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The Multiscale Stochastic Model of Fractional Hereditary Materials (FHM)

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Abstract

In a recent paper the authors proposed a mechanical model corresponding, exactly, to fractional hereditary materials (FHM). Fractional derivation index $\beta \in [0, 1/2]$ corresponds to a mechanical model composed by a column of massless newtonian fluid resting on a bed of independent linear springs. Fractional derivation index $\beta \in [1/2, 1]$, corresponds, instead, to a mechanical model constituted by massless, shear-type elastic column resting on a bed of linear independent dashpots. The real-order of derivation is related to the exponent of the power-law decay of mechanical characteristics. In this paper the authors aim to introduce a multiscale fractance description of FHM in presence of stochastic fluctuations of model parameters. In this setting the random multiscale fractance may be used to capture the fluctuations of material parameters observed in experimental tests by means of proper analytical evaluation of the model statistics.

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Keywords: Multiscale Fractance; Fractional Hereditariness; Fractional Derivatives; Random models

1. Introduction

Advanced engineering materials as polymeric foams, aerogels, rubber-like materials show the presence of a marked time-dependence experienced in creep and relaxation static tests. Analogous considerations may be withdrawn from the observation of dynamic tests that results in the introduction of the relaxation/retardation spectrum of materials and a first mechanical explanation of the time-dependent material response has been provided at the mid-fifties of the last century [1]. In that study the molecular interactions in coiled polymers have been taken into account and a rheological model involving a discrete set of relaxation/retardation times have been obtained. This result led several applied scientists and engineering to model the time-dependence of material response with appropriate sequences of springs and dashpots that is with combinations of Maxwell and Voigt

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mechanical elements possessing a finite number of relaxation/retardation times. At the same time, experimental relaxation/creep tests conducted in [2, 3] showed that the time-dependent material behavior of many, polymeric-type materials was well described by real-exponent power-laws, that is the residual stress decays as $\sigma(t) \propto t^{-\beta}$ with $0 \le \beta \le 1$. This consideration has been recently observed from the continuous relaxation/retardation spectra [4, 5] observed in real polymeric materials and foams as well as in biological hard and soft tissues such as trabecular bone tissue [6, 7], muscles and ligaments [8] and the time-dependent behavior of phospho-lipid biological membranes [9]. The presence of power-laws in the time-evolution of residual stress and of the strain flow yields rheological stress-strain relations based on fractional-order integrals and derivatives [10] that are mechanically represented by the springpot element [11] (after Scott-Blair [11]) that has been widely studied in the last decades to generalize the classical rheological models by Maxwell and Voigt to deal with fractional relaxation.

Spring-pot element has been used in several contexts of linear and non-linear dynamics and it shows the main drawback to require a careful numerical treatment and therefore some approximate mechanical scheme capable to represent this behavior have been introduced in scientific literature [12, 13]

Recently, the authors have introduced the exact mechanical description of the spring-pot element in terms of a multiscale mechanical fractance that, at the limit, restitutes a rheological description in terms of fractional-order integrals and derivatives [14, 15]. Experimental observation conducted on real-type materials showed that fractional relaxation/creep function is very dependent on the considered material specimen yielding strong fluctuations of material coefficients.

In this study the authors aim to study the time evolution of FHM assuming that the proportionality coefficient the relaxation/creep function is modeled by a random variable with prescribed probability density function. The mechanical response of the material is obtained with the aid of the multiscale mechanical models of FHM yielding a multiscale mechanical fractance with random relaxation times.

2. The Multiscale Mechanical Model of Fractional Hereditary Materials

In this section we will introduce the fundamental relations about fractional-order hereditariness in terms of relaxation and/or creep functions (sec.2.1); The multiscale model of FHM will be outlined in sec.(2.2).

2.1. The power-law model of hereditariness: Fractional integrals and derivatives

Creep and relaxation function of complex materials has proved to be well described by power-laws with real-order exponent in the form [2]:

$$G(t) = c(\beta)t^{-\beta}/\Gamma(1-\beta) \quad ; \quad G(t) = t^{\beta}/c(\beta)\Gamma(\beta) \tag{1 a, b}$$

The expressions of the creep and relaxation functions reported in eqs.(1 a,b) are related, in Laplace domain by means of the relation:

$$\hat{G}(s)\hat{J}(s) = \frac{1}{s^2} \tag{2}$$

Nomenclature

t	Time variable
z	Spatial coordinate along the material column
A	Unitary Cross-section of the column
$\sigma(z,t)$	Shear/normal stress in mechanical model
$\gamma(z,t)$	Shear/normal strain in the mechanical model
$\sigma(t)=\sigma(0,t)$	Applied shear/normal stress at the top of the column
$\gamma(t)=\gamma(0,t)$	Applied Shear/normal strain at the top of the column
$(I_0^\beta \cdot)(t)$	Riemann-Liouville fractional integral of order β
$(D_0^\beta \cdot)(t)$	Riemann-Liouville fractional derivative of order β
$({}_C D_0^\beta \cdot)(t)$	Caputo fractional derivative of order β
$c(\beta)$	Anomalous proportionality coefficient: $[c(\beta)] = F/LT^\beta$
G_0	Parent shear modulus of the mechanical fractance: $[G_0] = F/L^2$
η_0	Parent viscosity coefficient of the mechanical fractance: $[\eta_0] = F/L^2T$
$\Gamma(\cdot)$	Euler-Gamma function
$G(t)$	Relaxation function
$J(t)$	Creep function
$Y_\beta(\cdot)$; $K_\beta(\cdot)$	First and second modified Bessel functions

where we denoted s the Laplace transform variable. The use of Boltzmann superposition principle yields, upon substitution of Eqs.(1 a,b), the stress/strain evolution oh FHM in terms of fractional-order operators as:

$$\sigma(t) = \frac{c(\beta)}{\Gamma(1-\beta)} \int_0^t G(t-\tau)\dot{\gamma}(\tau)\,d\tau = c(\beta)\left({}_C D_0^\beta \gamma\right)(t) \tag{3a}$$

$$\gamma(t) = \frac{1}{c(\beta)\Gamma(\beta)} \int_0^t G(t-\tau)\sigma(\tau)\,d\tau = \left(I_0^\beta \sigma\right)(t) \Big/ c(\beta) \tag{3b}$$

where the formal definition of fractional-order integrals and derivatives read:

$$\left(I_0^\beta f\right)(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f(\tau)\, d\tau \tag{4a}$$

$$\left(D_0^\beta f\right)(t) = \frac{d}{dt}\left(I_0^{1-\beta} f\right)(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-\tau)^{-\beta} f(\tau)\, d\tau = \left({}_C D_0^\beta f\right)(t) + \frac{f(0)}{\Gamma(1-\beta)\, t^\beta} \tag{4b}$$

The rheological description of creep and stress relaxation of FHM has been represented, from a mechanical viewpoint, introducing a proper device dubbed springpot [12], with an intermediate behavior among a linear elastic spring and a linear viscous dashpots. Recently the authors introduced a mechanical equivalent model that restitutes, exactly, the rheological stress-strain relation involved by the springpot element and mathematically described in Eqs.(3 a,b) [14].

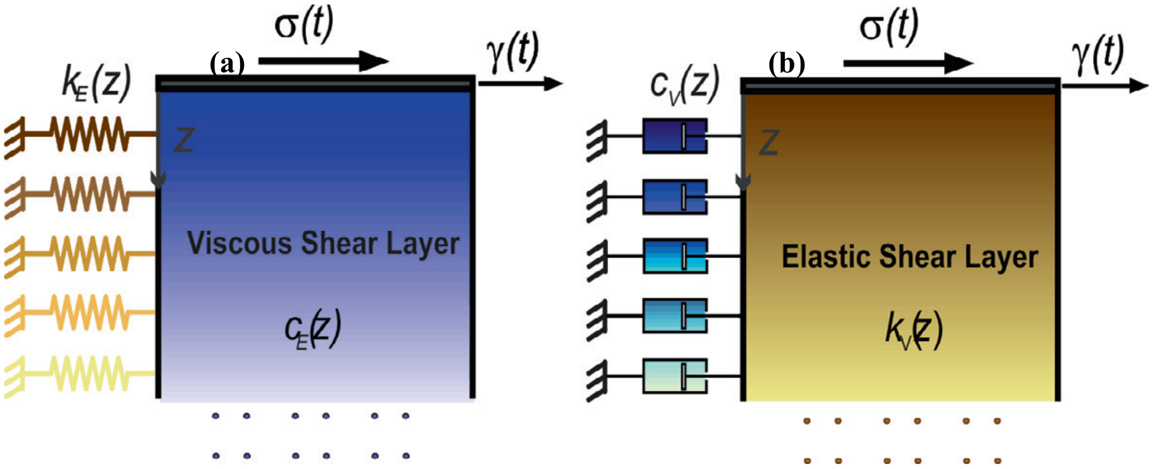

Figure 1: Mechanical analogue of fractional hereditary materials: (a) Elasto-viscous material; (b) Visco-elastic material

In this perspective two kinds of material behavior have been identified:

- The so-called elasto-viscous (EV) materials in which the elastic, solid, phase prevails at the beginning of the creep test and that are characterized by derivation indexes $0 \le \beta \le 1/2$.
- The visco-elastic (VE) materials in which the fluid, viscous phase prevails over the solid behavior at the beginning of the creep test that are characterized by $1/2 \le \beta \le 1$.

EV and VE mechanical model of FHM corresponds to different mechanical models that have been represented, respectively, in Fig.(1 a ,b).

In more detail EV materials are mechanically represented by a unbounded viscous shear column externally restrained on a bed of linear independent elastic springs (Fig.1a); The mechanical model of VE materials is represented by an elastic shear layer resting on a bed of linearly independent dashpots (Fig.1b). Springs elastic constants and viscous coefficients of EV material, under the assumption of unitary cross-section $(A = 1)$ of the column are expressed as:

$$k_E(z) = G_E(z)A = \frac{G_0 z^{-\alpha}}{\Gamma(1+\alpha)} \quad ; \quad c_E(z) = \eta_E(z)A = \frac{\eta_0 z^{-\alpha}}{\Gamma(1-\alpha)} \tag{5 a, b}$$

with $0 \le \alpha \le 1$ whereas, for the mechanical model of VE materials the decay of shear modulus and viscosity reads:

$$k_V(z) = G_V(z)A = \frac{G_0 z^{-\alpha}}{\Gamma(1+\alpha)} \quad ; \quad c_V(z) = \eta_V(z)A = \frac{\eta_0 z^{-\alpha}}{\Gamma(1-\alpha)} \tag{6 a, b}$$

The governing equation of the mechanical model of EV material in (fig. 1a) reads:

$$\frac{\partial}{\partial z}\left[\eta_E(z)\frac{\partial \dot{\gamma}}{\partial z}\right] = G_E(z)\gamma(z,t) \tag{7}$$

The solution of such differential equation have been obtained in Laplace domain resorting yielding the shear stress in terms of the strain field (spatial gradient of the transverse displacement) in the form:

$$\hat{\gamma}(z,s) = z^{\beta}\left[B_1 Y_{\beta}\left(\frac{z}{\sqrt{\tau_{\alpha}^{(E)} s}}\right) + B_2 K_{\beta}\left(\frac{z}{\sqrt{\tau_{\alpha}^{(E)} s}}\right)\right] \tag{8}$$

where integration constant in eq.(7) must be obtained by position of the relevant boundary conditions that reads:

$$\lim_{z\to 0}\hat{\gamma}(z,s) = \hat{\gamma}_0(s) \quad ; \quad \lim_{z\to\infty}\hat{\gamma}(z,s) = 0 \tag{9 a, b}$$

with $\beta = (\alpha + 1)/2$ and the relaxation time $\tau_{\alpha}^{(E)} = -\tau_0 \Gamma(\alpha)/\Gamma(-\alpha)$. Solution of the boundary value problem in eq.(7) accounting for the boundary conditions in eq.(9 a, b) yields the transverse displacement field and the shear stress field in the form:

$$\sigma(t) = C_E(\bar{\beta})\left({}_C D_0^{\bar{\beta}}\gamma\right)(t) \tag{10}$$

where $\bar{\beta} = 1 - \beta$ and the force coefficient $C_E(\bar{\beta}) = G_0\Gamma(\bar{\beta})/\left(2^{1-2\bar{\beta}}\Gamma(2-2\bar{\beta})\Gamma(1-\bar{\beta})\right)$ leading to conclude that eq.(10) coalesces with the rheological stress-strain relation of FHM reported in eq.(3). Similar considerations may be withdrawn for the mechanical model of VE materials reported in fig.(1 b) with governing equation ruled by the expression:

$$\frac{\partial}{\partial z}\left[G_V(z)\frac{\partial \gamma}{\partial z}\right] = \eta_V(z)\dot{\gamma}(z,t) \tag{11}$$

with shear stress expressed at the top of the elastic shear layer in the form:

$$\sigma(t) = C_V(\beta)\left({}_C D_0^{\beta}\gamma\right)(t) \tag{12}$$

The continuous mechanical model reported in this section involves a continuous retardation/relaxation spectra [4,5] of the material. This consideration led to conclude that a discretized version of the model will be represented by a mechanical fractance with scaling relaxation times and, as a consequence, to time-order multiscale model of hereditariness. Indeed the discretized model of EV and VE material is represented in fig.(2 a, b) obtained considering an uniform grid of rigid layers located at uniform distances Δz connected by elastic springs and dashpots that read, for EV and VE materials, respectively:

$$k_{Ej} = \frac{G_0 z_j^{-\alpha}}{\Gamma(1+\alpha)}\Delta z\ ;\ c_{Ej} = \frac{\eta_0 z_j^{-\alpha}}{\Gamma(1-\alpha)\Delta z}\ ;\ k_{Vj} = \frac{G_0 z_j^{-\alpha}}{\Gamma(1-\alpha)\Delta z}\ ;\ c_{Vj} = \frac{\eta_0 z_j^{-\alpha}\Delta z}{\Gamma(1+\alpha)} \qquad \text{(13 a, b, c ,d)}$$

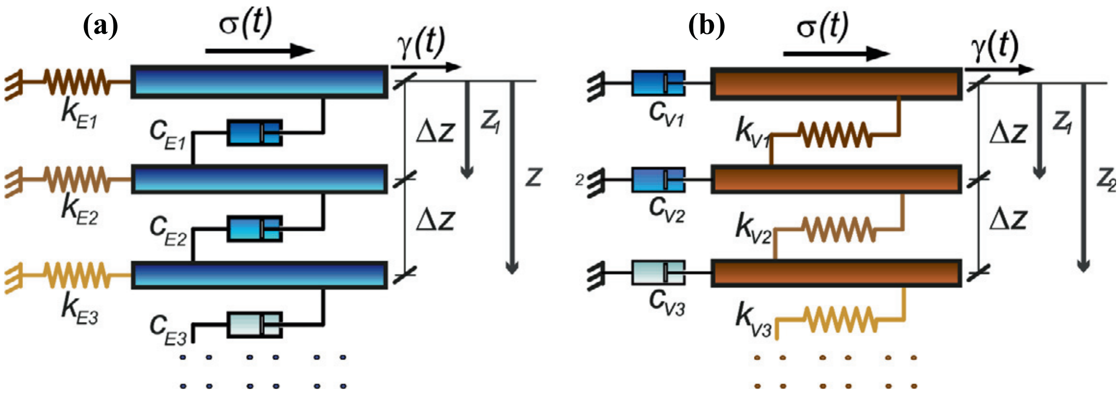

Figure 2: Fractance model of fractional hereditary materials: (a) EV-material model; (b) VE-material model

with $j = 1,2,...\infty$. Collecting the transverse degree of freedom of the discrete model in fig.(3) in a N-dimensional vector $\boldsymbol{\gamma}(t) = \left[\begin{matrix}\gamma_1(t) & \gamma_2(t) & ... & ... & \gamma_N(t)\end{matrix}\right]^T$ the dynamic equilibrium equation of the discretized system reported in figs.(3a) is ruled by the differential equations:

$$p_E \mathbf{A}\dot{\boldsymbol{\gamma}} + q_E \mathbf{B}\boldsymbol{\gamma} = \mathbf{v}\sigma(t) \tag{14}$$

with:

$$\mathbf{A} = \begin{bmatrix} 1^{-\alpha} & -1^{-\alpha} & 0 & ... & 0 \\ -1^{-\alpha} & 1^{-\alpha}+2^{-\alpha} & -2^{-\alpha} & ... & 0 \\ 0 & -2^{-\alpha} & 2^{-\alpha}+3^{-\alpha} & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & (N-1)^{-\alpha}+N^{-\alpha} \end{bmatrix} \qquad \text{(15 a)}$$

$$\mathbf{B} = \begin{bmatrix} 1^{-\alpha} & 0 & 0 & ... & 0 \\ 0 & 2^{-\alpha} & 0 & ... & 0 \\ 0 & 0 & 3^{-\alpha} & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & N^{-\alpha} \end{bmatrix} \tag{15 b}$$

with coefficient p_E and q_E in Eq.(14) expressed as $p_E = \eta_0 \Delta z^{-(1+\alpha)} / \Gamma(1-\alpha), q_E = G_0 \Delta z^{(1-\alpha)} / \Gamma(1+\alpha)$, and the vector load $\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \end{bmatrix}^T$. Similar arguments holds for the VE model of FHM described in fig.(3 b) with governing equations that may be written as:

$$p_V \mathbf{A}\dot{\boldsymbol{\gamma}} + q_V \mathbf{B}\boldsymbol{\gamma} = \mathbf{v}\sigma(t) \tag{16}$$

with $p_V = \eta_0 \Delta z^{(1-\alpha)} / \Gamma(1+\alpha), q_V = G_0 \Delta z^{-(1+\alpha)} / \Gamma(1-\alpha)$. System of equations defined for EV and VE materials in eqs.(14, 16) may be solved introducing a coordinate transform $\mathbf{x}(t) = \mathbf{B}^{1/2}\boldsymbol{\gamma}(t)$ yielding for EV and VE model, respectively:

$$p_E \mathbf{D}\dot{\mathbf{y}} + q_E \mathbf{y} = \bar{\mathbf{v}}\sigma(t); \; p_V \mathbf{D}\dot{\mathbf{y}} + q_V \mathbf{y} = \bar{\mathbf{v}}\sigma(t) \tag{17 a, b}$$

where $\bar{\mathbf{v}} = \mathbf{B}^{-1/2}\mathbf{v}$ and the symmetric and positive definite dynamic equilibrium matrix $\mathbf{D}$ expressed in the form:

$$\mathbf{D} = \begin{bmatrix} 1 & -(2/1)^{\frac{\alpha}{2}} & 0 & ... & 0 \\ -(2/1)^{\frac{\alpha}{2}} & 1+(2/1)^{\alpha} & -(3/2)^{\frac{\alpha}{2}} & ... & 0 \\ 0 & -(3/2)^{\frac{\alpha}{2}} & 1+(3/2)^{\alpha} & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & -(N/N-1)^{\frac{\alpha}{2}} & 1+(N/N-1)^{\alpha} \end{bmatrix} \tag{18}$$

Solution of eqs.(17 a, b) will be obtained in the modal coordinate space introducing the modal transform $\mathbf{x}(t) = \boldsymbol{\Phi}\mathbf{y}(t)$ with the eigenvector matrix $\boldsymbol{\Phi} = \begin{bmatrix} \phi_1 & \phi_2 & ... & \phi_N \end{bmatrix}$ and ϕ_j the generic eigenvector of matrix $\mathbf{D}$ associated to eigenvalue λ_j. The generic dynamic equilibrium equation of the discretized mechanical model in the modal coordinate space reads, respectively for the EV and VE materials:

$$\dot{y}_j + \rho_j y_j = \frac{\phi_{1,j}}{p_E \lambda_j}\sigma(t) \quad ; \quad \delta_j \dot{y}_j + y_j = \frac{\phi_{1,j}}{q_V \lambda_j}\sigma(t) \quad ; \quad j = 1,2,3,....N \tag{19 a, b}$$

with, $\phi_{1,j}$ the first component of the j^{th} eigenvector and the coefficients ρ_j and δ_j in the form:

$$\rho_j = q_E\Big/(p_E\lambda_j) > 0 \quad ; \quad \delta_j = p_V\Big/(q_V\lambda_j) > 0 \tag{20 a, b}$$

3. The Multiscale and Multiphase Stochastic Model of Fractional Hereditary Materials

Mechanical model corresponding to fractional rheological stress-strain relation may be extended to case of multiphase rheological materials that are characterized by multiple power-law decays in relaxation functions. Indeed it may be observed (see fig.1) that long-term behavior of materials may involve a non-zero value of the residual stress so that, as $t \to \infty$, $G(t) \to g_\infty \neq 0$. In this context the mathematical expression of the relaxation function is provided by $G(t) = c(\beta)t^{-\beta} + g_\infty$ and it corresponds to the rheological model represented by a parallel arrangement of springpot and a linear spring with stiffness g_∞ that is known as fractional-order Kelvin model [16]. Stress and strain evolution may be obtained by means of the relation:

$$\sigma(t) = c(\beta)\left({}_C D_0^\beta \gamma\right)(t) + g_\infty; \; \gamma(t) = \frac{1}{c(\beta)}\int_0^t (t-\tau)^{\beta-1} E_{\beta,1}\left(-\frac{g_\infty}{c(\beta)}(t-\tau)^\beta\right)\sigma(\tau)d\tau \tag{21 a, b}$$

with $E_{\beta,1}(\cdot)$ the one parameter Mittag-Leffler function. An alternative description of fractional Kelvin model may be obtained by means of the exact mechanical model of fractional-order elements as reported in fig.(3 a, b) with $k_\infty = g_\infty A/\Delta z$ involving only slight changes to matrices $\mathbf{A}, \mathbf{B}$ [15].

The observation of experimental data as reported in fig.(3 b) shows that the mechanical parameters in the relaxation functions, namely, $c(\beta)$ and g_∞ depends on the material specimen considered and, therefore, they may be modeled as independent random variables. In this section we will denote random variables modeling mechanical parameters $c(\beta)$ and g_∞ with capital letters as $c(\beta) \to C(\beta)$ and $g_\infty \to G_\infty$. The probability density function of random variables $C(\beta)$ and G_∞ will be denoted as $p_C(c(\beta))$ and $p_G(g_\infty)$, respectively with the joint probability density $p_{CG}(c(\beta), g_\infty) = p_G(g_\infty)p_C(c(\beta))$. The random model of the mechanical parameters in the relaxation function of the material yields a random description of the direct and inverse stress-strain relations yielding, for first and second-order moments of the stress the relations:

$$\begin{aligned} E\left[\sigma(t)\right] &= E\left[C(\beta)\right]\left({}_C D_0^\beta \gamma\right)(t) + E\{G_\infty\} \\ E\left[\sigma(t)^2\right] &= E\left[C(\beta)^2\right]\left({}_C D_0^\beta \gamma\right)(t) + E\left[G_\infty^2\right] \end{aligned} \tag{22 a, b}$$

First and second-order expectations of the strain involves all-order statistics of the random variables $C(\beta)$ and G_∞ as it may be observed casting eq.(20) in an alternative form:

$$\gamma(t) = \sum_{j=1}^{\infty} (-G_\infty)^j \left(\frac{1}{C(\beta)}\right)^{j+1} \left(I_0^{\beta(j+1)}\sigma\right)(t) \tag{23}$$

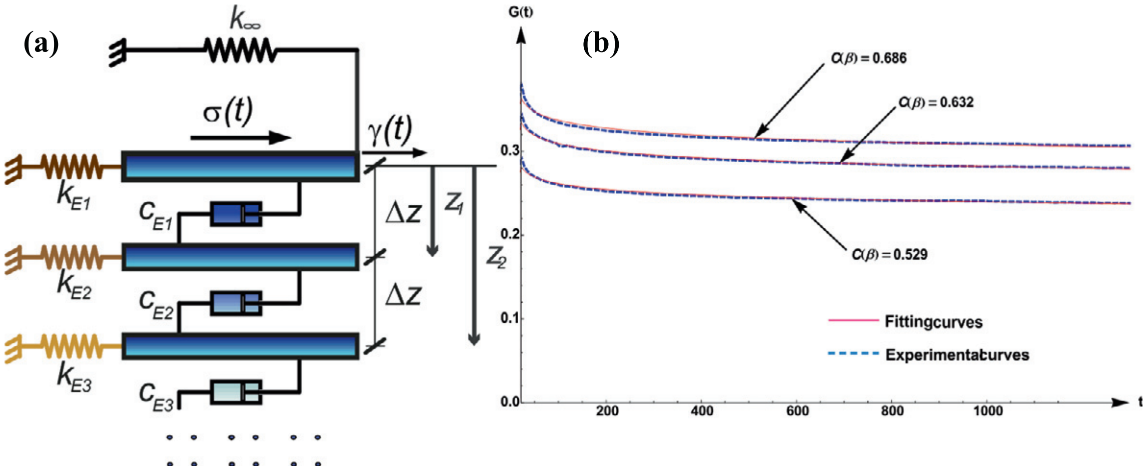

Figure 3: (a) Fractance model of EV-fractional Voigt element; (b) Experimental relaxation function for different material specimen (Aerstop VX5, Material laboratory of the DICAM, Università degli studi di Palermo, I-90128, Palermo, Italy, reprinted with permission).

yielding the first and second-order statistics of the strain evolution in the form:

$$E\left[\gamma(t)\right] = \sum_{j=1}^{\infty} E\left[(-G_\infty)^j\right] E\left[\left(\frac{1}{C(\beta)}\right)^{j+1}\right] \left(I_0^{\beta(j+1)}\sigma\right)(t)$$

$$E\left[\gamma^2(t)\right] = \sum_{j=1}^{\infty}\sum_{r=1}^{\infty} E\left[(-G_\infty)^{j+r}\right] E\left[\left(\frac{1}{C(\beta)}\right)^{j+r+2}\right] \left(I_0^{\beta(j+1)}\sigma\right)(t)\left(I_0^{\beta(r+1)}\sigma\right)(t) \qquad (24\text{ a, b})$$

The expressions reported in eqs.(24 a, b) for the first and second-order statistics of the strain field evolution involves all-order statistics of the random variables $C(\beta)$ and G_∞ so that, as their probability density deviates from the normal distribution accurate description of random variables requires the introduction of several mathematical expectation in eq.(23 a, b). Such a consideration, beside the mathematical aspect, involves, however, creep functions in terms of power-laws with exponents larger than one that is not representative of a viscoelastic material. As an alternative, the mechanical description of fractional Kelvin model with random coefficients may be used to characterize the time evolution of the strain field from a multiscale physical perspective. This is a very interesting consideration that will be addressed in future studies.

4. Conclusions

In this paper the authors aim study multiscale time-dependent hereditariness observed in the so-called FHM. Indeed fractional rheological models are characterized by a continuous relaxation/retardation spectrum that corresponds, in a mechanical context to an intermediate device, denoted springpot. In previous studies the authors showed that springpot element possesses an exact mechanical equivalence with a mechanical fractance of linear springs and dashpots with variable coefficients. It has been shown that the governing equation of the fractance may approximate, with a desidered order, the relaxation as well as the retardation effect of any fractional hereditary materials. Moreover the model is also capable to handle the case of multiple power-law relations in the creep/relaxation function as observed in multiphase fractional hereditary materials, as it may be experienced in long-term behavior of several time-dependent materials. In this study the authors extended the concept of time-scales to the case of random dependence of the material parameters on the considered specimen. It has been shown that assuming a random variation of the material coefficients with prescribed probability density function involves the same statistical dependence in the direct rheological model whereas the inverse relation depends upon every-order statistics of the material random parameters. The dependence is a serious drawback in the analysis of time dependent response of such material behavior since material random parameters may deviate, significantly, from the Normal distribution and they may require consideration of several stochastic moments to achieve an accurate probabilistic description. Such a consideration is highlighted observing that the inverse stress-strain relation, namely the strain evolution due to a prescribed load history, involves a series expansion in terms of fractional integrals of increasing order of the stress evolution. It is believed that the mathematical description of random multiphase fractional hereditary materials may be simplified by means of the mechanical models of fractional springpot that will be addressed in future studies.

Moreover the presence of multiaxial stress state applied to a FHM specimen may be also dealt with the proposed mechanical model as we introduce a different arrangement of springs and dashpots, for EV and VE materials, respectively. The precise arrangement of springs and dashpots must be set as the functional class of the constitutive relations, in terms of the constitutive material tensor, has been set at the macroscale.

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