{\boldsymbol{N}} \xi \overline{\xi}_c^{\,F} \boldsymbol{M}_{ci}^E + 2\, a\, \boldsymbol{S} \lambda_{ci}^F - \boldsymbol{S y}_i \geq \boldsymbol{0} \quad \text{in } (0,R), \quad \forall\, i \in I(b) \tag{13h'}$$

$$\lambda_{ci}^F \geq \boldsymbol{0}\,, \quad \tilde{\lambda}_{ci}^F \left(a\, \boldsymbol{M}_y \eta - a\, \tilde{\boldsymbol{N}} \xi \overline{\xi}_c^{\,F} \boldsymbol{M}_{ci}^E + 2\, a\, \boldsymbol{S} \lambda_{ci}^F - \boldsymbol{S y}_i \right) = 0 \quad \text{in } (0,R), \quad \forall\, i \in I(b) \tag{13i'}$$

$$-\boldsymbol{f}_i^{\,I} + \boldsymbol{S\lambda}_i^I \geq \boldsymbol{0} \quad \text{in } (0,R), \quad \forall\, i \in I(b) \tag{13j'}$$

$$g(r) = -\gamma + \xi \left(\tilde{\boldsymbol{\lambda}}_0 \boldsymbol{B K}_0^E + \sum_{i=1}^{b} \tilde{\boldsymbol{\lambda}}_{ci} \boldsymbol{B K}_{ci}^E \right) \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}H} + \tilde{\boldsymbol{\eta}} \sum_{i=1}^{b} \left(\boldsymbol{\lambda}_i^E + \boldsymbol{\lambda}_i^S + \boldsymbol{y}_i - a \boldsymbol{\lambda}_{ci}^F + \boldsymbol{\lambda}_i^F + \boldsymbol{\lambda}_i^I \right) \frac{\mathrm{d}M_y}{\mathrm{d}H} \leq 0 \quad \text{in } [0,R] \tag{13k'}$$

$$H \geq 0\,, \quad Hg = 0 \quad \text{in } [0,R] \tag{13ℓ'}$$

The physical meaning of the Lagrange multipliers can be easily recognized, namely: $\boldsymbol{y}_i = \boldsymbol{\lambda}_{ci}^F$, $\forall\, i \in I(b)$, as results by the comparison of Eqns. (13o,p) and Eqns. (13h',i') with, as usual, $a = 1$. In addition, with the above position for *a*, Eqns. (13w,d',e',k') transform, respectively:

$$\sum_{i=1}^{b} \int_0^R \left(\overline{\xi}_0^{\,E} \tilde{\boldsymbol{M}}_0^E \boldsymbol{N\lambda}_i^E + \overline{\xi}_c^{\,E} \tilde{\boldsymbol{M}}_{ci}^E \boldsymbol{N\lambda}_i^E + \overline{\xi}_0^{\,S} \tilde{\boldsymbol{M}}_0^E \boldsymbol{N\lambda}_i^S + \overline{\xi}_c^{\,S} \tilde{\boldsymbol{M}}_{ci}^E \boldsymbol{N\lambda}_i^S + \overline{\xi}_0^{\,F} \tilde{\boldsymbol{M}}_0^E \boldsymbol{N\lambda}_i^F + \overline{\xi}_c^{\,F} \tilde{\boldsymbol{M}}_{ci}^E \boldsymbol{N\lambda}_i^F + \overline{\xi}_0^{\,I} \tilde{\boldsymbol{M}}_0^E \boldsymbol{N\lambda}_i^I + \overline{\xi}_c^{\,I} \tilde{\boldsymbol{M}}_{ci}^E \boldsymbol{N\lambda}_i^I \right) r\, dr = c \tag{14a}$$

$$\chi_{cri} + \boldsymbol{N}_r \left[\overline{\xi}_c^{\,E} \boldsymbol{\lambda}_i^E + \overline{\xi}_c^{\,S} \boldsymbol{\lambda}_i^S + \overline{\xi}_c^{\,F} \boldsymbol{\lambda}_i^F + \overline{\xi}_c^{\,I} \boldsymbol{\lambda}_i^I \right] + \left(u_i \right)'' = 0 \quad \text{in } (0,R), \quad \forall\, i \in I(b) \tag{14b}$$

$$\chi_{c\vartheta i} + \boldsymbol{N}_\vartheta \left[\overline{\xi}_c^{\,E} \boldsymbol{\lambda}_i^E + \overline{\xi}_c^{\,S} \boldsymbol{\lambda}_i^S + \overline{\xi}_c^{\,F} \boldsymbol{\lambda}_i^F + \overline{\xi}_c^{\,I} \boldsymbol{\lambda}_i^I \right] + \frac{1}{r} \left(u_i \right)' = 0 \quad \text{in } (0,R), \quad \forall\, i \in I(b) \tag{14c}$$

$$g(r) = -\gamma + \xi \left(\tilde{\boldsymbol{\chi}}_0 \boldsymbol{B} \boldsymbol{K}_0^E + \sum_{i=1}^{b} \tilde{\boldsymbol{\chi}}_{ci} \boldsymbol{B} \boldsymbol{K}_{ci}^E \right) \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}H} \tag{14d}$$

$$+ \tilde{\boldsymbol{\eta}} \sum_{i=1}^{b} \left(\boldsymbol{\lambda}_i^E + \boldsymbol{\lambda}_i^S + \boldsymbol{\lambda}_i^F + \boldsymbol{\lambda}_i^I \right) \frac{\mathrm{d}M_y}{\mathrm{d}H} \leq 0 \quad \text{in } [0,R], \quad \forall i \in I(b)$$

Furthermore, $\boldsymbol{\lambda}_i^E$, $\boldsymbol{\lambda}_i^S$, $\boldsymbol{\lambda}_i^F$ and $\boldsymbol{\lambda}_i^I$, $\forall i \in I(b)$, are plastic coefficients related to the elastic limit, the elastic shakedown limit, the plastic shakedown limit and instantaneous collapse, respectively, compatible with the plastic curvature vectors:

$$\boldsymbol{\chi}_0^p = \boldsymbol{N} \sum_{i=1}^{b} \left(\overline{\xi}_0^E \boldsymbol{\lambda}_i^E + \overline{\xi}_0^S \boldsymbol{\lambda}_i^S + \overline{\xi}_0^F \boldsymbol{\lambda}_i^F + \overline{\xi}_0^I \boldsymbol{\lambda}_i^I \right) \tag{15a}$$

$$\boldsymbol{\chi}_{ci}^p = \boldsymbol{N} \left(\overline{\xi}_c^E \boldsymbol{\lambda}_i^E + \overline{\xi}_c^S \boldsymbol{\lambda}_i^S + \overline{\xi}_c^F \boldsymbol{\lambda}_i^F + \overline{\xi}_c^I \boldsymbol{\lambda}_i^I \right) \quad \forall i \in I(b) \tag{15b}$$

In Eqns. (13b'-e'); $\boldsymbol{u}_0$ and $\boldsymbol{u}_i$, $\forall i \in I(b)$, are deflections, while, in Eqns. (13z,a'), $\boldsymbol{\chi}_0$ and $\boldsymbol{\chi}_{ci}$, $\forall i \in I(b)$, are elastic curvature vectors associated to the bending moment vectors $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_{ci}$, $\forall i \in I(b)$.

The constant c, that has the role of scaling factor for all the kinematical Lagrange multipliers, cannot vanish because it equals the external work in Eqns. (13w), and thus, by Eqns. (9a) it results $\xi = \overline{\xi}$ and, as usual, $c = 1$ can be stated.

In Eqns. (13n,s,v,f',g',j') $\boldsymbol{f}^S$, $\boldsymbol{f}^F$ and $\boldsymbol{f}_i^I$, $\forall i \in I(b)$, represent plastic potentials related to the optimal design at the limit state of elastic shakedown, plastic shakedown and instantaneous collapse, respectively.

Finally, Eqns. (13k',ℓ') provide the relevant optimality conditions for the design; actually they describe the featuring properties of the optimal design. They take into account the interaction between the different behavioural constraints through the common argument H. In the particular case that the solution to problem (11) implies that $H = 0$ within a plate ring, then the behavioural constraint interaction is not active $g(r) < 0$; otherwise, in the relevant case of $H > 0$ within the ring, then, in $[0,R]$, the following equation holds:

$$H(r) > 0, \quad g(r) = 0 \tag{16a,b}$$

$$\tilde{\boldsymbol{\eta}} \sum_{i=1}^{b} \left(\boldsymbol{\lambda}_i^E + \boldsymbol{\lambda}_i^S + \boldsymbol{\lambda}_i^F + \boldsymbol{\lambda}_i^I \right) \frac{\mathrm{d}M_y}{\mathrm{d}H} = \gamma - \xi \left(\tilde{\boldsymbol{\chi}}_0 \boldsymbol{B} \boldsymbol{K}_0^E + \sum_{i=1}^{b} \tilde{\boldsymbol{\chi}}_{ci} \boldsymbol{B} \boldsymbol{K}_{ci}^E \right) \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}H} \tag{16c}$$

where the first member in Eqn. (16c) represents the sensitivity, with respect to the design variable H, of the plastic dissipation density and the second member in the same equation is the analogous sensitivity of the modified cost per unit area of the plate middle surface, the latter being the sum of the standard unit cost and an additional energy term accounting for the interaction between the primary elastic curvatures $\boldsymbol{K}_0^E$ and $\boldsymbol{K}_{ci}^E$, and the conjugate elastic curvatures $\boldsymbol{\chi}_0$ and $\boldsymbol{\chi}_{ci}$. Equations (17) turn out to be a generalization to the present contest of the well known theorem of Drucker-Prager-Shield-Rozvany [20] of optimal plastic design.

5 APPLICATION

Let us consider the simply supported steel circular plate of Figure 5 subjected to an axisymmetric load condition constituted by a concentrated fixed load Q applied on the centre of the plate and to a perfectly cyclic ring radial couple k, $-\bar{k} \leq k \leq +\bar{k}$, applied on the outer edge of the plate.

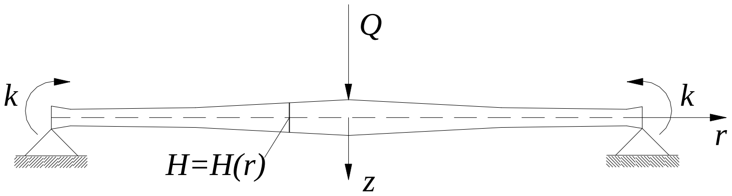

Figure 5: Assigned circular plate and relevant load condition.

In order to numerically obtain an optimal design of the above described circular plate with the constraints considered at the previous section, it is necessary to perform a suitable discretization of the structure and, consequently, of the problem itself. Actually, the solution of the minimum volume problem (11) or of the related Euler-Lagrange equations (13) is a very hard task. In the following, a discretization of the relevant minimum volume problem is developed and it represents an extension at the present case of previously proposed discretization models [17].

First of all, the plate is discretized into λ ring finite elements identified by the values of λ selected radii; consequently, the design variables are represented by the λ thicknesses of the relevant rings. In the typical ring the thickness can vary according to suitably chosen shape functions. Therefore, the plate thickness can be expressed in a discrete form by:

$$H(r) = \boldsymbol{\Theta}_H(r)\boldsymbol{Y}_H \quad \text{in } [0,R] \tag{17}$$

where $\boldsymbol{\Theta}_H$ is the suitably chosen shape function vector and

$$\tilde{\boldsymbol{Y}}_H = \left[Y_{H,1} \quad Y_{H,2} \quad \ldots \quad Y_{H,\lambda} \right] \tag{18}$$

is the vector of thickness evaluated at λ selected radii (design variables).

At the same time, the deflection fields related to the fixed and the cyclic basic loads must be discretized by choosing suitable vectors $\left(\boldsymbol{W}_0^E, \boldsymbol{W}_{ci}^E\right)$, the components of which represent the relevant deflections evaluated at the q nodes of the discretized structure $\left(q = l + 1\right)$. Therefore, the compatible deflection field in Eqns. (11d,e,h,i) related to the fixed and cyclic loads, respectively, can be expressed as:

$$w_0^E(r) = \boldsymbol{\Theta}_w(r)\boldsymbol{W}_0^E \quad \text{in } [0,R] \tag{19a}$$

$$w_{ci}^E(r) = \boldsymbol{\Theta}_w(r)\boldsymbol{W}_{ci}^E \quad \text{in } [0,R], \quad \forall i \in I(b) \tag{19b}$$

where $\boldsymbol{\Theta}_w$ is the suitably chosen shape function vector and

$$\tilde{\boldsymbol{W}}_0^E = \left[W_{0,1}^E \quad W_{0,2}^E \quad \ldots \quad W_{0,q}^E \right] \tag{20a}$$

$$\tilde{\boldsymbol{W}}_{ci}^{E} = \left\lfloor W_{ci,1}^{E} \quad W_{ci,2}^{E} \quad \ldots \quad W_{ci,q}^{E} \right\rfloor \tag{20b}$$

are the defined deflection vectors.

Finally, even the plastic activation vector, $\boldsymbol{\lambda}_{ci}^{F}$ related to the plastic response of the plate just to the amplified cyclic loads and the fictitious plastic activations $\boldsymbol{\lambda}_{0}^{S}$, $\boldsymbol{\lambda}_{0}^{F}$, $\boldsymbol{\lambda}_{0i}^{I}$, $\forall\, i \in I(b)$, related to the elastic shakedown limit, to the plastic shakedown limit and to the instantaneous collapse, all functions of radius *r*, must be represented as functions of discrete parameters; following the same criteria as before, they can be expressed, respectively, as:

$$\boldsymbol{\lambda}_{ci}^{F}(r) = \boldsymbol{\Theta}_{\lambda}(r)\boldsymbol{\Lambda}_{ci}^{F} \qquad \forall\, i \in I(b) \tag{21a}$$

$$\boldsymbol{\lambda}_{0}^{S}(r) = \boldsymbol{\Theta}_{\lambda}(r)\boldsymbol{\Lambda}_{0}^{S} \tag{21b}$$

$$\boldsymbol{\lambda}_{0}^{F}(r) = \boldsymbol{\Theta}_{\lambda}(r)\boldsymbol{\Lambda}_{0}^{F} \tag{21c}$$

$$\boldsymbol{\lambda}_{0i}^{I}(r) = \boldsymbol{\Theta}_{\lambda}(r)\boldsymbol{\Lambda}_{0i}^{I} \qquad \forall\, i \in I(b) \tag{21d}$$

being $\boldsymbol{\Theta}_{\lambda} = \boldsymbol{\Theta}_{\lambda}(r)$ the suitable chosen $6 \times 6q$ shape function matrix and

$$\tilde{\boldsymbol{\Lambda}}_{ci}^{F} = \left\lfloor \boldsymbol{\Lambda}_{ci,1}^{F} \quad \boldsymbol{\Lambda}_{ci,2}^{F} \quad \ldots \quad \boldsymbol{\Lambda}_{ci,q}^{F} \right\rfloor \qquad \forall\, i \in I(b) \tag{22a}$$

$$\tilde{\boldsymbol{\Lambda}}_{0}^{S} = \left[\boldsymbol{\Lambda}_{0,1}^{S} \quad \boldsymbol{\Lambda}_{0,2}^{S} \quad \ldots \quad \boldsymbol{\Lambda}_{0,q}^{S} \right] \tag{22b}$$

$$\tilde{\boldsymbol{\Lambda}}_{0}^{F} = \left[\boldsymbol{\Lambda}_{0,1}^{F} \quad \boldsymbol{\Lambda}_{0,2}^{F} \quad \ldots \quad \boldsymbol{\Lambda}_{0,q}^{F} \right] \tag{22c}$$

$$\tilde{\boldsymbol{\Lambda}}_{0i}^{I} = \left[\boldsymbol{\Lambda}_{0i,1}^{I} \quad \boldsymbol{\Lambda}_{0i,2}^{I} \quad \ldots \quad \boldsymbol{\Lambda}_{0i,q}^{I} \right] \qquad \forall\, i \in I(b) \tag{22d}$$

the chosen discrete parameter vectors.

Making use of the virtual work principle, taking into account Eqns. (19), the continuous equilibrium Eqns. (11f,j) transform into algebraical systems composed by q equations for each basic load condition, namely:

$$\int_0^R \tilde{\boldsymbol{\Theta}}_w \left[\left(r M_{r0}^{E}\right)'' - \left(M_{\vartheta 0}^{E}\right)' + r P_0 \right] r dr = \mathbf{0} \tag{23a}$$

$$\int_0^R \tilde{\boldsymbol{\Theta}}_w \left[\left(r M_{rci}^{E}\right)'' - \left(M_{\vartheta ci}^{E}\right)' + r P_{ci} \right] r dr = \mathbf{0} \qquad \forall\, i \in I(b) \tag{23b}$$

With the above positions, problem (11) transforms into the following mathematical programming problem:

$$\min_{\left(\boldsymbol{Y}_H,\xi,\boldsymbol{W}_0^E,\boldsymbol{W}_{ci}^E,\boldsymbol{K}_0^E,\boldsymbol{K}_{ci}^E,\boldsymbol{M}_0^E,\boldsymbol{M}_{ci}^E,\boldsymbol{\Lambda}_0^S,\boldsymbol{\Lambda}_{ci}^F,\boldsymbol{\Lambda}_0^F,\boldsymbol{\Lambda}_{0i}^I\right)} \frac{1}{2\pi} V = \min_{\left(\boldsymbol{Y}_H,\xi,\boldsymbol{W}_0^E,\boldsymbol{W}_{ci}^E,\boldsymbol{K}_0^E,\boldsymbol{K}_{ci}^E,\boldsymbol{M}_0^E,\boldsymbol{M}_{ci}^E,\boldsymbol{\Lambda}_0^S,\boldsymbol{\Lambda}_{ci}^F,\boldsymbol{\Lambda}_0^F,\boldsymbol{\Lambda}_{0i}^I\right)} \left(\int_0^R \boldsymbol{\Theta}_H(r)\, r\, dr\right) \boldsymbol{Y}_H \tag{24a}$$

subject to:

$$\boldsymbol{Y}_H \geq \mathbf{0} \tag{24b}$$

$$\bar{\xi} - \xi \leq 0 \tag{24c}$$

$$K_{r0}^E + \left(\boldsymbol{\Theta}_w\right)'' \boldsymbol{W}_0^E = 0 \quad \text{in } (0,R) \tag{24d}$$

$$K_{\vartheta 0}^E + \frac{1}{r}\left(\boldsymbol{\Theta}_w\right)' \boldsymbol{W}_0^E = 0 \quad \text{in } (0,R) \tag{24e}$$

$$\int_0^R \tilde{\boldsymbol{\Theta}}_w \left[\left(r M_{r0}^E\right)'' - \left(M_{\vartheta 0}^E\right)' + r P_0 \right] r\, dr = \mathbf{0} \tag{24f}$$

$$\boldsymbol{M}_0^E = \mathbf{I}\, \boldsymbol{B} \boldsymbol{K}_0^E \quad \text{in } [0,R] \tag{24g}$$

$$K_{rci}^E + \left(\boldsymbol{\Theta}_w\right)'' \boldsymbol{W}_{ci}^E = 0 \quad \text{in } (0,R), \quad \forall i \in I(b) \tag{24h}$$

$$K_{\vartheta ci}^E + \frac{1}{r}\left(\boldsymbol{\Theta}_w\right)' \boldsymbol{W}_{ci}^E = 0 \quad \text{in } (0,R), \quad \forall i \in I(b) \tag{24i}$$

$$\int_0^R \tilde{\boldsymbol{\Theta}}_w \left[\left(r M_{rci}^E\right)'' - \left(M_{\vartheta ci}^E\right)' + r P_{ci} \right] r\, dr = \mathbf{0} \quad \forall i \in I(b) \tag{24j}$$

$$\boldsymbol{M}_{ci}^E = \mathbf{I}\, \boldsymbol{B} \boldsymbol{K}_{ci}^E \quad \text{in } [0,R], \quad \forall i \in I(b) \tag{24k}$$

$$-\boldsymbol{\Phi}_i^{ED} = \int_0^R \tilde{\boldsymbol{\Theta}}_\lambda \left[M_y \boldsymbol{\eta} - \tilde{\mathbf{N}}\left(\xi \bar{\xi}_0^E \boldsymbol{M}_0^E + \xi \bar{\xi}_c^E \boldsymbol{M}_{ci}^E\right)\right] r\, dr \geq \mathbf{0} \quad \forall i \in I(b) \tag{24ℓ}$$

$$-\boldsymbol{\Phi}_i^{SD} = \int_0^R \tilde{\boldsymbol{\Theta}}_\lambda \left[M_y \boldsymbol{\eta} - \tilde{\mathbf{N}}\left(\xi \bar{\xi}_0^S \boldsymbol{M}_0^E + \xi \bar{\xi}_c^S \boldsymbol{M}_{ci}^E\right) + \boldsymbol{S} \boldsymbol{\Theta}_\lambda \boldsymbol{\Lambda}_0^S \right] r\, dr \geq \mathbf{0} \quad \forall i \in I(b) \tag{24m}$$

$$\boldsymbol{\Lambda}_0^S \geq \mathbf{0} \tag{24n}$$

$$\boldsymbol{Z}_{ci}^D \equiv -\boldsymbol{\Phi}_{ci}^{FD} = \int_0^R \tilde{\boldsymbol{\Theta}}_\lambda \left(M_y \boldsymbol{\eta} - \tilde{\mathbf{N}} \xi \bar{\xi}_c^F \boldsymbol{M}_{ci}^E + \boldsymbol{S} \boldsymbol{\Theta}_\lambda \boldsymbol{\Lambda}_{ci}^F \right) r\, dr \quad \forall i \in I(b) \tag{24o}$$

$$\boldsymbol{Z}_{ci}^D \geq \mathbf{0}, \quad \boldsymbol{\Lambda}_{ci}^F \geq \mathbf{0}, \quad \tilde{\boldsymbol{Z}}_{ci}^D \boldsymbol{\Lambda}_{ci}^F = 0 \quad \forall i \in I(b) \tag{24p}$$

$$-\boldsymbol{\Phi}_i^{FD} = \int_0^R \tilde{\boldsymbol{\Theta}}_\lambda \left[\boldsymbol{Z}_{ci}^D - \tilde{\mathbf{N}} \xi \bar{\xi}_0^F \boldsymbol{M}_0^E + \boldsymbol{S} \boldsymbol{\Theta}_\lambda \boldsymbol{\Lambda}_0^F \right] r\, dr \geq \mathbf{0} \quad \forall i \in I(b) \tag{24q}$$

$$\boldsymbol{\Lambda}_0^F \geq \mathbf{0} \tag{24r}$$

$$-\boldsymbol{\Phi}_i^{ID} = \int_0^R \tilde{\boldsymbol{\Theta}}_\lambda \left[M_y \boldsymbol{\eta} - \tilde{\boldsymbol{N}} \left(\xi \overline{\xi}_0^I \boldsymbol{M}_0^E + \xi \overline{\xi}_c^I \boldsymbol{M}_{ci}^E \right) + \boldsymbol{S} \boldsymbol{\Theta}_\lambda \boldsymbol{\Lambda}_{0i}^I \right] r dr \geq \mathbf{0} \quad \forall i \in I(b) \tag{24s}$$

$$\boldsymbol{\Lambda}_{0i}^I \geq \mathbf{0} \quad \forall i \in I(b) \tag{24t}$$

where $\boldsymbol{\Phi}_i^{ED}, \boldsymbol{\Phi}_i^{SD}$, $\boldsymbol{\Phi}_{ci}^{FD}$, $\boldsymbol{\Phi}_i^{FD}$ and $\boldsymbol{\Phi}_i^{ID}$, $\forall i \in I(b)$, represent the discretized forms of the relevant plastic potentials expressed in an appropriate integrated form.

As it is possible to observe, even if discretized, the above reported search problem (24) is a strongly non-linear mathematical programming problem; actually it involves the steady state elastic plastic response of the structure under cyclic loading. As a consequence, a suitable linearization must be adopted and an appropriate iterative technique must be utilized for reaching the solution.

Therefore, reference will be made to a special technique, already utilized by the same authors in previous papers [31], based on the main assumption that all the quantities depending on the design variables can be expressed at each step as linear functions of these variables, i.e., in particular, as the sum of their values at the previous step plus the product of their partial derivatives with respect to the design variables times the increments of the design variables.

At each step, the computational procedure consists of four different phases characterized by the circumstance that in each phase some variables of the original problem are assumed as known values, while in the same phase other variable values are brought up to date. In particular, the first phase is related to a suitable assumption for the design variable values, the second one is devoted to the determination of the purely elastic response of the structure to fixed and cyclic loads, the third one is devoted to the computation of the increments of the design variables, while in the last fourth phase the design variables are brought up to date. Obviously, when all the described phases have been effected all the variable values are brought up to date and, therefore, a new step can start. The procedure is stopped when the design variable values computed at two successive steps differ less than a suitably prefixed tolerance.

Even the described iterative technique implies the solution of linear and non linear problems, but it results much less onerous than the solution of problem (24), in terms of computational cost. It is worth noticing that the solution obtained by means of the described iterative procedure fulfil all the Kuhn-Tucker conditions of problem (24), as it is possible to prove, but this procedure is skipped for the sake of brevity.

In order to apply the described technique at the simply supported circular plate plotted in Figure 5, let us discretized it into five ring elements (Figure 6) with constant thickness Y_{Hj} $(j = 1, \ldots, 5)$. The following data have been considered for the numerical computation: $R = 15\,\text{cm}$, $\sigma_y = 40\,\text{kN/cm}^2$, $E = 21000\,\text{kN/cm}^2$, $\nu = 0.3$. For computational purposes the concentrated fixed load Q applied on the centre of the plate has been simulated by an uniformly distributed equivalent load acting on a sufficiently small circular area, and it has been assumed $Q = 8\,\text{kN}$. Furthermore, for the perfect cyclic ring radial couple k, $-\overline{k} \leq k \leq +\overline{k}$, applied on the outer edge of the plate, $\overline{k} = 0.5308\,\text{kNcm/cm}$ has been assumed.

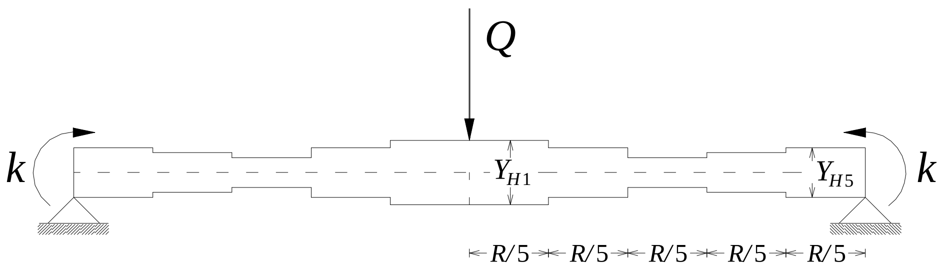

Figure 6: Discretized circular plate and load condition.

Although in the described problem (24) constraints on all the different possible structure limit behaviours have been considered, in the present application, constraints just on the elastic shakedown limit and on the instantaneous collapse limit will be simultaneously imposed, in order to take into account the elastic behaviour of the plate in serviceability conditions and, at the same time, to utilize its ductility features and for preventing the collapse in the cases of exceptional very high loads. The following values have been assigned to the load multipliers: $\bar{\xi}_0^S = \bar{\xi}_c^S = 1$, $\bar{\xi}_0^I = 1$, $\bar{\xi}_c^I = 2$ and $\bar{\xi} = 1.2$.

The optimal thicknesses of the elements deduced by the solution to problem (25) are: $Y_{H1} = 1.310\,\text{cm}$,, $Y_{H2} = 0.591\,\text{cm}$, $Y_{H3} = 0.452\,\text{cm}$, $Y_{H4} = 0.478\,\text{cm}$, $Y_{H5} = 0.486\,\text{cm}$.

6 CONCLUSIONS

The optimal design of minimum volume of circular axisymmetric steel plates subjected to loads, suitable combinations of fixed and perfect cyclic ones, varying inside a given domain has been studied, taking contemporaneously into account the different resistance criteria related to all the possible structure limit behaviour, as described in the space of the fixed and cyclic load multipliers (Bree diagram). A continuous elastic perfectly plastic model has been considered for the plate and the loads have been thought as quasi statically variable inside the given domain. The classical Love-Kirchhoff plate model has been employed and the hypothesis of small displacements and strains has been assumed. Furthermore, no ductility limits have here been considered. The optimal design problem has been formulated as the search for the minimum volume design whose elastic limit load multiplier, elastic shakedown limit load multiplier, plastic shakedown limit load multiplier and instantaneous limit load multiplier be not smaller than suitably assigned values. For each criterion, a corresponding suitably selected safety factor has been imposed. The Euler-Lagrange equations related to the optimization problems have been found and interpreted, so that the special features of the optimal design have been pointed out. In particular, the optimality condition for the plate turn out to be a generalization to the present contest of a well known theorem of optimal plastic design.

For computational purposes, a discretization of the relevant minimum volume problem has been developed. It utilizes suitable shape functions to describe the plate thickness field, the compatible deflection field and the plastic activation fields. The discretized minimum problem has the form of a non-linear mathematical programming one, and its solution has been reached by using an iterative approach, based on the linear programming. It is possible to prove that the design and behavioural variables iteratively obtained satisfy the Kuhn-Tucker conditions related to the original discretized problem, i.e. the obtained solution satisfies the original problem.

A simple application has been worked out; the obtained results confirmed the theoretical expectations in terms of behavioural features of the obtained optimal designs.

REFERENCES

[1] E. Melan, *Der Spannungszustand eines Hencky-Mises'shen continuum bei verändilicher Belastung*, Stz. Ber. AK. Wiss Wien, 147, 73 (1938).

[2] W.T. Koiter, *General theorem for elastic-plastic solids*, In Progress in Solid Mechanics, Ed. I.N. Sneddon and R. Hill, North Holland Vol. I, 165-219, (1960).

[3] C. Polizzotto, "A unified approach to quasi-static shakedown problem for elastic-plastic solids with a piecewise linear yield surface", *Meccanica*, **13**, 109-120, (1978).

[4] M. Capurso, L. Corradi and G. Maier, Bounds on deformations and displacements in shakedown theory, *Matériaux et structures sous chargement cyclique*, Ass. amicale des ingénieurs anciens eléves de l'E.N.P.C., Paris, 231-244, (1979).

[5] J. A. König, *On upper bounds to shakedown loads*, Z. Angew. Math., 59, 349-354, (1979).

[6] C. Polizzotto, "A unified treatment of shakedown theory and related bounding techniques", *S. M. Archives*, 7, 1, 19-75, (1982).

[7] J. A. König, *Shakedown of elastic plastic structures*, PWN-Polish Scientific Publishers, Warsaw and Elsevier, Amsterdam, (1987).

[8] C. Cinquini, G. Guerlement and D. Lamblin, "Finite element iterative methods for optimal elastic design of circular plates", *Comp. & Struct.*, **12** (1), 85-92, (1980).

[9] C. Cinquini and G. Sacchi, "A note on the optimal elastic design for given deflection", *Optimization of distributed parameter structures*, in E. J. Haug and J. Cea ed, **1**, 383-398, (1981).

[10] G. Maier, A. Zavelani Rossi and D. Benedetti, "A finite element approach to optimal design of plastic structures in plane stress", *Int. J. Num. Meth. Engrg.*, **4**, 455-473, (1972).

[11] M. Z. Cohn and S. R. Parimi, "Optimal design of plastic structures for fixed and shakedown loadings", *J. Appl. Mech.*, 595-599, (1973).

[12] L. Corradi and A. Zavelani Rossi, "A linear programming approach to shakedown analysis of structures", *Comp. Meth. Appl. Mech. Eng.*, **3**, 37-53, (1974).

[13] J. A., König, "On optimum shakedown design", In A. Sawczuk and Z. Mroz ed., *Optimization in structure design*, Springer-Verlag, Berlin, Germany, 405-414, (1975).

[14] A. R. S. Ponter, *Shakedown and ratchetting below the creep range*, Rep. EUR8702 EN, Commission of European Communities, Brussels, Belgium, (1983).

[15] C. Polizzotto, "Shakedown analysis and design in presence of limited ductility behaviour", *Eng. Struct.*, **6**, 80-87, (1984).

[16] F. Giambanco, L. Palizzolo and C. Polizzotto, "Optimal shakedown design of beam structures", *Struct. Optimiz.*, 8, 156-167, (1994a).

[17] F. Giambanco, L. Palizzolo and C. Polizzotto, "Optimal shakedown design of circular plates", *Journal of Engineering Mechanics*, **120** (12), 2535-2555, (1994b).

[18] F. Giambanco and L. Palizzolo, "Optimality conditions for shakedown design of trusses", *Comp. Mech.*, **16** (6), 369-378, (1995).

[19] F. Giambanco, L. Palizzolo and L. Cirone, "Computational methods for optimal shakedown design of FE structures", *Structural Optimization*, **15** (3/4), 284-295, (1998).

[20] G.I.N. Rozvany, *Optimal design of flexural systems*, Pergamon Press, Oxford, England, (1976).

[21] M. Save and W. Prager, *Structural Optimization*, Plenum Press, New York, (1985).
[22] G.I.N. Rozvany, *Structural design via optimality criteria*, The Nederlands, Kluwer Academic Publishers, Dordrecht, (1989).
[23] G. Maier, R. Srinivasan, and M. Save, “Optimal plastic design”, *J. Structural Mechanics*, **4**, 349-378, (1976).
[24] F. Giambanco, L. Palizzolo and A. Caffarelli, “Optimal design of elastic plastic structures: elastic and plastic shakedown formulations”, *Intensive Workshop – Optimal Design of Materials and Structures*, Paris, France, (2002).
[25] F. Giambanco, L. Palizzolo and A. Caffarelli, “Computational procedures for plastic shakedown design of structures”, *Journal of Structural and Multidisciplinary Optimization*, **28**, 5, 317-329, (2004).
[26] L. Palizzolo, “Optimal design of trusses according to a plastic shakedown criterion”, *J. of Appl. Mech.*, **71**, 2, 240-246, (2004).
[27] S. Kaliszky and J. Lógó, “Optimal design of elasto-plastic structures subjected to normal and extreme loads”, *Computers and Structures*, **84**, 1770-1779, (2006).
[28] F. Giambanco, “Multicriteria optimal design of circular plates”, *IASS–IACM 2000 Fourth International Colloquium on Computation of Shell & Spatial Structures*, Chania-Crete, June 4-7, (2000).
[29] L. Palizzolo, “Multicriteria optimal design of continuous beams”, *ECCOMAS 2000, European Congress on Computational Methods in Applied Sciences and Engineering*, Barcelona, September, 11-14, (2000).
[30] A. Caffarelli, F. Giambanco and L. Palizzolo, “Optimal Design: a comparison between different criteria”, *The Second International Conference on Advances in Structural Engineering and Mechanics*, Busan, Korea, (2002).
[31] L. Palizzolo, “Optimization of continuous elastic perfectly plastic beams”, *Computers & Structures*, **82**, 4-5, 397-411, (2004).
[32] L. Palizzolo, A. Caffarelli and M. Zito, “Optimal design of circular plates according to different criteria”, *Lightweight structures in civil engineering (LSCE)*, Warsaw, (2005).
[33] C. Polizzotto, “A unified treatment of shakedown theory and related bounding techniques”, *S.M. Archives*, **7**, 19-75, (1982).
[34] C. Polizzotto, G. Borino and P. Fuschi, “On the steady-state response of elastic perfectly plastic solids to cyclic loads”, *Inelastic Solids and Structures*, Edit by M. Kleiber and J. A. König, Pineridge Press, Swansea, U.K., 473-488, (1990).
[35] C. Polizzotto, “On elastic plastic structures under cyclic loads”, *Eur. J. Mech., A/Solids*, **13**, 4, 149-173, (1994a).
[36] C. Polizzotto, “Steady states and sensitivity analysis in elastic-plastic structures subjected to cyclic loads”, *Int. J. Solids Struct.*, **31**, 953-970, (1994b).
[37] J. Zarka and J. Casier, “Elastic-plastic response of structure to cyclic loading: practical rules”, *In Nemat Nasser, S (ed.) Mechanics Today*, Oxford: Pergamon Press., 93-198, (1979).
[38] J. Zarka, J. Frelat, G. Inglebert and P. Kasnat-Navidi, *A new approach in inelastic analysis of structures*, Ecole Polytechnique Palaisau, France, (1990).
[39] C. Polizzotto, G. Borino and P. Fuschi, “Assessment of the elastic/plastic shakedown load boundary for structures subjected to cyclic loading”, *XV Congresso Nazionale AIMETA*, Taormina, Italy, (2001).