Algebraic $(2, 2)$-transformation groups

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Abstract

In this paper we determine all algebraic transformation groups G, defined over an algebraically closed field k, which operate transitively, but not primitively, on a variety Ω, provided the following conditions are fulfilled. We ask that the (non-effective) action of G on the variety of blocks is sharply 2-transitive, as well as the action on a block Δ of the normalizer G_Δ. Also we require sharp transitivity on pairs (X, Y) of independent points of Ω, i.e. points contained in different blocks.

Although classifications of imprimitive permutation groups appeared already at beginning of the last century (see [10]) and imprimitive actions play an important role in geometry, the corresponding literature is actually less well-developed than the one concerning primitive groups. For finite groups some classification has been done (see for instance [1], [5] and [11]). In [1] by using wreath products, the best-known construction principle to get imprimitive groups, a classification of finite imprimitive groups, acting highly transitively on blocks and satisfying conditions very common in geometry, is achieved.

The present paper arises with the aim to obtain classifications for infinite imprimitive groups belonging to well-studied categories. We start with an imprimitive algebraic group G, over an algebraically closed field k, operating on an algebraic variety Ω of positive dimension in such a way that the induced actions on the set $\overline{\Omega}$ of blocks and on a block Δ are both sharply 2-transitive. Moreover we ask the group to act sharply transitively on pairs of points lying in different blocks. The latter condition, frequently occurring in geometry (see for instance [2]), avoids a too general context. For the classification we do not need the group actions be bi-regular morphisms but we just ask that the orbit maps be separable morphisms. It turns out that G is the semidirect product of a 3-dimensional unipotent connected group G_u by a 1-dimensional connected torus T, both acting on the points of an affine plane over k with a full set of parallel lines as the blocks.

There are two subgroups which play a fundamental role for the classification: the kernel $G_{[\overline{\Omega}]}$ of the representation on $\overline{\Omega}$ (the so-called *inertia subgroup*) and its stabilizer $G_{[\overline{\Omega}]_O}$ of a fixed point O, which turns out to be even the point-wise stabilizer of the block containing O. There exists a G-invariant transversal L of G with respect to $G_{[\overline{\Omega}]_O}$ which is essential for the classification. L is a subgroup precisely if $G_u/\mathfrak{z}(G_u)$ is commutative, in such a case $G_u/\mathfrak{z}(G_u)$ is even a vector group. Fixing the structure of L, the classification (see the main theorem) depends on four (not necessarily independent) integer parameters which distinguish the isomorphism class of G. But if the $\operatorname{char} \mathsf{k}$ is positive, then for suitable values of the integer parameters it happens that L could be both a vector group and a non-commutative group.

We refer to [13] for well-known results about non-affine algebraic groups and to [9] about affine algebraic groups.

§**1**. Throughout the paper G will denote an algebraic group defined over an algebraically closed field k, operating effectively on the points of a variety Ω of positive dimension. We assume that the orbit maps $g \mapsto g(X)$ are separable morphisms $G \to \Omega$ and G acts transitively with a nontrivial system of imprimitivity $\overline{\Omega}$. Moreover, putting

- the *normalizer* $G_\Delta := \{g \in G : g(\Delta) = \Delta\}$ of $\Delta \in \overline{\Omega}$,
- the *centralizer* $G_{[\,\Delta\,]} := \{g \in G_\Delta : g(X) = X \ \forall X \in \Delta\}$ of $\Delta \in \overline{\Omega}$,
- the *inertia* subgroup $G_{[\,\overline{\Omega}\,]} := \big\{g \in G : g(\Delta) = \Delta \ \forall \Delta \in \overline{\Omega}\big\}$,

we require the following transitivities:

1. $G_\Delta/G_{[\,\Delta\,]}$ acts sharply 2-transitively on Δ,
2. $G/G_{[\,\overline{\Omega}\,]}$ acts sharply 2-transitively on $\overline{\Omega}$,
3. G acts sharply transitively on $\Lambda := \big\{(X,Y) \in \Omega^2 : \Delta_X \neq \Delta_Y\big\}$, where $\Delta_Z \in \overline{\Omega}$ denotes the block containing $Z \in \Omega$.

We call such a triple $\mathsf{G} = (G, \Omega, \overline{\Omega})$ a $(2,2)$-*imprimitive algebraic group*. Since the stabilizer of a point is not trivial, conditions 3 and 1 guarantiy that the centre of G consists just of the identity. Hence the algebraic group G must be affine.

1 . Proposition:

i) Every block $\Delta \in \overline{\Omega}$ is closed and $G_\Delta = G_{[\,\overline{\Omega}\,]}G_X$ for any $X \in \Delta$;

ii) the inertia subgroup $G_{[\,\overline{\Omega}\,]}$ is closed.

Proof : Every block $\Delta \in \overline{\Omega}$ is a constructible set as the union, for $X \in \Delta$, of two G_X-orbits, $\{X\}$ and $\Delta \setminus \{X\}$, so Δ is closed by Theorem 1.6 in [7]. Then $G_{[\,\overline{\Omega}\,]}$ is the intersection of all closed subgroups G_Δ. Finally $G_\Delta = G_{[\,\overline{\Omega}\,]}G_X$ follows from the fact that the normal subgroup $G_{[\,\overline{\Omega}\,]}$ acts transitively on Δ. □

2 . *Remark* : As orbit maps are separable morphisms $G \to \Omega$, by the universal mapping property we may identify Ω with the homogeneous space G/G_O for a fixed stabilizer $G_O = \big\{g \in G : g(O) = O\big\}$, $O \in \Omega$. As well as, in view of Proposition 1, we may identify $\overline{\Omega}$ with the homogeneous space G/G_Δ.

3 . Proposition: *For all $X \in \Omega$ the centralizer $G_{[\,\overline{\Omega}\,]_X} = \big\{g \in G_{[\,\overline{\Omega}\,]} : g(X) = X\big\}$ is contained in $G_{[\,\Delta_X]}$ and $G_{[\,\overline{\Omega}\,]} = G_{[\,\overline{\Omega}\,]_X} \times G_{[\,\overline{\Omega}\,]_Y}$ for any $(X,Y) \in \Lambda$.*

Proof : $G_{[\,\overline{\Omega}\,]_X}$ acts (effectively and) sharply transitively on the block Δ_Y, the centralizer $G_{X,Y}$ being trivial. If blocks contain finitely many points the order of $G_{[\,\overline{\Omega}\,]_X}$ is $|\Delta|$. In such a case $G_{[\,\overline{\Omega}\,]_X}$ operates non-effectively on $\Delta_X \setminus \{X\}$ with orbits of the same length θ, since $G_{[\,\overline{\Omega}\,]_{X,X'}} = G_{[\,\overline{\Omega}\,]} \cap G_{[\,\Delta_X\,]}$ for any $X' \in \Delta \setminus \{X\}$. But $\gcd(|\Delta|-1, |\Delta|) = 1$ forces $\theta = 1$.

If blocks contain infinitely many points, $G_{[\,\overline{\Omega}\,]_X}$ acts on Δ_Y as the kernel of the Frobenius group $G_{\Delta_Y}/G_{[\,\Delta_Y\,]}$. So $G_{[\,\overline{\Omega}\,]_X}$ is a 1-dimensional connected unipotent group by [7] (Theorem 1.10), hence must act trivially on Δ_X by Proposition 1 in [8]. Therefore in any case $G_{[\,\overline{\Omega}\,]_X} < G_{[\,\Delta_X]}$ and this forces $G_{[\,\overline{\Omega}\,]_X}$ to be a normal subgroup of $G_{[\,\overline{\Omega}\,]}$. The last claim follows from the sharply transitivity of G on Λ. □

4 . Proposition:

a) $\overline{\Omega}$ *contains infinitely many blocks and every block contains infinitely many points;*

b) G_O *is the semidirect product of the* 1*-dimensional connected unipotent subgroup* $G_{[\overline{\Omega}]_O}$ *by a* 1*-dimensional connected torus* T*;*

c) $G/G_{[\overline{\Omega}]}$ *is a* 2*-dimensional Frobenius algebraic group with complement* $\simeq T$*;*

d) For all $\Delta \in \overline{\Omega}$, $G_\Delta/G_{[\Delta]}$ *is a* 2*-dimensional Frobenius algebraic group whose* 1*-dimensional kernel is isomorphic to* $G_{[\overline{\Omega}]_X}$ *for any* $X \in \Omega \setminus \Delta$.

Proof : The group $G_O/G_{[\overline{\Omega}]_O}$ acts effectively and sharply transitively on $\overline{\Omega} \setminus \{\Delta_O\}$ and maps surjectively onto $G_O/G_{[\Delta_O]}$ by Proposition 3. Thus $|\overline{\Omega}| < \infty$ implies $|\Delta_O| < \infty$ and Ω would be of finite cardinality. So infinitely many blocks occur and the kernel of the Frobenius algebraic group $G/G_{[\overline{\Omega}]}$ is a 1-dimensional connected unipotent group ([7], Theorems 1.8 and 1.10) with a 1-dimensional connected torus as the complement $G_{\Delta_O}/G_{[\overline{\Omega}]} = G_{[\overline{\Omega}]}G_O/G_{[\overline{\Omega}]} \simeq G_O/G_{[\overline{\Omega}]_O}$ ([8], Proposition 1).

Finally the non-trivial factor group $G_O/G_{[\Delta_O]}$, as a continuous epimorphic image of $G_O/G_{[\overline{\Omega}]_O}$, must be a 1-dimensional connected torus, as well. So G_O must split over the unipotent group $G_{[\overline{\Omega}]_O}$ by a 1-dimensional connected torus T. □

5 . Proposition: *G is a solvable connected affine group of dimension* 4 *and* G *is the semidirect product of its unipotent radical* G_u *by the torus* T*. Moreover the centre* $\mathfrak{z}(G_u)$ *of* G_u *is contained in* $G_{[\overline{\Omega}]}$ *and for any* $X \in \Omega$ *we have* $G_{[\overline{\Omega}]} = \mathfrak{z}(G_u) \times G_{[\overline{\Omega}]_X}$.

Proof : As $G_{[\overline{\Omega}]}$ is a 2-dimensional connected unipotent group by Propositions 4.*d* and 3 and $G/G_{[\overline{\Omega}]}$ is a connected solvable 2-dimensional group by Proposition 4.*c*, the unipotent radical G_u has codimension 1 and acts transitively on Ω. We have $\mathfrak{z}(G_u) < G_{[\overline{\Omega}]}$ since $\mathfrak{z}(G_u)$ centralizes each $G_{[\overline{\Omega}]_X}$. Finally $\mathfrak{z}(G_u)$ is transitive on every block Δ, hence sharply transitive, the group $G_\Delta/G_{[\Delta]}$ being primitive. □

6 . *Remark* : If we denote by g_u and g_s the images of $g \in G$ under the projections $G_u \times T \to G_u$ and $G_u \times T \to T$, respectively, the mapping $\pi : G \to G_u/G_{[\overline{\Omega}]_O}$ with $\pi(g) = g_u G_{[\overline{\Omega}]_O}$ turns out to be a separable morphism of algebraic varieties. The fibres of π are precisely the cosets gG_O, so $gG_O \mapsto g_u G_{[\overline{\Omega}]_O}$ yields an isomorphism $G/G_O \to G_u/G_{[\overline{\Omega}]_O}$. So we may take the homogeneous space $G_u/G_{[\overline{\Omega}]_O}$ as Ω and

$$\left(g, hG_{[\overline{\Omega}]_O}\right) \mapsto ghg_s^{-1}G_{[\overline{\Omega}]_O} \quad (g \in G, h \in G_u)$$

as the action of G on Ω since $(g_1g_2)_u = (g_1)_u(g_1)_s(g_2)_u(g_1)_s^{-1}$. In particular $\Omega \simeq G_u/G_{[\overline{\Omega}]_O}$ is a 2-dimensional (irreducible affine) variety with

$$\overline{\Omega} = \bigcup_{g \in G_u} \Delta_{g(O)} \simeq \bigcup_{g \in G_u} g\mathfrak{z}(G_u)G_{[\overline{\Omega}]_O}.$$

§**2**. Let $G = U \rtimes T$ be a semidirect product of an n-dimensional connected unipotent group U by a 1-dimensional connected torus T. According to Serre [14], p. 172, the group U has a representation on the affine space k^n in such a way the subspaces

$$U_i = \left\{(x_1, \ldots, x_n) \in \mathsf{k}^n : x_{i+1} = \ldots = x_n = 0)\right\}$$

are normal subgroups of G, the product is given by $(x_1, \ldots, x_n)(y_1, \ldots, y_n) =$

$$\big(x_1 + y_1 + \psi_1(x_2, \ldots, x_n, y_2, \ldots, y_n), \ldots, x_{n-1} + y_{n-1} + \psi_{n-1}(x_n, y_n), x_n + y_n\big),$$

for suitable polynomials $\psi_j \in \mathsf{k}[x_j, \ldots, x_n, y_{j+1}, \ldots, y_n]$, and the automorphism of U induced by an element $\tau \in T$ maps $(x_1, \ldots, x_n)$ to

$$\left(a_\tau^{e_1} x_1 + \varphi_1^{(\tau)}(x_2, \ldots, x_n), \ldots, a_\tau^{e_{n-1}} x_{n-1} + \varphi_{n-1}^{(\tau)}(x_n), a_\tau^{e_n} x_n\right)$$

with $a_\tau \in \mathsf{k}^*$, an element depending bi-regularly on τ, the map $\varphi_j^{(\tau)}$ a morphism $U_n/U_j \to U_j/U_{j-1}$ and e_j a fixed integer.

7 . Lemma: *Let $n \geq 2$. Then for any $\tau \in T$ the morphism $\varphi_{n-1}^{(\tau)}$ yields a group homomorphism $U_n/U_{n-1} \to U_{n-1}/U_{n-2}$. Moreover we may take as ψ_{n-1}*

a) the zero polynomial, *if U_n/U_{n-2} is a vector group,*

b) $\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x_n^{ip^r} y_n^{(p-i)p^r}$, *if U_n/U_{n-2} is an Abelian group of exponent p^2,*

c) $x_n^{p^r} y_n^{p^s}$, *if U_n/U_{n-2} is not commutative,*

where, in the cases b) and c), $p = \operatorname{char} \mathsf{k} > 0$, r and s are nonnegative integers such that $r < s$ and $e_{n-1} = e_n \deg(\psi_{n-1})$.

Proof : We may take as $\psi_{n-1}(x_n, y_n)$ (see for instance Lemma 7.1 in [6])

- 0, if U_n/U_{n-2} is a vector group,
- $b\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x_n^{ip^r} y_n^{(p-i)p^r}$, if U_n/U_{n-2} is Abelian but not a vector group,
- $bx_n^{p^r} y_n^{p^s}$, if U_n/U_{n-2} is not commutative,

for some non-negative integers r, s with $r < s$ and a non-zero scalar b, that may assumed 1 thanks to the isomorphism

$$(\ldots, x_{n-1}, x_n)U_{n-2} \mapsto (\ldots, bx_{n-1}, x_n)U_{n-2}.$$

Now the fact that τ operates on U_n as an automorphism group implies that the co-boundary

$$\delta^1\left(\varphi_{n-1}^{(\tau)}\right)(x_n, y_n) = \varphi_{n-1}^{(\tau)}(y_n) - \varphi_{n-1}^{(\tau)}(x_n + y_n) + \varphi_{n-1}^{(\tau)}(x_n)$$

is one of the following

- 0, if U_n/U_{n-2} is a vector group;
- $\left(a_\tau^{e_{n-1}} - a_\tau^{e_n \deg \psi_{n-1}}\right) \psi_{n-1}(x_n, y_n)$, otherwise.

In the latter case the fact that ψ_{n-1} is not a co-boundary forces each a_τ to be a root of the polynomial $\mathrm{T}^{e_n \deg(\psi_{n-1})} - \mathrm{T}^{e_{n-1}}$ and this forces the condition $e_{n-1} = e_n \deg(\psi_{n-1})$. As a consequence $\delta^1\left(\varphi_{n-1}^{(\tau)}\right)$ must be in any case the zero polynomial, which means that $\varphi_{n-1}^{(\tau)}$ yields a group homomorphism $U_n/U_{n-1} \to U_{n-1}/U_{n-2}$. $\square$

8 . *Remark* : It follows from [3] that the action of a 1-dimensional torus on a 2-dimensional connected unipotent group U may be given by diagonal $(2\times 2)-$matrices with entries in k. The following lemma, which generalizes both the lemma on p. 109 in [12] and Corollary 2.9 in [7], shows that this can be done without destroying the group structure of U.

9 . Lemma: *Let $\varphi_2^{(\tau)} = \ldots = \varphi_{n-1}^{(\tau)} = 0$ and assume $\varphi_1^{(\tau)}$ is a group homomorphism $U_n/U_{n-1} \to U_1$. Then there exists a bi-regular section $\sigma : U_n/U_{n-1} \to U_n$ such that $\sigma\left(x_n U_{n-1}\right) = \left(f(x_n), 0, \ldots, 0, x_n\right)$ with $\delta^1(f) = 0$ and $\sigma\left(U_n/U_{n-1}\right)$ invariant under T.*

Proof : We may suppose $\varphi_1^{(\tau)} \in \mathsf{k}[x_n]$ with $\varphi_1^{(\tau)}(x_n) = \sum_{i\in I,\, j\in J} c_{ij} a_\tau^j x_n^i$ for some finite sets I and J of integers with

$$I = \begin{cases} \{1\}, & \text{if } \operatorname{char} \mathsf{k} = 0; \\ \text{a finite set of } p\text{-powers}, & \text{if } \operatorname{char} \mathsf{k} = p > 0. \end{cases}$$

The product $\tau_1\tau_2$ of two elements of T gives

$$\varphi_1^{(\tau_1\tau_2)}(x_n) = a_{\tau_1}^{e_1}\varphi_1^{(\tau_2)}(x_n) + \varphi_1^{(\tau_1)}\big(a_{\tau_2}^{e_n} x_n\big),$$

hence for each $i \in I$

$$\sum_{j\in J} c_{ij} a_{\tau_1}^j a_{\tau_2}^j = \sum_{j\in J} c_{ij}\left(a_{\tau_1}^{e_1} a_{\tau_2}^j + a_{\tau_1}^j a_{\tau_2}^{ie_n}\right).$$

By comparing we infer that just c_{i,e_1} and c_{i,ie_n} can occur as nonzero entries. So

$$c_{i,e_1} a_{\tau_1}^{e_1} a_{\tau_2}^{e_1} + c_{i,ie_n} a_{\tau_1}^{ie_n} a_{\tau_2}^{ie_n} = c_{i,e_1}\big(a_{\tau_1}^{e_1} a_{\tau_2}^{e_1} + a_{\tau_1}^{e_1} a_{\tau_2}^{ie_n}\big) + c_{i,ie_n}\big(a_{\tau_1}^{e_1} a_{\tau_2}^{ie_n} + a_{\tau_1}^{ie_n} a_{\tau_2}^{ie_n}\big),$$

or $c_{i,e_1} + c_{i,ie_n} = 0$. Therefore $\varphi_1^{(\tau)}(x_n) = \sum_{i\in I} c_{i,e_1}(a_\tau^{e_1} - a_\tau^{ie_n})x_n^i$ and

$$\big\{\big(-\textstyle\sum_{i\in I} c_{i,e_1} x_n^i, 0, \ldots, 0, x_n\big) : x_n \in \mathsf{k}\big\}$$

turns out to be T-invariant with $\delta^1 : \sum_{i\in I} c_{i,e_1} \mathrm{T}^i \mapsto 0$. □

Set $M := \big\{(0, \ldots, 0, x_n) : x_n \in \mathsf{k}\big\}$ and let $\upsilon = (0, \ldots, 0, u) \in M$. We have

10 . Lemma: *Let $n \geq 3$. Assume the centralizer $\mathfrak{C}_{U_{n-1}}(\upsilon)$ of υ in U_{n-1} satisfies the condition $\mathfrak{C}_{U_{n-1}}(\upsilon) = U_{n-2} \mod U_{n-3}$ for all $\upsilon \in M$. Then the automorphism ρ_υ of U_{n-1}/U_{n-3} induced by conjugation by υ maps*

$$(\ldots, x_{n-2}, x_{n-1}, 0)U_{n-3} \longmapsto \big(\ldots, x_{n-2} + u^h x_{n-1}^k, x_{n-1}, 0\big)U_{n-3}$$

with h and k p-powers if $\operatorname{char} \mathsf{k} = p > 0$, *$h = k = 1$ otherwise.*

Proof : As $U_{n-1}/U_{n-2} \leq \mathfrak{z}(U_n/U_{n-2})$, we have

$$\rho_\upsilon : (\ldots, x_{n-2}, x_{n-1}, 0)U_{n-3} \longmapsto \big(\ldots, x_{n-2} + \sigma(u, x_{n-1}), x_{n-1}, 0\big)U_{n-3}$$

for some additive polynomial $\sigma \in \mathsf{k}[u, x_{n-1}]$, which turns out to be monomial because $\mathfrak{C}_{U_{n-1}}(\upsilon) = U_{n-2} \bmod U_{n-3}$ forces ρ_υ to act fixed-point freely on U_{n-2}/U_{n-3}. Thus $\sigma(u, x_{n-1}) = cu^h x_{n-1}^k$ for some integers h, k and scalar $c \in \mathsf{k}^*$ that we may assume 1, up to the isomorphism $(\ldots x_{n-2}, x_{n-1}, x_n)U_{n-3} \mapsto \ldots x_{n-2}, c^{-\frac{1}{k}} x_{n-1}, x_n)U_{n-3}$. Clearly the integers h and k have to satisfy the claimed conditions. □

In the remaining part of the paper we ask the torus T to act sharply transitively on U_n/U_{n-1}. This means

$$e_n = \begin{cases} 1, & \text{if } \operatorname{char} \mathsf{k} = 0; \\ \text{a } p\text{-power}, & \text{if } \operatorname{char} \mathsf{k} = p > 0. \end{cases} \tag{1}$$

§**3**. Now we go back to the the $(2,2)$-imprimitive algebraic group $\mathsf{G} = (G, \Omega, \overline{\Omega})$. This section is devoted to the case where the 2-dimensional factor group $G_u/\mathfrak{z}(G_u)$ is *commutative.*

11 . Proposition: *$G_u/\mathfrak{z}(G_u)$ is a vector group.*

Proof : Assume it is not, then $\operatorname{char}\mathsf{k} \neq 0$ and $G_{[\,\overline{\Omega}\,]}/\mathfrak{z}(G_u)$ coincides with the unique 1-dimensional connected algebraic subgroup of $G_u/\mathfrak{z}(G_u)$. Consequently $G_{[\,\overline{\Omega}\,]}$ is the unique 2-dimensional connected algebraic normal subgroup of G_u containing $\mathfrak{z}(G_u)$. Furthermore $G_u/\mathfrak{z}(G_u)$ commutative and $\dim\mathfrak{z}(G_u) = 1$ require that $\mathfrak{z}(G_u)$ is the commutator subgroup of G_u, hence that each commutator morphism $\sigma_g : x \mapsto [g, x]$ is a group homomorphism $G_u \to \mathfrak{z}(G_u)$, whose kernel must have dimension ≥ 2. So $\ker\sigma_g \geq G_{[\,\overline{\Omega}\,]}$ for any $g \in G_u$, a contradiction since $\bigcap_{g\in G_u}\ker\sigma_g = \mathfrak{z}(G_u)$. $\square$

12 . Proposition: *There exists a T-invariant normal subgroup L of G_u containing the centre $\mathfrak{z}(G_u)$ and $G_u = L \rtimes G_{[\,\overline{\Omega}\,]_O}$.*

Proof : By [12] (Lemma on p. 109) the T-invariant subgroup $G_{[\,\overline{\Omega}\,]}/\mathfrak{z}(G_u)$ of $G_u/\mathfrak{z}(G_u)$ has a T-invariant complement, say $L/\mathfrak{z}(G_u)$ for some T-invariant normal subgroup L of G_u containing $\mathfrak{z}(G_u)$. $\square$

According to the notation of §2 we may take $U_1 = \mathfrak{z}(G_u)$, $U_2 = G_{[\,\overline{\Omega}\,]}$, $U_3 = G_u$. In addition we may choose

$$G_{[\,\overline{\Omega}\,]_O} = \big\{(0, x_2, 0) : x_2 \in \mathsf{k}\big\},$$

the subgroup $G_{[\,\overline{\Omega}\,]_O}$ being T-invariant. Observing that the normal subgroup L of G is not contained in U_2, we may also put

$$L = \big\{(x_1, 0, x_3) : x_1, x_3 \in \mathsf{k}\big\}.$$

Thus the product $(x_1, 0, x_3)(y_1, 0, y_3)$ of two elements of L is given by

$$\big(x_1 + y_1 + \beta(x_3, y_3), 0, x_3 + y_3\big)$$

and by Lemma 7 we may take

$$\beta(x_3, y_3) = \begin{cases} 0, & \text{if } L \text{ is a vector group,} \\ \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x_3^{ip^r} y_3^{(p-i)p^r}, & \text{if } L \text{ is commutative of exponent } p^2, \\ x_3^{p^r} y_3^{p^s}, & \text{if } L \text{ is not commutative,} \end{cases} \tag{2}$$

for some nonnegative integers r, s with $r < s$. Besides an element $\upsilon = (0, 0, u) \in L$ moves the block Δ_O to a different block $\Delta_{\upsilon(O)}$ (Remark 6), so υ centralizes no element in $G_{[\,\overline{\Omega}\,]_O}$, the intersection $G_{[\,\overline{\Omega}\,]_O} \cap G_{[\,\overline{\Omega}\,]_{\ell(O)}}$ being trivial. Then Lemma 10 applies and, up to the isomorphism $(x_1, x_2, x_3) \mapsto (x_1, c^{\frac{1}{h_2}} x_2, x_3)$, we may claim

13 . Proposition: *The product $(x_1, x_2, x_3)(y_1, y_2, y_3)$ in G_u may be defined through*

$$\Big(x_1 + y_1 + y_2^{h_2} x_3^{h_3} + \beta(x_3, y_3),\ x_2 + y_2,\ x_3 + y_3\Big),$$

where β is given by (2) and each exponent h_i is a p-power in case $\operatorname{char}\mathsf{k} = p > 0$, $h_i = 1$ otherwise. $\square$

As we observed in Remark 8, there is no loss of generality if we assume the action of the torus T on the affine plane L given by diagonal $(2 \times 2)-$matrices. But $G_{[\,\overline{\Omega}\,]_O}$ occurs as a further T-invariant subgroup of dimension 1, so the diagonal action of each $\tau \in T$ extends to the whole group G_u via

$$(x_1, x_2, x_3) \mapsto \big(a_\tau^{e_1} x_1,\ a_\tau^{e_2} x_2,\ a_\tau^{e_3} x_3\big). \tag{3}$$

The value of the exponent e_3 was given by (1), whereas the possible relationship occurring between e_1 and e_3 was stated in Lemma 7. Now by imposing that τ is a group homomorphism we find

$$e_1 = e_2 h_2 + e_3 h_3 \tag{4}$$

with h_i arising from the product of G_u given in Proposition 13.

§**4**. Assume now the factor group $G_u/\mathfrak{z}(G_u)$ to be *not commutative*. This requires $\operatorname{char} \mathsf{k} = p > 0$ and we are going to see that even $p > 2$ holds.

Referring to the notation of §2 we may take again $U_3 = G_u$, $U_2 = G_{[\overline{\Omega}]}$, $U_1 = \mathfrak{z}(G_u)$ and

$$G_{[\overline{\Omega}]_O} = \{(0, x_2, 0) : x_2 \in \mathsf{k}\}.$$

Also, by Lemma 7,

$$\psi_2 : (x_3, y_3) \mapsto x_3^{p^m} y_3^{p^n},$$

for some integer p-powers p^m and p^n such that $m < n$. Furthermore, looking at Remark 6, we see that an element $\upsilon = (0, 0, x_3)$ moves the block Δ_O to a different block $\Delta_{\upsilon(O)}$. So υ does not centralize any element of $G_{[\overline{\Omega}]_O}$ because the intersection $G_{[\overline{\Omega}]_O} \cap G_{[\overline{\Omega}]_{\upsilon(O)}}$ is assumed to be trivial. So Lemma 10 applies and, up to an isomorphism, we may assume that the automorphism induced on $G_{[\overline{\Omega}]}$ by an element $(0, 0, x_3)$ maps

$$(y_1, y_2, 0) \mapsto (y_1 + y_2^{h_2} x_3^{h_3}, y_2, 0)$$

for suitable integer p-powers $h_i = p^{l_i}$, $i = 2, 3$. If we represent G_u as a non-central extension of the vector group $G_{[\overline{\Omega}]}$ by $G_u/G_{[\overline{\Omega}]}$ using the cross section $(x_1, x_2, x_3)G_{[\overline{\Omega}]} \mapsto (0, 0, x_3)$, the product $(x_1, x_2, x_3)(y_1, y_2, y_3)$ of two elements in G_u can also be given by

$$(x_1 + y_1 + y_2^{h_2} x_3^{h_3} + \beta(x_3, y_3), x_2 + y_2 + x_3^{p^m} y_3^{p^n}, x_3 + y_3)$$

with β in $\mathsf{k}[x_3, y_3]$ such that $\beta(0, y_3) = \beta(x_3, 0) = 0$ and G_u is determined by taking

$$\psi_1(x_1, x_2, y_1, y_2) = y_2^{h_2} x_3^{h_3} + \beta(x_3, y_3).$$

Now associative law forces the polynomial

$$\delta^2(\beta)(z_1, z_2, z_3) = \beta(z_1, z_2) + \beta(z_1 + z_2, z_3) - \beta(z_2, z_3) - \beta(z_1, z_2 + z_3)$$

to be

$$\delta^2(\beta)(z_1, z_2, z_3) = z_1^{p^{l_3}} z_2^{p^{l_2+m}} z_3^{p^{l_2+n}} \tag{5}$$

and we can state

14 . **Proposition:** *A necessary and sufficient condition in order that G_u can be constructed as an extension of $\mathfrak{z}(G_u)$ by a non-commutative connected unipotent group is that there exists a polynomial $\beta \in \mathsf{k}[x_3, y_3]$ satisfying* (5) *with $\beta(0, y_3) = \beta(x_3, 0) =$ 0. In such a case we may take $\psi_1(x_2, x_3, y_2, y_3) = y_2^{p^{l_2}} x_3^{p^{l_3}} + \beta(x_3, y_3)$.* $\square$

The crucial question now is under what conditions such a polynomial β there exists. Using a universal property of the operator δ^2 we have

$$\sum_{\pi \in \mathsf{S}_n} \operatorname{sign}(\pi)\, \delta^2(\beta)(z_{\pi(1)}, z_{\pi(2)}, z_{\pi(3)}) = 0$$

and this, in view of (5), is equivalent to

$$l_3 - l_2 = m, \text{ or } l_3 - l_2 = n. \tag{6}$$

Assume now char $\mathsf{k} = 2$ and $l_2 + m > 0$ and denote by β_j the homogeneous component of β of degree j. As in our case the operator δ^2 is additive, (5) says that $\delta^2(\beta) = \delta^2(\beta_k)$, where $k = 2^{l_2}(2^q + 2^m + 2^n)$ with either $q = m$, or $q = n$ according as whether $l_3 - l_2 = m$, or $l_3 - l_2 = n$. Let

$$\beta_k(y_1, y_2) = \textstyle\sum_{i=0}^{k} a_i y_1^{k-i} y_2^i.$$

Then (5) becomes

$$\textstyle\sum_{i=0}^{k} a_i \big(z_1^{k-i} z_2^i + (z_1 + z_2)^{k-i} z_3^i + z_2^{k-i} z_3^i + z_1^{k-i}(z_2 + z_3)^i\big) = z_1^{2^{l_2+q}} z_2^{2^{l_2+m}} z_3^{2^{l_2+n}}.$$

Deriving this identity with respect to z_1 and evaluating at $(0, y_1, y_2)$ we obtain

$$a_{k-1} y_1^{k-1} + \frac{\partial}{\partial y_1} \beta_k(y_1, y_2) + a_{k-1}(y_1 + y_2)^{k-1} = 0, \tag{7}$$

whereas deriving with respect to z_3 and evaluating at $(y_1, y_2, 0)$ we get

$$a_1(y_1 + y_2)^{k-1} + a_1 y_2^{k-1} + \frac{\partial}{\partial y_2} \beta_k(y_1, y_2) = 0. \tag{8}$$

As char $\mathsf{k} = 2$, $\frac{\partial}{\partial y_1}\beta_k(y_1, y_2)$ and $\frac{\partial}{\partial y_2}\beta_k(y_1, y_2)$ are polynomials in y_1^2 and y_2^2, respectively, the identities (7) and (8) force $a_{k-1} = a_1 = 0$, hence

$$\frac{\partial}{\partial y_1} \beta(y_1, y_2) = \frac{\partial}{\partial y_2} \beta(y_1, y_2) = 0$$

and this yields $a_i = 0$ for all odd i. Thus we may do the substitution $(z_1, z_2, z_3) \mapsto (z_1^2, z_2^2, z_3^2)$, hence $(z_1, z_2, z_3) \mapsto (z_1^{2^{l_2+m}}, z_2^{2^{l_2+m}}, z_3^{2^{l_2+m}})$ by iterating the process. So (5) turns into

$$\delta^2(\gamma)(z_1, z_2, z_3) = z_1^{2^{q-m}} z_2 z_3^{2^{n-m}} \tag{9}$$

with $\gamma\big(y_1^{2^{l_2+m}}, y_2^{2^{l_2+m}}\big) = \beta(y_1, y_2)$. Let γ_t be the homogeneous component of degree $t := 1 + 2^{n-m} + 2^{q-m}$ of γ and let $\gamma_t(y_1, y_2) = \sum_{i=0}^{t} b_i y_1^{t-i} y_2^i$. Then (9) says that $\delta^2(\gamma) = \delta^2(\gamma_t)$, hence

$$\sum_{i=0}^{t} b_i \Big(z_1^{t-i} z_2^i + (z_1 + z_2)^{t-i} z_3^i + z_2^{t-i} z_3^i + z_1^{t-i}(z_2 + z_3)^i\Big) = z_1^{2^{q-m}} z_2 z_3^{2^{n-m}}.$$

Likewise above we obtain

$$\begin{cases} b_{t-1} y_1^{t-1} + \frac{\partial}{\partial y_1} \gamma_t(y_1, y_2) + b_{t-1}(y_1 + y_2)^{t-1} = (1 - \epsilon) y_1 y_2^{2^{n-m}}, \\ b_1(y_1 + y_2)^{t-1} + b_1 y_2^{t-1} + \frac{\partial}{\partial y_2} \gamma_t(y_1, y_2) = 0, \end{cases}$$

where either $\epsilon = 0$, or $\epsilon = 1$ according as whether $q = m$, or $q = n$. So Euler's identity says that $t\gamma_t(y_1, y_2)$ is the polynomial

$$b_{t-1} y_1 \Big(y_1^{t-1} + (y_1 + y_2)^{t-1} + (1 - \epsilon) y_1 y_2^{2^{n-m}}\Big) + b_1 y_2 \Big((y_1 + y_2)^{t-1} + y_2^{t-1}\Big)$$

or the polynomial

$$b_{t-1}\left(y_1^{1+2^{n-m}}y_2^{2^{q-m}}+y_1^{1+2^{q-m}}y_2^{2^{n-m}}+y_1y_2^{2^{n-m}+2^{q-m}}+(1-\epsilon)\,y_1^2y_2^{2^{n-m}}\right)+$$
$$+b_1\left(y_1^{2^{n-m}+2^{q-m}}y_2+\ y_1^{2^{n-m}}y_2^{1+2^{q-m}}+y_1^{2^{q-m}}y_2^{1+2^{n-m}}\right).$$

Let $q=m$. Then we have the polynomial identity

$$\big(b_{t-1}+b_1\big)\left(y_1^{1+2^{n-m}}y_2+y_1y_2^{1+2^{n-m}}\right)+b_1y_1^{2^{n-m}}y_2^2=0$$

which asks $b_{t-1}=b_1=0$ and, consequently, $\frac{\partial}{\partial y_1}\gamma_t(y_1,y_2)=y_1y_2^{2^{n-m}}$, a contradiction. Let $q=n$. Then

$$\gamma_t(y_1,y_2)=b_{t-1}y_1y_2^{2^{n-m+1}}+b_1y_1^{2^{n-m+1}}y_2$$

and $\delta^2(\gamma_t)(x_1,x_2,x_3)=0$. This contradicts (9) and $p\neq 2$ follows.

Actually, if $p\neq 2$ the polynomials

$$\beta(x_3,y_3)=\begin{cases}\frac{1}{2}x_3^{2p^{l_3}}y_3^{p^{l_2+n}} & \text{if } l_3-l_2=m;\\ x_3^{p^{l_3}+p^{l_2+m}}y_3^{p^{l_3}}+\frac{1}{2}x_3^{p^{l_2+m}}y_3^{2p^{l_3}} & \text{if } l_3-l_2=n.\end{cases}\tag{10}$$

satisfy the conditions required in Proposition 14. Any other polynomial satisfying the conditions of Proposition 14 differs from (10) for a co-cycle $\kappa(x_3,y_3)$ for a central extension of k_+ by k_+ that we are going to show it is a co-boundary.

It follows from Remark 8 that we may assume any element $\tau\in T$ acts on G_u via

$$(x_1,x_2,x_3)\mapsto\big(a_\tau^{e_1}x_1+\varphi_1^{(\tau)}(x_3),a_\tau^{e_2}x_2,a_\tau^{e_3}x_3\big),$$

with the morphism $\varphi_1^{(\tau)}$ depending only on x_3 because $G_{[\overline{\Omega}]_O}$ is T-invariant. By imposing that τ operates as a group homomorphism we obtain first

$$e_2=e_3(p^m+p^n)\text{ and }e_1=e_2h_2+e_3h_3=e_3\big(p^{l_3}+p^{l_2+m}+p^{l_2+n}\big),\tag{11}$$

but also

$$a_\tau^{e_1}\beta(x_3,y_3)-\beta\big(a_\tau^{e_3}x_3,a_\tau^{e_3}y_3\big)+a_\tau^{e_1}\kappa(x_3,y_3)-\kappa\big(a_\tau^{e_3}x_3,a_\tau^{e_3}y_3\big)=\delta^1\big(\varphi_1^{(\tau)}\big)(x_3,y_3),$$

or

$$a_\tau^{e_1}\kappa(x_3,y_3)-\kappa\big(a_\tau^{e_3}x_3,a_\tau^{e_3}y_3\big)=\delta^1\big(\varphi_1^{(\tau)}\big)(x_3,y_3),\tag{12}$$

because $e_1=e_3\deg\beta$ in view of (11). Since e_3 is a p-power and $p>2$, the integer e_1 can be, by (11), neither a p-power, nor the sum of two p-powers. Thus Theorem 4.6 in [4] guaranties that κ is a co-boundary, i.e. $\kappa=\delta^1(g)$ for some polynomial $g\in\mathsf{k}[\mathrm{T}]$, that may be eliminated using the substitution $x_1\mapsto x_1-g(x_3)$. Such a replacement yields $\delta^1\big(\varphi_1^{(\tau)}\big)(x_3,y_3)=0$, i.e. $\varphi_1^{(\tau)}$ is additive, and we may assume the action of T given by diagonal matrices, as Lemma 9 claims.

§**5.** Now we collect all information achieved in the previous sections and classify G according to the structure of the transversal L. With the aid of Remark 6 we can state:

15 . Main Theorem: *Every $(2,2)$-imprimitive algebraic group $\mathsf{G} = (G, \Omega, \overline{\Omega})$ can be constructed on the affine variety $\mathsf{k}^3 \times \mathsf{k}^*$ as follows:*

- *define the unipotent radical G_u on the affine space k^3 through the product*

  $$(x_1, x_2, x_3)(y_1, y_2, y_3) = \big(x_1 + y_1 + \psi_1(x_3, y_2, y_3), x_2 + y_2 + \psi_2(x_3, y_3), x_3 + y_3\big),$$

  where either

  $\psi_2(x_3, y_3) = 0$ *and* $\psi_1(x_3, y_2, y_3) = y_2^{h_2} x_3^{h_3} + \beta(x_3, y_3)$ *with each* h_i *an integer* p*-power* p^{l_i} *in case* $\operatorname{char} \mathsf{k} = p > 0$, $h_i = 1$ *otherwise, and* $\beta(x_3, y_3)$ *one of the polynomials*
  - 0;
  - $\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x_3^{ip^r} y_3^{(p-i)p^r}$;
  - $x_3^{p^r} y_3^{p^s}$;

  for suitable nonnegative integers r, s *such that* $r < s$,

  or

  $\psi_2(x_3, y_3) = x_3^{p^m} y_3^{p^n}$, *with* $p = \operatorname{char} \mathsf{k} > 2$ *and* m, n *non-negative integers such that* $m < n$, *and* $\psi_1(x_3, y_2, y_3)$ *as above with*

  $$\beta(x_3, y_3) = \begin{cases} \frac{1}{2} x_3^{2p^{l_3}} y_3^{p^{l_2+n}}, & \text{if } l_3 - l_2 = m; \\ x_3^{p^{l_3}+p^{l_2+m}} y_3^{p^{l_3}} + \frac{1}{2} x_3^{p^{l_2+m}} y_3^{2p^{l_3}}, & \text{if } l_3 - l_2 = n; \end{cases}$$

- *leave $a \in \mathsf{k}^*$ operate on k^3 via*

  $$(u_1, u_2, u_3) \mapsto \big(a^{e_1} u_1, a^{e_2} u_2, a^{e_3} u_3\big)$$

  where
  - $e_1 = e_2 h_2 + e_3 h_3$, *but also* $e_1 = e_3 \deg \beta$ *if β is not the zero polynomial;*
  - $e_2 = e_3 \frac{\deg \beta - h_3}{h_2}$ *if β is not the zero polynomial;*
  - e_3 *is a positive integer* p*-power in case* $\operatorname{char} \mathsf{k} = p > 0$, $e_3 = 1$ *otherwise;*

- *identify Ω with the affine plane k^2 with the parallel lines $y = k$ giving the set $\overline{\Omega}$ of blocks. Then a transformation $(u_1, u_2, u_3, a) \in G$ moves the point $(x, y) \in \Omega$ to the point*

  $$\big(u_1 + a^{e_2 h_2 + e_3 h_3} x + \psi_1(u_3, 0, a^{e_3} y),\ u_3 + a^{e_3} y)\big) \qquad \square$$

The canonical representation of G given through the main theorem depends on the polynomial β as well as on the integer parameters e_2, e_3, h_2, h_3, though h_2 and h_3 could already be determined by β, e_2 and e_3. Labelling G as $\mathsf{G}_\beta^{(e_2,\, e_3,\, h_2,\, h_3)}$, we ask whether an isomorphism

$$\Phi : \mathsf{G}_\beta^{(e_2,\, e_3,\, h_2,\, h_3)} \longrightarrow \mathsf{G}_{\beta'}^{\left(e_2',\, e_3',\, h_2',\, h_3'\right)}$$

between two $(2,2)$-imprimitive algebraic groups with different parameters exists. Of course we may assume the same sets of points and blocks for both groups, so Φ is a pair (Φ_1, Φ_2) with Φ_1 a group isomorphism $G_\beta^{(e_2,\, e_3,\, h_2,\, h_3)} \longrightarrow G_{\beta'}^{\left(e'_2,\, e'_3, h'_2,\, h'_3\right)}$ and $\Phi_2 : \mathsf{k}^2 \to \mathsf{k}^2$ a bijective morphism of the affine plane k^2 transforming horizontal lines into horizontal lines such that

$$\Phi_2\big(g(P)\big) = \Phi_1(g)\big(\Phi_2(P)\big) \quad \big(g \in G_\beta^{(e_2,\, e_3,\, h_2,\, h_3)}, P \in \mathsf{k}^2\big).$$

As G_u is transitive on Ω, up to inner automorphisms we may assume that Φ_2 leaves the point $O = (0, 0)$ of Ω fixed, hence the line $y = 0$ stable. Then the stabilizer of O, as well as the normalizer and centralizers of Δ_O correspond; in particular

$$\begin{aligned} \Phi_1\big((0, u_2, 0)\big) &= (0, b_2u_2, 0) \quad (b_2 \in \mathsf{k}^*), \\ \Phi_1\big((u_1, 0, 0)\big) &= (b_1u_1, 0, 0) \quad (b_1 \in \mathsf{k}^*), \end{aligned} \tag{13}$$

and, moreover,

$$\begin{aligned} \Phi_1\big((0, 0, u_3)\big) &= \big(f_1(u_3), f_2(u_3), b_3u_3)\big) \quad (b_3 \in \mathsf{k}^*), \\ \Phi_2\big((x, y)\big) &= \big(b_1x + f_1(y),\, b_3y\big), \end{aligned} \tag{14}$$

for suitable polynomials $f_j \in \mathsf{k}[\mathrm{T}]$ such that

$$\begin{aligned} \delta^1(f_2)(x_3, y_3) &= b_2\psi_2(x_3, y_3) - \psi'_2(b_3x_3, b_3y_3) & (x_3, y_3 \in \mathsf{k}), \\ \delta^1(f_1)(x_3, y_3) &= b_1\psi_1(x_3, 0, y_3) - \psi'_1\big(b_3x_3, f_2(y_3), b_3y_3\big) & (x_3, y_3 \in \mathsf{k}). \end{aligned} \tag{15}$$

Manifestly tori fixing the point O correspond under Φ_1; in particular we have $\Phi_1\big(T_\beta^{(e_2,\, e_3,\, h_2,\, h_3)}\big) = T_{\beta'}^{\left(e'_2,\, e'_3, h'_2,\, h'_3\right)}$ since tori are conjugated under G_u. This means

$$(u_1, u_2, u_3)^{\Phi_1(\tau)} = \big(a_\tau^{\varepsilon e'_1}u_1, a_\tau^{\varepsilon e'_2}u_2, a_\tau^{\varepsilon e'_3}u_3\big),$$

with $\varepsilon = \pm 1$. The identity $\Phi_1\big((0, 0, u_3)^\tau\big) = \big(\Phi_1(0, 0, u_3)\big)^{\Phi_1(\tau)}$ and the first part of (14) yield $\varepsilon = 1$, $e_3 = e'_3$ and $f_j\big(a_\tau^{e_3}u_3\big) = a_\tau^{e'_j}f_j(u_3)$, $j = 1, 2$, whereas $\Phi_1\big((u_1, u_2, 0)^\tau\big) = \big(\Phi_1(u_1, u_2, 0)\big)^{\Phi_1(\tau)}$ and (13) give $e_1 = e'_1$ and $e_2 = e'_2$. So the polynomials f_j must be monomials and consequently, in case $f_j \neq 0$,

$$e_j = e_3 \deg(f_j) \quad (j = 1, 2). \tag{16}$$

Therefore $f_j(\mathrm{T}) = d_j\mathrm{T}^{\frac{e_j}{e_3}}$, $d_j \in \mathsf{k}$, $j = 1, 2$. Furthermore imposing the condition $\Phi_1(0, 0, u_3)\Phi_1(0, v_2, 0) = \Phi_1\big((0, 0, u_3)(0, v_2, 0)\big)$ we obtain $b_2^{h'_2}b_3^{h'_3}u_3^{h'_3}v_2^{h'_2} = b_1u_3^{h_3}v_2^{h_2}$, i.e. $(h'_2, h'_3) = (h_2, h_3)$ and

$$b_1 = b_2^{h_2}b_3^{h_3}. \tag{17}$$

So the first step is achieved:

16 . Proposition: *Let* $\mathsf{G}_{\beta'}^{\left(e'_2,\, e'_3, h'_2,\, h'_3\right)}$ *and* $\mathsf{G}_\beta^{(e_2,\, e_3,\, h_2,\, h_3)}$ *isomorphic as algebraic permutation groups. Then*

$$\big(e'_2, e'_3, h'_2, h'_3\big) = (e_2, e_3, h_2, h_3).$$ □

Theorem 4.6 in [4] says that the first of (15) occurs precisely if

$$\delta^1(f_2) = \psi_2 - \psi'_2 = 0. \tag{18}$$

Also the fact that $e_1 = e_3 \deg(\beta)$ if β is not the zero polynomial confines matters to examine the case where $\operatorname{char} \mathsf{k} = p > 0$, $\beta = 0$ and either $\beta'(x_3, y_3) = \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x_3^{ip^r} y_3^{(p-i)p^r}$, or $\beta'(x_3, y_3) = x_3^{p^r} y_3^{p^s}$: by (16) we have $\deg \beta' = \frac{e_1}{e_3} = \deg f_1$ in case $d_1 \neq 0$. Then the second identity of (15) turns into

$$\delta^1(f_1)(x_3, y_3) = -\beta'(b_3 x_3, b_3 y_3) - b_3^{h_3} x_3^{h_3} f_2(y_3)^{h_2} \tag{19}$$

and again Theorem 4.6 in [4] excludes the possibility that f_2 is the zero polynomial. Then f_2 is an additive monomial by (18) and (16) forces e_2 to be a p-power. Thus, in view of the main theorem, both e_1 and $\deg \beta'$, are the sum of two p-powers. So just the following two possibilities can occur: either $\beta'(x_3, y_3) = x_3^{p^r} y_3^{p^s}$, or $\operatorname{char} \mathsf{k} = 2$ and $\beta'(x_3, y_3) = x_3^{2^r} y_3^{2^r}$. Thus the main theorem gives either $e_2 h_2 + e_3 h_3 = e_3(p^r + p^s)$, or $e_2 h_2 + e_3 h_3 = e_3 2^{r+1}$, which means that the pair of p-powers (h_2, h_3) is one of the following

$$\begin{aligned} &1. \quad (h_2, h_3) = \left(\tfrac{e_3}{e_2} p^r, p^s\right); \\ &2. \quad (h_2, h_3) = \left(\tfrac{e_3}{e_2} p^s, p^r\right); \\ &3. \quad (h_2, h_3) = \left(\tfrac{e_3}{e_2} 2^r, 2^r\right). \end{aligned} \tag{20}$$

As the right side of (19) must be a co-boundary, (20.1) gives, (20.2) and (20.3), lead respectively to

$$\begin{aligned} &1. \quad d_1 = b_3^{p^r + p^s} = b_3^{p^s} d_2^{\frac{e_3}{e_2} p^r}, && \text{hence } d_2 = b_3^{\frac{e_2}{e_3}}; \\ &2. \quad d_1 = 0 \text{ and } b_3^{p^r + p^s} = -b_3^{p^r} d_2^{\frac{e_3}{e_2} p^s}, && \text{hence } d_2 = -b_3^{\frac{e_2}{e_3}}; \\ &3. \quad b_3^{2^{r+1}} = b_3^{2^r} d_2^{\frac{e_3}{e_2} 2^r}, && \text{hence } d_2 = b_3^{\frac{e_2}{e_3}}. \end{aligned}$$

Now it is straightforward calculation to verify that, for any $b_1, b_2, d_3 \in \mathsf{k}$, the maps

$$\begin{aligned} &1. \begin{cases} G_0^{\left(e_2,\, e_3,\, \frac{e_3}{e_2} p^r,\, p^s\right)} \to G_{x^{p^r} y^{p^s}}^{\left(e_2,\, e_3,\, \frac{e_3}{e_2} p^r,\, p^s\right)}, \\ (u_1, u_2, u_3, a) \mapsto \left(b_2^{\frac{e_3}{e_2} p^r} b_3^{p^s} u_1 + (b_3 u_3)^{p^r + p^s},\, b_2 u_2 + (b_3 u_3)^{\frac{e_2}{e_3}}, b_3 u_3,\, a\right); \end{cases} \\ &2. \begin{cases} G_0^{\left(e_2,\, e_3,\, \frac{e_3}{e_2} p^s,\, p^r\right)} \to G_{x^{p^r} y^{p^s}}^{\left(e_2,\, e_3,\, \frac{e_3}{e_2} p^s,\, p^r\right)}, \\ (u_1, u_2, u_3, a) \mapsto \left(b_2^{\frac{e_3}{e_2} p^s} b_3^{p^r} u_1,\, b_2 u_2 - (b_3 u_3)^{\frac{e_2}{e_3}},\, b_3 u_3,\, a\right); \end{cases} \\ &3. \begin{cases} G_0^{\left(e_2,\, e_3,\, \frac{e_3}{e_2} 2^r,\, 2^r\right)} \to G_{x^{2^r} y^{2^r}}^{\left(e_2,\, e_3,\, \frac{e_3}{e_2} 2^r,\, 2^r\right)}, \\ (u_1, u_2, u_3, a) \mapsto \left(b_2^{\frac{e_3}{e_2} 2^r} b_3^{2^r} u_1 + d_1 u_3^{2^{r+1}},\, b_2 u_2 + (b_3 u_3)^{\frac{e_2}{e_3}}, b_3 u_3,\, a\right); \end{cases} \end{aligned} \tag{21}$$

are group isomorphisms in correspondence to the values $(20.\,i)$ of the pair of p-powers (h_2, h_3). Manifestly such isomorphisms supply isomorphisms for the associated permutation groups. Summing up we have

17 . Theorem: *The integer parameters e_2, e_3, h_2, h_3 and the polynomial β determine uniquely the isomorphy class of the $(2,2)$-imprimitive algebraic group G, except the cases where the pair (h_2, h_3) takes one of the (integer) values $(20.\,i)$ which produces the corresponding isomorphisms $(21.\,i)$.* □

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