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APPROXIMATION OF FIXED POINTS OF ASYMPTOTICALLY G-NONEXPANSIVE MAPPINGS

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ABSTRACT

In this paper we consider a notion of asymptotically g-strongly regular mappings and we use this notion for studying the problem of approximation of common fixed points of asymptotically g-nonexpansive mappings.

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1. Introduction.

For a given mapping, $f : X \to X$, every solution of the equation $f(x) = x$ is called a fixed point of f. Let (X, d) be a metric space and $f : X \to X$. f is a contraction if there exists $k \in [0, 1[$ such that $d(f(x), f(y)) \le k\, d(x, y)$ for all $x, y \in X$. The Banach contraction principle states that if $f : X \to X$ is a contraction and X is a complete metric space, then f has a unique fixed point. The Banach contraction principle plays a very important role in nonlinear analysis and has many generalizations, see [4]. Let $f,\ g : X \to X$ two mappings. $x \in X$ is a coincidence point for f and g if $f(x) = g(x)$ and a fixed point for f and g if $f(x) = g(x) = x$. Also the study of common fixed point of mappings satisfying contractive type condition has been a very active field of research during the last decades.

The purpose of this paper is to prove a uniqueness and existence theorem of fixed point for two mappings and to use this result for studying the problem of approximation of fixed points of mappings satisfying asymptotically type conditions.

Let A be a nonempty subset of a metric space (X, d) and let

$f, g : A \to A$. The map f is called *nonexpansive* if

$$d(f(x), f(y)) \leq d(x, y)$$

for all $x, y \in A$; *g-nonexpansive* if

$$d(f(x), f(y)) \leq d(g(x), g(y))$$

for all $x, y \in A$; *asymptotically g-nonexpansive* if there exists a sequence $(k_n) \subset [1, +\infty[$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(f^n(x), f^n(y)) \leq k_n d(g(x), g(y)),$$

where f^n is the n-th iterate of f, for all $x, y \in A$ and $n = 1, 2, \dots$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3] as a generalization of the class of nonexpansive mappings. They proved that a self-mapping defined in a nonempty closed convex bounded subset of a real uniformly convex Banach space has a fixed point if it is asymptotically nonexpansive.

Recently, Cho, Sahu end Jung [2] proved strong convergence of almost fixed points $x_n = \mu_n f^n(x_n) + (1 - \mu_n)u$ to the fixed point of asymptotically pseudocontractive mappings in normed spaces.

Beg, Sahu and Diwan [1] introduce a new class of uniformly R-subweakly commuting mappings and used it to study the problem of approximation of common fixed points of asymptotically g-nonexpansive mappings in the setting of Banach space with uniformly Gâteaux differentiable norm.

In this paper we consider a notion of asymptotically g-strongly regular mappings and we use this notion for studying the problem of approximation of common fixed points of asymptotically g-nonexpansive mappings.

2 Preliminaries.

Let A be a nonempty subset of a metric space (X, d) and let $f, g : A \to A$. The map f is called a *g-contraction* if there exists a constant $k \in [0, 1[$ such that

$$d(f(x), f(y)) \leq kd(g(x), g(y)) \qquad \forall x, y \in A;$$

f is a *g-weakly contraction* if there exists a continuous function $r : [0, +\infty[\to [0, +\infty[$ such that $r(t) < t$ for all $t > 0$ and

$$d(f(x), f(y)) \leq r(d(g(x), g(y))) \quad \forall x, y \in A.$$

The map f is *uniformly asymptotically regular on* A if, for all $\eta > 0$ there exists n_η such that

$$d(f^n(x), f^{n+1}(x)) < \eta \quad \forall x \in A, \quad n \geq n_\eta.$$

The map f is *g-regular* in $x \in A$ if there exists a constant $R > 0$ such that

$$d(g(f(x)), f(x)) \leq Rd(f(x), g(x)).$$

Let X be a normed space with norm $\|\cdot\|$ and let A be an u-starshaped subset of X. The map f is *g-strongly regular* in $x \in A$ if there exists a constant $R > 0$ such that

$$d(g(f(x)), f(x)) \leq Rd(g(x), [f(x), u]),$$

where $[f(x), u]$ is the segment with endpoints $f(x)$, u and d is the metric induced by the norm. The map f is *asymptotically g-strongly regular* on A if there exists a sequence $(R_n) \subset \mathbb{R}_+$ such that, for every $n \in \mathbb{N}$, the mapping f^n is g-strongly regular, with constant R_n, in every point $x \in A$ where $f^n(x) = g(x)$ holds.

In the sequel, for every $h : X \to X$ we set $F(h) = \{x \in X : h(x) = x\}$.

3. Common fixed points in metric spaces.

The following lemma gives a result of existence and uniqueness of a common fixed point for g-weakly contractions.

LEMMA 1. *Let (X, d) be a metric space and let A be a nonempty subset of X. Let $f, g : A \to A$ be mappings, $u \in F(g)$, $f(A \setminus \{u\}) \subset g(A) \setminus \{u\}$ and $f(A \setminus \{u\}) \cup \{u\}$ or $g(A)$ be complete. Suppose that f is a g-weakly contraction. If f is g-regular in each $z \in A \setminus \{u\}$ where $f(z) = g(z)$, then $F(f) \cap F(g)$ is singleton.*

Proof. First we establish the uniqueness of the common fixed point. Assume that u_1, $u_2 \in F(f) \cap F(g)$ with $u_1 \neq u_2$, from

$$\begin{aligned} d(u_1, u_2) &= d(f(u_1), f(u_2)) \leq r(d(g(u_1), g(u_2))) \\ &= r(d(u_1, u_2)) < d(u_1, u_2), \end{aligned}$$

we obtain a contradiction.

If $u \in F(f)$ the thesis follows from the uniqueness.

We suppose that $u \notin F(f)$. Let $x_0 \in A \setminus \{u\}$. We define a sequence $(x_n) \subset A \setminus \{u\}$ by $g(x_n) = f(x_{n-1})$ for every positive integer n, this is possible being $f(A \setminus \{u\}) \subset g(A) \setminus \{u\}$. It is not restrictive to suppose that $f(x_n) \neq f(x_{n-1})$ for every positive integer n.

From

$$\begin{aligned} d(f(x_{n+1}), f(x_n)) &\leq r(d(g(x_{n+1}), g(x_n))) \\ &= r(d(f(x_n), f(x_{n-1}))) < d(f(x_n), f(x_{n-1})), \end{aligned}$$

we deduce that the sequence $(d(f(x_{n+1}), f(x_n)))$ is decreasing, then there exists

$$\lim_{n\to\infty} d(f(x_{n+1}), f(x_n)) = l \geq 0.$$

We observe that $l = 0$, in fact, if $l > 0$ by using

$$d(f(x_{n+1}), f(x_n)) \leq r(d(g(x_{n+1}), g(x_n))) = r(d(f(x_n), f(x_{n-1}))),$$

taking the limit as $n \to +\infty$, the above inequality yields $l \leq r(l) < l$ which is absurd.

To prove that $(f(x_n))$ is a Cauchy sequence, it is sufficient to show that the subsequence $(f(x_{2n}))$ of $(f(x_n))$ is a Cauchy sequence. For convenience, let $y_n = f(x_n)$ for $n = 0, 1, \ldots$. We suppose that (y_{2n}) is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that for every positive integer k, there exist m_k and n_k with $m_k > n_k \geq k$ such that

$$d(y_{2m_k}, y_{2n_k}) \geq \epsilon.$$

We suppose that m_k is the last integer greater than n_k satisfying the above inequality. From

$$\begin{aligned} \epsilon \leq d(y_{2m_k}, y_{2n_k}) &\leq d(y_{2m_k}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2m_k-2}) + d(y_{2m_k-2}, y_{2n_k}) \\ &< d(y_{2m_k}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2m_k-2}) + \epsilon, \end{aligned}$$

as $k \to +\infty$, we obtain

$$d(y_{2m_k}, y_{2n_k}) \to \epsilon.$$

From

$$|d(y_{2m_k-1}, y_{2n_k}) - d(y_{2m_k}, y_{2n_k})| \leq d(y_{2m_k-1}, y_{2m_k}),$$

we deduce

$$d(y_{2m_k-1}, y_{2n_k}) \to \epsilon.$$

Consequently

$$\epsilon \leq d(y_{2m_k}, y_{2n_k}) \leq d(y_{2m_k}, y_{2n_k+1}) + d(y_{2n_k}, y_{2n_k+1})$$

$$\leq d(y_{2n_k}, y_{2n_k+1}) + r(d(y_{2m_k-1}, y_{2n_k})),$$

as $k \to +\infty$, we deduce

$$\epsilon \leq r(\epsilon) < \epsilon$$

and this is a contradiction.

Now, if $f(A \setminus \{u\}) \cup \{u\}$ or $g(A)$ is complete, being $(f(x_n))$ a Cauchy sequence there exists $y \in f(A \setminus \{u\}) \cup \{u\}$ or $y \in g(A)$, such that

$$f(x_n) \to y \quad \text{and} \quad g(x_n) \to y.$$

We observe that $y \neq u$. In fact, if $y = u$ from

$$d(f(x_n), f(u)) \leq r(d(g(x_n), g(u))) = r(d(g(x_n), u)),$$

taking the limit as $n \to +\infty$, we have

$$d(u, f(u)) \leq r(d(u, u)) = r(0) = 0$$

and then $f(u) = u$ that contradicts the hypothesis.

Moreover, as $y \in g(A)$, there exists $z \in A \setminus \{u\}$ such that $g(z) = y$.

To prove that $y \in F(f) \cap F(g)$ we observe that

- $f(z) = g(z) = y$, in fact

$$d(f(x_n), f(z)) \leq r(d(g(x_n), g(z)))$$

and, as $n \to +\infty$,

$$d(y, f(z)) \leq r(d(y, g(z))) = r(0) = 0.$$

- $g(y) = y$, in fact

$$d(g(y), y) = d(g(f(z)), f(z)) \leq Rd(f(z), g(z)) = 0.$$

- $f(y) = y$, in fact

$$d(f(y), y) = d(f(y), f(z)) \leq r(d(g(y), g(z))) = r(d(y, y)) = 0.$$

From the uniqueness follows that $\{y\} = F(f) \cap F(g)$. □

4 Approximation of common fixed points.

Let X be a normed space with norm $\|\cdot\|$, let A be a subset of X and $f, g : A \to A$. We suppose that $u \in F(g)$ and A is u-starshaped, then to each asymptotically g-nonexpansive mapping $f : A \to A$ we can associate a sequence (f_n) of g-contractions defined in A. In fact, let (k_n) be the sequence in the definition of asymptotically g-nonexpansive mapping and fix a sequence $(\lambda_n) \subset]0, 1[$ with $\lambda_n \to 1$. For every positive integer n, let $f_n : A \to A$ be defined by

$$f_n(x) = \mu_n f^n(x) + (1 - \mu_n)u, \quad \forall x \in A,$$

where $\mu_n = \lambda_n / k_n$. It is easy to verify that f_n is a g-contraction, in fact for every $x, y \in A$, we have

$$\begin{aligned} \|f_n(x) - f_n(y)\| &= \|\mu_n f^n(x) + (1 - \mu_n)u - (\mu_n f^n(y) + (1 - \mu_n)u)\| \\ &= \mu_n \|f^n(x) - f^n(y)\| \leq \lambda_n \|g(x) - g(y)\|. \end{aligned}$$

Moreover, if for each $n \in \mathbb{N}$ the mapping f^n is g-strongly regular with constant R_n and g is linear, then f_n is g-regular. In fact,

$$\begin{aligned} \|g(f_n(x)) - f_n(x)\| &= \|g(\mu_n f^n(x) + (1 - \mu_n)u) - (\mu_n f^n(x) + (1 - \mu_n)u)\| \\ &= \|\mu_n g(f^n(x)) + (1 - \mu_n)g(u) - \mu_n f^n(x) - (1 - \mu_n)u)\| \\ &= \|\mu_n g(f^n(x)) - \mu_n f^n(x)\| \leq \mu_n R_n d(g(x), [f^n(x), u]) \\ &\leq \mu_n R_n \|g(x) - \mu_n f^n(x) - (1 - \mu_n)u\| = \mu_n R_n \|g(x) - f_n(x)\|. \end{aligned}$$

The previous observations and the Lemma 1 give the following lemma:

LEMMA 2. *Let X be a normed space and A a nonempty subset of X. Let $f, g : A \to A$ be mappings, A u-starshaped, $g(A) = A$, $u \in F(g)$, $f(A \setminus \{u\}) \subset g(A) \setminus \{u\}$. Suppose that $f(A \setminus \{u\}) \cup \{u\}$ or $g(A)$ is a complete subspace of X and f is asymptotically g-nonexpansive, asymptotically g-strongly regular and g is linear. If (f_n) is a sequence of g-contractions associate to the mapping f, then f_n and g have a unique common fixed point x_n in A for all $n \in \mathbb{N}$.*

We observe that the hypothesis $g(A) = A$ assures that $f_n(A \setminus \{u\}) \subset g(A \setminus \{u\})$ for all $n \in \mathbb{N}$.

The following theorem gives a result of approximation of a common fixed point of asymptotically g-nonexpansive mappings through the fixed points of a sequence (f_n) of g-contractions associate to f.

THEOREM 1. *Let X be a Banach space and A a nonempty closed subset of X. Let $f, g : A \to A$ be mappings, A u-starshaped, $g(A) = A$, $u \in F(g)$, $f(A \setminus \{u\}) \subset g(A) \setminus \{u\}$ and $\overline{f(A \setminus \{u\})}$ be compact.*

If f is uniformly asymptotically regular, asymptotically g-nonexpansive and asymptotically g-strongly regular on $A \setminus \{u\}$ and if g is continuous and linear, then f and g have a common fixed point on A.

Proof. Let (f_n) be a sequence of g-contractions associate to the mapping f. By Lemma 2, $\forall n \geq 1$ there exists an only point $x_n \in A$ such that

$$g(x_n) = x_n = \mu_n f^n(x_n) + (1 - \mu_n)u.$$

We have

$$\begin{aligned} ||x_n - f^n(x_n)|| &= ||\mu_n f^n(x_n) + (1 - \mu_n)u - f^n(x_n)|| \\ &= (1 - \mu_n)||f^n(x_n) - u||. \end{aligned}$$

Since $f(A \setminus \{u\})$ is bounded,

$$||x_n - f^n(x_n)|| \to 0 \quad \text{as} \quad n \to +\infty.$$

Being g continuous and linear and f asymptotically g-nonexpansive, we obtain that

$$\begin{aligned}\|f^{n+1}(x_n)-f(x_n)\| = \|f(f^n(x_n))-f(x_n)\| &\leq K_1\|g(f^n(x_n))-g(x_n)\| \\ &= K_1\|g(f^n(x)-x_n)\| \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Now, from

$$\begin{aligned}||x_n - f(x_n)|| \leq ||x_n - f^n(x_n)|| &+ ||f^n(x_n) - f^{n+1}(x_n)|| \\ &+ ||f^{n+1}(x_n) - f(x_n)||,\end{aligned}$$

since f is uniformly asymptotically regular, we deduce that

$$||x_n - f(x_n)|| \to 0$$

and so

$$\lim_{n\to+\infty} (x_n - f(x_n)) = 0.$$

Since $\overline{f(A \setminus \{u\})}$ is compact and A is closed, there exists a subsequence (x_{n_h}) of (x_n) such that $x_{n_h} \to y \in A$ as $h \to +\infty$.

We observe that $f(x_{n_h}) \to y$ as $h \to +\infty$ and being $g(A) = A$ there exists $z \in A$ such that $g(z) = y$. Then we prove that

- $f(z) = g(z) = y$:

$$||f(x_{n_h}) - f(z)|| \leq k_1||g(x_{n_h}) - g(z)||$$

  and for $h \to +\infty$

$$||y - f(z)|| \leq k_1||y - g(z)|| = 0.$$

- $g(y) = y$:

$$||g(y) - y|| = ||g(f(z)) - f(z)|| \leq R||f(z) - g(z)|| = 0.$$

- $f(y) = y$:

$$||f(y)-y|| = ||f(y)-f(z)|| \leq k_1||g(y)-g(z)|| = k_1||y-y|| = 0.$$

It follows that $y \in F(f) \cap F(S)$.

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