Advances In Boundary Element Techniques XI

Edited by

Ch Zhang
M H Aliabadi
M Schanz

EC ltd

Published by EC, Ltd, UK

Phone (+44) 2380 260334

ELASTOPLASTIC ANALYSIS FOR ACTIVE MACRO-ZONES VIA MULTIDOMAIN SYMMETRIC BEM

Panzeca T.[1,a], Parlavecchio E.[1,b], Terravecchia S.[1,c], Zito L.[1,d]

[1]Disag, Viale delle Scienze, 90128 Palermo, Italy,

[a]tpanzeca@tiscali.it, [b]parlavecchio@unipa.it,

[c]silvioterravecchia@gmail.com ,[d]zitoliborio@tiscali.it

Keywords: SBEM; multidomain; elastoplasticity; active macro-zones; return mapping algorithm.

Abstract. In this paper a strategy to perform elastoplastic analysis by using the Symmetric Boundary Element Method (SBEM) for multidomain type problems is shown. This formulation uses a self-stresses equation to evaluate the trial stress in the predictor phase, and to provide the elastoplastic solution in the corrector one. Since the solution is obtained through a return mapping involving simultaneously all the plastically active bem-elements, the proposed strategy does not depend on the path of the plastic strain process and it is characterized by computational advantages due the considerable decrease of the plastic iterations number. This procedure has been developed inside Karnak.sGbem code [1] by introducing an additional module.

Introduction

A multi-domain SBEM strategy [2], based on an initial strain approach, is applied for the analysis of 2D structures, in the hypothesis of elastic-perfectly plastic behaviour, von Mises model, associated flow rules and strain plane state. Let us start from the discretization of the domain in substructures (in analogy with the finite elements methods), called bem-elements, where the plastic strain accumulation have to be valuated. Then let us impose the regularity conditions, in strong form on the displacements (nodal compatibility) and in weak form on the tractions (generalized equilibrium) both evaluated on the interfaces boundary, and let us effectuate a strong variable condensation. This procedure provides a self-equilibrium stresses equation governing the elatoplastic problem and connecting stresses, valuated on the each bem-e strain points, to plastic strains, treated as volumetric distortions, through an influence matrix (stiffness matrix), negative semi-definite as for the finite elements. The same equation is used both in order to valuate the predictors within the elastic phase, and to correct the elastic solution.

In the first phase the use of only self-stresses equation offers the advantage to evaluate the predictor in simple way. Indeed this equation contains influence coefficients depending on both known imposed plastic strains and the external actions amplified by load multiplier. For the generic load increment, it permits to locate all the bem-elements in which the plastic admissibility condition is violated, i.e. to define the active macro-zones which require correction techniques. Then, in the second phase, the trial solution is corrected by a return mapping algorithm, which is defined in according at the extremal paths theory [3],,in this approach used within a discrete problem.

The proposed algorithm permits the simultaneous correction of the elastic solution in all the plastically active bem-elements and utilizes the same self-stresses equation in a nonlinear global system of 4xa equations in 4xa unknowns, where *a* is the active bem-elements number. In the present approach the approximate solution is easily obtained by using the well-known standard Newton-Raphson procedure, just used in elastoplastic problems within the Bem formulations by some authors [4,5].

In order to prove the efficiency of the proposed strategy, a numeral test, performed by the Karnak.sGbem code [1], is shown at the end of this paper.

1. Self-stresses equation via multidomain SBEM

In this section the procedure utilized to obtain, by using the SBEM for multi-domain type problems, the equation connecting the stresses to the imposed volumetric strains, through a stiffness matrix involving all the bem-elements in the discretized system, is shown.
Let us consider a bi-dimensional body having domain Ω and boundary Γ, subjected to actions acting in its plane:

- forces $\overline{\mathbf{f}}$ at the portion Γ_2 of free boundary,
- displacements $\overline{\mathbf{u}}$ imposed at the portion Γ_1 of constrained boundary,
- body forces $\overline{\mathbf{b}}$ and plastic strains $\boldsymbol{\varepsilon}^p$ in Ω.

The external actions $\overline{\mathbf{f}}, \overline{\mathbf{u}}, \overline{\mathbf{b}}$ may increase separately or simultaneously through the multiplier β.
In the hypothesis that the physical and geometrical characteristics of the body are zone-wise variables, an appropriate subdivision of the domain in bem-elements is introduced. This subdivision involves the introduction of an interface boundary Γ_0 between contiguous bem-elements and, as a consequence, two new unknown quantities rising in the analysis problem, i.e. the displacements $\mathbf{u}_0$ and the tractions $\mathbf{t}_0$ vectors, both referred to interface boundaries.
The adopted strategy [2] contemplates the study of each bem-e embedded in a unlimited domain having the same physical properties and the same thickness of the examining bem-e. It is necessary to distinguish the boundary as Γ of Ω or as Γ^+ of the complementary domain $\Omega_\infty \backslash \Omega$. As a consequence the boundary quantities take on a different meaning: the forces acting on the boundary must be interpreted as layered force distribution, whereas the displacements must be thought as a double layered displacement one.

1.1 Characteristic equations of the bem-e

Let us start by imposing for each bem-e the following Dirichlet and Neumann conditions:

$$\mathbf{u}_1 = \overline{\mathbf{u}}_1 \quad \text{on } \Gamma_1, \qquad \mathbf{t}_2 = \overline{\mathbf{f}}_2 \quad \text{on } \Gamma_2 \tag{1a,b}$$

and introducing the Somigliana Identities (S.I.) of the displacements and of the tractions in the previous eqs.(1a,b). The following integral equation system is obtained:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{u}_1[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0] + \beta\cdot\mathbf{u}_1[\overline{\mathbf{f}}_2,-\overline{\mathbf{u}}_1,\overline{\mathbf{b}}] + \mathbf{u}_1[\boldsymbol{\varepsilon}^p] && \text{on } \Gamma_1\\
\mathbf{t}_2 &= \mathbf{t}_2[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0] + \beta\cdot\mathbf{t}_2[\overline{\mathbf{f}}_2,-\overline{\mathbf{u}}_1,\overline{\mathbf{b}}] + \mathbf{t}_2[\boldsymbol{\varepsilon}^p] && \text{on } \Gamma_2\\
\mathbf{u}_0 &= \mathbf{u}_0[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0] + \beta\cdot\mathbf{u}_0[\overline{\mathbf{f}}_2,-\overline{\mathbf{u}}_1,\overline{\mathbf{b}}] + \mathbf{u}_0[\boldsymbol{\varepsilon}^p] && \text{on } \Gamma_0\\
\mathbf{t}_0 &= \mathbf{t}_0[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0] + \beta\cdot\mathbf{t}_0[\overline{\mathbf{f}}_2,-\overline{\mathbf{u}}_1,\overline{\mathbf{b}}] + \mathbf{t}_0[\boldsymbol{\varepsilon}^p] && \text{on } \Gamma_0
\end{aligned} \tag{2a-d}$$

where the vector $\boldsymbol{\varepsilon}^p$ represents the inelastic strains due to thermal or plastic actions, whose presence requires domain integrals having singular kernels, suitably studied [6,7].
The eqs.(2a-c) have to be interpreted as the response of the body on the boundaries Γ_1^+, Γ_2^+, Γ_0^+, respectively, with the free terms opposite in sign, whereas eq.(2d) has the meaning of traction valued on the actual interface boundary Γ_0.

$$\begin{aligned}
&\mathbf{u}_1^+[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0] + \beta\cdot\mathbf{u}_1^+[\overline{\mathbf{f}}_2,-\overline{\mathbf{u}}_1,\overline{\mathbf{b}}] + \mathbf{u}_1^+[\boldsymbol{\varepsilon}^p] = \mathbf{0} && \text{on } \Gamma_1^+\\
&\mathbf{t}_2^+[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0] + \beta\cdot\mathbf{t}_2^+[\overline{\mathbf{f}}_2,-\overline{\mathbf{u}}_1,\overline{\mathbf{b}}] + \mathbf{t}_2^+[\boldsymbol{\varepsilon}^p] = \mathbf{0} && \text{on } \Gamma_2^+\\
&\mathbf{u}_0^+[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0] + \beta\cdot\mathbf{u}_0^+[\overline{\mathbf{f}}_2,-\overline{\mathbf{u}}_1,\overline{\mathbf{b}}] + \mathbf{u}_0^+[\overline{\boldsymbol{\varepsilon}}^p] = \mathbf{0} && \text{on } \Gamma_0^+\\
&\mathbf{t}_0 = \mathbf{t}_0[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0] + \beta\cdot\mathbf{t}_0[\overline{\mathbf{f}}_2,-\overline{\mathbf{u}}_1,\overline{\mathbf{b}}] + \mathbf{t}_0[\boldsymbol{\varepsilon}^p] && \text{on } \Gamma_0
\end{aligned} \tag{3a-d}$$

In addiction, let us introduce the stress vector:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}[\mathbf{f}_1,-\mathbf{u}_2,\mathbf{f}_0,-\mathbf{u}_0] + \beta\cdot\boldsymbol{\sigma}[\overline{\mathbf{f}}_2,-\overline{\mathbf{u}}_1,\overline{\mathbf{b}}] + \boldsymbol{\sigma}[\boldsymbol{\varepsilon}^p] \qquad \text{on } \Omega \tag{3e}$$

and the boundary discretization into boundary elements by making the following modelling of all the known and unknown quantities:

$$\mathbf{f}_1 = \boldsymbol{\Psi}_t\,\mathbf{F}_1,\quad \overline{\mathbf{f}}_2 = \boldsymbol{\Psi}_t\,\overline{\mathbf{F}}_2,\quad \mathbf{t}_0 = \boldsymbol{\Psi}_t\,\mathbf{F}_0,\quad \mathbf{u}_2 = \boldsymbol{\Psi}_u\,\mathbf{U}_2,\quad \overline{\mathbf{u}}_1 = \boldsymbol{\Psi}_u\,\overline{\mathbf{U}}_1,\quad \mathbf{u}_0 = \boldsymbol{\Psi}_u\,\mathbf{U}_0,\quad \boldsymbol{\varepsilon}^p = \boldsymbol{\Psi}_p\,\mathbf{p} \tag{4a-g}$$

where $\boldsymbol{\Psi}_t$ and $\boldsymbol{\Psi}_u$ are shape functions regarding the boundary quantities, while $\boldsymbol{\Psi}_p$ are domain shape functions used to model plastic strains $\mathbf{p}$ connected to the Gauss points of the bem-e. Besides, the capital letters $\mathbf{F}$ and $\mathbf{U}$ indicate the nodal vectors of the forces ($\mathbf{F}_1$ on Γ_1 and $\mathbf{F}_0$ on Γ_0) and of the displacements ($\mathbf{U}_2$ on Γ_2 and $\mathbf{U}_0$ on Γ_0) defined on the boundary elements.
Let us perform the weighting of all the coefficients of the eqs.(3a-d). For this purpose, the same shape functions as those modelling the causes have been employed, but introduced in an energetically dual way in according to the Galerkin approach. In this way it is possible to obtain the following block system:

$$\left|\begin{array}{c}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \hline \mathbf{P}_0\end{array}\right| = \left|\begin{array}{ccc:c}\mathbf{A}_{u1,u1} & \mathbf{A}_{u1,f2} & \mathbf{A}_{u1,u0} & \mathbf{A}_{u1,f0}\\ \mathbf{A}_{f2,u1} & \mathbf{A}_{f2,f2} & \mathbf{A}_{f2,u0} & \mathbf{A}_{f2,f0}\\ \mathbf{A}_{u0,u1} & \mathbf{A}_{u0,f2} & \mathbf{A}_{u0,u0} & \bar{\mathbf{A}}_{u0,f0}\\ \hdashline \mathbf{A}_{f0,u1} & \mathbf{A}_{f0,f2} & \bar{\mathbf{A}}_{f0,u0} & \mathbf{A}_{f0,f0}\end{array}\right| \left|\begin{array}{c}\mathbf{F}_1\\ -\mathbf{U}_2\\ \mathbf{F}_0\\ \hline -\mathbf{U}_0\end{array}\right| + \left|\begin{array}{c}\mathbf{A}_{u1,\sigma}\\ \mathbf{A}_{f2,\sigma}\\ \mathbf{A}_{u0,\sigma}\\ \hline \mathbf{A}_{f0,\sigma}\end{array}\right| |\,\mathbf{p}\,| + \beta \left|\begin{array}{c}\hat{\mathbf{W}}_1\\ \hat{\mathbf{P}}_2\\ \hat{\mathbf{W}}_0\\ \hline \hat{\mathbf{L}}_0\end{array}\right| \qquad (5a\text{-}d)$$

where the first and second rows represent the Dirichlet and Neumann conditions (1a,b) written in weighted form $\int_{\Gamma_1^+} \tilde{\boldsymbol{\psi}}_f \mathbf{u}_1^+ = \mathbf{0}$ and $\int_{\Gamma_2^+} \tilde{\boldsymbol{\psi}}_u \mathbf{t}_2^+ = \mathbf{0}$. The third and fourth rows regard the weighting of the displacements $\int_{\Gamma^+} \tilde{\boldsymbol{\psi}}_f \mathbf{u}_c^{1+} = \mathbf{0}$ and of the tractions at the interface zone. In particular the vector $\mathbf{P}_0 = \int_{\Gamma_0} \boldsymbol{\psi}_u^T \mathbf{t}_0$ in eq.(3d) collects the generalized tractions defined on the boundary elements of Γ_0.
The influence matrix, containing 4x4 block matrices, is symmetric. The introduced coefficient β is the multiplier of the boundary $(-\bar{\mathbf{U}}_1)$, $\bar{\mathbf{F}}_2$ and domain $\bar{\mathbf{b}}$ actions.
Eq.(3e) defines the field stress, obtained through the S.I., i.e.:

$$|\,\boldsymbol{\sigma}\,| = \left|\begin{array}{ccc:c}\mathbf{a}_{\sigma,u1} & \mathbf{a}_{\sigma,f2} & \mathbf{a}_{\sigma,u0} & \mathbf{a}_{\sigma,f0}\end{array}\right| \left|\begin{array}{c}\mathbf{F}_1\\ -\mathbf{U}_2\\ \mathbf{F}_0\\ \hline -\mathbf{U}_0\end{array}\right| + |\,\mathbf{a}_{\sigma,\sigma}\,|\,|\,\mathbf{p}\,| + \beta\,|\,\hat{\mathbf{l}}_\sigma\,| \,. \qquad (5e)$$

The eqs.(5a-e) may be expressed in compact form in the following way:

$$\begin{aligned}\mathbf{0} &= \mathbf{A}\mathbf{X} + \mathbf{A}_0(-\mathbf{U}_0) + \mathbf{A}_\sigma \mathbf{p} + \beta \cdot \hat{\mathbf{L}}\\ \mathbf{P}_0 &= \tilde{\mathbf{A}}_0 \mathbf{X} + \mathbf{A}_{00}(-\mathbf{U}_0) + \mathbf{A}_{0\sigma}\mathbf{p} + \beta \cdot \hat{\mathbf{L}}_0\\ \boldsymbol{\sigma} &= \mathbf{a}_\sigma \mathbf{X} + \mathbf{a}_{\sigma 0}(-\mathbf{U}_0) + \mathbf{a}_{\sigma\sigma}\mathbf{p} + \beta \cdot \hat{\mathbf{l}}_\sigma\end{aligned} \qquad (6a\text{-}c)$$

where the vector $\mathbf{X}$ collects the sub-vectors $\mathbf{F}_1$, $(-\mathbf{U}_2)$ and $\mathbf{F}_0$, whereas the $(-\mathbf{U}_0)$ and $\mathbf{p}$ vectors characterize the displacements of the nodes in the interface zone, changed in sign, and the nodal plastic strains, respectively.
The vector $\mathbf{P}_0$ represents the generalized (or weighted) traction vector defined in the boundary elements of the interface zone, obtained as a weighted response to all the known, amplified by β, and unknown actions, regarding boundary and domain quantities. The vector $\boldsymbol{\sigma}$ represents the stress, valued at the Gauss points, due to the all the known, amplified by β, and unknown actions.
By performing a variables condensation through the replacement of the $\mathbf{X}$ vector extracted from eq.(6a) into eqs.(6b,c), one obtains:

$$\begin{aligned}\mathbf{P}_0 &= \mathbf{D}_{00}\mathbf{U}_0 + \mathbf{D}_{0\sigma}\mathbf{p} + \beta \cdot \hat{\mathbf{P}}_0\\ \boldsymbol{\sigma} &= \mathbf{d}_{\sigma 0}\mathbf{U}_0 + \mathbf{d}_{\sigma\sigma}\mathbf{p} + \beta \cdot \hat{\boldsymbol{\sigma}}\end{aligned} \qquad (7a,b)$$

These latter are the equations characteristic of each bem-e. They relate the generalized (or weighted) tractions $\mathbf{P}_0$ defined on the interface zone Γ_0 and the stresses $\boldsymbol{\sigma}$ at the bem-e domain to the nodal displacements $\mathbf{U}_0$, to the plastic strains $\mathbf{p}$ and the two load terms $\hat{\mathbf{P}}_0$ and $\hat{\boldsymbol{\sigma}}$ amplified by β, respectively. These latter represent the generalized tractions vector along the interface boundary and the stresses vector in the domain with reference to each bem-e, as elastic response. Moreover, $\mathbf{D}_{00}$, $\mathbf{D}_{0\sigma}$, $\mathbf{d}_{\sigma 0}$, $\mathbf{d}_{\sigma\sigma}$ are the stiffness matrices of the bem-e, being $\mathbf{D}_{00}$ and $\mathbf{d}_{\sigma\sigma}$ square matrices, $\mathbf{D}_{0\sigma}$ and $\mathbf{D}_{\sigma 0}$ rectangular ones.

1.2 Assembled system and self-stresses equation

Let us subdivide the body in *m* bem-elements and consider for each of these the eqs.(7a,b). Thus we obtain two global relations connecting all the generalized tractions and the stresses related to the bem-elements considered, formally equal to the same eqs.(7a,b), but regarding the constitutive equations of the assembled system.

Let us introduce the compatibility among the nodal displacements of the adjacent bem-elements:

$$\mathbf{U}_0 = \mathbf{H}\,\boldsymbol{\xi}_0 \tag{8}$$

where $\mathbf{H}$ is a topological matrix and $\boldsymbol{\xi}_0$ the nodal displacements vector of the assembled system, and the equilibrium condition among generalized tractions at the interface boundaries.

$$\mathbf{H}^T\,\mathbf{P}_0 = \mathbf{0} \tag{9}$$

Using the previous eqs.(8,9), the eqs.(7a,b) become:

$$\begin{aligned} &\mathbf{K}_{00}\boldsymbol{\xi}_0 + \mathbf{K}_{0\sigma}\,\mathbf{p} + \beta\cdot\hat{\mathbf{f}}_0 = \mathbf{0} \\ &\boldsymbol{\sigma} = \mathbf{k}_{\sigma 0}\,\boldsymbol{\xi}_0 + \mathbf{k}_{\sigma\sigma}\,\mathbf{p} + \beta\cdot\hat{\boldsymbol{\sigma}} \end{aligned} \tag{10a,b}$$

By performing a new variables condensation through the replacement of the $\boldsymbol{\xi}_0$ vector extracted from eq.(10a) into eq.(10b), it is obtained:

$$\boldsymbol{\sigma} = \mathbf{Z}\,\mathbf{p} + \beta\cdot\hat{\boldsymbol{\sigma}}_s \tag{11}$$

where

$$\mathbf{Z} = -\mathbf{k}_{\sigma 0}\,\mathbf{K}_{00}^{-1}\mathbf{K}_{0\sigma} + \mathbf{k}_{\sigma\sigma},\;\; \hat{\boldsymbol{\sigma}}_s = -\mathbf{k}_{\sigma 0}\,\mathbf{K}_{00}^{-1}\,\hat{\mathbf{f}}_0 + \hat{\boldsymbol{\sigma}} \tag{12a,b}$$

The eq.(11) provides the stress at the strain points of each bem-e in function of the volumetric plastic strain $\mathbf{p}$ and of the external actions $\hat{\boldsymbol{\sigma}}_s$, the latter amplified by β. The matrix $\mathbf{Z}$, defined self-stresses influence matrix of the assembled system, is a square matrix having 3*m*x3*m* dimensions with *m* bem-elements number, full, non symmetric and semi-defined negative. The evaluation of this matrix involves only the elastic characteristic of the material and the structure geometry.

The matrix $\mathbf{Z}$ permits to evaluate the elastic response in the Gauss points of all the bem-elements due to the plastic stain vector $\mathbf{p}$, whereas the vector $\hat{\boldsymbol{\sigma}}_s$ collects the influence coefficients, as response to the known external actions $\overline{\mathbf{F}}_2, -\overline{\mathbf{U}}_1, \overline{\mathbf{b}}$.

2. Active macro-zones analysis

In this section the strategy to compute the plastic strains for each loading step and at every bem-e is shown. These approaches utilize eq.(11) both to evaluate the predictors and during the corrector phase, here after shown.

Let us start computing the trial stresses, i.e. the purely elastic response at the instant $n+1$ in each *m* bem-elements of the discretized body.

For thus purpose, eq.(11) provides all the predictors $\boldsymbol{\sigma}^*_{(n+1)}$ as function of the plastic strain $\mathbf{p}_{(n)}$, stored up at the previous step and then imposed as volumetric distortions, and of load increment $\beta_{(n+1)}$:

$$\boldsymbol{\sigma}^*_{(n+1)} = \mathbf{Z}\,\mathbf{p}_{(n)} + \beta_{(n+1)}\cdot\hat{\boldsymbol{\sigma}}_s \tag{13}$$

where $\mathbf{Z}$ matrix is full and regards all the bem-elements, obtained by the discretization.

The check of the plastic consistency condition of the stresses computed on appropriately chosen points is performed by using the yield condition expressed in this context through the von Mises law for each bem-e:

$$F[\boldsymbol{\sigma}_{(n+1)}] = \tfrac{1}{2}\boldsymbol{\sigma}^T_{(n+1)}\;\boldsymbol{M}\;\boldsymbol{\sigma}_{(n+1)} - \sigma_y^2 \le 0\,. \tag{14}$$

In the *a* bem-elements (with $a \le m$) where this latter inequality is violated, a return mapping phase occurs to evaluate the plastic strains and the direction of the plastic flow.

This phase, called corrector phase, uses the same eq.(11) to obtain the elastoplastic solution at every bem-e where the plastic consistency condition is violated. In this phase the vector $\boldsymbol{\sigma}$, representing the end step stress, as well as the volumetric plastic strain vector $\mathbf{p}$ are unknown quantities. This latter is the plastic strain to impose at every active plastically bem-e in order to have the stress on the yield boundary of the elastic domain, through which the direction of the plastic flow may be defined. Obviously, inside of each loading step the corrector phase has to be repeated until all the predictors do not satisfy the plastic consistency conditions.

In detail eq.(11), written for every *h* bem-elements ($h = 1,...,a$), is utilized to perform the elastoplastic analysis at $n+1$ load step simultaneously in all the plastically active macro-zones individuated in the previous predictor phase, i.e.:

$$\boldsymbol{\sigma}_a - \boldsymbol{\sigma}_a^* - \mathbf{Z}_{aa}\,\mathbf{p}_a = \mathbf{0} \tag{15}$$

where the subscript $n+1$ has been omitted for convenience.

The $\mathbf{Z}_{aa}$ matrix coefficients derive from the $\mathbf{Z}$ matrix present in eq.(15), by extracting the blocks relative to the *a* plastically active bem-elements. The double index specifies the bem-elements where the plastic strains (cause) and the related stresses (effect) arise.

Let us introduce the plastic admissibility conditions for the *a* bem-elements:

$$\boldsymbol{F}[\boldsymbol{\sigma}_a] \le \mathbf{0}\,, \quad \boldsymbol{\Lambda}_a \ge \mathbf{0}\,, \quad \boldsymbol{\Lambda}_a \boldsymbol{F}[\boldsymbol{\sigma}_a] = \mathbf{0} \tag{16a-c}$$

In the hypothesis that, for each *h*-th bem-e, the shape function definite in eq.(4g) is the same of the shape function related to the plastic multiplier, i.e. $\lambda_h = \psi_p \Lambda_h$ with $\psi_p \ge 0$, the plastic strain for the *h*-th active bem-e is expressed as:

$$\mathbf{p}_h = \Lambda_h \frac{\partial F}{\partial \boldsymbol{\sigma}_h} = \Lambda_h\, \boldsymbol{M}\, \boldsymbol{\sigma}_h \tag{17}$$

The solving non linear system for all the active bem-elements is the following:

$$\begin{cases} \boldsymbol{f}_{Ih} \equiv \boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^* - \sum_{k=1}^{a} \Lambda_k\, \mathbf{Z}_{hk}\, \boldsymbol{M}\, \boldsymbol{\sigma}_k = \mathbf{0} \\ f_{IIh} \equiv \frac{1}{2} \boldsymbol{\sigma}_h^T\, \boldsymbol{M}\, \boldsymbol{\sigma}_h - \sigma_y^2 = 0 \quad \textit{with } h = 1,...a \end{cases} \tag{18a,b}$$

where $\boldsymbol{\sigma}_h$ is the stress solution located on the yield surface of the elastic domain, $\boldsymbol{\sigma}_h^*$ the elastic predictor, $\Lambda_h\, \mathbf{Z}_{hh}\, \boldsymbol{M}\, \boldsymbol{\sigma}_h$ the direct corrective components (stress in the *h*-th bem-e due to distortion $\mathbf{p}_h$ applied on the same bem-e) and $\sum_{k=1}^{a \notin h} \Lambda_k\, \mathbf{Z}_{hk}\, \boldsymbol{M}\, \boldsymbol{\sigma}_k$ the indirect corrective components (stress in the *h*-th bem-e due to distortion $\mathbf{p}_k$ applied on the *k*-th bem-e) respectively.

The eqs.(18a,b) comprises a system of 4xa non linear equations in 4xa unknowns (three stress components $\boldsymbol{\sigma}_h$ and a plastic multiplier Λ_h for each active bem-e).

The approximate solution of this nonlinear problem involving all the plastically active bem-elements is here obtained by applying the Newton-Raphson procedure as follows:

$$\left|\begin{array}{ccc|ccc} \mathbf{I} - \Lambda_1^j \mathbf{Z}_{11} \boldsymbol{M} & \cdots & -\Lambda_a^j \mathbf{Z}_{1a} \boldsymbol{M} & -\mathbf{Z}_{11} \boldsymbol{M} \boldsymbol{\sigma}_1^j & \cdots & -\mathbf{Z}_{1a} \boldsymbol{M} \boldsymbol{\sigma}_a^j \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\Lambda_a^j \mathbf{Z}_{a1} \boldsymbol{M} & \cdots & \mathbf{I} - \Lambda_a^j \mathbf{Z}_{aa} \boldsymbol{M} & -\mathbf{Z}_{a1} \boldsymbol{M} \boldsymbol{\sigma}_1^j & \cdots & -\mathbf{Z}_{aa} \boldsymbol{M} \boldsymbol{\sigma}_a^j \\ \hline (\boldsymbol{\sigma}_1^j)^T \boldsymbol{M} & \cdots & \mathbf{0} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & (\boldsymbol{\sigma}_a^j)^T \boldsymbol{M} & 0 & \cdots & 0 \end{array}\right| \cdot \left|\begin{array}{c} \boldsymbol{\sigma}_1^{j+1} - \boldsymbol{\sigma}_1^j \\ \vdots \\ \boldsymbol{\sigma}_a^{j+1} - \boldsymbol{\sigma}_a^j \\ \hline \Lambda_1^{j+1} - \Lambda_1^j \\ \vdots \\ \Lambda_a^{j+1} - \Lambda_a^j \end{array}\right| = \left|\begin{array}{c} -\boldsymbol{f}_{I1}^j \\ \vdots \\ -\boldsymbol{f}_{Ia}^j \\ \hline -f_{II1}^j \\ \vdots \\ -f_{IIa}^j \end{array}\right| \tag{19}$$

which, written in compact form, becomes:

$$\boldsymbol{X}_a^{j+1} = \boldsymbol{X}_a^j - \mathbf{J}_a (\boldsymbol{X}_a^j)^{-1}\, \boldsymbol{f}(\boldsymbol{X}_a^j)\ . \tag{20}$$

The Jacobian matrix $\mathbf{J}_a$ contains the derivatives of the functions defined in eqs.(18a,b), $\boldsymbol{X}_a^{j+1}$ is the vector of the unknowns, $\boldsymbol{X}_a^j$ and $\boldsymbol{f}(\boldsymbol{X}_a^j)$ are the known vectors evaluated in the *j*-th step.
The vector $\boldsymbol{X}_a$ in the j+1-th step is the solution in terms of stress and plastic multipliers evaluated on the Gauss points of all the plastically active bem-elements.
Since the Jacobian operator $\mathbf{J}_a(\boldsymbol{X}_a^j)$ usually has big dimensions which coefficients have different meaning, its inverse and update could require high computational cost. In order to overcome these disadvantage, the following strategy, able to reduce the computational burden in the iterative process, is been developed. Let us consider the system of eqs.(19) here rewritten in compact form:

$$\left|\begin{array}{c|c} \mathbf{J}_{\sigma\sigma}^j & \mathbf{J}_{\sigma\Lambda}^j \\ \hline \mathbf{J}_{\Lambda\sigma}^j & \mathbf{0} \end{array}\right| \cdot \left|\begin{array}{c} \boldsymbol{\sigma}^{j+1} - \boldsymbol{\sigma}^j \\ \hline \boldsymbol{\Lambda}^{j+1} - \boldsymbol{\Lambda}^j \end{array}\right| = \left|\begin{array}{c} -\mathbf{F}_{\sigma}^j \\ \hline -\mathbf{F}_{\Lambda}^j \end{array}\right| \tag{21}$$

and in explicit form:

$$\begin{cases} \mathbf{J}_{\sigma\sigma}^j(\boldsymbol{\sigma}^{j+1} - \boldsymbol{\sigma}^j) + \mathbf{J}_{\sigma\Lambda}^j(\boldsymbol{\Lambda}^{j+1} - \boldsymbol{\Lambda}^j) = -\mathbf{F}_{\sigma}^j \\ \mathbf{J}_{\Lambda\sigma}^j(\boldsymbol{\sigma}^{j+1} - \boldsymbol{\sigma}^j) = -\mathbf{F}_{\Lambda}^j \end{cases} \tag{22a,b}$$

Let us perform a condensation of variables by extracting the vector $(\boldsymbol{\sigma}^{j+1} - \boldsymbol{\sigma}^j)$ from eq.(22a):

$$(\boldsymbol{\sigma}^{j+1} - \boldsymbol{\sigma}^j) = (\mathbf{J}_{\sigma\sigma}^j)^{-1}[\mathbf{J}_{\sigma\Lambda}^j(\boldsymbol{\Lambda}^{j+1} - \boldsymbol{\Lambda}^j) - \mathbf{F}_{\sigma}^j] \tag{23}$$

and replacing it into eq.(22b). It is obtained:

$$\tilde{\mathbf{J}}_{\Lambda\Lambda}^j(\boldsymbol{\Lambda}^{j+1} - \boldsymbol{\Lambda}^j) = \tilde{\mathbf{F}}_{\Lambda}^j \tag{24}$$

where:

$$\tilde{\mathbf{J}}_{\Lambda\Lambda}^j = -\mathbf{J}_{\Lambda\sigma}^j(\mathbf{J}_{\sigma\sigma}^j)^{-1}\mathbf{J}_{\sigma\Lambda}^j\,, \quad \tilde{\mathbf{F}}_{\Lambda}^j = \mathbf{J}_{\Lambda\sigma}^j(\mathbf{J}_{\sigma\sigma}^j)^{-1}\mathbf{F}_{\sigma}^j - \mathbf{F}_{\Lambda}^j \tag{25a,b}$$

The proposed algorithm shows high computational efficiency because the inversion is related to only two blocks $\tilde{\mathbf{J}}_{\Lambda\Lambda}^j$ and $\mathbf{J}_{\sigma\sigma}^j$ of reduced dimensions.
$\mathbf{Z}$ is a square matrix. It is written only once and its dimension depends on the bem-elements number as a result of the discretization. This matrix is used twice: as $\mathbf{Z}$ having n dimension in the predictor phase second its origininary form, as $\mathbf{Z}_{aa}$ having $a \le h$ dimension during the return mapping phase with reference to the plastically active bem-elements, i.e. those bem-elements where the predictor do not satisfy the yield condition.
The peculiarity of the shown approach is that the return mapping process is used simultaneously in the plastically active bem-elements, avoiding the return mapping strategy for single bem-e, it showing arbitrariness and very onerous computational burden.
In addition the proposal to perform simultaneously on the plastically active bem-elements involves iterative return mapping process with a very high saving in the computational times.

3. Numerical results

In order to show the efficiency of the proposed method, a traction test, by using the SBEM code Karnak.sGbem [1], has been performed. In the present section a square plate with circular hole is subjected to tensile load $\mathrm{q} = 100000\ \mathrm{daN/m}$, as shown in Fig.1. The material characteristics are the Young's modulus $\mathrm{E} = 200000\ \mathrm{daN/cmq}$ and the Poisson's ratio $\nu = 0.29$, whereas the uniaxial yield value is $\sigma_{\mathrm{y}} = 4500\ \mathrm{daN/cmq}$. The plate geometry, shown in Fig.1, has unit thickness.
The load-displacement curve is shown in Fig.2 and the solution was compared to the strongly iterative solution in the sphere of SBEM [8].

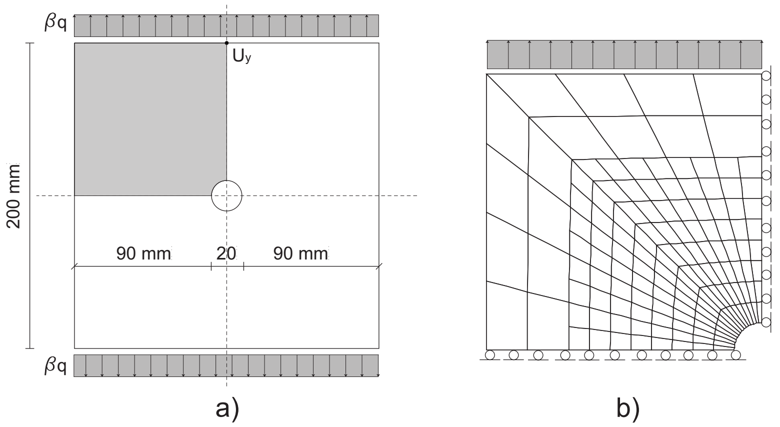

Fig. 1. Plate with circular hole: a) problem description; b) adopted mesh.

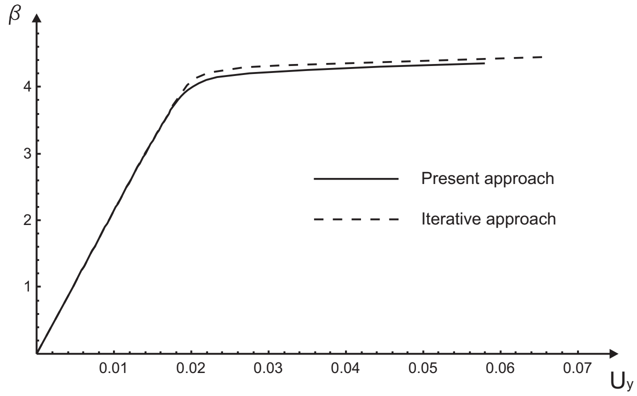

Fig. 2. Load – displacement curve.

References

[1] Cucco F, Panzeca T, Terravecchia S.,(2002). The program Karnak.sGbem Release 2.1.

[2] Panzeca T., Cucco F., Terravecchia S., (2002). Symmetric boundary element method versus Finite element method. *Comp. Meth. Appl. Mech. Engng.*; 191: 3347-3367.

[3] Ponter A.R.S., Martin J.B., (1972). Some extremal properties and energy theorems for inelastic materials and their relationship to the deformation theory of plasticity. *J. Mech. Phys. Solids*; 20: 281-300.

[4] Bonnet M., Mukherjee S., (1996). Implicit BEM formulations for usual and sensitivity problems in elastoplasticity using the consistent tangent operator concept. *Int. J. Solids & Struct.*; 33: 4461-4480.

[5] Mallardo V., Alessandri C., (2004). Arc-length procedures with BEM in physically nonlinear problems. *Engng. Anal. Bound. Elem.*; 28: 547-559.

[6] Panzeca T., Terravecchia S., Zito L., (2010). Computational aspects in 2D SBEM analysis with domain inelastic actions. *Int. J. Num. Meth. Engng.*; 82: 184-204.

[7] Gao X.W., Davies T.G., (2000). An effective boundary element algorithm for 2D and 3D elastoplastic problem. *Int. J. Solids & Struct.*; 37: 4987-5008.

[8] Panzeca T., Cucco F., Parlavecchio E., Zito L. (2009). Elastoplastic analysis by the Multidomain Symmetric Boundary Element Method. *Xth International Conference on Boundary Element Techniques*, (Beteq 2009).