Selective writing and read-out of a register of static qubits

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Selective writing and read-out of a register of static qubits

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Abstract. We propose a setup comprising an arbitrarily large array of static qubits (SQs), which interact with a flying qubit (FQ). The SQs work as a quantum register, which can be written or read out by means of the FQ through quantum state transfer (QST). The entire system, including the FQ's motional degrees of freedom, behaves quantum mechanically. We demonstrate a strategy allowing for selective QST between the FQ and a single SQ chosen from the register. This is achieved through a perfect mirror located beyond the SQs and suitable modulation of the inter-SQ distances.

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Contents

1. Introduction

A prominent paradigm in quantum information processing (QIP) [1] is to employ flying qubits (FQs) and static qubits (SQs) as the carriers and registers of quantum information, respectively [2]. Key to such an idea is the ability to write and read out the information content of a SQ by means of a FQ. By this, here we mean that an efficient quantum state transfer (QST) between these two types of qubits must be possible on demand. In this picture, *control over memory allocation* appears to be a desirable if not indispensable requirement. For instance, one can envisage the situation where only one or a few SQs are available, e.g. because the remaining ones are encoding some information to save. On the other hand, one may need to carry away only the information saved in certain specific SQs. Alternatively, only a restricted area of the register of SQs may be interfaced with some external processing network where one would like to eventually convey information or from which output data are to be received. In such cases, the ability of *selecting* the exact location where the information content of the FQ should be uploaded or downloaded is demanded. Ideally, according to the schematics in figure 1, one would like the FQ to reach the specific target SQ, then fully transfer its quantum state to this and eventually fly away. Evidently, this picture is implicitly based on the assumption that, firstly, the motional degrees of freedom (MDOFs) of the FQ are in fact fully classical and, secondly, these can be accurately controlled. Despite its simplicity, although interesting research along these lines is being carried out mostly through the so-called surface acoustic waves (see e.g. [3] and references therein), such an approach calls for a very high level of control.

If set within a fully quantum framework, the most natural situation to envisage is the one where the FQ, besides bearing an internal spin, moves in a quantum mechanical way and hence propagates as a wave-like object. Such a circumstance substantially complicates the dynamics in that, besides the complex spin–spin interactions, intricate wave-like effects such as multiple reflections between the many SQs occur as well. This appears to be an adverse environment to accomplish *selective* QST: while ideally one would like to focus the FQ's wave packet right on the target SQ, the former is expected to spread throughout the SQs' register. Thus, not only is it non-trivial what strategy would enable selective QST but even the mere possibility that this could occur can be questioned.

In this work, we consider a paradigmatic Hamiltonian memory read-out model where the FQ propagates along a one-dimensional (1D) line comprising a collection of (fixed) spatially
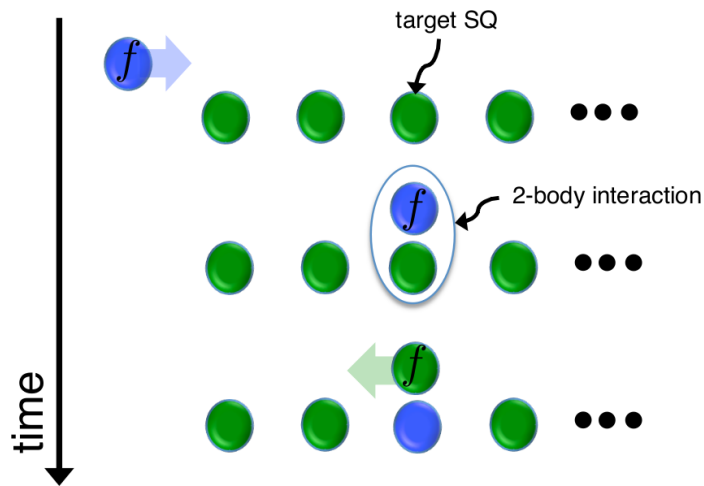
Figure 1. Selective QST between an FQ and a register of SQs. The FQ reaches the target SQ, exchanges its information content with it and eventually leaves the register.
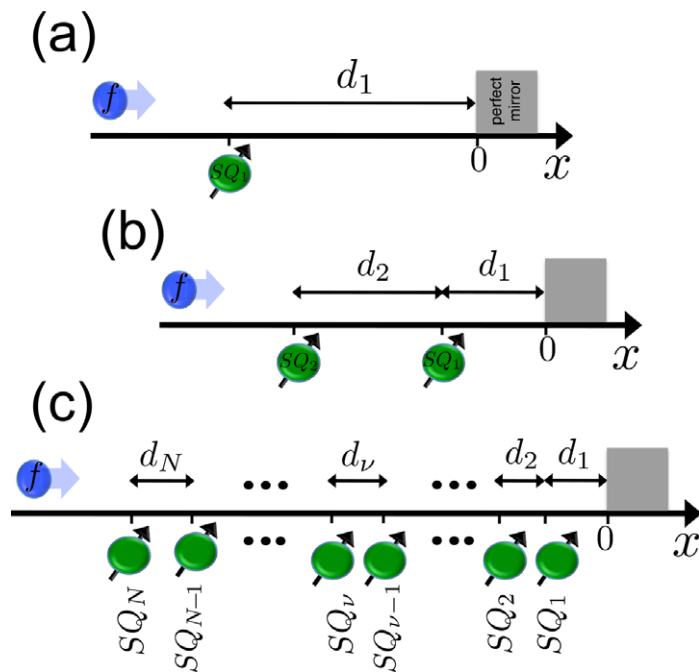
Figure 2. Sketch of the setup in the case of one (a), two (b) and arbitrary N (c) SQs. The FQ f incomes from the left with a given wave vector k, undergoes multiple scattering between the SQs and the perfect mirror and eventually moves away from the register with the same k.

separated non-interacting SQs and couples to them via contact-type spin–spin Heisenberg interactions (see figure 2). We start with a single SQ and prove that a unitary SWAP between the itinerant and static spins is unattainable. The insertion of a perfect mirror along the 1D line, however, makes it possible. At the same time, since the transmission channel is suppressed there

is no uncertainty over the final path followed by the FQ. Next, we find that even for a pair of SQs this can be achieved with either of the two SQs through an ad hoc setting of distances and coupling strengths. Surprisingly enough, this means that Feynman paths entering multiple reflections can combine so as to effectively decouple one SQ while enabling at the same time a unitary swap involving the other one. Even more surprisingly, the working principle behind this phenomenon is such that it is naturally generalized to the case of an arbitrarily large register of SQs, as we rigorously prove.

2. Read-out of a single static memory qubit

Consider the case where a single memory static qubit SQ_1 lies on the x-axis close to position $x = 0$. To read out the quantum information stored in SQ_1 (or write it there), an FQ f is injected along the axis with momentum k, say from the left-hand side. We model the f–SQ_1 interaction as a contact-type spin-dependent scattering potential having the Heisenberg coupling form. The system Hamiltonian can thus be expressed as $\hat{H} = \hat{p}^2/2 + \hat{V}$, where $\hat{p}$ is the momentum operator of f (its mass being set equal to 1 for simplicity) and

$$\hat{V} = G(\hat{\boldsymbol{\sigma}}_f \cdot \hat{\boldsymbol{\sigma}}_1)\delta(x) \tag{1}$$

is the coupling potential with the associated strength G.[5] Here, x is the spatial coordinate of f, while $\hat{\boldsymbol{\sigma}}_f$ and $\hat{\boldsymbol{\sigma}}_1$ are the spin operators of qubits f and SQ_1, respectively, i.e. $\hat{\boldsymbol{\sigma}} = (\sigma_x, \sigma_y, \sigma_z)$ with $\hat{\sigma}_{\beta=x,y,z}$ having eigenvalues $\pm 1/2$ (we set $\hbar = 1$ throughout). We ask whether or not, when f will emerge from the scattering process, the internal degree of freedom (i.e. the spin) of the two qubits has been exchanged according to the mapping

$$\rho_{f1} \rightarrow \rho_{f1}^{(\text{SWAP})} = \hat{W}_{f1} \rho_{f1} \hat{W}_{f1}^\dagger, \tag{2}$$

where ρ_{f1} is the (joint) input spin state of f and SQ_1, while $\hat{W}_{ij}$ is the usual swap two-qubit unitary operator exchanging the states of qubits i and j [1]. While there are, in fact, counterexamples [4, 5] showing that this is impossible[6], we give next the general proof that such a swap operation cannot occur. For this purpose, let us define $|\Psi^\pm\rangle_{f1} = (|\uparrow\downarrow\rangle_{f1} \pm |\downarrow\uparrow\rangle_{f1})/\sqrt{2}$, where for each qubit, either flying or static, $|\uparrow\rangle$ and $|\downarrow\rangle$ stand for the eigenstates of $\hat{\sigma}_z$ with eigenvalues $1/2$ and $-1/2$, respectively (from now on, we omit particle subscripts whenever unnecessary). The state $|\Psi^-\rangle$ is the well-known singlet, while the triplet subspace is spanned by $\{|\uparrow\uparrow\rangle, |\Psi^+\rangle, |\downarrow\downarrow\rangle\}$. Using the identity $\hat{\boldsymbol{\sigma}}_f \cdot \hat{\boldsymbol{\sigma}}_1 = (\hat{S}_{f1}^2 - \hat{\sigma}_f^2 - \hat{\sigma}_1^2)/2$, where $\hat{S}_{f1} = \hat{\boldsymbol{\sigma}}_f + \hat{\boldsymbol{\sigma}}_1$, the interaction Hamiltonian can be written as $\hat{V} = (G/2)(\hat{S}_{f1}^2 - 3/2)\delta(x)$, entailing $[\hat{H}, \hat{S}_{f1}^2] = 0$ [5–7]. Within the singlet (triplet) subspace, the effective interaction is thus *spinless* and reads $\hat{V}_s = -(3G/4)\delta(x)$ [$\hat{V}_t = (G/4)\delta(x)$]: the problem is reduced to a scattering from a (spin-independent) δ-barrier. For a δ-potential step $\Gamma\delta(x)$ and a particle incoming with momentum k, the reflection and transmission probability amplitudes $r^{(0)}(\gamma)$ and $t^{(0)}(\gamma)$, respectively, are found through a textbook calculation to be

$$r^{(0)}(\gamma) = t^{(0)}(\gamma) - 1 = -i\gamma/(1 + i\gamma), \tag{3}$$

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[5] The assumption of the δ-shaped potential is a standard one, and for the present setup it relies on the usually met condition that the FQ's wavelength is significantly larger than the characteristic SQ size.

[6] In [5], it was proven that, given the initial spin state $|\uparrow\downarrow\rangle_{f1}$, the scattering process between f and SQ_1 can never lead to $\langle\hat{\sigma}_{1z}\rangle = 1/2$. Owing to the conservation of $\hat{\sigma}_{fz} + \hat{\sigma}_{1z}$, this is equivalent to stating that the transformation $|\uparrow\downarrow\rangle_{f1} \rightarrow |\downarrow\uparrow\rangle_{f1}$ is unattainable.

where we have introduced the rescaled parameter $\gamma = \Gamma/k$. These functions allow to calculate the reflection coefficient for the singlet and triplet sectors as

$$r_{\rm s} = t_{\rm s} - 1 = r^{(0)}(-3g/4) \quad \text{(singlet)}, \tag{4}$$

$$r_{\rm t} = t_{\rm t} - 1 = r^{(0)}(g/4) \qquad \text{(triplet)}, \tag{5}$$

where we have set $g = G/k$. Evidently, $|r_{\rm t}| \neq |r_{\rm s}|$ for any $G \neq 0$. This is the very reason which forbids one from using the above scattering process for implementing *any* unitary gate on the spin degree of freedom of f and SQ_1, hence, in particular, the SWAP gate (2) enabling perfect writing/read-out of SQ_1. Observe indeed that, once the orbital degree of freedom of the FQ are traced out, the final spin state ρ'_{f1} of the joint system f–SQ_1 can be related to the initial one ρ_{f1} (in general, mixed) through the completely positive, trace-preserving map [1]

$$\rho_{f1} \rightarrow \rho'_{f1} = \hat{T}_{f1}\rho_{f1}\hat{T}^{\dagger}_{f1} + \hat{R}_{f1}\rho_{f1}\hat{R}^{\dagger}_{f1}, \tag{6}$$

where the first contribution refers to the f-wave component emerging from the right of the 1D line (transmission channel), while the second contribution to the one emerging from the left (reflection channel). The Kraus operators [1, 10] $\hat{T}_{f1}$ and $\hat{R}_{f1}$ describing these two complementary events are provided, respectively, by the transmission and reflection operators of the model, namely

$$\hat{R}_{f1} = r_{\rm s}\hat{\Pi}^{(\rm s)}_{f1} + r_{\rm t}\hat{\Pi}^{(\rm t)}_{f1}, \quad \hat{T}_{f1} = t_{\rm s}\hat{\Pi}^{(\rm s)}_{f1} + t_{\rm t}\hat{\Pi}^{(\rm t)}_{f1}, \tag{7}$$

where $\hat{\Pi}^{(\rm s)}_{f1} = |\Psi^-\rangle_{f1}\langle\Psi^-|$ and $\hat{\Pi}^{(\rm t)}_{f1} = \hat{I}_{f1} - \hat{\Pi}^{(\rm s)}_{f1}$ are the projector operators associated with the singlet and triplet subspaces, respectively, of the f–SQ_1 system. Note that in the computational basis $\{|\alpha_f\alpha_1\rangle\}$ ($\alpha_f, \alpha_1 = \uparrow, \downarrow$), a matrix element $\langle\alpha'_f\alpha'_1|\hat{R}_{f1}|\alpha_f\alpha_1\rangle$ yields the probability amplitude that, given the initial joint spin state $|\alpha'_f\alpha'_1\rangle$, f is reflected back and the final spin state is $|\alpha_f\alpha_1\rangle$ [8, 9] (an analogous statement holds for $\hat{T}_{f1}$). Via the identities (4) and (5), one can easily verify that equation (7) immediately entails the proper normalization condition $\hat{T}^{\dagger}_{f1}\hat{T}_{f1} + \hat{R}^{\dagger}_{f1}\hat{R}_{f1} = \hat{I}_{f1}$. Furthermore, expressed in this form it is now easy to see why the mapping (6) is never unitary: in fact for this to happen, $\hat{R}_{f1}$ and $\hat{T}_{f1}$ should be mutually proportional, i.e. $r_{\rm s(t)} = \xi t_{\rm s(t)}$. This is impossible since it requires $r_{\rm s}/t_{\rm s} = r_{\rm t}/t_{\rm t}$, which can be fulfilled only if $r_{\rm s} = r_{\rm t}$ (conflicting with $|r_{\rm s}| \neq |r_{\rm t}|$ proven above).

A strategy to get around this hindrance is to insert a *perfect mirror* at $x = 0$ beyond the SQ located at $x = x_1$ at a distance d_1 as sketched in figure 2(a) (this is inspired by [9], where, however, a somewhat different system was addressed). First of all, such a modified geometry suppresses the transmission channel eliminating the uncertainty in the direction along which f propagates after interacting with SQ_1. Specifically, in the presence of the perfect mirror we have $\hat{T}^{(\rm m)}_{f1} = 0$ and equation (6) thus reduces to

$$\rho_{f1} \rightarrow \rho'_{f1} = \hat{R}^{(\rm m)}_{f1}\rho_{f1}\hat{R}^{(\rm m)\dagger}_{f1}, \tag{8}$$

where now the reflection matrix $\hat{R}^{(\rm m)}_{f1}$ is always unitary $\hat{R}^{(\rm m)\dagger}_{f1}\hat{R}^{(\rm m)}_{f1} = \hat{R}^{(\rm m)}_{f1}\hat{R}^{(\rm m)\dagger}_{f1} = \hat{I}_{f1}$. More interestingly, equation (8) allows for the perfect SWAP gate (2) to be implemented. To see this, observe that since the squared total spin is still a conserved quantity as in the no-mirror case, the problem reduces to a spinless particle scattering from a *spinless* barrier $\Gamma\delta(x - x_1)$ and a perfect mirror which, via a simple textbook calculation, gives the reflection amplitude

$$r^{(\rm m)}(\gamma) = -\left[\mathrm{i}\gamma + (1 - \mathrm{i}\gamma)\mathrm{e}^{2\mathrm{i}kd_1}\right]/\left[1 + \mathrm{i}\gamma(1 - \mathrm{e}^{2\mathrm{i}kd_1})\right] \tag{9}$$
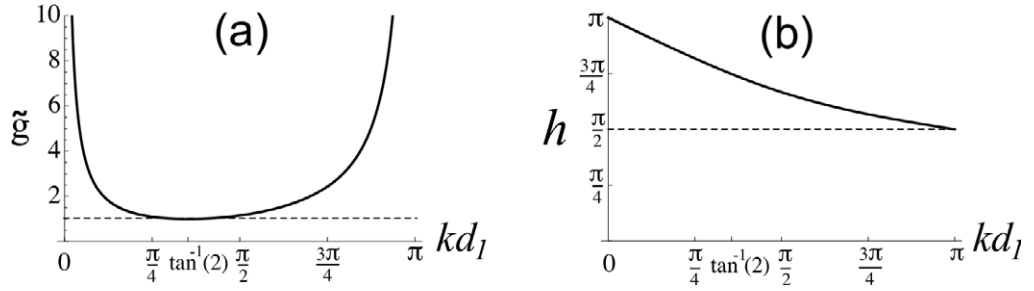
Figure 3. Plots of the functions $\tilde{g}(kd_1)$ in equation (13) (panel (a)) and $h(kd_1)$ in equation (16) (panel (b)), which set the conditions for perfect SWAP between f and the static memories. Either function is periodic of period π. Note, in particular, that as the optical distance kd_1 approaches $n\pi$ ($n = 1, 2, \ldots$) condition (13) can be satisfied only in the asymptotic limit of infinite spin–spin coupling. Moreover, there is a threshold $g_{th} = 1$ (dashed line in panel (a)) that g must exceed to ensure the existence of values of kd_1 allowing for the implementation of the SWAP gate.

(recall that $\gamma = \Gamma/k$). Therefore, a reasoning fully analogous to the previous case leads to

$$\hat{R}_{f1}^{(m)} = r_s^{(m)}\hat{\Pi}_{f1}^{(s)} + r_t^{(m)}\hat{\Pi}_{f1}^{(t)} \tag{10}$$

with

$$r_s^{(m)} = r^{(m)}(-3g/4) \quad \text{(singlet)}, \tag{11}$$

$$r_t^{(m)} = r^{(m)}(g/4) \qquad \text{(triplet)}. \tag{12}$$

Observe that $\hat{R}_{f1}^{(m)}$ is unitary because $r^{(m)}(\gamma)$ has unit modulus. To work out the conditions for realizing an f–SQ$_1$ SWAP gate (2), we use the fact that this unitary can be written as $\hat{W}_{f1} = -\hat{\Pi}_{f1}^{(s)} + \hat{\Pi}_{f1}^{(t)}$. Evidently, $\hat{R}_{f1}^{(m)}$ can be made coincident with $\hat{W}_{f1}$ (up to an irrelevant global phase factor) if and only if $r_s^{(m)} = -r_t^{(m)}$. This identity is fulfilled provided that g and kd_1 are related to each other according to the function

$$g = \tilde{g}(kd_1) = \frac{2}{3}\left(\sqrt{3 + 4\cot^2 kd_1} - \cot kd_1\right) \tag{13}$$

which is plotted in figure 3(a). Interestingly, $\tilde{g}(kd_1) \geqslant 1$ means that g must exceed the threshold $g_{th} = 1$ to ensure occurrence of the SWAP. To summarize, in the presence of a single SQ and for a given spin–spin coupling strength, for any $0 < kd_1 < \pi$ (see figure 3(a)) there always exists a corresponding coupling constant $G \geqslant k$ ensuring the occurrence of the f–SQ$_1$ SWAP. Conversely, as long as G is strictly larger than k, there are always two distinct values of kd_1 enabling the perfect SWAP between f and SQ$_1$.

Before concluding this section, we point out that, based on the form of $r_{s(t)}^{(m)}$, when the optical distance kd_1 is an integer multiple of π (i.e. $kd_1 = n\pi$) the above coefficients reduce to $r_s^{(m)} = r_t^{(m)} = -1$ and hence $\hat{R}_{f1}^{(m)} = -\hat{I}_{f1}$ independently of the coupling strength. This situation is indeed equivalent to *moving the mirror to SQ$_1$'s location*: the chance for the FQ to be found at such a position then vanishes and its spin is thus unable to couple to the SQs. More generally, the property that two objects whose optical separation is an integer multiple of π behave *as if they were at the same place* will be exploited repeatedly in this work.
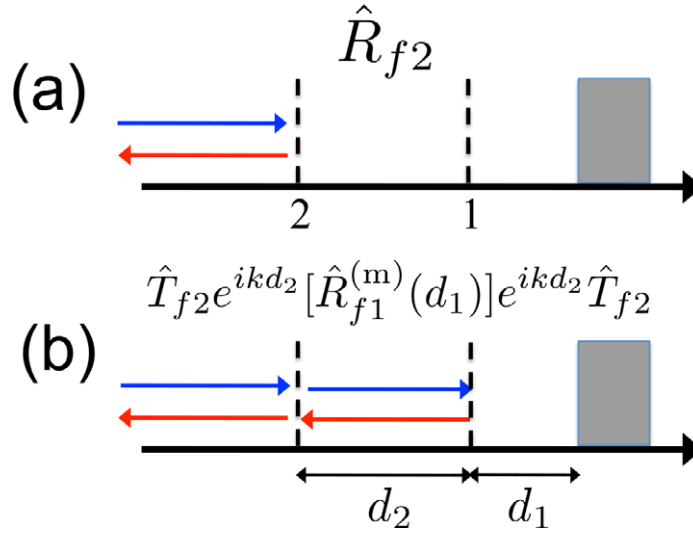
Figure 4. The first-order (a) and the second-order (b) Feynman paths contributing to equation (15).

3. Two static qubits

In addition to SQ_1 and the perfect mirror, the setup now comprises a further SQ, dubbed SQ_2, located on the left of 1 at a distance d_2 from it as shown in figure 2(b). Hence, the spin–spin coupling term in $\hat{H}$ now reads

$$\hat{V} = G \sum_{i=1,2} (\hat{\boldsymbol{\sigma}}_f \cdot \hat{\boldsymbol{\sigma}}_i) \delta(x - x_i), \tag{14}$$

where $x_1 = -d_1$ and $x_2 = -(d_1 + d_2)$. We aim to implement either a f–SQ_1 or a f–SQ_2 SWAP operation, i.e. either the unitary $\hat{W}_{f1} \otimes \hat{I}_2$ or $\hat{I}_1 \otimes \hat{W}_{f2}$, respectively (note that in any case we require one of the two SQs to be unaffected). Analogously to the single-SQ case, the mirror suppresses the transmission channel and thereby one can define a unitary reflection operator $\hat{R}_{f12}$ within the eight-dimensional (8D) overall spin space that fully describes the interaction process output. In the spirit of scattering matrices combination via the sum over different Feynman paths [13], the scattering operator $\hat{R}_{f12}$ results from a superposition of all possible paths, the first of which are sketched in figure 4. The overall sum is obtained in terms of a geometric series as

$$\hat{R}_{f12} = \hat{R}_{f2} + \hat{T}_{f2} (\hat{I}_{f12} - \hat{R}_{f1}^{(m)} \hat{R}_{f2} \, e^{i2kd_2})^{-1} \hat{R}_{f1}^{(m)} \hat{T}_{f2} \, e^{2ikd_2}, \tag{15}$$

where although not shown by our notation, while it involves qubits f and $SQ_{1(2)}$, each reflection or transmission operator on the right-hand side is intended as an extension to the present 8D spin space. Also, note that $\hat{R}_{f1}^{(m)}$ is a function of kd_1.

The present setup ensures QST between f–SQ_1 and f–SQ_2, respectively, in the regimes

$$f - SQ_1 \quad QST: \quad kd_2 = h(kd_1), \quad g = \tilde{g}(kd_1), \tag{16}$$

$$f - SQ_2 \quad QST: \quad kd_1 = n\pi, \quad g = \tilde{g}(kd_2), \tag{17}$$

where $n = 1, 2, \ldots$, while $h(kd_1) = \pi - \arg[r_s^{(m)}(\tilde{g})]/2$ is a periodic function of period π plotted in figure 3(b). Condition (17) is easily understood: we have already discussed (see the previous section) that when $kd_1 = n\pi$ the optical distance between SQ$_1$ and the mirror is effectively zero; hence, it is as if the mirror lies at $x = x_1$ so as to inhibit the f–SQ$_1$ coupling. We are thus left basically with the same setup as the one in the previous section, which shows that if condition $g = \tilde{g}(kd_2)$ is fulfilled (cf equation (13)), then $\hat{R}_{f12} = \hat{I}_1 \otimes \hat{W}_{f2}$.

To prove equation (16), which is key to the central findings in this paper, it is convenient to introduce the coupled spin basis arising from the coupling of $\hat{\sigma}_f$, $\hat{\sigma}_1$ and $\hat{\sigma}_2$. We define $\hat{S}_{fi} = \hat{\sigma}_f + \hat{\sigma}_i$ ($i = 1, 2$) and the total spin $\hat{S} = \hat{\sigma}_f + \sum_{i=1,2} \hat{\sigma}_i$. It is then straightforward to check that equation (14) can be expressed as

$$\hat{V} = (G/2) \sum_{i=1,2} (\hat{S}_{fi}^2 - 3/2)\delta(x - x_i) \tag{18}$$

and thus $[\hat{H}, \hat{S}^2] = 0$ (owing to $[\hat{S}_{fi}^2, \hat{S}^2] = 0$). Also, $[\hat{H}, \hat{S}_z] = 0$. Note, however, that neither $\hat{S}_{f1}^2$ nor $\hat{S}_{f2}^2$ is conserved since $[\hat{S}_{f1}^2, \hat{S}_{f2}^2] \neq 0$. Using now the coupling scheme where $\hat{\sigma}_f$ is first summed to $\hat{\sigma}_1$,[7] the coupled basis reads $\mathcal{B}_{f1} = \{|s_{f1}; s, m\rangle\}$, where s_{f1}, s and $m = -s, \ldots, s$ are the quantum numbers associated with $\hat{S}_{f1}^2$, $\hat{S}^2$ and $\hat{S}_z$, respectively. As $s_{f1} = 0, 1$ (singlet and triplet, respectively) the possible values for s are $s = 1/2, 3/2$. In the subspace $s = 3/2$ only $s_{f1} = 1$ occurs, while for $s = 1/2$, s_{f1} can be both 0 and 1. It should be clear now that given that s and m are good quantum numbers ($\hat{S}^2$ and $\hat{S}_z$ are conserved) $\hat{R}_{f12}$ is block diagonal in the basis $\mathcal{B}_{f1}$: four blocks are 1D, each identified by one of the vectors $\{|s_{f1} = 1; s = 3/2, m = -3/2, \ldots, 3/2\rangle\}$; two blocks are instead two-dimensional (2D), each spanned by $\{|s_{f1} = 0; s = 1/2, m\rangle, |s_{f1} = 1; s = 1/2, m\rangle\}$ and labeled by $m = -1/2, 1/2$. Due to symmetry reasons, for fixed s the effective form of $\hat{R}_{f12}$ in each block is independent of m. Let us first begin with the two $s = 1/2$ blocks. In light of the previous section, for both of them, independently of the value of m, we can write $\hat{R}_{f1}^{(m)} = r_s^{(m)}|0\rangle\langle 0| + r_t^{(m)}|1\rangle\langle 1|$, where we have introduced the concise notation $|s_{f1}\rangle = |s_{f1}; s = 1/2, m\rangle$. As to $\hat{R}_{f2} = \hat{T}_{f2} - \hat{I}_{f2}$, one has to solve an effective scattering problem in a 2D spin space in the presence of the spin-dependent potential barrier $(G/2)(\hat{S}_{f2}^2 - 3/4 - q_{s_2})\delta(x - x_2)$, where s_2 is the quantum number associated with $\hat{\sigma}_2^2$ and we have introduced the discrete function $q_j = j(j + 1)$ (here, although $s_2 = 1/2$, we leave such a quantum number unspecified for reasons that will become clear later on). Such a task can be carried out easily, as we show in the appendix. Next, by requiring condition (13), which ensures that $\hat{R}_{f1}^{(m)}$ implements a QST between f and SQ$_1$ by setting $r_s^{(m)} = -r_t^{(m)}$ and plugging $\hat{R}_{f1}^{(m)}$ and $\hat{R}_{f2}$ into equation (15), the matrix elements of $\hat{R}_{f12}$ in the $s = 1/2$ block $r_{s'_{f1} s_{f1}} = \langle s'_{f1}|\hat{R}_{f12}|s_{f1}\rangle$ are calculated as

$$r_{00} = -[\tilde{g}^2 q_{s_2} - 2(2 - \mathrm{i}\tilde{g})r_s^{(m)}\mathrm{e}^{2\mathrm{i}kd_2} - \mathrm{i}\tilde{g}(2 - \mathrm{i}q_{s_2}\tilde{g})r_s^{(m)2}\mathrm{e}^{4\mathrm{i}kd_2}]/\Delta, \tag{19}$$

$$r_{11} = -[\mathrm{i}\tilde{g}(2 + \mathrm{i}q_{s_2}\tilde{g}) - 2(2 + \mathrm{i}\tilde{g})r_s^{(m)}\mathrm{e}^{2\mathrm{i}kd_2} + q_{s_2}\tilde{g}^2 r_s^{(m)2}\mathrm{e}^{4\mathrm{i}kd_2}]/\Delta, \tag{20}$$

$$r_{01} = r_{10} = 2\mathrm{i}\sqrt{q_{s_2}}\,\tilde{g}(1 - r_s^{(m)2}\mathrm{e}^{4\mathrm{i}kd_2})/\Delta \tag{21}$$

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[7] See any basic textbook dealing with the sum of three angular momenta, e.g. [14].

with

$$\Delta = -4 + i\tilde{g}(1 - r_s^{(m)}e^{2ikd_2})[2 + iq_{s_2}\tilde{g}(1 + r_s^{(m)}e^{2ikd_2})] \tag{22}$$

(for compactness of notation the dependence of $\tilde{g}$ on kd_1 is not shown). To realize an f–SQ_1 SWAP, i.e. $\hat{R}_{f12} = \hat{I}_2 \otimes \hat{W}_{f1}$, $|s_{f1} = 0\rangle$ and $|s_{f1} = 1\rangle$ must be eigenstates of $\hat{R}_{f12}$ with opposite eigenvalues, namely $r_{00} = -r_{11}$ must hold. Thereby, off-diagonal entries r_{01} must vanish, which yields the condition $r_s^{(m)} = e^{-2ikd_2}$, i.e. $kd_2 = \pi - \arg[r_s^{(m)}(\tilde{g})]/2 = h(kd_1)$, according to our definition of the h function (see above)[8]. By replacing this into equations (19) and (20), we immediately end up with $r_{00} = -r_{11} = 1$.

Since for the 1D blocks $s = 3/2$, as mentioned, s_{f1} can only take value 1 and the same occurs for s_{f2} as is easily seen. Hence, $s_{f1} = s_{f2} = 1$ and the interaction Hamiltonian is given by $\hat{V} = (G/2)\sum_{i=1,2}(q_{s_{fi}} - 3/2)\delta(x - x_i) \equiv (G/4)\sum_{i=1,2}\delta(x - x_i)$, i.e. it is effectively spinless. It should be clear then that the corresponding entry of $\hat{R}_{f12}$, denoted by $r^{(3/2)}$, can be found from equation (15) through the formal replacements $\hat{R}_{f1} \to r_t^{(m)}$ and $\hat{R}_{f2} \to r_t$ (see the previous section). The formerly introduced condition $r_t^{(m)} = -r_s^{(m)} = -e^{-2ikd_2}$ immediately yields $r^{(3/2)} = -1$ (matching the value found for r_{11} as it must be given that they both correspond to $s_{f1} = 1$). This demonstrates that, up to an irrelevant global phase factor, the f–SQ_1 SWAP indeed occurs under condition (16). It is important to stress that this result is *independent* of the value taken by r_t. In other words, the same result is achieved by replacing $(G/4)\delta(x - x_2)$ with $\Gamma\delta(x - x_2)$ with an arbitrary Γ.

4. An arbitrary number of static qubits

We now address the case where an arbitrary number N of SQs are present, the νth one lying at $x = x_\nu$ in a way that $d_\nu = x_{\nu-1} - x_\nu$ is the distance between the νth and $(\nu - 1)$th ones (see figure 2(c)). Hence, now

$$\hat{V} = G\sum_{i=1}^{N}(\hat{\boldsymbol{\sigma}}_f \cdot \hat{\boldsymbol{\sigma}}_i)\delta(x - x_i). \tag{23}$$

Again, we aim at implementing a selective SWAP between f and SQ_ν ($\nu = 1, \dots, N$). Selective QST is achieved for

$$\nu < N: \quad kd_{i \neq \nu, \nu+1} = n_i\pi, \quad kd_{\nu+1} = h(kd_\nu), \quad g = \tilde{g}(kd_\nu), \tag{24}$$

$$N: \quad kd_{i < N} = n_i\pi, \quad g = \tilde{g}(kd_N), \tag{25}$$

where n_i can be any positive integer. Regime (25) is immediately explained since it entails that $|x_{N-1}|$, namely the distance between SQ_{N-1} and the mirror, is a multiple integer of π; hence the mirror behaves as if it lied at $x = x_{N-1}$. All the SQs from SQ_1 to SQ_{N-1} are thus decoupled from f. We, in fact, retrieve the case of one SQ at a distance d_N from the mirror, where QST is ensured by condition (13) (with the replacement $d_1 \to d_N$).

The case in equation (24) is explained as follows. The mirror is effectively positioned at $x = x_{\nu-1}$ since each $kd_{i \leqslant \nu-1}$ is a multiple integer of π. On the other hand, $kd_{i > \nu+1} = n_i\pi$ holds

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[8] Strictly speaking, the solution is $kd_2 = n\pi - \arg[r_s^{(m)}(\tilde{g})]/2$ for $n = 1, 2, \dots$ (n integer). All these solutions are physically equivalent. Lower values of n, i.e. $n \leqslant 0$, are to be discarded since they would make kd_2 negative.

as well: the SQs indexed by i such that $\nu + 1 \leqslant i \leqslant N$ behave as if they were *all* located at $x = x_{\nu+1}$. Thereby, effectively, $\hat{V} = G \sum_{i=\nu+1}^{N} (\hat{\boldsymbol{\sigma}}_f \cdot \hat{\boldsymbol{\sigma}}_i) \delta(x - x_{\nu+1}) + G(\hat{\boldsymbol{\sigma}}_f \cdot \hat{\boldsymbol{\sigma}}_\nu) \delta(x - x_\nu)$ (subject to a hard-wall boundary condition at $x = x_{\nu-1}$). Let $\hat{\boldsymbol{\sigma}}_{\text{eff}} = \sum_{i=\nu+1}^{N} \hat{\boldsymbol{\sigma}}_i$ be the total spin of the $N - \nu$ SQs effectively located at $x = x_{\nu+1}$ and s_{eff} the quantum number associated with $\hat{\boldsymbol{\sigma}}_{\text{eff}}^2$. For $N - \nu$ even, $s_{\text{eff}} = 0, 1, \ldots, (N - \nu)/2$, while for $N - \nu$ odd $s_{\text{eff}} = 1/2, 3/2, \ldots, (N - \nu)/2$. As, clearly, s_{eff} is a good quantum number, in each subspace of fixed s_{eff} an effective static spin-s_{eff} particle lies at $x = x_{\nu+1}$.[9] By coupling this spin to f and SQ_ν, we find that the total quantum number can take values $s = s_{\text{eff}} - 1, s_{\text{eff}}, s_{\text{eff}} + 1$ (we can assume $s_{\text{eff}} \geqslant 1$ since the case $s_{\text{eff}} = 1/2$ has been analyzed in the previous section). Among these, only $s = s_{\text{eff}}$ is degenerate since in the corresponding eigenspace either $\hat{S}_{f\nu}^2$ or $\hat{S}_{fe}^2 = (\hat{\boldsymbol{\sigma}}_f + \hat{\boldsymbol{\sigma}}_{\text{eff}})^2$ can take *two* possible values, i.e. $s_{f\nu} = 0, 1$ and $s_{fe} = s_{\text{eff}} \pm 1/2$ (s_{fe} is the quantum number associated with $\hat{S}_{fe}^2$). The reflection matrix for the system is thus block-diagonal, where each block corresponding to either $s = s_{\text{eff}} - 1$ or $s = s_{\text{eff}} + 1$ is 1D, while a block corresponding to $s = s_{\text{eff}}$ is 2D. In the latter case, the corresponding reflection amplitudes in the basis $\{|s_{f1}; s, m_s\rangle = |s_{f1}\rangle\}$ can then be worked out in full analogy with the $s = 1/2$ subspace in the case of two SQs (see the previous section). Hence, they are given by equations (19)–(21) under the simple replacements $s_2 \to s_{\text{eff}}$, $d_1 \to d_\nu$ and $d_2 \to d_{\nu+1}$. Thereby, f–ν QST occurs for $r_t^{(m)} = -r_s^{(m)} = -e^{-2ikd_{\nu+1}}$, which holds provided that $g = \tilde{g}(kd_\nu)$ and $kd_{\nu+1} = h(kd_\nu)$. On the other hand, for $s = s_{\text{eff}} - 1$ ($s = s_{\text{eff}} + 1$), we have $s_{fe} = s_{\text{eff}} - 1/2$ ($s_{fe} = s_{\text{eff}} + 1/2$), while $s_{f\nu} = 1$. Hence, similarly to the $s = 3/2$ case in the previous section, in either of these subspaces the interaction Hamiltonian has the spinless effective form $\hat{V} = (G/2)(q_{s_{\text{eff}}\pm 1/2} - 3/4 - q_{s_{\text{eff}}}) \delta(x - x_{\nu+1}) + (G/4)\delta(x - x_\nu)$. The condition $r_t^{(m)} = -r_s^{(m)} = -e^{-2ikd_{\nu+1}}$ then ensures that in each case the corresponding overall reflection amplitude equals -1 (see the comment at the end of the previous section). A SWAP operation between f and SQ_ν is therefore implemented.

5. Working conditions

Based on the above findings, in particular, equation (24), the following working conditions for achieving selective writing/read-out of the static register can be devised. Firstly, one fixes once for all the desired coupling strength $g = g_0$ (provided that it exceeds the threshold value $g_{\text{th}} = 1$, equivalent to $G = k$; see figure 3(a)). Next, we choose one of the two different distances (in units of k^{-1}) that correspond to $g = g_0$ according to the function $\tilde{g}(kd)$ (see figure 3(a)). Let us call such a distance d_a, which therefore fulfills $\tilde{g}(kd_a) \equiv g_0$. A further distance $d_b = h(kd_a)/k$ (cf figure 3(b)) is then univocally identified. All the nearest-neighbor distances are set equal to an integer multiple of π (in units of k^{-1}) but the νth and $(\nu + 1)$th ones, which are set to d_a and d_b, respectively. In a practical implementation, such a tunable setting of nearest-neighbor distances could be achieved by fabricating the setup in such a way that the FQ can propagate along three possible paths instead of a single one (similarly to the geometry of the well-known Aharonov–Bohm rings). If the paths have different lengths, the actual path followed by the FQ can be chosen by means of tunable beam splitters, in fact setting the effective SQ–SQ distance.

In practice, unavoidable static disorder will affect the ideal pattern of nearest-neighbor SQ distances. Through a proof-of-principle resilience analysis, we have assessed that, by assuming

[9] Unlike a very spin-s_{eff} particle, in our case a given value of s_{eff} can exhibit degeneracies (e.g. for $N = 3$ the value $s_{\text{eff}} = 1/2$ is two fold degenerate). Yet, such degeneracies do not play any role here and can, in fact, be ignored.

Gaussian noise and in the case of a single SQ, an uncertainty in its position of the order of about 10% yields a process fidelity above the 95% threshold. This witnesses an excellent level of tolerance, in line with similar tests [9, 15]. Preliminary studies for the cases of two and three SQs have been carried out as well, confirming comparable performances. A comprehensive conclusive characterization of the effects of static disorder in the case of an arbitrary number of SQs, however, requires a rather involved analysis and thus goes beyond the scope of this paper.

6. Conclusions

We have considered a typical scenario envisaged in distributed quantum information, where writing and read-out of a register of SQs is performed through an FQ. In a fully quantum theory, the MDOFs of the FQ should be treated as quantum, which is expected to substantially complicate the dynamics. By taking a paradigmatic Hamiltonian, we have discovered that, as long as the f–SQ coupling is above a certain threshold value (i.e. $G \geqslant k$ with k being the input momentum of the FQ), for an arbitrary number of SQs *selective* QST can be achieved on demand by tuning only two SQ distances.

Throughout, as is customary in scattering-based theories, we have assumed to deal with a perfectly monochromatic plane wave for the FQ. In practice, clearly, this is a narrow-bandwidth wavepacket centered at a carrier wave vector k_0. A detailed resilience study of the performances of our protocol in such conditions is beyond the scope of the present paper. Yet, similarly to [9, 15, 16], it is reasonable to expect the gate fidelity to be only mildly affected owing to the smoothness of the functions $\tilde{g}(kd)$ and $h(kd)$ (cf figure 3). In our model, we assumed a Heisenberg-type spin–spin interaction. As already stressed, our attitude here was to take this well-known coupling as a paradigmatic model to show the possibility that selective writing/read-out is in principle achievable. However, there exist setups where the Heisenberg-type coupling occurs so as to make them potential candidates for realizing our protocol. For instance (see also [19]), this is the case for an electron propagating along a semiconducting carbon nanotube [20] and scattered from single-electron quantum dots or molecular spin systems featuring unpaired electrons, such as Sc@C82 [21]. Alternatively, one can envisage a photon propagating in a 1D waveguide to embody the FQ in such a way that its spin is encoded in the polarization degrees of freedom. A three-level Λ-type atom could then work as the SQ, where the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis is encoded in the ground doublet, while each transition to the excited state requires orthogonal photonic polarizations; see [22, 23]. Although similar, the corresponding (pseudo) spin–spin coupling, yet, is not equivalent to a Heisenberg-type one. We found some numerical evidence that this alternative coupling model could work as well, at least in the few-SQ case. An analytical treatment, however, is quite involved and thus no definite answer can be given. This is connected to the question of whether some specific symmetry is a necessary prerequisite for such remarkable effects to take place (in passing, note that the Heisenberg model conserves the squared total spin, which was crucial to carry out our proofs). All these issues are the focus of ongoing investigations.

It is worth mentioning that in a recent work [24], Ping *et al* proposed a protocol for imprinting the quantum state of a 'writing' FQ on an array of SQs and retrieving it through a 'reading' FQ at a next stage [24]. There, information is intentionally encoded over the *entire* register, which has some advantages, while MDOFs are in fact treated as classical. Significantly enough, here we have shown that the inclusion of quantum MDOFs can allow for control over

local encoding/decoding. In line with other works [16–18], such an apparent complication appears instead to be a powerful resource to carry out refined QIP tasks.

Appendix. Derivation of $\hat{R}_{f2}$ in the basis $\{|s_{f1}\rangle\}$

Here, we derive the matrix elements of the operator $\hat{R}_{f2}$ in the degenerate subspace $s = s_2$ (which is $s = 1/2$ in the case of $N = 2$ SQs), namely all the reflection coefficients $\bar{r}_{s'_{f1}s_{f1}} = \langle s'_{f1}|\hat{R}_{f2}|s_{f1}\rangle$ in terms of the basis $\mathcal{B}_{f1} = \{|s_{f1} = 0, 1\rangle\}$, where $|s_{f1}\rangle = |s_{f1}; s = s_2, m\rangle$. In line with the main text, we give the proof without specifying s_2 (which can thus be any positive integer or semi-integer number). The Hamiltonian reads $\hat{H} = \hat{p}^2/2 + \hat{V}$ with $\hat{V} = (G/2)(\hat{S}^2_{f2} - 3/4 - q_{s_2})\delta(x)$ (we have set $x_2 = 0$ since the result is evidently independent of x_2). The key task is to work out the matrix representation of $\hat{S}^2_{f2}$ in the basis of eigenstates of $\hat{S}^2_{f1}$, $\mathcal{B}_{f1} = \{|s_{f1} = 0, 1\rangle\}$. We first observe that in the present $s = s_2$ subspace $s_{f2} = s_2 \pm 1/2$. Accordingly, the scheme where f is first coupled to 2 leads to the alternative basis $\mathcal{B}_{f2} = \{|s_{f2} = s_2 \pm 1/2\rangle\}$ such that $\hat{S}^2_{f2}|s_{f2} = s_2 \pm 1/2\rangle = q_{s_{f2}}|s_{f2} = s_2 \pm 1/2\rangle$. Thereby, in the basis $\mathcal{B}_{f2}$, $\hat{S}^2_{f2}$ has the diagonal matrix representation $\mathrm{diag}(q_{s_2-1/2}, q_{s_2+1/2})$. The transformation matrix between the two basis can be calculated through $6j$ coefficients [14] (see footnote 6) as

$$\langle s_{f2}|s_{f1}\rangle = (-1)^{s_2+1}\sqrt{(2s_{f1}+1)(2s_{f2}+1)} \begin{Bmatrix} s_2 & 1/2 & s_{f2} \\ 1/2 & s_2 & s_{f1} \end{Bmatrix}. \tag{A.1}$$

Using these then yields $\hat{S}^2_{f2}$ in the basis $\mathcal{B}_{f1}$ as

$$\langle 0|\hat{S}^2_{f2}|0\rangle = -\frac{3}{8} + \frac{q_{s_2}}{2}, \qquad \langle 1|\hat{S}^2_{f2}|1\rangle = -\frac{7}{8} + \frac{q_{s_2}}{2}, \tag{A.2}$$

$$\langle 0|\hat{S}^2_{f2}|1\rangle = \langle 1|\hat{S}^2_{f2}|0\rangle = \frac{\sqrt{q_{s_2}}}{2}. \tag{A.3}$$

Next, in close analogy with [11, 12], we search for a stationary state $|\Psi_{s'_{f1}}\rangle = \varphi_{s'_{f1}0}(x)|0\rangle + \varphi_{s'_{f1}1}(x)|1\rangle$ such that $\hat{H}|\Psi_{s'_{f1}}\rangle = (k^2/2)|\Psi_{s'_{f1}}\rangle$, where $s'_{f1} = 0, 1$ labels the initial spin state (prior to the interaction process). Each function φ has the form

$$\varphi_{s'_{f1}s_{f1}}(x) = (\delta_{s'_{f1}s_{f1}}\mathrm{e}^{ikx} + \bar{r}_{s'_{f1}s_{f1}}\mathrm{e}^{-ikx})\theta(-x) + \bar{t}_{s'_{f1}s_{f1}}\mathrm{e}^{ikx}\theta(x). \tag{A.4}$$

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The unknown coefficients, including $\{\bar{r}_{s'_{f1}s_{f1}}\}$, i.e. the entries of $\hat{R}_{f2}$, can be found by imposing the continuity condition of $\varphi_{s'_{f1}0}(x)$ and $\varphi_{s'_{f1}1}(x)$ at $x = 0$ and the two constraints

$$\Delta\varphi'_{s'_{f1},0}(0) = G\sqrt{q_{s_2}}\,\varphi_{s'_{f1},1}(0), \tag{A.5}$$

$$\Delta\varphi'_{s'_{f2},1}(0) = -G\varphi_{s'_{f1},1}(0) + G\sqrt{q_{s_2}}\,\varphi_{s'_{f1},0}(0), \tag{A.6}$$

where $\Delta\varphi'_{s'_{f1},s_{f1}}(0)$ is the jump of the derivative at $x = 0$. With the help of equations (A.2) and (A.3), equations (A.5) and (A.6) can be straightforwardly obtained from the Schrödinger equation by integrating it across $x = 0$ and then projecting onto $|0\rangle$ and $|1\rangle$ [12]. By solving the linear system in the cases $s'_{f1} = 0, 1$, we thus end up with

$$\bar{r}_{00} = \langle 0|\hat{R}_{f2}|0\rangle = q_{s_2}g^2/\Delta_{s_2}, \quad \bar{r}_{11} = -ig(2 + iq_{s_2}g)/\Delta_{s_2}, \tag{A.7}$$

$$\bar{r}_{01} = \langle 0|\hat{R}_{f2}|1\rangle = \langle 1|\hat{R}_{f2}|0\rangle^* = 2i\sqrt{q_{s_2}}\,g/\Delta_{s_2}, \tag{A.8}$$

where $\Delta_{s_2} = -4 + 2ig - q_{s_2}g^2$.

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