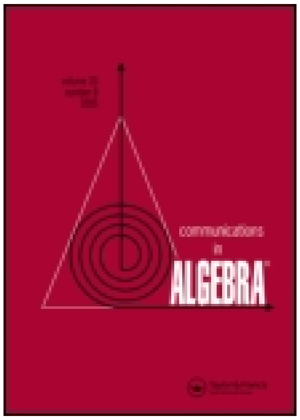

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Computing the $\mathbb{Z}_2$-Cocharacter of 3×3 Matrices of Odd Degree

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COMPUTING THE $\mathbb{Z}_2$-COCHARACTER OF 3 × 3 MATRICES OF ODD DEGREE

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Let F be a field of characteristic 0 and $A = M_{2,1}(F)$ the algebra of 3×3 matrices over F endowed with the only non trivial $\mathbb{Z}_2$-grading. Aver'yanov in [1] determined a set of generators for the T_2-ideal of graded identities of A. Here we study the identities in variables of homogeneous degree 1 via the representation theory of the symmetric group, and we determine the decomposition of the corresponding character into irreducibles.

Key Words: Cocharacter; Grading; Polynomial identity.

2010 Mathematics Subject Classification: Primary: 16R10; 16R50; 16W55.

1. INTRODUCTION

Let F be a field of characteristic 0 and $M_k(F)$ the algebra of $k \times k$ matrices over F. It is possible to give a structure of $\mathbb{Z}_2$-graded algebra to $M_k(F)$ and all possible $\mathbb{Z}_2$-gradings are well-known (see [8]). Given a $\mathbb{Z}_2$-grading on $M_k(F)$, an important problem in the theory of polynomial identities is that of determining a set of generators for the T_2-ideal of graded identities of $M_k(F)$. In case $k = 2$, there is only one possible $\mathbb{Z}_2$-grading and Di Vincenzo in [3] gave a set of generators for the corresponding T_2-ideal. Also in case $k = 3$ the algebra $M_3(F)$ has only one non trivial $\mathbb{Z}_2$-grading. Such algebra is denoted by $A = M_{2,1}(F)$, and its grading is given by $A = A_0 \oplus A_1$, where $A_0 = \begin{pmatrix} F & F & 0 \\ F & F & 0 \\ 0 & 0 & F \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & F \\ F & F & 0 \end{pmatrix}$.

Recently Aver'yanov in [1] gave a set of generators for the T_2-ideal of graded identities of such algebra in terms of concordant polynomials. The graded identities in variables of homogeneous degree 0 coincide with the ordinary identities of 2×2 matrices. It follows that the corresponding $\mathbb{Z}_2$-cocharacter coincides with the cocharacter of 2×2 matrices and this was computed by Drensky in [4] and Procesi in [9]. Here we want to do an analogous study for polynomials in variables of homogeneous degree 1. In fact, the main objective of this paper is that of giving a description, through the representation theory of the symmetric group, of the space of multilinear graded polynomial identities in variables of homogeneous degree 1.

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More precisely, by mean of concordant polynomials we study four different spaces of multilinear polynomials on which two copies of the symmetric group act. We then determine the decomposition of the corresponding character into irreducibles. For instance, let $V_{r,r+1} = \mathrm{span}_F\{z_{\sigma(1)}y_{\tau(1)}\dots z_{\sigma(r)}y_{\tau(r)}z_{\sigma(r+1)} \mid \sigma \in S_{r+1}, \tau \in S_r\}$ and let $Id^{\mathbb{Z}_2}(A)$ be the T_2-ideal of graded polynomial identities of $A = M_{2,1}(F)$. Then by considering the permutation action of $S_r \times S_{r+1}$ on $V_{r,r+1}$ modulo $Id^{\mathbb{Z}_2}(A)$, we completely determine the following decomposition of the corresponding character, denoted by $\chi_{r,r+1}(A)$, into irreducibles:

$$\chi_{r,r+1}(A) = \sum_{p=0}^{\lfloor \frac{r}{2} \rfloor}(\chi_{(r-p,p)} \otimes \chi_{(r-p+1,p)} + \chi_{(r-p,p)} \otimes \chi_{(r-p,p+1)}).$$

2. PRELIMINARIES

Throughout this article, we shall denote by F a field of characteristic zero. Recall that, if A is an associative algebra over F, a $\mathbb{Z}_2$-*grading* on A is a vector space decomposition $A = A_0 \oplus A_1$ such that

$$A_0A_0 + A_1A_1 \subseteq A_0 \qquad \text{and} \qquad A_0A_1 + A_1A_0 \subseteq A_1.$$

The elements of A_0 and A_1 are called homogeneous of degree zero (or even elements) and of degree one (or odd elements), respectively.

Now, let $A = M_{2,1}(F)$ be the algebra of 3×3 matrices over F with the only nontrivial $\mathbb{Z}_2$-grading such that

$$A_0 = \begin{pmatrix} F & F & 0 \\ F & F & 0 \\ 0 & 0 & F \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & F \\ F & F & 0 \end{pmatrix}.$$

We write

$$A = A^{00} \oplus A^{01} \oplus A^{10} \oplus A^{11}, \tag{1}$$

where $A^{00} = \mathrm{span}_F\{e_{11}, e_{12}, e_{21}, e_{22}\}$, $A^{01} = \mathrm{span}_F\{e_{13}, e_{23}\}$, $A^{10} = \mathrm{span}_F\{e_{31}, e_{32}\}$, and $A^{11} = \mathrm{span}_F\{e_{33}\}$.

Let $F\langle X\rangle$ be the free associative algebra on a countable set X, and write X as the disjoint union of two sets: $X = Y \cup Z$. If we denote by F^0 the subspace of $F\langle Y \cup Z\rangle$ spanned by all monomials in the variables of X having an even number of variables of Z and by F^1 the subspace spanned by all monomials with an odd number of variables in Z, then $F\langle Y \cup Z\rangle = F^0 \oplus F^1$ is a graded algebra with grading (F^0, F^1).

Recall that a polynomial $f(y_1, \dots, y_n, z_1, \dots, z_m) \in F\langle Y \cup Z\rangle$ is a $\mathbb{Z}_2$-*graded identity* of A if $f(a_1, \dots, a_n, b_1, \dots, b_m) = 0$ for all $a_1, \dots, a_n \in A_0$ and $b_1, \dots, b_m \in A_1$. We denote by $Id^{\mathbb{Z}_2}(A)$ the ideal of graded identities of A. Clearly, $Id^{\mathbb{Z}_2}(A)$ is a T_2-*ideal* of $F\langle X\rangle$, i.e., an ideal invariant under all endomorphisms φ of $F\langle X\rangle$ such that $\varphi(F^0) \subseteq F^0$ and $\varphi(F^1) \subseteq F^1$.

It is well known [7] that in characteristic zero every T-ideal is generated, as a T-ideal, by its multilinear polynomials. Hence if we denote, for every $n \geq 1$, by

$$P_n^{gr} = \text{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n,\ x_i = y_i \text{ or } x_i = z_i,\ i = 1, \ldots, n\}$$

the vector *space of multilinear polynomials* of degree n in $y_1, z_1, \ldots, y_n, z_n$ (i.e. y_i or z_i appears in each monomial at degree 1), the study of $Id^{\mathbb{Z}_2}(A)$ is equivalent to the study of $P_n^{gr} \cap Id^{\mathbb{Z}_2}(A)$, for all $n \geq 1$.

Next, following [1] we define concordant polynomials.

Let u be a multilinear monomial in $F\langle X\rangle$. Denote by $Z_0(u)$ ($Z_1(u)$, respectively) the set of variables z appearing in u such that $u = vzw$ and $\deg_Z(v)$ is even (odd, respectively). Further, denote by $Y_0(u)$ ($Y_1(u)$, respectively) the set of variables y appearing in u such that $u = vyw$ and $\deg_Z(v)$ is even (odd, respectively). Denote by $K(u)$ the quadruple $(Y_0(u), Z_0(u), Y_1(u), Z_1(u))$.

For instance, if $u = y_1z_1z_2y_2z_3y_3y_4$, then $Y_0(u) = \{y_1, y_2\}$, $Z_0(u) = \{z_1, z_3\}$, $Y_1(u) = \{y_3, y_4\}$, and $Z_1(u) = \{z_2\}$.

Let $f(y_1, \ldots, y_p, z_1, \ldots, z_q) \in Id^{\mathbb{Z}_2}(A)$ be a multilinear polynomial. Denote by f_K the sum of all monomials u of f for which $K(u) = K$. Thus we write

$$f = \sum_K f_K,$$

where $K = (Y_0, Z_0, Y_1, Z_1)$, $Y_0, Y_1 \subset Y$, $Z_0, Z_1 \subset Z$.

It is easy to show that $f \in Id^{\mathbb{Z}_2}(A)$ if and only if $f_K \in Id^{\mathbb{Z}_2}(A)$, for all K. A multilinear *polynomial* f of the form $f = f_K$ for some $K = (Y_0, Z_0, Y_1, Z_1)$ is called *K-homogeneous* (or *concordant*). Moreover, a *substitution* under which the variables belonging to Y_0, Z_0, Y_1, and Z_1 are replaced by elements of A^{00}, A^{01}, A^{11} and A^{10} respectively or of A^{11}, A^{10}, A^{00}, and A^{01}, respectively, is called *K-concordant*.

We have the following proposition.

Proposition 1 ([1]). *Let f be a multilinear polynomial. Then $f \in Id^{\mathbb{Z}_2}(A)$ if and only if all its K-homogeneous components vanish under any K-concordant substitution.*

According to this proposition the study of the polynomial identities of A is reduced to that of K-homogeneous polynomials and recently Aver'yanov ([1]) found a basis of K-homogeneous polynomials for the T_2-ideal $Id^{\mathbb{Z}_2}(A)$. Here we want to study the corresponding graded cocharacter in some special cases.

Notice that since $A_0 = M_2(F) \oplus F$, we have that

$$Id(A_0) = Id(M_2(F)) \cap Id(F) = Id(M_2(F)). \tag{2}$$

Hence if P_n^0 is the space of multilinear polynomials in n variables of homogeneous degree 0 and we regard $P_n^0/(P_n^0 \cap Id^{\mathbb{Z}_2}(A))$ as an S_n-module under the permutation action of the variables, its character coincides with the ordinary cocharacter of $M_2(F)$. Such cocharacter has been computed by Drensky [4] and Procesi [9].

The purpose of this article is to describe, through the representation theory of the symmetric group, the space of multilinear polynomials in odd variables modulo the $\mathbb{Z}_2$-graded identities of A.

To this end, let us denote by y_i and z_i the variables in Z_0 and Z_1, respectively. In what follows we shall assume, as we may, that the variables y_i are evaluated in A^{01}, and the variables z_i in A^{10}. In fact, if f is a polynomial only in odd variables, according to Proposition 1, we can consider f as a K-homogeneous polynomial with $K = (\emptyset, \{y_i\}, \emptyset, \{z_j\})$.

Notice that if G is any group with an element $g \in G$ such that $g^2 \neq 1$, we can consider the elementary grading on $B = M_3(F)$ determined by $(1, 1, g)$. In this case if $B = \bigoplus_{g\in G} B_g$ is the decomposition of B into its homogeneous components, we have $B_1 = A^{00} \oplus A^{11}$, $B_g = A^{10}$, $B_{g^{-1}} = A^{01}$, and $B_h = 0$, for all $h \neq g$ in G. Hence the polynomial identities of B on variables of homogeneous degree g and g^{-1} coincide with the $\mathbb{Z}_2$-graded polynomial identities of A in variables y_i and z_i. Such identities were previously computed by Bahturin and Drensky ([2]), and we next state their result in this setting.

We adopt the convention of marking with the same symbol ($\bar{}$, $\tilde{}$, etc.) variables which are alternating. Hence, let $f = f(x_1, \dots, x_n)$ be a multilinear polynomial alternating on $x_1, \dots, x_i$, we shall write

$$f(\bar{x}_1, \dots, \bar{x}_i, x_{i+1}, \dots, x_n) = \sum_{\sigma\in S_i} (\operatorname{sgn} \sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(i)}, x_{i+1}, \dots, x_n).$$

We shall also write $[x_1, x_2] = x_1x_2 - x_2x_1$.

Theorem 1 ([2], Theorem 5.3). *The $\mathbb{Z}_2$-graded polynomial identities of $A = M_{2,1}(F)$ which depend only on odd variables are consequences of the following graded identities*:

$$y_1y_2 \equiv 0, \tag{3}$$

$$z_1z_2 \equiv 0, \tag{4}$$

$$[z_1y_1, z_2y_2] \equiv 0, \tag{5}$$

$$\tilde{y}_1z_1\tilde{y}_2z_2\tilde{y}_3 \equiv 0, \tag{6}$$

$$\tilde{z}_1y_1\tilde{z}_2y_2\tilde{z}_3 \equiv 0. \tag{7}$$

3. A GENERAL SETTING

For any $r \geq 1$, define the following four spaces:

$$V'_{r,r} = \operatorname{span}_F\{z_{\sigma(1)}y_{\tau(1)} \dots z_{\sigma(r)}y_{\tau(r)} \mid \sigma, \tau \in S_r\},$$
$$V''_{r,r} = \operatorname{span}_F\{y_{\sigma(1)}z_{\tau(1)} \dots y_{\sigma(r)}z_{\tau(r)} \mid \sigma, \tau \in S_r\},$$
$$V_{r,r+1} = \operatorname{span}_F\{z_{\sigma(1)}y_{\tau(1)} \dots z_{\sigma(r)}y_{\tau(r)}z_{\sigma(r+1)} \mid \sigma \in S_{r+1}, \tau \in S_r\},$$
$$V_{r,r-1} = \operatorname{span}_F\{y_{\sigma(1)}z_{\tau(1)} \dots y_{\sigma(r-1)}z_{\tau(r-1)}y_{\sigma(r)} \mid \sigma \in S_r, \tau \in S_{r-1}\}.$$

Since $y_1y_2 \equiv 0$ and $z_1z_2 \equiv 0$ are identities of A, it is clear that if $f \notin V'_{r,r} \oplus V''_{r,r} \oplus V_{r,r+1} \oplus V_{r,r-1}$, then f is a graded identity of A. Moreover, by multihomogeneity, an obvious evaluation on A shows that if $f_1 \in V'_{r,r}$, $f_2 \in V''_{r,r}$, $f_3 \in V_{r,r+1}$, and $f_4 \in V_{r,r-1}$ are such that $f_1 + f_2 + f_3 + f_4 \in Id^{\mathbb{Z}_2}(A)$, then both f_i lies in

$Id^{\mathbb{Z}_2}(A)$ for $i = 1, 2, 3, 4$. Therefore, we can study the four spaces above separately, and we shall do so through the representation theory of the symmetric group.

Let the group $S_r \times S_r$ act on $V'_{r,r}$ by permuting the variables $y_1, \ldots, y_r$ and $z_1, \ldots, z_r$ separately. Hence if $(\sigma, \tau) \in S_r \times S_r$ and $f(y_1, \ldots, y_r, z_1, \ldots, z_r) \in V'_{r,r}$, we define $(\sigma, \tau)f(y_1, \ldots, y_r, z_1, \ldots, z_r) = f(y_{\sigma(1)}, \ldots, y_{\sigma(r)}, z_{\tau(1)}, \ldots, z_{\tau(r)})$. This action preserves $V'_{r,r} \cap Id^{\mathbb{Z}_2}(A)$ (i.e., the $\mathbb{Z}_2$-graded identities of A lying in $V'_{r,r}$), and so, we consider the $S_r \times S_r$-module $V'_{r,r}(A) = V'_{r,r}/(V'_{r,r} \cap Id^{\mathbb{Z}_2}(A))$. Our aim is to decompose its character into irreducibles. Recall that the irreducible S_n-characters are indexed by partitions λ of n (we write $\lambda \vdash n$). Hence if we let $\chi'_{r,r}(A)$ be the $S_r \times S_r$-character of $V'_{r,r}(A)$, by complete reducibility one writes

$$\chi'_{r,r}(A) = \sum_{\lambda, \mu \vdash r} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu \tag{8}$$

where $\chi_\lambda \otimes \chi_\mu$ is the irreducible $S_r \times S_r$-character corresponding to the pair of partitions (λ, μ) and $m_{\lambda,\mu} \geq 0$ is the multiplicity of $\chi_\lambda \otimes \chi_\mu$ in $\chi'_{r,r}(A)$.

Similarly, we define the $S_r \times S_r$-module $V''_{r,r}(A) = V''_{r,r}/(V''_{r,r} \cap Id^{\mathbb{Z}_2}(A))$, the $S_r \times S_{r+1}$-module $V_{r,r+1}(A) = V_{r,r+1}/(V_{r,r+1} \cap Id^{\mathbb{Z}_2}(A))$, and the $S_r \times S_{r-1}$-module $V_{r,r-1}(A) = V_{r,r-1}/(V_{r,r-1} \cap Id^{\mathbb{Z}_2}(A))$. We let $\chi''_{r,r}(A)$, $\chi_{r,r+1}(A)$ and $\chi_{r,r-1}(A)$ be the corresponding characters.

Let $\lambda, \mu \vdash r$ and T_λ, T_μ be Young tableaux of shape λ and μ, respectively. Let us denote by $e_{T_\lambda} e_{T_\mu}$ the corresponding essential idempotent of $F(S_r \times S_r)$. Recall that we can write $e_{T_\lambda} = C^-_{T_\lambda} R^+_{T_\lambda}$ where $C^-_{T_\lambda} = \sum_{\sigma \in C_{T_\lambda}} (\mathrm{sgn}\sigma)\sigma$, $R^+_{T_\lambda} = \sum_{\tau \in R_{T_\lambda}} \tau$, and C_{T_λ}, R_{T_λ} are the column and row stabilizer of T_λ, respectively. If $M_{\lambda,\mu}$ is an irreducible $S_r \times S_r$-submodule of $V'_{r,r}$, then $M_{\lambda,\mu} = F(S_r \times S_r) e_{T_\lambda} e_{T_\mu} f$, for some polynomial $f \in V'_{r,r}$ and Young tableaux T_λ, T_μ (see [7]).

Since by (6) any graded polynomial alternating in 3 variables y_i is an identity of A, it follows that $m_{\lambda,\mu} = 0$ in (8) (and in the analogous $\chi''_{r,r}(A)$, $\chi_{r,r+1}(A)$ and $\chi_{r,r-1}(A)$) as soon as the partition λ has more than 2 parts. Similarly, by (7), $m_{\lambda,\mu} = 0$ if μ has more than 2 parts.

Therefore, we have the following decomposition for the character

$$\chi'_{r,r}(A) = \sum_{\substack{\lambda, \mu \vdash r \\ h(\lambda), h(\mu) < 3}} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu, \tag{9}$$

where $h(\lambda)$ and $h(\mu)$ is the number of parts of λ and μ, respectively.

We can apply similar remarks to the spaces $V''_{r,r}(A)$, $V_{r,r+1}(A)$, $V_{r,r-1}(A)$, and deduce a decomposition similar to (9) for the corresponding characters.

In the computation of the above characters we shall actually use the representation theory of GL as follows. We consider

$$F_2^r = \mathrm{span}\{z_{i_1} y_{j_1} \ldots z_{i_r} y_{j_r} \mid 1 \leq i_1, \ldots, i_r, j_1, \ldots, j_r \leq 2\}$$

a space of homogeneous polynomials of degree $2r$. Then if $U = \mathrm{span}_F\{y_1, y_2\}$ and $V = \mathrm{span}_F\{z_1, z_2\}$, we let $GL(U) \times GL(V)$ act diagonally on F_2^r. Then the quotient space $F_2^r(A) = F_2^r/(F_2^r \cap Id^{\mathbb{Z}_2}(A))$ is a $GL(U) \times GL(V)$-module, and we let $\psi_r(A)$ be its character.

It is well known that there is a duality between S_n-representations and GL-representations. Let ψ_λ denote the GL-character corresponding to the partition $\lambda \vdash r$. Then if $\psi_r(A) = \sum_{\lambda,\mu\vdash r} m_{\lambda,\mu}\psi_\lambda \otimes \psi_\mu$, we have $\chi'_{r,r}(A) = \sum_{\lambda,\mu\vdash r} m_{\lambda,\mu}\chi_\lambda \otimes \chi_\mu$, i.e., the multiplicities in $\psi_r(A)$ and $\chi'_{r,r}(A)$ are the same (see [6]). Similar decompositions, with the obvious changes hold for the other characters $\chi''_{r,r}(A)$, $\chi_{r,r+1}(A)$, $\chi_{r,r-1}(A)$.

We recall that any irreducible submodule of $F_2^r(A)$ corresponding to the pair (λ, μ) of partitions of r is cyclic and is generated by a nonzero polynomial $f_{\lambda,\mu}(y_1, \ldots, y_a, z_1, \ldots, z_b)$ corresponding to a pair of Young tableaux, where $0 \leq a, b \leq 2$ are the number of parts of λ and μ, respectively, called the highest weight vector of the module (see [5], Theorem 12.4.12). Moreover, any highest weight vector $f_{\lambda,\mu}$ can be expressed uniquely as a linear combination of the highest weight vectors f_{T_λ,T_μ} corresponding to standard tableaux T_λ and T_μ (see [5], Proposition 12.4.14). In particular, let $\lambda = (r-p, p)$, $\mu = (r-q, q)$, with $0 \leq p, q \leq \lfloor \frac{r}{2} \rfloor$, and let $\lambda' = (2^p, 1^{r-2p})$ and $\mu' = (2^q, 1^{r-2q})$ be the conjugate partitions of λ and μ, respectively. Then f_{T_λ,T_μ} can be written, mod $Id^{\mathbb{Z}_2}(A)$, both as a linear combination of polynomials of the type

$$(z_{h_1}\dot{y}_1 z_{k_1}\dot{y}_2)\ldots(z_{h_p}\ddot{y}_1 z_{k_p}\ddot{y}_2)z_{l_1}y_1\ldots z_{l_{r-2p}}y_1$$

or as a linear combination of polynomials of the type

$$(\tilde{z}_1 y_{a_1}\tilde{z}_2 y_{b_1})\ldots(\bar{z}_1 y_{a_q}\bar{z}_2 y_{b_q})z_1 y_{c_1}\ldots z_1 y_{c_{r-2q}}.$$

Moreover, we recall the following remark.

Remark 1. If

$$\psi_r(A) = \sum_{\substack{\lambda,\mu\vdash r\\ h(\lambda),h(\mu)<3}} m_{\lambda,\mu}\psi_\lambda \otimes \psi_\mu$$

is the $GL_2 \times GL_2$-character of $F_2^r(A)$, then $m_{\lambda,\mu} \neq 0$ if and only if there exists a pair of tableaux (T_λ, T_μ) such that the corresponding highest weight vector f_{T_λ,T_μ} is not a graded polynomial identity for A.

Next we state some basic relations that we shall use throughout the article.

Remark 2. The following equalities hold modulo $Id^{\mathbb{Z}_2}(A)$.

$$z_1\bar{y}_1 z_2\bar{y}_2 \equiv \bar{z}_1 y_1\bar{z}_2 y_2 \tag{10}$$

$$\bar{z}_1\dot{y}_1\bar{z}_2\dot{y}_2 \equiv 2\ \tilde{z}_1 y_1\tilde{z}_2 y_2 \tag{11}$$

$$(z_1\dot{y}_1\tilde{z}_2\dot{y}_2)\tilde{z}_1 y_1 \equiv (\tilde{z}_1 y_1\tilde{z}_2 y_2)z_1 y_1 \tag{12}$$

$$(\bar{z}_1\dot{y}_1 z_2\dot{y}_2)(z_1\ddot{y}_1\bar{z}_2\ddot{y}_2) \equiv (\dot{z}_1 y_1\dot{z}_2 y_2)(\ddot{z}_1 y_1\ddot{z}_2 y_2) \tag{13}$$

$$(\bar{z}_1\dot{y}_1\bar{\bar{z}}_2\dot{y}_2)(\bar{\bar{z}}_1\ddot{y}_1\bar{z}_2\ddot{y}_2) \equiv 2\ (\bar{z}_1 y_1\bar{z}_2 y_2)(\bar{\bar{z}}_1 y_1\bar{\bar{z}}_2 y_2). \tag{14}$$

Proof. The equalities (10), (11), and (12), clearly follow from (5). Also (14) follows from (13). Concerning the last one, we have

$$(\bar{z}_1\dot{y}_1z_2\dot{y}_2)(z_1\ddot{y}_1\bar{z}_2\ddot{y}_2) \equiv (z_1\dot{y}_1z_2\dot{y}_2)(z_1\ddot{y}_1z_2\ddot{y}_2) \equiv (\dot{z}_1y_1\dot{z}_2y_2)(\ddot{z}_1y_1\ddot{z}_2y_2). \qquad \square$$

4. COMPUTING $\mathbb{Z}_2$-COCHARACTERS

If $\chi'_{r,r}(A)$ is the character of the $S_r \times S_r$-module $V'_{r,r}(A)$, we have the following result which was proved in ([2], Theorem 3.1). Here we give a proof based on an easy computation of GL-characters. Recall that for any real α, $\lfloor \alpha \rfloor$ is the integer part of α.

Theorem 2.

$$\chi'_{r,r}(A) = \sum_{p=0}^{\lfloor \frac{r}{2} \rfloor} \chi_{(r-p,p)} \otimes \chi_{(r-p,p)}.$$

Proof. Write $\chi'_{r,r}(A) = \sum_{\substack{\lambda,\mu\vdash r \\ h(\lambda),h(\mu)<3}} m_{\lambda,\mu}\chi_\lambda \otimes \chi_\mu$ and let $\lambda = (r-p, p)$, $\mu = (r-q, q)$, $0 \le p, q \le \lfloor \frac{r}{2} \rfloor$.

First we claim that $m_{\lambda,\mu} \neq 0$ implies $p = q$ (i.e. $\lambda = \mu$). In fact, by Remark 1, $m_{\lambda,\mu} \neq 0$ if there exist two tableaux T_λ and T_μ such that the corresponding highest weight vector f_{T_λ,T_μ} is not a graded polynomial identity for A.

We can write f_{T_λ,T_μ}, mod $Id^{\mathbb{Z}_2}(A)$, as a linear combination of polynomials of the type:

$$(\tilde{z}_1y_{a_1}\tilde{z}_2y_{b_1})\dots(\tilde{z}_1y_{a_q}\tilde{z}_2y_{b_q})z_1y_{c_1}\dots z_1y_{c_{r-2q}}. \tag{15}$$

Since $\tilde{z}_iy\tilde{z}_jy = [z_iy, z_jy] \equiv 0$, we get that the indexes a_i and b_i, $1 \le i \le q$, must all be distinct. Therefore, we get that $p \ge q$.

On the other hand, f_{T_λ,T_μ} can be also written, mod $Id^{\mathbb{Z}_2}(A)$, as a linear combination of polynomials of the type

$$(z_{h_1}\dot{y}_1z_{k_1}\dot{y}_2)\dots(z_{h_p}\ddot{y}_1z_{k_p}\ddot{y}_2)z_{l_1}y_1\dots z_{l_{r-2p}}y_1. \tag{16}$$

Then by using the relation $z\tilde{y}_iz\tilde{y}_j \equiv 0$, as above we get that $q \ge p$. Thus $p = q$ and the claim is proved.

Now let $g = f_{T_\lambda,T'_\lambda}$ be the highest weight vector corresponding to a pair of Young tableaux (T_λ, T'_λ). Then, by (16), since the indexes h_i and k_i, $1 \le i \le p$, are distinct, we write g as as a linear combination of polynomials of the type

$$\underbrace{(z_1\dot{y}_1z_2\dot{y}_2)\dots(z_1\ddot{y}_1z_2\ddot{y}_2)}_{p}\underbrace{(z_1y_1)\dots(z_1y_1)}_{r-2p}. \tag{17}$$

We observe that p of the variables z_1 in (17) must alternate with the z_2's. Then, by using the equalities of Remark 2, we get that

$$g \equiv C\underbrace{(z_1\bar{y}_1z_2\bar{y}_2)\dots(z_1\bar{\bar{y}}_1z_2\bar{\bar{y}}_2)}_{p}\underbrace{(z_1y_1)\dots(z_1y_1)}_{r-2p} \qquad (\text{mod } Id^{\mathbb{Z}_2}(A)),$$

for some nonzero integer C. This proves that $m_{(r-p,p),(r-p,p)} \le 1$.

If we now consider the substitution $z_i = e_{3i}$, $y_i = e_{i3}$, $i = 1, 2$, we obtain that $g \not\equiv 0$. Hence $m_{(r-p,p),(r-p,p)} = 1$, $0 \leq p \leq \lfloor \frac{r}{2} \rfloor$, and the proof is complete. □

We recall that we act on the space $V_{r,r+1}(A)$ with the group $S_r \times S_{r+1}$, and we let $\chi_{r,r+1}(A)$ be its character.

Write

$$\chi_{r,r+1}(A) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash r+1}} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu.$$

Theorem 3.

$$\chi_{r,r+1}(A) = \sum_{p=0}^{\lfloor \frac{r}{2} \rfloor} (\chi_{(r-p,p)} \otimes \chi_{(r-p+1,p)} + \chi_{(r-p,p)} \otimes \chi_{(r-p,p+1)}).$$

Proof. Let T_λ and T_μ be two tableaux such that the corresponding highest weight vector f_{T_λ,T_μ} is not a graded identity for A, and let $\lambda = (r-p, p)$, $\mu = (r+1-q, q)$, $0 \leq p, q \leq \lfloor \frac{r}{2} \rfloor$.

First we claim that $m_{\lambda,\mu} \neq 0$ implies $q = p$ or $q = p+1$.

We write f_{T_λ,T_μ}, mod $Id^{\mathbb{Z}_2}(A)$, as a linear combination of polynomials of the type

$$(z_{h_1} \dot{y}_1 z_{k_1} \dot{y}_2) \dots (z_{h_p} \ddot{y}_1 z_{k_p} \ddot{y}_2) z_{l_1} y_1 \dots z_{l_{r-2p}} y_1 z_{l_{r-2p+1}}. \tag{18}$$

Then by using the relation $z\tilde{y}_i z\tilde{y}_j \equiv 0$ as in Theorem 2, we get that $q \geq p$.

On the other hand, f_{T_λ,T_μ} can also be written, mod $Id^{\mathbb{Z}_2}(A)$, as a linear combination of polynomials of the type

$$(\bar{z}_1 y_{a_1} \bar{z}_2 y_{b_1}) \dots (\bar{\bar{z}}_1 y_{a_{q-1}} \bar{\bar{z}}_2 y_{b_{q-1}}) z_1 y_{c_1} \dots z_1 y_{c_{r-2q}} w, \tag{19}$$

where $w = z_1 y_s (\tilde{z}_1 y_t \tilde{z}_2)$ or $w = (\tilde{z}_1 y_{a_q} \tilde{z}_2 y_{b_q}) z_1$ according to if the rightmost variable alternates or does not alternate, respectively. Then, by using the relation $\tilde{z}_i y \tilde{z}_j y \equiv 0$ as above, we get that $p \geq q-1$. Thus it follows that $p \leq q \leq p+1$, and the claim is proved.

Next we show that $m_{\lambda,\mu} = 1$ if $q = p$ or $q = p+1$.

Suppose first that $\lambda = (r-p, p)$, $\mu = (r-p+1, p)$, and let $g = f_{T_\lambda,T_\mu}$ be the highest weight vector corresponding to a pair of Young tableaux (T_λ, T_μ). Then, by (18), since the indexes h_i and k_i, $1 \leq i \leq p$, are distinct, we can write g as as a linear combination of polynomials of the type

$$\underbrace{(z_1 \dot{y}_1 z_2 \dot{y}_2) \dots (z_1 \ddot{y}_1 z_2 \ddot{y}_2)}_{p} \underbrace{(z_1 y_1) \dots (z_1 y_1)}_{r-2p} z_1. \tag{20}$$

We observe that p of the variables z_1 in (20) must alternate with the z_2's. Then, by using the equalities of Remark 2, we get that

$$g \equiv C \underbrace{(\bar{z}_1 y_1 \bar{z}_2 y_2) \dots (\bar{\bar{z}}_1 y_1 \bar{\bar{z}}_2 y_2)}_{p} \underbrace{(z_1 y_1) \dots (z_1 y_1)}_{r-2p} z_1 \pmod{Id^{\mathbb{Z}_2}(A)},$$

for some nonzero integer C.

Let now $\lambda = (r-p, p)$ and $\mu = (r-p, p+1)$. By applying the above argument, we get that

$$g \equiv C \underbrace{(\bar{z}_1 y_1 \bar{z}_2 y_2) \dots (\bar{\bar{z}}_1 y_1 \bar{\bar{z}}_2 y_2)}_{p} \underbrace{(z_1 y_1) \dots (z_1 y_1)}_{r-2p-1} \tilde{z}_1 y_1 \tilde{z}_2 \qquad (\text{mod } Id^{\mathbb{Z}_2}(A)),$$

for some nonzero integer C.

This proves that in both cases, $m_{\lambda,\mu} \leq 1$.

If we now consider the substitution $z_i = e_{3i}$, $y_i = e_{i3}$, $i = 1, 2$, we obtain that $g \not\equiv 0$, and the proof is complete. □

Recall that $V_{r,r-1}(A)$ is the space of multilinear polynomials in r variables y_i and $r-1$ variables z_i modulo $Id^{\mathbb{Z}_2}(A)$. We act on this space with $S_r \times S_{r-1}$, and we let its character be

$$\chi_{r,r-1}(A) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash r-1}} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu.$$

We have the following theorem.

Theorem 4.

$$\chi_{r,r-1}(A) = \sum_{p=0}^{\lfloor \frac{r}{2} \rfloor} (\chi_{(r-p,p)} \otimes \chi_{(r-p-1,p)} + \chi_{(r-p,p)} \otimes \chi_{(r-p,p-1)}).$$

Proof. As in the previous theorems, we let T_λ and T_μ be two tableaux such that $f_{T_\lambda,T_\mu} \notin Id^{\mathbb{Z}_2}(A)$. Let also $\lambda = (r-p, p)$ and $\mu = (r-1-q, q)$, $0 \leq p, q \leq \lfloor \frac{r}{2} \rfloor$.

We claim that $m_{\lambda,\mu} \neq 0$ implies $q = p$ or $q = p-1$.

Then we write f_{T_λ,T_μ}, mod $Id^{\mathbb{Z}_2}(A)$, as a linear combination of polynomials of the type

$$y_d(\bar{z}_1 y_{a_1} \bar{z}_2 y_{b_1}) \dots (\bar{\bar{z}}_1 y_{a_q} \bar{\bar{z}}_2 y_{b_q}) z_1 y_{c_1} \dots z_1 y_{c_{r-2q-1}}. \tag{21}$$

By using the relation $\tilde{z}_i y \tilde{z}_j y \equiv 0$ as before, we get that $p \geq q$.

On the other hand f_{T_λ,T_μ} can be also written, mod $Id^{\mathbb{Z}_2}(A)$, as a linear combination of polynomials of the type

$$w(z_{h_1} \dot{y}_1 z_{k_1} \dot{y}_2) \dots (z_{h_{p-1}} \ddot{y}_1 z_{k_{p-1}} \ddot{y}_2) z_{l_1} y_1 \dots z_{l_{r-2p-1}} y_1, \tag{22}$$

where $w = (\tilde{y}_1 z_t \tilde{y}_2) z_s y_1$ or $w = y_1(z_{h_p} \tilde{y}_1 z_{k_p} \tilde{y}_2)$ according if the leftmost variable does or does not alternate, respectively. Then, by using the relation $z\tilde{y}_i z\tilde{y}_j \equiv 0$ as above, we get that $q \geq p-1$. Thus it follows that $p-1 \leq q \leq p$, and the claim is proved.

Next we show that $m_{\lambda,\mu} = 1$ if $q = p$ or $q = p-1$.

Suppose first that $\lambda = (r-p, p)$, $\mu = (r-p-1, p)$, and let $g = f_{T_\lambda,T_\mu} \notin Id^{\mathbb{Z}_2}(A)$ be the highest weight vector corresponding to a pair of Young tableaux

(T_λ, T_μ). Then, by (21), since the indexes a_i and b_i, $1 \le i \le p$, are distinct, we can write g as a linear combination of polynomials of the type

$$y_1 \underbrace{(\bar{z}_1 y_1 \bar{z}_2 y_2) \dots (\bar{\bar{z}}_1 y_1 \bar{\bar{z}}_2 y_2)}_{p} \underbrace{(z_1 y_1) \dots (z_1 y_1)}_{r-2p-1}. \tag{23}$$

We observe that p of the variables y_1 in (23) must alternate with p variables y_2. Then, by using the equalities of Remark 2, we get that

$$g \equiv C\, y_1 \underbrace{(z_1 \bar{y}_1 z_2 \bar{y}_2) \dots (z_1 \bar{\bar{y}}_1 z_2 \bar{\bar{y}}_2)}_{p} \underbrace{(z_1 y_1) \dots (z_1 y_1)}_{r-2p-1} \quad (\text{mod } Id^{\mathbb{Z}_2}(A)),$$

for some nonzero integer C.

Let now $\lambda = (r-p, p)$ and $\mu = (r-p, p-1)$. As in the previous case, we get

$$g \equiv C\tilde{y}_1 z_1 \tilde{y}_2 \underbrace{(z_1 \dot{y}_1 z_2 \dot{y}_2) \dots (z_1 \ddot{y}_1 z_2 \ddot{y}_2)}_{p-1} \underbrace{(z_1 y_1) \dots (z_1 y_1)}_{r-2p} \quad (\text{mod } Id^{\mathbb{Z}_2}(A)),$$

for some nonzero integer C.

It follows that in both cases $m_{\lambda,\mu} \le 1$.

If we now consider the substitution $z_i = e_{3i}$, $y_i = e_{i3}$, $i = 1, 2$, we obtain that $g \not\equiv 0$, and the proof is complete. □

Recall that $V''_{r,r} = \text{span}_F\{y_{\sigma(1)} z_{\tau(1)} \dots y_{\sigma(r)} z_{\tau(r)} \mid \sigma, \tau \in S_r\}$ and $V''_{r,r}(A) = V''_{r,r}/(V''_{r,r} \cap Id^{\mathbb{Z}_2}(A))$ has a structure of $S_r \times S_r$-module whose character is

$$\chi''_{r,r}(A) = \sum_{\lambda,\mu \vdash r} m_{\lambda,\mu} \chi_\lambda \otimes \chi_\mu.$$

Next we compute $\chi''_{r,r}(A)$.

Theorem 5.

$$\chi''_{r,r}(A) = \chi_{(r)} \otimes \chi_{(r)} + \chi_{(r)} \otimes \chi_{(r-1,1)} + \sum_{p=1}^{\lfloor \frac{r}{2} \rfloor} (\chi_{(r-p,p)} \otimes \chi_{(r-p+1,p-1)}$$
$$+ \chi_{(r-p,p)} \otimes \chi_{(r-p-1,p+1)} + 2\chi_{(r-p,p)} \otimes \chi_{(r-p,p)}).$$

Proof. As in the previous theorems we let T_λ and T_μ be two tableaux such that $f_{T_\lambda,T_\mu} \notin Id^{\mathbb{Z}_2}(A)$. Let also $\lambda = (r-p, p)$ and $\mu = (r-q, q)$, $0 \le p, q \le \lfloor \frac{r}{2} \rfloor$.

First we claim that $m_{\lambda,\mu} \ne 0$ implies $p - 1 \le q \le p + 1$.

We can write f_{T_λ,T_μ}, mod $Id^{\mathbb{Z}_2}(A)$, as a linear combination of polynomials of the type

$$y_d(\bar{z}_1 y_{a_1} \bar{z}_2 y_{b_1}) \dots (\bar{\bar{z}}_1 y_{a_{q-1}} \bar{\bar{z}}_2 y_{b_{q-1}}) z_1 y_{c_1} \dots z_1 y_{c_{r-2q-1}} w, \tag{24}$$

where $w = z_1 y_s (\tilde{z}_1 y_t \tilde{z}_2)$ or $w = (\tilde{z}_1 y_{a_q} \tilde{z}_2 y_{b_q}) z_1$ if the rightmost variable does or does not alternate, respectively. Then, by using the relation $\tilde{z}_i y \tilde{z}_j y \equiv 0$ as in the previous theorems, we get that $p \ge q - 1$.

On the other hand f_{T_λ,T_μ} can be also written, mod $Id^{\mathbb{Z}_2}(A)$, as a linear combination of polynomials of the type

$$u(z_{h_1}\dot{y}_1 z_{k_1}\dot{y}_2)\ldots(z_{h_{p-1}}\ddot{y}_1 z_{k_{p-1}}\ddot{y}_2)z_{l_1}y_1\ldots z_{l_{r-2p-1}}y_1 z_{l-2p}, \tag{25}$$

where $u = (\tilde{y}_1 z_t \tilde{y}_2)z_s y_1$ or $u = y_1(z_{h_p}\tilde{y}_1 z_{k_p}\tilde{y}_2)$ if the leftmost variable does or does not alternate, respectively. Then, by using the relation $z\tilde{y}_i z\tilde{y}_j \equiv 0$ as above, we get that $q \geq p-1$, and the claim follows.

Now we shall compute $m_{\lambda,\mu}$ in the three possible cases: $q = p-1$, $q = p+1$, and $q = p$.

Suppose first that $q = p-1$, and observe that $1 \leq p \leq \lfloor \frac{r}{2} \rfloor$. Thus, let $\lambda = (r-p, p)$, $\mu = (r-p+1, p-1)$, and let $g = f_{T_\lambda,T_\mu}$ be the highest weight vector corresponding to a pair of Young tableaux (T_λ, T_μ). Then, by (25), since the indexes h_i and k_i, $1 \leq i \leq p-1$, are distinct and since (25) must contain only $p-1$ z_2's, we can write g as as a linear combination of polynomials of the type

$$\tilde{y}_1 z_1 \tilde{y}_2 \underbrace{(z_1\dot{y}_1 z_2\dot{y}_2)\ldots(z_1\ddot{y}_1 z_2\ddot{y}_2)}_{p-1}\underbrace{(z_1y_1)\ldots(z_1y_1)}_{r-2p} z_1. \tag{26}$$

We observe that $p-1$ of the variables z_1 in (26) must alternate with corresponding z_2's. Then, by using the equalities of Remark 2, we get that

$$g \equiv C\tilde{y}_1 z_1 \tilde{y}_2 \underbrace{(z_1\dot{y}_1 z_2\dot{y}_2)\ldots(z_1\ddot{y}_1 z_2\ddot{y}_2)}_{p-1}\underbrace{(z_1y_1)\ldots(z_1y_1)}_{r-2p} z_1 \quad (\text{mod } Id^{\mathbb{Z}_2}(A)),$$

for some nonzero integer C.

This proves that $m_{\lambda,\mu} \leq 1$. If we now consider the substitution $z_i = e_{3i}$, $y_i = e_{i3}$, $i = 1, 2$, we obtain that $g \not\equiv 0$ and so, $m_{(r-p,p),(r-p+1,p-1)} = 1$, $1 \leq p \leq \lfloor \frac{r}{2} \rfloor$.

Consider now the case $q = p+1$, i.e., let $\lambda = (r-p, p)$ and $\mu = (r-p-1, p+1)$, $0 \leq p \leq \lfloor \frac{r}{2} \rfloor$. Let also $g = f_{T_\lambda,T_\mu}$ correspond to a pair of Young tableaux (T_λ, T_μ). Then, by (24), since the indexes a_i and b_i, $1 \leq i \leq p$, are distinct and since (24) must contain only p y_2's, we can write g as as a linear combination of polynomials of the type

$$y_1 \underbrace{(\bar{z}_1 y_1 \bar{z}_2 y_2)\ldots(\bar{\bar{z}}_1 y_1 \bar{\bar{z}}_2 y_2)}_{p}\underbrace{(z_1y_1)\ldots(z_1y_1)}_{r-2p-2} \tilde{z}_1 y_1 \tilde{z}_2. \tag{27}$$

As in the previous case, since p of the variables y_1 in (27) must alternate with the y_2's, we get that

$$g \equiv Cy_1 \underbrace{(\bar{z}_1 y_1 \bar{z}_2 y_2)\ldots(\bar{\bar{z}}_1 y_1 \bar{\bar{z}}_2 y_2)}_{p}\underbrace{(z_1y_1)\ldots(z_1y_1)}_{r-2p-2} \tilde{z}_1 y_1 \tilde{z}_2 \quad (\text{mod } Id^{\mathbb{Z}_2}(A)),$$

for some nonzero integer C. Thus, by considering the same substitution of the previous case, we get that also $m_{(r-p,p),(r-p-1,p+1)} = 1$, $0 \leq p \leq \lfloor \frac{r}{2} \rfloor$.

Finally, we consider the case $q = p$ (i.e., $\lambda = \mu$), and let $g = f_{T_\lambda,T'_\lambda}$ be the highest weight vector corresponding to a pair of Young tableaux (T_λ, T'_λ).

It is clear that if $p = 0$, i.e., $\lambda = \mu = (r)$, then $g \notin Id^{\mathbb{Z}_2}(A)$. In fact in this case, we get $g = \underbrace{(y_1 z_1) \ldots (y_1 z_1)}_{r}$, and by evaluating for instance z_1 to e_{31} and y_1 to e_{13}, we get a nonzero value. Hence $m_{(r),(r)} = 1$.

Suppose now that $\lambda = \mu = (r - p, p)$, $1 \leq p \leq \lfloor \frac{r}{2} \rfloor$. We want to show that in this case $m_{\lambda,\lambda} = 2$.

We start by considering (24) (similarly, (25)), and we apply arguments similar to the previous ones concerning the number of y_2's and z_2's into (24). By using the equalities of Remark 2, we get that g is a linear combination of the following polynomials:

$$g_1 = \dot{y}_1 v z_1 y_1 \tilde{z}_1 \dot{y}_2 \tilde{z}_2 \qquad g_2 = \dot{y}_1 v \tilde{z}_1 \dot{y}_2 \tilde{z}_2 y_1 z_1 \qquad g_3 = \dot{y}_1 v z_1 \dot{y}_2 \tilde{z}_1 y_1 \tilde{z}_2$$

$$g_4 = \dot{y}_1 v \tilde{z}_1 y_1 \tilde{z}_2 \dot{y}_2 z_1 \qquad g_5 = y_1 v \tilde{z}_1 \dot{y}_1 \tilde{z}_2 \dot{y}_2 z_1 \qquad g_6 = y_1 v z_1 \dot{y}_1 \tilde{z}_1 \dot{y}_2 \tilde{z}_2,$$

where $v = \underbrace{(\bar{z}_1 y_1 \bar{z}_2 y_2) \ldots (\bar{\bar{z}}_1 y_1 \bar{\bar{z}}_2 y_2)}_{p-1} \underbrace{(z_1 y_1) \ldots (z_1 y_1)}_{r-2p-1}$.

It is easy to check that

$$g_6 \equiv -g_4 \equiv g_2 \equiv -y_1 v \tilde{z}_1 y_1 \tilde{z}_2 y_2 z_1 \qquad g_5 \equiv -2 g_2 \qquad \text{and} \qquad g_3 \equiv g_1 - g_2.$$

Now the polynomials g_1 and g_2 are linearly independent. In fact, let $f = \alpha\, g_1 + \beta\, g_2 \equiv 0 \pmod{Id^{\mathbb{Z}_2}(A)}$. If we consider the usually substitution $z_i = e_{3i}$, $y_i = e_{i3}$, $i = 1, 2$, we obtain $f = -(\alpha + \beta) e_{11} - \alpha\, e_{22} = 0$. Hence $\alpha = 0$ and also $\beta = 0$. Then it follows that $m_{(r-p,p),(r-p,p)} = 2$, $1 \leq p \leq \lfloor \frac{r}{2} \rfloor$, and the proof is complete. □

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